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Tutte paths and long cycles in circuit graphs

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ABSTRACT

Thomassen proved that 4-connected planar graphs are Hamilton connected by showing that every 2-connected planar graph G contains a Tutte path P between any two given vertices, that is, every component of $G - P$ has at most three neighbors on P . In this paper, we prove a quantitative version of this result for circuit graphs, a natural class of planar graphs which includes all 3-connected planar graphs, by further controlling the number of components in $G - P$. We also give an application of this result by providing a best possible bound for the circumference of essentially 4-connected planar graphs.

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1. Introduction

The Four Color Theorem [1,2] (also see [9]) states that every plane graph is 4-face-colorable. All known proofs of the Four Color Theorem require the use of a computer. However, if a plane graph has a Hamilton cycle then one can properly four color all its faces easily.

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Tait [10] conjectured that every 3-connected cubic planar graph contains a Hamilton cycle, which, if true, would imply the Four Color Theorem. However, Tutte [13] discovered a counterexample in 1946 and, since then, families of counterexamples have been constructed, see for instance [4]. On the other hand, Whitney [15] proved that every planar triangulation without separating triangles is Hamiltonian, which was extended by Tutte [14] to all 4-connected planar graphs. Later, Thomassen [11] showed that 4-connected planar graphs are in fact Hamilton-connected, i.e., there is a Hamilton path between any two vertices in the graph. Thomas and the second author [12] further extended Thomassen's technique to show that all 4-connected projective-planar graphs contain Hamilton cycles. There has also been work on long cycles in graphs on other surfaces, see, for instance, [3] and [16].

In [11], Thomassen prove a more general result for 2-connected planar graphs. To state that result, we need the following concepts and notation. Let G be a graph and $H \subseteq G$ (i.e., H a subgraph of G). An H -bridge of G is a subgraph of G which is either induced by an edge in $E(G) \setminus E(H)$ with both incident vertices on H , or induced by the edges of G that are incident with one or two vertices in a single component of $G - H$ (the graph obtained from G by deleting all vertices in $V(H)$ and all incident edges). For any H -bridge B of G , a vertex in $V(B \cap H)$ is called an *attachment* of B on H . We say that H is a *Tutte subgraph* of G if every H -bridge of G has at most three attachments on H . Moreover, for any subgraph $F \subseteq G$, H is said to be an F -Tutte subgraph of G if H is a Tutte subgraph of G and every H -bridge of G containing an edge of F has at most two attachments on H . A *Tutte cycle* (respectively, *Tutte path*) is a Tutte subgraph that is a cycle (respectively, path).

Thomassen [11] showed that if G is a 2-connected plane graph and C is a facial cycle of G , then for any $e \in E(C)$, $u \in V(C)$, and $v \in V(G) \setminus \{u\}$, G has a C -Tutte path P between u and v and through e . We prove a similar result in which we also control the number of P -bridges. For a graph G and a subgraph P of G , let

$$\beta_G(P) = |\{B : B \text{ is a } P\text{-bridge of } G \text{ and } |V(B)| \geq 3\}|.$$

To state our result, we need additional notation. For two graphs G and H , we use $G \cup H$ and $G \cap H$ to denote the union and intersection of G and H , respectively. For any positive integer k and any graph G , a k -separation in G is a pair (G_1, G_2) of subgraphs of G such that $|V(G_1 \cap G_2)| = k$, $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, and $G_i \not\subseteq G_{3-i}$ for $i = 1, 2$. A k -cut in G is a set $S \subseteq V(G)$ with $|S| = k$ such that there exists a separation (G_1, G_2) in G with $V(G_1 \cap G_2) = S$ and $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$.

Given a plane graph G and a cycle C in G , we say that (G, C) is a *circuit graph* if G is 2-connected, C is the *outer cycle* of G (i.e., C bounds the infinite face of G), and, for any 2-cut T in G , each component of $G - T$ must contain a vertex of C . Note that C has a clockwise orientation and a counterclockwise orientation, and we may use the symmetry between these two orientations. For any distinct elements $x, y \in V(C) \cup E(C)$, we use xCy to denote the subpath of C in clockwise order from x to y such that $x, y \notin E(xCy)$.

We say that xCy is *good* if G has no 2-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{s, t\}$ such that x, s, t, y occur on xCy in order, $sCt \subseteq G_2$, and $|G_2| \geq 3$. Moreover, let

$$\tau_{Gxy} = \begin{cases} 2/3, & xCy \text{ is not good;} \\ 2/3, & |\{x, y\} \cap E(C)| = 1 \text{ and } x \text{ and } y \text{ are incident;} \\ 1/3, & |\{x, y\} \cap E(C)| = 1 \text{ and } |V(xCy)| = 2; \\ 0, & \text{otherwise.} \end{cases}$$

If there is no danger of confusion, we may drop the reference to G in the subscripts. Our main result can be stated as follows.

Theorem 1.1. *Let $n \geq 3$ be an integer, let (G, C) be a circuit graph on n vertices, let $u, v \in V(C)$ be distinct, and let $e \in E(C)$, such that u, e, v occur on C in clockwise order. Then G has a C -Tutte path P between u and v such that $e \in E(P)$ and*

$$\beta(P) \leq (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

The proof of Theorem 1.1 follows the ideas in [11], but many adjustments are needed to complete the work. In Section 2, we deal with some special cases of Theorem 1.1 when there exist certain 2-cuts in the graph. In Section 3, we complete the proof of Theorem 1.1. In Section 4, we use Theorem 1.1 to derive a bound on the circumference of essentially 4-connected planar graphs.

We conclude this section with useful notation. We often use $|G|$ to denote the number of vertices in G , and represent a path by a sequence of vertices (with consecutive vertices being adjacent). Let G be a graph. For any $S \subseteq V(G)$, $G - S$ denotes the subgraph of G obtained from G by deleting all vertices in S and all edges of G incident with S . We often write $G - H$ for $G - V(G \cap H)$. Moreover, for any family T of 2-element subsets of $V(G)$ we use $G + T$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup T$. When $T = \{\{u, v\}\}$, we write $G + uv$ instead of $G + \{\{u, v\}\}$.

2. Special cases

To help the reader get familiar with the notation involved in the statement of Theorem 1.1, we illustrate them by considering two simple cases: $e = uv$, and $|G| = 3$.

Lemma 2.1. *Theorem 1.1 holds when $e = uv$ or $|G| = 3$.*

Proof. As G is 2-connected, we have $|G| \geq 3$. First, suppose $e = uv$. Then vCu is not good because of the 2-separation $(uCv, G - uv)$; so $\tau_{vu} = 2/3$. Moreover, since u, v are both incident with e , $\tau_{ue} = \tau_{ev} = 2/3$. Hence, $P := vu$ gives the desired C -Tutte path as $\beta(P) = 1 \leq (|G| - 6)/3 + \tau_{vu} + \tau_{ev} + \tau_{ue}$.

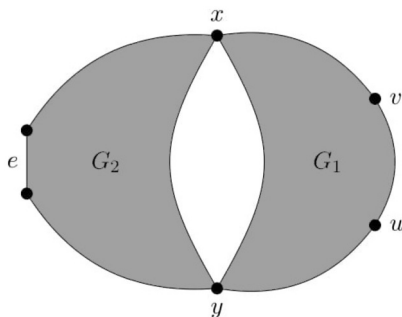


Fig. 1. The separation (G_1, G_2) in G .

