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On the rainbow matching conjecture for 3-uniform hypergraphs

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Abstract Aharoni and Howard and, independently, Huang et al. (2012) proposed the following rainbow version of the Erdős matching conjecture: For positive integers n, k and m with $n \geqslant km$, if each of the families $F_1, \ldots, F_m \subseteq \binom{[n]}{k}$ has size more than $\max\{\binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k}\}$, then there exist pairwise disjoint subsets $e_1, \ldots, e_m$ such that $e_i \in F_i$ for all $i \in [m]$. We prove that there exists an absolute constant n_0 such that this rainbow version holds for $k = 3$ and $n \geqslant n_0$. We convert this rainbow matching problem to a matching problem on a special hypergraph H. We then combine several existing techniques on matchings in uniform hypergraphs: Find an absorbing matching M in H; use a randomization process of Alon et al. (2012) to find an almost regular subgraph of $H - V(M)$; find an almost perfect matching in $H - V(M)$. To complete the process, we also need to prove a new result on matchings in 3-uniform hypergraphs, which can be viewed as a stability version of a result of Łuczak and Mieczkowska (2014) and might be of independent interest.

Keywords rainbow matching conjecture, Erdős matching conjecture, stability

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1 Introduction

For a positive integer k and a set V, let $[k] := \{1, \ldots, k\}$ and

$$\binom{V}{k} := \{A \subseteq V : |A| = k\}.$$

A hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq 2^{V(H)}$. A hypergraph H is *k-uniform* if all its edges have size k and we call it a *k-graph* for short. Throughout this paper, we often identify $E(H)$ with H when there is no confusion and, in particular, denote by $|H|$ the number of edges in H. Given a set T of edges in H, we use $V(T)$ to define $\bigcup_{e\in T} e$. Given a vertex subset $S \subseteq V(H)$ in H, we use $H[S]$ to denote the subgraph of H induced by S, and let $H - S = H[V(H) \setminus S]$.

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A *matching* in a hypergraph H is a set of pairwise disjoint edges in H. We use $\nu(H)$ to define the maximum size of a matching in H. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of hypergraphs on the same vertex set. A set of m pairwise disjoint edges is called a *rainbow matching* for $\mathcal{F}$ if each edge is from a different F_i. If such a matching exists, then we also say that $\mathcal{F}$ *admits a rainbow matching*.

A classical problem in extremal set theory asks for the maximum number of edges in n-vertex k-graphs H with $\nu(H) < m$. Let n, k and m be positive integers with $n \geqslant km$. The k-graphs

$$S(n,m,k) := \binom{[n]}{k} \Big\backslash \binom{[n]\backslash[m-1]}{k}$$

and $D(n,m,k) := \binom{[km-1]}{k}$ on the same vertex set $[n]$ do not have matchings of size m. Erdős [6] conjectured in 1965 that among all the k-graphs with no matching of size m, $S(n,m,k)$ or $D(n,m,k)$ has the maximum number of edges: Any n-vertex k-graph H with $\nu(H) < m$ contains at most

$$f(n,m,k) := \max\left\{\binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k}\right\}$$

edges. This is often referred to as the *Erdős matching conjecture* in the literature, and there has been extensive research on this conjecture (see, for example, [3, 5, 8–11, 13, 22]). In particular, the special case for $k = 3$ was settled for large n by Łuczak and Mieczkowska [22] and completely resolved by Frankl [9].

The following analogous conjecture, known as the *rainbow matching conjecture*, was made by Aharoni and Howard [1] and, independently, by Huang et al. [15]. For related topics on rainbow type problems, we refer the interested readers to [16, 18, 20, 23].

Conjecture 1.1 (See [1,15])**.** Let n, k and m be positive integers with $n \geqslant km$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k-graphs on the same vertex set $[n]$ such that $|F_i| > f(n,m,k)$ for all $i \in [m]$. Then $\mathcal{F}$ admits a rainbow matching.

The case $k = 2$ of this conjecture is in fact a direct consequence of an earlier result of Akiyama and Frankl [2] (which was restated in [7]). The following was obtained by Huang et al. [15].

Theorem 1.2 (See [15, Theorem 3.3])**.** *Conjecture* 1.1 *holds when* $n > 3k^2m$.

Keller and Lifshitz [17] proved that Conjecture 1.1 holds when $n \geqslant f(m)k$ for some large constant $f(m)$ which only depends on m, and this was further improved to $n = \Omega(m \log m)k$ by Frankl and Kupavskii [12]. Both proofs use the junta method. Very recently, Lu et al. [19] showed that Conjecture 1.1 holds when $n \geqslant 2km$ and n is sufficiently large.

The following is our main result, which proves Conjecture 1.1 for $k = 3$ and sufficiently large n.

Theorem 1.3. *There exists an absolute constant* n_0 *such that the following holds for all* $n \geqslant n_0$*. For any positive integers* n *and* m *with* $n \geqslant 3m$*, let* $\mathcal{F} = \{F_1, \dots, F_m\}$ *be a family of* 3*-graphs on the same vertex set* $[n]$ *such that* $|F_i| > f(n,m,3)$ *for all* $i \in [m]$*. Then* $\mathcal{F}$ *admits a rainbow matching.*

Our proof of Theorem 1.3 uses some new ideas and combines different techniques from Alon et al. [3], Łuczak and Mieczkowska [22], and Lu et al. [21]. (For a high level description of our proof, we refer the readers to Section 2 and/or Section 7.) In the process, we prove a stability result on 3-graphs (see Lemma 4.2) that plays a crucial role in our proof and might be of independent interest: If the number of edges in an n-vertex 3-graph H with $\nu(H) < m$ is close to $f(n,m,3)$, then H must be close to $S(n,m,3)$ or $D(n,m,3)$.

The rest of the paper is organized as follows. In Section 2, we introduce additional notation, and state and/or prove a few lemmas for later use. In Section 3, we deal with the families $\mathcal{F}$ in which most 3-graphs are close to the same 3-graph, i.e., $S(n,m,3)$ or $D(n,m,3)$. To deal with the remaining families, we need the above mentioned stability result for matchings in 3-graphs, which is done in Section 4. In Section 5, we show that there exists an absolute constant $c > 0$ such that Theorem 1.3 holds for $m > (1-c)n/3$. The proof of Theorem 1.3 for $m \leqslant (1-c)n/3$ is completed in Section 6. Finally, we complete the proof of Theorem 1.3 in Section 7.

2 Previous results and lemmas

In this section, we define saturated families and stable hypergraphs, and state several lemmas that we will use frequently. We begin with some notation. Suppose that H is a hypergraph and U and T are subsets of $V(H)$. Let

$$N_H(T) := \{A : A \subseteq V(H) \setminus T \text{ and } A \cup T \in E(H)\}$$

be the *neighborhood* of T in H, and let $d_H(T) := |N_H(T)|$. We write $d_H(v)$ for $d_H(\{v\})$. Let

$$\Delta(H) := \max_{v \in V(H)} d_H(v) \quad \text{and} \quad \Delta_2(H) := \max_{T \in \binom{V(H)}{2}} d_H(T).$$

In the case $T \subseteq U$, we often identify $d_{H[U]}(T)$ with $d_U(T)$ when there is no confusion.

It will be helpful to consider "maximal" counterexamples to Conjecture 1.1. Let n, k and m be positive integers with $n \geqslant km$ and let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the same vertex set $[n]$. We say that $\mathcal{F}$ is **saturated**, if $\mathcal{F}$ does not admit a rainbow matching, but for every $F \in \mathcal{F}$ and $e \notin F$, the new family $\mathcal{F}(e, F) := (\mathcal{F} \backslash \{F\}) \cup \{F \cup \{e\}\}$ admits a rainbow matching. The following lemma says that the vertex degrees of every k-graph in a saturated family are typically small.

Lemma 2.1. *Let n, k and m be positive integers with $n \geqslant km$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a saturated family of k-graphs on the same vertex set $[n]$. Then for each $v \in [n]$ and each $i \in [m]$,*

$$d_{F_i}(v) \leqslant \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1} \quad \text{or} \quad d_{F_i}(v) = \binom{n-1}{k-1}.$$

Proof. Suppose $d_{F_i}(v) < \binom{n-1}{k-1}$, where $v \in [n]$ and $i \in [m]$. Then there exists $e \in \binom{[n]}{k} \setminus F_i$ such that $v \in e$. Since $\mathcal{F}$ is saturated, the family $\mathcal{F}(e, F_i)$ admits a rainbow matching, say $M \cup \{e\}$, with M being a rainbow matching for the family $\mathcal{F} \setminus \{F_i\}$.

If

$$d_{F_i}(v) > \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1} = \left| \binom{[n] \backslash \{v\}}{k-1} \Big\backslash \binom{[n] \backslash (\{v\} \cup V(M))}{k-1} \right|,$$

then there exists an edge $f \in F_i$ such that $v \in f$ and $f \cap V(M) = \emptyset$. Now $M \cup \{f\}$ is a rainbow matching for $\mathcal{F}$, which leads to a contradiction. So $d_{F_i}(v) \leqslant \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}$. □

We will remove vertices of degree $\binom{n-1}{k-1}$ and use Lemma 2.1 to produce the saturated family $\mathcal{F} = \{F_1, \ldots, F_m\}$ of k-graphs such that for each $v \in V(F_i)$ and each $i \in [m]$,

$$d_{F_i}(v) \leqslant \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}.$$

Next, we define stable hypergraphs. Let n and k be positive integers with $n \geqslant k$. Let $e = \{a_1, \ldots, a_k\}$ and $f = \{b_1, \ldots, b_k\}$ be members of $\binom{[n]}{k}$ with $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. We write $e \leqslant f$ if $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant k$, and $e < f$ if $e \leqslant f$ and $e \neq f$.

A k-graph $F \subseteq \binom{[n]}{k}$ is said to be **stable** if $e < f \in F$ implies $e \in F$. A family $\mathcal{F}$ of k-graphs on the same vertex set $[n]$ is **stable** if each k-graph in $\mathcal{F}$ is stable.

The following result of Huang et al. [15] will be used frequently, which enables us to work with stable families when proving Conjecture 1.1.

