Journal of Combinatorial Theory, Series B 162 (2023) 118–133

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

journal homepage: www.elsevier.com/locate/jctb

Polynomial χ-binding functions for t-broom-free graphs

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ARTICLE INFO

Article history:
Received 16 June 2021
Available online xxxx

Keywords:
Polynomial χ-boundedness
Polynomial binding function
t-broom-free graphs
Chair-free graphs

ABSTRACT

For any positive integer t, a t-*broom* is a graph obtained from $K_{1,t+1}$ by subdividing an edge once. In this paper, we show that, for graphs G without induced t-brooms, we have $\chi(G) = o(\omega(G)^{t+1})$, where $\chi(G)$ and $\omega(G)$ are the chromatic number and clique number of G, respectively. When $t = 2$, this answers a question of Schiermeyer and Randerath. Moreover, for $t = 2$, we strengthen the bound on $\chi(G)$ to $7\omega(G)^2$, confirming a conjecture of Sivaraman. For $t \geq 3$ and $\{t\text{-broom}, K_{t,t}\}$-free graphs, we improve the bound to $o(\omega^t)$.

1. Introduction

A class of graphs $\mathcal{G}$ is called *hereditary* if every induced subgraph of any graph in $\mathcal{G}$ also belongs to $\mathcal{G}$. One important and well-studied hereditary graph class is the family of H-*free* graphs, i.e., graphs that have no induced subgraph isomorphic to a fixed graph

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[1] This author was partially supported by NSF Grant DMS-1856645.
[2] This author was partially supported by NSF Grant DMS-1856645.
[3] This author was partially supported by NSF Grant DMS-1954134.

https://doi.org/10.1016/j.jctb.2023.04.005
0095-8956/

H. Given a class of graphs $\mathcal{H}$, we say that a graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$.

For a graph G, $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G, respectively. Tutte [10,11] showed that for any n, there exists a triangle-free graph with chromatic number at least n. (See [21] for another construction, and see [24] for more constructions.) Hence in general there exists no function of $\omega(G)$ that gives an upper bound on $\chi(G)$ for all graphs G. A hereditary class of graphs $\mathcal{G}$ is called *χ-bounded* if there is a function f (called a *χ-binding function*) such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. If f is additionally a polynomial function, then we say that $\mathcal{G}$ is *polynomially χ-bounded*. Graph classes with polynomial χ-binding functions are important, as they satisfy the Erdős-Hajnal conjecture [13] on the Ramsey number of H-free graphs.

One well-known hereditary χ-bounded graph class is the class of *perfect graphs* (i.e., graphs G such that every induced subgraph H of G satisfies $\chi(H) = \omega(H)$), which is a class of graphs for which the identity function is a χ-binding function. A *hole* in a graph G is an induced cycle in G of length at least four. An *antihole* of G is an induced subgraph of G whose complement graph is a cycle of length at least four. Chudnovsky, Robertson, Seymour, and Thomas [7] characterized perfect graphs as the set of graphs that have neither odd holes nor odd antiholes, known as the Strong Perfect Graph Theorem.

One important research direction in the area of χ-boundedness is about determining graph families $\mathcal{H}$ such that the class of $\mathcal{H}$-free graphs is χ-bounded, as well as finding the smallest possible χ-binding function for such hereditary class of graphs. By a probabilistic construction of Erdős [12], if $\mathcal{H}$ is finite and none of the graphs in $\mathcal{H}$ is acyclic, then the family of $\mathcal{H}$-free graphs is not χ-bounded. Gyárfás [15] and Sumner [26] independently conjectured that for every tree T, the class of T-free graphs is χ-bounded. This conjecture has been confirmed for some special trees (see, for example, [8,15,16,18,19,23,24]), but remains open in general.

There is a natural connection between χ-boundedness and the classical Ramsey number $R(m, n)$, the smallest integer N such that every graph on at least N vertices contains an independent set on m vertices or a clique on n vertices. Gyárfás [15] showed that the class of $K_{1,t}$-free graphs is χ-bounded with the smallest χ-binding function $f^*(\omega)$ satisfying $\frac{R(t,\omega+1)-1}{t-1} \leq f^*(\omega) \leq R(t, \omega)$. It is shown in [1–3,20] that $R(3, n) = \Theta(n^2/\log n)$ and, for fixed $t > 3$, $c_1(\frac{n}{\log n})^{\frac{t+1}{2}} \leq R(t, n) \leq c_2\frac{n^{t-1}}{\log^{t-2} n}$, where c_1 and c_2 are absolute constants. Hence for $K_{1,3}$-free (also known as *claw-free*) graphs G, we have $\chi(G) = O(\omega(G)^2/\log\omega(G))$. Chudnovsky and Seymour [9] showed that if G is a connected claw-free graph with independence number $\alpha(G) \geq 3$ then $\chi(G) \leq 2\omega(G)$.

In this paper, we consider a slightly larger class of graphs. For a positive integer t, a *t-broom* is the graph obtained from $K_{1,t+1}$ by subdividing an edge once. See the graph on the right in Fig. 1, and we denote that t-broom by $(u_0, v_1v_2, u_1, u_2, \ldots, u_t)$ or by (u_0, v_1v_2, S), where $S = \{u_1, \ldots, u_t\}$. (Note that u_0 is the vertex of degree $t+1$, v_1 is the vertex of degree 2 whose neighbors are u_0 and v_2, and $u_1, \ldots, u_t$ are the remaining neighbors of u_0.)

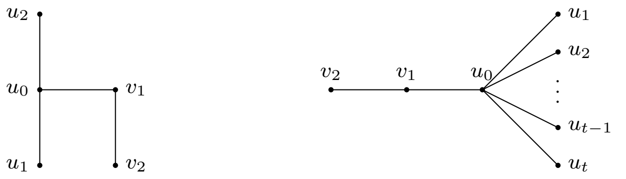

Fig. 1. The chair graph and the t-broom.

A 2-broom is also known as a *chair* graph or *fork* graph, see the left graph in Fig. 1. The class of chair-free graphs is an immediate superclass of claw-free graphs and P_4-free graphs, both of which are polynomially χ-bounded. Although it has been known [18] that the class of t-broom-free graphs is χ-bounded, it is unknown whether this class is polynomially χ-bounded. Esperet [14] conjectured that any hereditary χ-bounded graph family admits a polynomial χ-binding function. Very recently, extending an idea from a recent result of Carbonero, Hompe, Moore, and Spirkl [5], Briański, Davies and Walczak [4] disproved the conjecture. For the family of chair-free graphs, Schiermeyer and Randerath [22] asked the following:

Question 1 *(Schiermeyer and Randerath [22]). Does there exist a polynomial χ-binding function for the family of chair-free graphs?*

There has been recent work on polynomial χ-binding functions for certain subclasses of chair-free graphs. See [6,17], where the chair graph is referred to as a *fork* graph. In this paper, we show that the class of t-broom-free graphs is polynomially χ-bounded.

