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RAINBOW PERFECT MATCHINGS FOR 4-UNIFORM HYPERGRAPHS[*]

HONGLIANG LU[†], YAN WANG[‡], AND XINGXING YU[§]

Abstract. Let n be a sufficiently large integer with $n \equiv 0 \pmod 4$, and let $F_i \subseteq \binom{[n]}{4}$, where $i \in [n/4]$. We show that if each vertex of F_i is contained in more than $\binom{n-1}{3} - \binom{3n/4}{3}$ edges, then $\{F_1, \ldots, F_{n/4}\}$ admits a rainbow matching, i.e., a set of $n/4$ edges consisting of one edge from each F_i. This generalizes a deep result of Khan *J. Combin. Theory Ser. B*, 116 (2016), pp. 333–366. on perfect matchings in 4-uniform hypergraphs.

Key words. rainbow matching, perfect matching, hypergraph, absorbing method

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1. Introduction.

A *hypergraph* is a family of subsets (called *edges*) of a nonempty set whose elements are the *vertices* of the hypergraph. For a hypergraph H, we use $V(H)$ to denote its vertex set and $E(H)$ to denote its edge set and let $e(H) := |E(H)|$. We say that a hypergraph H is k*-uniform* for some positive integer k if all edges of H have the same size k. A k-uniform hypergraph is also known as a k*-graph*.

A *matching* in a hypergraph H is a set of pairwise disjoint edges of H. Finding maximum matchings in k-graphs is NP-hard for $k \geq 3$; see [21]. Hence, it is of interest to find tight sufficient conditions for the existence of a large matching in k-graphs. The most well-known open problem in this area is the following conjecture made by Erdős [9] in 1965: For positive integers k, n, t, if H is a k-graph of order n and $e(H) > \max\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\}$, then H has a matching of size t. This bound on $e(H)$ is tight because of the complete k-graph on $kt-1$ vertices and the k-graph on n vertices in which every edge intersects a fixed set of $t-1$ vertices. There have been recent activities on this conjecture; see [5, 6, 10, 11, 12, 13, 18, 32].

One type of condition that has been used to ensure the existence of large matchings is the so-called Dirac-type conditions, which involve degrees of sets of vertices. Our work in this paper falls into this category. For convenience, let $[k] := \{1, \ldots, k\}$ for any positive integer k, and let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$ for any set S and positive integer k. Let H be a k-graph. For any $T \subseteq V(H)$, the *degree* of T in H, denoted by $d_H(T)$, is the number of edges of H containing T. For any integer $0 \leq l \leq k-1$, $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ denotes the minimum l*-degree* of H. Hence, $\delta_0(H) = e(H)$. Note that $\delta_1(H)$ is often called the *mini-*

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[†]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China (luhongliang@mail.xjtu.edu.cn).
[‡]School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, China (yan.w@sjtu.edu.cn).
[§]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA (yu@math.gatech.edu).

mum vertex degree of H. For $x \in V(H)$, we define the neighborhood of x to be $N_H(x) := \{e \in \binom{V(H)\backslash\{x\}}{k-1} : e \cup \{x\} \in E(H)\}$. When there is no confusion, we also use $N_H(x)$ to denote the $(k-1)$-graph with vertex set $V(H)\backslash\{x\}$ and edge set $N_H(x)$.

For integers n, k, s, d satisfying $0 \le d \le k-1$, $n \equiv 0 \pmod k$, and $0 \le s \le n/k$, $m_d^s(k,n)$ denotes the minimum integer m such that every k-graph H on n vertices with $\delta_d(H) \ge m$ has a matching of size s. Rödl, Ruciński, and Szemerédi [36] determined $m_{k-1}^{n/k}(k,n)$ for large n, which has motivated a large amount of work; see [16, 17, 26, 27, 28, 38]. For instance, Treglown and Zhao [38] extended this result by determining $m_d^{n/k}(k,n)$ for all $d \ge k/2$. On the other hand, it seems more difficult to determine $m_d^{n/k}(k,n)$ when $d < k/2$. Kühn, Osthus, and Treglown [28] and, independently, Khan [26] determined $m_1^{n/3}(3,n)$. Khan [27] further determined $m_1^{n/4}(4,n)$. The main work in this paper is to prove a more general result which implies Khan's result and uses different techniques.

Let $\mathcal{F} = \{F_1, \ldots, F_t\}$ be a family of hypergraphs. A set of t pairwise disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$. (In this case, we also say that $\mathcal{F}$ or $\{F_1, \ldots, F_t\}$ *admits* a rainbow matching.) There has been a lot of interest in studying rainbow versions of matching problems; see [1, 2, 3, 4, 12, 18, 19, 20, 24, 25, 31, 33, 35]. For instance, Aharoni and Howard [3] made the following conjecture, which first appeared in Huang, Loh, and Sudakov [18]: Let t be a positive integer and $\mathcal{F} = \{F_1, \ldots, F_t\}$ such that, for $i \in [t]$, $F_i \subseteq \binom{[n]}{k}$ and $e(F_i) > \{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\}$; then $\mathcal{F}$ admits a rainbow matching. Huang, Loh, and Sudakov [18] showed that this conjecture holds for $n > 3k^2t$. Frankl and Kupavskii [12] improved this lower bound to $n \ge 12tk\log(e^2t)$, which was further improved by Lu, Wang, and Yu [30] to $n \ge 2kt$. Keevash et al. [22, 23] independently verified this conjecture for $n > Ckt$ for some (large and unspecified) constant C. Recently, Kupavskii [29] gave the concrete dependencies on the parameters by showing the conjecture holds for $n > 3ekt$ with $t > 10^7$.

For 3-graphs, Lu, Yu, and Yuan [31] proved that, for sufficiently large n with $n \equiv 0 \pmod 3$, if $\delta_1(F_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ for $i \in [n/3]$, then $\mathcal{F}$ has a rainbow matching. This implies the result of Kühn, Osthus, and Treglown [28] and Khan [27] on perfect matchings in 3-graphs.

In this paper, we prove the following result on rainbow matchings in 4-graphs, which gives Khan's result [27] on perfect matchings in 4-graphs as a special case.

THEOREM 1.1. *Let n be a sufficiently large integer with $n \equiv 0 \pmod 4$. Let $\mathcal{F} = \{F_1, \ldots, F_{n/4}\}$ such that $F_1, \ldots, F_{n/4}$ are 4-graphs on a common vertex set of cardinality n, and for $i \in [n/4]$, $\delta_1(F_i) > \binom{n-1}{3} - \binom{3n/4}{3}$. Then $\mathcal{F}$ admits a rainbow matching.*

The bound on $\delta_1(F_i)$ in Theorem 1.1 is sharp. To see this, let k, m, n be positive integers such that $k \ge 2$ and $2 \le m < n/k$. Let $H_k(n,m)$ be a k-graph such that

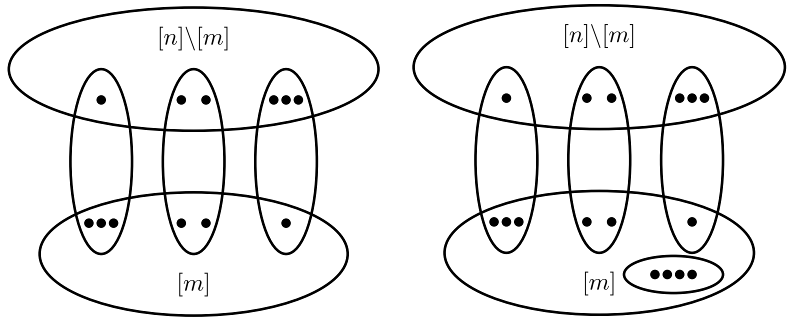
$$V(H_k(n,m)) = [n],$$

$$E(H_k(n,m)) = \left\{ e \in \binom{[n]}{k} : e \cap [m] \neq \emptyset \text{ and } e \cap ([n] \setminus [m]) \neq \emptyset \right\},$$

and let $H_k^*(n,m)$ be a k-graph such that

$$V(H_k^*(n,m)) = [n],$$

$$E(H_k^*(n,m)) = \left\{ e \in \binom{[n]}{k} : e \cap [m] \neq \emptyset \right\}.$$



(a) $H_4(n,m)$ (b) $H_4^*(n,m)$

FIG. 1. *Illustrations of $H_k(n,m)$ and $H_k^*(n,m)$ when $k=4$.*

Figure 1 illustrations of $H_k(n,m)$ and $H_k^*(n,m)$ when $k=4$. Then $\delta_1(H_k(n,m)) = \delta_1(H_k^*(n,m)) = \binom{n-1}{k-1} - \binom{n-1-m}{k-1}$. Observe that neither $H_k(n,m)$ nor $H_k^*(n,m)$ has no matching of size $m+1$. It follows that when $n \equiv 0 \pmod k$, we have $\delta_1(H_k(n,n/k-1)) = \binom{n-1}{k-1} - \binom{n-n/k}{k-1}$, and $\{H_k(n,n/k-1),\dots,H_k(n,n/k-1)\}$ admits no rainbow matching.