Lemma 2.2 (See [15, Lemma 2.1]). *Let n, k and m be positive integers with $n \geqslant km$. If the family $\{F_1, \ldots, F_m\}$ of k-graphs with $V(F_i) = [n]$ for all $i \in [m]$ does not admit a rainbow matching, then there exists a stable family $\{F'_1, \ldots, F'_m\}$ of k-graphs with $|F_i| = |F'_i|$ and $V(F'_i) = [n]$ for all $i \in [m]$ which still preserves this property.*

Corollary 2.3. *Let n, k and m be positive integers with $n \geqslant km$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the vertex set $[n]$ that does not admit a rainbow matching. Then there exists a family $\mathcal{F}' = \{F'_1, \ldots, F'_m\}$ of k-graphs on the same vertex set $[n]$ such that $\mathcal{F}'$ is both stable and saturated and $|F'_i| \geqslant |F_i|$ for $i \in [m]$.*

Proof. Let $\mathcal{F}^* = \{F_1^*, \ldots, F_m^*\}$ be a family of k-graphs on the same vertex set $[n]$ such that $\mathcal{F}^*$ admits no rainbow matching, $|F_i^*| \geqslant |F_i|$ for $i \in [m]$, and subject to these, $\sum_{i\in[m]} |F_i^*|$ is maximum.

Then $\mathcal{F}^*$ is saturated. Now applying Lemma 2.2 to $\mathcal{F}^*$, we obtain a stable family $\mathcal{F}' = \{F_1', \ldots, F_m'\}$ of k-graphs on the vertex set $[n]$ such that $\mathcal{F}'$ admits no rainbow matching, and $|F_i'| = |F_i^*|$ for $i \in [m]$. By the choice of $\mathcal{F}^*$, we see that $\mathcal{F}'$ is also saturated. □

We now describe an operation that converts a rainbow matching problem to a matching problem on a single hypergraph. Let n, k, m and r be non-negative integers with $r = \lfloor n/k \rfloor - m$ and $m \geqslant 1$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the same vertex set $[n]$, and let $\mathcal{V} = \{v_1, \ldots, v_m\}$ and $\mathcal{U} = \{u_1, \ldots, u_r\}$ be two disjoint sets such that $(\mathcal{V} \cup \mathcal{U}) \cap [n] = \emptyset$. We use $H(\mathcal{F})$ to define the $(k+1)$-graph with the vertex set $[n] \cup \mathcal{V}$ and the edge set $\bigcup_{i=1}^m \{e \cup \{v_i\} : e \in F_i\}$, and use $H^*(\mathcal{F})$ to define the $(k+1)$-graph with the vertex set $[n] \cup \mathcal{V} \cup \mathcal{U}$ and the edge set

$$E(H(\mathcal{F})) \cup \bigcup_{i=1}^{r} \left\{ e \cup \{u_i\} : e \in \binom{[n]}{k} \right\}.$$

If $F_1 = \cdots = F_m = S(n,m,k)$ (resp. $F_1 = \cdots = F_m = D(n,m,k)$), then we write $H(\mathcal{F})$ as $H_S(n,m,k)$ (resp. $H_D(n,m,k)$).

It is easy to see that $\mathcal{F}$ admits a rainbow matching if and only if $H(\mathcal{F})$ has a matching of size m, which is also if and only if $H^*(\mathcal{F})$ has a matching of size $m+r$. This allows us to access existing approaches and tools invented for matching problems. For example, we take the approach by considering whether or not the hypergraphs $H(\mathcal{F})$ in question are close to the extremal configurations $H_S(n,m,k)$ and $H_D(n,m,k)$. We will see in Section 3 that if $H(\mathcal{F})$ is close to $H_D(n,m,k)$ and $\mathcal{F}$ is stable, then $\mathcal{F}$ admits a rainbow matching.

Here, we give an easy lemma concerning a case where $H(\mathcal{F})$ is not close to $H_S(n,m,k)$, which will be used along with Lemma 2.1. Let H_1 and H_2 be two k-graphs on the same vertex set V and let ϵ be some positive real number; we say that H_2 is **ϵ-close** to H_1 if $|E(H_1) \setminus E(H_2)| \leqslant \epsilon |V|^k$.

Lemma 2.4. *For any given integer $k \geqslant 3$, let ϵ and c be real numbers such that $0 < \epsilon \ll c \ll 1$[1]. Let n and m be integers such that $n/3k^2 \leqslant m \leqslant (1-c)n/k$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the vertex set $[n]$. If for every $i \in [m]$ and $v \in [n]$,*

$$d_{F_i}(v) \leqslant \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

then $H(\mathcal{F})$ is not ϵ-close to $H_S(n,m,k)$.

Proof. We note that $S(n,m,k)$ has $m-1$ vertices of degree $\binom{n-1}{k-1}$. Since for every $i \in [m]$ and $v \in [n]$,

$$d_{F_i}(v) \leqslant \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

we have

$$|E(H_S(n,m,k)) \setminus E(H(\mathcal{F}))| \geqslant m \cdot (m-1) \cdot \binom{n-k(m-1)-1}{k-1} \cdot \frac{1}{k} > \frac{n^2}{10k^5} \binom{cn}{k-1} > \epsilon (n+m)^{k+1},$$

where the second inequality is due to $n/3k^2 \leqslant m \leqslant (1-c)n/k$ and the third inequality follows from $\epsilon \ll c$. This shows that $H(\mathcal{F})$ is not ϵ-close to $H_S(n,m,k)$. □

To deal with the case where $H(\mathcal{F})$ is not close to $H_D(n,m,3)$, we first find a small matching M in $H^*(\mathcal{F})$ such that M can "absorb" small vertex sets and $H^*(\mathcal{F}) - V(M)$ has an almost perfect matching. When $\mathcal{F}$ is stable, the matching M can be found very easily by the following lemma and its proof.

[1] Here and throughout the rest of the paper, the notation $a \ll b$ means that a is sufficiently small compared with b which need satisfy finitely many inequalities in the proof.

Lemma 2.5. *Let k be a fixed positive integer and let $0 < \gamma' \ll \gamma \ll c \ll 1$ be real numbers. Let n and m be positive integers with $n/3k^2 \leqslant m \leqslant (1-c)n/k$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a stable family of k-graphs such that $V(F_i) = [n]$ and $|F_i| > f(n,m,k)$ for all $i \in [m]$. Then for sufficiently large n, $H^*(\mathcal{F})$ has a matching M with $|M| \leqslant \gamma n$ such that for any set $S \subseteq V(H^*(\mathcal{F})) \setminus V(M)$ with $|S| \leqslant \gamma' n$ and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$, $H^*(\mathcal{F})[V(M) \cup S]$ has a perfect matching.*

Proof. Recall that $\mathcal{V} = \{v_1, \ldots, v_m\}$ and $\mathcal{U} = \{u_1, \ldots, u_r\}$, where $r = \lfloor n/k \rfloor - m$. Fix an integer t satisfying $\gamma' n < t < \gamma n$. Then $t < \gamma n \leqslant \lfloor cn/k \rfloor \leqslant \lfloor n/k \rfloor - m = r$. Let $s = \lceil n/3k^2 \rceil - 1$.

By Theorem 1.2 (viewing all the k-graphs as the same k-graph), since $|F_i| > f(n,m,k) \geqslant f(n,s,k)$ for all $i \in [m]$, every F_i has a matching of size s. Since F_i is stable, $F_i[[s]]$ is a complete k-graph. Hence,

(i) for any $i_1, i_2, \ldots, i_k \leqslant kt \leqslant k\gamma n < s$ and $j \in [m]$, we have $\{v_j, i_1, i_2, \ldots, i_k\} \in H^*(\mathcal{F})$.

From the definition of $H^*(\mathcal{F})$, we have

(ii) for any $i_1, i_2, \ldots, i_k \in [n]$ and $j \in [r]$, $\{u_j, i_1, i_2, \ldots, i_k\} \in H^*(\mathcal{F})$.

Since $t < r$, we may choose a matching M of size t in $H^*(\mathcal{F})$ with $V(M) = \{u_1, \ldots, u_t\} \cup [kt]$. Note that $|M| = t \leqslant \gamma n$. We claim that this M is the desired matching. To see this, consider any subset S with $S \cap V(M) = \emptyset$, $|S| \leqslant \gamma' n$ and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$. Let $t' = |S \cap (\mathcal{V} \cup \mathcal{U})|$. So $t' \leqslant \gamma' n < t$. Then by (i) and (ii), there is a perfect matching M_1 in $H^*(\mathcal{F})[S \cap (\mathcal{V} \cup \mathcal{U}) \cup [kt']]$. By (ii), there exists a perfect matching M_2 in $H^*(\mathcal{F})[(V(M) \cup S) \setminus V(M_1)]$. So $M_1 \cup M_2$ is a perfect matching in $H^*(\mathcal{F})[V(M) \cup S]$. □

For the "absorbing" matching M in $H^*(\mathcal{F})$ in Lemma 2.5, we also want $H^*(\mathcal{F}) - V(M)$ to have an almost perfect matching. For this we need to use the following result of Frankl and Rödl [14].

Theorem 2.6 (See [14])**.** *For every integer $k \geqslant 2$ and any real number $\sigma > 0$, there exist $\tau = \tau(k, \sigma)$ and $d_0 = d_0(k, \sigma)$ such that for every integer $n \geqslant D \geqslant d_0$ the following holds: Every n-vertex k-graph H with*

$$(1-\tau)D < \Delta_1(H) < (1+\tau)D$$

and $\Delta_2(H) < \tau D$ contains a matching covering all but at most σn vertices.

In order to obtain a k-graph H satisfying Theorem 2.6, we use the approach from [3] by conducting two rounds of randomization on $H^*(\mathcal{F}) - V(M)$. We summarize part of the proof in [3] (more precisely, their proof of Claim 4.1) as a lemma. A *fractional matching* in a k-graph H is a function $w : E(H) \to [0,1]$ such that for any $v \in V(H)$, $\sum_{\{e \in E(H): v \in e\}} w(e) \leqslant 1$. A fractional matching is called *perfect* if $\sum_{e \in E(H)} w(e) = |V(H)|/k$.

Lemma 2.7 (See [3], retained from their proof of Claim 4.1)**.** *Let $k \geqslant 3$ and H be a k-graph on at most $2n$ vertices. Suppose that there are subsets $R^i \subseteq V(H)$ for $i = 1, \ldots, n^{1.1}$ satisfying the following:*

(a) *every vertex $v \in V(H)$ satisfies that $|\{i : v \in R^i\}| = (1+o(1))n^{0.2}$,*

(b) *every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i,*

(c) *every edge $e \in H$ is contained in at most one set R^i, and*

(d) *for every $i = 1, \ldots, n^{1.1}$, R^i has a perfect fractional matching w^i.*

Then H has a spanning subgraph H' such that

$$d_{H'}(v) = (1+o(1))n^{0.2}$$

for all $v \in V(H')$ and $\Delta_2(H') \leqslant n^{0.1}$.

