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Approximating TSP walks in subcubic graphs

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We prove that every simple 2-connected subcubic graph on n vertices with n_2 vertices of degree 2 has a TSP walk of length at most $\frac{5n+n_2}{4}-1$, confirming a conjecture of Dvořák, Král', and Mohar. This bound is best possible; there are infinitely many subcubic and cubic graphs whose minimum TSP walks have lengths $\frac{5n+n_2}{4}-1$ and $\frac{5n}{4}-2$ respectively. We characterize the extremal subcubic examples meeting this bound. We also give a quadratic-time combinatorial algorithm for finding such a TSP walk. In particular, we obtain a $\frac{5}{4}$-approximation algorithm for the graphic TSP on simple cubic graphs, improving on the previously best known approximation ratio of $\frac{9}{7}$.

1. Introduction

The famous *Traveling Salesperson Problem* (TSP) asks for a spanning cycle of minimum length in an edge-weighted complete graph. It is not possible to approximate the TSP within any constant factor of the optimum unless $P = NP$; otherwise, one could solve the Hamiltonian cycle problem, one of Karp's original NP-complete problems [13]. An important special case which admits a constant factor approximation is the *metric*

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TSP in which the edge weights form a metric, a natural assumption for many applications. A further specialization of the metric TSP is the *graphic TSP* in which the edge weights form the distance function in some underlying connected graph G on the same vertex set. This is equivalent to finding a spanning closed walk (a *TSP walk*) in G with the minimum number of edges. Let us denote this minimum length by $\mathrm{tsp}(G)$.

The graphic TSP still contains the Hamiltonian cycle problem, and is thus NP-hard to solve exactly. On the other hand, Christofides [5] and independently Serdyukov [20,21] gave a $\frac{3}{2}$-approximation for the metric TSP in 1976 and 1978 respectively. For many years, this had remained the best approximation ratio for any nontrivial special case of the metric TSP. The first improvement to this ratio was made in 2005 by Gamarnik, Lewenstein, and Sviridenko [9] who gave a $(\frac{3}{2} - \frac{5}{389})$-approximation algorithm for the special case of the graphic TSP on 3-connected cubic graphs (a graph is *cubic* if all of its vertices have degree 3). Following this result, Gharan, Saberi, and Singh [10] gave a $(\frac{3}{2} - \epsilon)$-approximation algorithm for the general graphic TSP. Then Mömke and Svensson [17] gave a novel approach for a 1.461-approximation algorithm for the graphic TSP, which was shown to be in fact a $\frac{13}{9}$-approximation by Mucha [18]. Later, Sebő and Vygen [19] presented a new algorithm for an improved $\frac{7}{5}$-approximation for the graphic TSP. For the metric TSP, the $\frac{3}{2}$ ratio was only very recently improved by Karlin, Klein, and Gharan [12] to $(\frac{3}{2} - \varepsilon)$ for some constant $\varepsilon > 10^{-36}$.

A further special case of the graphic TSP, namely on subcubic graphs, has received significant attention (a graph is *subcubic* if all of its vertices have degree at most 3). Subcubic and cubic graphs are among the simplest classes of graphs which retain the inapproximability of the metric TSP; the general metric and graphic TSPs are NP-hard to approximate within a $\frac{123}{122}$ and $\frac{185}{184}$-factor of the optimum respectively [14,16]. Even when restricted to subcubic and cubic graphs, it remains NP-hard to approximate within a $\frac{685}{684}$ and $\frac{1153}{1152}$-factor respectively [15]. Furthermore, subcubic graphs are known to exhibit the worst-case behavior in the well-known "$\frac{4}{3}$-integrality gap conjecture" from the 80's (see [11]), which asserts that the standard "subtour elimination" linear program relaxation for the metric TSP has an integrality gap of $\frac{4}{3}$. This $\frac{4}{3}$-integrality gap can be asymptotically realized by a family of subcubic graphs (e.g. [2]).

Note that a polynomial-time constructive proof of the $\frac{4}{3}$-integrality gap would yield a $\frac{4}{3}$-approximation algorithm. Motivated by this, Aggarwal, Garg, and Gupta [1] gave a $\frac{4}{3}$-approximation for 3-connected cubic graphs. This approximation ratio was extended to 2-connected cubic graphs by Boyd et al. [3], and to 2-connected subcubic graphs by Mömke and Svensson [17]. The $\frac{4}{3}$ ratio was then slightly improved for cubic graphs to $(\frac{4}{3} - \frac{1}{61326})$ by Correa, Larreé, and Soto [7] and independently to $(\frac{4}{3} - \frac{1}{8754})$ by Zuylen [22], which was further improved to 1.3 by Candráková and Lukot'ka [4], and later to $\frac{9}{7}$ by Dvořák, Král', and Mohar [8].

Let G be a simple 2-connected subcubic graph. We write $n(G)$ to denote the number of vertices in G, and $n_2(G)$ to denote the number of degree 2 vertices in G. Dvořák, Král', and Mohar [8] showed that G has a TSP walk of length at most $\frac{9n(G)+2n_2(G)}{7} - 1$. They also constructed infinitely many subcubic (respectively, cubic) graphs whose minimum

TSP walks have lengths $\frac{5n(G)+n_2(G)}{4}-1$ (respectively, $\frac{5n(G)}{4}-2$), and conjectured that $\frac{5n(G)+n_2(G)}{4}-1$ is the right bound. In this paper, we prove this conjecture.

Theorem 1.1. *Let G be a 2-connected simple subcubic graph. Then $\mathrm{tsp}(G) \leq \frac{5n(G)+n_2(G)}{4}-1$. Moreover, a TSP walk of length at most $\frac{5n(G)+n_2(G)}{4}-1$ can be found in $O(n(G)^2)$ time.*

In particular, we obtain a $\frac{5}{4}$-approximation algorithm for the graphic TSP on simple cubic graphs. We remark that our algorithm is purely combinatorial and deterministic. We also characterize the extremal examples of Theorem 1.1; that is, the 2-connected simple subcubic graphs G such that $\mathrm{tsp}(G) = \frac{5n(G)+n_2(G)}{4}-1$ (see Theorem 5.5). As pointed out by Dvořák et al. [8], Theorem 1.1 is false for non-simple graphs. This can be seen from the graph obtained from three internally disjoint paths between two vertices, each of length $2k+1$, by the addition of parallel edges so that it becomes cubic.

As in [8], rather than working with Eulerian multigraphs obtained from spanning connected subgraphs by adding multiple edges (as often done in the literature), we consider spanning subgraphs F of G in which every vertex has degree 0 or 2. That is, F is a spanning subgraph consisting of vertex-disjoint cycles and isolated vertices. We call such a subgraph F an *even cover* of G. Let $c(F)$ denote the number of cycles in F and $i(F)$ denote the number of isolated vertices in F. Define the *excess* of F to be

$$\mathrm{exc}(F) = 2c(F) + i(F).$$

For a graph G, let $\mathcal{E}(G)$ denote the set of even covers of G, and define the *excess* of G as

$$\mathrm{exc}(G) = \min_{F\in\mathcal{E}(G)} \mathrm{exc}(F).$$

For example, consider the graph Θ which consists of three internally disjoint paths between two vertices, each path with k vertices of degree 2. It is easy to see that an even cover consisting of a cycle and k isolated vertices obtains the minimum excess. Thus for $k \geq 1$,

$$\mathrm{exc}(\Theta) = 2 + k \leq \frac{(3k+2)+3k}{4} + 1 = \frac{n(\Theta)+n_2(\Theta)}{4} + 1,$$

with equality when $k = 1$ (in which case $\Theta \cong K_{2,3}$).

It is observed in [8] that if G is a subcubic graph, then there is an exact relation between $\mathrm{tsp}(G)$ and $\mathrm{exc}(G)$:

Proposition 1.2 *(Dvořák et al. [8]). Let G be a subcubic graph. Then*

$$\mathrm{tsp}(G) = \mathrm{exc}(G) - 2 + n(G). \tag{1}$$

Moreover, an even cover $F \in \mathcal{E}(G)$ can be converted into a TSP walk in G of length $\mathrm{exc}(F) - 2 + n(G)$ *in linear time.*

Thus, to prove Theorem 1.1, it suffices to show that

$$\mathrm{exc}(G) \leq \frac{n(G) + n_2(G)}{4} + 1, \tag{2}$$

and that an even cover F of G satisfying this bound can be found in quadratic time. Indeed, we will see that (2) follows from a more technical result (Theorem 2.4) that bounds $\mathrm{exc}(F) - \frac{n(G)+n_2(G)}{4}$ for certain sets of even covers F of G. In Section 2, we develop our key definitions and state Theorem 2.4. In Section 3, we provide some technical lemmas on the structure of the extremal graphs for Theorem 2.4, which we call *θ-chains*. We complete the proof of Theorem 2.4 in Section 4. In Section 5, we characterize extremal graphs for Theorem 1.1. In Section 6, we outline a quadratic-time algorithm that finds an even cover F in simple 2-connected subcubic graphs G with $\mathrm{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$.

We end this section with some notation. For a positive integer k, let $[k] = \{1, \ldots, k\}$. If G and H are graphs, we write $G \cup H$ (respectively, $G \cap H$) to denote the graph with vertex set $V(G) \cup V(H)$ (respectively, $V(G) \cap V(H)$) and edge set $E(G) \cup E(H)$ (respectively, $E(G) \cap E(H)$). Let G be a graph. If S is a set of vertices or a set of edges, we let $G - S$ denote the subgraph of G obtained by deleting elements of S as well as edges incident with a vertex in S. When $S = \{s\}$ is a singleton, we simply write $G - s$. For a collection of 2-element subsets of $V(G)$, we write $G + S$ for the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. However, for $x, y \in V(G)$ we use $G + xy$ to denote the graph obtained from G by adding a (possibly parallel) edge between x and y. For a subgraph $H \subseteq G$ and a set $S \subseteq V(G)$, we let $H + S$ denote the subgraph of G such that $V(H + S) = V(H) \cup S$ and $E(H + S) = E(H)$. For $S \subseteq V(G)$, we use $N(S)$ to denote the neighborhood of S in G. If $S = \{s\}$ is a singleton, we simply write $N(s)$. When $|N(S)| \in \{1, 2\}$, *suppressing* S means deleting S and adding a (possibly loop or parallel) edge between the vertices of $N(S)$. When $S = \{s\}$ is a singleton, *suppressing* s means suppressing $\{s\}$.

2. Subcubic chains

In order to help with induction, we consider even covers which contain or avoid a specified edge. Let G be a graph and let $e \in E(G)$. We write $\mathcal{E}(G, e)$ to denote the set of even covers of G containing e, and $\widehat{\mathcal{E}}(G, e)$ to denote the set of even covers of G not containing e. Define

$$\mathrm{exc}(G, e) := \min_{F \in \mathcal{E}(G,e)} \mathrm{exc}(F) - 2$$

$$\widehat{\mathrm{exc}}(G, e) := \min_{F \in \widehat{\mathcal{E}}(G,e)} \mathrm{exc}(F)$$

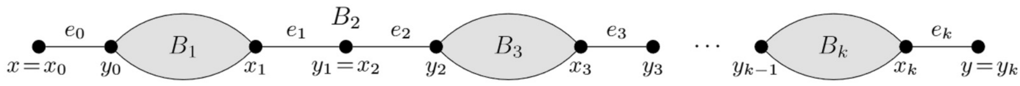

Fig. 1. A subcubic chain.

Clearly, we have $\mathrm{exc}(G) = \min\{\mathrm{exc}(G,e)+2, \widehat{\mathrm{exc}}(G,e)\}$ for any edge $e \in E(G)$. The "-2" in the definition of $\mathrm{exc}(G,e)$ leads to a natural interpretation of the quantities $\delta(G,e)$ and $\widehat{\delta}(G,e)$ defined below, and also results in simpler calculations as it accounts for the fact that the cycle C of F containing e will often only be used as a path $C-e$ as part of a larger cycle (see Propositions 2.1 and 2.2).

To prove (2), it will be convenient to define the following parameters for a graph G and an edge $e \in E(G)$:

$$\delta(G,e) := \mathrm{exc}(G,e) - \frac{n(G)+n_2(G)}{4},$$

$$\widehat{\delta}(G,e) := \widehat{\mathrm{exc}}(G,e) - \frac{n(G)+n_2(G)}{4}.$$

Note that if every vertex of G has degree 2 or 3 (for instance, if G is subcubic and 2-connected), then $\delta(G,e)$ and $\widehat{\delta}(G,e)$ are always half-integral since $n(G)+n_2(G) = (n(G)-n_2(G)) + 2n_2(G)$ where $(n(G)-n_2(G))$ is the number of vertices of odd degree in G, which is always even.

A *subcubic chain* C is a simple connected subcubic graph, written as an alternating sequence $C = xe_0B_1e_1B_2\dots B_ke_ky$ for some nonnegative integer k, satisfying the following properties (see Fig. 1):

- $\{e_0,\dots,e_k\}$ is the set of cut-edges of C,
- $\{B_0,B_1,\dots,B_k,B_{k+1}\}$ is the set of connected components of $C-\{e_0,\dots,e_k\}$, where $V(B_0)=\{x\}$ and $V(B_{k+1})=\{y\}$,
- B_i is either a single vertex or 2-connected for all $i \in [k]$, and
- each e_i has one endpoint in B_i and one endpoint in B_{i+1} for all $i=0,\dots,k$.

We say that C has *end points* x,y and has *end edges* e_0 and e_k. A subcubic chain is *trivial* if $k=0$ (that is, C is an edge xy), and *nontrivial* otherwise.

Let $C = xe_0B_1e_1B_2\dots B_ke_ky$ be a nontrivial subcubic chain. For $i \in [k]$, let x_i denote the endpoint of e_{i-1} in B_i and let y_i denote the endpoint of e_i in B_i. (Note that $x_i \neq y_i$ when $n(B_i) \neq 1$, as C is subcubic.) We define $\overline{B_i} = B_i + \overline{e_i}$ where $\overline{e_i} = x_iy_i$, and $\overline{C} = C - \{x,y\} + e_C$ where $e_C = x_1y_k$. We call each $(\overline{B_i},\overline{e_i})$ a *chain-block* of C, and $\overline{C}$ the *closure* of C. Note that the closure of a nontrivial subcubic chain C is a subcubic graph with no cut-vertex such that $\overline{C}-e_C$ is simple. If C is a trivial subcubic chain, we define $\mathrm{exc}(\overline{C},e_C) = \widehat{\mathrm{exc}}(\overline{C},e_C) = \delta(\overline{C},e_C) = \widehat{\delta}(\overline{C},e_C) = 0$.

Proposition 2.1. *Let $C = xe_0B_1e_1B_2 \dots B_ke_ky$ be a subcubic chain, and let $\{(\overline{B_i}, \overline{e_i}) : i \in [k]\}$ denote the chain-blocks of C. Then*

- $\mathrm{exc}(\overline{C}, e_C) = \sum_{i=1}^{k} \mathrm{exc}(\overline{B_i}, \overline{e_i})$,
- $\widehat{\mathrm{exc}}(\overline{C}, e_C) = \sum_{i=1}^{k} \widehat{\mathrm{exc}}(\overline{B_i}, \overline{e_i})$,
- $\delta(\overline{C}, e_C) = \sum_{i=1}^{k} \delta(\overline{B_i}, \overline{e_i})$, *and*
- $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^{k} \widehat{\delta}(\overline{B_i}, \overline{e_i})$.