Theorem 1.1. *Let t be a positive integer. For t-broom-free graphs G, $\chi(G) = o(\omega(G)^{t+1})$.*

When $t = 1$, a t-broom-free graph is a P_4-free graph and, hence, perfect; so the assertion of Theorem 1.1 holds. When $t = 2$, Theorem 1.1 answers Question 1 in the affirmative. Indeed, we prove a quadratic bound for the case when $t = 2$, confirming a conjecture of Sivaraman mentioned in [17].

Theorem 1.2. *For all chair-free graphs G, $\chi(G) \leq 7\omega(G)^2$.*

We remark that after the submission of our paper, Scott, Seymour and Spirkl [25] extended Theorem 1.1 by showing that for any fixed double star T, the class of T-free graphs is polynomially χ-bounded, where a *double star* is a tree in which at most two vertices have degree more than one. In the case of t-broom-free graphs, our χ-binding function is much smaller.

Schiermeyer and Randerath [22] informally conjectured that the smallest χ-binding functions for the class of chair-free graphs and the class of claw-free graphs are asymptotically the same. Very recently, Chudnovsky, Huang, Karthick, and Kaufmann [6] proved that every $\{\text{chair}, K_{2,2}\}$-free graph G satisfies that $\chi(G) \leq \lceil \frac{3}{2}\omega(G) \rceil$. Here, we consider $\{t\text{-broom}, K_{t,t}\}$-free graphs for $t \geq 3$ and prove the following

Theorem 1.3. *Let $t \geq 3$ be an integer. For all $\{t$-broom, $K_{t,t}\}$-free graphs G, $\chi(G) = o(\omega(G)^t)$.*

In Section 2, we obtain useful structural information on t-broom-free graphs, by taking an induced complete multipartite subgraph and studying subgraphs induced by vertices at certain distance from this mulitpartite graph. In Section 3, we complete the proofs of Theorems 1.1 and 1.2. We prove Theorem 1.3 in Section 4.

In the remainder of this section, we describe notation and terminology used in the paper. For a positive integer k, we use $[k]$ to denote the set $\{1, \dots, k\}$. We denote a path by a sequence of vertices in which consecutive vertices are adjacent. For a graph G and $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. We use $\alpha(G)$ to denote the independence number of G.

Let G be a graph. For any $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v and $d_G(v) = |N_G(v)|$ is the degree of v in G. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of G, respectively. For any positive integer i, let $N^i_G(S) := \{u \in V(G)\backslash S : \min\{d_G(u,v) : v \in S\} = i\}$, where $d_G(u,v)$ is the distance between u and v in G. Then $N^1_G(S) = N_G(S)$ is the neighborhood of S in G. Moreover, we let $N^{\geq i}_G(S) := \cup_{j=i}^{\infty} N^j_G(S)$. When $S = \{s\}$ we write $N^i_G(s)$ instead of $N^i_G(\{s\})$. For any subgraph H of G, we write $N^i_G(H)$ for $N^i_G(V(H))$ and $N^{\geq i}_G(H)$ for $N^{\geq i}_G(V(H))$. When G is clear from the context, we ignore the subscript G.

Let G be a graph and let S, T be disjoint subsets of $V(G)$. For a vertex $v \in V(G)\backslash S$, we say that v is *complete* to S in G if $vs \in E(G)$ for all $s \in S$; v is *anticomplete* to S if $vs \notin E(G)$ for all $s \in S$; and v is *mixed* on S if v is neither complete nor anticomplete to S. We say that S is *complete* (respectively, *anticomplete*) to T if all vertices in S are complete (respectively, anticomplete) to T.

2. Structure of t-broom-free graphs

In our proofs of the three results stated in Section 1, we work with an induced complete multipartite subgraph Q of G and bound the chromatic numbers of subgraphs induced by vertices at certain distance from Q. In this section, we prove a few lemmas on the structure of those subgraphs. (Since the statements of the lemmas are somewhat technical, the interested reader may want to read Sections 3 and 4 before this section.) The first lemma concerns $G[N^{\geq 2}(Q)]$.

Lemma 2.1. *Let t, q be integers with $t \geq 2$ and $q \geq 2$. Let G be a t-broom-free connected graph and $V_1, \dots, V_q$ be pairwise disjoint independent sets in G, such that $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph. Suppose for every $v \in N(Q)$, v is not complete to $V(Q)$. Then $\Delta(G[N^{\geq 2}(Q)]) < 3R(t, \omega)$.*

Proof. For convenience, let $\omega = \omega(G)$. Suppose there exists $z_k \in N^k(Q)$ for some $k \geq 2$ such that $|N(z_k) \cap N^{\geq 2}(Q)| \geq 3R(t, \omega)$. Then G has a path $z_k z_{k-1} \cdots z_1 z_0$, such that

$z_0 \in V(Q)$ and $z_i \in N^i(Q)$ for $i \in [k]$. Note that this path is induced. Since $z_1 \in N(Q)$, there exist distinct $i, j \in [q]$ such that $z_1 a_i \in E(G)$ for some $a_i \in V_i$, and $z_1 a_j \notin E(G)$ for some $a_j \in V_j$.

If $G[(N(z_k) \cap N^{\geq 2}(Q)) \cap N(z_{k-2})]$ has an independent set of size t, say T, then $k \geq 3$ and $(z_1, a_i a_j, T)$ (when $k = 3$) or $(z_{k-2}, z_{k-3} z_{k-4}, T)$ (when $k \geq 4$) is an induced t-broom in G, a contradiction. So such T does not exist. Then $|(N(z_k) \cap N^{\geq 2}(Q)) \cap N(z_{k-2})| < R(t, \omega)$, which implies $|(N(z_k) \cap N^{\geq 2}(Q)) \backslash N(z_{k-2})| \geq 2R(t, \omega)$. Therefore, $|(N(z_k) \cap N^{\geq 2}(Q) \backslash N(z_{k-2})) \backslash N(z_{k-1})| \geq R(t, \omega)$ or $|(N(z_k) \cap N^{\geq 2}(Q) \backslash N(z_{k-2})) \cap N(z_{k-1})| \geq R(t, \omega)$.

In the former case, let I_1 be an independent set in $G[(N(z_k) \cap N^{\geq 2}(Q) \backslash N(z_{k-2})) \backslash N(z_{k-1})]$ with $|I_1| = t$; then $(z_k, z_{k-1} z_{k-2}, I_1)$ is an induced t-broom in G, a contradiction. In the latter case, let I_2 be an independent set of size t in $G[(N(z_k) \cap N^{\geq 2}(Q) \backslash N(z_{k-2})) \cap N(z_{k-1})]$. Then $(z_1, a_i a_j, I_2)$ (when $k = 2$) or $(z_{k-1}, z_{k-2} z_{k-3}, I_2)$ (when $k \geq 3$) is an induced t-broom in G, a contradiction. $\square$

The next lemma describes the structure of subgraphs of G induced by certain subsets of $N(Q)$.