We will also need to control the independence number of random subgraphs of $H^*(\mathcal{F}) - V(M)$. The intuition is that when $H(\mathcal{F})$ is not close to $H_D(n,m,k)$ or $H_S(n,m,k)$, $H^*(\mathcal{F}) - V(M)$ does not have very large independence number. The following lemma in [21] was proved by Lu et al. using the container method.

Lemma 2.8 (See [21, Lemma 5.4])**.** *Let d, ϵ' and α be positive real numbers and let k and n be positive integers. Let H be an n-vertex k-graph such that $e(H) \geqslant dn^k$ and $e(H[S]) \geqslant \epsilon' e(H)$ for all $S \subseteq V(H)$ with $|S| > \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then for any positive real number $\gamma \ll \alpha$, the size of maximum independent sets in $H[R]$ is at most $(\alpha + \gamma)n^{0.1}$ with probability at least $1 - (n^{O(1)} \mathrm{e}^{-\Omega(n^{0.1})})$.*

We need an inequality on the function $f(n,m,k)$ proved by Frankl [9].

Lemma 2.9 (See [9, Proposition 5.1]). *Let n, m and k be positive integers with $n \geqslant km-1$. Then*

$$f(n,m,k) \geqslant f(n-1,m-1,k) + \binom{n-1}{k-1}.$$

We conclude this section with the well-known Chernoff inequality.

Lemma 2.10 (Chernoff inequality [4]). *Suppose that $X_1,\ldots,X_n$ are independent random variables taking values in $\{0,1\}$. Let $X=\sum_{i=1}^n X_i$ and $\mu=\mathbb{E}(X)$. Then for any $0<\delta\leqslant 1$,*

$$\mathbb{P}[X \geqslant (1+\delta)u] \leqslant \mathrm{e}^{-\delta^2 u/3} \quad \textit{and} \quad \mathbb{P}[X \leqslant (1-\delta)u] \leqslant \mathrm{e}^{-\delta^2 u/3}. \tag{2.1}$$

In particular, if $X\sim \mathrm{Bin}(n,p)$ and $\lambda<\frac{3}{2}np$, then

$$\mathbb{P}([|X-np|] \geqslant \lambda) \leqslant \mathrm{e}^{-\Omega(\lambda^2/np)}. \tag{2.2}$$

3 Extremal configuration $H_D(n,m,3)$

From Lemmas 2.1 and 2.4, we see that if $\mathcal{F}$ is a saturated family of k-graphs on the vertex set $[n]$ and $H(\mathcal{F})$ is close to the extremal configuration $H_S(n,m,k)$, then there exist $F\in\mathcal{F}$ and $v\in[n]$ such that $d_F(v)=\binom{n-1}{k-1}$. Such vertices v can be removed from all the k-graphs in $\mathcal{F}\setminus\{F\}$ to obtain a smaller family $\mathcal{F}'$, so that if $\mathcal{F}'$ admits a rainbow matching, then $\mathcal{F}$ admits a rainbow matching.

In this section, we consider the case where $H(\mathcal{F})$ is close to $H_D(n,m,3)$ and $\mathcal{F}$ is stable.

Lemma 3.1. *Let ϵ and c be real numbers such that $0<\epsilon\ll c\ll 1$. Let n and m be positive integers such that $n/27\leqslant m\leqslant (1-c)n/3$. Let $\mathcal{F}=\{F_1,\ldots,F_m\}$ be a stable family of 3-graphs on the vertex set $[n]$ such that $|F_i|>f(n,m,3)$ for all $i\in[m]$. If $H(\mathcal{F})$ is ϵ-close to $H_D(n,m,3)$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. Let $b=6\epsilon^{1/6}n$. If F_i is $\sqrt{\epsilon}$-close to $D(n,m,3)$, then F_i contains a complete subgraph of size $3m-b$; for, otherwise, as F_i is stable, we have

$$|E(D(n,m,3))\setminus E(F_i)| \geqslant \binom{b}{3} > \sqrt{\epsilon}n^3,$$

which leads to a contradiction.

We claim that for any $i\in[m]$ and $j\in\{0,\ldots,b\}$, $\{2j+1,2j+2,3m-j\}\in F_i$. To prove this claim we fix $i\in[m]$. Suppose for a contradiction that there exists an integer t with $0\leqslant t\leqslant b$ such that $\{2t+1,2t+2,3m-t\}\notin F_i$. Since $|F_i|>\binom{3m-1}{3}$ and F_i is stable, we have $\{1,2,3m\}\in F_i$. So $t\geqslant 1$. We now count the edges in F_i: Let q_1 be the number of edges of F_i in $[3m-1]$, and q_2 be the number of edges of F_i not contained in $[3m-1]$. Since F_i is stable and $\{2t+1,2t+2,3m-t\}\notin F_i$, we see that $\{a,b,c\}\notin F_i$ when $2t+2\leqslant a<b<3m-t\leqslant c\leqslant 3m-1$. So

$$q_1 \leqslant \binom{3m-1}{3} - t\binom{3m-3t-3}{2}.$$

Since $\{2t+1,2t+2,3m-t\}\notin F_i$, we have that for any $e\in F_i$ with $e\cap([n]\setminus[3m-1])\neq\emptyset$, $e\cap[2t]\neq\emptyset$. This shows $q_2\leqslant 2t(n-3m+1)n$. First suppose that $n\leqslant 7m/2$. Then we have

$$\begin{aligned}|F_i| &\leqslant \binom{3m-1}{3} - t\binom{3m-3t-3}{2} + 2tn(n-3m+1)\\ &\leqslant \binom{3m-1}{3} - t\left[\binom{3m-3t-3}{2} - 7m(m/2+1)\right] < \binom{3m-1}{3},\end{aligned}$$

where the second inequality holds since $n \leqslant 7m/2$, and the last inequality holds since $t \leqslant b = 6\epsilon^{1/6} n \ll m$, which leads to a contradiction. So we may assume $n > 7m/2$. Let $m = \alpha n$. Then $1/27 \leqslant \alpha < 2/7$. We assert that

$$\binom{n}{3} - \binom{n-m+1}{3} > \binom{3m-1}{3} + 2tn^2.$$

To see this, let $f(x) = 1 - (1-x)^3 - (3x)^3$, and then

$$\frac{6}{n^3}\left(\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3}\right) = f(\alpha) + o(1).$$

Since $f'(x) = 3(1 - 2x - 26x^2)$ is decreasing in $[1/27, 2/7]$ with $f'(1/27) > 0$ and $f'(2/7) < 0$, we have

$$f(\alpha) \geqslant \min\{f(1/27), f(2/7)\} = f(2/7) = \frac{2}{343}$$

for $1/27 \leqslant \alpha < 2/7$. This shows that

$$\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} = \frac{f(\alpha)}{6} n^3 + o(n^3) \geqslant 2tn^2,$$

as asserted. Then it follows that

$$|F_i| \leqslant \binom{3m-1}{3} - t\binom{3m-3t-3}{2} + 2tn(n-3m+1) < \binom{3m-1}{3} + 2tn^2 < \binom{n}{3} - \binom{n-m+1}{3},$$

which leads to a contradiction as $|F_i| > f(n,m,3)$. This finishes the proof of the claim.

Recall $\mathcal{V} = \{v_1, \dots, v_m\}$ from the definition of $H(\mathcal{F})$. By the above claim,

$$M_1 := \{\{v_i, 2i-1, 2i, 3m-i+1\} : i \in [b]\}$$

is a matching in $H(\mathcal{F})$. Without loss of generality, let $F_1, \dots, F_a$ be all the k-graphs in $\mathcal{F}$ which are not $\sqrt{\epsilon}$-close to $D(n,m,3)$. Since $H(\mathcal{F})$ is ϵ-close to $H_D(n,m,3)$, we have $a \leqslant \sqrt{\epsilon} n < b$. Then for any $j \in [m] \backslash [b]$, since F_j is $\sqrt{\epsilon}$-close to $D(n,m,3)$, F_j contains a complete subgraph with size at least $3m - b$. Hence we have $\{2j-1, 2j, 3m-j+1\} \in F_j$. So $M_2 := \{\{v_j, 2j-1, 2j, 3m-j+1\} : b < j \leqslant m\}$ is a matching in $H(\mathcal{F})$ which is disjoint from M_1. Then $M_1 \cup M_2$ forms a matching of size m in $H(\mathcal{F})$. So $\mathcal{F}$ admits a rainbow matching, completing the proof of Lemma 3.1. □

4 A stability lemma

In this section, we prove a result for stable 3-graphs, which may be viewed as a stability version of the following result of Łuczak and Mieczkowska proved in [22].

Theorem 4.1 (See [22]). *There exists a positive integer n_1 such that for integers m and n with $n \geqslant n_1$ and $1 \leqslant m \leqslant n/3$, if H is an n-vertex 3-graph with $e(H) > f(n,m,3)$, then $\nu(H) \geqslant m$.*

Building on the proof in [22], we prove the following lemma.

Lemma 4.2. *For any real number $\epsilon > 0$, there exists a positive integer $n_1(\epsilon)$ such that the following holds. Let m and n be integers with $n \geqslant n_1(\epsilon)$ and $1 \leqslant m \leqslant n/3$, and let H be a stable 3-graph on the vertex set $[n]$. If $e(H) > f(n,m,3) - \epsilon^4 n^3$ and $\nu(H) < m$, then H is ϵ-close to $S(n,m,3)$ or $D(n,m,3)$.*

Proof. Suppose that $e(H) > f(n,m,3) - \epsilon^4 n^3$ and $s := \nu(H) < m$. Let $M = \{(i_\ell, j_\ell, k_\ell) : \ell \in [s]\}$ be a largest matching in H and the partition $V(M) = I \cup J \cup K$ such that every edge $(i,j,k) \in E(M)$ with $i < j < k$ satisfies $i \in I, j \in J$ and $k \in K$. Since H is stable, we may choose $V(M)$ to be $[3s]$.

Let $V' = [n] \backslash [3s]$. For $x \in [3s]$, let $e(x)$ denote the edge in M containing x. Let

$$F_1 = \left\{\{v\} \in \binom{[3s]}{1} : d_{V'}(v) \geqslant 20n\right\}, \quad F_2 = \left\{\{v,w\} \in \binom{[3s]}{2} : e(v) \neq e(w) \text{ and } d_{V'}(v,w) \geqslant 20\right\}$$

and

$$F_3=\left\{\{u,v,w\}\in\binom{[3s]}{3}: e(u), e(v) \text{ and } e(w) \text{ are pairwise distinct}\right\}.$$

Let $H^*=([3s],F)$ be the hypergraph with the vertex set $[3s]$ and the edge set $F=M\cup F_1\cup F_2\cup F_3$.

