Journal of Combinatorial Theory, Series B 163 (2023) 83–111

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

journal homepage: www.elsevier.com/locate/jctb

Co-degree threshold for rainbow perfect matchings in uniform hypergraphs

Hongliang Lu [a,1], Yan Wang [b,2], Xingxing Yu [c,3]

[a] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China*
[b] *School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, China*
[c] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, United States of America*

A R T I C L E   I N F O

Article history:
Received 7 December 2021
Available online xxxx

Keywords:
Rainbow matching
Co-degree threshold
Absorbing method

A B S T R A C T

Let k and n be two integers, with $k \geq 3$, $n \equiv 0 \pmod{k}$, and n sufficiently large. We determine the $(k-1)$-degree threshold for the existence of a rainbow perfect matchings in n-vertex k-uniform hypergraph. This implies the result of Rödl, Ruciński, and Szemerédi on the $(k-1)$-degree threshold for the existence of perfect matchings in n-vertex k-uniform hypergraphs. In our proof, we identify the extremal configurations of closeness, and consider whether or not the hypergraph is close to the extremal configuration. In addition, we also develop a novel absorbing device and generalize the absorbing lemma of Rödl, Ruciński, and Szemerédi.

© 2023 Elsevier Inc. All rights reserved.

E-mail address: yan.w@sjtu.edu.cn (Y. Wang).

[1] Partially supported by the National Natural Science Foundation of China under grant No. 12271425.

[2] Partially supported by National Key R&D Program of China under Grant No. 2022YFA1006400, National Natural Science Foundation of China under grant No. 12201400 and Explore X project of Shanghai Jiao Tong University.

[3] Partially supported by NSF grant DMS-1954134.

https://doi.org/10.1016/j.jctb.2023.07.003
0095-8956/© 2023 Elsevier Inc. All rights reserved.

1. Introduction

A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. Let H_1 and H_2 be two hypergraphs. If $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$, then H_1 is said to be a *subgraph* of H_2 and we denote this by $H_1 \subseteq H_2$. Let k be a positive integer and write $[k] = \{1, \dots, k\}$. For a set S, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A hypergraph H is k-*uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and a k-uniform hypergraph is also called a k-*graph*. Given $T \subseteq V(H)$, let $H - T$ denote the subgraph of H with vertex set $V(H) \setminus T$ and edge set $\{e \in E(H) \ : \ e \subseteq V(H) \setminus T\}$.

Let H be a hypergraph and $S \subseteq V(H)$. The *neighborhood* of S in H is $N_H(S) = \{T \subseteq V(H) \setminus S : T \cup S \in E(H)\}$ and the *degree* of S in H is $d_H(S) = |N_H(S)|$. For any positive integer l, $\delta_l(H) := \min\{d_H(S) : S \in \binom{V(H)}{l}\}$ is the *minimum l-degree* of H. Note that $\delta_1(H)$ is called the minimum *vertex degree* of H. If H is a k-graph then $\delta_{k-1}(H)$ is known as the minimum *co-degree* of H. For a subset $M \subseteq E(H)$, we let $V(M) = \bigcup_{e \in M} e$.

A *matching* in a hypergraph H is a subset of $E(H)$ consisting of pairwise disjoint edges, which is *perfect* if $V(M) = V(H)$. While a maximum matching in a graph can be found in polynomial time [5], it is NP-hard to find even for 3-graphs [13]. Much effort has been devoted to finding good sufficient conditions for the existence of a large matching in uniform hypergraphs, including Dirac type conditions. A celebrated result in this area is due to Rödl, Ruciński, and Szemerédi [28], which refines the analysis in [27]. They determined the minimum co-degree threshold function that ensures a perfect matching in n-vertex k-graphs. For integers k, n, with $n \geq k \geq 3$ and $n \equiv 0 \pmod k$, let

$$t(n,k) := \begin{cases} n/2 + 2 - k, & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ n/2 + 3/2 - k, & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd,} \\ n/2 + 1/2 - k, & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even,} \\ n/2 + 1 - k, & \text{otherwise.} \end{cases}$$

Rödl, Ruciński, and Szemerédi [28] proved the following result.

Theorem 1.1 *(Rödl, Ruciński, and Szemerédi 2009). Let k, n be integers, with $k \geq 3$, $n \equiv 0 \pmod k$, and n sufficiently large. Let H be a k-graph on n vertices such that $\delta_{k-1}(H) > t(n,k)$. Then H has a perfect matching.*

Codegree condition $\delta_{k-1}(H) > t(n,k)$ is best possible because of the following k-graphs $H(n,k)$ on vertex set $[n]$ from [20] (for odd k, see Lemma 15) and [28] (for even k, see Definition 3.2): When k is odd, $[n]$ has a partition A, B such that $|A|$ is the unique odd integer from the set $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\}$ and $E(H(n,k)) = \left\{e \in \binom{V(H(n,k))}{k} : |e \cap A| \equiv 0 \pmod 2\right\}$. When k is even, $V(H(n,k))$ has a partition A, B such that

$$|A| := \begin{cases} n/2-1, & \text{if } n/k \text{ is even,} \\ n/2-1, & \text{if } n/k \text{ is odd and } n/2 \text{ is odd,} \\ n/2, & \text{if } n/k \text{ is odd and } n/2 \text{ is even,} \end{cases}$$

and $E(H(n,k)) = \{e \in \binom{V(H(n,k))}{k} : |e \cap A| \equiv 1 \pmod 2\}$. Note that the sets A, B are called *partition classes* of $H(n,k)$.

Let $F_1, \dots, F_t$ be t hypergraphs and let $\mathcal{F} = \{F_i\}_{i \in [t]}$ denote a family of hypergraphs (we use multiset $\{F_1, \dots, F_t\}$ to denote $\mathcal{F}$ when there is no confusion); a set of pairwise disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$. In this case, we also say that $\mathcal{F}$ *admits* a rainbow matching. There has been effort to extend results on matchings in hypergraphs to rainbow matchings, see for instance, [2,6,7,10,16–19,22,24–26]. The main result in this paper is a rainbow version of Theorem 1.1.x

Theorem 1.2. *Let k, n be integers with $k \geq 3$, $n \equiv 0 \pmod k$, and n sufficiently large. Let $\{F_1, \dots, F_{n/k}\}$ be a family of k-graphs on the common vertex set $[n]$, such that $\delta_{k-1}(F_i) > t(n,k)$ for $i \in [n/k]$. Then $\{F_1, \dots, F_{n/k}\}$ admits a rainbow perfect matching.*

It is easy to see that we derive Theorem 1.1 from Theorem 1.2 by setting $F_1 = \dots = F_{n/k} = H$. Moreover, if $F_i = H(n,k)$ for $i \in [n/k]$ then $\{F_1, \dots, F_{n/k}\}$ admits no rainbow perfect matching. So the co-degree bound in Theorem 1.2 is best possible. We point out that Theorem 1.2 for $k = 2$ is a result of Joos and Kim [12] and Akiyama and Frankl [1].

For $n \equiv 0 \pmod k$, let $\{F_1, \dots, F_{n/k}\}$ be a family of k-graphs on the same vertex set $[n]$. Let $X = \{x_1, \dots, x_{n/k}\}$ be disjoint from $[n]$. We consider the hypergraph $\mathcal{F}(n,k)$ with vertex set $X \cup [n]$ and edge set $\bigcup_{i=1}^{n/k} \{\{x_i\} \cup e \ : \ e \in E(F_i)\}$. We denote this hypergraph by $\mathcal{H}(n,k)$ when $F_i = H(n,k)$ for $i \in [n/k]$ with same partition classes A, B of $[n]$, and refer to $\mathcal{H}(n,k)$ as *extremal configuration*. It is easy to see the following is true.

Observation. $\delta_{k-1}(F_i) > t(n,k)$ for $i \in [n/k]$ implies that $d_{\mathcal{F}(n,k)}(S) > t(n,k)$ for any $S \in \binom{V(\mathcal{F}(n,k))}{k}$ with $|S \cap X| = 1$. $\{F_1, \dots, F_{n/k}\}$ admits a rainbow matching if, and only if, $\mathcal{F}(n,k)$ has a perfect matching.

So we will show that $\mathcal{F}(n,k)$ has a perfect matching. Indeed, we consider a more general class of hypergraphs. Let Q, V be two disjoint sets. A $(k+1)$-graph H with vertex set $Q \cup V$ is said to be $(1,k)$-*partite* with *partition classes* Q, V if, for each edge $e \in E(H)$, $|e \cap Q| = 1$ and $|e \cap V| = k$. A $(1,k)$-partite $(k+1)$-graph H with partition classes Q, V is *balanced* if $|V| = k|Q|$. We say that a subset $S \subseteq V(H)$ is *balanced* if $|S \cap V| = k|S \cap Q|$. Theorem 1.2 follows from the following result.

Theorem 1.3. *Let k, n be integers with $k \geq 3$, $n \equiv 0 \pmod k$, and $n \gg k$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$, such that for any $S \in \binom{V(\mathcal{F})}{k}$ with $|S \cap X| = 1$, $d_{\mathcal{F}}(S) > t(n,k)$. Then $\mathcal{F}$ admits a perfect matching.*

To prove Theorem 1.3, we consider whether or not $\mathcal{F}$ is "close" to the extremal configuration $\mathcal{H}(n,k)$. In Section 2, we describe several properties of the extremal configurations. In Section 3, we prove Theorem 1.3 for the case when $\mathcal{F}$ is close to the extremal configuration. In Section 4, we study absorbing devices for perfect matchings, and in Section 5, we study an absorbing device for near perfect matchings. We deal with the case when $\mathcal{F}$ is not close to the extremal configuration in Section 6 and offer some concluding remarks in Section 7.

2. Properties of extremal configurations

We will often use the following $(1,k)$-partite $(k+1)$-graphs as intermediate configuration to compare $(1,k)$-partite $(k+1)$-graphs with $\mathcal{H}(n,k)$. Suppose W,U form a partition of $[n]$ such that $|W| = (1/2 \pm o(1))n$ and $|U| = (1/2 \pm o(1))n$. For $i \in \{0,1\}$, let $H^i_{n,k}(W,U)$ denote the k-graph with vertex set $[n]$ and edge set $\{S \in \binom{[n]}{k} : |S \cap W| \equiv i \pmod 2\}$. When $|W| = \lfloor n/2 \rfloor$, we denote $H^i_{n,k}(W,U)$ by $H^i_{n,k}$.

We need the following definition to quantify the difference between $\mathcal{F}(n,k)$ and $\mathcal{H}(n,k)$. Let $\varepsilon > 0$ be a real number. Given two k-graphs H_1, H_2 with $V(H_1) = V(H_2)$, we say that H_2 is *strongly ε-close* to H_1 if $|E(H_1) \setminus E(H_2)| \le \varepsilon |V(H_1)|^k$. We say that H_2 is *weakly ε-close* to H_1 if $c(H_1,H_2) \le \varepsilon |V(H_1)|^k$, where $c(H_1,H_2)$ be the minimum of $|E(H_1) \setminus E(H_2')|$ taken over all isomorphic copies H_2' of H_2 with $V(H_2') = V(H_2)$. It is easy to see that the following is true.

Lemma 2.1. *Let $\varepsilon > 0$ be a real number. Let k,n be integers with $k \ge 3$, $n \equiv 0 \pmod k$ and n is sufficiently large. Let W,U be a partition of $[n]$ with $|W| = (1/2 \pm o(1))n$ and $|U| = (1/2 \pm o(1))n$. Then the following statements hold.*

- *(i) If k is even then $H^i_{n,k}(U,W) = H^i_{n,k}(W,U)$ for $i \in \{0,1\}$.*
- *(ii) If k is odd then $H^i_{n,k}(U,W)$ and $H^j_{n,k}(W,U)$, $i,j \in \{0,1\}$, are weakly ε-close to each other.*

Proof. First consider the case when k is even; so n is even. By the definition of $H^i_{n,k}(U,W)$, we have

$$E(H^i_{n,k}(W,U)) = \{e \in \binom{[n]}{k} \mid |e \cap W| \equiv |e \cap U| \equiv i \pmod 2\} = E(H^i_{n,k}(U,W)).$$

So we have $H^i_{n,k}(U,W) = H^i_{n,k}(W,U)$.

Now consider the case when k is odd. For $i \in \{0,1\}$,

$$\begin{aligned} E(H^i_{n,k}(W,U)) &= \{e \in \binom{[n]}{k} \mid |e \cap W| \equiv i \pmod 2\} \\ &= \{e \in \binom{[n]}{k} \mid |e \cap U| \equiv 1-i \pmod 2\} \end{aligned}$$

$$= E(H^{1-i}_{n,k}(U, W)).$$

So we have $H^1_{n,k}(W, U) = H^0_{n,k}(U, W)$ and $H^0_{n,k}(W, U) = H^1_{n,k}(U, W)$. Hence, it suffices to show that $H^1_{n,k}(U, W)$ is weakly ε-close $H^1_{n,k}(W, U)$.

If $|W| \geq |U|$, let $R \subseteq W$ such that $|R| = |U|$. One can see that $H^1_{n,k}(R, [n] \setminus R) \cong H^1_{n,k}(U, W)$. Note that

$$|E(H^1_{n,k}(W, U)) \setminus E(H^1_{n,k}(R, [n] \setminus R))| \leq |W \setminus R| \binom{n}{k-1} = (|W| - |U|) \binom{n}{k-1} < \varepsilon n^k.$$

So $H^1_{n,k}(U, W)$ is weakly ε-close $H^1_{n,k}(W, U)$.