Lemma 2.2. *Let t, q be integers with $t \geq 2$ and $q \geq 2$. Let G be a t-broom-free graph and $V_1, \ldots, V_q$ be pairwise disjoint independent sets in G, such that $|V_q| = t$ and $Q := G[\cup_{i \in [q]} V_i]$ is a complete q-partite graph. Let*

- $Z = \{v \in N(Q) : v \text{ is complete to } V_q\}$,
- $W = \{v \in N(Q) : v \text{ is anticomplete to } V_q\}$, *and,*
- *for each* $I \subseteq \cup_{i \in [q-1]} V_i$, *let*

$$W_I := \{v \in W : \ v \text{ is complete to } I \text{ and anticomplete to } \cup_{i \in [q-1]} V_i \backslash I\}.$$

Then the following statements hold:

(i) *For any distinct subsets I, I' of $\cup_{i \in [q-1]} V_i$, W_I is anticomplete to $W_{I'}$.*
(ii) *For each $z \in Z$ and for any component X of $G[W]$, z is complete to $V(X)$ or z is anticomplete to $V(X)$.*
(iii) *For any component X of $G[W]$ with $\alpha(X) \geq t$, let $Z_X := \{z \in Z : \ z \text{ is complete to } V(X)\}$; then Z_X is complete to $Z \backslash Z_X$.*

Proof. Let $I, I' \subseteq \cup_{i \in [q-1]} V_i$ such that $I \neq I'$, and assume that W_I is not anticomplete to $W_{I'}$. Then there exist $w \in W_I$ and $w' \in W_{I'}$ such that $ww' \in E(G)$. Since $I \neq I'$, we may assume that there exists $a \in I \backslash I'$. Then (a, ww', V_q) is an induced t-broom in G, a contradiction. So (i) holds.

Next, let $z \in Z$ and $w, w' \in V(X)$ such that $ww' \in E(G)$. If $zw \in E(G)$ and $zw' \notin E(G)$, then (z, ww', V_q) is an induced t-broom in G, a contradiction. Hence (ii) holds.

To prove (iii), suppose there exist $z \in Z_X$ and $z' \in Z \backslash Z_X$ such that $zz' \notin E(G)$. By assumption, X contains an independent set of size t, say T. By (ii), z' is anticomplete to $V(X)$; so z' is anticomplete to T. Choose $a \in V_q$. Now (z, az', T) is an induced t-broom in G, a contradiction. Thus we have (iii). $\square$

We now consider a specific type of complete multipartite subgraphs Q, as well as the vertices in $N(Q)$ that are complete to all but the last part of Q. We can bound the maximum degree of the subgraph of G induced by such vertices.

Lemma 2.3. *Let t, q be integers with $t \geq 2$ and $q \geq 2$. Let G be a t-broom-free graph and $V_1, \ldots, V_q$ be pairwise disjoint independent sets in G, such that $|V_i| = 1$ for $i \in [q-1]$, $|V_q| = t$, and $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph. Suppose such Q is chosen to maximize q. Let*

$$B := \{v \in N(Q) : v \text{ is complete to } V(Q)\backslash V_q\}.$$

Then $\Delta(G[B]) < R(t, \omega(G))$.

Proof. Suppose, otherwise, there exists $v \in B$ such that $d_{G[B]}(v) \geq R(t, \omega(G))$. Observe that $\omega(G[N_{G[B]}(v)]) \leq \omega(G) - 1$. Hence, $G[N_{G[B]}(v)]$ contains an independent set of size t, say T. By the definition of B, both v and T are complete to V_j for all $j \in [q-1]$. Let $V'_j = V_j$ for $j \in [q-1]$, $V'_q = \{v\}$, and $V'_{q+1} = T$. Then $G[\cup_{j\in[q+1]} V'_j]$ is an induced complete $(q+1)$-partite subgraph in G, contradicting the choice of Q. $\square$

We also need to consider the vertices in $N(Q)$ that are mixed on the last part of Q and not complete to some other part of Q. We can bound the number of such vertices.

Lemma 2.4. *Let t, q be integers with $t \geq 2$ and $q \geq 2$. Let G be a t-broom-free graph and $V_1, \ldots, V_q$ be pairwise disjoint independent sets in G, such that $|V_i| = 1$ for $i \in [q-1]$, $|V_q| = t$, and $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph. Let*

$$A := \{v \in N(Q) : \ v \text{ is mixed on } V_q \text{ and } v \text{ is not complete to } V(Q)\backslash V_q\}.$$

Then $|A| < t^2\omega(G)R(t, \omega(G))$.

Proof. For each $i \in [q-1]$, let v_i be the unique vertex in V_i. Let $v \in A$ be arbitrary. By definition there exists $j \in [q-1]$ and there exist $a_q, b_q \in V_q$, such that $va_q \in E(G), vb_q \notin E(G)$, and $vv_j \notin E(G)$. If there are multiple such choices of (a_q, b_q, j), we pick an arbitrary one and assign the vertex v the label (a_q, b_q, j). Since $q \leq \omega(G)$ and $|V_q| = t$, there are in total at most $t^2\omega(G)$ such labels.

Thus, if $|A| \geq t^2\omega(G)R(t,\omega(G))$, then there exists a set $A' \subseteq A$ such that $|A'| \geq R(t,\omega(G))$ and all vertices in A' receive the same label, say, (a_q, b_q, j). Since all vertices in A' are adjacent to a_q, it follows that $\omega(G[A']) < \omega(G)$. Hence, $G[A']$ must contain an independent set of size t, say T. Now $(a_q, v_j b_q, T)$ is an induced t-broom in G, a contradiction. $\square$

When $t = 2$, we can substitute Lemma 2.4 with the following result. Its proof is the only place where we use the Strong Perfect Graph Theorem [9]: A graph is perfect if and only if it contains no odd hole or odd antihole. (Recall that a graph G is *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs H of G.)

Lemma 2.5. *Let q be an integer with $q \geq 2$. Let G be a chair-free graph and let $V_1, \dots, V_q$ be pairwise disjoint independent sets in G, such that $|V_i| = 1$ for $i \in [q-1]$, $|V_q| = 2$, and $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph. Let $V_q = \{v_q, v'_q\}$, $V_i = \{v_i\}$ for $i \in [q-1]$, and*

$$A := \{v \in N(Q) : \ v \textit{ is mixed on } V_q \textit{ and } v \textit{ is not complete to } V(Q)\backslash V_q\}.$$

Then both $G[A \cap N(v_q)]$ and $G[A \cap N(v'_q)]$ are perfect.

Proof. Let $A' = A \cap N(v_q)$ and $A'' = A \cap N(v'_q)$. By the definition of A, $|N_G(v) \cap V_q| = 1$ for all $v \in A$; so $A' \cap A'' = \emptyset$ and $A' \cup A'' = A$. By symmetry, it suffices to prove that $G[A']$ is perfect.