Now assume $e \neq uv$ and $|G| = 3$. Further assume by symmetry that u is not incident with e . Then $\tau_{vu} = 0$, $\tau_{ue} = 1/3$, and $\tau_{ev} = 2/3$. Hence, $P := C - uv$ gives the desired C -Tutte path as $\beta(P) = 0 = (|G| - 6)/3 + \tau_{vu} + \tau_{ue} + \tau_{ev}$. $\square$

We now deal with two special cases when the plane graph G in Theorem 1.1 has certain 2-cuts. In the first case, G has a 2-cut separating $\{u, v\}$ from e . We formulate it as a lemma.

Lemma 2.2. *Suppose $n \geq 4$ is an integer and Theorem 1.1 holds for graphs on at most $n - 1$ vertices. Let (G, C) be a circuit graph on n vertices, $u, v \in V(C)$ be distinct, and $e \in E(C)$, such that u, e, v occur on C in clockwise order.*

If G has a 2-separation (G_1, G_2) such that $\{u, v\} \subseteq V(G_1)$, $\{u, v\} \not\subseteq V(G_2)$, $e \in E(G_2)$, and $|G_2| \geq 3$, then G has a C -Tutte path P between u and v such that $e \in E(P)$ and $\beta(P) \leq (n - 6)/3 + \tau_{vu} + \tau_{ue} + \tau_{ev}$.

Proof. Let $V(G_1 \cap G_2) = \{x, y\}$ with $x \in V(eCv)$ and $y \in V(uCe)$. See Fig. 1. Let $G'_i = G_i + xy$ for $i \in \{1, 2\}$ such that G'_1 is a plane graph with outer cycle $C_1 := xCy + yx$ and G'_2 is a plane graph with outer cycle $C_2 := yCx + xy$. Note that both (G'_1, C_1) and (G'_2, C_2) are circuit graphs. Let $e_1 := xy$, $n_1 := |G'_1|$, and $n_2 := |G'_2|$. Then $n_1 + n_2 = n + 2$. Since $\{u, v\} \not\subseteq V(G_2)$, we may assume by symmetry that $u \neq y$.

By assumption, G'_1 has a C_1 -Tutte path between u and v such that $e_1 \in E(P_1)$ and

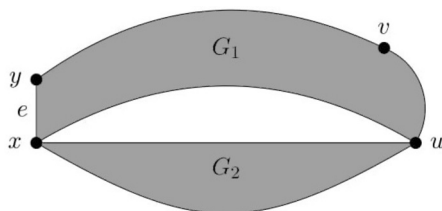
$$\beta_{G'_1}(P_1) \leq (n_1 - 6)/3 + \tau_{G'_1vu} + \tau_{G'_1ue_1} + \tau_{G'_1e_1v},$$

and G'_2 has a C_2 -Tutte path P_2 between x and y such that $e \in E(P_2)$ and

$$\beta_{G'_2}(P_2) \leq (n_2 - 6)/3 + \tau_{G'_2xy} + \tau_{G'_2ye} + \tau_{G'_2ex}.$$

Note that $P := (P_1 \cup P_2) - e_1$ is a C -Tutte path in G between u and v such that $e \in E(P)$. Moreover, $\tau_{G'_1vu} = \tau_{Gvu}$ and $\tau_{G'_2xy} = 0$. Thus,

$$\beta_G(P) = \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2)$$

Fig. 2. The separation (G_1, G_2) in G .

$$\leq (n-6)/3 - 4/3 + \tau_{Gvu} + \tau_{G'_1ue_1} + \tau_{G'_1e_1v} + \tau_{G'_2ye} + \tau_{G'_2ex}.$$

We claim that $\tau_{G'_1e_1v} + \tau_{G'_2ex} \leq \tau_{Gev} + 2/3$. This is clear if $\tau_{Gev} = 2/3$. If $\tau_{Gev} = 1/3$ then $|eCv| = 2$ and, hence, $|e_1C_1v| = |xCv| = 2$ or $|eC_2x| = |eCx| = 2$; so $\tau_{G'_1e_1v} = 1/3$ or $\tau_{G'_2ex} = 1/3$, and the inequality holds as well. Now assume $\tau_{Gev} = 0$. Then $|eCv| \geq 3$ and eCv is good in G . So $|e_1C_1v| \geq 3$ and e_1C_1v is good in G'_1 , or $|eC_2x| \geq 3$ and eC_2x is good in G'_2 , or $|e_1C_1v| = |eC_2x| = 2$. Hence, $\tau_{G'_1e_1v} = 0$, or $\tau_{G'_2ex} = 0$, or $\tau_{G'_1e_1v} = \tau_{G'_2ex} = 1/3$. Again we see that the inequality holds.

Similarly, $\tau_{G'_1ue_1} + \tau_{G'_2ye} \leq \tau_{Gue} + 2/3$. So $\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$. $\square$

The next lemma deals with a different type of 2-cuts in the graph G in Theorem 1.1.

Lemma 2.3. Suppose $n \geq 4$ is an integer and Theorem 1.1 holds for graphs on at most $n-1$ vertices. Let (G, C) be a circuit graph on n vertices, $u, v \in V(C)$ be distinct, and $e = xy \in E(C)$, such that u, x, y, v occur on C in clockwise order.

If $\{u, x\}$ or $\{v, y\}$ is a 2-cut in G then G has a C -Tutte path P between u and v such that $e \in E(P)$ and $\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$.

Proof. Suppose $\{u, x\}$ or $\{v, y\}$ is a 2-cut in G , say $\{u, x\}$ by symmetry. See Fig. 2. Then G has a 2-separation (G_1, G_2) such that $xCu \subseteq G_1$, $uCx \subseteq G_2$, and $|G_2| \geq 3$. We choose (G_1, G_2) so that G_2 is maximal. Then $ux \notin E(G_1)$. Note that $\tau_{Gue} = 2/3$.

Case 1. G_1 is 2-connected.

Then let C_1 denote the outer cycle of G_1 . Since (G, C) is a circuit graph, (G_1, C_1) is a circuit graph. By assumption, G_1 has a C_1 -Tutte path P between u and v such that $e \in E(P)$ and

$$\beta_{G_1}(P) \leq (|G_1| - 6)/3 + \tau_{G_1vu} + \tau_{G_1ue} + \tau_{G_1ev}.$$

Note that $\tau_{G_1vu} = \tau_{Gvu}$, $\tau_{G_1ue} = 0$ (as $ux \notin E(G_1)$), and $\tau_{G_1ev} = \tau_{Gev}$. So

$$\beta_G(P) = \beta_{G_1}(P) + 1 \leq (|G| - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev},$$

and P is the desired path.

Case 2. G_1 is not 2-connected.

Let $G'_1 := G_1 + ux$ be the plane graph with outer cycle $C_1 := xCu + ux$, and let $G'_2 := G_2 + xu$ be the plane graph with outer cycle $C_2 := uCx + xu$. Since (G, C) is a circuit graph, we see that both (G'_1, C_1) and (G'_2, C_2) are circuit graphs.

Note that $\tau_{G'_1vu} = \tau_{Gvu}$, $\tau_{G'_1ue} = 1/3$, and $\tau_{G'_1ev} = \tau_{Gev}$. By assumption, G'_1 has a C_1 -Tutte path P_1 between u and v such that $e \in E(P_1)$ and

$$\beta_{G'_1}(P_1) \leq (|G'_1| - 6)/3 + \tau_{G'_1vu} + \tau_{G'_1ue} + \tau_{G'_1ev} = (|G_1| - 6)/3 + \tau_{Gvu} + (\tau_{Gue} - 1/3) + \tau_{Gev}.$$

Since G_1 is not 2-connected, $ux \in E(P_1)$.

