

Making Sense of Algebraic Expressions in Context

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Making Sense of Algebraic Expressions in Context

Making sense of algebraic expressions in context is multi-faceted and extends

beyond representing real-world situations using algebra. The Common Core State Mathematics Practice Standards (NGA Center and CCSSO 2010) speak to the importance of “attending to the meaning of *quantities*, not just how to compute them” (p. 6). We follow Smith and Thompson (2008) in defining a *quantity* as a person’s conception of measurable aspects of objects or events, such as the height of a building or the distance a car travels. Reasoning with quantities is important for being able to connect arithmetic with algebra.

As part of Project MathTalk (www.mathtalk.org), we created videos that feature a pair of Grade 9 Algebra 1 students—Haleemah and Elijah (who goes by ET)—coming to understand algebraic expressions in context. The students found three aspects of making sense of algebraic expressions challenging: (a) having quantitative meaning for arithmetic operations in algebraic expressions; (b) being able to interpret algebraic expressions from multiple perspectives, and (c) demonstrating reversibility of thought. Correspondingly, this article is organized into three major sections—one for each aspect. In each section, we elaborate the mathematical idea, describe the task given to Haleemah and ET, and present video clips of the students encountering each challenge and making progress. Lastly, we reflect on teaching actions that support the development of each aspect of making sense of algebraic expressions in context.

Quantitative Meaning for Arithmetic Operations in Algebraic Expressions

Developing meaning for arithmetic operations is important for understanding algebraic expressions in context. When working with algebraic expressions, students may be able to define their variables and explain where certain numbers come from, but they may not have adequate meaning for the arithmetic operations involved in their expressions. Beyond being an action that

one carries out to arrive at an answer, an arithmetic operation can have one or more *quantitative* meanings depending on the context.

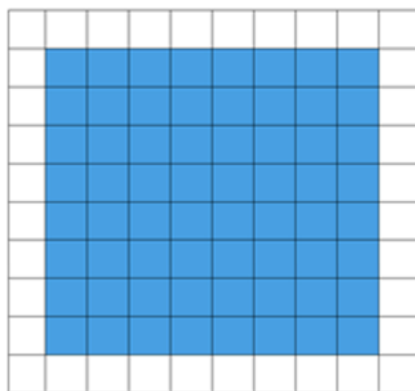
According to Thompson and Saldanha (2003), quantitative meaning of an operation is “about what one *sees*, not what one does, and about reasoning that is grounded in meaning” (p. 37). For instance, multiplication has meaning beyond the action one carries out to find a product. Thinking of multiplication as “groups of” is an example of one quantitative meaning of multiplication. In this section, we show how our two students, Haleemah and ET, initially struggled to make sense of multiplication in an algebraic equation but made progress when they developed a “groups of” conception.

The Pool Task and Students’ Initial Work

We presented Haleemah and ET with the Pool Task (figure 1), which shows a square swimming pool with a border that has 10 tiles on a side and asks students to determine the number of tiles in the border. This task is part of an instructional sequence where students use the visual representation of the pool to create an arithmetic method and then generalize the method using algebra. Haleemah and ET’s arithmetic method for finding the number of tiles in the border was $(10 \cdot 4) - 4 = 36$. They were also able to generalize this method and express it using the algebraic equation $(x \cdot 4) - 4 = B$, where x is the number of tiles on a side, and B is the total number of tiles in the border.

Fig. 1

The picture below represents a tiled swimming pool. The water is shown in blue, and the border is in white. This border has 10 tiles on one side. Determine the number of tiles in the border of the pool.