We partition A' into $q-1$ pairwise disjoint sets (possibly empty) as follows. Let $A_1 := \{v \in A' : \ vv_1 \notin E(G)\}$. Suppose for some $i \in [q-1]$, we have defined $A_1, \dots, A_i$. If $i = q-1$, we are done; if $i < q-1$, let $A_{i+1} := \{v \in A' \backslash \bigcup_{j\in[i]} A_j : \ vv_{i+1} \notin E(G)\}$. Hence, by the definition of A, $A' = \cup_{i\in[q-1]} A_i$.

Observe that if $i \geq 2$ and $A_i \neq \emptyset$, then for all $j \in [i-1]$, v_j is complete to A_i. We claim that for each $i \in [q-1]$ with $A_i \neq \emptyset$, $G[A_i]$ is a clique; indeed, suppose there exist $u_1, u_2 \in A_i$ such that $u_1u_2 \notin E(G)$, then $(v_q, v_i v'_q, \{u_1, u_2\})$ is an induced chair in G, a contradiction.

Now assume for a contradiction that $G[A']$ is not perfect. Then, by the Strong Perfect Graph Theorem, $G[A']$ contains an odd hole or an odd antihole. Let $\{a_1, a_2, \dots, a_{2k+1}\}$ be the vertex set of an odd hole or odd antihole in $G[A']$ (so $k \geq 2$). For each $i \in [2k+1]$, since $a_i \in A$, there exists $\sigma(i) \in [q-1]$ such that $a_i \in A_{\sigma(i)}$. By symmetry, we may assume that $\sigma(1) = \min\{\sigma(i) : i \in [2k+1]\}$.

Suppose $G[\{a_1, \dots, a_{2k+1}\}]$ is an odd hole. Without loss of generality, assume for $i, j \in [2k+1]$, $a_ia_j \in E(G)$ if $|i-j| \equiv 1 \pmod{2k-1}$, and $a_ia_j \notin E(G)$ otherwise. Since $a_2a_{2k+1} \notin E(G)$ and $G[A_{\sigma(1)}]$ is a clique, $a_2 \notin A_{\sigma(1)}$ or $a_{2k+1} \notin A_{\sigma(1)}$. By symmetry, we may assume that $a_2 \notin A_{\sigma(1)}$. Hence, $\sigma(2) > \sigma(1)$ and $a_2v_{\sigma(1)} \in E(G)$. Similarly, $a_{2k} \notin A_{\sigma(1)}$ as $a_{2k}a_1 \notin E(G)$; so $a_{2k}v_{\sigma(1)} \in E(G)$. Since $a_1a_{2k}, a_2a_{2k} \notin E(G)$, $(v_{\sigma(1)}, a_2a_1, \{a_{2k}, v'_q\})$ is an induced chair in G, a contradiction.

Thus, $G[\{a_1, \dots, a_{2k+1}\}]$ is an odd antihole. Without loss of generality, we may assume that, for $i, j \in [2k+1]$, $a_i a_j \notin E(G)$ if $|i-j| \equiv 1 \pmod{2k-1}$, and $a_i a_j \in E(G)$ otherwise. Note that $a_1 a_3 \in E(G)$ and $a_1 a_2, a_2 a_3 \notin E(G)$. Hence $a_2 \notin A_{\sigma(1)}$ and $a_2 v_{\sigma(1)} \in E(G)$. If $a_3 \notin A_{\sigma(1)}$, then $a_3 v_{\sigma(1)} \in E(G)$; now $(v_{\sigma(1)}, a_3 a_1, \{a_2, v_q'\})$ is an induced chair in G, a contradiction. So $a_3 \in A_{\sigma(1)}$. Proceeding inductively, we see that $a_{2j+1} \in A_{\sigma(1)}$ for all $j \in [k]$. Thus $a_1 a_{2k+1} \in E(G)$ as $G[A_{\sigma(1)}]$ is a clique. This gives a contradiction since $a_1 a_{2k+1} \notin E(G)$. $\square$

We now prove a lemma about $\{t$-broom, $K_{t,t}\}$-free graphs, using a similar idea as in the proof of Lemma 2.5 above, and show that $\chi(G[A])$ admits a degenerate structure. A graph H is said to be *d-degenerate*, where d is a positive integer, if the vertices of H may be labeled as $u_1, \dots, u_n$ such that $|N_H(u_i) \cap \{u_{i+1}, \dots, u_n\}| \leq d$ for all $i \in [n-1]$. It is not hard to see that the chromatic number of a d-generate graph is at most $d+1$ by coloring its vertices greedily with respect to this ordering.

Lemma 2.6. *Let t, q be integers with $t \geq 2$ and $q \geq 2$. Let G be a $\{t$-broom, $K_{t,t}\}$-free graph and let $V_1, \dots, V_q$ be pairwise disjoint independent sets in G, such that $|V_i| = 1$ for $i \in [q-1]$, $|V_q| = t$, and $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph. Let $V_i = \{v_i\}$ for $i \in [q-1]$, let $a_1, a_2 \in V_q$, and let*

$$A := \{v \in N(Q) :\ va_1 \in E(G),\ va_2 \notin E(G), \text{ and } v \text{ is not complete to } V(Q)\backslash V_q\}.$$

Then there exists $X \subseteq A$ with $|X| \leq \omega(G)(t+2)R(t-1, \omega(G))$ such that $G[A\backslash X]$ is $(2R(t, \omega(G)) - 1)$-degenerate.

Proof. We partition A into $q-1$ pairwise disjoint sets (possibly empty) as follows. Let $A_1 := \{v \in A :\ vv_1 \notin E(G)\}$. Suppose for some $i \in [q-1]$, we have defined $A_1, \dots, A_i$. If $i = q-1$, we are done; if $i < q-1$, let $A_{i+1} := \{v \in A \backslash \bigcup_{j\in[i]} A_j :\ vv_{i+1} \notin E(G)\}$. By the definition of A, we have $A = \cup_{i\in[q-1]} A_i$ and, if $i \geq 2$ and $A_i \neq \emptyset$, then A_i is complete to $\{v_j\}$ for all $j \in [i-1]$. For simplicity of notation, if $x \in A_i$ for some $i \in [q-1]$, we use A_x to refer to A_i.

Let $\omega := \omega(G)$. Since a_1 is complete to A, $\omega(G[A]) \leq \omega - 1$. Observe that $\alpha(G[A_i]) < t$; for, if T is an independent set of size t in $G[A_i]$ then $(a_1, v_i a_2, T)$ is an induced t-broom in G, a contradiction. Thus, $|A_i| < R(t, \omega)$.

We define a *pre-order* $(A, \preceq)$ (i.e., a binary relation that is reflexive and transitive) on vertices in A. For any two vertices $u \in A_i$ and $v \in A_j$, let $u \preceq v$ if $i \leq j$. Moreover, we say $u \prec v$ if $i < j$. For any vertex $x \in A$, define $F_x = \{y \in A : x \preceq y\}$ and $F_{>x} = \{y \in A : x \prec y\}$.