Proof. If C is trivial then the proposition is true by definition (an empty sum is defined to be 0), so we may assume that C is nontrivial. Note that a cycle in $\overline{C}$ contains e_C if and only if it contains all of $e_1, \dots, e_{k-1}$. This gives a natural bijective correspondence between even covers $F \in \mathcal{E}(\overline{C}, e_C)$ and tuples of even covers $(F_1, \dots, F_k)$ where $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for each $i \in [k]$. Indeed, this correspondence is obtained by "splitting" the cycle D of F containing e_C into k cycles, $(D \cap \overline{B_i}) + \overline{e_i}$ for $i \in [k]$. With this correspondence, we have $\mathrm{exc}(F) = 2 + \sum_{i=1}^{k}(\mathrm{exc}(F_i) - 2)$. Hence,

$$\begin{aligned}
\mathrm{exc}(\overline{C}, e_C) &= \min_{F \in \mathcal{E}(\overline{C}, e_C)} \mathrm{exc}(F) - 2 \\
&= \sum_{i=1}^{k} \min_{F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})} (\mathrm{exc}(F_i) - 2) \\
&= \sum_{i=1}^{k} \mathrm{exc}(\overline{B_i}, \overline{e_i}).
\end{aligned}$$

Since $n(\overline{C}) = \sum_{i=1}^{k} n(\overline{B_i})$ and $n_2(\overline{C}) = \sum_{i=1}^{k} n_2(\overline{B_i})$, this also implies $\delta(\overline{C}, e_C) = \sum_{i=1}^{k} \delta(\overline{B_i}, \overline{e_i})$.

Similarly, there is a natural bijective correspondence between even covers $F \in \widehat{\mathcal{E}}(\overline{C}, e_C)$ and tuples $(F_1, \dots, F_k)$ where $F_i \in \widehat{\mathcal{E}}(\overline{B_i}, \overline{e_i})$ for each $i \in [k]$. That is, F_i is the restriction of F on B_i for all $i \in [k]$. Moreover, $\mathrm{exc}(F) = \sum_{i=1}^{k} \mathrm{exc}(F_i)$. Hence,

$$\begin{aligned}
\widehat{\mathrm{exc}}(\overline{C}, e_C) &= \min_{F \in \widehat{\mathcal{E}}(\overline{C}, e_C)} \mathrm{exc}(F) \\
&= \sum_{i=1}^{k} \min_{F_i \in \widehat{\mathcal{E}}(\overline{B_i}, \overline{e_i})} \mathrm{exc}(F_i) \\
&= \sum_{i=1}^{k} \widehat{\mathrm{exc}}(\overline{B_i}, \overline{e_i}).
\end{aligned}$$

This similarly gives $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^{k} \widehat{\delta}(\overline{B_i}, \overline{e_i})$. □

The parameters $\delta(\overline{C}, e_C)$ and $\widehat{\delta}(\overline{C}, e_C)$ can be interpreted as the "difference" in the δ or $\widehat{\delta}$ of the overall graph G made by the presence of the subcubic chain C compared to a trivial chain (a single edge). This is formalized in the next proposition.

Let G be a graph containing a nontrivial subcubic chain $C = xe_0B_1 \dots B_ke_ky$ such that $C - \{x, y\}$ is a connected component of $G - \{e_0, e_k\}$. In this case, we say that C is a *subcubic chain of* G. If C is a subcubic chain of G, we write G/C to denote the graph obtained by suppressing $V(C) \setminus \{x, y\}$, and write $e_{G/C}$ to denote the resulting edge. We say that G/C is obtained from G by *suppressing* C. A cycle in G containing the edge e_0 (hence all of $\{e_0, \dots, e_k\}$) is said to be a *cycle through* C, and an *even cover through* C is an even cover of G containing a cycle through C.

Proposition 2.2. *Let C be a subcubic chain of a graph G, and let e be a cut-edge of C. Then $\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C)$ and $\widehat{\delta}(G, e) = \widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)$.*

Proof. Given an even cover $F \in \mathcal{E}(G, e)$, e is contained in some cycle D in F. By splitting D into two cycles $(D \cap G/C) + e_{G/C}$ and $(D \cap C) + e_C$, we obtain from F two even covers $F' \in \mathcal{E}(G/C, e_{G/C})$ and $F_C \in \mathcal{E}(\overline{C}, e_C)$ satisfying $\mathrm{exc}(F) = \mathrm{exc}(F') + \mathrm{exc}(F_C) - 2$. This bijective correspondence gives

$$\begin{aligned}
\mathrm{exc}(G, e) &= \min_{F \in \mathcal{E}(G,e)} \mathrm{exc}(F) - 2 \\
&= \min_{F' \in \mathcal{E}(G/C, e_{G/C})} (\mathrm{exc}(F') - 2) + \min_{F_C \in \mathcal{E}(\overline{C}, e_C)} (\mathrm{exc}(F_C) - 2) \\
&= \mathrm{exc}(G/C, e_{G/C}) + \mathrm{exc}(\overline{C}, e_C).
\end{aligned}$$

Similarly, for any even cover $F \in \widehat{\mathcal{E}}(G, e)$, its restriction on G/C is in $\widehat{\mathcal{E}}(G/C, e_{G/C})$ and its restriction on $\overline{C}$ is in $\widehat{\mathcal{E}}(\overline{C}, e_C)$; and we have $\widehat{\mathrm{exc}}(G, e) = \widehat{\mathrm{exc}}(G/C, e_{G/C}) + \widehat{\mathrm{exc}}(\overline{C}, e_C)$.

Since $n(G) = n(G/C) + n(\overline{C})$ and $n_2(G) = n_2(G/C) + n_2(\overline{C})$, the proposition follows from the definitions of δ and $\widehat{\delta}$. $\square$

We will show in Theorem 2.4 that $\delta(G, e) + \widehat{\delta}(G, e) \leq 0$ for every 2-connected subcubic graph G and every edge $e \in E(G)$ for which $G - e$ is simple. If $\delta(G, e) + \widehat{\delta}(G, e) = 0$, then we say that (G, e) is *tight*. A subcubic chain C is *tight* if its closure $(\overline{C}, e_C)$ is tight.

The next proposition states that a subcubic chain is tight if and only if all of its chain-blocks are tight.

Proposition 2.3. *Let $C = xe_0B_1e_1B_2 \dots B_ke_ky$ be a subcubic chain, and assume $\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) \leq 0$ for all i. Then $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$, with equality if and only if $\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) = 0$ for all $i \in [k]$.*

Proof. Since $\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) \leq 0$ for all i, we have by Proposition 2.1,

$$\delta(\overline{C}, e_C) = \sum_{j=1}^{k} \delta(\overline{B_i}, \overline{e_i}) \leq \sum_{j=1}^{k} (-\widehat{\delta}(\overline{B_i}, \overline{e_i})) = -\widehat{\delta}(\overline{C}, e_C).$$

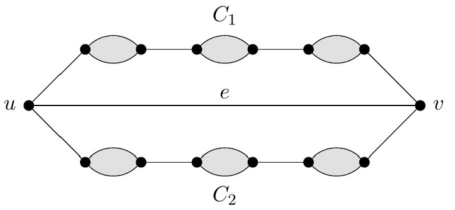

Fig. 2. A rooted θ-chain.

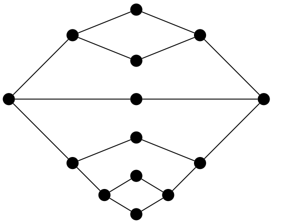

Fig. 3. A minimal θ-chain.

Hence, $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$, with equality if and only if $\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) = 0$ for all i. $\square$

We say that a subcubic chain C is *minimal* if it is tight and $\delta(\overline{C}, e_C) = -\frac{1}{2}$, and that C is *near-minimal* if it is tight and $\delta(\overline{C}, e_C) \in \{-\frac{1}{2}, -1\}$. Two subcubic chains C_1 and C_2 are *balanced* if $\delta(\overline{C_1}, e_{C_1}) = \delta(\overline{C_2}, e_{C_2})$.

A *θ-chain* is a graph G that is the union of three internally disjoint subcubic chains C_1, C_2, C_3 with common endpoints. We call C_1, C_2, C_3 the *chains of* G. Note that the choices of the three chains C_1, C_2, C_3 may not be unique (consider the graph obtained from two disjoint 4-cycles by adding two edges joining them so that the endpoints of the two edges are nonadjacent in each 4-cycle). A *rooted θ-chain* is a pair (G, e) where G is a graph and $e = uv \in E(G)$ such that $G - e$ is the union of two internally disjoint subcubic chains C_1, C_2 with common endpoints $\{u, v\}$. We call C_1, C_2 the *chains of* (G, e). See Fig. 2.

A (rooted) θ-chain is *balanced* if all pairs of its chains are balanced, *tight* if the closures of its chains are all tight, and *(near) minimal* if all of its chains are (near) minimal. Note that a (near) minimal (rooted) θ-chain is also balanced and tight by definition. See Fig. 3.

We can now state our main result, which immediately implies (2). For inductive purposes, we allow the graph G to be a loop e on a single vertex and we also allow one edge of $G - e$ to be parallel to e. In all cases however, $G - e$ is a simple subcubic graph.

Theorem 2.4. *Let G be a 2-connected subcubic graph and let $e = uv$ be an edge of G such that $G - e$ is simple. Then the following statements hold:*

(T1) *$\delta(G,e) \le -\frac{1}{2}$, with equality if and only if either G is a loop or (G,e) is a balanced tight rooted θ-chain.*

(T2) *If $G-e$ is 2-connected, then $\widehat{\delta}(G,e) \le \frac{3}{2}$, with equality if and only if $G-e$ is a minimal θ-chain.*

(T3) *If $\delta(G,e) = -1$, then either*

(a) *$G \cong K_4$, or*

(b) *e has a parallel edge, and suppressing $\{u,v\}$ to an edge e' results in a graph G' such that either G' is a loop or (G',e') is a near-minimal rooted θ-chain, or*

(c) *there exists $e' \in E(G)$ such that $\{e,e'\}$ is a 2-edge-cut in G, and suppressing either subcubic chain C of G with end edges e,e' yields either a loop or a balanced tight rooted θ-chain $(G/C, e_{G/C})$, or*

(d) *(G,e) is a rooted θ-chain such that $\min_{i\in[2]}\Big(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\Big) = -\frac{1}{2}$.*

(T4) *$\delta(G,e) + \widehat{\delta}(G,e) \le 0$.*

One immediate consequence of Theorem 2.4 is that if C is a subcubic chain, then $\delta(\overline{C}, e_C) \le -\frac{1}{2}$ unless C is trivial, in which case $\delta(\overline{C}, e_C) = 0$ by definition. In particular, $\delta(G,e) \le -\frac{1}{2}$ for every nonempty 2-connected subcubic graph G and $e \in E(G)$ such that $G-e$ is simple. Hence, if C is a minimal subcubic chain, then by Proposition 2.1, it has exactly one chain-block $(\overline{B}, \overline{e_B})$, and this chain-block satisfies $\delta(\overline{B}, \overline{e_B}) = -\frac{1}{2}$.

3. Properties of θ-chains

In this section, we derive useful properties of balanced, tight, or minimal θ-chains assuming Theorem 2.4 for smaller graphs. We begin by proving statements **(T1)** and **(T3)** of Theorem 2.4, assuming Theorem 2.4 for smaller graphs, for the special case where (G,e) is a rooted θ-chain (equivalently, G is simple and $\{u,v\}$ forms a cut in G). The proof is a relatively straightforward but illustrative demonstration of our techniques.

Lemma 3.1. *Let (G,e) be a simple rooted θ-chain, and let C_1, C_2 denote the two chains of (G,e). Assume that Theorem 2.4 holds for graphs with fewer than $n(G)$ vertices. Then*

(i) *$\delta(G,e) = -\frac{1}{2} + \min_{i\in[2]}\Big(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\Big) \le -\frac{1}{2}$, with equality if and only if (G,e) is a balanced tight rooted θ-chain,*

(ii) *$\widehat{\delta}(G,e) \le \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}) \le \frac{1}{2}$,*

(iii) *$(\delta(G,e), \widehat{\delta}(G,e)) = (-\frac{1}{2}, \frac{1}{2})$ if and only if (G,e) is a minimal rooted θ-chain, and*

(iv) *if $\delta(G,e) = -1$ then $\min_{i\in[2]}\Big(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\Big) = -\frac{1}{2}$.*

Proof. An even cover $F \in \mathcal{E}(G,e)$ corresponds to a pair (F_1, F_2) where $F_i \in \mathcal{E}(\overline{C_i})$ for each $i \in [2]$ and $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for exactly one $i \in [2]$. This correspondence gives

$\mathrm{exc}(F) = \mathrm{exc}(F_1) + \mathrm{exc}(F_2)$. Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 2$ and $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, we have

$$\begin{aligned}
\mathrm{exc}(G,e) &= \min_{F\in\mathcal{E}(G,e)} \mathrm{exc}(F) - 2 \\
&= \min_{i\in[2]} \left(\min_{F_i\in\mathcal{E}(\overline{C_i},e_{C_i})} (\mathrm{exc}(F_i) - 2) + \min_{F_{3-i}\in\widehat{\mathcal{E}}(\overline{C_{3-i}},e_{C_{3-i}})} \mathrm{exc}(F_{3-i}) \right) \\
&= \min_{i\in[2]} \left(\mathrm{exc}(\overline{C_i}, e_{C_i}) + \widehat{\mathrm{exc}}(\overline{C_{3-i}}, e_{C_{3-i}}) \right) \\
&= \min_{i\in[2]} \left(\frac{n(\overline{C_i}) + n_2(\overline{C_i})}{4} + \delta(\overline{C_i}, e_{C_i}) + \frac{n(\overline{C_{3-i}}) + n_2(\overline{C_{3-i}})}{4} + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}) \right) \\
&= \min_{i\in[2]} \left(\frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \delta(\overline{C_i}, e_{C_i}) + \delta(\overline{C_{3-i}}, e_{C_{3-i}}) \right).
\end{aligned}$$

Therefore,

$$\delta(G,e) = -\frac{1}{2} + \min_{i\in[2]} \left(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}) \right), \tag{3}$$

whence for $i \in [2]$,

$$\delta(G,e) \le -\frac{1}{2} + \delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}). \tag{4}$$

By assumption, Theorem 2.4 holds for $(\overline{C_i}, e_{C_i})$; so $\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_i}, e_{C_i}) \le 0$ for each $i \in [2]$. Adding the two inequalities of (4) gives

$$2\delta(G,e) \le -1 + \sum_{i\in[2]} \left(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_i}, e_{C_i}) \right) \le -1.$$

Hence,

$$\delta(G,e) \le -\frac{1}{2}. \tag{5}$$

Moreover, $\delta(G,e) = -\frac{1}{2}$ if and only if all of the above inequalities are tight, which means $(\overline{C_1}, e_{C_1})$ and $(\overline{C_2}, e_{C_2})$ are tight, and

$$0 = \delta(\overline{C_1}, e_{C_1}) + \widehat{\delta}(\overline{C_2}, e_{C_2}) = \delta(\overline{C_1}, e_{C_1}) - \delta(\overline{C_2}, e_{C_2}).$$

In other words, C_1, C_2 are balanced. Together with (3) and (5), this proves (i).