Let $\mathcal{F} = \{F_1,\dots,F_{n/4}\}$ as defined in Theorem 1.1. Note that $V(\mathcal{F}) = [n]$. We prove Theorem 1.1 by working with a 5-graph $H(\mathcal{F})$ obtained from $\mathcal{F}$: The vertex set of $H(\mathcal{F})$ is $[n]\cup\{x_1,\dots,x_{n/4}\}$, and the edge set of $H(\mathcal{F})$ is $\bigcup_{i=1}^{n/4}\{e\cup\{x_i\} : e \in E(F_i)\}$. Clearly, $\mathcal{F}$ admits a rainbow matching if and only if $H(\mathcal{F})$ has a perfect matching.

For convenience, we say that a $(k+1)$-graph H is $(1,k)$-*partite* if there exists a partition of $V(H)$ into sets V_1, V_2 (called *partition classes*) such that, for any $e \in E(H)$, $|e \cap V_1| = 1$ and $|e \cap V_2| = k$. A $(1,k)$-partite $(k+1)$-graph with partition classes V_1, V_2 is *balanced* if $k|V_1| = |V_2|$. Thus, for instance, $H(\mathcal{F})$ above is a balanced (1,4)-partite 5-graph with partition classes $X, [n]$.

More generally, let $\mathcal{F} = \{F_1,\dots,F_m\}$ be a family of n-vertex k-graphs on a common vertex set V and let $X = \{x_1,\dots,x_m\}$ be a set disjoint from V. We use $\mathcal{H}_{n,m}^k(\mathcal{F})$ to represent the balanced $(1,k)$-partite $(k+1)$-graph with partition classes X, V and edge set $\bigcup_{i=1}^m\{e\cup\{x_i\} : e \in E(F_i)\}$. If $F_i = H_k(n,m)$ (or $H_k^*(n,m)$) for all $i \in [m]$, then we write $\mathcal{H}_k(n,m)$ (or $\mathcal{H}_k^*(n,m)$) for $\mathcal{H}_{n,m}^k(\mathcal{F})$ (or $\mathcal{H}_k^*(n,m)$). Now Theorem 1.1 is a direct consequence of the following result.

THEOREM 1.2. *Let n be an integer such that $n \equiv 0 \pmod 4$ and n is sufficiently large. Let H be a balanced $(1,4)$-partite 5-graph with partition classes $X, [n]$ such that $\delta_1(N_H(x)) > \binom{n-1}{3} - \binom{3n/4}{3}$ for all $x \in X$. Then H admits a perfect matching.*

Our proof of Theorem 1.2 is divided into two parts by considering whether H is close to $\mathcal{H}_4(n,n/4)$ or not. For any real $\varepsilon > 0$ and two k-graphs H_1, H_2 on the same vertex set V, we say that H_2 is ε-*close* to H_1 if there exists an isomorphic copy H_2' of H_2 with $V(H_2') = V$ such that $|E(H_1) \setminus E(H_2')| < \varepsilon|V(H_1)|^k$.

We show the following lemma when H is close to $\mathcal{H}_4(n,n/4)$. In fact, we are able to prove the following lemma for $(1,k)$-partite $(k+1)$-graphs that are close to $\mathcal{H}_k(n,n/k)$ for all $k \geq 2$.

LEMMA 1.3. *Let $k \geq 3$ be an integer, $0 < \varepsilon < (10k)^{-6}$, and let n be an integer with $n \equiv 0 \pmod k$ and $n \geq 20k^2$. Let H be a balanced $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$ and $V(H) = V(\mathcal{H}_k(n,n/k))$. If H is ε-close to $\mathcal{H}_k(n,n/k)$ and $\delta_1(N_H(x)) > \binom{n-1}{k-1} - \binom{n-n/k}{k-1}$ for all $x \in X$, then H has a perfect matching.*

When H is not close to $\mathcal{H}_4(n, n/4)$, we prove the following.

LEMMA 1.4. *Let $0 < \varepsilon \ll 1$, and let n be a sufficiently large integer with $n \equiv 0 \pmod 4$. Let H be a balanced $(1,4)$-partite 5-graph with partition classes $X, [n]$ and $V(H) = V(\mathcal{H}_4(n, n/4))$. If H is not ε-close to $\mathcal{H}_4(n, n/4)$ and $\delta_1(N_H(x)) > \binom{n-1}{3} - \binom{3n/4}{3}$ for all $x \in X$, then H has a perfect matching.*

It is easy to see that Theorem 1.2 follows immediately from Lemmas 1.3 and 1.4.

In section 2, we prove Lemma 1.3. To prove Lemma 1.4, we will need to find a small "absorbing" matching in H, and this part is done in section 3. In section 4, we show that if H is not close to $\mathcal{H}_4(n, n/4)$, then we can find a subgraph of H that is almost regular (in terms of vertex degree) and has maximum 2-degree bounded above by $n^{0.1}$. We make use of a recent stability result of Gao et al. [15] for 3-graphs (see Lemma 4.2) and another result there on almost regular spanning subgraphs. We then complete the proof using a result of Pippenger and Spencer [34].

2. Hypergraphs close to $\mathcal{H}_k(n, n/k)$.

In this section, we prove Lemma 1.3 for the case when $H^k_{n,n/k}(\mathcal{F})$ is ε-close to $\mathcal{H}_k(n, n/k)$ for some sufficiently small ε.

We first prove Lemma 1.3 for those balanced $(1,k)$-partite $(k+1)$-graphs H in which, for each vertex $v \in V(H)$, most edges of H containing v also belong to $\mathcal{H}_k(n, n/k)$. More precisely, given $\alpha > 0$ and two $(k+1)$-graphs H_1, H_2 on the same vertex set, a vertex $v \in V(H_1)$ is *α-bad* with respect to H_2 if $|N_{H_2}(v) \setminus N_{H_1}(v)| > \alpha |V(H_1)|^k$. (A vertex $v \in V(H_1)$ is *α-good* with respect to H_2 if it is not α-bad with respect to H_2.) So if v is α-good with respect to H_2, then all but at most $\alpha |V(H_1)|^k$ of the edges containing v in H_2, also lie in H_1.

LEMMA 2.1. *Let $k \geq 2$ be an integer, $0 < \alpha < (10^k k^k (k+1)!)^{-1}$, and let n be an integer with $1/n \ll \alpha$ and $n \equiv 0 \pmod k$. If H is a balanced $(1,k)$-partite $(k+1)$-graph on the same vertex set as $\mathcal{H}_k(n, n/k)$ and every vertex of H is α-good with respect to $\mathcal{H}_k(n, n/k)$, then H has a perfect matching.*

Proof. Let $X, [n]$ denote the partition classes of H, and let $W = [n/k]$ and $U = [n] \setminus W$. Let M denote a matching in H such that $|e \cap X| = |e \cap W| = 1$ for each $e \in M$, and subject to this, $|M|$ is maximum. Let $U' = U \setminus V(M), W' = W \setminus V(M)$, and $X' = X \setminus V(M)$. We may assume $|M| < n/k$; for otherwise, the assertion of the lemma is true.

Note that $|M| \geq n/2k$. Suppose $|M| < n/2k$. Then $|U'|/(k-1) = |W'| = |X'| \geq n/2k$. By the maximality of $|M|$, H has no edge contained in $X' \cup W' \cup U'$ containing exactly one vertex from X' and exactly one vertex from W'. Hence, for any $u \in U'$, we have

$$\begin{aligned}
&|N_{\mathcal{H}_k(n,n/k)}(u) \setminus N_H(u)| \\
&\geq |X'||W'| \binom{|U'|}{k-2} \\
&\geq (n/2k)(n/2k)((k-1)n/2k - k + 3)^{k-2}/(k-2)! \quad (\text{since } n \geq 20k^2) \\
&\geq \frac{n^k}{4k^2 5^{k-2}(k-2)!} \quad (\text{since } k \geq 2) \\
&> \alpha \left(\frac{(k+1)n}{k} \right)^k = \alpha |V(H)|^k \quad (\text{since } \alpha < k^k/(4k^2 5^{k-2}(k-2)!(k+1)^k)).
\end{aligned}$$

Thus, u is not α-good with respect to $\mathcal{H}_k(n, n/k)$, a contradiction.

Fix $x \in X'$, $u_1, \ldots, u_{k-1} \in U'$, and $w \in W'$. Write $S = \{x, w, u_1, \ldots, u_{k-1}\}$. If there exists distinct $e_1, \ldots, e_k \in M$ such that $H[S \cup (\cup_{i=1}^k e_i)]$ has a matching M' of size $k+1$ such that, for any $f \in M'$, $|f \cap X| = 1 = |f \cap W|$, then $(M \setminus \{e_i : i \in [k]\}) \cup M'$ contradicts the choice of M. So such M' does not exist for any choice of distinct $e_1, \ldots, e_k \in M$. This implies that there exists a $(k+1)$-subset f of $V(H)$ such that $f \subseteq S \cup (\bigcup_{i=1}^k e_i)$, $|f \cap X'| = 1 = |f \cap W'|$, $|f \cap e_i| = 1$ for $i \in [k]$, but $f \notin E(H)$.