Now we may assume that $|W| < |U|$. Let $T \subseteq [n]$ such that $|T| = |U|$ and $W \subseteq T$. Then we have $H^1_{n,k}(U, W) \cong H^1_{n,k}(T, [n] \setminus T)$. Note that

$$|E(H^1_{n,k}(W, U)) \setminus E(H^1_{n,k}(T, [n] \setminus T))| \leq |T \setminus W| \binom{n}{k-1} = (|U| - |W|) < \varepsilon n^k.$$

Hence $H^1_{n,k}(U, W)$ is weakly ε-close $H^1_{n,k}(W, U)$. This completes the proof. $\square$

Let k, n be integers with $k \geq 3$ and $n \equiv 0 \pmod k$. Let m_0, m_1 be integers between 0 and n/k (inclusive). For convenience, define $\mathcal{H}^{m_0,m_1}_{n,k}(W, U)$ as the $(1, k)$-partite $(k+1)$-graph $\mathcal{H}$ with partition classes X and $[n]$ and a partition W, U of $[n]$, such that $|X| = n/k$ and $|\{x \in X : N_{\mathcal{H}}(x) = H^i_{n,k}(W, U)\}| = m_i$ for $i \in \{0, 1\}$. For $i \in \{0, 1\}$, if $m_i = m$ and $m_0 + m_1 = n/k$ then we denote $\mathcal{H}^{m_0,m_1}_{n,k}(W, U)$ by $\mathcal{H}^i_{n,k}(W, U; m)$.

In the remainder of this section, we study $(1, k)$-partite $(k+1)$-graphs that $\mathcal{F}$ are close to some $\mathcal{H}(n, k)$ and consider those vertices in $\mathcal{F}$ that are contained in lots of edges of $\mathcal{H}(n, k)$. We introduce the following concept. Let k, n be integers with $k \geq 3$, $n \equiv 0 \pmod k$, and n sufficiently large. Let $\mathcal{F}$ and $\mathcal{H}$ be $(1, k)$-partite $(k+1)$-graphs with partition classes $X, [n]$. A vertex v of $\mathcal{F}$ is said to be *α-good with respect to $\mathcal{H}$* if $|N_{\mathcal{H}}(v) \setminus N_{\mathcal{F}}(v)| < \alpha |V(\mathcal{F})|^k$. Otherwise, v is said to be *α-bad* with respect to $\mathcal{H}$.

The following lemma shows that the number of bad vertices with respect to $\mathcal{H}^i_{n,k}(W, U; m)$ in $\mathcal{F}$ is small if $\mathcal{F}$ is close to $\mathcal{H}^i_{n,k}(W, U; m)$ for some $i \in \{0, 1\}$.

Lemma 2.2. *Let k, n be integers with $k \geq 3$ and $n \equiv 0 \pmod k$, and let ε be a constant. Let $\mathcal{F}$ be a $(1, k)$-partite $(k+1)$-graph with partition classes X and $[n]$ where $|X| = n/k$. Let $0 \leq m \leq n/k$ be an integer. If $\mathcal{F}$ is strongly ε-close to some $\mathcal{H}^i_{n,k}(W, U; m)$, then the number of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$ with respect to $\mathcal{H}^i_{n,k}(W, U; m)$ is at most $(1 + 1/k)(k + 1)\varepsilon^{1/3} n$.*

Proof. Let N be the set of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$ with respect to $\mathcal{H}^i_{n,k}(W, U; m)$. If $|N| > ((1 + 1/k)(k + 1)\varepsilon^{1/3} n$ then

$$|E(\mathcal{H}^i_{n,k}(W, U; m)) \setminus E(\mathcal{F})| > \frac{1}{k+1} |N| \varepsilon^{2/3} |V(\mathcal{F})|^k \geq \varepsilon \left(n + \frac{n}{k}\right)^{k+1}.$$

This contradicts the assumption that $\mathcal{F}$ is strongly ε-close to $\mathcal{H}^i_{n,k}(W,U;m)$. $\square$

The next lemma says that we can find an edge in $\mathcal{F}$ which serves as "parity breaker". This is the only place in the proof of Theorem 1.3 where we require $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n,k)$ for all $x \in X$.

Lemma 2.3. *Let k, n be integers with $k \geq 3$, $n \equiv 0 \pmod k$, and $n \geq 2k$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$, such that $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n,k)$ for all $x \in X$. Let W, U be a partition of $[n]$ with $\min\{|W|,|U|\} \geq k$. Let $i \in \{0,1\}$ be an integer such that $|\{e \in \mathcal{F} \mid |e \cap W| \equiv i \pmod 2\}| \geq |\mathcal{F}|/4$. Then for any proper subset X' of X, there exists e_0 with $e_0 \in E(\mathcal{F})$ or $e_0 = \emptyset$, such that*

(i) if $i = 0$ then $|W \setminus e_0| \equiv |X' \setminus e_0| \pmod 2$;
(ii) if $i = 1$ then $|W \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod 2$;
(iii) if $k - i$ is even then $|U \setminus e_0| \equiv |X' \setminus e_0| \pmod 2$;
(iv) if $k - i$ is odd then $|U \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod 2$.

Proof. Fix $x \in X \setminus X'$ and $x' \in X'$ if $X' \neq \emptyset$. Then $\delta_{k-1}(N_{\mathcal{F}}(x)) \geq t(n,k)+1$. In all cases below, we may assume the assertion of this lemma does not hold for $e_0 = \emptyset$.

Case 1: $i = 0$ and $k - i = k$ is even.

We have either $|W| \not\equiv |X'| \pmod 2$ or $|U| \not\equiv |X'| \pmod 2$. As n is even (because k divides n and, by assumption, k is even) and $|U| + |W| = n$, we have that $|U|$ and $|W|$ must have the same parity. Therefore, it follows that

$$|W| \equiv |U| \not\equiv |X'| \pmod 2. \tag{1}$$

Suppose that there exists e_0 such that $x \in e_0$ and either $|e_0 \cap W| = 1$ or $|e_0 \cap U| = 1$. As $e_0 \cap X = \{x\}$ (recall that for $e \in \mathcal{F}$ we have $|e \cap X| = 1$) and $x \in X \setminus X'$, we have $|X' \setminus e_0| = |X'|$. Now note that if $|e_0 \cap W| = 1$, then $|e_0 \cap U| = k-1$. As k is even, then $|e_0 \cap W|$ and $|e_0 \cap U|$ are odd. Obviously, the same holds when $|e_0 \cap U| = 1$. Therefore, $|W|$ and $|W \setminus e_0|$ have different parities as well as $|U|$ and $|U \setminus e_0|$. By (1), it follows that $|W \setminus e_0| \equiv |X'| \equiv |U \setminus e_0| \pmod 2$. As $|X' \setminus e_0| = |X'|$, we conclude that (i) and (iii) holds.

Now we may assume that, for every $e \in E(\mathcal{F})$ with $x \in e$, $|e \cap W| \neq 1$ and $|e \cap U| \neq 1$. Then for any $(k-1)$-set $S \subseteq U$, $N_{N_{\mathcal{F}}(x)}(S) \subseteq U \setminus S$. Thus, $|U - S| \geq |N_{N_{\mathcal{F}}(x)}(S)| \geq t(n,k) + 1 \geq n/2 + 2 - k$. Thus $|U| \geq n/2 + 1$. Similarly, we derive $|W| \geq n/2 + 1$. This leads to a contradiction as $n = |U| + |W|$.

Case 2: $i = 0$ and $k - i = k$ is odd.

We have $|W| \not\equiv |X'| \pmod 2$ or $|U| \not\equiv n/k - |X'| \pmod 2$. Since k is odd, $n/k - |X'| \equiv n - |X'| \pmod 2$. Note that $|U| + |W| = n$. Thus we have

$$|U| \not\equiv n - |X'| \pmod 2 \text{ and } |W| \not\equiv |X'| \pmod 2. \tag{2}$$

Suppose that there exists $e_0 \in \mathcal{F}$ such that $x \in e_0$ and $|e_0 \cap W| = 1$ or $|e_0 \cap W| = k-2$. Then $X \cap e_0 = \{x\}$, $|W \setminus e_0| \neq |W| \pmod 2$ and $|U \setminus e_0| = |U| \pmod 2$. Note that $X' \setminus e_0 = X'$ and so $|X \setminus (X' \cup e_0)| \neq |X \setminus X'| \pmod 2$. Thus by (2), we have $|W \setminus e_0| \equiv |X'| \equiv |X' \setminus e_0| \pmod 2$ and

$$|U \setminus e_0| \equiv |U| \equiv n - |X'| - 1 \equiv n/k - |X'| - 1 \equiv |X \setminus X'| - 1 \equiv |X \setminus (X' \cup e_0)| \pmod 2.$$

Hence we may conclude that (i) and (iv) holds for this e_0. So assume that for any $e \in E(\mathcal{F})$ with $x \in e$, $|e \cap W| \neq 1$ and $|e \cap W| \neq k-2$.

Subcase 2.1: n is even; so n/k is even.

Then for any $(k-1)$-subset $S \subseteq U$, $N_{N_{\mathcal{F}}(x)}(S) \subseteq U \setminus S$. Thus, $|U \setminus S| \geq |N_{N_{\mathcal{F}}(x)}(S)| \geq t(n,k)+1 \geq n/2+2-k$; so $|U| \geq n/2+1$. Similarly, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap W| = k-2$, $N_{N_{\mathcal{F}}(x)}(T) \subseteq W \setminus T$. Thus, $|W \setminus T| \geq |N_{N_{\mathcal{F}}(x)}(T)| \geq t(n,k)+1 \geq n/2+2-k$; so $|W| \geq n/2$. However, this is a contradiction as $n = |W|+|U| \geq n+1$.

Subcase 2.2. n is odd; so n/k is odd.

Then for any $(k-1)$-subset $S \subseteq U$, $N_{N_{\mathcal{F}}(x)}(S) \subseteq U \setminus S$ which implies $|U \setminus S| \geq |N_{N_{\mathcal{F}}(x)}(S)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|U| \geq (n+1)/2$. Similarly, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap W| = k-2$, $N_{N_{\mathcal{F}}(x)}(T) \subseteq W \setminus T$ and, hence, $|W \setminus T| \geq |N_{N_{\mathcal{F}}(x)}(T)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|W| \geq (n-1)/2$ and the equality holds only when $(n-1)/2$ is even. Thus since $n = |W|+|U|$,

$$|U| = (n+1)/2 \text{ and } |W| = (n-1)/2 \equiv 0 \pmod 2. \tag{3}$$

By (2) and (3), we have $|X'| \equiv 1 \pmod 2$.

Now we have $X' \neq \emptyset$, and fix $x' \in X'$. Suppose that there exists an edge $e_0 \in E(\mathcal{F})$ such that $x' \in e_0$ and $|e_0 \cap W| = 2$ or $|e_0 \cap W| = k-1$. Then we have $|W \backslash e_0| \equiv |W| \pmod 2$, $|U \backslash e_0| \neq |U| \pmod 2$ and $|X' \backslash e_0| \neq |X'|$. By (2), one can see that $|W \backslash e_0| \equiv |X' \backslash e_0| \pmod 2$,

$$|X \setminus (X' \cup e_0)| = |X \setminus X'| = n/k - |X'| \text{ and } n/k - |X'| \equiv n - |X'| \equiv |U| - 1 \equiv |U \backslash e_0|.$$

Thus we may conclude that (i) and (iv) hold with this e_0. So we may assume that for any $e \in E(\mathcal{F})$ with $x' \in e$, $|e \cap W| \neq 2$ and $|e \cap W| \neq k-1$. Then for any $(k-1)$-subset $T \subseteq W$, $N_{N_{\mathcal{F}}(x')}(T) \subseteq W \setminus T$ and, hence, $|W \setminus T| \geq |N_{N_{\mathcal{F}}(x')}(T)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|W| \geq (n+1)/2$, which a contradicts to (3).

Case 3: $i = 1$ and $k-i = k-1$ is odd (i.e., k is even).

Then $|W| \not\equiv n/k - |X'| \pmod 2$ or $|U| \not\equiv n/k - |X'| \pmod 2$. Note that n is even as k is even. Since $n = |W|+|U|$, $|W|$ and $|U|$ have the same parity, i.e.,

$$|W| \equiv |U| \not\equiv n/k - |X'| \pmod 2. \tag{4}$$

Firstly, consider $|X'| \equiv 1 \pmod 2$. Then $X' \neq \emptyset$, and fix $x' \in X'$. Suppose that there exists $e_0 \in E(\mathcal{F})$ such that $x' \in e_0$, and $|e_0 \cap W| = 1$ or $|e_0 \cap W| = k-1$, then $|W| \neq |W \setminus e_0| \pmod 2$ and $|U| \neq |U \setminus e_0| \pmod 2$. Recall that $|X \cap e_0| = |X' \cap e_0| = 1$. So $|X' \setminus e_0| \neq |X'|$ and $|X \setminus (X' \cup e_0)| = |X \setminus X'| = n/k - |X'| \pmod 2$. By (4), we have

$$|W \setminus e_0| \equiv |U \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod 2.$$

So we may conclude (ii) and (iv) holds for this e_0, in this case. So assume that, for any $e \in E(\mathcal{F})$ with $x' \in e$, $|e \cap W| \neq 1$ and $|e \cap W| \neq k-1$. Hence, for any $(k-1)$-subset $S \subseteq [n]$ with $S \subseteq W$, $N_{N_{\mathcal{F}}(x')}(S) \subseteq W \setminus S$ and, hence, $|W \setminus S| \geq |N_{N_{\mathcal{F}}(x')}(S)| \geq t(n,k)+1 \geq n/2+2-k$; so $|W| \geq n/2+1$. Similarly, $|U| \geq n/2+1$. This is a contradiction since $n = |W| + |U| \geq n+2$.

Secondly, we may assume $|X'| \equiv 0 \pmod 2$. Suppose that there exists $e_0 \in E(\mathcal{F})$ with $x \in e_0$ such that $|e_0 \cap W| = 2$ or $|e_0 \cap U| = 2$. Then $|W| \equiv |W \setminus e_0| \equiv |U \setminus e_0| \equiv |U| \pmod 2$. Recall that $|X \setminus (X' \cup e_0)| \neq |X \setminus X'| \pmod 2$ and $|X \setminus X'| = n/k - |X'|$. By (4), we have $|W \setminus e_0| \equiv |U \setminus e_0| \equiv |X \setminus (X' \cup e_0)| \pmod 2$. Thus (ii) and (iv) holds with this e_0. Hence, we may assume that, for any $e \in E(\mathcal{F})$ with $x \in e$, $|e \cap W| \neq 2$ and $|e \cap U| \neq 2$.