If $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for each $i \in [2]$ then, by merging the cycles in F_i containing e_{C_i} for $i \in [2]$, we obtain an even cover $F \in \widehat{\mathcal{E}}(G,e)$ with $\mathrm{exc}(F) = \mathrm{exc}(F_1) + \mathrm{exc}(F_2) - 2$. So

$$\begin{aligned}
\widehat{\mathrm{exc}}(G,e) &\le \min_{F\in\widehat{\mathcal{E}}(G,e)} \mathrm{exc}(F) \\
&\le \min_{F_1\in\mathcal{E}(\overline{C_1},e_{C_1})} \mathrm{exc}(F_1) + \min_{F_2\in\mathcal{E}(\overline{C_2},e_{C_2})} (\mathrm{exc}(F_2)-2) \\
&= (\mathrm{exc}(\overline{C_1},e_{C_1})+2) + \mathrm{exc}(\overline{C_2},e_{C_2}) \\
&= \frac{n(\overline{C_1})+n_2(\overline{C_1})}{4} + \delta(\overline{C_1},e_{C_1}) + \frac{n(\overline{C_2})+n_2(\overline{C_2})}{4} + \delta(\overline{C_2},e_{C_2}) + 2 \\
&= \frac{n(G)+n_2(G)}{4} + \frac{3}{2} + \delta(\overline{C_1},e_{C_1}) + \delta(\overline{C_2},e_{C_2}).
\end{aligned}$$

Hence,

$$\widehat{\delta}(G,e) \le \frac{3}{2} + \delta(\overline{C_1},e_{C_1}) + \delta(\overline{C_2},e_{C_2}).$$

Since G is simple, each C_i is a nontrivial chain; so $\delta(\overline{C_i},e_{C_i}) \le -\frac{1}{2}$ by the assumption that Theorem 2.4 holds for $(\overline{C_i},e_{C_i})$. This gives $\widehat{\delta}(G,e) \le \frac{1}{2}$ and proves (ii).

To prove (iii), suppose $(\delta(G,e),\widehat{\delta}(G,e)) = (-\frac{1}{2},\frac{1}{2})$. Then $\delta(\overline{C_1},e_{C_1})+\delta(\overline{C_2},e_{C_2}) = -1$ by (ii). Since $\delta(\overline{C_i},e_{C_i}) \le -\frac{1}{2}$ for $i\in[2]$ (by assumption), $\delta(\overline{C_i},e_{C_i}) = -\frac{1}{2}$ for each $i\in[2]$. Moreover, each $(\overline{C_i},e_{C_i})$ is tight (by (i)), so (G,e) is a minimal rooted θ-chain.

Finally, note that (iv) follows from (i). $\square$

The next lemma says that given a choice of adding an edge uv_1 or uv_2 to a 2-connected subcubic graph Z, the two quantities $\delta(Z+uv_1,uv_1)$ and $\delta(Z+uv_2,uv_2)$ cannot both be large.

Lemma 3.2. *Let Z be a 2-connected simple subcubic graph and let u,v_1,v_2 be three distinct vertices of degree 2 in Z. Assume Theorem 2.4 holds for graphs with at most $n(Z)$ vertices. Then $\delta(Z+uv_1,uv_1)+\delta(Z+uv_2,uv_2) \le -2$.*

Proof. By the assumption that Theorem 2.4 holds for graphs with at most $n(Z)$ vertices, we have $\delta(Z+uv_i,uv_i) \le -\frac{1}{2}$ for each $i\in[2]$, with equality if and only if $(Z+uv_i,uv_i)$ is a balanced tight rooted θ-chain. If both $\delta(Z+uv_1,uv_1)\le -1$ and $\delta(Z+uv_2,uv_2)\le -1$, then there is nothing to prove. So we may assume by symmetry that $\delta(Z+uv_1,uv_1) = -\frac{1}{2}$; thus $(Z+uv_1,uv_1)$ is a balanced tight rooted θ-chain. Note that it suffices to show that $\delta(Z+uv_2,uv_2) \le -\frac{3}{2}$.

Let C_1,C_2 denote the two chains of $(Z+uv_1,uv_1)$. Let us assume without loss of generality that $v_2\in V(C_1)$. Write $C_1 = v_1e_0B_1e_1B_2\dots B_ke_ku$ (where $k\ge 1$) and write its chain-blocks $(\overline{B_i},\overline{e_i})$ for all $i\in[k]$. Since C_1,C_2 are balanced, we have $\delta(\overline{C_1},e_{C_1}) = \delta(\overline{C_2},e_{C_2})$, and since they are both tight, we have $\delta(\overline{C_i},e_{C_i})+\widehat{\delta}(\overline{C_i},e_{C_i}) = 0$ for $i\in[2]$. So by Proposition 2.3 and the assumption that Theorem 2.4 holds for each $(\overline{B_i},\overline{e_i})$, we have

$$\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) = 0 \qquad \text{for all } i \in [k]. \tag{6}$$

Let $\ell \in [k]$ be the unique index such that $v_2 \in B_\ell$. (Note ℓ is well defined as Z is subcubic and v_2 has degree 2 in Z.) Let v' denote the vertex of B_ℓ incident with $e_{\ell-1}$.

Then there is an even cover $F \in \mathcal{E}(Z+uv_2, uv_2)$ obtained from a tuple $(F', F_1, \ldots, F_k)$ where $F' \in \mathcal{E}(\overline{C_2}, e_{C_2})$, $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for each $i \in [\ell-1]$, $F_\ell \in \mathcal{E}(B_\ell + v'v_2, v'v_2)$, and $F_j \in \widehat{\mathcal{E}}(\overline{B_j}, \overline{e_j})$ for each $j = \ell+1, \ldots, k$. This gives $\mathrm{exc}(F) - 2 = (\mathrm{exc}(F') - 2) + \sum_{i=1}^{\ell}(\mathrm{exc}(F_i) - 2) + \sum_{j=\ell+1}^{k} \mathrm{exc}(F_j)$. Moreover, since $n(B_\ell + v'v_2) = n(\overline{B_\ell})$ and $n_2(B_\ell + v'v_2) = n_2(\overline{B_\ell})$, we have

$$n(Z + uv_2) = 2 + n(\overline{C_2}) + \sum_{i=1}^{\ell-1} n(\overline{B_i}) + n(B_\ell + v'v_2) + \sum_{j=\ell+1}^{k} n(\overline{B_j}),$$

$$n_2(Z + uv_2) = n_2(\overline{C_2}) + \sum_{i=1}^{\ell-1} n_2(\overline{B_i}) + n_2(B_\ell + v'v_2) + \sum_{j=\ell+1}^{k} n_2(\overline{B_j}).$$

This gives

$$\begin{aligned}\mathrm{exc}(Z + uv_2, uv_2) &\le \mathrm{exc}(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \mathrm{exc}(\overline{B_i}, \overline{e_i}) \\ &\quad + \mathrm{exc}(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^{k} \widehat{\mathrm{exc}}(\overline{B_j}, \overline{e_j}) \\ &= \frac{n(Z+uv_2) + n_2(Z+uv_2)}{4} - \frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \\ &\quad + \delta(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}),\end{aligned}$$

whence

$$\begin{aligned}\delta(Z + uv_2, uv_2) \le -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) + \delta(B_\ell + v'v_2, v'v_2) \\ + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e}^j).\end{aligned}$$

Note that $\widehat{\mathrm{exc}}(\overline{B_\ell}, \overline{e_\ell}) = \widehat{\mathrm{exc}}(B_\ell + v'v_2, v'v_2)$ since both quantities are equal to the minimum excess of an even cover of B_ℓ. This implies $\widehat{\delta}(\overline{B_\ell}, \overline{e_\ell}) = \widehat{\delta}(B_\ell + v'v_2, v'v_2)$. Using (6) and that $\delta(B_\ell + v'v_2, v'v_2) + \widehat{\delta}(B_\ell + v'v_2, v'v_2) \le 0$ as Theorem 2.4 holds for $(B_\ell + v'v_2, v'v_2)$ (by assumption), we have

$$\begin{aligned}
\delta(Z+uv_2, uv_2) &\le -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \big(-\widehat{\delta}(\overline{B_i}, \overline{e_i})\big) \\
&\qquad + \big(-\widehat{\delta}(B_\ell + v'v_2, v'v_2)\big) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \big(-\widehat{\delta}(\overline{B_i}, \overline{e_i})\big) + \big(-\widehat{\delta}(\overline{B_\ell}, \overline{e_\ell})\big) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{j=1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_i}, \overline{e_i}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \widehat{\delta}(\overline{C_1}, e_{C_1}) - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \qquad \text{(by Proposition 2.1)} \\
&= -\frac{1}{2} - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \qquad \text{(as } C_1 \text{ and } C_2 \text{ are balanced and tight)} \\
&\le -\frac{3}{2},
\end{aligned}$$

since $-\widehat{\delta}(\overline{B_j}, \overline{e_j}) = \delta(\overline{B_j}, \overline{e_j}) \le -1/2$ for all $j \in [k]$ by (6) and the assumption that Theorem 2.4 holds for $(\overline{B_j}, \overline{e_j})$. $\square$

We can now prove the following lemma for θ-chains.

Lemma 3.3. *Let G be a subcubic graph with $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected. Assume that Theorem 2.4 holds for graphs with fewer than $n(G)$ vertices. Let G_u be the graph obtained from $G - e$ by suppressing u into an edge f_u, and assume that (G_u, f_u) is a rooted θ-chain. Then*

- (i) $\widehat{\delta}(G, e) \le \frac{3}{2}$, *with equality if and only if $G - e$ is a minimal θ-chain whose three minimal chains can be chosen to have common endpoints $N(u) \setminus \{v\}$,*
- (ii) $\delta(G, e) \le -\frac{3}{2}$, *and*
- (iii) $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{3}{2}, \frac{3}{2})$ *if and only if $G - e$ is a minimal θ-chain and e joins two nonadjacent vertices of a 4-cycle in $G - e$.*

Proof. Let $N(u) \setminus \{v\} = \{x, y\}$, the set of endpoints of f_u. Let C_1, C_2 denote the two chains of (G_u, f_u) with common endpoints $\{x, y\}$, and let C_3 denote the subcubic chain $x(xu)u(uy)y$. Note that $n(G) = 2 + \sum_{i=1}^{3} n(\overline{C_i})$, $n_2(G) = -2 + \sum_{i=1}^{3} n_2(\overline{C_i})$ (since the $\overline{C_i}$'s do not account for the edge e), and $\overline{C_3}$ is a loop. Let i_1, i_2, i_3 be a permutation of $[3]$ such that $\delta(\overline{C_{i_1}}, e_{C_{i_1}}) \le \delta(\overline{C_{i_2}}, e_{C_{i_2}}) \le \delta(\overline{C_{i_3}}, e_{C_{i_3}})$.

Consider a triple (F_1, F_2, F_3) such that $F_{i_1} \in \mathcal{E}(\overline{C_{i_1}}, e_{C_{i_1}})$, $F_{i_2} \in \mathcal{E}(\overline{C_{i_2}}, e_{C_{i_2}})$, and $F_{i_3} \in \widehat{\mathcal{E}}(\overline{C_{i_3}}, e_{C_{i_3}})$. Let $F \in \widehat{\mathcal{E}}(G, e)$ be obtained from $F_1 \cup F_2 \cup F_3$ by merging the cycles in F_{i_1}, F_{i_2} through $e_{C_{i_1}}, e_{C_{i_2}}$. Then $\text{exc}(F)-2 = (\text{exc}(F_{i_1})-2)+(\text{exc}(F_{i_2})-2)+\text{exc}(F_{i_3})$; so

$$\begin{aligned}\widehat{\text{exc}}(G,e)-2 &= \text{exc}(\overline{C_{i_1}}, e_{C_{i_1}}) + \text{exc}(\overline{C_{i_2}}, e_{C_{i_2}}) + \widehat{\text{exc}}(\overline{C_{i_3}}, e_{C_{i_3}}) \\ &= \frac{n(G)+n_2(G)}{4} + \delta(\overline{C_{i_1}}, e_{C_{i_1}}) + \delta(\overline{C_{i_2}}, e_{C_{i_2}}) + \widehat{\delta}(\overline{C_{i_3}}, e_{C_{i_3}}).\end{aligned}$$

Since Theorem 2.4 holds for $(\overline{C_i}, e_{C_i})$ for each $i \in [3]$ (by assumption), we have $\widehat{\delta}(\overline{C_{i_3}}, e_{C_{i_3}}) \le -\delta(\overline{C_{i_3}}, e_{C_{i_3}}) \le -\delta(\overline{C_{i_2}}, e_{C_{i_2}})$ and $\delta(\overline{C_i}, e_{C_i}) \le -\frac{1}{2}$ for $i \in [3]$, which gives

$$\widehat{\text{exc}}(G,e)-2 \le \frac{n(G)+n_2(G)}{4} + \delta(\overline{C_{i_1}}, e_{C_{i_1}}) \le \frac{n(G)+n_2(G)}{4} - \frac{1}{2}.$$

Therefore, $\widehat{\text{exc}}(G,e) \le \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$, and $\widehat{\delta}(G,e) \le \frac{3}{2}$.

Suppose $\widehat{\delta}(G,e) = \frac{3}{2}$. Then the above inequalities hold with equality. Hence, $-\frac{1}{2} = \delta(\overline{C_{i_1}}, e_{C_{i_1}}) = \delta(\overline{C_{i_2}}, e_{C_{i_2}}) = \delta(\overline{C_{i_3}}, e_{C_{i_3}})$. Since Theorem 2.4 holds for all $(\overline{C_i}, e_{C_i})$ (by assumption), $(\overline{C_i}, e_{C_i})$ is tight (hence minimal) for all $i \in [3]$. Therefore, $G-e$ is a minimal θ-chain with its three chains having common endpoints $N(u) \setminus \{v\}$.

Now suppose $G-e$ is a minimal θ-chain with the three minimal chains C_1, C_2, C_3 with common endpoints $N(u)\setminus\{v\}$. Let $F \in \widehat{\mathcal{E}}(G,e)$. If F contains a cycle through two of C_1, C_2, C_3, then the above argument shows $\text{exc}(F) = \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$. So we just need to show that if F does not contain a cycle through any of C_1, C_2, C_3, then $\text{exc}(F) \ge \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$. Indeed, such F when restricted to $(\overline{C_i}, e_{C_i})$ for $i \in [3]$ gives a triple (F_1, F_2, F_3) such that $F_i \in \widehat{\mathcal{E}}(\overline{C_i}, e_{C_i})$ for each $i \in [3]$, and $\text{exc}(F) = 2 + \sum_{i=1}^{3} \text{exc}(F_i)$ (since the two vertices of $N(u)\setminus\{v\}$ are isolated in F). So

$$\begin{aligned}\text{exc}(F) &\ge 2 + \sum_{i=1}^{3} \widehat{\text{exc}}(\overline{C_i}, e_{C_i}) \\ &= 2 + \sum_{i=1}^{3}\left(\frac{n(\overline{C_i}) + n_2(\overline{C_i})}{4} + \widehat{\delta}(\overline{C_i}, e_{C_i})\right) \\ &= \frac{n(G)+n_2(G)}{4} + 2 + \sum_{i=1}^{3} \widehat{\delta}(\overline{C_i}, e_{C_i}) \\ &= \frac{n(G)+n_2(G)}{4} + \frac{7}{2}.\end{aligned}$$

The last equality holds since $\widehat{\delta}(\overline{C_i}, e_{C_i}) = \frac{1}{2}$ for each $i \in [3]$, completing the proof of (i).