The Pool Task

Challenge

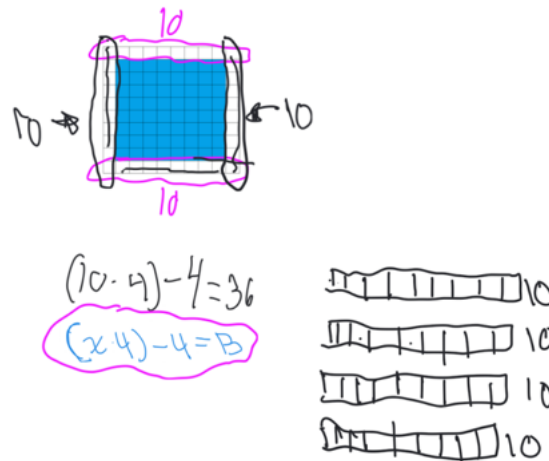
ET and Haleemah were then asked to engage in quantitative reasoning, first with the arithmetic equation $(10 \cdot 4) - 4 = 36$. When asked about where $10 \cdot 4$ was in the pool drawing, they encountered a challenge. They expressed two instances where they were not reasoning with a meaning of multiplication as “groups of.” ET said, “it’s $[10 \cdot 4]$ referring to the whole border.” Haleemah responded, “I think it’s the number of tiles on the border and the corners being multiplied twice.” In the first instance, ET was thinking that $10 \cdot 4$ is represented by the whole border. In the second instance, Haleemah referred to multiplication, but in a way that was not appropriate. Even though they were able to apply and generalize their method using algebra, they found it hard to make sense of multiplication in the expression $(10 \cdot 4)$ in the pool context.

Progress

In response, the teacher asked Haleemah and ET to draw the 40 tiles separate from the pool. In creating a drawing, they were eventually able to represent 40 tiles as four *groups of* 10 tiles (figure 2). Watch video 1 ([link online](#)) to see Haleemah and ET make meaning of the arithmetic operation of multiplication in a new way! Reflecting on the meaning of $10 \cdot 4$ in the

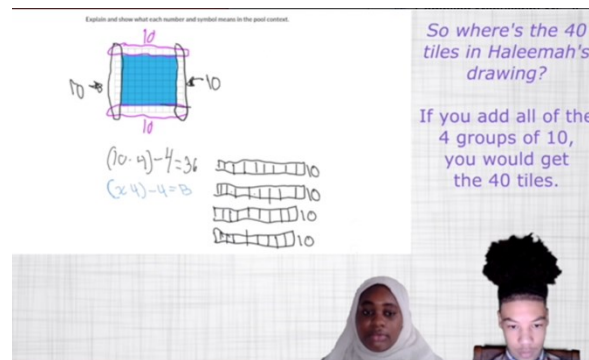
pool context, Haleemah described it as “4 groups of 10” and ET explained, “If you add all of the 4 groups of 10, you would get the 40 tiles.”

Fig. 2



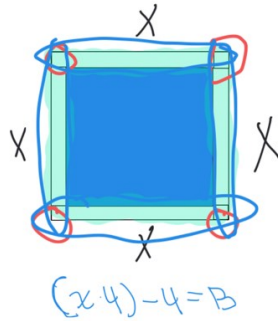
Drawing of 40 tiles separate from the pool

Video 1 Making Progress with a Drawing



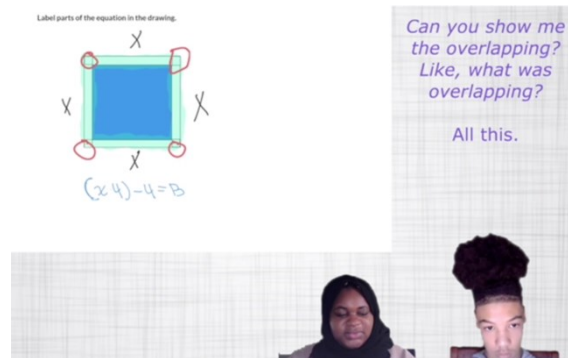
Once the students had quantitative meaning for multiplication, they were able to identify where parts of their algebra equation were represented in a new drawing (figure 3). As video 2 shows (link online), Haleemah and ET drew on their meaning for multiplication as *groups of* to explain the meaning of $x \cdot 4$ as 4 *groups of* x . This meaning allowed for a richer quantitative meaning for their entire algebraic equation, $(x \cdot 4) - 4 = B$.