Hence, for any $(k-1)$-subset $S \subseteq [n]$ with $|W \cap S| = k-2$, $N_{N_{\mathcal{F}}(x)}(S) \subseteq W \setminus S$ and, hence, $|W \setminus S| \geq |N_{N_{\mathcal{F}}(x)}(S)| \geq t(n,k)+1 \geq n/2+2-k$. So $|W| \geq n/2$ and equality holds only when n/k is even or when n/k is odd and $k/2$ is odd. Similarly, $|U| \geq n/2$ and equality holds only when n/k is even or when n/k is odd and $k/2$ is odd. Since $n = |W| + |U|$, $|W| = |U| = n/2$. If n/k is even then $|W| = n/2$ is even (as k is even); however, $|W|$ is odd since $|W| \not\equiv n/k - |X'| \pmod 2$, a contradiction. If n/k is odd then $k/2$ is odd and, hence, $|W| = n/2$ is odd. However, $|W|$ is even, since $|W| \not\equiv n/k - |X'| \pmod 2$, a contradiction.

Case 4: $i = 1$ and $k - i = k - 1$ is even (i.e., k is odd).

Then $|W| \not\equiv n/k - |X'| \pmod 2$ or $|U| \not\equiv |X'| \pmod 2$. Note that $n = |W| + |U|$ and $n/k - |X'| \equiv n - |X'| \pmod 2$. So

$$|W| \not\equiv n - |X'| \pmod 2 \text{ and } |U| \not\equiv |X'| \pmod 2. \tag{5}$$

Suppose that there exists $e_0 \in E(\mathcal{F})$ with $x \in e_0$ such that $|e_0 \cap U| = 1$ or $|e_0 \cap U| = k-2$. Then $|e_0 \cap W| \equiv 0 \pmod 2$. Thus $|W \setminus e_0| \equiv |W| \pmod 2$ and $|U \setminus e_0| \neq |U| \pmod 2$. Recall that $|X' \setminus e_0| = |X'|$ and $|X \setminus (X' \cup e_0)| \neq |X \setminus X'| \pmod 2$. By (5), we have $|U \setminus e_0| \equiv |X' \setminus e_0| \pmod 2$ and

$$|W \setminus e_0| \equiv |W| \equiv n - |X'| - 1 \equiv n/k - |X'| - 1 \equiv |X \setminus X'| - 1 \equiv |X \setminus (X' \cup e_0)| \pmod 2.$$

Thus (ii) and (iii) hold for this e_0. So assume that for any $e \in E(\mathcal{F})$ with $x \in e$, $|e \cap U| \neq 1$ and $|e \cap U| \neq k-2$.

Subcase 4.1: n is even.

Then, for any $(k-1)$-subset $S \subseteq W$, $N_{N_{\mathcal{F}(x)}}(S) \subseteq W \setminus S$ and, hence, $|W \setminus S| \geq |N_{N_{\mathcal{F}(x)}}(S)| \geq t(n,k)+1 \geq n/2+2-k$; so $|W| \geq n/2+1$. Similarly, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap U| = k-2$, $N_{N_{\mathcal{F}(x)}}(T) \subseteq U \setminus T$ and, hence, $|U \setminus T| \geq |N_{N_{\mathcal{F}(x)}}(T)| \geq t(n,k)+1 \geq n/2+2-k$; so $|U| \geq n/2$. Now $n = |W|+|U| \geq n+1$, a contradiction.

Subcase 4.2: n is odd.

Then for any some $(k-1)$-subset $S \subseteq W$, $N_{N_{\mathcal{F}(x)}}(S) \subseteq W \setminus S$ and, hence, $|W \setminus S| \geq |N_{N_{\mathcal{F}(x)}}(S)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|W| \geq n/2+1/2$. Also, for any $(k-1)$-subset $T \subseteq [n]$ with $|T \cap U| = k-2$, $N_{N_{\mathcal{F}(x)}}(T) \subseteq U \setminus T$ and, hence, $|U \setminus T| \geq |N_{N_{\mathcal{F}(x)}}(T)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|U| \geq n/2-1/2$ and the equality holds only when $(n-1)/2$ is even. Since $n = |W|+|U|$, $|W| = (n+1)/2$ and

$$|U| = (n-1)/2 \text{ and } (n-1)/2 \text{ is even.} \tag{6}$$

By (5), $|U| \not\equiv |X'| \pmod 2$ and so $|X'|$ is odd.

Now fix $x' \in X'$. If there exists $e_0 \in E(\mathcal{F})$ with $x' \in e_0$ such that $|e_0 \cap W| = 1$ or $|e_0 \cap W| = k-2$, then with similar discussion as above, (ii) and (iii) hold for this e_0. So assume that for any $e \in E(\mathcal{F})$ with $x' \in e$, $|e \cap W| \neq 1$ and $|e_0 \cap W| \neq k-2$. Then for any $(k-1)$-subset $S \subseteq U$, $N_{N_{\mathcal{F}(x')}}(S) \subseteq U \setminus S$ and, hence, $|U \setminus S| \geq |N_{N_{\mathcal{F}(x')}}(S)| \geq t(n,k)+1 \geq n/2+3/2-k$; so $|U| \geq (n+1)/2$, contradicting to (6). $\square$

3. Hypergraphs close to extremal configurations

We often need to move some vertices between two sets and keep track of their degrees. The following notation will be convenient. Let H be a k-graph. For $j \in \{0,1\}, v \in V(H)$, and $S \subseteq V(H)$, we define

$$d^j_{H,S}(v) := |\{e \in E(H) : v \in e \text{ and } |e \cap S| \equiv j \pmod 2\}|.$$

We begin with a lemma that allows us to find a matching in the balanced $(1,k)$-partite $(k+1)$-graph $\mathcal{F}$ referred in Theorem 1.3 covering any small fixed set of vertices. For convenience, we set the following parameters for a given integer k for the remainder of this section: $\eta = 1/(4k!)$, $c = 1/(8(k+1)!)$, $\varepsilon = 1/(80^k k^{k-5}(k!)^k((k+1)!)^k)^{3/2}$, and $\gamma = \frac{\varepsilon^{2/3}2^k}{c^k k^k}$.

Lemma 3.1. *Let k,n be integers with $k \geq 3$, $n \equiv 0 \pmod k$, and $n \gg 1/\varepsilon$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X,[n]$, and assume $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n,k)$ for all $x \in X$. Let W,U be a partition of $[n]$ such that $\min\{|W|,|U|\} \geq k$, and let X_0,X_1 be a partition of X such that $d^j_{\mathcal{F},W}(x) \geq \eta n^k$ for $j \in \{0,1\}$ and $x \in X_j$. Suppose there exists $i \in \{0,1\}$ such that $d^i_{\mathcal{F}-X_{1-i},W}(v) \geq \eta n^k$ for all $v \in [n]$.*

Then for any e_0 satisfying the conclusion of Lemma 2.3 and for any $N \subseteq X \cup [n]$ with $|N| < 2cn$, there exists a matching M in $\mathcal{F}$ such that $N \subseteq V(M)$, $e_0 \in M$ when $e_0 \neq \emptyset$, $|V(M)| \leq (k+1)2cn$, and $|W \setminus V(M)| \equiv |X_1 \setminus V(M)| \pmod 2$.

Proof. Let $N_0 = N \cap X_0$ and $N_1 = N \cap X_1$. Let e_0 be the edge satisfying the conclusion of Lemma 2.3. If $N \subseteq e_0$, then let $M := \emptyset$ when $e_0 = \emptyset$, and $M = \{e_0\}$ when $e_0 \neq \emptyset$. So assume $N \nsubseteq e_0$. Divide $N \setminus e_0$ into three pairwise disjoint sets: $N_0 \setminus e_0 = \{v_1, \dots, v_r\}$, $N_1 \setminus e_0 = \{v_{r+1}, \dots, v_s\}$, and $(N \setminus e_0) \cap [n] = \{v_{s+1}, \dots, v_t\}$. We find the desired matching M by covering the vertices in $N \setminus e_0$ greedily.

Suppose we have found the matching $M_u := \{e_0, e_1, ..., e_u\}$ for some $u \geq 0$ such that for $1 \leq j \leq u$, $\{v_j\} = e_j \cap N$ and for $1 \leq i \leq \min\{u, r\}$, $|e_j \cap W| \equiv 0 \pmod 2$, for $r+1 \leq j \leq \min\{u, s\}$, $|e_j \cap W| \equiv 1 \pmod 2$, and for $s+1 \leq j \leq \min\{u, t\}$, $|e_j \cap W| \equiv i \pmod 2$ and $e_j \cap X \subseteq X_i$.

If $N \subseteq \cup_{i=0}^{u} e_i$, then M_u is the desired matching. Otherwise, let $v_{u+1} \in N \setminus \cup_{i=0}^{u} e_i$. Since $u + 1 < |N| \leq 2cn$, the number of edges in $\mathcal{F}$ containing $v_{u+1} \in N$ and a vertex from $\cup_{i=0}^{u} e_i \cup N$ is less than

$$(|N| + (u+1)(k+1))\, n^{k-1} \leq (k+1)2cn^k.$$

By assumption, $d^0_{\mathcal{F},W}(v) \geq \eta n^k$ for $v \in N_0$, $d^1_{\mathcal{F},W}(v) \geq \eta n^k$ for $v \in N_1$ and $d^i_{\mathcal{F}-X_{1-i},W}(v) \geq \eta n^k$ for $v \in (N \setminus e_0) \cap [n]$. Thus there exists an edge e_{u+1} in $\mathcal{F} - \cup_{i=0}^{u} e_i$ such that $\{v_{u+1}\} = e_{u+1} \cap N$, $|e_{u+1} \cap W| \equiv 0 \pmod 2$ when $u+1 \leq r$, $|e_{u+1} \cap W| \equiv 1 \pmod 2$ when $r < u+1 \leq s$, and $|e_{u+1} \cap W| \equiv i \pmod 2$ and $e_j \cap X \subseteq X_i$ when $s+1 < u+1 \leq t$. Continuing this process for at most $|N \setminus e_0|$ steps, we obtain the desired matching M. □

Let $\mathcal{H}(W, U; r)$ denote the balanced $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$ and a partition W, U of $[n]$ such that for every $x \in X$, $N_{\mathcal{H}(W,U;r)}(x) = \{e \in \binom{[n]}{k} : |e \cap W| = r\}$. Now we show that if all the vertices of a $(1, k)$-partite $(k+1)$-graph $\mathcal{F}$ are γ-good with respect to $\mathcal{H}(W, U; r)$, then there exists a perfect matching in $\mathcal{F}$ consisting of edges intersecting W exactly r times.

Lemma 3.2. *Let k, n, r be integers such that $k \geq 3$, $0 \leq r \leq k$, $n \equiv 0 \pmod k$, and $n \gg k$. Let $\mathcal{F}$ be a balanced $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$, and let W, U be a partition of $[n]$ with $|W| = rn/k$. Suppose all vertices in $\mathcal{F}$ are γ-good with respect to $\mathcal{H}(W, U; r)$. Then there exists a perfect matching M in $\mathcal{F}$ such that $|e \cap W| = r$ for all $e \in M$.*

Proof. As $|W| = rn/k$, we have $|U| = (k-r)n/k$ and $|e \cap W| = r$ if and only if $|e \cap U| = k - r$. By symmetry between W and U, we may assume $|W| \geq |U|$ and so $r \geq k/2$. Let M be a maximum matching in $\mathcal{F}$ such that, for every $e \in M$, $|e \cap W| = r$. Let $W_0 := W \setminus V(M)$ and $U_0 := U \setminus V(M)$. Then $|W_0| = |W| - r|M| = r(n/k - m)$ and $|U_0| = |U| - (k-r)|M| = (k-r)(n/k - m)$. Since $r \geq k/2$, $|W_0| \geq |U_0|$.

Suppose $|M| < \frac{n}{4k}$. Then $|W_0| \geq \frac{n}{4}$. By maximality of M, for $v \in W_0$, we have

$$|N_{H(W,U;r)}(v) \setminus N_{\mathcal{F}}(v)| \geq |X \setminus V(M)| \binom{|W_0|}{r-1} \binom{|U_0|}{k-r}.$$

Thus, if $r = k$ then

$$|N_{H(W,U;r)}(v) \setminus N_{\mathcal{F}}(v)| \geq \frac{3n}{4k}\binom{|W_0|}{k-1} \geq \gamma(n+n/k)^k.$$

Now suppose $r \leq k-1$. Then

$$|U_0| = \frac{k-r}{r}|W_0| \geq \frac{k-r}{r}(n/4) \geq \frac{n}{4(k-1)}.$$

So we have

$$\begin{aligned}|N_{H(W,U;r)}(v) \setminus N_{\mathcal{F}}(v)| &\geq \frac{3n}{4k}(|W_0| - r + 2)^{r-1}(|U_0| - k + r + 1)^{k-r}/k! \\ &\geq \gamma(n+n/k)^k\end{aligned}$$

contradicting the fact that v is γ-good with $\mathcal{H}(W,U;r)$.

We may assume $|M| \geq \frac{n}{4k}$. Now, suppose for a contradiction that M is not a perfect matching. There exist $x_{k+1} \in X \backslash V(M)$, distinct $v_{k+1,1}, ..., v_{k+1,r} \in W_0$, and distinct $v_{k+1,r+1}, ..., v_{k+1,k} \in U_0$. Let $\{e_1, e_2, ..., e_k\} \in \binom{M}{k}$ and write $e_i := \{x_i, v_{i,1}, ..., v_{i,k}\}$, such that, for $i \in [k]$, $x_i \in X$, $v_{i,j} \in W$ for $j \in [r]$, and $v_{i,j} \in U$ for $j \in [k] \setminus [r]$. For $j \in [k+1]$, let $f_j := \{x_j, v_{j+1,1}, v_{j+2,2}, ..., v_{j+1+k,k}\}$, with the addition in the subscripts modulo $k+1$ (except we write $k+1$ for 0). Note that $f_1, ..., f_{k+1}$ are pairwise disjoint and $|f_j \cap W| = r$ for $j \in [k+1]$.