Suppose for all $x \in A$, $|N(x) \cap F_x| \leq 2R(t, \omega) - 1$. Consider an ordering $u_1, \dots, u_{|A|}$ of the vertices in A such that $u_i \preceq u_{i+1}$ for all $i \in [|A| - 1]$. By the above assumption, $|N(u_i) \cap \{u_{i+1}, \dots, u_{|A|}\}| \leq |N(u_i) \cap F_{u_i}| \leq 2R(t, \omega) - 1$ for all $i \in [|A| - 1]$. Thus, $G[A]$ is $(2R(t, \omega) - 1)$-degenerate, and the assertion of Lemma 2.6 holds with $X = \emptyset$.

So we may assume that there exists some $x_1 \in A$ such that $|N(x_1) \cap F_{x_1}| \geq 2R(t,\omega)$. Choose a minimal such x_1 (with respect to $\preceq$). Let $H_1 = N(x_1) \cap F_{x_1}$. Suppose for some positive integer $k \geq 2$, we have defined $H_1, \ldots, H_{k-1}$ such that $H_i \subseteq N(x_i) \cap F_{x_i}$ for all $i \in [k-1]$. If $|N_{H_{k-1}}(x) \cap F_x| \leq 2R(t,\omega) - 1$ for all $x \in H_{k-1}$, then we terminate this process. Otherwise pick a minimal $x_k \in H_{k-1}$ (with respect to $\preceq$) such that $|N_{H_{k-1}}(x_k) \cap F_{x_k}| \geq 2R(t,\omega)$. Then let $H_k := N_{H_{k-1}}(x_k) \cap F_{x_k}$. When the above process terminates, we obtain a sequence, say $(x_k, H_k), k = 1, \ldots, s$. Note that $\{x_k : k \in [s]\}$ induces a clique in G. Moreover, by definition, each x_k is adjacent to $a_1 \in V_q$. Hence $s < \omega$.

Let $X := \cup_{k\in[s]} ((H_{k-1} \backslash H_k) \cap F_{x_k})$ where $H_0 = A$. Then $A \backslash X = \bigcup_{k\in[s+1]} (H_{k-1} \backslash F_{x_k})$, where $F_{x_{s+1}} = \emptyset$. It suffices to show that $|(H_{k-1} \backslash H_k) \cap F_{x_k}| \leq (t+2)R(t-1,\omega)$ for all $k \in [s]$ and that $G[A \backslash X]$ is $(2R(t,\omega) - 1)$-degenerate.

First we consider $G[A \backslash X]$. For $k \in [s]$, by the minimality of x_k (with respect to $\preceq$), for all $u \in H_{k-1}$ such that $u \prec x_k$, we have $|N_{H_{k-1}}(u) \cap F_u| \leq 2R(t,\omega) - 1$. Observe that $N_{A\backslash X}(u) \cap F_u \subseteq N_{H_{k-1}}(u) \cap F_u$. Hence $|N_{A\backslash X}(u) \cap F_u| \leq 2R(t,\omega) - 1$. By the terminating condition, $|N_{A\backslash X}(u) \cap F_u| \leq |N_{H_s}(u) \cap F_u| \leq 2R(t,\omega) - 1$ for all $u \in H_s$. Therefore, we can order the vertices in $A \backslash X$ as $u_1, \ldots, u_{|A\backslash X|}$, such that $u_i \preceq u_{i+1}$ for all $i \in [|A\backslash X| - 1]$; then $|N(u_i) \cap \{u_{i+1}, \ldots, u_{|A\backslash X|}\}| \leq 2R(t,\omega) - 1$ for all $i \in [|A\backslash X| - 1]$. Thus, $G[A \backslash X]$ is $(2R(t,\omega) - 1)$-degenerate.

To show that $|(H_{k-1} \backslash H_k) \cap F_{x_k}| \leq (t+2)R(t-1,\omega)$ for any $k \in [s]$, we see that no vertex in $(H_{k-1} \backslash H_k) \cap F_{x_k}$ is adjacent to x_k. So it suffices to bound the number of non-neighbors of x_k in $H_{k-1} \cap F_{x_k}$. Note that $F_{x_k} = A_{x_k} \cup F_{>x_k}$. Observe that x_k has at most $R(t-1,\omega) - 1$ non-neighbors in A_{x_k}; otherwise the graph induced on the vertex set of non-neighbors of x_k in A_{x_k} has an independent set of size $t-1$ and this implies that $\alpha(G[A_{x_k}]) \geq t$, a contradiction. Thus, it suffices to bound the number of the non-neighbors of x_k in $H_{k-1} \cap F_{>x_k}$, i.e., $|(H_{k-1} \cap F_{>x_k}) \backslash N(X_k)|$, from above by $(t+1)R(t-1,\omega)$.

Recall that x_k has at least $2R(t,\omega)$ neighbors in $H_{k-1} \cap F_{x_k}$ (as $|N_{H_{k-1}}(x_k) \cap F_{x_k}| \geq 2R(t,\omega)$). Since $|A_{x_k}| < R(t,\omega)$, $|N_{H_{k-1}}(x_k) \cap F_{>x_k}| \geq R(t,\omega)$. Hence there exists an independent set Y_k of size t in $G[N_{H_{k-1}}(x_k) \cap F_{>x_k}]$.

We claim that for each $y \in Y_k$, y and x_k has at most $R(t-1,\omega)$ common non-neighbors in $H_{k-1} \cap F_{>x_k}$. Otherwise there is an independent set T of size $t-1$ in $G[(H_{k-1} \cap F_{>x_k}) \backslash (N(x_k) \cup N(y))]$. Let u be the vertex in $\{v_1, \ldots, v_{q-1}\}$ that is anti-complete to A_{x_k} and complete to $F_{>x_k}$. Then $(u, yx_k, T \cup \{a_2\})$ is an induced t-broom in G, a contradiction.

Thus, each $y \in Y_k$ has at most $R(t-1,\omega)$ non-neighbors in $(H_{k-1} \cap F_{>x_k}) \backslash N(x_k)$. Suppose $|(H_{k-1} \cap F_{>x_k}) \backslash N(x_k)| \geq (t+1)R(t-1,\omega)$. Then there exists a set $S \subseteq (H_{k-1} \cap F_{>x_k}) \backslash N(x_k)$ with $|S| \geq (t+1)R(t-1,\omega) - |Y_k|R(t-1,\omega) = R(t-1,\omega)$ such that S is complete to Y_k. Since $|S| \geq R(t-1,\omega)$, $G[S]$ has an independent set of size $t-1$, say S'. Now $(S' \cup \{x_k\}) \cup Y_k$ induces a $K_{t,t}$ in G, a contradiction. Hence, $|(H_{k-1} \cap F_{>x_k}) \backslash N(x_k)| < (t+1)R(t-1,\omega)$. □

3. Proofs of Theorems 1.1 and 1.2

When $t = 1$, a t-broom is a path on 4 vertices. So Theorem 1.1 holds for $t = 1$ since a P_4-free graph is perfect. Hence we may assume $t \geq 2$. To prove Theorems 1.1 and 1.2, we apply induction on $\omega(G)$. The proofs are the same, except in the case when $t = 2$ we bound $\chi(G[A])$ by using Lemma 2.5 (instead of Lemma 2.4).