We now prove (ii) and (iii). Let us assume without loss of generality that $v \in V(C_1)$, and write $C_1 = xe_0B_1e_1B_2\dots B_ke_ky$ with chain-blocks $(\overline{B_i}, \overline{e_i})$. Let $\ell \in [k]$ denote the unique index such that $v \in V(B_\ell)$. By symmetry, we may assume that $\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \le$

$\sum_{j=\ell+1}^{k} \delta(\overline{B_j}, \overline{e_j})$. Then, by the assumption that Theorem 2.4 holds for each $(\overline{B_j}, \overline{e_j})$, we have

$$\sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \leq \sum_{j=\ell+1}^{k} (-\delta(\overline{B_j}, \overline{e_j})) \leq -\left(\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i})\right). \tag{7}$$

Consider the tuple of even covers $(F_1, \ldots, F_k, F^2)$, where $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for $i \in [\ell-1]$, $F_\ell \in \mathcal{E}(B_\ell + x'v, x'v)$ where x' is the endpoint of $e_{\ell-1}$ in B_ℓ, $F_j \in \widehat{\mathcal{E}}(\overline{B_j}, e_{B_j})$ for $j = \ell+1, \ldots, k$, and $F^2 \in \mathcal{E}(\overline{C_2}, e_{C_2})$. This corresponds to an even cover $F \in \mathcal{E}(G, e)$ containing a cycle through all of $xe_0B_1 \ldots B_{\ell-1}e_{\ell-1}$, e, uy, and C_2, such that

$$\mathrm{exc}(F) - 2 = \sum_{i=1}^{\ell-1}(\mathrm{exc}(F_i) - 2) + (\mathrm{exc}(F_\ell) - 2) + \sum_{j=\ell+1}^{k} \mathrm{exc}(F_j) + (\mathrm{exc}(F^2) - 2).$$

Since

$$\begin{aligned} n(G) &= \sum_{i=1}^{\ell-1} n(\overline{B_i}) + n(B_\ell + x'v) + \sum_{j=\ell+1}^{k} n(\overline{B_j}) + n(\overline{C_2}) + 3, \text{ and} \\ n_2(G) &= \sum_{i=1}^{\ell-1} n_2(\overline{B_i}) + n_2(B_\ell + x'v) + \sum_{j=\ell+1}^{k} n_2(\overline{B_j}) + n_2(\overline{C_2}) - 1, \end{aligned}$$

we have

$$\begin{aligned} \mathrm{exc}(G, e) &\leq \sum_{i=1}^{\ell-1} \mathrm{exc}(\overline{B_i}, \overline{e_i}) + \mathrm{exc}(B_\ell + x'v, x'v) + \sum_{j=\ell+1}^{k} \widehat{\mathrm{exc}}(\overline{B_j}, \overline{e_j}) + \mathrm{exc}(\overline{C_2}, e_{C_2}) \\ &= \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \left(\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i})\right) + \delta(B_\ell + x'v, x'v) \\ &\quad + \left(\sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j})\right) + \delta(\overline{C_2}, e_{C_2}) \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \delta(B_\ell + x'v, x'v) + \delta(\overline{C_2}, e_{C_2}) \qquad \text{(by (7))} \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{3}{2}, \end{aligned}$$

where the last inequality follows as by our assumption Theorem 2.4 holds for $(B_\ell + x'v, x'v)$ and $(\overline{C_2}, e_{C_2})$. Hence $\delta(G, e) \leq -\frac{3}{2}$ and (ii) holds.

To prove (iii), suppose $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{3}{2}, \frac{3}{2})$. Then equality holds above, so we have $\delta(B_\ell + x'v, x'v) = \delta(\overline{C_2}, e_{C_2}) = -\frac{1}{2}$. Moreover, C_1 and C_2 are minimal chains (by (i)), which implies $k = \ell = 1$ and $\delta(\overline{B_\ell}, \overline{e_\ell}) = \delta(\overline{C_1}, e_{C_1}) = -\frac{1}{2}$ (by Proposition 2.1).

So $\delta(\overline{B_\ell}, \overline{e_\ell}) + \delta(B_\ell + x'v, x'v) = -1$. Now B_ℓ is a single vertex; otherwise, by applying Lemma 3.2 to B_ℓ, x', the other endpoint y' of $\overline{e_\ell}$, and v, we obtain $\delta(\overline{B_\ell}, \overline{e_\ell}) + \delta(B_\ell + x'v, x'v) = \delta(B_\ell + x'y', x'y') + \delta(B_\ell + x'v, x'v) \leq -2$, a contradiction. Therefore, we have $B_\ell = \{x'\} = \{v\}$, and e joins two nonadjacent vertices of the 4-cycle $xvyux$. □

We conclude this section with a lemma bounding $\widehat{\delta}(G, e)$, which proves statement **(T2)** of Theorem 2.4, assuming Theorem 2.4 for smaller graphs.

Lemma 3.4. *Let G be a 2-connected subcubic graph with $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected. Assume that Theorem 2.4 holds for graphs with fewer than $n(G)$ vertices. Then $\widehat{\delta}(G, e) \leq \frac{3}{2}$, with equality if and only if (G_u, f_u) is a minimal rooted θ-chain, where G_u is the graph obtained from $G - e$ by suppressing u into an edge f_u.*

Proof. Since $G - e$ is 2-connected, both u and v have degrees 3. Define G_u, f_u as stated in the lemma. We claim that

$$\widehat{\delta}(G, e) = \min\{\delta(G_u, f_u) + 2, \widehat{\delta}(G_u, f_u) + 1\}. \tag{8}$$

Indeed, there is a bijective correspondence between $\widehat{\mathcal{E}}(G, e)$ and $\mathcal{E}(G_u)$ obtained as follows. If $F \in \widehat{\mathcal{E}}(G, e)$ contains a cycle through u, then we obtain $F_u \in \mathcal{E}(G_u, f_u)$ by suppressing u in F, and we have $\operatorname{exc}(F) = \operatorname{exc}(F_u)$. Otherwise, if u is an isolated vertex in F, then we obtain $F_u \in \widehat{\mathcal{E}}(G_u, f_u)$ by removing u from F, and we have $\operatorname{exc}(F) = \operatorname{exc}(F_u) + 1$. Since $n(G) + n_2(G) = n(G_u) + n_2(G_u)$, (8) follows from the definitions of $\delta, \widehat{\delta}$.

It follows from (8) that $\widehat{\delta}(G, e) \leq \delta(G_u, f_u) + 2 \leq \frac{3}{2}$ by the assumption that Theorem 2.4 holds for (G_u, f_u). Moreover, $\widehat{\delta}(G, e) = \frac{3}{2}$ if and only if $\delta(G_u, f_u) = -\frac{1}{2}$ and $\widehat{\delta}(G_u, f_u) = \frac{1}{2}$, which is equivalent to (G_u, f_u) being a minimal rooted θ-chain by Lemma 3.1. □

4. Proof of Theorem 2.4

We proceed by induction on $n(G)$. Note that **(T4)** is implied by **(T1)** and **(T2)**: If $\delta(G, e) \leq -1$ and $\widehat{\delta}(G, e) \leq 1$, then **(T4)** holds. Otherwise, we have $\delta(G, e) = -\frac{1}{2}$ or $\widehat{\delta}(G, e) = \frac{3}{2}$. In the former case, **(T4)** follows from **(T1)** and Lemma 3.1; in the latter case, **(T4)** follows from **(T2)** and Lemma 3.3. Also note that Lemmas 3.3 and 3.4 imply **(T2)**. Therefore, it suffices to prove **(T1)** and **(T3)**.

If $G - \{u, v\}$ is disconnected, then **(T1)** and **(T3)** both hold by Lemma 3.1. So we may assume that $G - \{u, v\}$ is connected. It now suffices to show that $\delta(G, e) \leq -1$ and that if equality holds, then one of the outcomes of **(T3)** holds.

Claim 4.0.1. *We may assume that G is simple.*

Proof. Since $G - e$ is simple, if G is not simple, then there is exactly one edge e^* parallel with e. Let G' be the graph obtained from G by suppressing $\{u, v\}$ to an edge e'.

Then $n(G) = n(G') + 2$ and $n_2(G) = n_2(G')$. By the inductive hypothesis, we have $\delta(G', e') \leq -\frac{1}{2}$. But every even cover $F' \in \mathcal{E}(G', e')$ gives an even cover $F \in \mathcal{E}(G, e)$ with the same excess, so

$$\begin{aligned}
\delta(G, e) &= \min_{F \in \mathcal{E}(G,e)} \operatorname{exc}(F) - 2 - \frac{n(G) + n_2(G)}{4} \\
&\leq \min_{F' \in \mathcal{E}(G',e')} \operatorname{exc}(F') - 2 - \frac{n(G') + n_2(G') + 2}{4} \\
&= \delta(G', e') - \frac{1}{2} \\
&\leq -1.
\end{aligned}$$

Now suppose $\delta(G, e) = -1$. Then both inequalities above are tight; in particular, we have $\delta(G', e') = -\frac{1}{2}$, and by the inductive hypothesis, G' is a loop or (G', e') is a balanced tight rooted θ-chain. If G' is a loop then (G, e) satisfies (b) of **(T3)**. So assume that (G', e') is a balanced tight rooted θ-chain, and let C_1, C_2 denote the two chains of (G', e').

Then a pair of even covers F_1, F_2 where $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for each $i \in [2]$ gives an even cover $F \in \mathcal{E}(G, e)$ by combining the two cycles of F_i through e_{C_i} and adding the cycle with edge set $\{e, e^*\}$, with

$$\operatorname{exc}(F) - 2 = (\operatorname{exc}(F_1) - 2) + (\operatorname{exc}(F_2) - 2) + 2.$$

Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 4$ and $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, we have

$$\begin{aligned}
\operatorname{exc}(G, e) &\leq \operatorname{exc}(\overline{C_1}, e_{C_1}) + \operatorname{exc}(\overline{C_2}, e_{C_2}) + 2 \\
&= \frac{n(G) + n_2(G)}{4} + 1 + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_1}, e_{C_1}),
\end{aligned}$$

so $\delta(G, e) \leq 1 + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2})$. Thus, we have $\delta(\overline{C_i}, e_{C_i}) \in \{-\frac{1}{2}, -1\}$ for each $i \in [2]$; in other words, (G', e') is a near-minimal rooted θ-chain. So (G, e) satisfies (b) of **(T3)**. ■

Claim 4.0.2. *We may assume that e is not in any 2-edge-cut of G.*

Proof. Suppose there is an edge e' such that $\{e, e'\}$ is a 2-edge-cut of G. Let C be a subcubic chain of G with end edges e, e'. By Proposition 2.2 and by the inductive hypothesis applied to $(G/C, e_{G/C})$ and $(\overline{C}, e_C)$, we have

$$\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \leq -1.$$

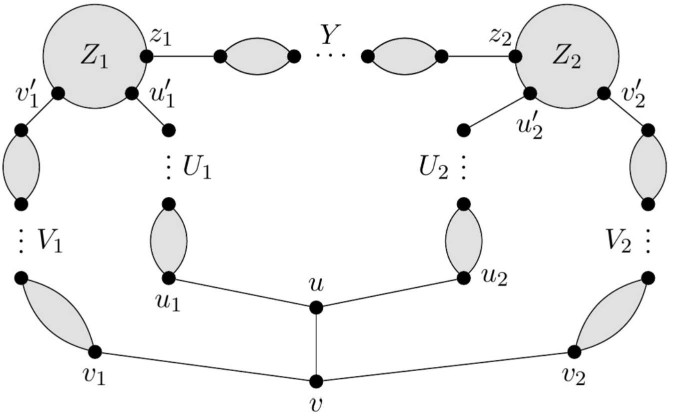

Fig. 4. $Z_1 \neq Z_2$.

Moreover, if $\delta(G, e) = -1$, then $\delta(G/C, e_{G/C}) = \delta(\overline{C}, e_C) = -\frac{1}{2}$, so $(G/C, e_{G/C})$ and $(\overline{C}, e_C)$ are loops or balanced tight rooted θ-chains and (c) of **(T3)** holds for (G, e). ■

By Claim 4.0.2, let u_1, u_2 denote the two neighbors of u distinct from v, and let v_1, v_2 denote the two neighbors of v distinct from u. Moreover, there exist two disjoint paths P_1, P_2 from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ in $G - \{u, v\}$. We may assume without loss of generality that the set of endpoints of P_i is $\{u_i, v_i\}$, $i \in [2]$.

Let S denote the set of all cut edges in $G - \{u, v\}$. Then each component of $G - \{u, v\} - S$ is either an isolated vertex or 2-connected.

Claim 4.0.3. *For each $i \in [2]$, there is a unique component Z_i of $G - \{u, v\} - S$, such that there are three paths in $G - \{u, v\}$ from Z_i to $\{u_i, v_i, u_{3-i}\}$, pairwise disjoint except possibly at their endpoints in Z_i, and there are three paths in $G - \{u, v\}$ from Z_i to $\{u_i, v_i, v_{3-i}\}$, pairwise disjoint except possibly at their endpoints in Z_i. See Figs. 4 and 5.*

Proof. By symmetry, it suffices to prove the claim for $i = 1$. First, we show that there is a unique component Z_1 of $G - \{u, v\} - S$ such that there are three paths in $G - \{u, v\}$ from Z_1 to $\{u_1, v_1, u_2\}$, pairwise disjoint except possibly at their endpoints in Z_1. Indeed, if there were two distinct such components Z, Z', they are by definition separated by a cut-edge $s \in S$ of $G - \{u, v\}$. But $G - \{u, v\} - s$ has exactly two connected components, one of which contains at least two of $\{u_1, v_1, u_2\}$, so one of Z, Z' is separated from two vertices of $\{u_1, v_1, u_2\}$ by a cut-edge, contradicting the assumptions on Z, Z'.

Similarly, there is a unique connected component Z'_1 of $G - \{u, v\} - S$ such that there are three paths in $G - \{u, v\}$ from Z'_1 to $\{u_1, v_1, v_2\}$, pairwise disjoint except possibly at their endpoints in Z'_1. We now show that $Z_1 = Z'_1$. Otherwise, there is a cut edge s of $G - \{u, v\}$ separating Z_1 from Z'_1. Then the two connected components of $G - \{u, v\} - s$

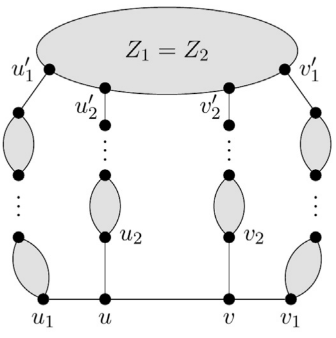

Fig. 5. $Z_1 = Z_2$.

each contain exactly one of $\{u_1, v_1\}$ and exactly one of $\{u_2, v_2\}$. But this implies that $\{e, s\}$ is a 2-edge-cut in G, contradicting Claim 4.0.2. ■

There are two cases to consider: either $Z_1 \neq Z_2$ or $Z_1 = Z_2$. For $i \in [2]$, let u_i' (respectively, v_i') denote the vertex of Z_i that is the endpoint of a (possibly trivial) path in $G - \{u, v\}$ from u_i (respectively, v_i) to Z_i that is internally disjoint from $Z_1 \cup Z_2$. Note that u_i' and v_i' are uniquely determined. For $i \in [2]$, let U_i denote the unique (possibly trivial) subcubic chain of $G - \{v, u_{3-i}\}$ with endpoints $\{u, u_i'\}$, and let V_i denote the unique subcubic chain of $G - \{u, v_{3-i}\}$ with endpoints $\{v, v_i'\}$.