If $f_j \in E(\mathcal{F})$ for all $j \in [k+1]$, then $M' := (M \cup \{f_1, ..., f_{k+1}\}) \backslash \{e_1, ..., e_k\}$ is matching in $\mathcal{F}$, contradicting the maximality of $|M|$. Hence, $f_j \notin E(\mathcal{F})$ for some $j \in [k+1]$. Note that there are $\binom{|M|}{k}$ choices of $\{e_1, ..., e_k\} \subseteq M$. Thus we have

$$\begin{aligned}&\Big|\{e \in E(H(W,U;r)) \setminus E(\mathcal{F}) : |e \cap \{v_{k+1,i} : i \in [k]\}| = 1\}\Big| \\ &\geq \binom{|M|}{k} \\ &\geq \left(\frac{n}{4k} - k + 1\right)^k/k! \quad (\text{since } |M| \geq \frac{n}{4k}) \\ &\geq \frac{1}{k!(5k)^k}n^k \quad (\text{since } n \geq 20k^2) \\ &> \gamma(k+1)(n+n/k)^k\end{aligned}$$

This implies that there exists $x \in \{x_{k+1}, v_{k+1,1}, \ldots, v_{k+1,k}\}$ such that

$$|N_{H(W,U;r)}(x) \setminus N_{\mathcal{F}}(x)| > \gamma(n+n/k)^k.$$

That is, x is not γ-good with respect to $H(W,U;r)$, a contradiction. $\square$

Note that Lemma 3.2 requires W, U to have specific sizes. After obtaining the matching M in $\mathcal{F}$ from Lemma 3.1, we need to find a perfect matching in $\mathcal{F} - V(M)$ to conclude the proof of Theorem 1.3 for the case when all the vertices of $\mathcal{F} - V(M)$ are good. The following lemma serves this purpose.

Lemma 3.3. *Let k, n be integers such that $k \geq 3$, $n \equiv 0 \pmod{k}$, and $1/n \ll \varepsilon \ll 1/k$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$. Let W', U' be a partition of $[n]$ with $\min\{|U'|, |W'|\} \geq 1.1n/k + k$, and let $i \in \{0,1\}$ such that*

(i) if $i = 0$ then $|W'| \equiv 0 \pmod{2}$,
(ii) if $i = 1$ then $|W'| \equiv n/k \pmod{2}$,
(iii) if $k - i \equiv 0 \pmod{2}$ then $|U'| \equiv 0 \pmod{2}$, and
(iv) if $k - i \equiv 1 \pmod{2}$ then $|U'| \equiv n/k \pmod{2}$.

If all vertices of $\mathcal{F}$ are γ-good with respect to $\mathcal{H}^i_{n,k}(W', U')$, then there is a perfect matching in $\mathcal{F}$.

Proof. First, suppose there exists an edge $e_1 \in E(\mathcal{F})$ such that $|W' \setminus e_1| = r_1 x + r_2 y$ and $|U' \setminus e_1| = (k - r_1)x + (k - r_2)y$, where x, y, r_1, r_2 are integers satisfying $x, y > 20k^2$ and $0 \leq r_1, r_2 \leq k$. We partition $W' \setminus e_1$ to W_1, W_2 such that $|W_1| = r_1 x$ and $|W_2| = r_2 y$, partition $U' \setminus e_1$ to U_1, U_2 such that $|U_1| = (k - r_1)x$ and $|U_2| = (k - r_2)y$, and partition $X \setminus e_1$ to X_1, X_2 such that $|X_1| = x$ and $|X_2| = y$. By assumption, for $i \in \{0,1\}$, $0 \leq r_i \leq k$. Hence, by Lemma 3.2, $\mathcal{F}[X_i \cup W_i \cup U_i]$ has a perfect matching, say M_i, consisting of edges containing exactly r_i vertices from W_i. Now $M_1 \cup M_2$ (when $e_1 = \emptyset$) or $M_1 \cup M_2 \cup \{e_1\}$ gives the desired matching.

Therefore, it suffices to prove the existence of such e_1. This is done in the following four cases.

Case 1. $i = 0$ and $k - i = k$ is even.

Then (i) and (iii) hold. So $|W'|$ and $|U'|$ are both even. Then, since all vertices of $\mathcal{F}$ are γ-good with respect to $\mathcal{H}^0_{n,k}(W', U')$, there is an edge $e_1 \in \mathcal{F}$ such that $|e_1 \cap W'| \equiv |W'| \pmod{k}$. Thus, $|W' \setminus e_1| \equiv 0 \pmod{k}$.

Since $n \equiv 0 \pmod{k}$, we have $|e_1 \cap U'| = k - |e_1 \cap W'| \equiv k - |W'| \pmod{k}$; hence, $|e_1 \cap U'| \equiv |U'| \pmod{k}$. So $|U' \setminus e_1| \equiv 0 \pmod{k}$.

Thus, we may take $r_1 = k$ and $r_2 = 0$. Note that $x = |W' \setminus e_1|/k > n/k^2 > 20k^2$ and $y = |U' \setminus e_1|/k > n/k^2 > 20k^2$.

Case 2. $i = 0$ and $k - i = k$ is odd.

Then (i) and (iv) hold. So $|W'|$ is even and $|U'| \equiv n/k \pmod{2}$. Since all vertices of $\mathcal{F}$ are γ-good with respect to $\mathcal{H}^0_{n,k}(W', U')$, there is an edge $e_1 \in \mathcal{F}$ such that $|e_1 \cap W'| \equiv |W'| \pmod{k-1}$.

Write $|W' \setminus e_1| = (k-1)x$ for some integer x. Then $|U' \setminus e_1| - x = n - x - |W'| - |e_1 \cap U'| = n - x - |W' \setminus e_1| - |W' \cap e_1| + |e_1 \cap U'| = n - k(x+1) \equiv 0 \pmod{k}$. So we take $r_1 = k-1$ and $r_2 = 0$.

Note that $x = |W' \setminus e_1|/(k-1) > n/k^2 > 20k^2$ and

$$\begin{aligned} y &= (|U' \setminus e_1| - x)/k \\ &= (|U' \setminus e_1| - |W' \setminus e_1|/(k-1))/k \\ &= (|U' \setminus e_1| - (n-k-|U' \setminus e_1|)/(k-1))/k \\ &= (|U' \setminus e_1| - n/k + 1)/(k-1) \\ &\geq 0.1n/k^2 > 20k^2. \end{aligned}$$

Case 3. $i = 1$ and $k - i = k - 1$ is odd.

Then (ii) and (iv) hold. So $|W'| \equiv n/k \pmod 2$ and $|U'| \equiv n/k \pmod 2$. Without loss generality, suppose that $|W'| \geq |U'|$. Since all vertices of $\mathcal{F}$ are γ-good with respect to $\mathcal{H}^1_{n,k}(W', U')$, there is an edge $e_1 \in E(\mathcal{F})$ such that $|W' \setminus e_1| \equiv n/k - 1 \pmod{k-2}$. So $|e_1 \cap W'| \equiv |W'| - n/k + 1 \pmod{k-2}$.

Let $|W' \setminus e_1| = (k-1)x + y$ and $x + y = n/k - 1$; then $x + (k-1)y = n - k - |W' \setminus e_1| = |U' \setminus e_1|$. Moreover, $x = (|W' \setminus e_1| - n/k + 1)/(k-2)$ and $y = (|U' \setminus e_1| - n/k + 1)/(k-2)$. So we may set $r_1 = k - 1$ and $r_2 = 1$.

Note that $x = (|W' \setminus e_1| - n/k + 1)/(k-2) > 0.1n/k^2 > 20k^2$ and $y = (|W' \setminus e_1| - n/k + 1)/(k-2) > 0.1n/k^2 > 20k^2$.

Case 4. $i = 1$ and $k - i = k - 1$ is even.

Then (ii) and (iii) hold. So $|W'| \equiv n/k \pmod 2$ and $|U'|$ is even. Since all vertices in $\mathcal{F}$ are γ-good with respect to $\mathcal{H}^1_{n,k}(W', U')$, there is an edge $e_1 \in E(\mathcal{F})$ such that $|e_1 \cap U'| \equiv |U'| \pmod{k-1}$.

Write $|U' \setminus e_1| = (k-1)y$ for some integer y. Then

$$\begin{aligned} |W' \setminus e_1| - y &= n - y - |U'| - |e_1 \cap W'| \\ &= n - y - |U' \setminus e_1| - (|e_1 \cap U'| + |e_1 \cap W'|) \\ &= n - y - (k-1)y - k \\ &= n - k(y+1) \equiv 0 \pmod k. \end{aligned}$$

So let $r_1 = k$ and $r_2 = 1$.

Note that

$$\begin{aligned} x &= (|W' \setminus e_1| - y)/k \\ &= (|W' \setminus e_1| - |U' \setminus e_1|/(k-1))/k \\ &= (|W' \setminus e_1| - (n-k-|W' \setminus e_1|)/(k-1))/k \\ &= (|W' \setminus e_1| - n/k + 1)/(k-1) \\ &\geq 0.1n/k^2 \\ &> 20k^2. \end{aligned}$$

Moreover $y = |U' \setminus e_1|/(k-1) > n/k^2 > 20k^2$. $\square$

We are ready to prove the main result in this section.

Lemma 3.4. *Let k, n be integers with $k \geq 3$, $n \equiv 0 \pmod{k}$, and $1/n \ll \varepsilon \ll 1/k$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$, such that $\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n,k)$ for all $x \in X$. Suppose $[n]$ has a partition W, U with $|W| = n/2 \pm o(n)$ and $|U| = n/2 \pm o(n)$ such that $\mathcal{F}$ is strongly ε-close to $\mathcal{H}^0_{n,k}(W,U;m)$ for some $m \in [n/k]$. Then $\mathcal{F}$ admits a perfect matching.*

Proof. Since $\mathcal{F}$ is strongly ε-close to $\mathcal{H}^0_{n,k}(W,U;m)$ for some $m \in [n/k]$, the number of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$ is at most $(k+1)\varepsilon^{1/3}(n/k+n)\,\varepsilon^{2/3}$ (see Lemma 2.2). Let B denote the set of $\varepsilon^{2/3}$-bad vertices in $\mathcal{F}$. Write $X_2 = X \cap B$ and $N = [n] \cap B$. For $i \in \{0,1\}$,

$$X_i = \{x \in X \backslash X_2 \mid N_{\mathcal{H}^0_{n,k}(W,U;m)}(x) \cong H^i_{n,k}(W,U)\}.$$

Write $m_i := |X_i|$ for $i \in \{0,1,2\}$. For $i \in \{0,1\}$ and $x \in X_i$, x is $\varepsilon^{2/3}$-good; hence,

$$|N_{\mathcal{H}^0(W,U;m)}(x) \setminus N_{\mathcal{F}}(x)| \leq \varepsilon^{2/3}(n+n/k)^k \leq (1+1/k)^k \varepsilon^{2/3} n^k.$$

So $N_{\mathcal{F}}(x)$ is strongly $(1+1/k)^k\varepsilon^{2/3}$-close to $H^i_{n,k}(W,U)$.

Recall the definition of constants $\eta = 1/(4k!)$, $c = 1/(8(k+1)!)$, $\varepsilon = 1/(80^k k^{k-5}(k!)^k((k+1)!)^k)^{3/2}$ and $\gamma = \frac{\varepsilon^{2/3}2^k}{c^k k^k}$. Let $\mathcal{F}_i := (\mathcal{F} - X_2) - X_{1-i}$ for $i \in \{0,1\}$. Define a partition W_0, U_0 of $[n]$ as follows: If $|X_0| \geq |X_1|$ then let

$$W_0 = (W \setminus \{v \in W \cap N : d^0_{\mathcal{F}_0,W}(v) \leq \eta n^k\}) \cup \{v \in U \cap N : d^1_{\mathcal{F}_0,W}(v) > \eta n^k\}$$

and

$$U_0 = (U \setminus \{v \in U \cap N : d^1_{\mathcal{F}_0,W}(v) > \eta n^k\}) \cup \{v \in W \cap N : d^0_{\mathcal{F}_0,W}(v) \leq \eta n^k\}.$$

If $|X_0| < |X_1|$ then let

$$W_0 = (W \setminus \{v \in W \cap N : d^1_{\mathcal{F}_1,W}(v) \leq \eta n^k\}) \cup \{v \in U \cap N : d^0_{\mathcal{F}_1,W}(v) > \eta n^k\}$$

and

$$U_0 = (U \setminus \{v \in U \cap N : d^0_{\mathcal{F}_1,W}(v) > \eta n^k\}) \cup \{v \in W \cap N : d^1_{\mathcal{F}_1,W}(v) \leq \eta n^k\}.$$

Let $X_{21} := \{x \in X_2 : d^0_{\mathcal{F},W_0}(x) \geq \eta n^k\}$ and $X_{22} = X_2 \backslash X_{21}$. We apply Lemma 2.3 to $\mathcal{F}$ with $X' = X_1 \cup X_{22}$ and $i = 0$ (if $|X_0| \geq |X_1|$), or $X' = X_0 \cup X_{21}$ and $i = 1$ (when $|X_0| < |X_1|$). So there exists e_0 such that $e_0 = \emptyset$ or $e_0 \in E(\mathcal{F})$ satisfying (i)(ii)(iii)(iv) of Lemma 2.3.

Define

$$N' := \begin{cases} B \cup X_1, & \text{if } |X_1| \le cn \\ B \cup X_0, & \text{if } |X_0| \le cn \\ B, & \text{otherwise.} \end{cases}$$

Then $|N'| \le 2cn$. Next, we apply Lemma 3.1 to $\mathcal{F}$ with W_0 and U_0 as a partition of $[n]$ and with N' as the N in Lemma 3.1. Note that if $|X_0| \ge |X_1|$, we set $i = 0$ and have $d^0_{\mathcal{F}-X_1,W_0}(v) \ge \eta n^k$ for all $v \in [n]$, and that if $|X_0| < |X_1|$, we set $i = 1$ and have $d^1_{\mathcal{F}-X_0,W_0}(v) \ge \eta n^k$ for all $v \in [n]$. Hence, there is a matching M_1 in $\mathcal{F}$ with $e_0 \in M_1$ (when $e_0 \ne \emptyset$), and $|V(M_1)| \le (k+1)2cn$ such that

(i) $N' \subseteq V(M_1)$;
(ii) $|W_0 \setminus V(M_1)| \equiv |X_1 \backslash V(M_1)| \pmod 2$.