Hence there exists $v \in S$ such that

$$|N_{\mathcal{H}_k(n,n/k)}(v) \setminus N_H(v)| > \frac{1}{k+1}\binom{n/2k}{k} > \frac{(n/2k-k+1)^k}{(k+1)!} > \frac{(n/3k)^k}{(k+1)!} > \alpha n^k$$

since $n > 6k(k-1)$ and $\alpha < (3^k k^k (k+1)!)^{-1}$. This is a contradiction. ∎

To achieve the goal of this section, we need Lemma 2.1 from [31].

LEMMA 2.2 (see Lu, Yu, and Yuan [31]). *Let n, t, k be positive integers such that $n > 2k^4 t$. For $i \in [t]$, let $G_i \subseteq \binom{[n]}{k}$ such that $\delta_1(G_i) > \binom{n-1}{k-1} - \binom{n-t}{k-1}$. Then $\{G_1, \ldots, G_t\}$ admits a rainbow matching.*

Proof of Lemma 1.3. Let $W = [n/k]$ and $U = [n] \setminus [n/k]$ be the partition classes of $H_k(n, [n/k])$ in $\mathcal{H}_k(n, n/k)$. Let B denote the set of $\sqrt{\varepsilon}$-bad vertices in H with respect to $\mathcal{H}_k(n, n/k)$. Since H is ε-close to $\mathcal{H}_k(n, n/k)$, we have $|B| \le 2(k+1)\sqrt{\varepsilon} n$; otherwise,

$$\begin{aligned}|E(\mathcal{H}_k(n,n/k)) \setminus E(H)| &\ge \frac{1}{k+1}\sum_{v \in V(H)} |N_{\mathcal{H}_k(n,n/k)}(v) \setminus N_H(v)| \\ &> 2(k+1)\sqrt{\varepsilon} n \cdot \frac{1}{k+1}\sqrt{\varepsilon}|V(H)|^k \ge \varepsilon |V(H)|^{k+1},\end{aligned}$$

a contradiction.

Let $U^b = U \cap B$, $X^b = X \cap B$ and $W^b = W \cap B$. Let $W^g = W \setminus W^b$. For convenience, write $q = |X^b|$ and $r = q + |W^b|$. Moreover, let $x_1, \ldots, x_r$ be distinct such that $X^b = \{x_1, \ldots, x_q\}$, let $W' \subseteq W^g$ be a set of size $n/k - r$, and let $G_i = N_H(x_i) - W^{'}$ for $i \in [r]$. Then, for $i \in [r]$,

$$\begin{aligned}\delta_1(G_i) &\ge \delta_1(N_H(x_i)) - \left(\binom{n-1}{k-1} - \binom{n-|W^{'}|-1}{k-1}\right) \\ &> \binom{n-|W^{'}|-1}{k-1} - \binom{n-n/k}{k-1} \\ &= \binom{n-n/k+r-1}{k-1} - \binom{n-n/k}{k-1}.\end{aligned}$$

Thus, by Lemma 2.2 (with $n - n/k + r$ as n and r as t), $\{G_1, \ldots, G_r\}$ admits a rainbow matching, say $M_0 = \{e_i \in E(G_i) : i \in [r]\}$. Now $M_0' = \{e_i \cup \{x_i\} : i \in [r]\}$ is a matching in H covering X^b. (Note this is the only place in this proof that requires the degree condition in the statement.)

Let $H_1 = H - V(M_0')$. Since $r \le |B| \le 2(k+1)\sqrt{\varepsilon} n$ and $\varepsilon < (10k)^{-6}$, every vertex in $X \setminus V(M_0')$ is $\varepsilon^{1/3}$-good with respect to $\mathcal{H}_k(n, n/k) - V(M_0')$. Choose η such that $0 < \varepsilon \ll \eta \ll 1/k$, and let

$$B' := \{v \in B \setminus V(M_0') : |\{e \in E(H) : v \in e \text{ and } |e \cap W^g| = 1\}| \ge \eta n^k\}.$$

Since $|B'| \leq |B| \leq 2(k+1)\sqrt{\varepsilon}n$ and $\varepsilon \ll \eta$, we may greedily pick a matching M_1 in $H - V(M_0')$ such that $B' \subseteq V(M_1)$ and every edge in M_1 contains at least one vertex from B' and exactly one vertex from W^g.

Now consider $H_2 = H_1 - V(M_1)$. Note that, since $n > 20k^2$,

$$\begin{aligned}\delta_1(H_2) &\geq \delta_1(H) - (k+1)|M_0' \cup M_1|n^{k-1}\\ &> \frac{n}{k}\left(\binom{n-1}{k-1} - \binom{n-n/k}{k-1}\right) - 2(k+1)^2\sqrt{\varepsilon}n^k.\end{aligned}$$

Thus, for each $v \in B \setminus V(M_0' \cup M_1)$, the number of edges of H_2 containing v and no vertex of W^g is at least

$$\begin{aligned}&\delta_1(H_2) - \eta n^k - \sum_{i=2}^{k-1}\frac{n}{k}\binom{|W^g|}{i}\binom{n-|W^g|-1-k|M_0'\cup M_1|}{k-1-i}\\ &\geq \frac{n}{k}\left(\binom{n-1}{k-1} - \binom{n-n/k}{k-1}\right) - 2(k+1)^2\sqrt{\varepsilon}n^k - \eta n^k\\ &\qquad\qquad - \frac{n}{k}\sum_{i=2}^{k-1}\binom{n/k}{i}\binom{n-n/k-1}{k-1-i}\\ &> \frac{n}{k}\left(\binom{n-1}{k-1} - \binom{n-n/k}{k-1}\right) - 2(k+1)^2\sqrt{\varepsilon}n^k - \eta n^k\\ &\quad - \frac{n}{k}\left(\binom{n-1}{k-1} - \binom{n-n/k-1}{k-1} - \binom{n/k}{1}\binom{n-n/k-1}{k-2}\right)\\ &= \frac{n}{k}\left(\frac{n}{k}-1\right)\binom{n-n/k-1}{k-2} - 2(k+1)^2\sqrt{\varepsilon}n^k - \eta n^k\\ &> \eta n^k \quad (\text{since } \varepsilon \ll \eta \ll 1/k).\end{aligned}$$

Hence, we may greedily pick a matching M_2 in H_2 such that every edge in M_2 contains at least one vertex from $B \setminus V(M_0' \cup M_1)$ and no vertex from W^g.

It is easy to see that $|M_0' \cup M_1 \cup M_2| \leq 2(k+1)\sqrt{\varepsilon}n$. Hence, every vertex of $H_2 - V(M_2)$ is $\varepsilon^{1/4}$-good with respect to $\mathcal{H}_k(n, n/k) - V(M_0' \cup M_1 \cup M_2)$. Thus for every vertex $u \in U \setminus V(M_0' \cup M_1 \cup M_2)$, the number of edges containing u and exactly two vertices of $W \setminus V(M_0' \cup M_1 \cup M_2)$ as well as avoiding $V(M_0' \cup M_1 \cup M_2)$ is at least

$$\frac{n}{k}\binom{n/k}{2}\binom{n-n/k-1}{k-3} - \varepsilon^{1/4}\left(n + \frac{n}{k}\right)^k - (k+1)|M_0' \cup M_1 \cup M_2|n^{k-1} > \eta n^k.$$

Thus we may greedily pick a matching M_2' such that $|M_2'| = |M_2|$ and every edge of M_2' contains exactly two vertices from W^g.

Put $M := M_0' \cup M_1 \cup M_2 \cup M_2'$ and $m := |M|$. Let $H_3 := H - V(M) = H_2 - V(M_2 \cup M_2')$. One can see that every vertex of H_3 is $\varepsilon^{1/5}$-good with respect to $\mathcal{H}_k(n - km, n/k - m) = \mathcal{H}_k(n, m) - V(M)$. By Lemma 2.1, H_3 contains a perfect matching, say M_3. Now $M_3 \cup M$ is a perfect matching in H. ∎

3. Absorbing matching. To deal with balanced $(1, k)$-partite $(k+1)$-graphs that are not close to $\mathcal{H}_k(n, n/k)$, we need to find a small matching that can "absorb" small sets of vertices. To find such a matching, we need to use the Chernoff inequality to bound deviations; see [7].

LEMMA 3.1 (Chernoff inequality for small deviation). *Let $X = \sum_{i=1}^{n} X_i$, where each random variable X_i has Bernoulli distribution with expectation p_i. For $\alpha \le 3/2$,*

$$\mathbb{P}(|X - \mathbb{E}X| \ge \alpha \mathbb{E}X) \le 2e^{-\frac{\alpha^2}{3}\mathbb{E}X}.$$

In particular, when $X \sim Bi(n,p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}(|X - np| \ge \lambda) \le e^{-\Omega(\lambda^2/(np))}.$$

We now prove a $(1,k)$-partite version of the absorption lemma for $(1,3)$-partite 4-graphs proved in [31]. Our proof follows along the same lines as in [31]. Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$. A set $S \subseteq V(H)$ is called *balanced* if $|S \cap [n]| = k|S \cap X|$.