Let $f(\omega)$ be a convex function satisfying

- $R(t, \omega) \leq f(\omega)$ and $1 \leq f(1)$, and
- $\big(t^2 \omega R(t, \omega) + 5R(t, \omega)\big) + f(\omega - 1) + f(1) \leq f(\omega)$.

By the generalized binomial theorem, we may choose $f(\omega)$ to be $C_t \omega^2 R(t, \omega)$ for some sufficiently large C_t depending on t. Hence, using the upper bound of $R(t, \omega)$, $f(\omega)$ may be chosen such that $f(\omega) = o(\omega^{t+1})$.

We will show that $\chi(G) \leq f(\omega(G))$. Note that the assertions of Theorems 1.1 and 1.2 are clearly true when $\omega(G) = 1$. Hence, let G be a t-broom-free graph with $\omega(G) = \omega \geq 2$, and assume that for all t-broom-free graphs H with $\omega(H) < \omega$, we have $\chi(H) \leq f(\omega(H))$.

We choose pairwise disjoint independent sets $V_1, \ldots, V_q$ in G, such that

(1) $|V_q| = t$ and $|V_i| = 1$ for $i \in [q-1]$,
(2) $Q := G[\cup_{i \in [q]} V_i]$ is a complete q-partite graph, and
(3) subject to (1) and (2), q is maximum.

Note that $q \geq 2$, otherwise G is $K_{1,t}$-free; so $\Delta(G) < R(t, \omega)$ (hence $\chi(G) \leq R(t, \omega) \leq f(\omega)$) and we are done. Clearly, $q \leq \omega$.

We study the structure of G by partitioning G into several vertex disjoint subgraphs and bounding the chromatic number of each subgraph. We partition $N(Q)$ as follows:

- $A := \{v \in N(Q) : \ v \text{ is mixed on } V_q \text{ and } v \text{ is not complete to } V(Q) \backslash V_q\}$.
- $B := \{v \in N(Q) : \ v \text{ is mixed on } V_q \text{ and } v \text{ is complete to } V(Q) \backslash V_q\}$.
- $C := N(Q) \backslash (A \cup B)$.

By the maximality of q, no vertex in $N(Q)$ is complete to $V(Q)$. Thus, for any $v \in C$, either v is anticomplete to V_q, or v is complete to V_q and not complete to $V(Q) \backslash V_q$. Note that

$$V(G) = V(Q) \cup N(Q) \cup N^{\geq 2}(Q) \text{ and } N(Q) = A \cup B \cup C.$$

Since there is no edge between Q and $N^{\geq 2}(Q)$, we can color Q and $G[N^{\geq 2}(Q)]$ with at most $\max\{\chi(Q), \chi(G[N^{\geq 2}(Q)])\}$ colors. Since $\chi(Q) = q$ and $\chi(G[N^{\geq 2}(Q)]) \leq 3R(t, \omega)$ by Lemma 2.1, we then obtain the following claim.

Claim 1. $\chi(G) \leq \max\{q, 3R(t,\omega)\} + \chi(G[A]) + \chi(G[B]) + \chi(G[C])$.

Since $\Delta(G[B]) < R(t,\omega)$ (by Lemma 2.3), we have $\chi(G[B]) \leq R(t,\omega)$. By Lemmas 2.4 and 2.5, $\chi(G[A]) \leq t^2\omega R(t,\omega)$ (when $t \geq 3$) and $\chi(G[A]) \leq 2\omega$ (when $t = 2$). Thus, in view of Claim 1, we need to bound $\chi(G[C])$.

Let $Z = N(V_q) \cap C$ and $W = C \backslash Z$; so Z is complete to V_q and W is anticomplete to V_q. We consider $G[W]$ first. For any component X of $G[W]$, if I is the set of all its neighbors in $V(Q)\backslash V_q$ (note that $I \subseteq \cup_{i\in[q-1]} V_i$ and $I \neq \emptyset$) then by (i) of Lemma 2.2, $V(X)$ is complete to I and anticomplete to $\cup_{i\in[q-1]} V_i \backslash I$. It follows that any component of $G[W]$ has clique number at most $\omega-1$. Thus, each component of $G[W]$ with independent number at most $t-1$ has at most $R(t,\omega)-1$ vertices. Let X_0 denote the union of all components of $G[W]$ with independence number at most $t-1$; so $\chi(X_0) \leq R(t,\omega)$.

Claim 2. $\chi(G[C] - X_0]) \leq f(\omega-1) + f(1)$.

Proof. For any component X of $G[W] - X_0$, let

$$Z_X := \{z \in Z : \ z \text{ is complete to } V(X)\}.$$

Let $S_1, S_2, \ldots, S_p$ be the sets that form the smallest partition of Z which refines all the bipartitions $Z_X, Z\backslash Z_X$ of Z for all components X of $G[W] - X_0$. By (iii) of Lemma 2.2, S_i is complete to S_j for all distinct $i, j \in [p]$. Let $\omega_i := \omega(G[S_i])$ for $i \in [p]$. For each component X of $G[W] - X_0$, let $F_X := \{k \in [p] : \ S_k \text{ is anticomplete to } V(X)\}$. Then by (ii) of Lemma 2.2, $[p]\backslash F_X = \{k \in [p] : S_k \text{ is complete to } V(X)\}$.

Observe that $\omega(G[Z]) < \omega$ and $\omega(X) < \omega$ for every component X of $G[W] - X_0$. Hence the induction hypothesis applies to all subgraphs of $G[Z]$ and all $G[X]$. We now describe, inductively, a coloring of $G[C] - X_0$ and show it uses at most $f(\omega-1) + f(1)$ colors.

- For each $i \in [p]$, inductively color the vertices of $G[S_i]$ with colors from a set R_i, where $|R_i| \leq f(\omega_i)$. We choose R_i, $i \in [p]$, such that $R_i \cap R_j = \emptyset$ whenever $i \neq j$.
- Let R be a fixed set of colors, such that R is disjoint from $\cup_{i\in[p]} R_i$ and $|R| = \max\{f(\omega(X)) - \sum_{k\in F_X} |R_k|, 0\}$, where the maximum is taken over all components X of $G[W] - X_0$.
- For each component X of $G[W] - X_0$, it follows from induction that $\chi(X) \leq f(\omega(X))$. Note that the vertex set of X is anticomplete to S_k if $k \in F_X$. Thus we can use the colors used on $\cup_{k\in F_X} G[S_k]$ (i.e., colors in $\cup_{k\in F_X} R_k$) to color X first. If $\chi(X) \leq \sum_{k\in F_X} |R_k|$, we are done. Otherwise we use at most $f(\omega(X)) - \sum_{k\in F_X} |R_k|$ colors from R to color X. Hence for each component X of $G[W] - X_0$, we assign the vertices of X with colors from $\cup_{k\in F_X} R_k$ and, if needed, some additional colors from R.