Let $W_1 := W_0 \setminus V(M_1)$ and $U_1 := U_0 \setminus V(M_1)$. Let $X_0' := X_0 \setminus V(M_1)$ and $X_1' = X_1 \setminus V(M_1)$. Note that $|X_i'| > cn/2$ or $X_i' = \emptyset$ for $i \in \{0,1\}$. If $|X_0| < cn$, then $X_0' = \emptyset$ and let $W_{11} := W_1$, $U_{11} := U_1$ and $W_{10} = U_{10} = \emptyset$; if $|X_1| < cn$, then $X_1' = \emptyset$ and let $W_{10} := W_1$, $U_{10} := U_1$ and $W_{11} = U_{11} = \emptyset$. Otherwise, let $W_{10} \cup W_{11}$ be a partition of W_1 and $U_{10} \cup U_{11}$ be a partition of U_1 such that for $i \in \{0,1\}$,

(iii) $|W_{10}| \equiv 0 \pmod 2$, $|W_{11}| \equiv |X_1'| \pmod 2$, $|W_{1i} \cup U_{1i}| = k|X_i'|$, and if $X_i' \ne \emptyset$ then $\min\{|W_{1i}|, |U_{1i}|\} \ge 1.1|X_i'| + k$.

Note that the existence of these partitions is guaranteed by (ii).

Let $\mathcal{F}_i' := \mathcal{F}[X_i' \cup W_{1i} \cup U_{1i}]$ for $i \in \{0,1\}$. Since every vertex in $V(\mathcal{F})\backslash B$ is $\varepsilon^{2/3}$-good with respect to $\mathcal{H}^0_{n,k}(W,U;m)$, every vertex of $\mathcal{F}_i'$ is γ-good with respect to $\mathcal{H}^i_{k|X_i'|,k}(W_{1i}, U_{1i})$. For, otherwise, without loss of generality, suppose $v \in V(\mathcal{F}_i')$ is γ-bad with respect to $\mathcal{H}^i_{k|X_i'|,k}(W_{1i}, U_{1i})$. Then

$$\begin{aligned} |N_{\mathcal{H}_{n,k}(W,U;m)}(v) \setminus N_{\mathcal{F}}(v)| &\ge |N_{\mathcal{H}^i_{k|X_i'|,k}(W_{1i},U_{1i})}(v) \setminus N_{\mathcal{F}_i'}(v)| \\ &> \gamma((k+1)|X_i'|)^k \\ &\ge \frac{\varepsilon^{2/3}2^k}{c^k k^k}\left(\frac{(k+1)cn}{2}\right)^k \\ &= \varepsilon^{2/3}(n+n/k)^k, \end{aligned}$$

contradicting that v in $\mathcal{F}$ is $\varepsilon^{2/3}$-good on $\mathcal{H}_{n,k}(W,U;m)$.

By Lemma 3.3, $\mathcal{F}_i'$ contains a perfect matching M_{2i} for $i = 0,1$ (and let $M_{2i} = \emptyset$ if $\mathcal{F}_i'$ is empty). So $M_1 \cup M_{20} \cup M_{21}$ is a perfect matching in $\mathcal{F}$. $\square$

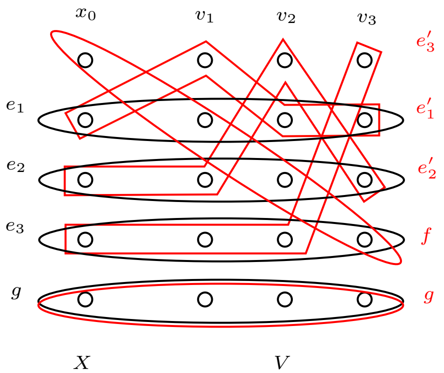

Fig. 1. Example of absorbing device I for $k = 3$.

4. Absorbing devices for perfect matchings

We need the following lemma from [28]. For subsets $N_1, \dots, N_k$ of the vertex sets of a k-graph H, let $E_H(N_1, \dots, N_k) := \{(v_1, \dots, v_k) : v_i \in N_i \text{ for } i \in [k] \text{ and } \{v_1, \dots, v_k\} \in E(H)\}$. Let $e_H(N_1, \dots, N_k) := |E_H(N_1, \dots, N_k)|$. Given a k-graph H, let $\overline{H}$ denote the k-graph with vertex set $V(H)$ and edge set

$$E(\overline{H}) = \left\{ e \in \binom{V(H)}{k} : e \notin E(H) \right\}.$$

The following lemma is Claim 5.1 in [28].

Lemma 4.1 *(Rödl, Ruciński, and Szemerédi [28]). Let n, k be two integers such that $n \gg k \geq 3$ and $n \equiv 0 \pmod{k}$. Let H be a k-graph on n vertices. If $\delta_{k-1}(H) \geq (1/2 - 1/\log n)n$ and H is not weakly ε-close to $H(n,k)$ or $\overline{H(n,k)}$, then at least one of the following holds.*

(i) For all $N_1, \dots, N_k \subseteq V(H)$ with $|N_i| \geq (1/2 - 1/\log n)n$, we have $e_H(N_1, \dots, N_k) \geq n^k/\log^3 n$.
(ii) $|\{(v_1, \dots, v_{k-1}) \in \binom{V(H)}{k-1} : d_H(\{v_1, \dots, v_{k-1}\}) > (1/2 + 2/\log n)n\}| \geq n^{k-1}/\log n$.

Next we define two types of absorbing devices for a given balanced set S of $k+1$ vertices. Both are $(k+1)$-matchings. The vertices of each device together with S induce a $(k+1)$-graph with a perfect matching.

Absorbing device I: Let $\mathcal{H}$ be a $(1,k)$-partite $(k+1)$-graph with partition classes X, V. Given a balanced set $S = \{x_0, v_1, \dots, v_k\}$ with $x_0 \in X$ and $v_i \in V$ for $i \in [k]$, a $(k+1)$-matching $\{e_1, \dots, e_k, g\}$ in $\mathcal{H}$ is said to be *S-absorbing* if $\mathcal{H}$ has a $(k+2)$-matching $\{e'_1, \dots, e'_k, f, g\}$ such that (see Fig. 1)

(i) $e'_i \cap e_j = \emptyset$ for all $i \neq j$,

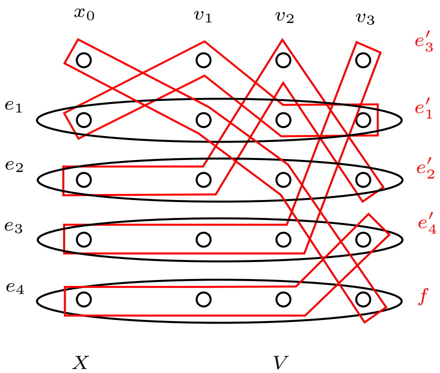

Fig. 2. Example of absorbing device II for $k = 3$.

(ii) $e'_i \setminus e_i = \{v_i\}$ and $|e_i \setminus e'_i| = 1$ for $i \in [k]$, and
(iii) $f = \{x_0\} \cup \bigcup_{i\in[k]}(e_i \setminus e'_i)$.

Note that the inclusion of the edge g is only for later convenience.

Absorbing device II: Let $\mathcal{H}$ be a $(1, k)$-partite $(k + 1)$-graph with partition classes X, V. Given a balanced set $S = \{x_0, v_1, \dots, v_k\}$ with $x_0 \in X$ and $v_i \in V$ for $i \in [k]$, a $(k+1)$-matching $\{e_1, \dots, e_{k+1}\}$ in $\mathcal{H}$ is said to be *S-absorbing* if $\mathcal{H}$ has a $(k+2)$-matching $\{e'_1, \dots, e'_{k+1}, f\}$ such that (see Fig. 2)

(i) $e_i \cap e'_j = \emptyset$ for $i, j \in [k]$, where $i \neq j$,
(ii) $e'_i \setminus e_i = \{v_i\}$ and $|e_i \setminus e'_i| = 1$ for $i \in [k]$,
(iii) $e'_{k+1} \cap e_k = e_k \setminus e'_k$, $|e_{k+1} \setminus e'_{k+1}| = 1$;
(iv) $f = \{x_0\} \cup \bigcup_{i\in[k+1]\setminus\{k\}}(e_i \setminus e'_i)$.

Next, we show that if (i) or (ii) of Lemma 4.1 holds for $\mathcal{F}(n, k)$, then for each balanced set S there are many S-absorbing devices in $\mathcal{F}(n, k)$.

Lemma 4.2. *Let n, k be two integers such that $n \gg k \geq 3$ and $n \equiv 0 \pmod k$. Let $\mathcal{F}$ be a $(1, k)$-partite $(k + 1)$-graph with partition classes $X, [n]$ such that $\delta_{k-1}(N_{\mathcal{F}}(x)) \geq (1/2 - 1/\log n)n$ for each $x \in X$. Let $R := \{x \in X : N_{\mathcal{F}}(x)$ is not weakly ε-close to $H(n, k)$ or $\overline{H(n, k)}\}$. Let $S = \{x_0, v_1, \dots, v_k\}$ be a balanced $(k + 1)$-set such that $x_0 \in R$.*

(i) If (i) of Lemma 4.1 holds for $N_{\mathcal{F}}(x_0)$, then the number of S-absorbing devices I in $\mathcal{F}$ is $\Omega(n^{(k+1)^2}/\log^3 n)$.
(ii) If $|R| = o(n)$ and $|\{x \in R \setminus \{x_0\} :$ (ii) of Lemma 4.1 holds for $N_{\mathcal{F}}(x)\}| \geq n/\log n$, then the number of S-absorbing devices II in $\mathcal{F}$ is $\Omega(n^{(k+1)^2}/\log^4 n)$.

Proof. First, suppose (i) of Lemma 4.1 holds for $N_{\mathcal{F}}(x_0)$. Since $\delta_{k-1}(N_{\mathcal{F}}(x_0)) \geq (1/2 - 1/\log n)n$, there are $\Omega(n^k)$ sets B_i, for each for $i \in [k]$, such that $B_i \cup \{v_i\} \in \mathcal{F}$.

Consequently, there are $\Omega(n^{k^2})$ choices of (pairwise disjoint) such sets $B_1, \dots, B_k$. Let $e'_i = B_i \cup \{v_i\}$ for $i \in [k]$.

Since $|N_{\mathcal{F}}(B_i)| \geq (1/2 - 1/\log n)n$ and (i) of Lemma 4.1 holds for $N_{\mathcal{F}}(x_0)$, we have

$$e_{\mathcal{F}}(x_0, N_{\mathcal{F}}(B_1), \dots, N_{\mathcal{F}}(B_k)) \geq n^k / \log^3 n.$$

So, there are at least $n^k/\log^3 n$ choices of edges $f = \{x_0, u_1, \dots, u_k\}$ such that $e_i := B_i \cup \{u_i\} \in E(\mathcal{F})$ for $i \in [k]$. Moreover, there are $\Omega(n^{k+1})$ choices g such that $g \in E((\mathcal{F} - S) - \cup_{i=1}^k e_i)$. Hence, altogether there are

$$(n^k/\log^3 n)\Omega(n^{k^2})\Omega(n^{k+1}) = \Omega(n^{(k+1)^2}/\log^3 n)$$

choices of S-absorbing $(k+1)$-matchings $\{e'_1, \dots, e'_k, f, g\}$.

Next we show (ii). As in the argument for (i), since $|R| = o(n)$, there are $\Omega(n^{k^2})$ choices of (pairwise disjoint) sets $B_1, \dots, B_k$ such that $R \cap (\cup_{i=1}^k B_i) = \emptyset$ and $e'_i = B_i \cup \{v_i\} \in E(\mathcal{F})$ for $i \in [k]$.

For $i \in [k-1]$, we choose $u_i \in N_{\mathcal{F}}(B_i)$, each in at least $(1/2 - 1/\log n)n$ ways, and let $e_i = B_i \cup \{u_i\}$. Let $y \in R \setminus \{x_0\}$ such that (ii) of Lemma 4.1 holds for $N_{\mathcal{F}}(y)$. By assumption, there are at least $n/\log n$ such y. We select a $(k-1)$-element sequence of vertices, say T, such that $d_{N_{\mathcal{F}(y)}}(T) \geq (1/2 + 2/\log n)n$ and T is disjoint from $S \cup B_k \cup \bigcup_{i=1}^{k-1} e_i$. Let $B_{k+1} = T \cup \{y\}$. Then we may pick u_k, u_{k+1} such that $B_k \cup \{u_k\}, B_{k+1} \cup \{u_k\}, B_{k+1} \cup \{u_{k+1}\}$, $\{u_0, u_1, \dots, u_{k-1}, u_{k+1}\} \in E(\mathcal{F})$. Note that $d_{\mathcal{F}}(B_{k+1}) \geq (1/2 + 2/\log n)n$. We have $|N_{\mathcal{F}}(B_{k+1}) \cap N_{\mathcal{F}}(B_k)| \geq n/\log n$ and $|N_{\mathcal{F}}(B_{k+1}) \cap N_{\mathcal{F}}(\{x_0, u_1 \dots, u_{k-1}\})\,| \geq n/\log n$. Thus there are at least $n/\log n$ choices for each of x_k, x_{k+1}. By Lemma 4.1 (ii), there are at least $n^{k-1}/\log n$ choices for T. Summarizing, we have chosen $B_1, \dots, B_k, B_{k+1}, u_1, \dots, u_{k-1}, T$ forming an S-absorbing $(k+1)$-matching, in

$$\Omega(n^{k^2})(n^{k-1}/\log n)n^{k-1}(n/\log n)^3) = \Omega(n^{(k+1)^2}/\log^4 n)$$

ways. $\square$

Absorbing device III: Let $\mathcal{H}$ be a $(1,k)$-partite $(k+1)$-graph with partition classes X, V. Given a balanced set $S = \{x_0, v_1, \dots, v_k\}$ with $x_0 \in X$ and $v_i \in V$ for $i \in [k]$, a $(k+1)$-matching $\{e_1, \dots, e_{k+1}\}$ in $\mathcal{H}$ is said to be S-*absorbing* if $\mathcal{H}$ has a $(k+2)$-matching $\{e'_1, \dots, e'_{k+1}, f\}$ such that (see Fig. 3)

(i) $e_i \cap e'_j = \emptyset$ for $i, j \in [k]$, where $i \neq j$,
(ii) $e'_i \setminus e_i = \{v_i\}$ and $|e_i \setminus e'_i| = 1$ for $i \in [k]$,
(iii) $e'_{k+1} \setminus e_{k+1} = \{x_0\}, |e_{k+1} \setminus e'_{k+1}| = 1$,
(iv) $f = \bigcup_{i \in [k+1]}(e_i \setminus e'_i)$.