Choose $e' \in E(uC_2x)$ such that $\tau_{G'_2e'x} = 1/3$ and $\tau_{G'_2ue'} \leq 2/3$. Note that $\tau_{G'_2xu} = 0$. By assumption, G'_2 has a C_2 -Tutte path P_2 between x and u such that $e' \in E(P_2)$ and

$$\beta_{G'_2}(P_2) \leq (|G'_2| - 6)/3 + \tau_{G'_2xu} + \tau_{G'_2ue'} + \tau_{G'_2e'x} \leq (|G_2| - 6)/3 + 1.$$

Now $P := (P_1 - ux) \cup P_2$ is a C -Tutte path in G between u and v such that $e \in E(P)$. Moreover,

$$\begin{aligned} \beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G_1| - 6)/3 + \tau_{Gvu} + (\tau_{Gue} - 1/3) + \tau_{Gev} + (|G_2| - 6)/3 + 1 \\ &< (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}. \end{aligned}$$

So P is the desired path. $\square$

3. Proof of Theorem 1.1

We apply induction on n . By Lemma 2.1 and by symmetry, we may assume that u is not incident with e , $|G| = n \geq 4$, and the assertion holds for graphs on at most $n - 1$ vertices. Let $e = v'v''$ such that u, v', v'', v occur on C in clockwise order. Then by Lemma 2.3,

(1) neither $\{u, v'\}$ nor $\{v, v''\}$ is a 2-cut in G .

Moreover, by Lemma 2.2, we may assume that G has no 2-cut T such that $T \neq \{u, v\}$ and T separates e from $\{u, v\}$. Thus, by planarity, uCe is contained in a block of $G - eCv$, which is denoted by H . See Fig. 3. Note that $H \cong K_2$ or H is 2-connected. We may assume that

(2) H is 2-connected.

For, suppose that $H \cong K_2$. Then v' must have degree 2 in G and $G - v'$ is 2-connected; for otherwise, by planarity, there would exist a vertex $z \in V(v''Cv)$ such that $\{v', z\}$ is a 2-cut in G separating e from $\{u, v\}$. Let $C' := v''Cu + uv''$ be the outer cycle of

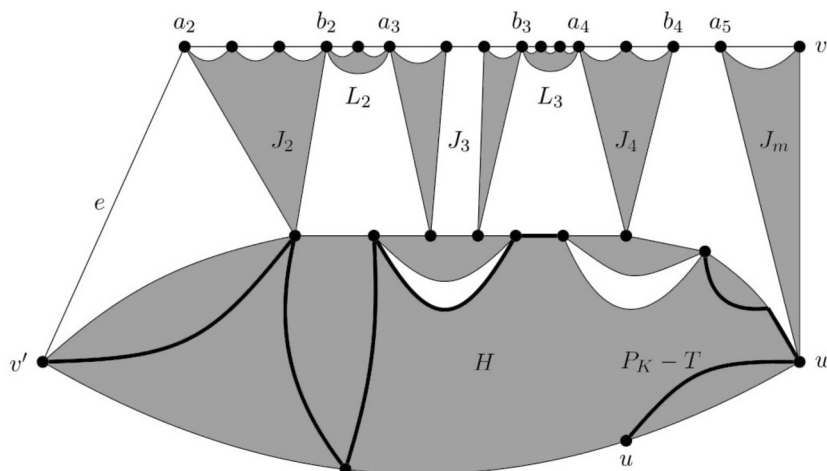


Fig. 3. The subgraph H of G and the bridges between eCv and H .

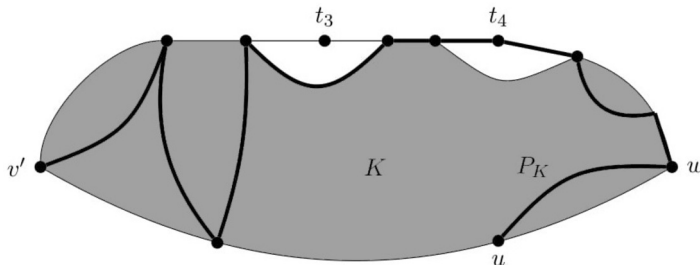
the plane graph $G' := (G - v') + uv''$, and let $e' := uv''$. Note that (G', C') is a circuit graph, $\tau_{G'ue'} = 2/3 = \tau_{ue} + 1/3$, $\tau_{G'e'v} = \tau_{Gev}$, and $\tau_{G'vu} = \tau_{Gvu}$. Hence, by induction hypothesis, G' has a C' -Tutte path P' between u and v such that $e' \in E(P')$ and

$$\beta_{G'}(P') \leq (|G'| - 6)/3 + \tau_{G'vu} + \tau_{G'ue'} + \tau_{G'e'v} = (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

Now $P := (P' - e') \cup uv'v''$ is a C -Tutte path in G between u and v such that $e \in E(P)$ and $\beta_G(P) \leq (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$. $\square$

By (2), let C' denote the outer cycle of H . Our strategy is to use induction hypothesis to find a path in H and extend it to the desired path in G along eCv . To do so, we need to avoid double counting too many vertices and, hence, we will need to contract some subgraphs of H . A 2-separation (H_1, H_2) in H with $V(H_1 \cap H_2) \subseteq V(v'C'u)$ is said to be *maximal* if there is no 2-separation (H'_1, H'_2) in H with $V(H'_1 \cap H'_2) \subseteq V(v'C'u)$, such that the subpath of $v'C'u$ between the two vertices in $V(H'_1 \cap H'_2)$ properly contains the subpath of $v'C'u$ between the two vertices in $V(H_1 \cap H_2)$.

Let K be obtained from H as follows: For every maximal 2-separation (H_1, H_2) in H with H_1 containing uCv' , contract H_2 to a single vertex (i.e., replace H_2 by a path of length 2 between the vertices of $V(H_1 \cap H_2)$). See Fig. 4. Let T denote the set of the new vertices resulted from such contractions. Note that each vertex of T has degree 2 in K . Let D be the outer cycle of K . Then $uDv' = uCv'$. Let $w \in V(vCu)$ such that $wC'v' = wCv'$ and, subject to this, wCv' is maximal. Let $w' = w$ if $w \in V(K)$; and otherwise let $w' \in T$ be the vertex resulted from the contraction of such an H_2 containing w . We may assume that

Fig. 4. The graph K .

(3) K contains a D -Tutte path P_K between u and v' such that $w' \in V(P_K)$ and

$$\beta_K(P_K) = \begin{cases} (|K| - 6)/3 + \tau_{Gue} + 1, & \text{if } |T| \geq 2, \\ (|K| - 6)/3 + \tau_{Gue} + 2/3, & \text{if } |T| \leq 1. \end{cases}$$

First, suppose $\tau_{Kuv'} < \tau_{Gue}$. Note that $|v'Du| \geq 3$ by (1) and (2) and by planarity. So we may choose $e' \in E(v'Du)$ with the following property: $|v'De'| \geq 2$; if $|T| \geq 2$ then e' is incident with w' ; and if $|T| = 1$ then e' is incident with the vertex in T and e' is incident with w' (if possible). Then $\tau_{Kv'e'} \leq 1/3$ when $|T| \leq 1$. By induction hypothesis, K contains a D -Tutte path P_K between u and v' such that $e' \in E(P_K)$ and

$$\begin{aligned} \beta_K(P_K) &\leq (|K| - 6)/3 + \tau_{Kuv'} + \tau_{Kv'e'} + \tau_{Ke'u} \\ &\leq \begin{cases} (|K| - 6)/3 + \tau_{Gue} + 1, & \text{if } |T| \geq 2, \\ (|K| - 6)/3 + \tau_{Gue} + 2/3, & \text{if } |T| \leq 1. \end{cases} \end{aligned}$$

By the choice of e' , we see that $w' \in V(P_K)$; so (3) holds.