Call an edge $e\in H$ *traceable* if $e\cap[3s]\in F$, and *untraceable* otherwise. Since M is a maximum matching in H, V' is independent in H. So the number of untraceable edges of H is bounded from above by

$$\binom{3s}{1}\cdot 20n+\left(\binom{s}{2}\binom{3}{1}\binom{3}{1}\times 19+\binom{s}{1}\binom{3}{2}n\right)+\binom{s}{1}\binom{3}{2}\binom{3s-3}{1}\leqslant 32n^2=o(n^3),$$

where we use $s<m\leqslant n/3$. We point out that those edges (there being $o(n^3)$ of them) will be negligible in the following proof.

Let T be a triple of edges from M. We say that T is *bad* if $V(T)$ contains three pairwise disjoint edges of H^* whose union intersects I in at most 2 vertices, and *good* otherwise. For each $i\in[3]$, let $f_i(T)$ denote the number of edges of F_i contained in $V(T)$. Note that $f_3(T)\leqslant 27$. The following two claims are explicit in [22].

Claim 1. There exist no three pairwise disjoint bad triples (of edges in M). Hence, there exist at most six edges in M such that each bad triple contains one of these edges.

Claim 2. Let T be a good triple.

(i) If $f_3(T)\geqslant 24$, then $f_1(T)=f_2(T)=0$.

(ii) If $f_3(T)=20$, then $f_1(T)\leqslant 1$ and $f_2(T)\leqslant 12$.

(iii) If $f_3(T)\leqslant 19$, then $f_1(T)\leqslant 3$ and $f_2(T)\leqslant 15$. Moreover, the only triples T for which $f_3(T)=19$, $f_2(T)=15$ and $f_1(T)=3$ are those in which each edge of H^* contained in $V(T)$ intersects I.

(iv) If $f_3(T)=21$, then $f_1(T)\leqslant 1$ and $f_2(T)\leqslant 10$.

(v) If $22\leqslant f_3(T)\leqslant 23$, then $f_1(T)=0$ and $f_2(T)\leqslant 7$.

We remove exactly six edges from M such that the resulting matching M' only contains good triples. Since H has at most $18n^2$ edges intersecting $V(M\backslash M')$ and $32n^2$ untraceable edges, we have

$$e(H)\leqslant |F_1|\binom{n-3s}{2}+|F_2|(n-3s)+|F_3|+50n^2.$$

To bound $|F_i|$, let us consider the summation of $f_i(T)$ over all $T\in\binom{M'}{3}$. Since each edge from F_i is counted exactly $\binom{(s-6)-i}{3-i}$ times in this sum, we have

$$|F_i|\binom{(s-6)-i}{3-i}=\sum_{T\in\binom{M'}{3}} f_i(T).$$

Therefore,

$$\begin{aligned}
e(H)&\leqslant \sum_{T\in\binom{M'}{3}}\left(f_1(T)\frac{\binom{n-3s}{2}}{\binom{s-7}{2}}+f_2(T)\frac{n-3s}{s-8}+f_3(T)\right)+50n^2\\
&\leqslant \sum_{T\in\binom{M'}{3}}\left(f_1(T)\frac{(n-3s)^2}{s^2}+f_2(T)\frac{n-3s}{s}+f_3(T)\right)+O(n^2).
\end{aligned}$$

Here, the last inequality is trivial when $s\leqslant 15$, and it holds when $s>15$ because the difference between the above two summations is at most

$$\sum_{T\in\binom{M'}{3}}\left(f_1(T)\frac{15(n-3s)^2}{s(s^2-15s)}+f_2(T)\frac{8(n-3s)}{s(s-8)}\right)\leqslant\binom{s-6}{3}\left(\frac{45(n-3s)^2}{s(s^2-15s)}+\frac{120(n-3s)}{s(s-8)}\right)=O(n^2),$$

where $3s<n$, $f_1(T)\leqslant 3$ and $f_2(T)\leqslant 15$ (from Claim 2).

To further bound $e(H)$, we partition good triples T depending on $f_3(T)$ and $f_1(T)$. Let

$$T_i = \left\{T \in \binom{M'}{3} : f_3(T) = i\right\}$$

for $i \in [27]$ and

$$X = \left\{T \in \binom{M'}{3} : f_1(T) = 3\right\}.$$

Consider any $T \in X$, and then T is a good triple[2)]. Since $f_1(T) = 3$, the three edges of F_1 contained in $V(T)$ are precisely the three vertices in $V(T) \cap I$, and each edge of H^* contained in $V(T)$ intersects I. Since H is stable and $V(M) = [3s]$, by using the definition of F_1, it is not hard to see that $X \subseteq T_{19}$.

Define

$$x_1 = \sum_{i=1}^{18} |T_i| + |T_{19} \setminus X|, \quad x_2 = |T_{20}|, \quad x_3 = |T_{21}|, \quad x_4 = |T_{22}| + |T_{23}|, \quad x_5 = \sum_{i=24}^{26} |T_i|, \quad x = |X|$$

and $y = |T_{27}|$. So

$$\sum_{i=1}^{5} x_i + x + y = \binom{s-6}{3}.$$

From now on, we let $t = (n-3s)/s$. By Claim 2 and the fact $X \subseteq T_{19}$, we can derive from the above upper bound on $e(H)$ that

$$\begin{aligned} e(H) \leqslant{} & (3x + 2x_1 + x_2 + x_3)t^2 + (15x + 15x_1 + 12x_2 + 10x_3 + 7x_4)t \\ & + (19x + 19x_1 + 20x_2 + 21x_3 + 23x_4 + 26x_5 + 27y) + O(n^2). \end{aligned}$$

For convenience, we write

$$f_t(x_1, x_2, x_3, x_4, x_5, x, y) = \sum_{i=1}^{5} \alpha_i(t) \cdot x_i + \beta_1(t) \cdot x + \beta_2(t) \cdot y,$$

where

$$\begin{aligned} &\alpha_1(t) = 2t^2 + 15t + 19, \quad \alpha_2(t) = t^2 + 12t + 20, \quad \alpha_3(t) = t^2 + 10t + 21, \\ &\alpha_4(t) = 7t + 23, \quad \alpha_5(t) = 26, \quad \beta_1(t) = 3t^2 + 15t + 19 \quad \text{and} \quad \beta_2(t) = 27. \end{aligned}$$

Then it follows that

$$e(H) \leqslant f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2).$$

Next, we derive properties of the functions $\alpha_i(t)$ and $\beta_j(t)$.

Claim 3. For any $t \geqslant 0$, $\max\{\beta_1(t), \beta_2(t)\} \geqslant \max\{\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)\} + 0.2$.

Proof. We have $\beta_2(t) = 27$. It is easy to see that for each $i \in [5]$, the functions $\alpha_i(t)$, $\beta_1(t) - \alpha_i(t)$ and $\beta_1(t)$ are increasing for $t \geqslant 0$. Note that $\beta_1(0.5) = 27.25$, $\alpha_2(0.5) = 26.25$, $\alpha_3(0.5) = 26.25$ and $\alpha_4(0.5) = 26.5$, and then $\max\{\beta_1(t), 27\} \geqslant \alpha_i(t) + 0.2$ for $t \geqslant 0$ and $i = 2, 3, 4$. Since $\beta_1(t) - \alpha_1(t) = t^2$ and $\alpha_1(\sqrt{0.2}) < 27 - 0.2$, we see $\max\{\beta_1(t), 27\} \geqslant \alpha_1(t) + 0.2$ for all $t \geqslant 0$. $\square$

Since

$$\beta_1(t)\binom{s-6}{3} \leqslant \frac{1}{2}(n-3s)^2 s + \frac{5}{2}(n-3s)s^2 + \frac{19}{6}s^3 = \frac{1}{6}n^3 - \frac{1}{6}(n-s)^3,$$

we see

$$\max\{\beta_1(t), \beta_2(t)\}\binom{s-6}{3} \leqslant \max\left\{\binom{n}{3} - \binom{n-s+1}{3}, \binom{3s-1}{3}\right\} + O(n^2) = f(n, s, 3) + O(n^2).$$

2) Since T is good, the union of any three disjoint edges of H^* in $V(T)$ must contain the three vertices in $V(T) \cap I$.

By Claim 3 and the fact that $\sum_{i=1}^{5} x_i + x + y = \binom{s-6}{3}$, we have

$$\begin{aligned} &f_t(x_1, x_2, x_3, x_4, x_5, x, y) \\ &\leqslant (\max\{\beta_1(t), \beta_2(t)\} - 0.2)\sum_{i=1}^{5} x_i + \beta_1(t)x + \beta_2(t)y \\ &\leqslant \max\{\beta_1(t), \beta_2(t)\}\binom{s-6}{3} - 0.2\sum_{i=1}^{5} x_i \leqslant f(n,s,3) - 0.2\sum_{i=1}^{5} x_i + O(n^2). \end{aligned} \tag{4.1}$$

Let $\cup X$ (resp. $\cup T_{27}$) denote the set of edges each of which belongs to some triple in X (resp. in T_{27}). Now we show the following claim.

Claim 4. $s > m - \epsilon n/4$, and $x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}$ or $y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}$.

Proof. If $s \leqslant m - \epsilon n/4$, then by (4.1) we have

$$\begin{aligned} e(H) &\leqslant f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leqslant f(n,s,3) + O(n^2) \\ &\leqslant f(n,m,3) - \binom{\epsilon/4n}{3} + O(n^2) \leqslant f(n,m,3) - \epsilon^4 n^3, \end{aligned}$$

which leads to a contradiction. So $s > m - \epsilon n/4$. First we see that $x + y > \binom{s-6}{3} - 10\epsilon^4 n^3$; for, otherwise, $\sum_{i=1}^{5} x_i \geqslant 10\epsilon^4 n^3$, which together with (4.1) implies

$$e(H) \leqslant f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leqslant f(n,m,3) - 2\epsilon^4 n^3 + O(n^2) \leqslant f(n,m,3) - \epsilon^4 n^3,$$

which leads to a contradiction. Now suppose that $x > \binom{\epsilon n/12}{3}$ and $y > \binom{\epsilon n/24}{3}$. Then $|\cup X| > \epsilon n/12$ and $|\cup T_{27}| > \epsilon n/24$. For any edge $e = (i,j,k) \in \cup X$ with $i < j < k$, by the previous discussion, we have $i \in F_1$. For any edge $e = (i,j,k) \in \cup T_{27}$ with $i < j < k$, by Claim 2 we see $i \notin F_1$. Thus $(\cup X) \cap (\cup T_{27}) = \emptyset$. The triples $T = \{e_1, e_2, e_3\}$ with $e_1 \in \cup X$ and $e_2, e_3 \in \cup T_{27}$ cannot satisfy both $f_3(T) = 27$ and $f_1(T) = 3$. This shows

$$x + y < \binom{s-6}{3} - |\cup X|\binom{|\cup T_{27}|}{2} \leqslant \binom{s-6}{3} - \frac{\epsilon n}{12}\binom{\epsilon n/24}{2},$$

contradicting that $x + y > \binom{s-6}{3} - 10\epsilon^4 n^3$. Hence, we have that either $x \leqslant \binom{\epsilon n/12}{3}$ or $y \leqslant \binom{\epsilon n/24}{3}$. □

Suppose

$$x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}.$$

So $x > \binom{s-6}{3} - \binom{\epsilon n/12}{3}$ and thus $|\cup X| > s - 6 - \epsilon n/12$. Recall that for any $T \in X$, T is a good triple, and hence each edge of H^* contained in $V(T)$ intersects I. Hence any traceable edge which intersects $V(\cup X)$ must also intersect I. Thus, the number of edges of H not intersecting I is at most

$$|V(M') \setminus V(\cup X)|\binom{n}{2} + 50n^2 \leqslant \frac{\epsilon n}{4}\binom{n}{2} + 50n^2 \leqslant \frac{\epsilon}{4}n^3.$$

As $|I| = s \leqslant m - 1$,

$$|E(S(n,m,3)) \backslash E(H)| = |E(H) \backslash E(S(n,m,3))| + e(S(n,m,3)) - e(H) \leqslant \frac{\epsilon}{4}n^3 + \epsilon^4 n^3 < \epsilon n^3.$$

So in this case, we see that H is ϵ-close to $S(n,m,3)$.