LEMMA 3.2. *Let $k \ge 3$ be an integer and $0 < b < 1/k$ be a constant. There exists an integer $n_1 = n_1(k,b)$ such that the following holds for any integer $n \ge n_1$: Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$ such that $|X| = n/k$ and $\delta_1(N_H(x)) > (1/2 + b)\binom{n-1}{k-1}$ for $x \in X$. Then for any c satisfying $0 < c < \min\{(\frac{k^k cn}{6b^k (k!)^k})^2, (2k^3(k+1)c^2)^{-10}\}$, there exists a matching M in H such that $|M| \le 2kcn$ and, for any balanced subset $S \subseteq V(H)$ with $|S| \le (k+1)c^{1.5}n/2$, $H[V(M) \cup S]$ has a perfect matching.*

Proof. For balanced $R \in \binom{V(H)}{k+1}$ and balanced $Q \in \binom{V(H)}{k(k+1)}$, we say that Q is *R-absorbing* if both $H[Q]$ and $H[Q \cup R]$ have perfect matchings. For each balanced $R \in \binom{V(H)}{k+1}$, let $\mathcal{L}(R)$ denote the collection of all R-absorbing sets in H.

Claim 1. For each balanced $R \in \binom{V(H)}{k+1}$, $|\mathcal{L}(R)| \ge b^k \binom{n}{k}^{k+1}/(2(k^2!))$.

Let $R \in \binom{V(H)}{k+1}$ be a fixed balanced set, and let $R \cap X = \{x\}$. Note that the number of edges in H containing x and intersecting $R \setminus \{x\}$ is at most $k\binom{n}{k-1}$. Thus, since $\delta_1(N_H(x)) > (1/2+b)\binom{n-1}{k-1}$, the number of edges $e \in E(H)$ with $e \cap R = \{x\}$ is at least

$$\frac{n(1/2+b)\binom{n-1}{k-1}}{k} - k\binom{n}{k-1} \ge \frac{1}{2}\binom{n}{k}.$$

Fix a choice of $e \in E(H)$ with $e \cap R = \{x\}$, and write $R \setminus \{x\} = \{u_1, \dots, u_k\}$ and $e \setminus \{x\} = \{v_1, \dots, v_k\}$. Let $W_0 = e \setminus \{x\}$. For each pair $\{u_j, v_j\}$ in order $j = 1, 2, \dots, k$, we choose a k-set U_j disjoint from $W_{j-1} \cup R$ such that both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H and let $W_j := U_j \cup W_{j-1}$. If W_k is defined, then W_k gives an absorbing $k(k+1)$-set for R.

Note that for $j \in [k]$ there are $k+1+jk$ vertices in $W_{j-1} \cup R$. Thus, the number of edges in H containing u_j (respectively, v_j) and another vertex in $W_{j-1} \cup R$ is at most $(k+1+jk)\binom{n}{k-2}\frac{n}{k} < (k+1)n\binom{n}{k-2}$. Since $\delta_1(N_H(x)) > (1/2+b)\binom{n-1}{k-1}$ for $x \in X$, the number of sets U_j for which $U_j \cap (W_{j-1} \cup R) = \emptyset$ and both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H is at least

$$\begin{aligned} &\frac{n}{k}\left(2(1/2+b)\binom{n-1}{k-1} - \binom{n-1}{k-1}\right) - 2(k+1)n\binom{n}{k-2} \\ &= 2b\binom{n}{k} - 2(k+1)n\binom{n}{k-2} \\ &> b\binom{n}{k} \end{aligned}$$

because n is sufficiently large.

To summarize, the number of W_k defined above from e is at least $\left(b\binom{n}{k}\right)^k$. Hence, there are at least $\frac{1}{2}\binom{n}{k}(b\binom{n}{k})^k$ absorbing, ordered $k(k+1)$-sets for R, with at most $(k^2!)$ of them corresponding to a single R-absorbing set. Therefore,

$$\mathcal{L}(R) \geq \frac{\frac{1}{2}\binom{n}{k}(b\binom{n}{k})^k}{k^2!} = \frac{b^k\binom{n}{k}^{k+1}}{2(k^2!)}.$$

This completes the proof of Claim 1.

Now, let c be a fixed constant with $0 < c < \min\{(\frac{b^k k^k}{6(k!)^k})^2, (2k^3(k+1)c^2)^{-10}\}$, and choose a family $\mathcal{G}$ of balanced $k(k+1)$-sets of $V(H)$ by selecting each of the $\binom{n/k}{k}\binom{n}{k^2}$ balanced $k(k+1)$-subsets of $V(H)$ independently with probability

$$p := \frac{cn}{\binom{n/k}{k}\binom{n}{k^2}}.$$

Then $\mathbb{E}(|\mathcal{G}|) = cn$ and $\mathbb{E}(|\mathcal{L}(R)\cap\mathcal{G}|) = p|\mathcal{L}(R)| \geq b^k\binom{n}{k}^{k+1}cn/(2(k^2!)\binom{n/k}{k}\binom{n}{k^2})$. It follows from Lemma 3.1 that, with probability $1 - o(1)$,

$$|\mathcal{G}| \leq 2cn \tag{3.1}$$

and, for all balanced $(k+1)$-sets R,

$$|\mathcal{L}(R)\cap\mathcal{G}| \geq p|\mathcal{L}(R)|/2 \geq \frac{b^k k^k cn}{6(k!)^k} \geq c^{1.5}n. \tag{3.2}$$

Furthermore, the expected number of intersecting pairs of $k(k+1)$-sets in $\mathcal{G}$ is at most

$$\begin{aligned}\binom{n/k}{k}\binom{n}{k^2}k(k+1)\left(\binom{n/k-1}{k-1}\binom{n}{k^2} + \binom{n/k}{k}\binom{n-1}{k^2-1}\right)p^2 &\leq 2k^3(k+1)c^2n\\ &\leq c^{1.9}n.\end{aligned}$$

Thus, using Markov's inequality, we derive that with probability at least 1/2

$$\mathcal{G} \text{ contains at most } 2c^{1.9}n \text{ intersecting pairs of } k(k+1)\text{-sets.} \tag{3.3}$$

Hence, there exists a family $\mathcal{G}$ satisfying (3.1), (3.2), and (3.3). Delete one $k(k+1)$-set from each intersecting pair in such a family $\mathcal{G}$, and remove all nonabsorbing $k(k+1)$-sets from $\mathcal{G}$. The resulting family, call it $\mathcal{G}'$, consists of pairwise disjoint balanced, absorbing $k(k+1)$-sets and satisfies

$$|\mathcal{L}(R)\cap\mathcal{G}'| \geq c^{1.5}n/2$$

for all balanced $(k+1)$-sets R.

Since $\mathcal{G}'$ consists only of absorbing $k(k+1)$-sets, $H[V(\mathcal{G}')]$ has a perfect matching, say M. By (3.1), $|M| \leq 2kcn$. For a balanced set $S \subseteq V(H)$ of size $|S| \leq (k+1)c^{1.5}n/2$, we partition S into balanced $(k+1)$-sets $R_1, \ldots, R_t$, where $t \leq c^{1.5}n/2$. Since $|\mathcal{L}(R_i)\cap\mathcal{G}'| \geq c^{1.5}n/2$, there is distinct absorbing $k(k+1)$-set $Q_1, \ldots, Q_t$ in $\mathcal{G}'$ such that Q_i is an R_i-absorbing set for $i \in [t]$. Now $\mathcal{H}[V(M)\cup S]$ has a perfect matching which consists of a perfect matching from each $H[Q_i \cup R_i]$. ∎

4. Fractional perfect matchings. To deal with hypergraphs that are not close to $\mathcal{H}_4(n, n/4)$, we need to control the independence number of those hypergraphs. This is done in the same way as in [31] by applying the hypergraph container result in [8, 37].

First, we need the following lemma, which is more general and slightly stronger than Lemma 4.2 in [31] but with a very similar proof. Let H be a hypergraph, $\lambda > 0$ be a real number, and $\mathcal{A}$ be a family of subsets of $V(H)$. We say that H is $(\mathcal{A}, \lambda)$-*dense* if $e(H[A]) \geq \lambda e(H)$ for every $A \in \mathcal{A}$.

LEMMA 4.1. *Let ε be a constant such that $0 < \varepsilon \ll 1$, and let n, k be integers such that $k \geq 3$ and $n \geq 40k^2/\varepsilon$. Let $a_1 = \varepsilon/(8k), a_2 = \varepsilon/(8k^3)$, and $a_3 < \varepsilon/(2^k \cdot k! \cdot 8k)$. Let H be a $(1, k)$-partite $(k+1)$-graph with vertex partition classes $X, [n]$ with $|X| = n/k$. Suppose $d_H(\{x, v\}) \geq \binom{n-1}{k-1} - \binom{n-n/k}{k-1} - a_3 n^{k-1}$ for any $x \in X$ and $v \in [n]$ and $|E(\mathcal{H}_k(n, n/k)) \setminus E(H_0)| \geq \varepsilon e(\mathcal{H}_k(n, n/k))$ for any isomorphic copy H_0 of H with $V(H_0) = V(\mathcal{H}_k(n, n/k))$. Then H is $(\mathcal{A}, a_1)$-dense, where $\mathcal{A} = \{A \subseteq V(H) : |A \cap X| \geq (1/k - a_1)n$ and $|A \cap [n]| \geq (1 - 1/k - a_2)n\}$.*

Proof. We prove this by way of contradiction. Suppose that there exists $A \subseteq V(H)$ such that $|A \cap X| \geq (1/k - a_1)n$, $|A \cap [n]| \geq (1 - 1/k - a_2)n$, and $e(H[A]) \leq a_1 e(H)$. Without loss of generality, we may choose A such that $|A \cap X| = (1/k - a_1)n$ and $|A \cap [n]| = (1 - 1/k - a_2)n$. Let $U \subseteq [n]$ such that $A \cap [n] \subseteq U$ and $|U| = n - n/k$. Let $A_1 = A \cap X$, $A_2 = X \setminus A$, $B_1 = A \cap [n]$, and $B_2 = U \setminus A$. Relabel the vertices of H in $[n]$ if necessary so that $U = [n] \setminus [n/k]$.