Thus, we may assume that $\tau_{Kuv'} \geq \tau_{Gue}$. Then $\tau_{Kuv'} = 0$ (so $\tau_{Gue} = 0$), or $\tau_{Kuv'} = 2/3$ and uCv' is not good (so $\tau_{Gue} = 2/3$). Hence, $\tau_{Kuv'} = \tau_{Gue} \neq 1/3$ and $|uCv'| \geq 3$.

Suppose $T \neq \emptyset$ and let $t \in T$ such that $t \in N_K(w') \cup \{w'\}$ whenever possible. Let $N_K(t) = \{x, y\}$ with v', x, t, y, u occurring on D in clockwise order. Let $K' := (K - t) + xy$ and $D' := yDx + xy$, such that K' is a plane graph and D' is its outer cycle. Then (K', D') is a circuit graph. Let $e' := xy$. Note that $\tau_{K'uv'} = \tau_{Kuv'} = \tau_{Gue}$. By induction hypothesis, K' contains a D' -Tutte path P' between u and v' such that $e' \in E(P')$ and

$$\begin{aligned} \beta_{K'}(P') &\leq (|K'| - 6)/3 + \tau_{K'uv'} + \tau_{K'v'e'} + \tau_{K'e'u} \\ &= (|K| - 6)/3 - 1/3 + \tau_{Gue} + \tau_{K'v'e'} + \tau_{K'e'u}. \end{aligned}$$

In particular, $\beta_{K'}(P') \leq (|K| - 6)/3 + \tau_{Gue} + 1$. Note that $w' \in V(P')$ by the choice of t . Let $P_K := (P' - e') \cup xty$. If $|T| \geq 2$ then P_K is the desired path for (3). Now assume $|T| = 1$. If $u \neq y$ then $\tau_{K'e'u} \leq 1/3$ (as $|T| = 1$); so $\beta_{K'}(P') \leq (|K| - 6)/3 + \tau_{Gue} + 2/3$ and P_K is the desired path for (3). So assume $u = y$. Then by (1), $x \neq v'$ and, hence,

$\tau_{K'v'e'} \leq 1/3$ (as $|T| = 1$). So $\beta_{K'}(P') \leq (|K| - 6)/3 + \tau_{Gue} + 2/3$; again P_K is the desired path for (3).

Hence, we may assume $T = \emptyset$. If $|v'Du| \geq 4$ then we may choose $e' \in E(v'Du)$ such that $\tau_{K'v'e'} = 0$ and $\tau_{Ke'u} \leq 2/3$; so by induction hypothesis, K has a D -Tutte path P_K between u and v' such that $e' \in E(P_K)$ and $\beta_K(P_K) \leq (|K| - 6)/3 + \tau_{Kuv'} + 2/3 = (|K| - 6)/3 + \tau_{Gue} + 2/3$, and (3) holds as $w' \in V(P_K)$ (since $T = \emptyset$). Thus, we may assume $|v'Du| = 3$ and let $x \in V(v'Du) \setminus \{u, v'\}$. Then $w' \in \{u, x\}$.

Since $|u Cv'| \geq 3$ (as $\tau_{Gue} \neq 1/3$), we may choose $f \in E(u Cv')$ such that $\tau_{Kfv'} = 1/3$ and $\tau_{Kuf} \leq 2/3$. Note that $\tau_{K'v'u} = 0$ (as $T = \emptyset$). So by induction hypothesis, K has a D -Tutte path P_K between u and v' such that $f \in E(P_K)$ and $\beta_K(P_K) \leq (|K| - 6)/3 + 1$. Note $w' \in V(P_K)$ as $T = \emptyset$. Thus, we may assume $\tau_{Gue} = 0$, for, otherwise, $\beta_K(P_K) \leq (|K| - 6)/3 + \tau_{Gue} + 2/3$, and (3) holds. So uCe is good in G .

Since $T = \emptyset$, x has a neighbor in $K - v'Du$. Let $K' := (K - v'Du) + uv'$ be the plane graph whose outer cycle D' consists of uv' and the path in the outer walk of $K - v'Du$ from v' to u and containing x . Then (K', D') is a circuit graph, since uCe is good in G . Let $e' := xu$. Then $\tau_{K'uv'} = 0$, $\tau_{K'v'e'} = 0$, and $\tau_{K'e'u} = 2/3$. By induction hypothesis, K' has a D' -Tutte path P_K between u and v' such that $e' \in E(P_K)$ and

$$\beta_{K'}(P_K) \leq (|K'| - 6)/3 + \tau_{K'uv'} + \tau_{K'v'e'} + \tau_{K'e'u} = (|K| - 6)/3 + \tau_{Gue} + 2/3.$$

Clearly, P_K is also a D -Tutte path in K and $\beta_K(P_K) = \beta_{K'}(P_K)$. Since $w' \in \{u, x\}$, $w' \in V(P_K)$; so P_K is the desired path for (3). $\square$

We wish to extend P_K along eCv to the desired path P in G . Thus we need a useful description of the structure of the part of G that lies between H and eCv . See Fig. 3 for an illustration.

Let $\mathcal{B}$ be the set of $(H \cup eCv)$ -bridges of G . Then $G = H \cup eCv \cup (\bigcup_{B \in \mathcal{B}} B)$. Since H is a block of $G - eCv$, $|B \cap H| \leq 1$ for all $B \in \mathcal{B}$. Note that each vertex $t \in T$ corresponds to a $(P_K - T)$ -bridge of H whose attachments on $P_K - T$ are the neighbors of t in P_K , and that all other $(P_K - T)$ -bridges of H are also P_K -bridges of K .

For $B_1, B_2 \in \mathcal{B}$ with $|B_1 \cap H| = |B_2 \cap H| = 1$, we write $B_1 \sim B_2$ if $V(B_1 \cap H) = V(B_2 \cap H) \subseteq V(P_K - T)$, or if there exists a $(P_K - T)$ -bridge B of H such that $V(B_1 \cap H) \cup V(B_2 \cap H) \subseteq V(B - P_K)$. Clearly, $\sim$ is an equivalence relation on $\mathcal{B}$. Let $\mathcal{B}_i$, $i = 1, \dots, m$, be the equivalence classes of $\mathcal{B}$ with respect to $\sim$, such that the sets $V(H) \cap (\bigcup_{B \in \mathcal{B}_i} V(B))$ occur on D from v' to w in order $i = 1, \dots, m$, with $v' \in V(B)$ for all $B \in \mathcal{B}_1$ and $w \in V(B')$ for some $B' \in \mathcal{B}_m$. Let $a_i, b_i \in V(eCv)$ such that

- (a) $a_i \in V(B)$ for some $B \in \mathcal{B}_i$ and $b_i \in V(B')$ for some $B' \in \mathcal{B}_i$ (possibly $B = B'$),
- (b) v'', a_i, b_i, v occur on eCv in order, and
- (c) subject to (a) and (b), $a_i C b_i$ is maximal.

Note that $v'' = a_1$ and $v = b_m$. Let J_i denote the union of $a_i C b_i$, all members of $\mathcal{B}_i$, those $(H \cup eCv)$ -bridges of G whose attachments are all contained in $a_i C b_i$, and, if

applicable, also the $(P_K - T)$ -bridge of H containing $B \cap H$ for all $B \in \mathcal{B}_i$. Note that $|J_i \cap (P_K - T)| \in \{1, 2\}$, and if $|J_i \cap (P_K - T)| = 2$ we let $t_i \in T$ be the vertex corresponding to the $(P_K - T)$ -bridge of H contained in J_i . For $1 \leq i < m$, let L_i denote the union of $b_i C a_{i+1}$ and those $(H \cup e C v)$ -bridges of G whose attachments are all contained in $b_i C a_{i+1}$. Let $\mathcal{L} = \{L_i : 1 \leq i < m\}$.