Therefore, we have $\chi(G[C] - X_0) = \chi(G[Z] \cup (G[W] - X_0)) \leq \sum_{i\in[p]} |R_i| + |R|$. Since S_i is complete to S_j for all distinct $i, j \in [p]$ and Z is complete to V_q, we have

$\sum_{i=1}^{p} \omega_i \leq \omega - 1$. Moreover, for each component X in $G[W] - X_0$, $V(X)$ is complete to S_k if $k \in [p] \backslash F_X$; therefore, $\omega(X) + \sum_{k \in [p] \backslash F_X} \omega_k \leq \omega$. Thus, by the convexity of the function f, we have

$$\sum_{i=1}^{p} f(\omega_i) \leq f(\omega - 1),$$

and

$$f(\omega(X)) + \sum_{k \in [p] \backslash F_X} f(\omega_k) \leq f(\omega - 1) + f(1).$$

Thus,

$$\begin{aligned}
\chi(G[C] - W_0) &\leq |R| + \sum_{i=1}^{p} |R_i| \\
&= \max_{\substack{X\text{: a component} \\ \text{of } G[W]-X_0}} \left\{ f(\omega(X)) - \sum_{i \in F_X} |R_i|, 0 \right\} + \sum_{i=1}^{p} |R_i| \\
&= \max_{\substack{X\text{: a component} \\ \text{of } G[W]-X_0}} \left\{ f(\omega(X)) + \sum_{i \in [p] \backslash F_X} |R_i|, \sum_{i=1}^{p} |R_i| \right\} \\
&\leq \max_{\substack{X\text{: a component} \\ \text{of } G[W]-X_0}} \left\{ f(\omega(X) + \sum_{i \in [p] \backslash F_X} f(\omega_i), \sum_{i=1}^{p} f(\omega_i) \right\} \\
&\leq f(\omega - 1) + f(1).
\end{aligned}$$

This completes the proof of Claim 2 $\square$

We can now bound $\chi(G)$. Since $\chi(X_0) \leq R(t, \omega)$, it follows from Claim 2 that

$$\chi(G[C]) \leq \chi(G[C] - X_0]) + \chi(X_0) \leq f(\omega - 1) + f(1) + R(t, \omega).$$

Hence, by Claim 1, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\chi(G) &\leq \max\{\omega, 3R(t, \omega)\} + t^2 \omega R(t, \omega) + R(t, \omega) + f(w - 1) + f(1) + R(t, \omega) \\
&\leq t^2 \omega R(t, \omega) + 5R(t, \omega) + f(\omega - 1) + f(1) \\
&\leq f(\omega),
\end{aligned}$$

by our choice of $f(\omega)$. This proves Theorem 1.1.

When $t = 2$, we have $R(2, \omega) = \omega$, and $\chi(G[A]) \leq 2\omega(G)$ (by Lemma 2.5). Thus, using those bounds in the above inequality, we obtain

$$\chi(G) \leq f(\omega - 1) + f(1) + 7\omega.$$

By choosing $f(\omega) = 7\omega^2$, we see that $f(\omega - 1) + f(1) + 7\omega \leq f(\omega)$. Hence, $\chi(G) \leq 7\omega^2$, completing the proof of Theorem 1.2.

4. Proof of Theorem 1.3

Let $g(\omega)$ be a convex function satisfying

- $1 \leq g(1)$,
- $\omega + R(t, \omega) + (t + 2)t^2\omega R(t - 1, \omega) + (2t^2 + 4)R(t, \omega) \leq g(\omega)$, and
- $g(\omega - 1) + \omega + R(t, \omega) \leq g(\omega)$.

Similar to the proof in Section 3, by the generalized Binomial Theorem, $g(\omega)$ can be chosen such that $g(\omega) = C_t(\omega R(t, \omega) + \omega R(t - 1, \omega))$ for some large constant C_t depending only on t. Hence we may choose $g(\omega)$ such that $g(\omega) = o(\omega^t)$.

We will show that $\chi(G) \leq g(\omega(G))$ by applying induction on $\omega(G)$. It is clear that the assertion of the theorem holds when $\omega(G) = 1$. Let G be a $\{t\text{-broom}, K_{t,t}\}$-free graph with $\omega(G) = \omega \geq 2$ and, for all $\{t\text{-broom}, K_{t,t}\}$-free graphs H with $\omega(H) < \omega$, we have $\chi(H) \leq g(\omega(H))$.

We choose pairwise disjoint independent sets $V_1, \dots, V_q$ in G, such that

(1) $|V_q| = t$ and $|V_i| = 1$ for $i \in [q - 1]$,
(2) $Q := G[\cup_{i\in[q]} V_i]$ is a complete q-partite graph, and
(3) subject to (1) and (2), q is maximum.

Such Q with $q \geq 2$ must exist, otherwise G is $K_{1,t}$-free and hence $\Delta(G) < R(t, \omega)$ and we are done. Clearly, $2 \leq q \leq \omega$. Let $V_i = \{v_i\}$ for $i \in [q - 1]$. We partition $N(Q)$ as follows.

- $A := \{v \in N(Q) :\ v \text{ is mixed on } V_q \text{ and } v \text{ is not complete to } V(Q)\backslash V_q\}$.
- $B := \{v \in N(Q) :\ v \text{ is mixed on } V_q \text{ and } v \text{ is complete to } V(Q)\backslash V_q\}$.
- $C := N(Q)\backslash(A \cup B)$.

Thus, for each $v \in C$, either v is complete to V_q or v is anticomplete to V_q. Let $Z = N(V_q) \cap C$ and $W = C\backslash Z$; so $Z = \{v \in C : v \text{ is complete to } V_q\}$ and $W := \{v \in C : v \text{ is anticomplete to } V_q\}$. Note that $N(Q)$ is the disjoint union of A, B, Z, W.

Claim 1. *$N(W) \cap N^2(Q) = \emptyset$, $\chi(G[A]) \leq t^2(2R(t, \omega) + (t + 2)\omega R(t - 1, \omega))$, $\Delta(G[B]) < R(t, \omega)$, and $|Z| \leq R(t, \omega)$.*

Proof. Suppose there exists $wy \in E(G)$ with $w \in W$ and $y \in N^2(Q)$. Choose $i \in [q-1]$ such that $wv_i \in E(G)$. Now (v_i, wy, V_q) is an induced t-broom in G, a contradiction. Hence, $N(W) \cap N^2(Q) = \emptyset$.

By Lemma 2.3, we have $\Delta(G[B]) < R(t,\omega)$; hence $\chi(G[B]) \leq R(t,\omega)$. For Z, recall that Z is complete to V_q. Hence $\omega(G[Z]) \leq \omega - 1$. Moreover, since $|V_q| = t$ and G is $K_{t,t}$-free, we have that $\alpha(G[Z]) < t$. So $|Z| \leq R(t,\omega)$.