By Claim 4, it remains to consider

$$y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}.$$

We claim that there exists a complete 3-graph K on more than $3m - 3\epsilon n/2$ vertices and $V(K) \subseteq V(M')$. Suppose to the contrary that $V(M')$ does not contain such a complete 3-graph K. Since

$$|V(M')| - (3m - 3\epsilon n/2) = 3(s-6) - 3m + 3\epsilon n/2 > \frac{\epsilon n}{2}$$

and H is stable, $V(M')$ contains an independent set of size $\frac{\epsilon n}{2}$, say A. Note that if $T = \{e_1, e_2, e_3\}$ with $e_i \cap A \neq \emptyset$ for all $i \in [3]$, then $f_3(T) < 27$. Since there are at least $|A|/3 \geqslant \epsilon n/6$ edges in M' which intersect with A, we see that $y \leqslant \binom{s-6}{3} - \binom{\epsilon n/6}{3}$, which leads to a contradiction.

Then

$$|E(D(n,m,3)) \backslash E(H)| \leqslant |E(D(n,m,3)) \backslash E(K)| \leqslant \frac{3}{2}\epsilon n \binom{n}{2} < \epsilon n^3,$$

i.e., H is ϵ-close to $D(n,m,3)$. This finishes the proof of Lemma 4.2. □

5 Almost perfect rainbow matchings

In this section, we prove a lemma about almost perfect rainbow matchings that we will need. In fact, this result holds for families of k-graphs, for any $k \geqslant 3$.

Lemma 5.1. *For any given integer $k \geqslant 3$, there exist positive real numbers c and n_2 such that the following holds. Let n and m be integers with $n \geqslant km$ and $n \geqslant n_2$, and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a stable family of k-graphs on the same vertex set $[n]$ such that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$. If $m > (1-c)n/k$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. We choose $c' = c'(k)$ and $c = c(k)$ small enough such that $0 < c \ll c' \ll 1$. Let n be sufficiently large and $n/k \geqslant m > (1-c)n/k$. Suppose to the contrary that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$ and $\mathcal{F}$ does not admit a rainbow matching.

By Corollary 2.3, we may additionally assume that $\mathcal{F}$ is saturated. Let U_i be the vertex set of a largest complete k-graph in F_i for $i \in [m]$. Since F_i is stable, we may choose $U_i = [|U_i|]$ such that $[n] \setminus U_i$ is an independent set in F_i. For each $i \in [m]$, we have $|U_i| > (1-c')km$; for, otherwise, we have the following contradiction for some $i \in [m]$:

$$|F_i| \leqslant \binom{n}{k} - \binom{c'km}{k} \leqslant \binom{n}{k} - (cn+1)\binom{n-1}{k-1} \leqslant \binom{n}{k} - (n-km+1)\binom{n-1}{k-1} < \binom{km-1}{k},$$

where the second inequality holds since $c \ll c' \ll 1$ and $m > (1-c)n/k$, the third inequality holds since $n - km < cn$, and the last inequality holds since

$$\binom{n}{k} - \binom{km-1}{k} = \sum_{i=1}^{n-km+1} \binom{n-i}{k-1} < (n-km+1)\binom{n-1}{k-1}.$$

Let $U = \bigcap_{i=1}^m U_i$. By the above paragraph, we see that $|U| \geqslant (1-c')km$. If $|U| \geqslant km$, then it is clear that $\mathcal{F}$ admits a rainbow matching. So we may assume that $U_m = U \subseteq [km-1]$. Because U_m is the vertex set of a largest complete k-subgraph of F_m and since F_m is stable and $|F_m| > \binom{km-1}{k}$, there exists some k-set $e \notin F_m$ such that $|e \cap U| = k-1$ and $km \in e$. Since $\mathcal{F}$ is saturated, there exists a rainbow matching M in $\mathcal{F} \setminus F_m$ such that $M \cup \{e\}$ is a rainbow matching in $\mathcal{F}(e, F_m)$. Since F_i is stable for each $i \in [m]$, we may assume that $V(M) \cup e = [km]$. Let $M' = \{e' \in M : e' \not\subseteq U\}$.

Claim 5. (a) $|M'| < c'km$,

(b) each edge of F_m is contained in U or intersects an edge of M', and

(c) for any $v \in V(M) \setminus U$, $d_{F_m[U]}(v) \leqslant c'k^2 m\binom{|U|}{k-2}$.

Proof. To prove (a), just observe that $|M'| \leqslant |V(M) \setminus U| = (km-1) - |U| < c'km$.

Suppose that (b) fails, i.e., there exists an edge $f \in F_m$ such that $f \backslash U \neq \emptyset$ and $f \cap V(M') = \emptyset$. Note that $f \cap (U \backslash V(M')) \neq \emptyset$, as $[n] \setminus U$ is independent in F_m. In particular, $|f \cap (U \backslash V(M'))| \leqslant k-1$. Let $|M'| = m - t$ for some $t \geqslant 1$. Recall that $U \cup V(M') = V(M) = [km-1]$. Hence $|U \backslash V(M')| = kt - 1$, and thus $U\backslash(V(M')\cup f)$ induces a common complete k-graph of size at least $k(t-1)$ in all F_i. Then we see that $M'\cup\{f\}$ together with a matching of size $t-1$ in $U\backslash(V(M')\cup f)$ form a rainbow matching for $\mathcal{F}$. So (b) holds.

Now we prove (c). For any $v\in V(M)\setminus U\subseteq[km]$, by the maximality of U, there exists $f\in\binom{[n]}{k}\setminus F_m$ such that $v\in f$ and $|f\cap U|=k-1$. So there exists a rainbow matching N in $\mathcal{F}\setminus F_m$ such that $N\cup\{f\}$ is a rainbow matching in $\mathcal{F}'(f,F_m)$. Since F_i is stable for $i\in[m]$, we may assume that $V(N)\cup f=[km]$. Let $N'=\{e'\in N: e'\nsubseteq U\}$. By applying (b) to N', every edge of F_m containing v intersects $V(N')$. Since

$$V(N')\leqslant k|N'|\leqslant k(km-|U|)\leqslant c'k^2m,$$

there are at most $c'k^2m\binom{|U|}{k-2}$ edges e' in F_m containing v such that $e'\subseteq U\cup\{v\}$. Hence (c) holds. This proves the claim. □

Note that $|e\cap U|=k-1$ and $V(M)\cup U=[km-1]$. Let q_1 be the number of edges of F_m contained in $[km-1]$, and q_2 be the number of edges of F_m with at least one vertex in $[n]\setminus[km-1]$. By (c), we have

$$q_1\leqslant\binom{km-1}{k}-|V(M)\setminus U|\binom{|U|}{k-1}+|V(M)\setminus U|\cdot c'k^2m\binom{|U|}{k-2}.$$

By (b), we see $q_2\leqslant|V(M')|\cdot(n-km+1)\binom{n-2}{k-2}$. So we have

$$\begin{aligned}
|F_m|&\leqslant\binom{km-1}{k}-|V(M)\setminus U|\left[\binom{|U|}{k-1}+c'k^2m\binom{|U|}{k-2}\right]+|V(M')|\,(n-km+1)\binom{n-2}{k-2}\\
&\leqslant\binom{km-1}{k}-|V(M)\setminus U|\left[\binom{|U|}{k-1}+c'k^2m\binom{|U|}{k-2}\right]+k|V(M)\setminus U|(cn+1)\binom{n-2}{k-2}\\
&=\binom{km-1}{k}-|V(M)\setminus U|\cdot\left[\binom{|U|}{k-1}-c'k^2m\binom{|U|}{k-2}-k(cn+1)\binom{n-2}{k-2}\right]\\
&<\binom{km-1}{k},
\end{aligned}$$

where the second inequality holds since $n-km<cn$ and $|M'|\leqslant|V(M)\setminus U|$, and the last inequality holds since c' and c are small enough and $|U|>(1-c')km>(1-c')(1-c)n$. This is a contradiction, finishing the proof of Lemma 5.1. □

6 Non-extremal configurations

Note that if there exist $F\in\mathcal{F}$ and $v\in[n]$ such that $d_F(v)=\binom{n-1}{k-1}$, then v can be removed from all the k-graphs in $\mathcal{F}\setminus\{F\}$ to obtain a smaller family $\mathcal{F}'$ so that $\mathcal{F}'$ admits a rainbow matching if and only if $\mathcal{F}$ admits a rainbow matching. Hence, if such vertex does not exist in a saturated family $\mathcal{F}$, then from Lemma 2.1, we see that $d_F(v)\leqslant\binom{n-1}{k-1}-\binom{n-k(m-1)-1}{k-1}$ for all $v\in F$ and $F\in\mathcal{F}$. This leads us to the following result.

Lemma 6.1. *Given real numbers $0<\epsilon\ll c\ll1$, let $n\geqslant n(\epsilon,c)$ be a sufficiently large integer and m be an integer such that $n/27<m<(1-c)n/3$. Let $\mathcal{F}=\{F_1,\dots,F_m\}$ be a stable family of 3-graphs on the vertex set $[n]$ such that for every $i\in[m]$, $|F_i|>f(n,m,3)$ and*

$$d_{F_i}(v)\leqslant\binom{n-1}{2}-\binom{n-3(m-1)-1}{2}$$

for each $v\in[n]$. If $H(\mathcal{F})$ is ϵ-close to neither $H_S(n,m,3)$ nor $H_D(n,m,3)$, then $\mathcal{F}$ admits a rainbow matching.