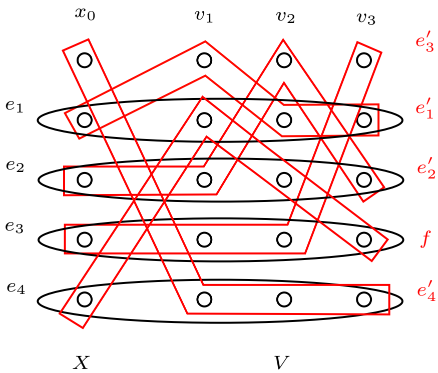

Fig. 3. Example of absorbing device III for $k = 3$.

In the following lemma, we show that any $(1, k)$-partite $(k + 1)$-graph is close to a $(k + 1)$-graph with same partition classes, or contains an absorbing matching.

Lemma 4.3. *Let n, k be two integers such that $n \equiv 0 \pmod{k}$ and let $1/n \ll \varepsilon \ll \beta \ll 1/k$. Let $\mathcal{H}$ be a balanced $(1, k)$-partite $(k + 1)$-graph with partition classes $X, [n]$. Let $R := \{x \in X : N_{\mathcal{H}}(x)$ is not weakly ε-close to $H^0_{n,k}$ or $H^1_{n,k}\}$. Let $x_0 \in X \setminus R$ such that $N_{\mathcal{H}}(x_0)$ is strongly ε-close to $H^i_{n,k}(W, U)$ for some $i \in \{0, 1\}$ and for some partition (W, U) of $[n]$ with $|W| = (1/2 \pm o(1))n = |U|$. Suppose $|R| \le n/\log n$ and $\delta_{k-1}(N_{\mathcal{H}}(x)) \ge (1/2 - 1/\log n)n$ for every $x \in X$. If $\mathcal{H}$ is not strongly β-close to $\mathcal{H}^0_{n,k}(W, U; m)$ for any integer $0 \le m \le n/k$, then for any balanced $(k + 1)$-set S with $x_0 \in S$, there are at least $\Omega(n^{(k+1)^2}/\log^4 n)$ S-absorbing devices I or III.*

Proof. First, let α be a number with $\varepsilon \ll \alpha \ll \beta$. We prove

Claim 1. Let $x \in X$ and let W_x, U_x be a partition of $[n]$ such that $|W_x| = (1/2 \pm o(1))n$ and $|U_x| = (1/2 \pm o(1))n$ and $N_{\mathcal{H}}(x)$ is strongly ε-close to $H^j_{n,k}(W_x, U_x)$ for some $j \in \{0, 1\}$. For any $N_1, \ldots, N_k \subseteq [n]$ with $|N_i| \ge (1/2 - 1/\log n)n$ for $i \in [k]$, if $e_{N_{\mathcal{H}}(x)}(N_1, N_2, \ldots, N_k) \le n^k/\log n$ then, for $j \in [k]$, $|U_x \setminus N_j| \le \alpha n$ or $|W_x \setminus N_j| \le \alpha n$.

Otherwise, suppose, without loss of generality, $|U_x \setminus N_1| \ge \alpha n$ and $|W_x \setminus N_1| \ge \alpha n$. Let D denote the set of $\sqrt{\varepsilon}$-bad vertices on $H^j_{n.,k}(W_x, U_x)$; then $|D| \le k\sqrt{\varepsilon}n$. Then, since $|N_1| \ge (1/2 - 1/\log n)n$, $|N_1 \setminus (U_x \cup D)| \ge \alpha n/2$ and $|N_1 \setminus (W_x \cup D)| \ge \alpha n/2$. Note for any $z \in N_1 \setminus (W_x \cup D)$ and $y \in N_1 \setminus (U_x \cup D)$, since $\varepsilon \ll \alpha$ and both z and y are $\sqrt{\varepsilon}$-good with respect to $H^j_{n,k}(W_x, U_x)$, we have $|N_{\mathcal{H}}(\{x, z\}) \cup N_{\mathcal{H}}(\{x, y\})| \ge \binom{n-1}{k-1} - \alpha n^{k-1}$. Hence, either $e_{N_{\mathcal{H}}(x)}(\{z\}, N_2, \ldots, N_k) \ge \frac{n^{k-1}}{2^k k!} - \alpha n^{k-1} \ge \frac{n^{k-1}}{2^{k+1}k!}$ or $e_{N_{\mathcal{H}}(x)}(y, N_2, \ldots, N_k) \ge \frac{n^{k-1}}{2^k k!} - \alpha n^{k-1} \ge \frac{n^{k-1}}{2^{k+1}k!}$. So by symmetry, we may assume that there exists $N'_1 \subseteq N_1 \setminus (W_x \cup D)$ such that $|N'_1| \ge |N_1 \setminus (W_x \cup D)|/2$ and for all $z \in N'_1$

$$e_{N_{\mathcal{H}}(x)}(\{z\}, N_2, \ldots, N_k) \ge \frac{n^{k-1}}{2^{k+1}k!}.$$

Hence,

$$e_{N_{\mathcal{H}}(x)}(N_1, N_2, \dots, N_k) \geq \frac{\alpha n^k}{2^{k+3}k!} > n^k/\log n,$$

contradicting the assumption of Claim 1 and completing its proof.

Let $A = \{x \in X \setminus R : |N_{\mathcal{H}}(x) \cap N_{\mathcal{H}}(x_0)| \geq \alpha n^k\}$ and let $B = X \setminus (A \cup R)$. Let $x \in B$. Then

$$\begin{aligned}|N_{\mathcal{H}}(x) \cap E(H^i_{n,k}(W,U))| &\leq |N_{\mathcal{H}}(x) \cap N_{\mathcal{H}}(x_0)| + |E(H^i_{n,k}(W,U)) \setminus N_{\mathcal{H}}(x_0)| \\ &\leq \alpha n^k + \varepsilon n^k.\end{aligned}$$

Hence, for any $x \in B$, we have

$$\begin{aligned}|E(\overline{H^i_{n,k}(W,U)}) \setminus E(N_{\mathcal{H}}(x))| &= e(\overline{H^i_{n,k}(W,U)}) \\ &\quad - (|N_{\mathcal{H}}(x)| - |N_{\mathcal{H}}(x) \cap E(H^i_{n,k}(W,U))|) \\ &\leq e(\overline{H^i_{n,k}}) - (e(H^i_{n,k}) - \alpha n^k - \varepsilon n^k) \\ &\quad + |N_{\mathcal{H}}(x) \cap E(H^i_{n,k}(W,U))| \\ &\leq \varepsilon n^k + \alpha n^k + \varepsilon n^k + \alpha n^k + \varepsilon n^k \leq 3\alpha n^k.\end{aligned}$$

Thus, we have

Claim 2. For any $x \in B$, $N_{\mathcal{H}}(x)$ is strongly (3α)-close to $\overline{H^i_{n,k}(W,U)}$, which is strongly 3α-close to $H^{1-i}_{n,k}(W,U)$. So

Claim 3. $|B| \leq (1-\beta)n/k$; so $|A| \geq \beta n/k - n/\log n$.

For, otherwise, $|B| > (1-\beta)n/k$. Then $|A| + |R| < \beta n/k$ and

$$\begin{aligned}|E(\mathcal{H}^{1-i}_{n,k}(W,U;|B|)) \setminus E(\mathcal{H})| &= \sum_{x \in B} |N_{\mathcal{H}^{1-i}_{n,k}(W,U;|B|)}(x) \setminus N_{\mathcal{H}}(x)| + (|A| + |R|)\binom{n}{k} \\ &< \frac{n}{k} 4\alpha n^k + \beta n/k \binom{n}{k} \leq \beta(n + n/k)^{k+1},\end{aligned}$$

a contradiction since $\mathcal{H}$ is not strongly β-close to $\mathcal{H}^{1-i}_{n,k}(W,U;|B|)$.

Claim 4. Let $\mathcal{H}' = \mathcal{H}[A \cup [n]]$. Then, for any $N_1, \dots, N_k \subseteq [n]$ with $|N_i| \geq (1/2 - 1/\log n)n$, either $e_{\mathcal{H}'}(\{x_0\}, N_1, \dots, N_k) \geq n^k/\log n$ or $e_{\mathcal{H}'}(A, N_1, \dots, N_k) \geq n^{k+1}/\log^3 n$.

Suppose on the contrary that there exist $N_1, \dots, N_k \subseteq [n]$ with $|N_i| \geq (1/2 - 1/\log n)n$, such that

$$e_{\mathcal{H}'}(\{x_0\}, N_1, \dots, N_k) < n^k/\log n \tag{7}$$

and

$$e_{\mathcal{H}'}(A, N_1, \dots, N_k) < n^{k+1}/\log^3 n. \tag{8}$$

Since $N_{\mathcal{H}}(x_0)$ is strongly ε-close to $H^i_{n,k}(W, U)$, the number of $\sqrt{\varepsilon}$-bad vertices in $N_{\mathcal{H}}(x)$ with respect to $H^i_{n,k}(W, U)$ is at most $k\sqrt{\varepsilon}n$.

By Claim 1 and (1), $|U \setminus N_i| \le \alpha n$ or $|W \setminus N_i| \le \alpha n$ for each $i \in [k]$. Let $A_1 := \{x \in A : e_{N_{\mathcal{H}}(x)}(N_1, \dots, N_k) \ge n^k/\log n\}$ and $A_2 = A \setminus A_1$. By (8), $|A_1| \le n/\log^2 n$.

Let $y \in A_2$. So $y \notin R$ and, thus, $N_{\mathcal{H}}(y)$ is weakly ε-close to $H^0_{n,k}$ or $H^1_{n,k}$. Let W_y, U_y be a partition of $[n]$ such that $N_{\mathcal{H}}(y)$ is strongly ε-close to $H^j_{n,k}(W_y, U_y)$ for some $j \in \{0, 1\}$. By Claim 1, we have for all $i \in [k]$, $|U_y \setminus N_i| \le \alpha n$ or $|W_y \setminus N_i| \le \alpha n$. Hence

$$|U_y \setminus U| \le |U_y \setminus N_i| + |N_i \setminus U| \le 2\alpha n \tag{9}$$

or

$$|U_y \setminus W| \le |U_y \setminus N_i| + |N_i \setminus W| \le 2\alpha n \tag{10}$$

We claim that $N_{\mathcal{H}}(y)$ is strongly (5α)-close to $H^j_{n,k}(W, U)$ if inequality (9) holds.

$$\begin{aligned}|E(H^j_{n,k}(W, U)) \setminus N_{\mathcal{H}}(y)| &\le |E(H^j_{n,k}(W_y, U_y)) \setminus N_{\mathcal{H}}(y)| \\ &\quad + |E(H^j_{n,k}(W, U)) \setminus E(H^j_{n,k}(W_y, U_y))| \\ &\le \varepsilon n^k + 4\alpha n^k \le 5\alpha n^k.\end{aligned}$$

If inequality (10) holds then

$$\begin{aligned}|E(H^j_{n,k}(U, W)) \setminus N_{\mathcal{H}}(y)| &\le |E(H^j_{n,k}(W_y, U_y)) \setminus N_{\mathcal{H}}(y)| \\ &\quad + |E(H^j_{n,k}(U, W)) \setminus E(H^j_{n,k}(W_y, U_y))| \\ &\le \varepsilon n^k + 4\alpha n^k \le 5\alpha n^k.\end{aligned}$$

Thus $N_{\mathcal{H}}(y)$ is strongly (5α)-close to $H^j_{n,k}(U, W)$.

Hence by Claim 2, for all $x \in X \setminus (A_1 \cup R)$, $N_{\mathcal{H}}(x)$ is strongly (5α)-close to $H^j_{n,k}(W, U)$ for some $j \in \{0, 1\}$. For $j \in \{0, 1\}$, let $X_j = \{x \in X : N_{\mathcal{H}}(x)$ is strongly (5α)-close to $H^j_{n,k}(W, U)\}$. Since $|A_1| \le n/\log^2 n$, and $|R| \le n/\log n$, we have $|X \setminus (X_0 \cup X_1)| = |A_1 \cup R| \le 2n/\log n$. Hence

$$\begin{aligned}|E(\mathcal{H}^0_{n,k}(W, U; |X_0|)) \setminus E(\mathcal{H})| &\le \sum_{j=0}^{1} \sum_{x \in X_j} |N_{\mathcal{H}^0_{n,k}(W,U;|X_0|)}(x) \setminus N_{\mathcal{H}}(x)| + (|A_1| + |R|)n^k \\ &\le \frac{n}{k} 5\alpha n^k + 2n^{k+1}/\log n \le \beta(n + n/k)^{k+1}.\end{aligned}$$

So $\mathcal{H}$ is strongly β-close $\mathcal{H}^0_{n,k}(W, U; |X_0|)$, a contradiction. This concludes the proof of Claim 4.