By Lemma 2.2, we have $|J_1| = 2$. Thus, letting $P_1 = J_1$, we have

$$(4) \quad \beta_{J_1}(P_1) = 0 = (|J_1| - 1)/3 - 1/3.$$

For $1 < i < m$, let

- $\mathcal{J}_1 = \{J_i : |J_i \cap (P_K - T)| = 1 \text{ and } a_i \neq b_i\}$,
- $\mathcal{J}_2 = \{J_i : |J_i \cap (P_K - T)| = 2 \text{ and } t_i \notin V(P_K)\}$, and
- $\mathcal{J}_3 = \{J_i : |J_i \cap (P_K - T)| = 2 \text{ and } t_i \in V(P_K)\}$.

- (5) For $J_i \in \mathcal{J}_1$, J_i has a path P_i between a_i and b_i such that $P_i \cup (J_i \cap P_K)$ is an $a_i C b_i$ -Tutte subgraph of J_i and

$$\beta_{J_i}(P_i \cup (J_i \cap P_K)) \leq \begin{cases} (|J_i| - 2)/3 - 1/3, & \text{if } e C v \text{ is good,} \\ (|J_i| - 2)/3, & \text{otherwise.} \end{cases}$$

Let $V(J_i \cap P_K) = \{x\}$. Consider the plane graph $J'_i := J_i + a_i x$ whose outer cycle C_i consists of $a_i C b_i$, the edge $e_i := x a_i$, and the path in the outer walk of J_i between b_i and x not containing a_i . Then (J'_i, C_i) is a circuit graph. Note that $\tau_{J'_i x e_i} = 2/3$ and $\tau_{J'_i b_i x} = 0$ (as $i < m$).

Hence, by induction hypothesis, J'_i has a C_i -Tutte path P'_i between x and b_i such that $e_i \in E(P'_i)$ and $\beta_{J'_i}(P'_i) \leq (|J_i| - 6)/3 + \tau_{J'_i e_i b_i} + 2/3$. Note that $\tau_{J'_i e_i b_i} \leq 2/3$ and if $e C v$ is good in G then $\tau_{J'_i e_i b_i} \leq 1/3$ (as $a_i \neq b_i$). So $P_i := P'_i - x$ gives the desired path for (5). $\square$

- (6) For $J_i \in \mathcal{J}_2$, J_i has a path P_i between a_i and b_i such that $P_i \cup (J_i \cap (P_K - T))$ is an $a_i C b_i$ -Tutte subgraph of J'_i and

$$\beta_{J_i}(P_i \cup (J_i \cap (P_K - T))) \leq \begin{cases} (|J_i| - 4)/3 + 1/3, & \text{if } e C v \text{ is good,} \\ (|J_i| - 4)/3 + 1, & \text{otherwise.} \end{cases}$$

Let $V(J_i \cap (P_K - T)) = \{x, y\}$ such that v', y, x, w occur on D in clockwise order. Let J'_i be the block of $J_i - \{x, y\}$ containing $a_i C b_i$, and let C_i be the outer cycle of J'_i . Note that $a_i C_i b_i = a_i C b_i$.

By planarity there exists a vertex $z \in V(b_i C_i a_i) \setminus \{a_i, b_i\}$ such that $b_i C_i z - z$ contains no neighbor of y and $z C_i a_i - z$ contains no neighbor of x .

First, suppose $b_i C_i a_i$ is good in J'_i . Let $e_i \in E(b_i C_i a_i)$ be incident with z such that $\tau_{J'_i e_i a_i} \leq 1/3$ or $\tau_{J'_i b_i e_i} \leq 1/3$. Now by induction hypothesis, J'_i has a C_i -Tutte path P_i between a_i and b_i such that $e_i \in E(P_i)$ and $\beta_{J'_i}(P_i) \leq (|J'_i| - 6)/3 + \tau_{J'_i a_i b_i} + 1$. If $J'_i \neq J_i - \{x, y\}$ then $|J'_i| \leq |J_i| - 3$ and, since $b_i C_i a_i$ is good in J'_i ,

$$\begin{aligned} \beta_{J_i}(P_i \cup (J_i \cap (P_K - T))) &= \beta_{J'_i}(P_i) + 1 \\ &\leq (|J_i| - 3 - 6)/3 + \tau_{J'_i a_i b_i} + 1 + 1 \\ &= (|J_i| - 4)/3 + \tau_{J'_i a_i b_i} + 1/3. \end{aligned}$$

If $J'_i = J_i - \{x, y\}$ then $|J'_i| = |J_i| - 2$ and

$$\begin{aligned} \beta_{J_i}(P_i \cup (J_i \cap (P_K - T))) &= \beta_{J'_i}(P_i) \\ &\leq (|J_i| - 2 - 6)/3 + \tau_{J'_i a_i b_i} + 1 \\ &= (|J_i| - 4)/3 + \tau_{J'_i a_i b_i} - 1/3. \end{aligned}$$

Since $\tau_{J'_i a_i b_i} = 0$ (if eCv is good) and $\tau_{J'_i a_i b_i} \leq 2/3$ (if eCv is not good), we see that P_i gives the desired path for (6).

Now assume that $b_i C_i a_i$ is not good in J'_i . Then let (M_1, M_2) be a 2-separation in J'_i such that $a_i C b_i \subseteq M_1$, $|M_2| \geq 3$, $z \in M_2$ (whenever possible), and, subject to these conditions, M_2 is minimal. Let $V(M_1 \cap M_2) = \{z_1, z_2\}$ such that a_i, b_i, z_1, z_2 occur on C_i in clockwise order. Let $M'_1 := M_1 + z_1 z_2$ be the plane graph with outer cycle $D_1 := z_2 C_i z_1 + z_1 z_2$, and let $M'_2 := M_2 + z_2 z_1$ be the plane graph with outer cycle $D_2 := z_1 C_i z_2 + z_2 z_1$. Then (M'_1, D_1) and (M'_2, D_2) are circuit graphs. Let $f := z_1 z_2$.

By induction hypothesis, M'_1 has a D_1 -Tutte path R_1 between a_i and b_i such that $f \in E(R_1)$ and

$$\beta_{M'_1}(R_1) \leq (|M'_1| - 6)/3 + \tau_{M'_1 a_i b_i} + 4/3 = (|M'_1| - 6)/3 + \tau_{J'_i a_i b_i} + 4/3.$$

Also by induction hypothesis and choosing an edge $f' \in E(z_1 C_i z_2)$ so that $\tau_{M'_2 z_1 f'} \leq 1/3$ or $\tau_{M'_2 f' z_2} \leq 1/3$, we see that M'_2 has a D_2 -Tutte path R_2 between z_1 and z_2 such that $f' \in E(R_2)$ and

$$\beta_{M'_2}(R_2) \leq (|M'_2| - 6)/3 + 1,$$

as $\tau_{M'_2 z_2 z_1} = 0$ (since (G, C) is a circuit graph). Let $P_i = (R_1 - f) \cup R_2$, which is a path in J'_i between a_i and b_i such that $P_i \cup (J_i \cap (P_K - T))$ is an $a_i C b_i$ -Tutte subgraph of J'_i . Note that $z \in V(P_i)$ by the choice of (M_1, M_2) (that $z_2 \in V(M_2)$ whenever possible and M_2 is minimal).