Case 1: $Z_1 \neq Z_2$.

There is a cut-edge separating Z_1 and Z_2 in $G - \{u, v\}$ and there is a unique subcubic chain Y of $G - \{u, v\}$ with an endpoint $z_i \in Z_i$ for each $i \in [2]$, internally disjoint from $Z_1 \cup Z_2$. Then G is the union of $U_1, U_2, V_1, V_2, Z_1, Z_2, Y$, and the edge $e = uv$. We have, for $i, j \in [2]$,

$$\begin{aligned} n(G) &= n(\overline{U_1}) + n(\overline{U_2}) + n(\overline{V_1}) + n(\overline{V_2}) + n(Z_1 + u_i'z_1) + n(Z_2 + v_j'z_2) + n(\overline{Y}) + 2, \\ n_2(G) &= n_2(\overline{U_1}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(\overline{V_2}) + n_2(Z_1 + u_i'z_1) \\ &\quad + n_2(Z_2 + v_j'z_2) + n_2(\overline{Y}) - 2. \end{aligned}$$

Suppose $F \in \mathcal{E}(G, e)$ goes through U_1, Y, and V_2. Then there is a correspondence between F and the tuple $(F_{U_1}, F_{Z_1}, F_Y, F_{Z_2}, F_{V_2}, F_{U_2}, F_{V_1})$, where

- $F_{U_1} \in \mathcal{E}(\overline{U_1}, e_{U_1})$, $F_{Z_1} \in \mathcal{E}(Z_1 + u_1'z_1, u_1'z_1)$, $F_Y \in \mathcal{E}(\overline{Y}, e_Y)$, $F_{Z_2} \in \mathcal{E}(Z_2 + v_2'z_2, v_2'z_2)$, $F_{V_2} \in \mathcal{E}(\overline{V_2}, e_{V_2})$, and
- $F_{U_2} \in \widehat{\mathcal{E}}(\overline{U_2}, e_{U_2})$, $F_{V_1} \in \widehat{\mathcal{E}}(\overline{V_1}, e_{V_1})$.

This gives

$$\begin{aligned} \operatorname{exc}(G, e) \leq\ & \operatorname{exc}(\overline{U_1}, e_{U_1}) + \operatorname{exc}(Z_1 + u_1'z_1, u_1'z_1) + \operatorname{exc}(\overline{Y}, e_Y) + \operatorname{exc}(Z_2 + v_2'z_2, v_2'z_2) \\ & + \operatorname{exc}(\overline{V_2}, e_{V_2}) + \widehat{\operatorname{exc}}(\overline{U_2}, e_{U_2}) + \widehat{\operatorname{exc}}(\overline{V_1}, e_{V_1}) \end{aligned}$$

$$= \frac{n(G)+n_2(G)}{4} + \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(\overline{V_2}, e_{V_2}) + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}),$$

hence

$$\begin{aligned}\delta(G,e) \le \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \\ + \delta(\overline{V_2}, e_{V_2}) + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}).\end{aligned} \tag{9}$$

Similarly, by considering an even cover in $\mathcal{E}(G,e)$ through U_2, Y, and V_1, we obtain

$$\begin{aligned}\delta(G,e) \le \delta(\overline{U_2}, e_{U_2}) + \delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1) \\ + \delta(\overline{V_1}, e_{V_1}) + \widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}).\end{aligned} \tag{10}$$

Now suppose $\delta(G,e) \ge -1$. Then

$$\begin{aligned}
-1 &\le \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(\overline{V_2}, e_{V_2}) \\
&\quad + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}) && \text{(by (9))} \\
&\le -\big(\widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}) + \delta(\overline{U_2}, e_{U_2}) + \delta(\overline{V_1}, e_{V_1})\big) && \text{(by inductive hypothesis)} \\
&\quad + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \\
&= -\big(\widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}) + \delta(\overline{U_2}, e_{U_2}) + \delta(\overline{V_1}, e_{V_1})\big) \\
&\quad - \big(\delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1)\big) \\
&\quad + \big(\delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1)\big) \\
&\quad + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \\
&\le 1 + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) && \text{(by (10))} \\
&\quad + \delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1).
\end{aligned}$$

This gives

$$\begin{aligned}
-2 &\le \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(Z_2 + u_2'z_2, u_2'z_2) \\
&\quad + \delta(Z_1 + v_1'z_1, v_1'z_1) + 2\delta(\overline{Y}, e_Y) \\
&\le -2,
\end{aligned}$$

since by inductive hypothesis, the all terms are each at most $-\frac{1}{2}$ except $\delta(\overline{Y}, e_Y) = 0$ when Y is a trivial chain. Hence, $\delta(G,e) = -1$,

$$\begin{aligned}\delta(Z_1 + u_1'z_1, u_1'z_1) = \delta(Z_2 + v_2'z_2, v_2'z_2) = \delta(Z_1 + u_2'z_2, u_2'z_2) \\ = \delta(Z_1 + v_1'z_1, v_1'z_1) = -\frac{1}{2},\end{aligned}$$

and $\delta(\overline{Y}, e_Y) = 0$ (i.e., Y is a trivial chain). By Lemma 3.2, Z_1 and Z_2 are single vertices. So for $i, j \in [2]$, $\delta(Z_i + u'_j z_i, u'_j z_i) = \delta(Z_i + v'_j z_i, v'_j z_i) = -\frac{1}{2}$. Hence, from (9) and (10), and by the inductive hypothesis, we have

$$\delta(\overline{U_i}, e_{U_i}) = \delta(\overline{V_j}, e_{V_j}) = 0$$

for each $i, j \in [2]$, so U_i, V_j are all trivial chains as well. This proves that $G \cong K_4$, satisfying (a) of **(T3)**.

Case 2: $Z_1 = Z_2$.

Let $Z := Z_1 = Z_2$. Then u'_1, u'_2, v'_1, v'_2 are distinct vertices (since G is subcubic and Z is 2-connected), and G is the union of U_1, U_2, V_1, V_2, Z, and the edge e. Note that

$$\begin{aligned} n(G) &= n(\overline{U_1}) + n(\overline{U_2}) + n(\overline{V_1}) + n(\overline{V_2}) + n(Z + u'_i v'_j) + 2 \\ n_2(G) &= n_2(\overline{U_1}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(\overline{V_2}) + n_2(Z + u'_i v'_j) - 2. \end{aligned}$$

For $i, j \in [2]$, let $F \in \mathcal{E}(G, e)$ be an even cover through U_i and V_j. This corresponds to a tuple $(F_{U_1}, F_{U_2}, F_{V_1}, F_{V_2}, F_Z)$ where

- $F_{U_i} \in \mathcal{E}(\overline{U_i}, e_{U_i}),\ F_{V_j} \in \mathcal{E}(\overline{V_j}, e_{V_j}),\ F_Z \in \mathcal{E}(Z + u'_i v'_j, u'_i v'_j)$, and
- $F_{U_{3-i}} \in \widehat{\mathcal{E}}(\overline{U_{3-i}}, e_{U_{3-i}}),\ F_{V_{3-j}} \in \widehat{\mathcal{E}}(\overline{V_{3-j}}, e_{V_{3-j}})$,

which gives

$$\begin{aligned} \operatorname{exc}(G, e) &\le \operatorname{exc}(\overline{U_i}, e_{U_i}) + \operatorname{exc}(\overline{V_j}, e_{V_j}) + \operatorname{exc}(Z + u'_i v'_j, u'_i v'_j) \\ &\quad + \widehat{\operatorname{exc}}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\operatorname{exc}}(\overline{V_{3-j}}, e_{V_{3-j}}) \\ &= \frac{n(G) + n_2(G)}{4} + \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \delta(Z + u'_i v'_j, u'_i v'_j) \\ &\quad + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}). \end{aligned}$$

Hence, for all $i, j \in [2]$,

$$\delta(G, e) \le \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \delta(Z + u'_i v'_j, u'_i v'_j) + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}) \tag{11}$$

We now show that $\delta(G, e) \le -\frac{3}{2}$, which completes the proof of Theorem 2.4. Suppose to the contrary that $\delta(G, e) \ge -1$. Then by (11) and the inductive hypothesis,

$$\begin{aligned} -1 &\le \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}) + \delta(Z + u'_i v'_j, u'_i v'_j) && \text{(by (11))} \\ &\le -\left(\widehat{\delta}(\overline{U_i}, e_{U_i}) + \widehat{\delta}(\overline{V_j}, e_{V_j}) + \delta(\overline{U_{3-i}}, e_{U_{3-i}}) + \delta(\overline{V_{3-j}}, e_{V_{3-j}})\right) && \text{(by (T4))} \\ &\quad + \delta(Z + u'_i v'_j, u'_i v'_j) \end{aligned}$$

$$\begin{aligned}
&= -\big(\widehat{\delta}(\overline{U_i}, e_{U_i}) + \widehat{\delta}(\overline{V_j}, e_{V_j}) + \delta(Z + u'_{3-i}v'_{3-j}, u'_{3-i}v'_{3-j}) + \delta(\overline{U_{3-i}}, e_{U_{3-i}}) \\
&\quad + \delta(\overline{V_{3-j}}, e_{V_{3-j}})\big) + \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i}v'_{3-j}, u'_{3-i}v'_{3-j}) \\
&\leq 1 + \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i}v'_{3-j}, u'_{3-i}v'_{3-j}). \qquad \text{(by (11))}
\end{aligned}$$

Hence for $i, j \in [2]$,

$$-2 \leq \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i}v'_{3-i}, u'_{3-j}v'_{3-j}) \tag{12}$$

On the other hand, applying Lemma 3.2 to u'_i, v'_1, v'_2 and v'_j, u'_1, u'_2, we have for all $i, j \in [2]$

$$\begin{aligned}
\delta(Z + u'_i v'_1, u'_i v'_1) + \delta(Z + u'_i v'_2, u'_i v'_2) &\leq -2 \text{ and} \\
\delta(Z + u'_1 v'_j, u'_1 v'_j) + \delta(Z + u'_2 v'_j, u'_2 v'_j) &\leq -2.
\end{aligned} \tag{13}$$

Now, setting $i = j = 1$ and setting $i = 1$ and $j = 2$ in (12), we have

$$-4 \leq \delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1).$$

On the other hand, setting $i = 1$ and $i = 2$ in the first inequality of (13), we have

$$\delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) \leq -4.$$

We thus have equality everywhere. In particular, $\delta(G, e) = -1$ and we have equality in (12) and (13), which implies that for all $i, j \in [2]$,

$$\delta(Z + u'_i v'_j, u'_i v'_j) = -1. \tag{14}$$

Since $Z + u'_i v'_j$ has at least two vertices of degree 2 (namely u'_{3-i} and v'_{3-j}), it is not isomorphic to K_4. Moreover, since Z is 2-connected, $u'_i v'_j$ is not contained in any 2-edge-cut in $Z + u'_i v'_j$. So each $(Z + u'_i v'_j, u'_i v'_j)$ satisfies (b) or (d) of **(T3)**.

We claim that $u'_i v'_j \notin E(Z)$ for all $i, j \in [2]$ (hence $(Z + u'_i v'_j, u'_i v'_j)$ satisfies (d) of **(T3)**). For, suppose without loss of generality that $u'_1 v'_1 \in E(Z)$. By the inductive hypothesis, (b) of **(T3)** holds for $(Z + u'_1 v'_1, u'_1 v'_1)$, so suppressing $\{u'_1, v'_1\}$ in Z to an edge e' results in a graph Z' such that (Z', e') is a near-minimal rooted θ-chain. Let C_1, C_2 denote the two chains of (Z', e'). Assume without loss of generality that $v'_2 \in V(C_1)$. Since v'_2 has degree 2 in Z, it is in the interior of C_1, and this implies that $Z - \{u'_1, v'_2\}$ is connected and $v'_2 u'_1 \notin E(Z)$. Then $(Z + u'_1 v'_2, u'_1 v'_2)$ satisfies (d) of **(T3)**, which implies that $Z - \{u'_1, v'_2\}$ is disconnected, a contradiction.

It follows that $(Z + u'_i v'_j, u'_i v'_j)$ satisfy (d) of **(T3)** for all $i, j \in [2]$, so $(Z + u'_i v'_j, u'_i v'_j)$ is a rooted θ-chain for all $i, j \in [2]$. Consider the rooted θ-chain $(Z + u'_1 v'_1, u'_1 v'_1)$. Since $(Z + u'_1 v'_2, u'_1 v'_2)$ (respectively, $(Z + u'_2 v'_1, u'_2 v'_1)$) is a rooted θ-chain, $\{v'_2\}$ (respectively, $\{u'_2\}$) is a block in one of the chains of $(Z + u'_1 v'_1, u'_1 v'_1)$. Let C_1 denote the subcubic chain of Z with end points $\{u_1', v_1'\}$ not containing v_2', and let C_2 denote the subcubic chain of Z with end points $\{u_1', v_2'\}$ not containing v_1'. Let D denote the subcubic chain of Z with end points $\{v_1', v_2'\}$ not containing u_1'.

Then for $j \in [2]$, $n(Z+u_1'v_j') = n(\overline{C_1})+n(\overline{C_2})+n(\overline{D})+3$ and $n_2(Z+u_1'v_j') = n_2(\overline{C_1})+n_2(\overline{C_2})+n_2(\overline{D})+1$. Thus for each $j \in [2]$, by forming an even cover in $\mathcal{E}(Z+u_1'v_j', u_1'v_j')$ using even covers from $\widehat{\mathcal{E}}(\overline{C_j}, e_{C_j})$, $\mathcal{E}(\overline{D}, e_D)$, and $\mathcal{E}(\overline{C_{3-j}}, e_{C_{3-j}})$, we obtain

$$\delta(Z + u_1'v_j', u_1'v_j') \leq -1 + \widehat{\delta}(\overline{C_j}, e_{C_j}) + \delta(\overline{D}, e_D) + \delta(\overline{C_{3-j}}, e_{C_{3-j}}).$$

Adding these two inequalities and using (14), we have

$$\begin{aligned}0 &\leq \delta(\overline{C_1}, e_{C_1}) + \widehat{\delta}(\overline{C_1}, e_{C_1}) + 2\delta(\overline{D}, e_D) + \delta(\overline{C_2}, e_{C_2}) + \widehat{\delta}(\overline{C_2}, e_{C_2}) \\ &\leq 2\delta(\overline{D}, e_D)\end{aligned}$$

by **(T4)** applied to $(\overline{C_i}, e_{C_i})$. It follows that D is a trivial chain, and $v_1'v_2' \in E(Z)$.

By symmetry, $u_1'u_2' \in E(Z)$. Thus, $\{u_1'u_2', v_1v_2'\}$ is a 2-edge-cut in Z. Let D_1, D_2 denote the connected components of $Z - \{u_1'u_2', v_1'v_2'\}$ and (by relabeling u_1', u_2' if necessary) assume $u_i', v_i' \in V(D_i)$ for $i \in [2]$. Then for $i, j \in [2]$, $n(Z + u_i'v_j', u_i'v_j') = n(\overline{D_1}, e_{D_1}) + n(\overline{D_2}, e_{D_2}) + 4$ and $n_2(Z+u_i'v_j', u_i'v_j') = n_2(\overline{D_1}, e_{D_1}) + n_2(\overline{D_2}, e_{D_2}) + 2$. Thus, by forming an even cover in $\mathcal{E}(Z+u_i'v_j', u_i'v_j')$ using even covers from $\mathcal{E}(\overline{D_k}, e_{D_k})$ and $\widehat{\mathcal{E}}(\overline{D_{3-k}}, e_{D_{3-k}})$ for $k \in [2]$, we get

$$\delta(Z + u_i'v_j', u_i'v_j') \leq -\frac{3}{2} + \delta(\overline{D_k}, e_{D_k}) + \widehat{\delta}(\overline{D_{3-k}}, e_{D_{3-k}})$$

Adding these two inequalities and using (14) and **(T4)**, we have

$$1 \leq \delta(\overline{D_1}, e_{D_1}) + \widehat{\delta}(\overline{D_1}, e_{D_1}) + \delta(\overline{D_2}, e_{D_2}) + \widehat{\delta}(\overline{D_2}, e_{D_2}) \leq 0,$$

a contradiction. This completes the proof of Theorem 2.4.