Let $S = \{x_0, v_1, \ldots, v_k\}$ with $v_1, \ldots, v_k \in [n]$ all distinct. For each $i \in [k]$, there are $\Omega(n^k)$ sets B_i such that $e'_i = B_i \cup \{v_i\} \in E(\mathcal{H})$, and there are $\Omega(n^{k^2})$ choices of (pairwise disjoint) such sets $B_1, \ldots, B_k$. For $i \in [k]$, we have $N_{\mathcal{H}'}(B_i) \geq (1/2 - 1/\log n)n$ by assumption. So there are $\Omega(n^{k^2})$ choices of (disjoint) sets $B_1, \ldots, B_k$ such that by Claim 4, one of the following two inequalities holds:

$$e_{\mathcal{H}'}(\{x_0\}, N_{\mathcal{H}'}(B_1), \ldots, N_{\mathcal{H}'}(B_k)) \geq n^k / \log n \tag{11}$$

$$e_{\mathcal{H}'}(A, N_{\mathcal{H}'}(B_1), \ldots, N_{\mathcal{H}'}(B_k)) \geq n^{k+1} / \log^3 n. \tag{12}$$

First, assume (11) holds. Then there at least $n^k / \log n$ choices $u_1, \ldots, u_k$ such that $e_i = B_i \cup \{u_i\} \in E(\mathcal{H}')$ for $i \in [k]$ and $\{x_0, u_1, \ldots, u_k\} \in E(\mathcal{H}')$. Moreover, there are $\Omega(n^{k+1})$ choices of $g \in E(\mathcal{H}')$ such that g is disjoint from we $\cup_{i=1}^k e_i$. Thus the number of S-absorbing devices I is at least

$$\Omega(n^{k^2})\Omega(n^k / \log n)\Omega(n^{k+1}) = \Omega(n^{(k+1)^2} / \log n).$$

Now assume (12) holds. Then there are $n^{k+1} / \log^3 n$ choices $\{y, u_1, \ldots, u_k\}$ such that $y \in A$ and, for $i \in [k]$, $B_i \cup \{u_i\} \in E(\mathcal{H}')$. By definition of A, there are $\Omega(n^k)$ choices B_{k+1} such that $B_{k+1} \in N_{\mathcal{H}}(y) \cap N_{\mathcal{H}}(x_0)$. Let $e_{k+1} = B_{k+1} \cup \{y\}$. So there are at least $\Omega(n^{k+1} / \log^3 n)\Omega(n^k)\Omega(n^{k^2}) = \Omega(n^{(k+1)^2} / \log^3 n)$ different choices of $\{e_1, \ldots, e_{k+1}\}$ such that $\{e_1, \ldots, e_{k+1}\}$ is an S-absorbing device III. $\square$

To prove another absorbing lemma, we need to use Chernoff bounds, see [3].

Lemma 4.4. *Suppose $X_1, ..., X_n$ are independent random variables taking values in $\{0, 1\}$. Let X denote their sum and $\mu = \mathbb{E}[X]$ denote the expected value of X. Then for any $0 < \delta \leq 1$,*

$$\mathbb{P}[X \leq (1 - \delta)\mu] < e^{-\frac{\delta^2 \mu}{2}}.$$

Lemma 4.5. *Let $k \geq 3$ be an integer and let $0 < \varepsilon \ll \beta \ll 1$. There exists $n_2 > 0$ such that the following holds for any integer $n > n_2$. Let $\mathcal{H}$ be a balanced $(1, k)$-partite $(k+1)$-graph with partition classes $X, [n]$ such that $\delta_{k-1}(N_{\mathcal{H}}(x)) > t(n, k)$ for all $x \in X$. Let $R := \{x \in X : N_{\mathcal{H}}(x)$ is not weakly ε-close to $H^0_{n,k}$ or $H^1_{n,k}\}$, and let $x_0 \in X$. Suppose one of the following three conditions holds for every $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$:*

- *(i) (i) of Lemma 4.1 holds for $N_{\mathcal{H}}(x_0)$, and $\mathcal{H}$ has $\Omega(n^{(k+1)^2} / \log^4 n)$ S-absorbing devices I.*
- *(ii) $|\{x \in R \setminus \{x_0\} :$ (ii) of Lemma 4.1 holds for $N_{\mathcal{H}}(x)\}| \geq n / \log n$, and $\mathcal{H}$ has $\Omega(n^{(k+1)^2} / \log^4 n)$ S-absorbing devices II.*
- *(iii) $|R| \leq n / \log n$, $x_0 \notin R$, and $N_{\mathcal{H}}(x_0)$ is strongly ε-close to $H^0_{n,k}(W, U)$ or $H^1_{n,k}(W, U)$ for some partition (W, U) of $[n]$ with $|W| = (1/2 + o(1))n = |U|$, $\mathcal{H}$ is not strongly*

β-close to $\mathcal{H}^0_{n,k}(W,U;m)$ for any $m \in [n/k]$, and $\mathcal{H}$ has $\Omega(n^{(k+1)^2}/\log^4 n)$ S-absorbing devices I or III.

Then there exists a matching M' in $\mathcal{H}$ such that $|M'| = O(\log^6 n)$ and, for each $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$, M' contains an S-absorbing $(k+1)$-matching.

Proof. For each balanced $(k+1)$-set $S \subseteq V(\mathcal{H})$ with $x_0 \in S$, let $\Gamma(S)$ be the collection of S-absorbing $(k+1)$-matchings. Then by Lemmas 4.2 and 4.3, $|\Gamma(S)| = \Omega(n^{(k+1)^2}/\log^4 n)$. So we may choose constant $\alpha := \alpha(k) > 0$ such that

$$|\Gamma(S)| \geq \alpha(k^2+k)!\binom{n/k}{k+1}\binom{n}{k(k+1)}/((k!)^{k+1}\log^4 n).$$

Let $\mathcal{M}$ be the family obtained by choosing a sequence of balanced $(k+1)$-sets $(S_1,\dots,S_{k+1})$ independently with probability

$$p = \frac{(k!)^{k+1}\log^6 n}{\binom{n/k}{k+1}\binom{n}{k(k+1)}(k^2+k)!}.$$

Note that $p < 1$ as we can choose n_2 large enough. Then

$$\mathbb{E}(|\mathcal{M}|) = p\binom{n/k}{k+1}\binom{n}{k(k+1)}(k^2+k)!/(k!)^{k+1} = O(\log^6 n),$$

and, for $(k+1)$-set $S \subseteq V(\mathcal{H})$ with $\{x_0\} = S \cap X$,

$$\mathbb{E}(|\mathcal{M}\cap\Gamma(S)|) \geq p\alpha(k^2+k)!\binom{n/k}{k+1}\binom{n}{k(k+1)}/((k!)^{k+1}\log^4 n) = \alpha\log^2 n.$$

By Lemma 4.4 and by choosing n_2 large enough, we have, for $n > n_2$ and for each $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$,

$$\mathbb{P}[|\mathcal{M}| > 2\alpha\log^6 n] = \mathbb{P}[|\mathcal{M}| > 2\mathbb{E}(|\mathcal{M}|)] \leq e^{-\mathbb{E}(|\mathcal{M}|)/3} = e^{-(\log^6 n)/3}.$$

So with probability at least $1 - o(1)$

$$|\mathcal{M}| \leq 2\alpha\log^6 n. \tag{13}$$

Again by Lemma 4.4 and by choosing n_2 large enough, we have, for $n > n_2$ and for each $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$,

$$\begin{aligned}\mathbb{P}[|\mathcal{M}\cap\Gamma(S)| \leq (\alpha\log^2 n)/2] &\leq \mathbb{P}[|\mathcal{M}\cap\Gamma(S)| \leq \mathbb{E}(|\mathcal{M}\cap\Gamma(S)|)/2] \\ &\leq e^{-\mathbb{E}(|\mathcal{M}\cap\Gamma(S)|)/8} \\ &\leq e^{-(\alpha\log^2 n)/8}.\end{aligned}$$

So by union bound and by choosing n_2 large, we have for $n > n_2$,

$$\begin{aligned}&\mathbb{P}[\exists S \in \binom{V(H)}{k+1} \text{ with } S \cap X = \{x_0\} : |\mathcal{M} \cap \Gamma(S)| \leq (\alpha \log^2 n)/2]\\ &\leq \binom{n}{k} e^{-(\alpha \log^2 n)/8}\\ &= 2n^{k-(\alpha \log n)/8} < 1/10.\end{aligned}$$

Thus, with probability at least 9/10, for all $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$, we have

$$|\mathcal{M} \cap \Gamma(S)| \geq (\alpha \log^2 n)/2 > 1. \tag{14}$$

Furthermore, the expected number of pairs of sequences $(S_1, \ldots, S_{k+1}), (T_1, \ldots, T_{k+1}) \in \mathcal{M}$ satisfying $(\cup_{i \in [k]} S_i) \cap (\cup_{i \in [k]} T_i) \neq \emptyset$ is at most

$$\begin{aligned}&\left(\frac{(k^2+k)!}{(k!)^k}\right)^2 \binom{n/k}{k+1} \binom{n}{k^2+k}\\ &\left(\binom{k+1}{1}\binom{n/k}{k}\binom{n}{k^2+k} + \binom{k^2+k}{1}\binom{n/k}{k}\binom{n}{k^2+k-1}\right) p^2 < 1/2.\end{aligned}$$

Thus, with probability at least 1/2 (by Markov's inequality), for all distinct $(S_1, \ldots, S_{k+1}) \in \mathcal{M}$ and $(T_1, \ldots, T_{k+1}) \in \mathcal{M}$,

$$\bigcup_{i \in [k+1]} S_i \text{ and } \bigcup_{i \in [k+1]} T_i \text{ are disjoint.} \tag{15}$$

Hence, with positive probability, $\mathcal{M}$ satisfies (13), (14), and (15). So we may assume that $\mathcal{M}$ satisfies (13), (14), and (15). Let M be the union of $\mathcal{M} \cap \Gamma(S)$ for all $S \in \binom{V(H)}{k+1}$ with $S \cap X = \{x_0\}$. Then M is the desired matching. $\square$

5. Absorbing devices for near perfect matchings

Let H be a $(1, k)$-partite $(k + 1)$-graph with partition classes $Q, [n]$. For a set $S \in \binom{V(H)}{k+2}$ with $|S \cap Q| = 1$, an edge $e \in E(H)$ is said to be *S-absorbing* if $H[e \cup S]$ has a matching of size 2.

Lemma 5.1. *Let $k \geq 3$, $0 < c < 1/(100k!)$, and n be a sufficiently large integer, and let H be a $(1, k)$-partite $(k + 1)$-graph with partition classes $Q, [n]$ such that $n/(100xk) \leq |Q| \leq (n - 1)/k$. If $\delta_{k-1}(N_H(v)) \geq cn$ for all $v \in Q$, then, for any $S \in \binom{V(H)}{k+2}$ with $|S \cap Q| = 1$, H has at least $c^4 n^{k+1}/2$ S-absorbing edges.*

Proof. Let $S = \{v\} \cup B$, where $v \in Q$ and $B = \{b_1, \ldots, b_k, b_{k+1}\} \in \binom{[n]}{k+1}$. Let $B' = B \setminus \{b_k, b_{k+1}\}$. Since $\delta_{k-1}(N_H(v)) \geq cn$, we have at least $cn - 2$ choices for $q \in [n] \setminus B$

such that $B' \cup \{v, q\} \in E(H)$. For a $(k-2)$-set A of $V(H) \setminus (S \cup \{q\})$ with $|A \cap Q| = 1$, since $d_H(A \cup \{b_k, b_{k+1}\}) \geq cn$, there are at least $cn - 2k$ choices $q' \in [n] \setminus (S \cup A \cup \{q\})$ such that $\{q'\} \cup \{b_k, b_{k+1}\} \cup A \in E(H)$. Since $d_H(A \cup \{q, q'\}) \geq cn$, there are at least $cn - 2k$ choices q'' such that $A \cup \{q, q', q''\} \in E(H)$. Clearly, any such $\{q, q', q''\} \cup A$ is S-absorbing.

Note that there are $(|Q|-1)\binom{n-k-2}{k-3}$ different choices for set A. Hence the number of S-absorbing edges is at least

$$(|Q|-1)\binom{n-k-2}{k-3}(cn-2)(cn-2k)(cn-2k) \geq c^4 n^{k+1}/2.$$

This completes the proof. $\square$

Analogous to the absorbing results in [8,23,28], we prove the existence of small absorbing matching for $(1,k)$-partite $(k+1)$-graphs with $|Q| \geq n/k - k^2$.

Lemma 5.2. *Let $k \geq 3$. For any constant $0 < c < 1/(100k!)$, there exists an integer $n_0 > 0$ with the following holds for every integer $n \geq n_0$: Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $Q, [n]$ such that $n/k - k^2 \leq |Q| \leq (n-1)/k$ and $\delta(N_H(v)) \geq cn$ for all $v \in Q$. Then there is a matching M in H such that $|M| \leq (32(k+3)/c^4)\log n$ and, for each $S \in \binom{V(H)}{k+2}$ with $|S \cap Q| = 1$, M contains at least $4(k+3)\log n$ S-absorbing edges.*

Proof. Let $C = 32(k+3)/c^4$. Let M be the family obtained by choosing each edge independently with probability $p = (C/2)n^{-(k+1)}\log n$. Thus $\mathbb{E}[|M'|] = |E(H)|p \leq n^{k+1}p = (C/2)\log n$.

The number of intersecting pairs of edges in $E(H)$ is at most $|Q|\binom{n}{k}^2 + |Q|^2 n\binom{n-1}{k-1}^2 \leq n^{2k+1}$; so the expected number of intersecting pairs of edges in M' is at most

$$n^{2k+1}p^2 \leq C^2 \log^2 n/(4n) = o(1).$$

By Markov's inequality, with probability strictly larger than 1/3, M is a matching of size at most $C \log n$.

For a set $S \in \binom{V(H)}{k+2}$ with $|S \cap Q| = 1$, let X_S denote the number of S-absorbing edges in M. Then by Lemma 5.1, we have

$$\mathbb{E}[X_S] \geq pc^4 n^{k+1}/2 = 8(k+3)\log n.$$

By Lemma 4.4,

$$\mathbb{P}[X_S \leq \ \mathbb{E}[X_S]/2] \leq \exp(-\mathbb{E}[X_S]/8) = \exp(-(k+2)\log n) = n^{-(k+2)}.$$

Note that there are at most $|Q|\binom{n}{k} < n^k/k$ sets $S \in \binom{V(H)}{k+1}$ with $|S \cap Q| = 1$. It follows from union bound that, with probability strictly larger than 1/4, $X_S \geq \mathbb{E}[X_S]/2 \geq 4(k+3)\log n$ for all $S \in \binom{V(H)}{k+1}$ with $|S \cap Q| = 1$. Thus, the desired M exists. □

6. Hypergraphs not close to extremal configurations

In this section, we prove Theorem 1.2 for hypergraphs that are not close to extremal configurations. For this, we need a result on almost perfect matchings in $(1, k)$-partite $(k + 1)$-graphs.