Fig. 3



Drawing for the equation $(x+4)-4=B$

Video 2 Extending Meaning to Algebra



Readers may worry about interpreting $10 \cdot 4$ as 4 groups of 10 or $x \cdot 4$ as 4 groups of x tiles. After all, some textbooks follow a pedagogical convention of interpreting $10 \cdot 4$ only as 10 groups of 4. However, this convention is cultural (e.g., in China and Japan the convention is to interpret $10 \cdot 4$ as 4 groups of 10), the convention is not universal even within the U.S. (e.g., Cotter 2015), and there is no mathematical imperative for such, primarily because of the commutative property of multiplication. Consequently, we treated Haleemah and ET's 4 groups of x as one correct interpretation of $x \cdot 4$.

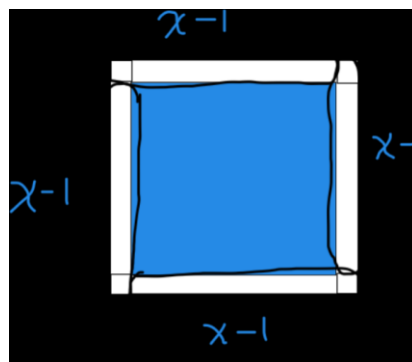
Teaching Reflection

Reflecting on video 2 from a teaching perspective, we can see that Haleemah and ET made progress after the teacher asked them to draw a picture of what the multiplication of $10 \cdot 4$ meant in the pool context. Such drawings are powerful because they require the arithmetic

operation to be illustrated in some way. In turn, this prompts students to shift from thinking of arithmetic operations as only a means of computing to considering quantitative conceptions of the operations.

In a classroom setting, students working with the Pool Task might encounter similar challenges in developing a quantitative meaning of multiplication. Consider another student-generated method of finding the number of tiles in the border: $(x-1) \cdot 4 = B$. A teacher could ask, “What does $(x-1) \cdot 4$ mean in the pool context?” A student might respond, as Haleemah and ET did, by saying that $(x-1) \cdot 4$ refers to the whole border. Another student might respond that $(x-1) \cdot 4$ means $(x-1)$ tiles and 4 sides. While both correct, neither response demonstrates a quantitative understanding of multiplication. In this situation, teachers can ask, “Would you draw a picture of a pool with x tiles on a side and label what $(x-1) \cdot 4$ means?” Employing these prompts may help students develop a *groups of* conception of multiplication and a richer understanding for the entire algebraic expression $(x-1) \cdot 4$, as 4 *groups of* $(x-1)$ tiles (figure 4).

Fig. 4



Drawing of 4 groups of $(x-1)$ tiles

Multiple Perspectives

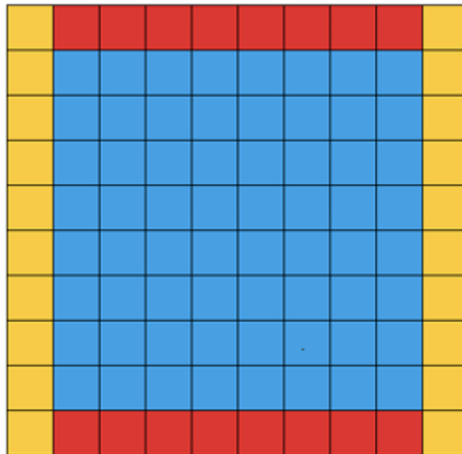
A second aspect of making sense of algebraic expressions in context is being able to interpret expressions from multiple perspectives. Sfard and Linchevski (1994) identify both

process and *object* perspectives as being important for algebraic reasoning. To characterize each perspective, consider Haleemah and ET's equation: $(x \cdot 4) - 4 = B$. From a process perspective, this equation means that the number of tiles on one side of the border, multiplied by 4, minus the 4 corner tiles, equals the total number of tiles in the border. From an object perspective, we consider the result of operating and look at the expression as an entity or whole. Here, $(x \cdot 4)$ is the collection of tiles in four groups of the number of tiles on one side, and $(x \cdot 4) - 4$ is the total number of tiles in the border. In this section, we present a challenge that Haleemah and ET encountered in shifting from a process to an object perspective for a new equation and how they made progress.