5. Extremal examples

In this section, we give a structural characterization of the extremal examples of Theorem 1.1. Recall that for a subcubic graph G and any edge $e \in E(G)$, we have

$$\begin{aligned}\mathrm{exc}(G) &= \min\{\mathrm{exc}(G, e) + 2,\ \widehat{\mathrm{exc}}(G, e)\} \\ &= \frac{n(G) + n_2(G)}{4} + \min\{\delta(G, e) + 2,\ \widehat{\delta}(G, e)\}.\end{aligned}$$

So if either $\delta(G, e) \leq -\frac{3}{2}$ or $\widehat{\delta}(G, e) \leq \frac{1}{2}$ for any edge $e \in E(G)$, then $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$. It follows that $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$ (equivalently, $\mathrm{tsp}(G) = \frac{5n(G)+n_2(G)}{4} - 1$) if and only if $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ for all $e \in E(G)$.

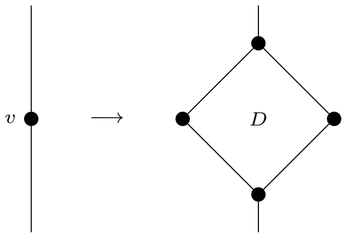

Fig. 6. The $\diamond$-operation.

Proposition 5.1. *Let G be a simple 2-connected subcubic graph and let e be an edge of G. Then $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$ if and only if either $G \cong K_4$ or G is a minimal θ-chain.*

Proof. Suppose $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$. Since $\delta(G,e) = -1$, one of the four outcomes of **(T3)** holds. If $G \cong K_4$ then we are done. Since G is simple, (b) of **(T3)** cannot occur. Moreover, (d) of **(T3)** does not hold; otherwise, (G,e) is a simple rooted θ-chain and, by Lemma 3.1 (ii), $\widehat{\delta}(G,e) \le \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}) \le 1/2$, a contradiction.

Thus (c) of **(T3)** holds: there exists $e' \in E(G)$ such that $\{e, e'\}$ is a 2-edge-cut in G and suppressing either subcubic chain C of G with end edges e, e' yields a loop or a balanced tight rooted θ-chain $(G/C, e_{G/C})$. Let C be a subcubic chain of G with end edges e, e'. Then by Proposition 2.2 and **(T4)**,

$$\begin{aligned} -1 = \delta(G,e) &= \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \le -\big(\widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)\big) \\ &= -\widehat{\delta}(G,e) = -1. \end{aligned}$$

This implies that $(\delta(G/C, e_{G/C}), \widehat{\delta}(G/C, e_{G/C})) = (\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) = (-\frac{1}{2}, \frac{1}{2})$, and thus $(\overline{C}, e_C)$ and $(G/C, e_{G/C})$ are minimal rooted θ-chains (by Lemma 3.1 (iii)). Therefore, by definition, G is a minimal θ-chain (since it is the internally disjoint union of C and the two chains of $(G/C, e_{G/C})$, all of which are minimal). $\square$

To give an alternate structural characterization of minimal (rooted) θ-chains, we now describe an operation introduced in [8]. Let H be a graph and $v \in V(H)$ be a vertex of degree 2. A *$\diamond$-operation on H at v* deletes v from H, adds a 4-cycle D disjoint from $H - v$, and adds a matching between the neighbors of v and two nonadjacent vertices in D. See Fig. 6. We say that a graph is *H-constructible* if it can be obtained from H by repeated $\diamond$-operations.

It is observed in [8] that after each $\diamond$-operation, the excess of the new graph increases by 1 and the new quantity $\frac{n(G)+n_2(G)}{4}$ also increases by 1. We will consider $K_{2,3}$-constructible graphs and K_4^--constructible graphs, where K_4^- is the graph obtained from the complete graph K_4 by removing an edge. Note that $\mathrm{exc}(K_{2,3}) = \frac{n(K_{2,3})+n_2(K_{2,3})}{4} + 1$; thus, if G is $K_{2,3}$-constructible then $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$.

Proposition 5.2 *(Dvořák et al. [8]). Let G be a simple 2-connected subcubic graph. If $G \cong K_4$ or G is $K_{2,3}$-constructible, then $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$.*

We show that the converse of Proposition 5.2 is also true, thereby giving a structural characterization of the extremal graphs for Theorem 1.1. First, we have an observation similar to Proposition 5.2. The *center* of K_4^- is the edge whose endpoints both have degree 3.

Proposition 5.3. *Let (G, e) be a simple minimal rooted θ-chain. Then G is K_4^--constructible, with the edge e corresponding to the center of K_4^-.*

Proof. By **(T1)** and Lemma 3.1 (iii), $(\delta(G,e), \widehat{\delta}(G,e)) = (-\frac{1}{2}, \frac{1}{2})$. Let C_1 and C_2 be the chains of (G,e). By the definition of a minimal rooted θ-chain, for each $i \in [2]$, we have $(\delta(\overline{C_i}, e_{C_i}), \widehat{\delta}(\overline{C_i}, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$, so $(\overline{C_i}, e_{C_i})$ is either a loop or a minimal rooted θ-chain by (**(T1)**) and Lemma 3.1. If $(\overline{C_i}, e_{C_i})$ is not a loop, then by induction, it is K_4^--constructible with e_{C_i} corresponding to the center of K_4^-. It follows that (G,e) is K_4^--constructible with e corresponding to the center of K_4^-. □

Proposition 5.4. *Let G be a simple minimal θ-chain. Then G is $K_{2,3}$-constructible.*

Proof. By definition, there exists a choice of three chains C_1, C_2, C_3 of G with common endpoints such that G is the internally disjoint union $C_1 \cup C_2 \cup C_3$, and we have $(\delta(\overline{C_i}, e_{C_i}), \widehat{\delta}(\overline{C_i}, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$ for each $i \in [3]$. If $G \cong K_{2,3}$, then we are done. So we may assume without loss of generality that $(\overline{C_1}, e_{C_1})$ is not a loop. Then it is a minimal rooted θ-chain by Lemma 3.1, and by Proposition 5.3, it is K_4^--constructible with the edge e_{C_1} corresponding to the center of K_4^-. On the other hand, $(G/C_1, e_{G/C_1})$ is by definition a minimal rooted θ-chain, so it is also K_4^--constructible by Proposition 5.3, with $e_{G/C}$ corresponding to the center of K_4^-. It follows that G is $K_{2,3}$-constructible. □

We thus have the following characterization of the extremal examples of Theorem 1.1.

Theorem 5.5. *Let G be a simple 2-connected subcubic graph. Then* $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + 1$*, with equality if and only if either $G \cong K_4$ or G is $K_{2,3}$-constructible.*

Proof. Let $e \in E(G)$. If $\delta(G,e) \leq -\frac{3}{2}$ or $\widehat{\delta}(G,e) \leq \frac{1}{2}$, then $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$. Otherwise, we have $(\delta(G,e), \widehat{\delta}(G,e)) = (-1, 1)$, or equivalently, $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$. Now if $G \cong K_4$ or G is $K_{2,3}$-constructible, then $(\delta(G,e), \widehat{\delta}(G,e)) = (-1, 1)$ by Proposition 5.2. Conversely, if $(\delta(G,e), \widehat{\delta}(G,e)) = (-1, 1)$, then by Propositions 5.1 and 5.4, either $G \cong K_4$ or G is $K_{2,3}$-constructible. □

6. Algorithm

We now provide an algorithm for finding a TSP walk of length at most $\frac{5n(G)+n_2(G)}{4} - 1$ in any simple 2-connected subcubic graph G. This is achieved by following the proof of Theorem 2.4 to construct an even cover F of G with $\mathrm{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$. As noted by Dvořák et al. [8], modifying this even cover to our desired TSP walk takes linear time.

In the proof of Theorem 2.4, we often have a choice of routing a cycle through certain subcubic chains and not through others. For each such chain C, we "save" $\delta(\overline{C}, e_C)$ by going through C and incur a "cost" $\widehat{\delta}(\overline{C}, e_C)$ by not going through C. The key idea of Theorem 2.4 is that these costs and savings are (at worst) balanced, i.e. $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$. Of course, for a given subcubic graph G and an edge e, we cannot efficiently compute $\delta(G, e)$ and $\widehat{\delta}(G, e)$ exactly (unless P=NP). Instead, we compute "worst-case" estimates

$$(\Delta(G, e), \widehat{\Delta}(G, e)) \in \left\{(-\tfrac{1}{2}, \tfrac{1}{2}),\ (-1, 1),\ (-\tfrac{3}{2}, \tfrac{3}{2})\right\}$$

such that $(\delta(G, e), \widehat{\delta}(G, e)) \leq (\Delta(G, e), \widehat{\Delta}(G, e))$ (coordinate-wise).

The natural approach would be to determine exactly when $\delta(G, e) = -\frac{1}{2}$ or $\widehat{\delta}(G, e) = \frac{3}{2}$ using our characterization of the extremal examples in Theorem 2.4, and assign $(\Delta(G, e), \widehat{\Delta}(G, e)) = (-\frac{1}{2}, \frac{1}{2})$ or $(-\frac{3}{2}, \frac{3}{2})$ respectively (and assign $(-1, 1)$ in all other cases). To check whether (G, e) is a minimal rooted θ-chain (for example), we would need to first check that it is a rooted θ-chain (which takes linear time) and then recursively check that each of its two chains are also minimal, taking quadratic time overall. This approach would result in a cubic algorithm to produce the desired even covers.

It turns out that a much simpler linear-time estimate is sufficient, and yields a quadratic-time algorithm to find the desired even covers. Indeed, by Lemma 3.1, if (G, e) is a rooted θ-chain (regardless of whether it is tight or balanced), then we have $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{1}{2}, \frac{1}{2})$. And by Lemma 3.3, if $G - e$ is simple and 2-connected and (G_u, f_u) is a rooted θ-chain (where G_u is obtained from $G - e$ by suppressing an endpoint u to an edge f_u), then we have $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{3}{2}, \frac{3}{2})$.

We thus define an algorithm $\texttt{Scan}(G, e)$ to estimate $(\delta(G, e), \widehat{\delta}(G, e))$ as follows. If G is a loop or $G - e$ is 2-connected, $\texttt{Scan}(G, e)$ will assign

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \begin{cases} (-\frac{1}{2}, \frac{1}{2}) & \text{if } (G, e) \text{ is a loop or a rooted } \theta\text{-chain}, \\ (-\frac{3}{2}, \frac{3}{2}) & \text{if } (G_u, f_u) \text{ is a rooted } \theta\text{-chain}, \\ (-1, 1) & \text{otherwise.} \end{cases}$$

If $G - e$ is not 2-connected (and it is not a loop), then (G, e) can be written as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ such that $k \geq 2$ (if $k = 1$, then $G - e = \overline{C} - e_C$ is 2-connected or an isolated vertex). In this case, our estimate on (G, e) will be the sum of the estimates of the chain-blocks $(\overline{B_i}, e_{B_i})$ of C:

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \sum_{i=1}^{k} (\Delta(\overline{B_i}, e_{B_i}), \widehat{\Delta}(\overline{B_i}, e_{B_i})).$$

For the remainder of this section, given a 2-connected subcubic graph G and an edge $e = uv \in E(G)$ such that $G - e$ is simple and has no cut-vertex, we let u_1, u_2 denote the two neighbors of u not equal to v, and denote by G_u the graph obtained by deleting e

and suppressing u to an edge $f_u = u_1u_2$. Note that computing G_u and f_u takes constant time. To resolve ambiguities in the choice of the vertex u in the edge $e = uv$ (in the case where $\widehat{\Delta}(G,e) = \frac{3}{2}$), we fix a linear ordering $\leq$ of the vertices throughout, and assume that $u \leq v$.

Proposition 6.1. *Let G be a subcubic graph and let $e = uv \in E(G)$ such that $G - e$ is simple. Then $\delta(G,e) \leq \Delta(G,e)$ and $\widehat{\delta}(G,e) \leq \widehat{\Delta}(G,e)$.*

Proof. First suppose G is a loop or $G - e$ is 2-connected. If (G,e) is a loop or a rooted θ-chain, then by Lemma 3.1, $(\delta(G,e), \widehat{\delta}(G,e)) \leq (-\frac{1}{2}, \frac{1}{2}) = (\Delta(G,e), \widehat{\Delta}(G,e))$. If (G_u, f_u) is a rooted θ-chain, then by Lemma 3.3, $(\delta(G,e), \widehat{\delta}(G,e)) \leq (-\frac{3}{2}, \frac{3}{2}) = (\Delta(G,e), \widehat{\Delta}(G,e))$. Otherwise, by Theorem 2.4, we have $(\delta(G,e), \widehat{\delta}(G,e)) \leq (-1,1) = (\Delta(G,e), \widehat{\Delta}(G,e))$.

Now suppose $G - e$ is not 2-connected. Then we can write (G,e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ where $k \geq 2$. By Proposition 2.1 and by induction, we have

$$\begin{aligned}(\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) &= \sum_{i=1}^{k} (\delta(\overline{B_i}, e_{B_i}), \widehat{\delta}(\overline{B_i}, e_{B_i})) \leq \sum_{i=1}^{k} (\Delta(\overline{B_i}, e_{B_i}), \widehat{\Delta}(\overline{B_i}, e_{B_i})) \\ &= (\Delta(G,e), \widehat{\Delta}(G,e)). \quad \square\end{aligned}$$

Checking whether (G,e) is a rooted θ-chain is equivalent to checking whether $G - \{u,v\}$ is disconnected, which can be done in linear time. More generally, we can determine the block structure of graphs with a depth first search (DFS) in $O(n(G) + |E(G)|)$ time (e.g. [6]), which is $O(n(G))$ when G is subcubic.

Algorithm 1: $\texttt{Scan}(G,e)$.

Input : A loop or a 2-connected subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple
Output: A half integral vector $(\Delta(G,e), \widehat{\Delta}(G,e)) \in \{(-\frac{1}{2}, \frac{1}{2}), (-1,1), (-\frac{3}{2}, \frac{3}{2})\}$.