Proof. Given $0<\epsilon\ll c\ll1$, let n' and m' be integers such that n' is sufficiently large and $n'/27<m'<(1-c)n'/3$. Let $\mathcal{F}=\{F_1,\dots,F_{m'}\}$ be a family of 3-graphs on the vertex set $[n']$ such that $|F_i|>f(n',m',3)$ and

$$d_{F_i}(v)\leqslant\binom{n'-1}{2}-\binom{n'-1-3(m'-1)}{2}$$

for $i \in [m']$ and $v \in [n']$. Suppose that $H(\mathcal{F})$ is not ϵ-close to $H_S(n', m', 3)$ or $H_D(n', m', 3)$. Our ultimate goal is to find a rainbow matching in $\mathcal{F}$.

Let $n' = 3m' + 3r' + s$, where $0 \leqslant s < 3$. Recall the definitions of $H(\mathcal{F})$ and $H^*(\mathcal{F})$ such that $V(H(\mathcal{F})) = [n'] \cup \mathcal{V}'$ and $V(H^*(\mathcal{F})) = [n'] \cup \mathcal{V}' \cup \mathcal{U}'$, where $|\mathcal{V}'| = m'$ and $|\mathcal{U}'| = r'$. By Lemma 2.5, for $0 < \gamma' \ll \gamma \ll \epsilon \ll c \ll 1$, there exists a matching M_a in $H^*(\mathcal{F})$ with $|M_a| \leqslant \gamma n'$ such that for any $S \subseteq V(H^*(\mathcal{F})) \backslash V(M_a)$ with $|S| \leqslant \gamma' n'$ and $3|S \cap (\mathcal{V}' \cup \mathcal{U}')| = |S \cap [n']|$, $H^*(\mathcal{F})[V(M_a) \cup S]$ has a perfect matching. In the rest of the proof, without loss of generality, we use the following notation:

$$\begin{aligned}&H = H^*(\mathcal{F}) - V(M_a), \quad [n] = [n'] \setminus V(M_a),\\ &\mathcal{V} = \mathcal{V}' \setminus V(M_a) = \{v_1, \dots, v_m\}, \quad \mathcal{U} = \mathcal{U}' \setminus V(M_a) = \{u_1, \dots, u_r\}.\end{aligned}$$

Then $n = 3m + 3r + s$. By using the above property of the matching M_a, it now suffices for us to find an almost perfect matching in H. To find this almost perfect matching, our plan is to show that there exists an almost regular subgraph of H with bounded maximum co-degree so that Theorem 2.6 can be applied. To that end, in what follows we will use the two-round randomization technique developed in [3].

Let R be chosen from $V(H)$ by taking each vertex independently of probability $n^{-0.9}$. We take $n^{1.1}$ independent copies of R and denote them by R^i for $1 \leqslant i \leqslant n^{1.1}$. For $S \subseteq V(H)$, denote $Y_S = |\{i : S \subseteq R^i\}|$. First we have the following claim.

Claim A. With probability $1 - o(1)$, the following hold:

(i) for every $v \in V(H)$, $Y_{\{v\}} = (1 + o(1))n^{0.2}$,

(ii) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i, and

(iii) every edge $e \in H$ is contained in at most one set R^i.

Proof. Note that $Y_S \sim \mathrm{Bin}(n^{1.1}, n^{-0.9|S|})$ for any $S \subseteq V(H)$. Thus, $\mathbb{E}[Y_{\{v\}}] = n^{0.2}$ for every $v \in V(H)$. By (2.2) in Lemma 2.10, we have $\mathbb{P}(|Y_{\{v\}} - n^{0.2}| > n^{0.15}) \leqslant \mathrm{e}^{-\Omega(n^{0.1})}$. By the union bound, we see that (i) holds. To prove (ii) and (iii), let

$$Z_2 = \left|\left\{\{u, v\} \in \binom{V(H)}{2} : Y_{\{u,v\}} \geqslant 3\right\}\right| \quad \text{and} \quad Z_3 = \left|\left\{S \in \binom{V(H)}{3} : Y_S \geqslant 2\right\}\right|.$$

Then

$$\mathbb{E}[Z_2] = \binom{|V(H)|}{2}\mathbb{P}(Y_{\{u,v\}} \geqslant 3) \leqslant \binom{n}{2}(n^{1.1})^3(n^{-1.8})^3 \leqslant 4n^{-0.1}$$

and

$$\mathbb{E}[Z_3] \leqslant \binom{n}{3}(n^{1.1})^2(n^{-2.7})^2 \leqslant 8n^{-0.2}.$$

By Markov's inequality, we have

$$\mathbb{P}(Z_2 = 0) > 1 - 4n^{-0.1} \quad \text{and} \quad \mathbb{P}(Z_3 = 0) > 1 - 8n^{-0.2}.$$

That implies that (ii) and (iii) hold with probability at least $1 - 4n^{-0.1}$ and $1 - 8n^{-0.2}$, respectively. □

Next, we want to prove that there exists a perfect (or, rather, maximum) fractional matching in each $H[R^i]$. To do so, we define a maximal subset $R'^i \subseteq R^i$ that satisfies $R'^i \cap [n] = 3|R'^i \cap (\mathcal{V} \cup \mathcal{U})|$ as follows. If $|R^i \cap [n]| \geqslant 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denoted by R'^i, which is chosen from R^i by deleting $|R^i \cap [n]| - 3|R^i \cap (\mathcal{V} \cup \mathcal{U}))|$ vertices in $R^i \cap [n]$ independently and uniformly at random. Otherwise $|R^i \cap [n]| < 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denoted by R'^i by the following step: First we delete at most 3 vertices (chosen independently and uniformly at random) in $R^i \cap [n]$ so that the number ℓ of the remaining vertices is a multiple of 3. Then we delete $|R^i \cap (\mathcal{V} \cup \mathcal{U})| - \ell/3$ vertices in $R^i \cap (\mathcal{V} \cup \mathcal{U}))$ independently and uniformly at random.

For $S \subseteq V(H)$, define $Y'_S = |\{i : S \subseteq R'^i\}|$. Note that $\mathbb{E}(|R^i \cap [n]|) = n^{0.1}$, $\mathbb{E}(|R^i \cap (\mathcal{V} \cup \mathcal{U})|) = n^{0.1}/3$ and $\mathbb{E}(|R^i \cap \mathcal{V}|) = n^{-0.9}m$. For each i, let A_i be the event $||R^i \cap [n]| - n^{0.1}| < n^{0.095}$, B_i be the event $||R^i \cap (\mathcal{V} \cup \mathcal{U})| - n^{0.1}/3| < n^{0.095}$, and C_i be the event $||R^i \cap \mathcal{V}| - n^{-0.9}m| < n^{0.095}$.

Claim B. With probability $1 - o(1)$, the following hold:

(i) $\bigwedge_i(A_i \wedge B_i \wedge C_i)$ holds,
(ii) for every $v \in V(H)$, $Y'_{\{v\}} = (1+o(1))n^{0.2}$,
(iii) every pair $\{u,v\} \subseteq V(H)$ is contained in at most two sets R'^i, and
(iv) every edge $e \in H$ is contained in at most one set R'^i.

Proof. Since $R'^i \subseteq R^i$, it is clear from Claim A that (iii) and (iv) hold with probability $1-o(1)$. Next, we consider (i). By (2.2) in Lemma 2.10 (with $\lambda = n^{0.095}$), for each $1 \leqslant i \leqslant n^{1.1}$, we have

$$\mathbb{P}(\overline{A_i}) \leqslant \mathrm{e}^{-\Omega(n^{0.09})}, \quad \mathbb{P}(\overline{B_i}) \leqslant \mathrm{e}^{-\Omega(3n^{0.09})} = \mathrm{e}^{-\Omega(n^{0.09})} \quad \text{and} \quad \mathbb{P}(\overline{C_i}) \leqslant \mathrm{e}^{-\Omega(\frac{n}{m}n^{0.09})} = \mathrm{e}^{-\Omega(n^{0.09})}.$$

Thus by the union bound, $\mathbb{P}(\bigwedge_i(A_i \wedge B_i \wedge C_i)) = 1-o(1)$, proving (i).

Assuming $A_i \wedge B_i \wedge C_i$, we see

$$|R^i \setminus R'^i| < \max\{|R^i \cap [n]| - 3|R^i \cap (\mathcal{V}\cup\mathcal{U})|, |R^i \cap (\mathcal{V}\cup\mathcal{U})| - \lfloor |R^i \cap [n]|/3 \rfloor + 3\} < 4n^{0.095}.$$

Then by the choice of R'^i, for all $v \in V(H)$, the probability $\mathbb{P}(\{v \in R^i \setminus R'^i \mid (A_i \wedge B_i \wedge C_i) \wedge (v \in R^i)\})$ is at most

$$\max\left\{\frac{|R^i \setminus R'^i|}{|R^i \cap [n]|}, \frac{|R^i \setminus R'^i|}{|R^i \cap (\mathcal{V}\cup\mathcal{U})|}\right\} \leqslant \frac{|R^i \setminus R'^i|}{|R^i \cap (\mathcal{V}\cup\mathcal{U})|} < \frac{4n^{0.095}}{n^{0.1}/3 - n^{0.095}} < 13n^{-0.005}.$$

Using coupling and applying (2.2) in Lemma 2.10 to $\mathrm{Bin}(|Y_v|, 13n^{-0.005})$ with $\lambda = 3n^{0.195}$, we have

$$\mathbb{P}\left(\left\{Y_{\{v\}} - Y'_{\{v\}} > 16n^{0.195} \,\middle|\, \bigwedge_i (A_i \wedge B_i \wedge C_i) \wedge (Y_{\{v\}} = (1+o(1))n^{0.2})\right\}\right) \leqslant \mathrm{e}^{-\Omega(n^{0.195})}.$$

Note that with probability $1-o(1)$, $\bigwedge_i(A_i \wedge B_i \wedge C_i)$ and $Y_{\{v\}} = (1+o(1))n^{0.2}$ hold for all $v \in V(H)$. By the union bound, we can derive that $0 \leqslant Y_{\{v\}} - Y'_{\{v\}} \leqslant 16n^{0.195} = o(n^{0.2})$ for all $v \in V(H)$ with probability $1-o(1)$. Hence (ii) holds with probability $1-o(1)$. This proves Claim B. □

Let $n_i = |R'^i \cap [n]|$ and $m_i = |R'^i \cap \mathcal{V}|$. Using Claim B(i), we see that with probability $1-o(1)$, $m_i = (1+o(1))mn^{-0.9} = \Theta(n^{0.1}) = \Theta(n_i)$ for all $1 \leqslant i \leqslant n^{1.1}$.