Let H_0 denote the isomorphic copy of H with the same partition classes $X, [n]$ as $\mathcal{H}_k(n, n/k)$. We derive a contradiction by showing that $|E(\mathcal{H}_k(n, n/k)) \setminus E(H_0)| < \varepsilon e(\mathcal{H}_k(n, n/k))$. Note that

$$e(\mathcal{H}_k(n, n/k)) = \frac{n}{k}\left(\binom{n}{k} - \binom{n-n/k}{k} - \binom{n/k}{k}\right) \geq \frac{n}{k}\binom{n}{k}/k$$

and, further,

$$e(\mathcal{H}_k(n, n/k)) \geq \frac{n}{k}\binom{n}{k}/k = \frac{n^2}{k^3}\binom{n-1}{k-1} > \frac{n^2}{k^3}\binom{n-n/k}{k-1} > \frac{n^3}{k^4}\binom{n-n/k}{k-2}. \tag{4.1}$$

Also, since $n > 2k$,

$$e(\mathcal{H}_k(n, n/k)) \geq \frac{n}{k}\binom{n}{k}/k > \frac{n^{k+1}}{2^k \cdot k! \cdot k^2}. \tag{4.2}$$

Consider $x \in A_1$ and $v \in [n] \setminus [n/k]$. Let $E_{H_0}(B_1, \{x, v\}) = \{e \in E(H_0) : \{x, v\} \subseteq e \subseteq B_1 \cup \{x, v\}\}$. Note that, for $v \in B_1$, we have

$$\begin{aligned}
&|\{e \in E(H_0) : \{x, v\} \subseteq e \text{ and } e \cap [n/k] \neq \emptyset\}| \\
&\geq d_{H_0}(\{x, v\}) \\
&\quad -|\{e \in E(H_0 - [n/k]) : \{x, v\} \subseteq e \text{ and } e \cap B_2 \neq \emptyset\}| - |E_{H_0}(B_1, \{x, v\})| \\
&\geq \left(\binom{n-1}{k-1} - \binom{n-n/k}{k-1} - a_3 n^{k-1}\right) - a_2 n \binom{n-n/k}{k-2} - |E_{H_0}(B_1, \{x, v\})|.
\end{aligned}$$

For $v \in B_2$, we have

$$\begin{aligned}
&|\{e \in E(H_0) : \{x, v\} \subseteq e \text{ and } e \cap [n/k] \neq \emptyset\}| \\
&\geq d_{H_0}(\{x, v\}) - |\{e \in E(H_0 - [n/k]) : \{x, v\} \subseteq e\}| \\
&\geq \left(\binom{n-1}{k-1} - \binom{n-n/k}{k-1} - a_3 n^{k-1}\right) - \binom{n-n/k}{k-1}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]} |\{e\in E(H_0),:\{x,v\}\subseteq e \text{ and } e\cap[n/k]\neq\emptyset\}| \\
\geq &\sum_{x\in A_1}\sum_{v\in B_1}\left(\binom{n-1}{k-1}-\binom{n-n/k}{k-1}-a_2n\binom{n-n/k}{k-2}-|E_{H_0}(B_1,\{x,v\})|\right) \\
&\sum_{x\in A_1}\sum_{v\in B_2}\left(\binom{n-1}{k-1}-\binom{n-n/k}{k-1}-a_3n^{k-1}-\binom{n-n/k}{k-1}\right) \\
\geq &\sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]}\left(\binom{n-1}{k-1}-\binom{n-n/k}{k-1}-a_3n^{k-1}-a_2n\binom{n-n/k}{k-2}\right) \\
&-\sum_{x\in A_1}\sum_{v\in B_2}\binom{n-n/k}{k-1}-\sum_{x\in A_1}\sum_{v\in B_1}|E_{H_0}(B_1,\{x,v\})| \\
= &\sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]}\left(\binom{n-1}{k-1}-\binom{n-n/k}{k-1}-a_3n^{k-1}-a_2n\binom{n-n/k}{k-2}\right)
\end{aligned}$$

$$-|A_1||B_2|\binom{n-n/k}{k-1}-|E(H_0[A])|. \tag{4.3}$$

Note that

$$\begin{aligned}
a_2\frac{n^3}{k^2}\binom{n-n/k}{k-2}-|A_1||B_2|\binom{n-n/k}{k-1} &> a_2\frac{n^2}{k}\left(\frac{n}{k}\binom{n-n/k}{k-2}-\binom{n-n/k}{k-1}\right) \\
&> a_2\frac{n^2}{k}\left(\frac{n}{k}\binom{n-n/k-1}{k-2}-\binom{n-n/k}{k-1}\right) \\
&= 0
\end{aligned}$$

That is,

$$a_2\frac{n^3}{k^2}\binom{n-n/k}{k-2}-|A_1||B_2|\binom{n-n/k}{k-1}>0. \tag{4.4}$$

Therefore, we have

$$\begin{aligned}
&|E(\mathcal{H}_k(n,n/k))\setminus E(H_0)| \\
= &\sum_{x\in A_1}|\{e\in E(\mathcal{H}_k(n,n/k))\setminus E(H_0): x\in e\}| \\
&+\sum_{x\in A_2}|\{e\in E(\mathcal{H}_k(n,n/k))\setminus E(H_0): x\in e\}| \\
\leq &\sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]}|\{e\in E(\mathcal{H}_k(n,n/k))\setminus E(H_0):\{x,v\}\subseteq e\}|+|A_2|\cdot e(H_k(n,n/k)) \\
\leq &\sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]}\left(\binom{n-1}{k-1}-\binom{n-n/k}{k-1}\right. \\
&\left.\qquad -|\{e\in E(H_0):\{x,v\}\subseteq e \text{ and } e\cap[n/k]\neq\emptyset\}|\right) \\
&+|A_2|\cdot e(H_k(n,n/k))
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{x\in A_1}\sum_{v\in[n]\setminus[n/k]} \left(a_3 n^{k-1} + a_2 n\binom{n-n/k}{k-2}\right) + |A_1||B_2|\binom{n-n/k}{k-1} + |E(H_0[A])| \\
&\quad +(a_1 n)\cdot e(\mathcal{H}_k(n,n/k))/(n/k) \quad \text{(by (4.3))} \\
&\le \frac{n}{k}\left(n-\frac{n}{k}\right)\left(a_3 n^{k-1} + a_2 n\binom{n-n/k}{k-2}\right) + |A_1||B_2|\binom{n-n/k}{k-1} \\
&\quad +a_1 e(H_0) + ka_1\cdot e(\mathcal{H}_k(n,n/k)) \\
&< \frac{n^2}{k}\left(a_3 n^{k-1} + a_2 n\binom{n-n/k}{k-2}\right) + a_1 e(H_0) + ka_1\cdot e(\mathcal{H}_k(n,n/k)) \quad \text{(by (4.4))} \\
&< a_1 e(H_0) + \left(2^k\cdot k!\cdot ka_3 + k^3 a_2 + ka_1\right)\cdot e(\mathcal{H}_k(n,n/k)) \quad \text{(by (4.1) and (4.2))} \\
&\le a_1\frac{n}{k}\binom{n}{k} + \left(2^k\cdot k!\cdot ka_3 + k^3 a_2 + ka_1\right)\cdot e(\mathcal{H}_k(n,n/k)) \\
&\le \left(ka_1 + 2^k\cdot k!\cdot ka_3 + k^3 a_2 + ka_1\right)\cdot e(\mathcal{H}_k(n,n/k)) \quad \text{(by (4.2))} \\
&\le \varepsilon\cdot e(\mathcal{H}_k(n,n/k)),
\end{aligned}
$$

a contradiction. $\square$

To prove a fractional matching lemma, we need a recent result of Gao et al. [15] on Erdős' matching conjecture for stable graphs. For a hypergraph H, let $\nu(H)$ denote the size of maximum matching in H. Łuczak and Mieczkowska [32] proved that there exists a positive integer n_1 such that, for integers m, n with $n \ge n_1$ and $1 \le m \le n/3$, if H is an n-vertex 3-graph with $e(H) > \max\{\binom{n}{3} - \binom{n-m+1}{3}, \binom{3m-1}{3}\}$, then $\nu(H) \ge m$. The result of Gao et al. [15] may be viewed as a stability version of this Łuczak–Mieczkowska result.