It remains to bound $\chi(G[A])$. Write $V_q = \{a_1, a_2, \ldots, a_t\}$. Since A is mixed on V_q, for each $v \in A$, there exist $i,j \in [t]$ such that $va_i \in E(G)$ and $va_j \notin E(G)$. Let $A^{(i,j)} = \{v \in A : va_i \in E(G), \text{ and } va_j \notin E(G)\}$. By Lemma 2.6, there exists $X \subseteq A^{(i,j)}$ such that $|X| \leq (t+2)\omega R(t-1,\omega)$ and $G[A^{(i,j)} \backslash X]$ is $(2R(t,\omega)-1)$-degenerate. Hence, $\chi(G[A^{(i,j)}]) \leq 2R(t,\omega) + |X| = 2R(t,\omega) + (t+2)\omega R(t-1,\omega)$. Thus, $\chi(G[A]) \leq t^2(2R(t,\omega) + (t+2)\omega R(t-1,\omega))$. $\square$

Next we consider $G[W]$. It follows from (i) of Lemma 2.2 that, for any component X of $G[W]$, $V(X)$ is complete to its neighborhood in $V(Q)\backslash V_q$. Let X_0 denote the union of all components of $G[W]$ with chromatic number at most $3R(t,\omega)$. By the definition of X_0 and the fact that every component of $G[W]$ has clique number at most $\omega - 1$, we have the following claim.

Claim 2. *$\chi(X_0) \leq 3R(t,\omega)$ and $\chi(G[W] - X_0) \leq g(\omega - 1)$.*

Claim 3. *$A \cup B$ is anticomplete to $W\backslash V(X_0)$.*

Proof. For any component X in $G[W] - X_0$, $\chi(X) > 3R(t,\omega)$ by the definition of X_0. This implies that $|V(X)| \geq \chi(X) > 3R(t,\omega)$; hence X contains an independent set of size t.

We claim that, for any distinct components X_1, X_2 of $G[W] - X_0$, $N(X_1) \cap V(Q) \subseteq N(X_2) \cap V(Q)$ or $N(X_2) \cap V(Q) \subseteq N(X_1) \cap V(Q)$. For, suppose there exist distinct $u_1, u_2 \in V(Q)$ such that $u_1 \in (N(X_1)\backslash N(X_2)) \cap V(Q)$ and $u_2 \in (N(X_2)\backslash N(X_1)) \cap V(Q)$. We know that X_2 has an independent set of size t, say T_2. Let x_1 be a vertex of X_1; then (u_2, u_1x_1, T_2) is an induced t-broom in G, a contradiction.

Thus, we choose a component X of $G[W] - X_0$ such that $N(X) \cap V(Q)$ is maximal. Observe that $N(X) \cap V(Q) = N(X) \cap (V(Q)\backslash V_q)$ which is a proper subset of $V(Q)\backslash V_q$; otherwise $V(X)$ is complete to $V(Q)\backslash V_q$ and $\Delta(X) < R(t,\omega)$ by Lemma 2.3, contradicting that $\chi(X) > 3R(t,\omega)$. So there exists $j \in [q-1]$ such that $v_j \notin N(X)$. Hence by the choice of X, $v_j \notin N(X')$ for any component X' of $G[W] - X_0$. This implies that v_j is anticomplete to $W\backslash V(X_0)$.

Now suppose there exists a vertex a in $A \cap B$ such that a is not anticomplete to $W\backslash V(X_0)$. Then there exists a component X of $G[W] - X_0$ and $w \in V(X)$ such that $aw \in E(G)$. Since a is mixed on V_q, we may assume without loss of generality that $a_1, a_2 \in V_q$ such that $aa_1 \in E(G)$ and $aa_2 \notin E(G)$. Note that $\chi(X) > 3R(t,\omega)$ and recall that v_j is anticomplete to $V(X)$.

If $av_j \in E(G)$ then $G' := G[\{v_j, a_2, a\} \cup V(X)]$ is a t-broom-free graph and $\omega(G') \leq \omega(G) = \omega$. Note that $Q' := G'[\{v_j, a_2\}]$ is a complete bipartite subgraph of G', $N_{G'}(Q') = \{a\}$, and $G'[N^{\geq 2}_{G'}(Q')] = X$. Since a is not complete to $V(Q')$ in G', we may apply Lemma 2.1 and conclude that $\Delta(G'[N^{\geq 2}_{G'}(Q')]) < 3R(t, \omega)$. Hence, $\chi(G'[N^{\geq 2}_{G'}(Q')]) \leq 3R(t, \omega)$, a contradiction as $X = G'[N^{\geq 2}_{G'}(Q')]$ and $\chi(X) > 3R(t, \omega)$.

Now assume that $av_j \notin E(G)$. Then $G'' := G[\{v_j, a_1, a\} \cup V(X)]$ is a t-broom-free graph and $Q'' := G''[\{v_j, a_1\}]$ is a complete bipartite subgraph of G'', $N_{G''}(Q'') = \{a\}$, and $X = G''[N^{\geq 2}_{G''}(Q'')]$. Since a is not complete to $V(Q'')$, we may apply Lemma 2.1 and conclude that $\Delta(G''[N^{\geq 2}_{G''}(Q'')]) < 3R(t, \omega)$. Hence, $\chi(G''[N^{\geq 2}_{G''}(Q'')]) \leq 3R(t, \omega)$, a contradiction as $X = G''[N^{\geq 2}_{G''}(Q'')]$ and $\chi(X) > 3R(t, \omega)$. $\square$

Note that $V(G) = V(Q) \cup N(Q) \cup N^{\geq 2}(Q)$ and $N(Q) = A \cup B \cup Z \cup V(X_0) \cup (W \backslash V(X_0))$ and $\chi(Q) = q$. Also note that $W \backslash V(X_0)$ is anticomplete to $A \cup B \cup N^{\geq 2}(Q)$ (by Claims 1 and 3), $V(X_0)$ is anticompete to $N^{\geq 2}(Q)$ (by Claim 1), and $W \backslash V(X_0)$ is anticomplete to $V(X_0)$ (by definition). Thus, we have

$$\chi(G) \leq q + |Z| \\ + \max\left\{\chi(G[W \backslash V(X_0)]), \chi(G[A]) + \chi(G[B]) + \max\{\chi(X_0), \chi(G[N^{\geq 2}(Q)])\}\right\}.$$

By the maximality of q, no vertex in $N(Q)$ is complete to V_j for all $j \in [q]$. Thus, by Lemma 2.1, $\chi(G[N^{\geq 2}(Q)]) \leq 3R(t, \omega)$. Therefore,

$$\chi(G) \leq \omega + R(t, \omega) + \max\{g(\omega - 1), (t + 2)t^2\omega R(t - 1, \omega) + (2t^2 + 4)R(t, \omega)\}.$$

Hence, by the choice of $g(\omega)$, we have $\chi(G) \leq g(\omega)$, completing the proof of Theorem 1.3.

Data availability

No data was used for the research described in the article.

Acknowledgment

We would like to sincerely thank the anonymous referees for their valuable comments and suggestions that greatly improved the manuscript.

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