The Pool Task Revisited and Students' Initial Work

Haleemah and ET were asked to find a second method for determining the number of tiles in the border of a pool with 10 tiles on a side (figure 1). Their new method was to add the number of tiles on the left and right sides and then add the number of tiles on the top and bottom of the border, expressed by their arithmetic equation $(10 \cdot 2) + (8 \cdot 2) = 36$ (figure 5). They also successfully generalized their method and expressed it using the algebraic equation $(x \cdot 2) + (x - 2) \cdot 2 = B$.

Fig. 5



$$(10 \cdot 2) + (8 \cdot 2) = 36$$

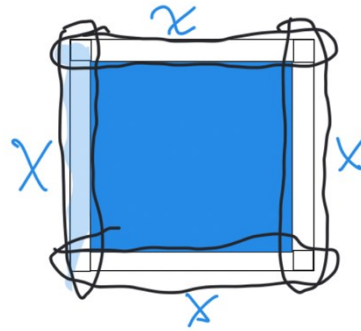
Haleemah and ET's second method

Challenge

Haleemah and ET were asked to explain the meaning of their equation in context and draw a picture (figure 6). They were able to correctly interpret the meaning of $x \cdot 2$ from both process and object perspectives (i.e., as x tiles multiplied by 2, and as the number of tiles in the left and right sides together). However, they found interpreting $x-2$ from an object perspective to be challenging, as shown in video 3 (link online). Haleemah initially interpreted $x-2$ from a process perspective. She indicated that $x-2$ consists of two parts: the number of tiles on the top of the border and the corners which are being taken away. But to label $x-2$ on the drawing, Haleemah needed to interpret $x-2$ from an object perspective, as the number of tiles left. When the teacher asked, “what are you left with [after taking away the corner tiles]?”, Haleemah struggled with the general case and explained that she was “not sure how to explain it...without the numbers.”

Fig. 6

This drawing represents a pool with any number of tiles on a side.
Label parts of the equation in the drawing.



$$(x-2) + (x-2)2 = 8$$

$$(10 \cdot 2) + (8 \cdot 2) = 36$$

Visualization of Haleemah and ET's equation in the pool context

Video 3 Puzzling Over Where $x-2$ is in a Drawing

This drawing represents a pool with any number of tiles on a side.
Label parts of the equation in the drawing.

B is the total number of tiles around the border

It is the number

$$(x-2) + (x-2)2 = 8$$

$$(10 \cdot 2) + (8 \cdot 2) = 36$$

What does the
-2 mean?

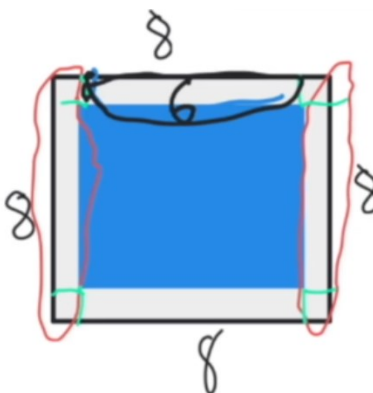
Um...

Progress

In response, the teacher asked Haleemah and ET to select a new value for the number of tiles on one side of the border for a new pool and then think about what $x-2$ means. They let $x=8$ and created the drawing shown in figure 7. They interpreted $8-2$ from a process perspective as starting with the number of tiles on top and then removing the 2 corner tiles. After some thought, Haleemah was able to also interpret $8-2$ from an object perspective as the number of tiles left

(which she labeled as “6” in her drawing; see figure 7). Then Haleemah and ET returned to the algebraic expression $x-2$.

Fig. 7



Haleemah labeled $8-2$ as 6 tiles left.

As shown in video 4 (link online), Haleemah was able to successfully interpret $x-2$ from an object perspective, as “what is left,” and label it on her drawing (figure 8). Now that Haleemah and ET can interpret $x-2$ from both process and object perspectives, they can make a richer set of connections between their algebraic equation and the pool context.