For sets $e = \{u_1, \ldots, u_k\} \subseteq [n]$ and $f = \{v_1, \ldots, v_k\} \subseteq [n]$ with $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for $i \in [k-1]$, we write $e \le f$ if $u_i \le v_i$ for all $i \in [k]$. A hypergraph H with $V(H) = [n]$ and $E(H) \subseteq \binom{[n]}{k}$ is said to be *stable* if, for any $e, f \in \binom{[n]}{k}$ with $e \le f$, $e \in E(H)$ implies $f \in E(H)$. The following is a special case (when $m = 3n/4$) of Lemma 4.2 in [15]. Note that one of the extremal configurations of Lemma 4.2 in [15], namely, $\mathcal{D}(n, m, 3)$ (the 3-graph with vertex set $[n]$ and edge set $\binom{\lfloor 3n/4-1\rfloor}{3}$), does not occur here.

LEMMA 4.2 (see Gao et al. [15]). *For any $\eta > 0$ there exists $n_0 > 0$ with the following properties: Let n be an integer with $n \ge n_0$, and let H be a stable 3-graph on the vertex set $[n]$. If $e(H) > \binom{n}{3} - \binom{3n/4}{3} - \eta^4 n^3$ and $\nu(H) < n/4$, then H is η-close to $H_3^*(n, n/4-1)$.*

We will use perfect fractional matchings in a hypergraph H not close to $\mathcal{H}_4(n, n/4-1)$ to obtain an almost regular subgraph of H. A *fractional matching* in H is a function $f : E(H) \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of nonnegative reals, such that $\sum_{v\in e} f(e) \le 1$ for all $v \in V(H)$, and it is *perfect* if $\sum_{v\in e} f(e) = 1$ for all $v \in V(H)$. We write $\nu_f(H) = \max\{\sum_{e\in E(H)} f(e) : f \text{ is a fractional matching in } H\}$.

LEMMA 4.3. *For any $\varepsilon > 0$ there exist $0 < \rho \ll \varepsilon$ and $n_0 = n_0(\varepsilon)$ such that, for any integer n with $n \ge n_0$ and $n \equiv 0 \pmod 4$, the following holds: Let H be a balanced $(1,4)$-partite 5-graph with partition classes $X, [n]$ such that $d_H(\{x, v\}) \ge \binom{n-1}{3} - \binom{3n/4}{3} - 3\rho n^3$ for any $x \in X$ and $v \in [n]$ and H contains no independent set S with $|S \cap X| \ge n/4 - \varepsilon n$ and $|S \cap [n]| \ge 3n/4 - \varepsilon n$; then H contains a fractional perfect matching.*

Proof. Let $X = \{x_1, \ldots, x_{n/4}\}$, and let $\omega : V(H) \to R^+$ be a minimum fractional vertex cover of H, that is, $\sum_{v\in e} \omega(v) \ge 1$ for all $e \in E(H)$, and subject to this,

$\omega(H) := \sum_{v\in V(H)} \omega(v)$ is minimum. We may assume that the vertices in X and $[n]$ are labeled such that $\omega(x_1) \geq \cdots \geq \omega(x_{n/4})$ and $\omega(1) \geq \cdots \geq \omega(n)$.

Let H' be the $(1,4)$-partite 5-graph with vertex set $V(H)$ and edge set

$$E(H) \cup \left\{ e \in \binom{V(H)}{5} : |e\cap X| = 1 \text{ and } \sum_{v\in e} w(v) \geq 1 \right\}.$$

Thus, by definition, ω is also a vertex cover of H'. Moreover, since $E(H) \subseteq E(H')$, ω is also a minimum fractional vertex cover of H'. Hence, by linear programming duality, we have $\nu_f(H) = \omega(H) = \omega(H') = \nu_f(H')$. Therefore, it suffices to show that H' has a perfect matching. First, we show that H' is stable.

(1) Let $T_1 = \{x_{i_1}, i_2, i_3, i_4, i_5\}$ and $T_2 = \{x_{j_1}, j_2, j_3, j_4, j_5\}$ be balanced 5-element subsets of $V(H)$, with $x_{i_1}, x_{j_1} \in X$ and $i_l \geq j_l$ for $l \in [5]$. Then $T_2 \in E(H')$ implies that $T_1 \in E(H')$.

Since $i_l \geq j_l$ for $l\in[5]$, we have $\omega(x_{i_1}) \geq \omega(x_{j_1})$ and $\omega(i_l) \geq \omega(j_l)$ for $2\leq l\leq 5$. Note that $\sum_{v\in T_2}\omega(v) \geq 1$ as $T_2 \in E(H')$. Thus $\sum_{v\in T_1}\omega(v) \geq \sum_{v\in T_2}\omega(v) \geq 1$. Hence, $T_1 \in E(H')$ by the definition of H', which completes the proof of (1).

Let G be the 3-graph with vertex set $[n-1]$ and edge set $N_H(\{x_{n/4}, [n]\})$. We may assume that

(2) G has no matching of size $n/4$.

Suppose G has a matching of size $n/4$, say $M = \{e_1, \ldots, e_{n/4}\}$. Partition $V(H)\setminus V(M)$ into 2-sets $f_1, \ldots, f_{n/4}$ such that $|f_i \cap X| = 1$ for all $i \in [n/4]$. By (1), $M \subseteq N_H(f_i)$. Thus $M' = \{e_i \cup f_i : i \in [n/4]\}$ is a perfect matching in H. So we may assume (2).

Since $d_H(\{x,v\}) \geq \binom{n-1}{3} - \binom{3n/4}{3} - 3\rho n^3$ for any $x\in X$ and $v\in[n]$, $e(G) > \binom{n-1}{3} - \binom{3n/4}{3} - 3\rho n^3$. Hence, by (2) and by Lemma 4.2, G is η-close to $H_3^*(n-1, n/4-1)$ with respect to the partition $[n-1]\setminus[n/4-1], [n/4-1]$, where $\eta = (3\rho)^{1/4}$. Let $Y = [n/4 - \lceil \eta n\rceil]$. We claim that

(3) for every $y \in Y$, $d_G(y) \geq \binom{n-1}{2} - 4\sqrt{\eta}n^2$.

Otherwise, since G is stable, $d_G(z) < \binom{n-1}{2} - 4\sqrt{\eta}n^2$ for $z \in Z := [n/4-1]\setminus[n/4 - \lceil \eta n\rceil]$. Hence,

$$\begin{aligned} |E(H_3^*(n-1, n/4-1)) \setminus E(G)| &\geq \frac{1}{3}\sum_{z\in Z}\left(\binom{n-1}{2} - d_G(z)\right) \\ &\geq \frac{1}{3}(\sqrt{\eta}n - 1)4\sqrt{\eta}n^2 > \eta(n-1)^3, \end{aligned}$$

a contradiction which completes the proof of (3).

Since H contains no independent set S such that $|S\cap X| \geq n/4 - \varepsilon n$ and $|S\cap[n]| \geq 3n/4 - \varepsilon n$, we may greedily find a matching M_1 of size $\lceil\sqrt{\eta}n\rceil$ in $H - Y$.

Next we greedily construct a matching M_2 of size $|Y|$ in $G - V(M_1)$ such that $|e\cap Y| = 1$ for all $e \in M_2$. For $y \in Y$, note that

$$|\{e \in E(G) : y \in e \text{ and } e\cap V(M_1) \neq \emptyset\}| \leq 3|M_1|n < 3\lceil\sqrt{\eta}n\rceil n \leq 4\lceil\sqrt{\eta}\rceil n^2.$$

By (3), $d_G(1) - 4\sqrt{\eta}n^2 \geq \binom{n-1}{2} - 8\sqrt{\eta}n^2 > 0$; so there exists an edge $e_1 \in E(G)\setminus V(M_1)$ such that $|e_1\cap Y| = 1$. Now suppose we have found a maximal matching $\{e_1, e_2, \ldots, e_r\}$ in $G - V(M_1)$ such that $|e_i \cap Y| = 1$ for all $i\in[r]$. If $r \geq |Y| = [n/4 - \lceil\sqrt{\eta}n\rceil]$, then $\{e_1, \ldots, e_r\}$ gives the desired matching M_2. So assume $r < |Y|$.

Write $G_r := (G - V(M_1)) - (\cup_{i=1}^r e_i)$. Let $v \in Y\backslash(V(M_1) \cup (\cup_{i=1}^r e_i))$. Note that $|[n]\backslash(Y \cup V(M_1) \cup (\cup_{i=1}^r e_i))| > n/4$. Since $d_G(v) > \binom{n-1}{2} - 4\sqrt{\eta}n^2$, the number of edges e in G with $v \in e$ and $e\backslash\{v\} \subseteq [n]\backslash(Y \cup V(M_1) \cup (\cup_{i=1}^r e_i))$ is at least

$$\binom{|[n]\backslash(Y \cup V(M_1) \cup (\cup_{i=1}^r e_i))|}{2} - 4\sqrt{\eta}n^2 > \binom{n/4}{2} - 4\sqrt{\eta}n^2 > 0.$$

So there exists an edge e_{r+1} in G_r such that $|e_{r+1} \cap Y| = 1$, contradicting the maximality of r.