Claim C. With probability $1-o(1)$, the following hold for all $1 \leqslant i \leqslant n^{1.1}$:
(a) $H[R'^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$-close to $H_S(n_i, m_i, 3)$ or $H_D(n_i, m_i, 3)$, and
(b) there exists a perfect fractional matching in $H[R'^i]$.

Proof. For each $T \in \binom{V(H)}{\leqslant 2}$, let

$$\mathrm{Deg}^i(T) := \left| N_H(T) \cap \binom{R'^i}{4-|T|} \right|.$$

By the definition of H, we have that

- for any $v_j \in \mathcal{V}$, $d_H(v_j) \geqslant f(n', m', 3) - (\gamma n')\binom{n'}{2} \geqslant f(n,m,3) - \gamma n^3$, and
- for any $T = \{v_j, u\}$ with $v_j \in \mathcal{V}$ and $u \in [n]$,

$$d_H(T) = d_{F_j}(u) \leqslant \binom{n'-1}{2} - \binom{n'-1-3(m'-1)}{2} \leqslant \binom{n-1}{2} - \binom{n-1-3(m-1)}{2} + \gamma n^2.$$

Assume that $\bigwedge_i(A_i \wedge B_i \wedge C_i)$ holds. Then $n_i = (1+o(1))n^{0.1}$ and $m_i = (1+o(1))mn^{-0.9}$. Since $R^i \setminus R'^i = o(n_i)$, for each $T \in \binom{V(R'^i)}{t}$ with $t \in [2]$, we have

$$\mathbb{E}[\mathrm{Deg}^i(T)] = (1+o(1))d_H(T)(n^{-0.9})^{4-t}.$$

Thus for any $v \in \mathcal{V} \cap R^i$,

$$\mathbb{E}[\mathrm{Deg}^i(v)] \geqslant (1+o(1))(f(n,m,3) - \gamma n^3)(n^{-0.9})^3 \geqslant f(n_i, m_i, 3) - 2\gamma n_i^3,$$

and for any $T=\{u,v\}$ with $v\in\mathcal{V}$ and $u\in[n]$, $\mathbb{E}[\mathrm{Deg}^i(T)]$ is at most

$$(1+o(1))\left[\binom{n-1}{2}-\binom{n-1-3(m-1)}{2}+\gamma n^2\right](n^{-0.9})^2\leqslant\binom{n_i-1}{2}-\binom{n_i-1-3(m_i-1)}{2}+2\gamma n_i^2.$$

We apply Janson's inequality (see[4, Theorem 8.7.2]) to bound the deviation of $\mathrm{Deg}^i(T)$ for $|T|\leqslant 2$. Write $\mathrm{Deg}^i(T)=\sum_{e\in N_H(T)}X_e$, where $X_e=1$ if $e\subseteq R'^i$ and $X_e=0$ otherwise. Let $t=|T|\in\{1,2\}$ and $p=n^{-0.9}$. Then

$$\begin{aligned}\Delta^*&=\sum_{e_i\cap e_j\neq\emptyset,e_i,e_j\in\binom{V(H)}{4-t}}\mathbb{P}(X_{e_i}=X_{e_j}=1)\\&\leqslant\sum_{\ell=1}^{4-t}p^{2(4-t)-\ell}\binom{n-t}{4-t}\binom{4-t}{\ell}\binom{n-4}{4-t-\ell}=O(n^{0.1(2(4-t)-1)}).\end{aligned}$$

By Janson's inequality, for $v\in\mathcal{V}\cap R^i$,

$$\mathbb{P}(\mathrm{Deg}^i(v)\leqslant(1-\gamma)\mathbb{E}[\mathrm{Deg}^i(v)])\leqslant\mathrm{e}^{-\gamma^2\mathbb{E}[\mathrm{Deg}^i(v)]/(2+\Delta^*/\mathbb{E}[\mathrm{Deg}^i(v)])}\leqslant\mathrm{e}^{-\Omega(n^{0.3}/(2+n^{0.5}/n^{0.3}))}=\mathrm{e}^{-\Omega(n^{0.1})},$$

and for the pair $\{v,u\}$ with $v\in\mathcal{V}$ and $u\in[n]$ (by considering the complement of H), we can have

$$\mathbb{P}(\mathrm{Deg}^i(\{v,u\})\geqslant(1+\gamma)\mathbb{E}[\mathrm{Deg}^i(\{v,u\})])\leqslant\mathrm{e}^{-\Omega(n^{0.1})}.$$

By the union bound, with probability $1-o(1)$ we derive from above that for all $1\leqslant i\leqslant n^{1.1}$,

(1) for any $v\in\mathcal{V}\cap R^i$, $\mathrm{Deg}^i(v)\geqslant(1-\gamma)\mathbb{E}[\mathrm{Deg}^i(v)]\geqslant f(n_i,m_i,3)-3\gamma n_i^3$, and

(2) for any pair $\{u,v_j\}\subseteq R'^i$ with $v_j\in\mathcal{V}$ and $u\in[n]$,

$$\mathrm{Deg}^i(\{u,v_j\})\leqslant\binom{n_i-1}{2}-\binom{n_i-1-3(m_i-1)}{2}+3\gamma n_i^2\leqslant\binom{n_i-1}{2}-\Omega(n_i^2),$$

which implies that $F_j[R'^i\cap[n]]$ is not $\epsilon^3/2$-close to $S(n_i,m_i,3)$, since $m_i=(1+o(1))mn^{0.9}$ and $m<(1-c)n/3$.

This shows that $H[R'^i\setminus\mathcal{U}]$ is not $\epsilon^4/4$-close to $H_S(n_i,m_i,3)$, where $\gamma\ll\epsilon$.

Let $\mathcal{V}_0:=\{v_i\in\mathcal{V}:\ F_i[[n]]$ is not ϵ-close to $D(n,m,3)\}$. We claim that $|\mathcal{V}_0|>\epsilon n$. Otherwise $|\mathcal{V}_0|\leqslant\epsilon n$, we have

$$|E(H_D(n',m',3))\setminus E(H(\mathcal{F}))|\leqslant\epsilon n\binom{n}{3}+(m-\epsilon n)\epsilon n^3+\gamma(n')^4\leqslant\epsilon(n')^4,$$

which leads to a contradiction as $H(\mathcal{F})$ is not ϵ-close to $H_D(n',m',3)$. As $|\mathcal{V}_0|>\epsilon n$, with probability $1-o(1)$ we have (by using Lemma 2.10) that

(3) $|R'^i\cap\mathcal{V}_0|\geqslant\frac{\epsilon n_i}{2}$ for all $1\leqslant i\leqslant n^{1.1}$.

For $v_j\in R'^i\cap\mathcal{V}_0$, we consider $F_j[[n]]$. Let G be the complement of $F_j[[n]]$. Then for any $S\subseteq V(G)$ with $|S|>3m-\epsilon n$, we have $e(G[S])\geqslant\epsilon e(G)$. Otherwise,

$$|E(D(n,m,3))\backslash E(F_j[[n]])|\leqslant\epsilon n\binom{n}{2}+\epsilon e(G)<\epsilon n^3,$$

contradicting $v_j\in\mathcal{V}_0$. By Lemma 2.8, the maximum size of the complete 3-graph in $F_j[R^i\cap[n]]$ is no more than $(3m/n-\epsilon+\gamma)n^{0.1}\leqslant 3m_i-\epsilon n_i/2$ with probability at least $1-(n^{O(1)}\mathrm{e}^{-\Omega(n^{0.1})})$. By assuming $\bigwedge_i(A_i\wedge B_i\wedge C_i)$, this implies that $F_j[R'^i\cap[n]]$ is not $\epsilon^3/2$-close to $D(n_i,m_i,3)$. By the union bound, with probability $1-o(1)$ we have

(4) for all $1\leqslant i\leqslant n^{1.1}$ and $v_j\in R'^i\cap\mathcal{V}_0$, $F_j[R'^i\cap[n]]$ is not $\epsilon^3/2$-closed to $D(n_i,m_i,3)$.

By (3) and (4), we see that with probability $1-o(1)$, $H[R'^i\backslash\mathcal{U}]$ is not $\epsilon^4/4$-close to $H_D(n_i,m_i,3)$, proving Claim C(a).

It remains to show Claim C(b), i.e., to construct a perfect fractional matching w_i in $H[R'^i]$ for each $1\leqslant i\leqslant n^{1.1}$. Our main tool is the stability result, Lemma 4.2.

Fix some $1 \leqslant i \leqslant n^{1.1}$. We write $R'^i \cap [n] = \{x_1^i, \dots, x_{n_i}^i\}$ with $x_1^i < x_2^i < \cdots < x_{n_i}^i$ and define $[d]_i := \{x_1^i, x_2^i, \dots, x_d^i\}$ for any integer d. We now state two simple inequalities for later use:

$$f(x,y,3) \geqslant f(x,y-a,3) + \binom{a}{3} \quad \text{and} \quad f(x,y,3) \geqslant f(x,y+a,3) - 3ax^2 \tag{6.1}$$

hold for any positive integers x, y and a with $a < y$.

To construct a perfect fractional matching w_i in $H[R'^i]$, first we consider $v_j \in R'^i \cap \mathcal{V}_0$ and assign weights to the edges of $H[R'^i]$ containing v_j. Using (1), by (6.1) and the fact that $\gamma \ll \epsilon \ll 1$, we have

$$|F_j[R'^i \cap [n]]| = \mathrm{Deg}^i(v_j) \geqslant f(n_i, m_i, 3) - 3\gamma n_i^3 \geqslant f(n_i, m_i + \epsilon^{20} n_i, 3) - \epsilon^{16} n_i^3.$$

By (2) and (4), $F_j[R'^i \cap [n]]$ is not $\epsilon^3/2$-close to $S(n_i, m_i, 3)$ or $D(n_i, m_i, 3)$. Since $|E(S(n_i, m_i+\epsilon^{20}n_i, 3)) \setminus E(S(n_i, m_i, 3))| \leqslant \epsilon^{20} n_i^3$ and $|E(D(n_i, m_i+\epsilon^{20}n_i, 3)) \setminus E(D(n_i, m_i, 3))| \leqslant 3\epsilon^{20} n_i^3$, we see that $F_j[R'^i \cap [n]]$ is not ϵ^4-close to $S(n_i, m_i+\epsilon^{20}n_i, 3)$ or $D(n_i, m_i+\epsilon^{20}n_i, 3)$. Then by Lemma 4.2 and the fact that F_j is stable, $F_j[R'^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i + 3\epsilon^{20}n_i]_i$. Now we assign weights $w_i(e)$ to all the edges e of $H[R'^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i + \epsilon^{20} n_i}$, and otherwise let $w_i(e) = 0$.