If $J'_i = J_i - \{x, y\}$ then $|J'_i| = |J_i| - 2$ and

$$\beta_{J_i}(P_i \cup (J_i \cap (P_K - T))) = \beta_{M'_1}(R_1) + \beta_{M'_2}(R_2)$$

$$\begin{aligned}
&\leq (|M'_1| - 6)/3 + \tau_{J'_i a_i b_i} + 4/3 + (|M'_2| - 6)/3 + 1 \\
&= (|J'_i| - 6)/3 + \tau_{J'_i a_i b_i} + 1 \\
&= (|J_i| - 4)/3 + \tau_{J'_i a_i b_i} - 1/3.
\end{aligned}$$

If $J'_i \neq J_i - \{x, y\}$ then $|J'_i| \leq |J_i| - 3$ and

$$\begin{aligned}
\beta_{J_i}(P_i \cup (J_i \cap (P_K - T))) &= \beta_{M'_1}(R_1) + \beta_{M'_2}(R_2) + 1 \\
&\leq (|M'_1| - 6)/3 + \tau_{J'_i a_i b_i} + 4/3 + (|M'_2| - 6)/3 + 2 \\
&= (|J'_i| - 6)/3 + \tau_{J'_i a_i b_i} + 2 \\
&\leq (|J_i| - 4)/3 + \tau_{J'_i a_i b_i} + 1/3.
\end{aligned}$$

Therefore, since $\tau_{J'_i a_i b_i} = 0$ if eCv is good and $\tau_{J'_i a_i b_i} \leq 2/3$ otherwise, we see that P_i is the desired path for (6). $\square$

- (7) For $J_i \in \mathcal{J}_3$, J_i has disjoint paths P_i, P'_i such that P_i is between a_i and b_i , P'_i is between the two vertices in $V(J_i \cap (P_K - T))$, $P_i \cup P'_i$ is an $a_i C b_i$ -Tutte subgraph of J_i , and

$$\beta_{J_i}(P_i \cup P'_i) \leq \begin{cases} (|J_i| - 4)/3 - 1/3, & \text{if } eCv \text{ is good,} \\ (|J_i| - 4)/3, & \text{otherwise.} \end{cases}$$

Let $V(J_i \cap (P_K - T)) = \{x, y\}$ and assume that v', y, x, w occur on D in clockwise order. Consider the plane graph $J'_i := J_i + b_i x$ with $a_i C b_i, e_i := b_i x, y$ occur on its outer cycle C_i in clockwise order. (Note that $xy \notin E(J_i)$ by the definition of J_i .) Then, since (G, C) is a circuit graph, (J'_i, C_i) is a circuit graph and $\tau_{J'_i e_i y} = \tau_{J'_i y a_i} = 0$.

Thus, by induction hypothesis, J'_i contains a C_i -Tutte path R_i between a_i and y such that $e_i \in E(R_i)$ and

$$\beta_{J'_i}(R_i) \leq (|J'_i| - 6)/3 + \tau_{J'_i a_i e_i} = (|J_i| - 4)/3 + \tau_{J'_i a_i e_i} - 2/3.$$

So $R_i - e_i$ is an $a_i C b_i$ -Tutte subgraph of J_i such that

$$\beta_{J_i}(R_i - e_i) = \beta_{J'_i}(R_i) \leq (|J_i| - 4)/3 + \tau_{J'_i a_i e_i} - 2/3.$$

Note that $\tau_{J'_i a_i e_i} \leq 1/3$ (if eCv is good) and $\tau_{J'_i a_i e_i} \leq 2/3$ (if eCv is not good). So $R_i - e_i$ gives the desired paths for (7). $\square$

Next, we consider J_m . Note that if $|J_m \cap (P_K - T)| = 2$ then $t_m = w' \in V(P_K) \cap T$, and if $|J_m \cap (P_K - T)| = 1$ then $w' \in V(J_m \cap (P_K - T))$. (Recall the definition of w' in the paragraph preceding (3).) Let $c = 2$ if $|J_m \cap (P_K - T)| = 1$, and $c = 4$ if $|J_m \cap (P_K - T)| = 2$. Note that when $c = 2$, c is the number of vertices double counted

by $|J_m|$ and $|K \cup (J_1 \cup L_1) \cup \dots \cup (J_{m-1} \cup L_{m-1})|$. In the case $c = 4$, c counts the vertex t_m and the vertices double counted by $|J_m|$ and $|K \cup (J_1 \cup L_1) \cup \dots \cup (J_{m-1} \cup L_{m-1})|$.

- (8) J_m has disjoint paths P_m, P'_m with P_m between a_m and $b_m = v$ and P'_m between the vertices in $V(J_m \cap (P_K - T))$, such that $P_m \cup P'_m$ is an $(a_m C u \cap J_m)$ -Tutte subgraph of J_m and

$$\beta_{J_m}(P_m \cup P'_m) \leq \begin{cases} (|J_m| - c)/3 + \tau_{Gvu} - 1/3, & \text{if } a_m \neq b_m \text{ and } eCv \text{ is good,} \\ (|J_m| - c)/3 + \tau_{Gvu}, & \text{otherwise.} \end{cases}$$

First, suppose $a_m = b_m$ and $|J_m \cap (P_K - T)| = 1$. Then $c = 2$, and let $P_m = a_m$ and $P'_m = w$. Now $\beta_{J_m}(P_m \cup P'_m) \leq 1$, with equality only when $|J_m| \geq 3$, in which case, vCu is not good in G and $\tau_{Gvu} = 2/3$. So $\beta_{J_m}(P_m \cup P'_m) \leq (|J_m| - c)/3 + \tau_{Gvu}$ as $c = 2$.

Now assume $a_m \neq b_m$ or $|J_m \cap (P_K - T)| = 2$. If $|J_m \cap (P_K - T)| = 2$ then let $V(J_m \cap (P_K - T)) = \{x, y\}$ such that v', y, w', x occur on D (the outer cycle of K) in clockwise order, and if $|J_m \cap (P_K - T)| = 1$ then let $y = x = w'$. Consider the plane graph $J_m^* := J_m + ya_m$ with outer cycle C_m containing $a_m C x$ and ya_m . Then (J_m^*, C_m) is a circuit graph. Let $e_m := ya_m$. Note that $\tau_{J_m^* b_m x} \leq \tau_{Gvu}$, if $x = y$ then $\tau_{J_m^* x e_m} = 2/3 = (4 - c)/3$, and if $x \neq y$ then $\tau_{J_m^* x e_m} = 0 = (4 - c)/3$.

By induction hypothesis, J_m^* has a C_m -Tutte path P_m^* between $b_m = v$ and x such that $e_m \in E(P_m^*)$ and

$$\begin{aligned} \beta_{J_m^*}(P_m^*) &\leq (|J_m^*| - 6)/3 + \tau_{Gvu} + (4 - c)/3 + \tau_{J_m^* e_m b_m} \\ &= (|J_m| - c)/3 - 2/3 + \tau_{Gvu} + \tau_{J_m^* e_m b_m}. \end{aligned}$$

Note that $\tau_{J_m^* e_m b_m} \leq 1/3$ (when eCv is good in G) and $\tau_{J_m^* e_m b_m} \leq 2/3$ (when eCv is not good in G). Hence, $P_m^* - ya_m$ gives the desired paths for (8). $\square$

Next, we consider the family $\mathcal{L} := \{L_i : 1 \leq i < m\}$, see its definition preceding (4).

- (9) For each $L_i \in \mathcal{L}$, L_i contains a $b_i C a_{i+1}$ -Tutte path Q_i from b_i to a_{i+1} such that $\beta_{L_i}(Q_i) \leq \max\{0, (|L_i| - 2)/3 - 1/3\}$.