1 **if** *$G - e$ has a cut-vertex* **then**
2 Write (G,e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$;
3 **return** $\sum_{i=1}^{k} \texttt{Scan}(\overline{B_i}, e_{B_i})$;
4 **if** *$G - \{u,v\}$ is disconnected or G is a loop* **then**
5 **return** $(-\frac{1}{2}, \frac{1}{2})$;
6 **else if** *$G_u - \{u_1, u_2\}$ is disconnected* **then**
7 **return** $(-\frac{3}{2}, \frac{3}{2})$;
8 **else**
9 **return** $(-1, 1)$;

Proposition 6.2. $\texttt{Scan}(G,e)$ *can be computed in $O(n(G))$ time.*

Proof. If $\texttt{Scan}(G,e)$ returns on lines 5, 7, or 9, then it performs at most three depth first searches, thus requiring $O(n(G))$ time. Now suppose $\texttt{Scan}(G,e)$ returns on line 3; that is, (G,e) is the closure of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ where $k \geq 2$. For all $i \in [k]$, $\overline{B_i} - e_{B_i}$ is either 2-connected or a single vertex, so $\texttt{Scan}(\overline{B_i}, e_{B_i})$ will not execute line 2. Thus $\texttt{Scan}(G,e)$ requires a depth first search on an input of size $n(G)$ on line 1 and at most three depth first searches for each $\overline{B_i}$, $i \in [k]$. As $\sum_{i=1}^{k} n(\overline{B_i}) < n(G)$, we have that in all cases, $\texttt{Scan}(G, e)$ requires $O(n(G))$ time. $\square$

We will define two algorithms $\texttt{EC}(G, e)$ and $\widehat{\texttt{EC}}(G, e)$ which will return an even cover F in $\mathcal{E}(G, e)$ and $\widehat{\mathcal{E}}(G, e)$ respectively such that $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$ and $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ respectively. For convenience, we wrap these two algorithms in a main algorithm Algo with preprocessing to handle the base case (where (G, e) is a loop) and the case where $G - e$ is not 2-connected.

Algorithm 2: $\texttt{Algo}(G, e, \text{flag})$.

Input : A loop or a 2-connected subcubic graph G and $e \in E(G)$ such that $G - e$ is simple, and a binary input flag

Output: $F \in \mathcal{E}(G, e)$ such that $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$ (if flag == true) or $F \in \widehat{\mathcal{E}}(G, e)$ such that $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ (if flag == false)

1 **if** G *is a loop* **then**
2   **if** *flag == true* **then**
3   **return** $F = G$;
4   **else**
5   **return** $F = G - e$;
6 **if** $G - e$ *is not 2-connected* **then**
7   Write (G, e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1B_2 \dots e_{k-1}B_ke_ky$;
8   Let $F_i = \texttt{Algo}(\overline{B_i}, e_{B_i}, \text{flag})$ for all $i \in [k]$;
9   **if** *flag == true* **then**
10   **return** $F = \bigcup_{i=1}^{k}(F_i - e_{B_i}) + e + \{e_i : i \in [k-1]\}$;
11   **else**
12   **return** $F = \bigcup_{i=1}^{k} F_i$;
13 Let $(\Delta, \widehat{\Delta}) = \texttt{Scan}(G, e)$;
14 **if** *flag == true* **then**
15   **return** $F = \texttt{EC}(G, e, \Delta)$;
16 **else**
17   **return** $F = \widehat{\texttt{EC}}(G, e)$;

For the remainder of the section, we let $f_{\texttt{Algo}} : \mathbb{N} \to \mathbb{N}$ denote a superadditive function (i.e. $f_{\texttt{Algo}}(n_1) + f_{\texttt{Algo}}(n_2) \le f_{\texttt{Algo}}(n_1 + n_2)$ for all $n_1, n_2 \in \mathbb{N}$) such that $\texttt{Algo}(G, e, \text{flag})$ takes at most $f_{\texttt{Algo}}(n)$ steps on inputs of size at most n. We will show in the end that we can take $f_{\texttt{Algo}}(n) = O(n^2)$.

We now give the algorithm $\widehat{\texttt{EC}}(G, e)$ used in line 17 of $\texttt{Algo}(G, e, \text{flag})$, which produces an even cover $F \in \widehat{\mathcal{E}}(G, e)$ with $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$. Recall that (G_u, f_u) is obtained from G and $e = uv$ by deleting e and suppressing u to an edge $f_u = u_1u_2$.

Algorithm 3: $\widehat{\texttt{EC}}(G, e)$.

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected

Output: An even cover $F \in \widehat{\mathcal{E}}(G, e)$ with $\operatorname{exc}(F) \le \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ where $\widehat{\Delta}(G, e) = \texttt{Scan}(G, e)_2$

Let $F' = \texttt{Algo}(G_u, f_u, \text{true})$;
return $F = (F' - f_u) + \{u\} + \{u_1u, uu_2\}$;

Proposition 6.3. *Suppose* Algo *is correct on inputs of size less than* n. *Then* $\widehat{\texttt{EC}}$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(1)$ *time for all inputs of size less than or equal to* n.

Proof. We clearly have $F \in \widehat{\mathcal{E}}(G,e)$. We claim that $\Delta(G_u, f_u)+2 \leq \widehat{\Delta}(G,e)$. If $\widehat{\Delta}(G,e) = \frac{3}{2}$, there is nothing to prove (since $\Delta \leq -\frac{1}{2}$). If $\widehat{\Delta}(G,e) = 1$, then (G_u, f_u) is not a rooted θ-chain, so $\Delta(G_u, f_u) \leq -1$. Finally, suppose $\widehat{\Delta}(G,e) = \frac{1}{2}$. Then (G,e) is a rooted θ-chain. This implies that (G_u, f_u) is the closure $(\overline{C}, e_C)$ of a subcubic chain C with at least three blocks, so $\Delta(G_u, f_u) = \Delta(\overline{C}, e_C) \leq -\frac{3}{2}$. It follows that

$$\operatorname{exc}(F) = \operatorname{exc}(F') \leq \frac{n(G)+n_2(G)}{4} + \Delta(G_u, f_u) + 2 \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G,e).$$

For the time complexity, note that `Algo` is called only once on (G_u, f_u), which takes $f_{\texttt{Algo}}(n(G_u)) = f_{\texttt{Algo}}(n-1)$ time. The remaining lines require constant time, thus $\widehat{\texttt{EC}}$ runs in $f_{\texttt{Algo}}(n-1) + O(1)$ time. □

We now give the algorithm $\texttt{EC}(G, e, \Delta)$ in line 15 of `Algo`, which produces an even cover $F \in \mathcal{E}(G,e)$ such that $\operatorname{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G,e) + 2$. For clarity of presentation, we split the algorithm into three cases depending on the value Δ. We first describe the case $\Delta = -\frac{1}{2}$.

Algorithm 4: $\texttt{EC}(G, e, -\frac{1}{2})$.

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected, and $\Delta(G,e) = -\frac{1}{2}$ (i.e. (G,e) is a rooted θ-chain)

Output: An even cover $F \in \mathcal{E}(G,e)$ with $\operatorname{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$

1 Determine the subcubic chains C_1 and C_2 of (G,e) with a DFS;

2 Let $(\Delta(C_1), \widehat{\Delta}(C_1)) = \texttt{Scan}(\overline{C_1}, e_{C_1})$ and let $(\Delta(C_2), \widehat{\Delta}(C_2)) = \texttt{Scan}(\overline{C_2}, e_{C_2})$;

3 Relabel if necessary so that $\Delta(C_1) + \widehat{\Delta}(C_2) \leq 0$;

4 Let $F_1 = \texttt{Algo}(\overline{C_1}, e_{C_1}, \text{true})$ and $F_2 = \texttt{Algo}(\overline{C_2}, e_{C_2}, \text{false})$;

5 Let v' be the neighbor of v in C_1 and let u' be the neighbor of u in C_1;

6 **return** $F = (F_1 - e_{C_1}) \cup F_2 + \{u, v\} + \{u'u, uv, vv'\}$;

Proposition 6.4. *Suppose* `Algo` *is correct on inputs of size less than* $n = n(G)$*. Then* $\texttt{EC}(G, e, -\frac{1}{2})$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(n)$ *time for all input graphs of size less than or equal to* n*.*

Proof. For correctness, first note that the relabeling step on line 3 is always possible as $\Delta(C_i) = -\widehat{\Delta}(C_i)$ for $i \in [2]$. Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 2$, $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, and $\operatorname{exc}(F) = \operatorname{exc}(F_1) + \operatorname{exc}(F_2)$, we have

$$\begin{aligned}
\operatorname{exc}(F) &= \operatorname{exc}(F_1) + \operatorname{exc}(F_2) \\
&\leq \frac{n(C_1)+n_2(C_1)}{4} + \Delta(C_1) + 2 + \frac{n(C_2)+n_2(C_2)}{4} + \widehat{\Delta}(C_2) \\
&\leq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}.
\end{aligned}$$

For the time complexity, line 1 requires $O(n)$ time. By Proposition 6.2, line 2 requires $O(n(\overline{C_1})) + O(n(\overline{C_2})) = O(n)$ time. By induction, line 4 takes $f_{\texttt{Algo}}(n(\overline{C_1})) +$

$f_{\texttt{Algo}}(n(\overline{C_2})) \leq f_{\texttt{Algo}}(n-1)$ time. Thus, in total, $\texttt{EC}(G, e, -\frac{1}{2})$ takes $f_{\texttt{Algo}}(n-1) + O(n)$ time of inputs of size n. $\square$

Before we handle the analysis of $\texttt{EC}(G, e, -1)$, we first give an important subroutine which is an algorithmic version of Lemma 3.2.

Algorithm 5: $\texttt{Subroutine}(Z, u, v_1, v_2)$.

Input : A simple 2-connected subcubic graph Z and distinct vertices u, v_1, v_2 of degree 2 in Z
Output: $F \in \mathcal{E}(Z + uv_i, uv_i)$ for some $i \in [2]$ with $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$

```
1 For each i ∈ [2], let (Δ_i, Δ̂_i) = Scan(Z + uv_i, uv_i);
2 if Δ_i ≤ −1 for some i ∈ [2] then
3 |  return F = Algo(Z + uv_i, uv_i, true);
4 Let C_{i,1}, C_{i,2} denote the two subcubic chains of (Z + uv_i, uv_i), i ∈ [2];
5 Let (Δ(C_{i,j}), Δ̂(C_{i,j})) = Scan(C̄_{i,j}, e_{C_{i,j}}) for i, j ∈ [2];
6 Relabel if necessary so that Δ(C_{1,1}) + Δ̂(C_{1,2}) ≤ −1/2;
7 Let F_1 = Algo(C̄_{1,1}, e_{C_{1,1}}, true) and F_2 = Algo(C̄_{1,2}, e_{C_{1,2}}, false);
8 Let u' be the neighbor of u in C_{1,1} and v' be the neighbor of v_1 in C_{1,1};
9 return F = (F_1 − e_{C_{1,1}}) ∪ F_2 + {u, v} + {u'u, uv_1, v_1v'};
```

Proposition 6.5. *Suppose* $\texttt{Algo}$ *is correct for all inputs of size less than or equal to* $n = n(Z)$*. Then* $\texttt{Subroutine}$ *is correct and takes* $f_{\texttt{Algo}}(n) + O(n)$ *time for all inputs of size less than or equal to* n.

Proof. We first analyze correctness. If we return on line 3, by correctness of $\texttt{Algo}$, we have $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$. So assume $\Delta_i = \Delta(Z + uv_i, uv_i) = -\frac{1}{2}$ for both $i \in [2]$. Thus both $(Z + uv_i, uv_i)$ are rooted θ-chains, which implies that v_{3-i} is a trivial block in one of the chains $C_{i,1}$ and $C_{i,2}$. This then implies that $\Delta(C_{i,1}) \neq \Delta(C_{i,2})$ for some $i \in [2]$. Thus the relabeling step on line 6 is always possible.

Now consider the even cover F returned on line 9. As $n(Z+uv_1) = n(\overline{C_{1,1}}) + n(\overline{C_{1,2}}) + 2$, $n_2(Z + uv_1) = n_2(\overline{C_{1,1}}) + n_2(\overline{C_{1,2}})$, and $\Delta(C_{1,1}) + \widehat{\Delta}(C_{1,2}) \leq -\frac{1}{2}$, we have

$$\begin{aligned} \text{exc}(F) &= \text{exc}(F_1) + \text{exc}(F_2) \\ &\leq \frac{n(\overline{C_{1,1}}) + n_2(\overline{C_{1,1}})}{4} + \Delta(\overline{C_{1,1}}) + 2 + \frac{n(\overline{C_{1,2}}) + n_2(\overline{C_{1,2}})}{4} + \widehat{\Delta}(\overline{C_{1,2}}) \\ &\leq \frac{n(Z + uv_1) + n_2(Z + uv_1)}{4} + \Delta(\overline{C_{1,1}}) + \widehat{\Delta}(\overline{C_{1,2}}) + \frac{3}{2} \\ &\leq \frac{n(Z + uv_1) + n_2(Z + uv_1)}{4} + 1. \end{aligned}$$

For the time complexity, as $n(\overline{C_{1,1}}) + n(\overline{C_{1,2}}) < n$, lines 3 and 7 both take at most $f_{\texttt{Algo}}(n)$ time. Furthermore, by Proposition 6.2, the remaining lines require $O(n)$ time. Since we call exactly one of line 3 or 7, $\texttt{Subroutine}(Z, u, v_1, v_2)$ takes $f_{\texttt{Algo}}(n) + O(n)$ time. $\square$

We are now ready to present $\texttt{EC}(G, e, -1)$.

Algorithm 6: $\texttt{EC}(G, e, -1)$.

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected, and $\Delta(G, e) = -1$.

Output: $F \in \mathcal{E}(G, e)$ with $\text{exc}(F) \le \frac{n(G)+n_2(G)}{4} + 1$

1 Let Z_1 and Z_2 be the blocks (or single vertices) of $G - \{u, v\}$ as defined in Claim 4.0.3;
2 Define vertices u_i, u'_i, v_j, v'_j and subcubic chains U_i, V_j for $i, j \in [2]$, as in the proof of Theorem 2.4;
3 Let $(\Delta(U_i), \widehat{\Delta}(U_i)) = \texttt{Scan}(\overline{U_i}, e_{U_i})$ and $(\Delta(V_j), \widehat{\Delta}(V_j)) = \texttt{Scan}(\overline{V_j}, e_{V_j})$ for $i, j \in [2]$;
4 **if** $Z_1 \neq Z_2$ **then**
5 Relabel vertices as necessary so that $\Delta(U_1) + \Delta(V_2) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_1) \le 0$;
6 Let $Z = Z_1 \cup Z_2 \cup Y$, where Y is the subcubic chain from Z_1 to Z_2;
7 Let $F_{U_1} = \texttt{Algo}(\overline{U_1}, e_{U_1}, \text{true})$, $F_{V_2} = \texttt{Algo}(\overline{V_2}, e_{V_2}, \text{true})$, $F_{U_2} = \texttt{Algo}(\overline{U_2}, e_{U_2}, \text{false})$, $F_{V_2} = \texttt{Algo}(\overline{V_1}, e_{V_1}, \text{false})$, and $F_Z = \texttt{Algo}(Z + u'_1v'_2, u'_1v'_2, \text{true})$;
8 **return** $F = (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} \cup (F_Z - u'_1v'_2) + \{u, v\} + \{u_1u, uv, vv_2\}$;
9 **else**
10 Relabel vertices as necessary so that $\Delta(U_1) + \Delta(V_i) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_{3-i}) \le 0$ for $i \in [2]$;
11 Let $F_Z = \texttt{Subroutine}(Z_1, u'_1, v'_1, v'_2)$;
12 Relabel so that $u'_1v'_2 \in F_Z$;
13 Let $F_{U_1} = \texttt{Algo}(\overline{U_1}, e_{U_1}, \text{true})$, $F_{V_2} = \texttt{Algo}(\overline{V_2}, e_{V_2}, \text{true})$, $F_{U_2} = \texttt{Algo}(\overline{U_2}, e_{U_2}, \text{false})$, and $F_{V_1} = \texttt{Algo}(\overline{V_1}, e_{V_1}, \text{false})$;
14 **return** $F = (F_Z - u'_1v'_2) \cup (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} + \{u, v\} + \{u_1u, uv, vv_2\}$;

Proposition 6.6. *Suppose* $\texttt{Algo}$ *is correct on all inputs of size less than* $n = n(G)$*. Then* $\widehat{\texttt{EC}}(G, e, -1)$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(n)$ *time for all inputs of size less than or equal to* n*.*

Proof. The proof of correctness follows the same structure of Section 4. The existence of Z_1 and Z_2 follows from Claim 4.0.3, and they can be determined from the block structure of $G - \{u, v\}$ in linear time. As $\Delta(U_i) = -\widehat{\Delta}(U_i)$ and $\Delta(V_i) = -\widehat{\Delta}(V_i)$ for $i \in [2]$, the relabeling on lines 5 and 10 are always possible. Furthermore, regardless of whether $Z_1 \neq Z_2$ or $Z_1 = Z_2$, we have

- $\text{exc}(F) - 2 = (\text{exc}(F_{U_1}) - 2) + (\text{exc}(F_{V_2}) - 2) + \text{exc}(F_{U_2}) + \text{exc}(F_{V_1}) + (\text{exc}(F_Z) - 2)$,
- $n(G) = n(\overline{U_1}) + n(\overline{V_2}) + n(\overline{U_2}) + n(\overline{V_1}) + n(Z + u'_1v'_2) - 2$, and
- $n_2(G) = n_2(\overline{U_1}) + n_2(\overline{V_2}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(Z + u'_1v'_2) + 2$.