Next, we consider $v_j \in R'^i \cap (\mathcal{V} \setminus \mathcal{V}_0)$. By (1) and (6.1), we have

$$|F_j[R'^i \cap [n]]| \geqslant f(n_i, m_i, 3) - 3\gamma n_i^3 \geqslant f(n_i, m_i - 6\gamma^{\frac{1}{3}} n_i, 3).$$

By Theorem 4.1 and the fact that F_j is stable, $F_j[R'^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i - 18\gamma^{\frac{1}{3}} n_i]_i$. Then we assign weights $w_i(e)$ to all the edges e of $H[R'^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i - 6\gamma^{1/3} n_i}$, and otherwise let $w_i(e) = 0$.

Note that for every $v_j \in R'^i \cap \mathcal{V}$, we have defined weights $w_i(e)$ for all the edges $e \in H[R'^i]$ with $v_j \in e$, whose total weights equal one. In the remaining proof, we want to extend this function w_i to the entire $H[R'^i]$ to form a perfect fractional matching. We complete this in two steps.

First, we define a perfect fractional matching w (as the *projection* of w_i) in the complete 3-graph K on the vertex set $R'^i \cap [n]$. Note that a function $w: E(K) \to [0,1]$ is a perfect fractional matching if and only if $w(v) := \sum_{v \in f \in K} w(f) = 1$ holds for every $v \in V(K)$. Initially, we define a function $w': E(K) \to [0,1]$ such that for each $f \in E(K)$, $w'(f) := \sum_e w_i(e)$ over all the edges $e \in H[R'^i]$ with $f \subseteq e$ and $|e \cap \mathcal{V}| = 1$. Since $|\mathcal{V}_0| > \epsilon n$ and $\gamma \ll \epsilon$, it follows from the above definitions on w_i that for any $v \in R'^i \cap [n]$,

$$w'(v) := \sum_{v \in f \in K} w'(f) \leqslant \frac{|\mathcal{V}_0|}{m_i + \epsilon^{20} n_i} + \frac{m_i - |\mathcal{V}_0|}{m_i - 6\gamma^{\frac{1}{3}} n_i} \leqslant \frac{\epsilon n_i}{m_i + \epsilon^{20} n_i} + \frac{m_i - \epsilon n_i}{m_i - 6\gamma^{\frac{1}{3}} n_i} < 1.$$

Since $\epsilon \ll c$, we have $3m_i + 3\epsilon^{20} n_i < n_i - 4$. So there exists a vertex set $\{a_1, a_2, a_3, a_4\}$ in K such that $w'(a_i) = 0$ for $i \in [4]$. Let K' be the 3-graph obtained from K by deleting vertices a_1, a_2, a_3 and a_4. Starting with $w := w'$, we increase w by using the following iterations: (i) pick a vertex v in $V(K')$ with the maximum $w(v)$[3)]; (ii) pick any edge $f \in K'$ containing v and update $w(f) \leftarrow w(f) + 1 - w(v)$; (iii) delete all the vertices $u \in V(K')$ with $w(u) = 1$ (which must include the vertex v) from K'; (iv) if $|V(K')| \leqslant 2$, then terminate; otherwise go to (i) again. This must terminate in finitely many iterations and when it terminates, we obtain a fractional matching w in K such that $w(a_i) = 0$ for $i \in [4]$ and $|V(K')| \leqslant 2$. So there exist two vertices b_1 and b_2 in $V(K) \setminus \{a_1, a_2, a_3, a_4\}$ such that for any vertex v in $V(K) \setminus \{a_1, a_2, a_3, a_4, b_1, b_2\}$, $w(v) = 1$. We may suppose $1 \geqslant w(b_1) \geqslant w(b_2)$. Let

$$w(a_1, a_2, b_1) = 1 - w(b_1), \quad w(a_1, a_2, b_2) = \frac{w(b_1) - w(b_2)}{2}, \quad w(a_3, a_4, b_2) = 1 - w(b_1) + \frac{w(b_1) - w(b_2)}{2}$$

and

$$w(a_1, a_2, a_3) = w(a_1, a_2, a_4) = w(a_1, a_3, a_4) = w(a_2, a_3, a_4) = \frac{w(b_1) + w(b_2)}{6}.$$

3) Note that this maximum $w(v)$ is strictly less than 1.

It is easy to check that w is a perfect fractional matching in K.

Now we notice that

$$\sum_{f\in K} w'(f) = \sum_{\{e\in H[R'^i]:|e\cap\mathcal{V}|=1\}} w_i(e) = |R'^i\cap\mathcal{V}|$$

and

$$\sum_{f\in K} w(f) = \frac{|R'^i\cap[n]|}{3} = |R'^i\cap(\mathcal{V}\cup\mathcal{U})|.$$

Moreover, the neighborhood of any $u_j\in R'^i\cap\mathcal{U}$ in $H[R'^i]$ is the complete 3-graph K. So we can partition the total weight $\sum_{f\in K}(w(f)-w'(f)) = |R'^i\cap\mathcal{U}|$ into $|R'^i\cap\mathcal{U}|$ copies of 1's (say each is represented by a set E_j of edges in K), and then for each $u_j\in R'^i\cap\mathcal{U}$, we assign the weight of each $f\in E_j$ to be $w_i(f\cup\{u_j\})$. One can easily check that we obtain a perfect fractional matching w_i in $H[R'^i]$. This completes the proof of Claim C. □

From Claims B and C, we see that the sets R'^i for $1\leqslant i\leqslant n^{1.1}$ satisfy (a)–(d) in Lemma 2.7. Then by Lemma 2.7, there exists a spanning subgraph H' of H such that for each $v\in V(H)$, $d_{H'}(v)=(1+o(1))n^{0.2}$ and $\Delta_2(H')\leqslant n^{0.1}$. By Theorem 2.6, H contains a matching M_b such that $S=V(H)\setminus V(M_b)$ contains at most $\gamma'n'$ vertices. Since $|S\cup M_a\cup M_b| = n' = 3r'+3m'+s$ where $0\leqslant s\leqslant 2$, we can delete at most s elements from S to get a subset S' such that $3|S'\cap(\mathcal{V}'\cup\mathcal{U}')| = |S'\cap[n']|$. By the setting at the beginning of the proof, Lemma 2.5 assures that $H^*(\mathcal{F})[V(M_a)\cup S']$ has a perfect matching, which together with M_b forms a matching in $H^*(\mathcal{F})$ of size $r'+m'$. Equivalently, this says that $\mathcal{F}$ admits a rainbow matching, finishing the proof of Lemma 6.1. □

7 Proof of Theorem 1.3

Let n be a sufficiently large integer. Let m be a positive integer with $n\geqslant 3m$ and let $\mathcal{F}=\{F_1,\dots,F_m\}$ be a family of 3-graphs on the same vertex set $[n]$ such that $|F_i|>f(n,m,3)$ for each $i\in[m]$. Suppose to the contrary that $\mathcal{F}$ does not admit a rainbow matching. In view of Lemma 2.2, we may assume that $\mathcal{F}$ is stable. Then by Lemma 5.1, there exists an absolute constant $c=c(3)>0$ such that $m\leqslant(1-c)n/3$. By Theorem 1.2, $m\geqslant n/27$. Hence,

$$n/27\leqslant m\leqslant(1-c)n/3. \tag{7.1}$$

We now apply the following algorithm. Initially, let $\mathcal{F}_0=\mathcal{F}$, $n_0=n$ and $m_0=m$. We repeat the following iterations. Suppose that we have defined $\mathcal{F}_i$, which contains m_i 3-graphs on the same vertex set $[n_i]$.

Step 1. Applying Corollary 2.3 to $\mathcal{F}_i$, we obtain a family $\mathcal{F}_{i+1}$ of 3-graphs on the vertex set $[n_i]$ that is both stable and saturated, and set $n_{i+1}=n_i$ and $m_{i+1}=m_i$.

Step 2. If for any $F\in\mathcal{F}_{i+1}$ and any $v\in[n_{i+1}]$, $d_F(v)<\binom{n_{i+1}-1}{2}$, then set $t:=i+1$ and output $\mathcal{F}_t$, n_t and m_t.

Step 3. If there exist $F\in\mathcal{F}_{i+1}$ and $v\in[n_{i+1}]$ such that $d_F(v)=\binom{n_{i+1}-1}{2}$, then set $n'_{i+1}=n_{i+1}-1$, $m'_{i+1}=m_{i+1}-1$ and $\mathcal{F}'_{i+1}:=\{F'-v : F'\in\mathcal{F}_i\setminus\{F\}\}$. Relabel the vertices if necessary so that all the 3-graphs in $\mathcal{F}'_{i+1}$ have the same vertex set $[n'_{i+1}]$. Set $\mathcal{F}_i:=\mathcal{F}'_{i+1}$, $n_i:=n'_{i+1}$, $m_i:=m'_{i+1}$ and go to Step 1.

Let $\mathcal{F}_t$ be the resulting family of 3-graphs, which contains m_t 3-graphs on the same vertex set $[n_t]$ and admits no rainbow matching. By (7.1), we see that $n_t\geqslant n-m>cn$ is sufficiently large. We also see from Lemma 2.9 that $|F|>f(n_t,m_t,3)$ holds for any $F\in\mathcal{F}_t$.

By definition, we see that $\mathcal{F}_t$ is stable and saturated such that for any $F\in\mathcal{F}_t$ and $v\in V_t$, $d_F(v)<\binom{n_t-1}{2}$. On the other hand, by Lemma 2.1, it further holds that

$$d_F(v)\leqslant\binom{n_t-1}{2}-\binom{n_t-1-3(m_t-1)}{2}\quad\text{for any } F\in\mathcal{F}_t \quad\text{and}\quad v\in V_t.$$

Since n_t is sufficiently large, using Lemma 5.1 and Theorem 1.2 again, we may assume that

$$n_t/27 \leqslant m_t \leqslant (1-c)n_t/3.$$

Now we choose $0 < \epsilon \ll c$. Since $\mathcal{F}_t$ satisfies the above properties, by applying Lemmas 2.4, 3.1 and 6.1, we can conclude that $\mathcal{F}_t$ admits a rainbow matching. This is a contradiction, completing the proof of Theorem 1.3. □

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