If $|b_i C a_{i+1}| \leq 2$ then let $Q_i := b_i C a_{i+1}$; we see that $\beta_{L_i}(Q_i) = 0$ as (G, C) is a circuit graph. So assume $|b_i C a_{i+1}| \geq 3$. Then consider the plane graph $L'_i := L_i + a_{i+1} b_i$ with outer cycle $D_i := b_i C a_{i+1} + a_{i+1} b_i$. Note that (L'_i, D_i) is a circuit graph. Choose an edge $e_i \in E(b_i C a_{i+1})$ so that $\tau_{L'_i b_i e_i} = 1/3$. Note that $\tau_{L'_i a_{i+1} b_i} = 0$ and $\tau_{L'_i e_i a_{i+1}} \leq 2/3$. So by induction hypothesis, L'_i contains a D_i -Tutte path Q'_i between b_i and a_{i+1} such that $e_i \in E(Q'_i)$ and $\beta_{L'_i}(Q'_i) \leq (|L'_i| - 6)/3 + 1 = (|L_i| - 2)/3 - 1/3$. $\square$

Let P be the union of $P_K - T$, $P_i \cup P'_i$ for $i = 1, \dots, m$ (where we let $P'_i = J_i \cap (P_K - T)$ when $J_i \in \mathcal{J}_1 \cup \mathcal{J}_2$), and Q_i for $i = 1, \dots, m - 1$. Clearly, P is a path between u and v and $e \in E(P)$.

It is easy to see that if B is a P -bridge of G then B is a P_K -bridge of K , or a $(P_i \cup P'_i)$ -bridge of J_i for some i with $1 \leq i \leq m$, or a Q_i -bridge of L_i for some i with $1 \leq i < m$, or $|B| = 2$ and $|B \cap eCv| = |B \cap (P_K - T)| = 1$. Thus, P is a C -Tutte path in G between u and v and containing e . Note that

$$\begin{aligned}\mathcal{J}_2 &= \{J_i : 1 < i < m, |J_i \cap (P_K - T)| = 2 \text{ and } P'_i = J_i \cap (P_K - T)\} \text{ and} \\ \mathcal{J}_3 &= \{J_i : 1 < i < m, |J_i \cap (P_K - T)| = 2 \text{ and } P'_i \neq J_i \cap (P_K - T)\}.\end{aligned}$$

If we extend $P_K - T$ from v' to v through $J_1, L_1, J_2, L_2, \dots, J_{m-1}, L_{m-1}, J_m$ in order, we see that

- J_1 and K double count 1 vertex (namely, v');
- when $|J_m \cap (P_K - T)| = 1$, J_m and $K \cup (J_1 \cup L_1) \cup \dots \cup (J_{m-1} \cup L_{m-1})$ double count $c = 2$ vertices (namely, a_m and w'); when $|J_m \cap (P_K - T)| = 2$, J_m and $K \cup (J_1 \cup L_1) \cup \dots \cup (J_{m-1} \cup L_{m-1})$ count $c = 4$ additional vertices (double counting a_m and the vertices in $V(J_m \cap P_K)$ and counting the additional vertex t_m);
- L_i and $K \cup (J_1 \cup L_1) \cup \dots \cup (J_{i-1} \cup L_{i-1}) \cup J_i$ double count 1 vertex, namely b_i ;
- for $1 < i < m$, if $J_i \in \mathcal{J}_1$ then J_i and $K \cup (J_1 \cup L_1) \cup \dots \cup (J_{i-1} \cup L_{i-1})$ double count 2 vertices: a_i and the vertex in $V(J_i \cap P_K)$;
- for $1 < i < m$, if $J_i \in \mathcal{J}_2 \cup \mathcal{J}_3$ then J_i and $K \cup (J_1 \cup L_1) \cup \dots \cup (J_{i-1} \cup L_{i-1})$ double count a_i and the vertices in $V(J_i \cap P_K)$ and count the additional vertex t_i .

Note that for each $J_i \in \mathcal{J}_2$, the P_K -bridge of K corresponding to the vertex $t_i \in T$ does not contribute to $\beta_G(P)$. Thus,

$$\begin{aligned}\beta_G(P) &= \beta_K(P_K) + \beta_{J_1}(P_1) + \sum_{J_i \in \mathcal{J}_1} \beta_{J_i}(P_i \cup P'_i) + \sum_{J_i \in \mathcal{J}_2} (\beta_{J_i}(P_i \cup P'_i) - 1) + \\ &\quad \sum_{J_i \in \mathcal{J}_3} \beta_{J_i}(P_i \cup P'_i) + \beta_{J_m}(P_m \cup P'_m) + \sum_{i=1}^{m-1} \beta_{L_i}(Q_i).\end{aligned}$$

We may assume

(10) eCv is good in G .

For, suppose eCv is not good in G . Then $\tau_{Gev} = 2/3$. Hence, by (4)–(9) and the above observation on double counting vertices, we have

$$\begin{aligned}\beta_G(P) &\leq \beta_K(P_K) + ((|J_1| - 1)/3 - 1/3) + \sum_{J_i \in \mathcal{J}_1} (|J_i| - 2)/3 + \sum_{J_i \in \mathcal{J}_2 \cup \mathcal{J}_3} (|J_i| - 4)/3 \\ &\quad + (|J_m| - c)/3 + \tau_{Gvu} + \sum_{L_i \in \mathcal{L}} \max\{0, (|L_i| - 2)/3 - 1/3\}\end{aligned}$$

$$\begin{aligned}
 &\leq (n-6)/3 - 1/3 + \tau_{Gvu} + (\beta_K(P_K) - (|K| - 6)/3) \\
 &\leq (n-6)/3 - 1/3 + \tau_{Gvu} + \tau_{Gue} + 1 \quad (\text{by (3)}) \\
 &= (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.
 \end{aligned}$$

So P is the desired path. $\square$

By (10), $|L_i| \leq 2$ for all $L_i \in \mathcal{L}$. By (4)–(10) and the above observation on double counting vertices, we have

$$\begin{aligned}
 \beta_G(P) &\leq \beta_K(P_K) + (|J_1| - 1)/3 - 1/3 + \sum_{J_i \in \mathcal{J}_1} ((|J_i| - 2)/3 - 1/3) + \\
 &\quad \sum_{J_i \in \mathcal{J}_2} ((|J_i| - 4)/3 + 1/3 - 1) + \\
 &\quad \sum_{J_i \in \mathcal{J}_3} ((|J_i| - 4)/3 - 1/3) + (|J_m| - c)/3 + \tau_{Gvu} \\
 &\leq (n-6)/3 + \tau_{Gvu} - |\mathcal{J}_1|/3 - (|T| + 1)/3 + (\beta_K(P_K) - (|K| - 6)/3),
 \end{aligned}$$

since $|T| = |\mathcal{J}_2 \cup \mathcal{J}_3|$. We may assume that

$$(11) \quad \mathcal{J}_i = \emptyset \text{ for } i = 1, 2, 3, \text{ and } |eCv| \geq 3.$$

First, we may assume $|T| \leq 1$. For, suppose $|T| \geq 2$. Then, since $\beta_K(P_K) \leq (|K| - 6)/3 + \tau_{Gue} + 1$ (by (3)),

$$\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev},$$

and P gives the desired path.

Therefore, $\beta_K(P_K) \leq (|K| - 6)/3 + \tau_{Gue} + 2/3$ by (3). We may also assume $\mathcal{J}_i = \emptyset$ for $i = 1, 2, 3$. For, otherwise, $|\mathcal{J}_1| \geq 1$ or $|T| \geq 1$; so

$$\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} - 2/3 + \tau_{Gue} + 2/3 \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev},$$

and P is the desired path.