Let $M_2 = \{e_1, \dots, e_{n/4-\lceil\sqrt{\eta}n\rceil}\}$. Note $V(H) \setminus (V(M_1) \cup V(M_2))$ contains $n/4 - \lceil\sqrt{\eta}n\rceil$ vertex-disjoint 2-set, say $f_1, \dots, f_{n/4-\lceil\sqrt{\eta}n\rceil}$, such that $|f_i \cap X| = 1$ for $i \in [n/4 - \lceil\sqrt{\eta}n\rceil]$. By (1), $M_2 \subseteq N_{H'}(f_i)$ for $i \in [n/4 - \lceil\sqrt{\eta}n\rceil]$. Write $M_2' = \{f_i \cup e_i : i \in [n/4 - \lceil\sqrt{\eta}n\rceil]\}$. Then $M_1 \cup M_2'$ is a perfect matching in H. ∎

5. Random rounding. We need a result of Lu, Yu, and Yuan [31] on the independence number of a subgraph of a balanced $(1,k)$-partite $(k+1)$-graph induced by a random subset of vertices. It is Lemma 4.3 in [31] which is stated for $(1,3)$-partite 4-graphs, but the same proof (which uses the hypergraph container result) also works for $(1,k)$-partite $(k+1)$-graphs.

LEMMA 5.1 (see Lu, Yu, and Yuan [31]). *Let $l, \varepsilon', \alpha_1, \alpha_2$ be positive reals, let $\alpha > 0$ with $\alpha \ll \min\{\alpha_1, \alpha_2\}$, let k, n be positive integers, and let H be a $(1,k)$-partite $(k+1)$-graph with partition classes Q, P such that $k|Q| = |P| = n$, $e(H) \geq ln^{k+1}$, and $e(H[F]) \geq \varepsilon' e(H)$ for all $F \subseteq V(H)$ with $|F \cap P| \geq \alpha_1 n$ and $|F \cap Q| \geq \alpha_2 n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then, with probability at least $1 - n^{O(1)}e^{-\Omega(n^{0.1})}$, every independent set J in $H[R]$ satisfies $|J \cap P| \leq (\alpha_1 + \alpha + o(1))n^{0.1}$ or $|J \cap Q| \leq (\alpha_2 + \alpha + o(1))n^{0.1}$.*

We also need Janson's inequality to provide an exponential upper bound for the lower tail of a sum of dependent zero-one random variables. See Theorem 8.7.2 in [7].

LEMMA 5.2 (see Janson's inequality [7]). *Let Γ be a finite set and $p_i \in [0,1]$ be a real for $i \in \Gamma$. Let Γ_p be a random subset of Γ such that the elements are chosen independently with $\mathbb{P}[i \in \Gamma_p] = p_i$ for $i \in \Gamma$. Let S be a family of subsets of Γ. For every $A \in S$, let $I_A = 1$ if $A \subseteq \Gamma_p$ and 0 otherwise. Define $X = \sum_{A \in S} I_A$, $\lambda = \mathbb{E}[X]$, and $\Delta = \frac{1}{2}\sum_{A \neq B}\sum_{A \cap B \neq \emptyset} \mathbb{E}[I_A I_B]$. Then, for $0 \leq t \leq \lambda$, we have*

$$\mathbb{P}[X \leq \lambda - t] \leq \exp\left(-\frac{t^2}{2\lambda + 4\Delta}\right).$$

Now, we use the Chernoff bound and Janson's inequality to prove a result on several properties of certain random subgraphs.

LEMMA 5.3. *Let n, k be integers such that $n \geq k \geq 3$, let H be a $(1,k)$-partite $(k+1)$-graph with partition classes A, B and $k|A| = |B| = n$, and let $A_3 \subseteq A$ and $A_4 \subseteq B$ with $|A_i| = n^{0.99}$ for $i = 3, 4$. Take $n^{1.1}$ independent copies of R, and denote them by R^i, $1 \leq i \leq n^{1.1}$, where R is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then deleting $O(n^{0.06})$ vertices so that $|R| \in (k+1)\mathbb{Z}$ and $k|R \cap A| = |R \cap B|$. For each $S \subseteq V(H)$, let $Y_S := |\{i : S \subseteq R^i\}|$. Then, with probability at least $1 - o(1)$, all of the following statements hold:*

(i) *$Y_{\{v\}} = (1 \pm n^{-0.01})n^{0.2}$ for all $v \in V(H)$.*
(ii) *$Y_{\{u,v\}} \leq 2$ for all $\{u,v\} \subseteq V(H)$.*
(iii) *$Y_e \leq 1$ for all $e \in E(H)$.*

(iv) *For all* $i = 1, \dots, n^{1.1}$, *we have* $|R_i \cap A| = (1/k \pm o(n^{-0.04}))n^{0.1}$ *and* $|R_i \cap B| = (1 \pm o(n^{-0.04}))n^{0.1}$.

(v) *Suppose* ρ *is a constant with* $0 < \rho < 1$ *such that* $d_H(\{x, v\}) \geq \binom{n-1}{k-1} - \binom{n-n/k}{k-1} - \rho n^{k-1}$ *for all* $x \in A$ *and* $v \in B$. *Then for* $1 \leq i \leq n^{1.1}$ *and for* $x \in R_i \cap A$ *and* $v \in R_i \cap B$, *we have*

$$d_{R_i}(\{x, v\}) > \binom{|R_i \cap B| - 1}{k-1} - \binom{|R_i \cap B| - |R_i \cap B|/k}{k-1} - 3\rho |R_i \cap B|^{k-1}.$$

(vi) $|R_i \cap A_j| = |A_j| n^{-0.9} \pm n^{0.06}$ *for* $1 \leq i \leq n^{1.1}$ *and* $j \in \{3, 4\}$.

Proof. For $1 \leq i \leq n^{1.1}$ and $j \in \{3, 4\}$, $\mathbb{E}[|R_i \cap A|] = n^{0.1}/k$, $\mathbb{E}[|R_i \cap B|] = n^{0.1}$ and $\mathbb{E}[|R_i \cap A_j|] = n^{-0.9}|A_j|$. Recall the assumptions $|A_3| = |A_4| = n^{0.99}$. By Lemma 3.1, we have

$$\mathbb{P}\left(\left||R_i \cap A| - n^{0.1}/k\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.02})},$$
$$\mathbb{P}\left(\left||R_i \cap B| - n^{0.1}\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.02})}, \text{ and}$$
$$\mathbb{P}\left(\left||R_i \cap A_j| - |A_j| n^{-0.9}\right| \geq n^{0.06}\right) \leq e^{-\Omega(n^{0.03})}.$$

Hence, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.02})}$, (iv) and (vi) hold.

For every $v \in V(H)$, $\mathbb{E}[Y_{\{v\}}] = n^{1.1} \cdot n^{-0.9} = n^{0.2}$. By Lemma 3.1 again,

$$\mathbb{P}\left(\left||Y_{\{v\}}| - n^{0.2}\right| \geq n^{0.19}\right) \leq e^{-\Omega(n^{0.18})}.$$

Hence, with probability at least $1 - O(n)e^{-\Omega(n^{0.18})}$, (i) holds.

For positive integers p, q, let $Z_{p,q} = |S \in \binom{V(H)}{p} : Y_S \geq q|$. Then

$$\mathbb{E}[Z_{p,q}] \leq \binom{n}{p}\binom{n^{1.1}}{q}(n^{-0.9})^{pq} \leq n^{p+1.1q-0.9pq}.$$

So $\mathbb{E}[Z_{2,3}] \leq n^{-0.1}$ and $\mathbb{E}[Z_{k,2}] \leq n^{2.2-0.8k} \leq n^{-0.2}$ for $k \geq 3$. Hence by Markov's inequality, (ii) and (iii) hold with probability at least $1 - o(1)$.

Finally we show (v). Suppose for all $x \in A$ and $v \in B$, $d_H(\{x, v\}) \geq \binom{n-1}{k-1} - \binom{n-n/k}{k-1} - \rho n^{k-1}$. We see that, for $1 \leq i \leq n^{1.1}$ and for $x \in R_i \cap A$ and $v \in R_i \cap B$,

$$\begin{aligned}\mathbb{E}[d_{R_i}(\{x, v\})] &> \binom{n-1}{k-1} n^{-0.9(k-1)} - \binom{n-n/k}{k-1} n^{-0.9(k-1)} - \rho n^{k-1} n^{-0.9(k-1)} \\ &> \binom{n^{0.1}-1}{k-1} - \binom{n^{0.1} - n^{0.1}/k}{k-1} - \rho n^{0.1(k-1)}.\end{aligned}$$

By (iv), with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.02})}$, for all $i = 1, \dots, n^{1.1}$, we have $|R_i \cap B| = (1 + o(n^{-0.04}))n^{0.1}$. Thus for all $x \in R_i \cap A$ and $v \in R_i \cap B$,

$$\mathbb{E}[d_{R_i}(\{x, v\})] > \binom{|R_i \cap B|}{k-1} - \binom{|R_i \cap B| - |R_i \cap B|/k}{k-1} - 2\rho |R_i \cap B|^{k-1}.$$