Video 4 Seeing $x-2$ as the Number of Tiles Left

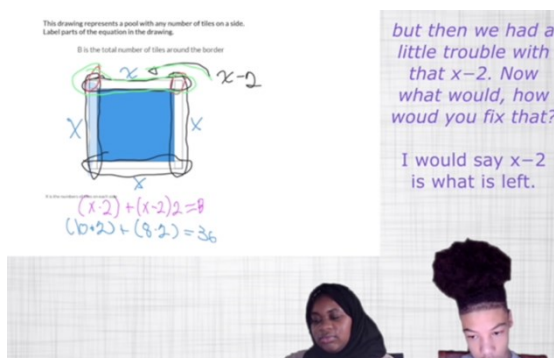
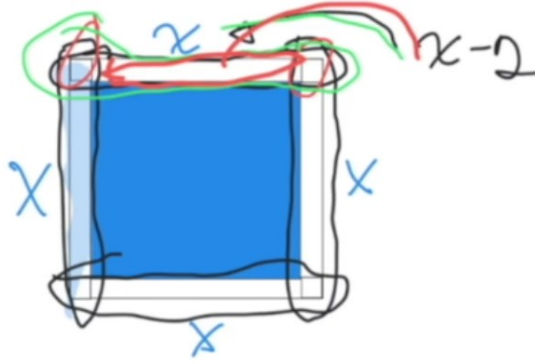


Fig. 8



Haleemah labeled $x-2$ in the drawing.

Teaching Reflection

To help Haleemah and ET develop both process and object perspectives for making meaning of algebraic expressions in context, the teacher asked them to mark where they saw $x-2$ from their equation, $(x \cdot 2) + (x-2)2 = B$ on the pool (figure 7). This question is important because being able to circle the amount of tiles represented by $x-2$ requires an object perspective. When Haleemah struggled, the teacher shifted to an easier arithmetic expression, $8-2$. To prompt the process perspective for interpreting $8-2$, the teacher asked, “What does the 8 mean in the pool context? the 2? the minus 2?” To prompt an object perspective, the teacher asked, “Once you’re done subtracting and you’ve gotten 6, what does the 6 mean in the pool context?”

Once Haleemah and ET had conceived of $8-2$ as the number of tiles left, the teacher returned to the algebraic expression and asked them to circle $x-2$ on the drawing. If students struggle, teachers can scaffold by providing a choice of quantities in context. For example, after asking students to circle the $x-2$, they can ask, “Is $x-2$ the number of tiles along one side, the corner tiles, some of the tiles on a side, or something else?” A scaffolding question provides different quantities for the student to consider without giving away the answer. Such a question also helps illuminate why using contextual tasks is important to the development of process and object perspectives. An interpretation of $x-2$ from a process perspective refers to different

quantities in context than the object perspective. This can help students differentiate between the two perspectives and can ground each perspective in quantitative meaning.

Reversibility

The final aspect of making sense of algebraic expressions in context is reversibility of thought. Reversibility refers to one's ability to take the results from an action as input and use the results to determine the actions that produced them (Steffe and Olive 2010). For example, consider the Pool Task (figure 1). Students were asked to create an arithmetic method for finding the number of tiles in the border, generalize their method, and express it using algebra.

Reversibility would be demonstrated by a student who can take a given algebraic expression, for example, $x^2 - (x-2)^2 = B$, and show by labeling a drawing of the pool with 10 tiles on a side, the corresponding method for finding the tiles in the border. If students are successful in reversing the direction, then they are demonstrating an understanding of the relationship between algebraic expressions and their meaning in context.

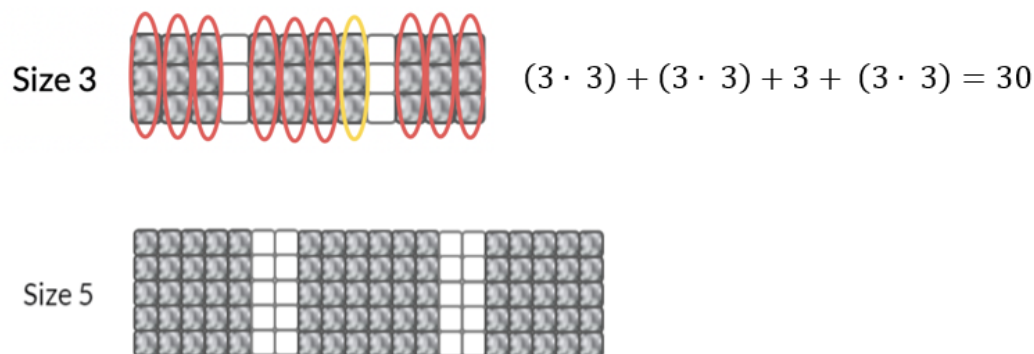
The Cobblestone Tasks and Students' Initial Work