If $|eCv| = 1$ then $\tau_{Gev} = 2/3$ and

$$\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + 2/3 = (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev};$$

P gives the desired path. If $|eCv| = 2$ then $\tau_{Gev} = 1/3$ and

$$\beta_G(P) \leq (n-6)/3 + \tau_{Gvu} - 1/3 + \tau_{Gue} + 2/3 = (n-6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev};$$

so P is the desired path. Therefore, we may assume $|eCv| \geq 3$. $\square$

Suppose $a_m \neq b_m$. Then by (3), (4), (8), (10), and (11), we have

$$\begin{aligned}\beta_G(P) &\leq \beta_K(P_K) + ((|J_1| - 1)/3 - 1/3) + ((|J_m| - c)/3 + \tau_{Gvu} - 1/3) \\ &\leq ((|K| - 6)/3 + \tau_{Gue} + 2/3) + ((|J_1| - 1)/3 - 1/3) \\ &\quad + ((|J_m| - c)/3 + \tau_{Gvu} - 1/3) \\ &\leq (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} \\ &= (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.\end{aligned}$$

So P is the desired path.

Thus, we may assume $a_m = b_m$. If $|J_m| = 2$ then

$$\begin{aligned}\beta_G(P) &= \beta_K(P_K) \\ &\leq (|K| - 6)/3 + \tau_{Gue} + 2/3 \quad (\text{by (3) and (11)}) \\ &\leq (n - 6)/3 + \tau_{Gue} - 1/3 \quad (\text{since } |eCv| \geq 3) \\ &\leq (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev},\end{aligned}$$

and P is the desired path. So assume $|J_m| \geq 3$. Then $\tau_{Gvu} = 2/3$. Since $|eCv| \geq 3$,

$$\begin{aligned}\beta_G(P) &= \beta_K(P_K) + 1 \\ &\leq (|K| - 6)/3 + \tau_{Gue} + 2/3 + 1 \quad (\text{by (3) and (11)}) \\ &\leq (n - 6)/3 + \tau_{Gue} + 2/3 \quad (\text{since } |eCv| \geq 3) \\ &\leq (n - 6)/3 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.\end{aligned}$$

Again, P gives the desired path. $\square$

4. Essentially 4-connected planar graphs

There has been interest in finding good lower bounds on the circumference of 3-connected planar graphs. (The *circumference* of a graph is the length of a longest cycle in that graph.) For instance, Chen and the second author [5] showed that the circumference of a 3-connected planar n -vertex graph is at least $n^{\log_3 2}$, which is best possible because of iterated planar triangulations $Tr(k)$: starting with $Tr(0) = K_3$, for each $k \geq 1$, add a vertex in each face of $Tr(k - 1)$ and connect it with an edge to each vertex on the boundary of that face.

A graph is *essentially 4-connected* if it is connected and, for any $S \subseteq V(G)$ with $|S| < 4$, $G - S$ is connected or has exactly two components one of which has exactly one vertex. Jackson and Wormald [7] proved that the circumference of any essentially 4-connected n -vertex planar graph is at least $(2n + 4)/5$. Very recently, this bound has been improved to $5(n + 2)/8$ by Fabrici, Harant, Mohr, and Schmidt [6]. Using Theorem 1.1,

we give a short proof of the following result. We mention that Kessler and Schmidt [8] announced an independent proof using a different technique.

Theorem 4.1. *Let $n \geq 6$ be an integer and let G be any essentially 4-connected n -vertex planar graph. Then the circumference of G is at least $\lceil (2n + 6)/3 \rceil$.*

Proof. First, suppose G is 4-connected. Fix a planar drawing of G and let T be the outer cycle of G . Let $uv, e \in E(T)$ be distinct. By applying Theorem 1.1, G has a T -Tutte path P between u and v such that $e \in E(P)$. Since G is 4-connected, $\beta_G(P) = 0$; so P is in fact a Hamilton path. Hence, $P + uv$ is a Hamilton cycle in G and, thus, has length n , which is at least $(2n + 6)/3$ (as $n \geq 6$).

Hence, we may assume that G is not 4-connected. Then, since G is essentially 4-connected, there exists $x \in V(G)$ such that x has degree 3 in G . So let $N_G(x) = \{u, v, w\}$ and let $H := G - x$ and assume that H is a plane graph with u, v, w on the outer cycle C of H in counter clockwise order. Note that (H, C) is a circuit graph.

Suppose two of $|uCw|, |wCv|, |vCu|$ is at least 3. Without loss of generality, we may assume that $|uCw| \geq 3$ and $|wCv| \geq 3$. Let $e \in E(uCw)$ be incident with w . Then $\tau_{Hvu} = 0$, $\tau_{Hue} \leq 1/3$, and $\tau_{Hev} = 0$. Hence, by Theorem 1.1, H has a C -Tutte path between u and v such that $e \in E(P)$ and $\beta_H(P) \leq (n - 7)/3 + 1/3 = (n - 6)/3$. Thus, $Q := P \cup uxy$ is a Tutte cycle in G such that $\beta_G(Q) \leq (n - 6)/3$. Since G is essentially 4-connected, every Q -bridge is a $K_{1,3}$. Hence, $|Q| \geq n - (n - 6)/3 = (2n + 6)/3$.

So we may assume that $|wCv| = |vCu| = 2$. Consider the plane graph $K := H - wv$ whose outer cycle D contains vCw . Since G is essentially 4-connected, K is 2-connected; so (K, D) is a circuit graph. We can choose $e \in E(wDv)$ incident with w . Now $\tau_{Kvu} = 0$, $\tau_{Kue} \leq 1/3$, and $\tau_{Kev} \leq 1/3$.

If $\tau_{Kue} = 0$ or $\tau_{Kev} = 0$, then by Theorem 1.1, K has a D -Tutte path P between u and v such that $e \in E(P)$ and $\beta_K(P) \leq (n - 7)/3 + 1/3 = (n - 6)/3$. Thus, $Q := P \cup uxy$ is a cycle in G with $|Q| \geq n - (n - 6)/3 = (2n + 6)/3$.

So assume $\tau_{Kue} = \tau_{Kev} = 1/3$ and, hence, $|wDv| = 3$ and $|uDw| = 2$. Since $n \geq 6$ and G is essentially 4-connected, one of $\{v, w\}$ has a neighbor inside D , say w by symmetry. Now consider the plane graph $J := H - uw$, which is 2-connected as G is essentially 4-connected. Let F denote the outer cycle of J , which contains $\{u, v, w\}$. Clearly, (J, F) is a circuit graph. Choose $f \in E(uFw)$ incident with w . Then $\tau_{Juf} \leq 1/3$, $\tau_{Jfv} = 0$, and $\tau_{Jvu} = 0$. Hence, by Theorem 1.1, J has an F -Tutte path between u and v such that $f \in E(P)$ and $\beta_J(P) \leq (n - 7)/3 + 1/3 = (n - 6)/3$. Thus, $Q := P \cup uxy$ is a cycle in G with $|Q| \geq n - (n - 6)/3 = (2n + 6)/3$. $\square$

Note that we need $n \geq 6$ in Theorem 4.1; however, when $n \leq 5$ the graph G is Hamiltonian. The bound in Theorem 4.1 is best possible in the following sense. Take a 4-connected triangulation T on k vertices, and inside each face of T add a new vertex and three edges from that new vertex to the three vertices in the boundary of that face. The resulting graph, say G , has $n := 3k - 4$ vertices. Now take an arbitrary cycle C

in G . For each $x \in V(C)$ with degree three in G , we delete x from C and add the edge of G between the two neighbors of x in C . This results in a cycle in T , say D . Clearly, $|D| \leq k$; which implies $|C| \leq 2k$. Hence, the circumference of G is at most $2k = 2(n+4)/3 = \lceil (2n+6)/3 \rceil$.

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