We wish to apply Lemma 5.2 with $\Gamma = B$, $\Gamma_p = R_i \cap B$, and $S = \binom{N_H(\{x,v\}) \cap B}{k-1}$. We define

$$\Delta = \frac{1}{2} \sum_{b_1, b_2 \in S, b_1 \neq b_2, b_1 \cap b_2 \neq \emptyset} \mathbb{E}[I_{b_1} I_{b_2}] \leq \frac{1}{2}|R_i \cap B|^{2k-3}$$

Thus,

$$
\begin{aligned}
&\mathbb{P}\left(d_{R_i}(\{x,v\}) \le \binom{|R_i \cap B|}{k-1} - \binom{|R_i \cap B| - |R_i \cap B|/k}{k-1} - 3\rho|R_i \cap B|^{k-1}\right)\\
\le&\mathbb{P}\left(d_{R_i}(\{x,v\}) \le \mathbb{E}[d_{R_i}(\{x,v\})] - \rho|R_i \cap B|^{k-1}\right)\\
\le&\exp(-\frac{(\rho|R_i \cap B|^{k-1})^2}{2\mathbb{E}(d_{R_i}(\{x,v\})) + 4\Delta}) \quad \text{(by Lemma 5.2)}\\
\le&\exp(-\frac{\rho^2|R_i \cap B|^{2k-2}}{2\binom{|R_i \cap B|}{k-1} + 2|R_i \cap B|^{2k-3}})\\
\le&\exp(-\Omega(n^{0.1})).
\end{aligned}
$$

Therefore, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.1})}$, (v) holds.

By applying union bound, $(i) - (vi)$ all hold with probability $1 - o(1)$. ∎

Now we prove that when the hypergraph H in Theorem 1.2 is not close to $\mathcal{H}_4(n, n/4)$ then H contains an almost regular spanning subgraph.

LEMMA 5.4. *Let $0 < \rho \ll \varepsilon \ll 1$ be reals and $n \equiv 0 \pmod 4$ be sufficiently large. Let H be a balanced $(1,4)$-partite 5-graph with partition classes $X, [n]$ such that $|X| = n/4$. Suppose that $d_H(\{x,v\}) > \binom{n-1}{3} - \binom{3n/4}{3} - \rho n^3$ for all $x \in X$ and $v \in [n]$. If H is not ε-close to $\mathcal{H}_4(n, n/4)$, then there exists a spanning subgraph H' of H such that the following conditions hold:*

(1) *For all $x \in V(H')$, with at most $n^{0.99}$ exceptions, $d_{H'}(x) = (1 \pm n^{-0.01})n^{0.2}$;*
(2) *For all $x \in V(H')$, $d_{H'}(x) < 2n^{0.2}$;*
(3) *For any two distinct $x, y \in V(H')$, $d_{H'}(\{x,y\}) < n^{0.19}$.*

Proof. Let $A_3 \subseteq X$ and $A_4 \subseteq [n]$ with $|A_i| = n^{0.99}$ for $i = 3, 4$. Let $R_1, \ldots, R_{n^{1.1}}$ be defined as in Lemma 5.3. By (iv) of Lemma 5.3 , we have, for all $i = 1, \ldots, n^{1.1}$,

$$|R_i \cap X| = (1/4 + o(n^{-0.04}))n^{0.1} \text{ and } |R_i \cap [n]| = (1 + o(n^{-0.04}))n^{0.1}.$$

By (v) of Lemma 5.3, we have, for $1 \le i \le n^{1.1}$ and for $x \in X \cap R_i$ and $v \in [n] \cap R_i$,

$$d_{R_i}(\{x,v\}) > \binom{|R_i \cap [n]|}{3} - \binom{3|R_i \cap [n]|/4}{3} - 3\rho|R_i \cap [n]|^3.$$

By (iv) and (vi) of Lemma 5.3, we may choose $I_i \subseteq R_i \cap (A_3 \cup A_4)$ such that, for $i = 1, \ldots, n^{1.1}$, $R'_i := R_i \setminus I_i$ is balanced and $|R'_i| = (1 - o(1))|R_i|$.

Let $a_1 = \varepsilon/32, a_2 = \varepsilon/512$, and $a_3 < \varepsilon(2^4 \cdot 4! \cdot 32)^{-1}$. By applying Lemma 4.1 to H, a_1, a_2, a_3, we see that H is $(\mathcal{F}, a_1)$-dense, where

$$\mathcal{F} = \{U \subseteq V(H) : |U \cap X| \ge (1/4 - a_1)n,\ |U \cap [n]| \ge (3/4 - a_2)n\}.$$

Now we apply Lemma 5.1 to H with $l = (3 \cdot 4^3 \cdot 4!)^{-1}$, $\alpha_1 = 1/4 - a_1$, $\alpha_2 = 3/4 - a_2$, and $\varepsilon' = a_1$. Therefore, with probability at least $1 - n^{O(1)}e^{-\Omega(n^{0.1})}$, for any independent set S of R'_i, $|S \cap R'_i \cap X| \le (1/4 - a_1 + o(1))n^{0.1}$ or $|S \cap R'_i \cap [n]| \le (3/4 - a_2 + o(1))n^{0.1}$.

By applying Lemma 4.3 to each $H[R'_i]$, we see that each $H[R'_i]$ contains a fractional perfect matching ω_i. Let $H^* = \cup_{i=1}^{n^{1.1}} R'_i$. We select a generalized binomial subgraph H' of H^* by letting $V(H') = V(H)$ and independently choosing edge e from $E(H^*)$, with probability $\omega_{i_e}(e)$ if $e \subseteq R'_{i_e}$. (By (iii) of Lemma 5.3, for each $e \in E(H^*)$, i_e is uniquely defined.)

Note that, since w_i is a fractional perfect matching of $H[R_i']$ for $1 \le i \le n^{1.1}$, $\sum_{e \ni v} w_i(e) \le 1$ for $v \in R_i'$. By (i) of Lemma 5.3 and by Lemma 3.1, $d_{H'}(v) = (1 \pm n^{-0.01})n^{0.2}$ for any vertex $v \in V(H) \backslash (\cup_{i=1}^{n^{1.1}} I_i)$, and $d_{H'}(v) \le (1 + n^{-0.01})n^{0.2} < 2n^{0.2}$ for vertex $v \in \cup_{i=1}^{n^{1.1}} I_i$. By (ii) of Lemma 5.3, $d_{H'}(\{x,y\}) \le 2 < n^{0.19}$ for any $\{x,y\} \in \binom{V(H)}{2}$. Therefore, H' is the desired hypergraph. ∎

We also need the following lemma attributed to Pippenger and Spencer [34] (see Theorem 4.7.1 in [7]), which extends a result of Frankl and Rödl [14].

LEMMA 5.5 (see Pippenger and Spencer [34]). *For every integer $k \ge 2$ and reals $r > 1$ and $a > 0$, there are $\gamma = \gamma(k, r, a) > 0$ and $d_0 = d_0(k, r, a)$ such that for every positive integer n and $D \ge d_0$ the following holds: Every k-uniform hypergraph $H = (V, E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the conditions that*

(1) *for all vertices $x \in V$ but at most γn of them, $d(x) = (1 \pm \gamma)D$;*
(2) *for all $x \in V$, $d(x) < rD$;*
(3) *for any two distinct $x, y \in V$, $d(x, y) < \gamma D$*

contains a matching of size at least $(1 - (k-1)a)(n/k)$.

6. Proof of Lemma 1.4.

Proof of Lemma 1.4. Let $0 < \rho' \ll \rho \ll \eta \ll \varepsilon \ll 1$. By Lemma 3.2, there exists a matching M_1 with $|M_1| \le \rho n$ such that, for any balanced set S with $|S| \le \rho' n$, $H[V(M_1) \cup S]$ has a perfect matching.

Let $H_1 = H - V(M_1)$. Then H_1 is not $(\varepsilon/2)$-close to $\mathcal{H}_4(n - 4|M_1|, n/4 - |M_1|)$. Write $n_1 = |[n] \backslash V(M_1)|$. Furthermore, for all $x \in V(H_1) \cap X$ and $v \in V(H_1) \cap [n]$,

$$d_{H_1}(\{x,v\}) \ge \binom{n-1}{3} - \binom{3n/4}{3} - 4|M_1|n^2 > \binom{n_1 - 1}{3} - \binom{3n_1/4}{3} - 10\rho n_1^3.$$

By Lemma 5.4, H_1 has a spanning subgraph H_1' such that the following conditions hold:

(1) For all $x \in V(H_1')$, with at most $n_1^{0.99}$ exceptions, $d_{H_1'}(x) = (1 \pm n_1^{-0.01})n_1^{0.2}$;
(2) For all $x \in V(H_1')$, $d_{H_1'}(x) < 2n_1^{0.2}$;
(3) For any two distinct $x, y \in V(H_1')$, $d_{H_1'}(\{x,y\}) < n_1^{0.19}$.

By Lemma 5.5, H_1' has a matching, say M_2, covering all but at most $\rho' n$ vertices. Write $S = V(H_1) \backslash V(M_2)$. Recall $H[V(M_1) \cup S]$ has a perfect matching M_1'. Now $M_1' \cup M_2$ gives the desired perfect matching. ∎

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