Haleemah and ET were asked to work on the Cobblestone Task (figure 9) and then the Reverse Cobblestone Task (figure 10). Both tasks are set in the context of making cobblestone patterns for sidewalks from white and gray stones. The Cobblestone Task provides students with the arithmetic method used by another student, Amir, to find the number of gray stones in a cobblestone pattern of Size 3. It asks students to apply Amir's method to a cobblestone pattern of Size 5 and then write an algebraic equation that generalizes Amir's method for an unknown Size Number.

Fig. 9

The drawing and equation below show how Amir found the number of gray stones in a cobblestone pattern of Size 3.

- A. Apply Amir's method to find the number of gray stones in Size 5
- B. Write an algebraic equation that generalizes Amir's method of finding the number of gray stones for any unknown Size Number



The Cobblestone Task

Haleemah and ET noticed that the Size Number matches the number of columns, as well as the number of gray stones in each column, for each section (e.g., there are 3 columns of 3 gray stones circled in red in each section of the Size 3 cobblestone pattern). Seeing this relationship allowed them to use their “groups of” meaning for multiplication in a rectangular array representation and correctly apply Amir’s method to a pattern of Size 5. They also generalized Amir’s method as $(x \bullet x) + (x \bullet x) + x + (x \bullet x) = T$, where each $x \bullet x$ in their equation represented x columns of x stones.

They were then asked to work on the Reverse Cobblestone Task (figure 10). Instead of working towards an algebraic equation, this reversibility task begins with an algebraic equation from a new student, Nicole: $x^2 + x \bullet (x+1) + x^2 = T$. Haleemah and ET were asked to circle and label the Size 3 drawing to show how Nicole saw the pattern and found the number of gray stones.

Fig. 10

Below is Nicole's algebraic equation for finding the number of gray stones in a cobblestone pattern for any unknown Size Number. Circle and label the Size 3 drawing to show how you think Nicole is thinking about the gray stones.

Size 3



$$3^2 + 3 \cdot 4 + 3^2 = 30$$

Nicole's equation:

$$x^2 + x \cdot (x + 1) + x^2 = T$$

The Reverse Cobblestone Task

Challenge

We expected that Haleemah and ET would illustrate Nicole's method by using Amir's method of circling columns of stones in the Size 3 drawing. Instead, they circled rows as well as columns of stones (figure 11). Watch video 5 (link online) to see the issue they encountered when probed to describe what the multiplication would mean. ET described the multiplication in $3 \cdot 3$ as related to "the equation that they're going to use to find the total number of tiles [stones] in that one section."

Fig. 11

Size 3



$$3^2 + 3 \cdot 4 + 3^2 = 30$$

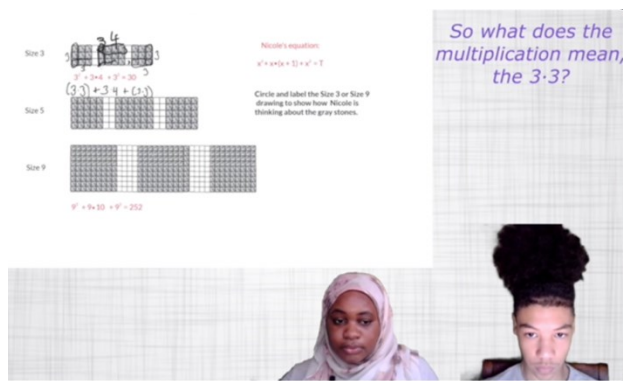
$$(3 \cdot 3) + 3 \cdot 4 + (3 \cdot 3)$$