By induction, we have $\text{exc}(F_{U_1}) - 2 \le \frac{n(\overline{U_1})+n_2(\overline{U_1})}{4} + \Delta(U_1)$, $\text{exc}(F_{V_2}) - 2 \le \frac{n(\overline{V_2})+n_2(\overline{V_2})}{4} + \Delta(V_2)$, $\text{exc}(F_{U_2}) \le \frac{n(\overline{U_2})+n_2(\overline{U_2})}{4} + \widehat{\Delta}(U_2)$, and $\text{exc}(F_{V_1}) \le \frac{n(\overline{V_1})+n_2(\overline{V_1})}{4} + \widehat{\Delta}(V_1)$. We argue now that in both cases we have

$$\text{exc}(F_Z) - 2 \le \frac{n(Z + u'_1v'_2) + n_2(Z + u'_1v'_2)}{4} - 1. \tag{15}$$

If $Z_1 = Z_2$, this follows from Proposition 6.5. If $Z_1 \neq Z_2$, then $(Z + u'_1v'_2, u'_1v'_2)$ is the closure of a subcubic chain with at least two blocks, namely Z_1 and Z_2. By induction on its chain-blocks, we have

$$\text{exc}(F_Z) - 2 \le \frac{n(Z + u'_1v'_2) + n_2(Z + u'_1v'_2)}{4} + \Delta(Z + u'_1v'_2, u'_1v'_2)$$

$$\leq \frac{n(Z+u_1'v_2')+n_2(Z+u_1'v_2')}{4}-1$$

and (15) holds in both cases. Thus,

$$\begin{aligned}\operatorname{exc}(F)-2 &= (\operatorname{exc}(F_{U_1})-2)+(\operatorname{exc}(F_{V_2})-2)+\operatorname{exc}(F_{U_2})+\operatorname{exc}(F_{V_1})+(\operatorname{exc}(F_Z)-2)\\ &\leq \frac{n(G)+n_2(G)}{4}+\Delta(U_1)+\Delta(V_2)+\widehat{\Delta}(U_2)+\widehat{\Delta}(V_1)+\Delta(Z+u_1v_2',u_1'v_2')\\ &\leq \frac{n(G)+n_2(G)}{4}-1.\end{aligned}$$

For the time complexity, note that we only call `Algo` and `Subroutine` on inputs whose sizes sum to less than n. As the remaining lines require $O(n)$ time by Proposition 6.2, we have that the entire algorithm requires $f_{\texttt{Algo}}(n-1)+O(n)$ time. □

We now present the final case for `EC`.

Algorithm 7: $\texttt{EC}(G,e,-\frac{3}{2})$.

Input : A subcubic graph G and $e=uv\in E(G)$ with $G-e$ is simple and 2-connected, and $\Delta(G,e)=-\frac{3}{2}$ (i.e. (G_u,f_u) is a rooted θ-chain)

Output: $F\in\mathcal{E}(G,e)$ with $\operatorname{exc}(F)\leq\frac{n(G)+n_2(G)}{4}+\frac{1}{2}$

Let C_1 and C_2 denote the chains of (G_u,f_u) with common endpoints $f_u=\{u_1,u_2\}$ and $v\in V(C_1)$;
Let $x_i\in V(C_2)$ be the neighbor of u_i for $i\in[2]$;
Write $C_1=u_1e_0B_1\dots e_{k-1}B_ke_ku_2$;
Let $\ell\in[k]$ be the unique index such that $v\in V(B_\ell)$;
Let v' denote the endpoint of $e_{\ell-1}$ in B_ℓ, and let v'' denote the endpoint of e_ℓ in B_ℓ;
Let D_1 and D_2 denote the chains of C_1 with end points $\{u_1,v'\}$ and $\{v'',u_2\}$ respectively;
For $i\in[2]$, let $(\Delta(D_i),\widehat{\Delta}(D_i))=\texttt{Scan}(\overline{D_i},e_{D_i})$;
Relabel if necessary so that $\Delta(D_1)+\widehat{\Delta}(D_2)\leq 0$;
Let $F_2=\texttt{Algo}(\overline{C_2},e_{C_2},\text{true})$, $F_{D,1}=\texttt{Algo}(\overline{D_1},e_{D_1},\text{true})$, $F_{D,2}=\texttt{Algo}(\overline{D_2},e_{D_2},\text{false})$, and $F_\ell=\texttt{Algo}(B_\ell+v'v,v'v,\text{true})$;
return
$F=(F_2-e_{C_2})\cup(F_{D,1}-e_{D_1})\cup F_{D,2}\cup(F_\ell-v'v)+\{u,u_1,u_2\}+\{e_0,e_{\ell-1},u_1x_1,uv,uu_2,u_2x_2\}$;

Proposition 6.7. *Suppose* `Algo` *is correct for all inputs of size less than $n=n(G)$. Then* $\texttt{EC}(G,e,-\frac{3}{2})$ *is correct and takes $f_{\texttt{Algo}}(n-1)+O(n)$ time for all inputs of size less than or equal to n.*

Proof. We first analyze the correctness of the returned even cover F. By induction, we have that $\operatorname{exc}(F_2)\leq\frac{n(\overline{C_2})+n_2(\overline{C_2})}{4}+\Delta(C_2)+2$, $\operatorname{exc}(F_{D,1})\leq\frac{n(\overline{D_1})+n_2(\overline{D_1})}{4}+\Delta(D_1)+2$, $\operatorname{exc}(F_{D,2})\leq\frac{n(\overline{D_2})+n_2(\overline{D_2})}{4}+\widehat{\Delta}(D_2)$, and $\operatorname{exc}(F_\ell)\leq\frac{n(B_\ell+v'v)+n_2(B_\ell+v'v)}{4}+\frac{3}{2}$. As $\operatorname{exc}(F)-2=(\operatorname{exc}(F_2)-2)+(\operatorname{exc}(F_{D,1})-2)+\operatorname{exc}(F_{D,2})+(\operatorname{exc}(F_\ell)-2)$, $n(G)=n(\overline{C_2})+n(\overline{D_1})+n(\overline{D_2})+n(B_\ell+v'v)+3$, and $n_2(G)=n_2(\overline{C_2})+n_2(\overline{D_1})+n_2(\overline{D_2})+n_2(B_\ell+v'v)-1$, we have

$$\begin{aligned}\operatorname{exc}(F)-2 &= (\operatorname{exc}(F_2)-2)+(\operatorname{exc}(F_{D,1})-2)+\operatorname{exc}(F_{D,2})+(\operatorname{exc}(F_\ell)-2)\\ &\leq \frac{n(G)+n_2(G)}{4}-\frac{1}{2}+\Delta(C_2)+\Delta(D_1)+\widehat{\Delta}(D_2)+\Delta(B_\ell+v'v,v'v)\end{aligned}$$

$$\leq \frac{n(G) + n_2(G)}{4} - \frac{3}{2},$$

since $\Delta(C_2), \Delta(B_\ell + v'v, v'v) \leq -\frac{1}{2}$ and $\Delta(D_1) + \widehat{\Delta}(D_2) \leq 0$. Thus $\text{exc}(F)$ satisfies our desired bound.

For the time analysis, as we only call `Algo` on inputs whose sizes sum to less than n, line 9 takes at most $f_{\texttt{Algo}}(n)$ time. Furthermore, by Proposition 6.2, the remaining lines require $O(n)$ time. Thus, $\texttt{EC}(G, e, -\frac{3}{2})$ takes $f_{\texttt{Algo}}(n-1) + O(n)$ time. □

To summarize, we have the following.

Corollary 6.8. `Algo` *is correct and takes* $O(n^2)$ *time.*

Proof. We show inductively that we can take $f_{\texttt{Algo}}(n) = O(n^2)$. First note that lines 1-5 take constant time. Line 6 takes linear time to check, and if executed, lines 7-12 take $O(n) + \sum_{i=1}^{k} f_{\texttt{Algo}}(n(\overline{B_i})) \leq O(n) + \sum_{i=1}^{k} O(n(\overline{B_i})^2) = O(n^2)$.

Line 13 take linear time by Proposition 6.2, and in lines 14-17, we execute exactly one of $\texttt{EC}(G, e, \Delta)$ and $\widehat{\texttt{EC}}(G, e)$, which takes $f_{\texttt{Algo}}(n-1) + O(n)$ time by Propositions 6.3, 6.4, 6.6, and 6.7. It follows that we can take $f_{\texttt{Algo}}(n) = O(n^2)$. □

Corollary 6.9. *Given a simple 2-connected subcubic graph G, we can find an even cover F of G with* $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$ *in quadratic time.*

Proof. Pick an arbitrary edge $e \in E(G)$. Run $\texttt{Algo}(G, e, \text{true})$ and $\texttt{Algo}(G, e, \text{false})$. One of the returned even covers will have excess at most $\frac{n(G)+n_2(G)}{4} + 1$. □

Let us now complete the proof of Theorem 1.1, restated here for the reader's convenience.

Theorem 1.1. *Let G be a 2-connected simple subcubic graph. Then* $\text{tsp}(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$. *Moreover, a TSP walk of length at most* $\frac{5n(G)+n_2(G)}{4} - 1$ *can be found in* $O(n(G)^2)$ *time.*

Proof. By Corollary 6.9, we can find an even cover F of G with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$ in quadratic time. Then by Proposition 1.2, we can convert F to a TSP walk of length $\text{exc}(F) - 2 + n(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$ in linear time. □

If the input graph G is cubic (i.e. $n_2(G) = 0$), then Theorem 1.1 finds a TSP walk of length at most $\frac{5n(G)}{4} - 1$ in quadratic time. Since every TSP walk trivially has length at least $n(G)$, this gives a $\frac{5}{4}$-approximation algorithm for TSP walks in 2-connected cubic graphs. For general subcubic graphs, Theorem 1.1 finds a TSP walk of length at most $\frac{3}{2}n(G)$ which trivially yields a $\frac{3}{2}$-approximation algorithm. The bound gets better for subcubic graphs with fewer vertices of degree 2; for example, if $n_2(G) \leq \frac{1}{3}n(G)$, then Theorem 1.1 yields a TSP walk of length at most $\frac{4}{3}n(G)$. We suspect that refining the ideas developed in this paper could lead to another $\frac{4}{3}$-approximation algorithm for subcubic graphs, matching the current best ratio by Mömke and Svensson [17], and possibly beyond.

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References

[1] N. Aggarwal, N. Garg, S. Gupta, A 4/3-approximation for tsp on cubic 3-edge-connected graphs, arXiv preprint arXiv:1101.5586, 2011.
[2] G. Benoit, S. Boyd, Finding the exact integrality gap for small traveling salesman problems, Math. Oper. Res. 33 (2008) 921–931.
[3] S. Boyd, R. Sitters, S. van der Ster, L. Stougie, The traveling salesman problem on cubic and subcubic graphs, Math. Program. 144 (2014) 227–245.
[4] B. Candráková, R. Lukot'ka, Cubic tsp-a 1.3-approximation, arXiv preprint arXiv:1506.06369, 2015.
[5] N. Christofides, Worst-case analysis of a new heuristic for the travelling salesman problem, Tech. Rep., Carnegie-Mellon Univ., Pittsburgh, PA, 1976, Management Sciences Research Group.
[6] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, Introduction to Algorithms, MIT Press, 2009.
[7] J. Correa, O. Larré, J.A. Soto, Tsp tours in cubic graphs: beyond 4/3, SIAM J. Discrete Math. 29 (2015) 915–939.
[8] Z. Dvořák, D. Král, B. Mohar, Graphic tsp in cubic graphs, in: 34th Symposium on Theoretical Aspects of Computer Science (STACS 2017), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.
[9] D. Gamarnik, M. Lewenstein, M. Sviridenko, An improved upper bound for the tsp in cubic 3-edge-connected graphs, Oper. Res. Lett. 33 (2005) 467–474.
[10] S.O. Gharan, A. Saberi, M. Singh, A randomized rounding approach to the traveling salesman problem, in: 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, IEEE, 2011, pp. 550–559.
[11] M.X. Goemans, Worst-case comparison of valid inequalities for the tsp, Math. Program. 69 (1995) 335–349.
[12] A.R. Karlin, N. Klein, S.O. Gharan, A (slightly) improved approximation algorithm for metric tsp, in: Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, 2021, pp. 32–45.
[13] R. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher (Eds.), Complexity of Computer Computations, Plenum, New York, 1972, pp. 85–103.
[14] M. Karpinski, M. Lampis, R. Schmied, New inapproximability bounds for tsp, J. Comput. Syst. Sci. 81 (2015) 1665–1677.
[15] M. Karpinski, R. Schmied, Approximation hardness of graphic tsp on cubic graphs, RAIRO Oper. Res. 49 (2015) 651–668.
[16] M. Lampis, Improved inapproximability for tsp, in: Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, Springer, 2012, pp. 243–253.
[17] T. Mömke, O. Svensson, Approximating graphic tsp by matchings, in: 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, IEEE, 2011, pp. 560–569.
[18] M. Mucha, $\frac{13}{9}$-approximation for graphic tsp, Theory Comput. Syst. 55 (2014) 640–657.
[19] A. Sebő, J. Vygen, Shorter tours by nicer ears: 7/5-approximation for the graph-tsp, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs, Combinatorica 34 (2014) 597–629.

[20] A.I. Serdyukov, O nekotorykh ekstremal'nykh obkhodakh v grafakh, Uprav. Sist. 17 (1978) 76–79.
[21] R. van Bevern, V.A. Slugina, A historical note on the 3/2-approximation algorithm for the metric traveling salesman problem, Hist. Math. 53 (2020) 118–127.
[22] A. van Zuylen, Improved approximations for cubic bipartite and cubic tsp, in: International Conference on Integer Programming and Combinatorial Optimization, Springer, 2016, pp. 250–261.