Lemma 6.1. *Let k, n be positive integers with $k \geq 3$. Let H be a $(1, k)$-partite $(k+1)$-graph with vertex partition classes $Q, [n]$, where $k + 1 \leq |Q| \leq n/k$. If $\delta_{k-1}(N_H(v)) > n/k$ for every $v \in Q$, then H has a matching covering all but at most $k - 1$ vertices of Q.*

Proof. Let M be a maximum matching in H. We may assume $|Q \setminus V(M)| \geq k$; for, otherwise, M gives the desired matching. Note

$$|[n] \setminus V(M)| = n - k|M| \geq k|Q| - k|M| \geq k|Q \setminus V(M)| \geq k^2.$$

So there exist k pair-disjoint k-sets in $[n]\setminus V(M)$, say $S_1, \dots, S_k$ such that $|S_i \cap Q| = 1$ for $i \in [k]$. By maximality of M, $N_H(S_i) \subseteq V(M)$ for $i \in [k]$.

We claim that $\sum_{i=1}^{k} |N_H(S_i) \cap e| \leq k$ for all $e \in M$. For otherwise, there exist $e \in M$ and distinct $u, v \in e$ such that $u \in N_H(S_i) \cap e$ and $v \in N_H(S_j) \cap e$. Since S_i and S_j are disjoint, $(M \setminus \{e\}) \cup \{S_i \cup \{u\}, S_j \cup \{v\}\}$ is a matching in H, contradicting the maximality of M.

Since $N_H(S_i) \subseteq V(M)$,

$$\sum_{i=1}^{k} |N_H(S_i)| = \sum_{e \in M} \sum_{i=1}^{k} |N_H(S_i) \cap e| \leq k|M| < n.$$

However, since $\delta_{k-1}(N_H(v)) > n/k$ for all $v \in Q$, we have $\sum_{i=1}^{k} |N_H(S_i)| \geq k\delta_{k-1}(N_H(v)) > n$, a contradiction. □

Lemma 6.2. *Let k, n be integers with $k \geq 3$. Let H be a $(1, k)$-partite $(k + 1)$-graph with partition classes $Q, [n]$, where $k + 1 \leq |Q| \leq (n - 1)/k$. If $\delta_{k-1}(N_H(v)) \geq n/k + 2^9 10^8 (k!)^4 (k + 3)k \log n$ for every $v \in Q$, then H has a matching covering Q.*

Proof. Let $c = 1/(200k!)$. Since $\delta_{k-1}(N_H(v)) \geq n/k + 2^9 10^8 (k!)^4 (k + 3)k \log n > cn$ for every $v \in Q$, by Lemma 5.2, H has a matching M of size at most $2^9 10^8 (k!)^4 (k + 3) \log n$ such that for any set $S \in \binom{V(H)}{k+2}$ with $|S \cap Q| = 1$, the number of S-absorbing edges in M is at least $k + 1$. Let $H' = H - V(M)$, with partition classes $Q \setminus V(M), [n] \setminus V(M)$. For every $v \in Q \setminus V(M)$, we have

$$\delta_{k-1}(N_{H'}(v)) \geq \delta_{k-1}(N_H(v)) - k|M| > n/k.$$

Thus by Lemma 6.1, H' has a matching M' covering all but at most k vertices of $Q \setminus V(M)$.

Let $M_0 := M$ and $M_0' = M'$. If $M_0 \cup M_0'$ covers Q, then we are done. Otherwise, there exists $S_1 \in \binom{V(H)\setminus V(M_0 \cup M_0')}{k+2}$ with $|S_1 \cap Q| = 1$. Recall M_0 contains an S_1-absorbing edge, say e_0; so $H[S_1 \cup e_0]$ contains a matching X_0 of size 2. Then $M_1' := M_0' \cup X_0$ is a matching in H. Let $M_1 := M_0 \setminus \{e_0\}$. If $M_1 \cup M_1'$ covers Q, then we are done. Otherwise, since $|Q \setminus V(M_1 \cup M_1')| \leq k$ and M has at least $k+1$ S-absorbing matching for each $S \in \binom{V(H)\setminus V(M_1 \cup M_1')}{k+2}$ with $|S \cap Q| = 1$, we may repeat the above procedure. Thus, we obtain a maximal sequence of pairs of matchings $M_0, M_0', M_1, M_1', \dots, M_t, M_t'$, a sequence of $(k+2)$-sets $S_1, \dots, S_t$ with $|S_i \cap Q| = 1$ for $i \in [t]$, and a sequence of matchings $X_0, X_1, \dots, X_t$. Now $M_t \cup M_t'$ is a matching of H covering Q. $\square$

Lemma 6.3. *Let $\varepsilon > 0$ be a constant and k, n be integers with $k \geq 3$ and $n \equiv 0 \pmod k$ such that $0 < 1/n \ll \varepsilon \ll 1/k$. Let $\mathcal{F}$ be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes X and $[n]$. Let $R := \{x \in X : N_{\mathcal{F}}(x)$ is not weakly ε-close to $H^0_{n,k}$ or $H^1_{n,k}\}$. Suppose*

- *$\delta_{k-1}(N_{\mathcal{F}}(x)) > t(n,k)$ for all $x \in X$;*
- *$|R| > n/\log n$, or $|R| \leq n/\log n$ and $\mathcal{F}$ is not strongly ε-close to $\mathcal{H}^0_{n,k}(W,U;m)$ for any $m \in [n/k]$ and for any partition W, U of $[n]$ with $|W| = n/2 \pm o(n)$ and $|U| = n/2 \pm o(n)$.*

Then $\mathcal{F}$ admits a perfect matching.

Proof. Note that the conclusion of Lemma 4.5 holds. We define $x_0 \in X$ as follows:

(i) If $|R| > n/\log n$ then choose $x_0 \in R$ such that, whenever possible, (i) of Lemma 4.1 holds for $N_{\mathcal{F}}(x_0)$.
(ii) If $|R| \leq n/\log n$ then let $x_0 \in X \setminus R$ and let W, U be a partition of $[n]$ with $|W| = n/2 \pm o(n) = |U|$ such that $N_{\mathcal{F}}(x_0)$ is strongly ε-close to $H^0_{n,k}(W,U)$ or $H^1_{n,k}(W,U)$.

By Lemma 4.5, there exists a matching M in $\mathcal{F}$ with $|M| = O(\log^6 n)$ such that for any balanced $(k+1)$-set $S \subseteq V(\mathcal{F})$ containing x_0, $\mathcal{F}[S \cup V(M)]$ has a perfect matching.

Let $\mathcal{F}' = \mathcal{F} - (V(M) \cup \{x_0\})$. Then $k+1 \leq |V(\mathcal{F}') \cap X| = |V(\mathcal{F}') \cap [n]|/k - 1$. Moreover, for every $v \in X \cap V(\mathcal{F}')$, we have

$$\begin{aligned}\delta_{k-1}(N_{\mathcal{F}'}(v)) &\geq \delta_{k-1}(N_{\mathcal{F}}(v)) - |V(M)| \geq n/2 - k - k|M| \\ &> n/2 - k - k\log^6 n > n/k + \log^2 n.\end{aligned}$$

Thus by Lemma 6.2, $\mathcal{F}'$ has a matching M' covering $X \backslash (V(M) \cup \{x_0\})$. Now it is easy to see that $S := V(\mathcal{F}) - V(M \cup M')$ is a balanced $(k+1)$-set with $\{x_0\} = S \cap X$. Note that $\mathcal{F}[S \cup V(M)]$ has a perfect matching M''. Therefore, $M' \cup M''$ is a perfect matching of $\mathcal{F}$. $\square$

7. Concluding remarks

First, we point out that Theorem 1.3 follows immediately from Lemmas 3.4 and 6.3, which in turn implies Theorem 1.2.

Thus, we proved a rainbow version of the result of Rödl, Ruciński, and Szemerédi [28] that determines the co-degree threshold function for the existence of a perfect matching in a k-graph.

There are many results on various Dirac type conditions for the existence of a matching of certain size. One can ask questions about whether similar rainbow versions hold for those results. Our method of converting the rainbow matching problem to a matching problem for a special class of hypergraphs provides a way for establishing the rainbow versions by using existing tools for matching problems.

We list some results that their rainbow version may be studied using our approach. Rödl, Ruciński, and Szemerédi [28] proved that, for $n \not\equiv 0 \pmod{k}$, the minimum co-degree threshold that ensures a matching M in a k-graph H with $|V(M)| \geq |V(H)| - k$ is between $\lfloor n/k \rfloor$ and $n/k + O(\log n)$, and conjectured that this threshold function is $\lfloor n/k \rfloor$. This conjecture was proved recently by Han [8]. Treglown and Zhao [29,30] determined the minimum l-degree threshold for perfect matchings in k-graphs for $k/2 \leq l \leq k-1$. Bollobás, Daykin, and Erdős [4] considered the minimum vertex degree for the appearance of matchings of certain size. They proved that for integer $k \geq 2$, if H is a k-graph of order $n \geq 2k^3(m-1)$ and $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then H has a matching of size at least m. The bound on n is improved to $n \geq 3k^2 m$ by Huang and Zhao [11] recently. For 3-graphs, Kühn, Osthus, and Treglown [21] and, independently, Khan [14] determined the minimum vertex degree threshold for perfect matchings in 3-graphs, which improves an earlier result by Hàn, Person, and Schacht [9]. The minimum vertex degree threshold for perfect matchings in 4-graphs is obtained by Khan in [15].

Data availability

No data was used for the research described in the article.

References

[1] J. Akiyama, P. Frankl, On the size of graphs with complete-factors, J. Graph Theory 9 (1985) 197–201.
[2] R. Aharoni, D. Howard, A rainbow k-partite version of the Erdős-Ko-Rado theorem, Comb. Probab. Comput. 26 (2017) 321–337.
[3] N. Alon, J. Spencer, The Probabilistic Method, third edition, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley Sons, Hoboken, NJ, 2008.

[4] B. Bollobás, D.E. Daykin, P. Erdős, Sets of independent edges of a hypergraphs, Q. J. Math. Oxford Ser. 27 (1976) 25–32.
[5] J. Edmonds, Path, trees and flowers, Can. J. Math. 17 (1965) 449–467.
[6] P. Frankl, A. Kupavskii, Simple juntas for shifted families, Discrete Anal. (2020) 14507.
[7] J. Gao, H. Lu, J. Ma, X. Yu, On the rainbow matching conjecture for 3-uniform hypergraphs, Sci. China Math. 65 (2022) 2423–2440.
[8] J. Han, Near perfect matching in k-uniform hypergraph, Comb. Probab. Comput. 24 (2015) 723–732.
[9] H. Hán, Y. Person, M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, SIAM J. Discrete Math. 23 (2009) 732–748.
[10] H. Huang, P. Loh, B. Sudakov, The size of a hypergraph and its matching number, Comb. Probab. Comput. 21 (2012) 442–450.
[11] H. Huang, Y. Zhao, Degree versions of the Erdős-Ko-Rado theorem and Erdős hypergraph matching conjecture, J. Comb. Theory, Ser. A 150 (2017) 233–247.
[12] F. Joos, J. Kim, On a rainbow version of Dirac's theorem, Bull. Lond. Math. Soc. 52 (2020) 498–504.
[13] R. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher, J.D. Bohlinger (Eds.), Complexity of Computer Computations, Plenum, New York, 1972, pp. 85–103.
[14] I. Khan, Perfect matchings in 3-uniform hypergraphs with large vertex degree, SIAM J. Discrete Math. 27 (2013) 1021–1039.
[15] I. Khan, Perfect matchings in 4-uniform hypergraphs, J. Comb. Theory, Ser. A 116 (2016) 333–366.
[16] P. Keevash, N. Lifshitz, E. Long, D. Minzer, Hypercontractivity for global functions and sharp thresholds, arXiv:1906.05568.
[17] P. Keevash, N. Lifshitz, E. Long, D. Minzer, Global hypercontractivity and its applications, arXiv: 2103.04604.
[18] N. Keller, N. Lifshitz, The junta method for hypergraphs and Chvátal's simplex conjecture, arXiv: 1707.02643.
[19] S. Kiselev, A. Kupavskii, Rainbow matchings in k-partite hypergraphs, Bull. Lond. Math. Soc. 53 (2021) 360?–369.
[20] D. Kühn, D. Osthus, Matchings in hypergraphs of large minimum degree, J. Graph Theory 51 (2006) 269–280.
[21] D. Kühn, D. Osthus, A. Treglown, Matchings in 3-uniform hypergraphs, J. Comb. Theory, Ser. B 103 (2013) 291–305.
[22] A. Kupavskii, Rainbow version of the Erdős matching conjecture via concentration, arXiv:2104.08083.
[23] H. Lu, Y. Wang, X. Yu, Almost perfect matchings in k-partite k-graphs, SIAM J. Discrete Math. 32 (2018) 522–533.
[24] H. Lu, Y. Wang, X. Yu, A better bound on the size of rainbow matchings, J. Comb. Theory, Ser. A 195 (105700) (2023).
[25] H. Lu, Y. Wang, X. Yu, Rainbow perfect matchings for 4-uniform hypergraphs, SIAM J. Discrete Math. 36 (2022) 1645–1662.
[26] H. Lu, X. Yu, X. Yuan, Rainbow matchings for 3-uniform hypergraphs, J. Comb. Theory, Ser. A 183 (105489) (2021).
[27] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, Eur. J. Comb. 27 (2006) 1333–1349.
[28] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, J. Comb. Theory, Ser. A 116 (2009) 616–636.
[29] A. Treglown, Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, J. Comb. Theory, Ser. A 119 (2012) 1500–1522.
[30] A. Treglown, Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, J. Comb. Theory, Ser. A 120 (2013) 1463–1482.