Nicole's equation:

$$x^2 + x \cdot (x + 1) + x^2 = T$$

Seeing rows and columns in the Reverse Cobblestone Task

Video 5 Using Rows and Columns to Illustrate Nicole's Method



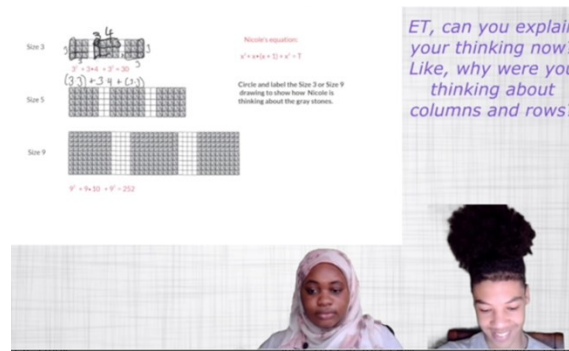
Calculationally, ET is correct. Unfortunately, he is missing quantitative meaning for the multiplication. He nicely described his thinking in terms of rows and columns of stones but carrying through this line of thinking to the operation can produce issues. What would a number of stones times a number of stones be—stones squared? Haleemah encountered a similar difficulty when she was asked to describe the middle section. She stated, “...for 3 times 4, I know, I think about it kinda like base times height.” Here, Haleemah’s meaning for multiplication is removed from the context. Previously, Haleemah and ET were able to reason quantitatively with multiplication by seeing “groups of” stones, but they seem to be approaching Nicole’s equation differently.

Progress

The students made progress after being prompted to think about their reasoning with Amir’s method. When asked how Amir was thinking about multiplication for a Size 11 cobblestone pattern, ET stated, “11 times 11 represents the 11 tiles [stones] in one column and 11 columns in one section.” Watch video 6 (link online) to see them realize that their “groups of” way of reasoning with Amir’s method could be applied to Nicole’s method. ET explained that “Amir’s method of using columns [could] still be used in Nicole’s equation.” He went on to say

that he could now see $3 \cdot 3$ as representing 3 columns with 3 stones in each column. This realization was key to applying the “groups of” meaning of multiplication.

Video 6 Reversing Their Thinking



Later this idea was extended to algebraic expressions. When asked to describe Nicole’s algebraic expression, ET said, “[the $x+1$ is] the number of columns,” and added that “ x is gonna be the Size Number” which represents the number of “stones in one column.” This breakthrough allowed them to reverse their thinking on the Reverse Cobblestone Task and show a conceptual understanding of the meaning of multiplication rooted in the context.

Teaching Reflection

The most important action teachers can take to help students develop reversible thinking is to regularly pose reversibility tasks. To create the Reverse Cobblestone Task, we simply flipped what was given and what students were asked to find from the Cobblestone Task. As a second example, reconsider the Pool Task (figure 1). The forward direction started with creating a method for finding the total number of tiles in the border by illustrating the method in the drawing of a pool and ended with representing the method using an algebra equation. To create a reversibility task, teachers can pose the Pool Task in the reverse direction. Begin with a new

algebraic equation, such as $(x-2) \cdot 4 + 4 = B$, and ask students to illustrate the pool to show a method for finding the number of tiles in the border that could be represented by the algebraic equation.

There are several steps that teachers can take to facilitate their students' progress toward reversible thinking. It is important that students are familiar with and comfortable working on a task in the forward direction. This was achieved for the Pool Task by having Haleemah and ET repeat the forward direction multiple times, each time creating a new method for finding the number of tiles in the pool and eventually representing that method by using algebra. Another way to scaffold students struggling with reversibility tasks is to remind students of their thinking during the forward direction. This was achieved on the Reverse Cobblestone Task by reminding Haleemah and ET of their work with Amir's method.

Conclusion

Teachers may find that their students experience similar challenges to Haleemah and ET in having quantitative meaning for arithmetic operations, developing multiple perspectives, and demonstrating reversible thinking. Project Mathtalk (www.mathtalk.org) has additional videos of Haleemah and ET deepening their quantitative reasoning, as well as other students making sense of algebraic expressions in context for exponential, logarithmic, and quadratic functions.

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