COUNTING INTERSECTION NUMBERS OF CLOSED GEODESICS ON SHIMURA CURVES

JAMES RICKARDS

Abstract. Let $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ correspond to the group of units of norm 1 in an Eichler order O of an indefinite quaternion algebra over $\mathbb{Q}$. Closed geodesics on $\Gamma\backslash\mathbb{H}$ correspond to optimal embeddings of real quadratic orders into O. The weighted intersection numbers of pairs of these closed geodesics conjecturally relates to the work of Darmon-Vonk on a real quadratic analogue to the difference of singular moduli. In this paper, we study the total intersection number over all embeddings of a given pair of discriminants. We precisely describe the arithmetic of each intersection, and produce a formula for the total intersection. This formula is a real quadratic analogue of the work of Gross and Zagier on factorizing the difference of singular moduli. The results are fairly general, allowing for a large class of non-maximal Eichler orders, and non-fundamental/non-coprime discriminants. The paper ends with some explicit examples illustrating the results of the paper.

Contents

1. Introduction

The $\mathrm{PSL}(2, \mathbb{Z})$-invariant $j-$function outputs algebraic numbers given quadratic imaginary inputs. These values generate certain ring class fields, and are known as singular moduli. In the celebrated work of Gross and Zagier in [GZ85], a formula for the factorization of a difference of singular moduli is given. More concretely, let D_1, D_2 be coprime negative fundamental discriminants, and they define the quantity $J(D_1, D_2)^2 \in \mathbb{Z}$,

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which is essentially the norm to $\mathbb{Q}$ of $j(\tau_1) - j(\tau_2)$, for quadratic imaginary τ_i of discriminant D_i. All primes p dividing $J(D_1, D_2)^2$ are shown to satisfy

$$p \mid \frac{D_1 D_2 - x^2}{4}$$

for some integer $x < \sqrt{D_1 D_2}$, and the exponents of such primes are computed in terms of Kronecker symbols (for more details, see [GZ85] or Section 2.5). One proof of this result involved counting endomorphisms between elliptic curves, which boiled down to an "intersection" computation on definite quaternion algebras. This work was generalized to allow for any distinct negative discriminants D_1, D_2 by Lauter and Viray in [LV15].

On the other hand, a satisfactory analogue of the difference of singular moduli for positive discriminants has remained elusive. A programme begun by Darmon and Vonk in [DV21] is to $p-$adically construct a quantity $J_p(D_1, D_2)$ for positive discriminants D_1, D_2, which is conjecturally algebraic and belonging to the compositum of ring class fields associated to D_1, D_2. Furthermore, this quantity appears to have a similar factorization to the formula of Gross and Zagier!

The conjectural analogue of $v_q\left(J(D_1, D_2)^2\right)$ is a weighted intersection number of a pair of closed geodesics on a Shimura curve. The aim of this paper is to explore these intersections in as much generality as possible. The main result is Theorem 63 (a simplified version is Theorem 10), which counts all intersections of geodesics corresponding to a pair of positive discriminants D_1, D_2. We work in a fairly general setting, and are also able to count intersections with extra arithmetic information. These formulae are a generalization of the main results in [Ric21a], which studied intersection numbers of closed geodesics in $\mathrm{PSL}(2, \mathbb{Z})\backslash\mathbb{H}$.

Along the way, we developed algorithms in PARI/GP ([PAR23]) to compute (weighted) intersection numbers of closed geodesics (see Section 2.6). These algorithms were crucial in providing evidence towards the connection with the work of Darmon and Vonk, as well as demonstrating and verifying the main counting results of this paper.

2. Overview of the paper

This section is dedicated to introducing the setup and notation required to precisely present the main results. A simplified version of the main result is Theorem 10, which adds a few conditions to achieve a nicer presentation. A detailed account of the connection to the work of Gross-Zagier and Darmon-Vonk is given in Section 2.5

2.1. General intersection numbers.

Let $\mathbb{H}$ denote the upper half plane and let Γ be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$, which acts on $\mathbb{H}$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}$. Geodesics on the orbifold $\Gamma\backslash\mathbb{H}$ are the image of geodesics on $\mathbb{H}$, and closed geodesics correspond to elements of Γ that are not a non-trivial power of another element of Γ, and have two distinct real roots. Call such elements *primitive hyperbolic.*

Let $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolic matrix. We label one root to be the first (attracting) root γ_f, and the other to be the second (repelling) root γ_s, via the equations

$$\lim_{n\to\infty} \gamma^n(x) := \gamma_f, \qquad \lim_{n\to\infty} \gamma^{-n}(x) := \gamma_s,$$

for any $x \in \mathbb{P}^1(\mathbb{R})$ that is not a root of γ. In particular, γ^{-1} has the same roots as γ, but with the first and second roots swapped.

For $z_1, z_2 \in \overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$, let ℓ_{z_1,z_2} denote the geodesic segment running from z_1 to z_2, where we do not include the endpoints z_1, z_2. Define $\dot{\ell}_{z_1,z_2} := \ell_{z_1,z_2} \cup \{z_1\}$. If $\gamma \in \Gamma$ is primitive and hyperbolic, define

$$\ell_\gamma := \ell_{\gamma_s,\gamma_f},$$

which is called the *root geodesic* of γ. For any $z \in \ell_\gamma$, the image of $\dot{\ell}_{z,\gamma z}$ in $\Gamma\backslash\mathbb{H}$ is a closed geodesic, denoted by $\tilde{\ell}_\gamma$. The image of ℓ_γ in $\Gamma\backslash\mathbb{H}$ runs over $\tilde{\ell}_\gamma$ infinitely many times.

If γ_1 is not conjugate to either γ_2 or γ_2^{-1} in Γ, then the closed geodesics $\tilde{\ell}_{\gamma_1}$ and $\tilde{\ell}_{\gamma_2}$ intersect in finitely many places. Otherwise, the geodesics completely overlap each other. To get rid of such issues, we refer to *transversal* intersections.

Definition 1. Given primitive hyperbolic matrices $\gamma_1, \gamma_2 \in \Gamma$, define

$$\tilde{\ell}_{\gamma_1} \pitchfork \tilde{\ell}_{\gamma_2}$$

to be the (finite) set of transversal intersections of $\tilde{\ell}_{\gamma_1}$ and $\tilde{\ell}_{\gamma_2}$ in $\Gamma\backslash\mathbb{H}$. Singular points (i.e. having non-trivial stabilizer in Γ) are counted with multiplicity: fix a local lift of $\tilde{\ell}_{\gamma_1}$, and the multiplicity is the number of local lifts of $\tilde{\ell}_{\gamma_2}$ that intersect transversely with the first lift. If $\tilde{\ell}_{\gamma_1}$ and $\tilde{\ell}_{\gamma_2}$ do not overlap, then this is the size of the stabilizer of the singular point.

Let f be any function defined on transversal intersections. The weighted intersection number of γ_1, γ_2 is defined to be

$$\mathrm{Int}_\Gamma^f(\gamma_1, \gamma_2) := \sum_{z \in \tilde{\ell}_{\gamma_1} \pitchfork \tilde{\ell}_{\gamma_2}} f(z).$$

In this paper, we consider the *unsigned* intersection number ($f = 1$), the *signed* intersection number ($f =$ the sign of intersection), and the *$q-$weighted* intersection number (see below Definition 7).

2.2. **Optimal embeddings.** Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$, let O be an Eichler order of level $\mathfrak{M}$, fix an embedding $\iota : B \to \mathrm{Mat}(2, \mathbb{R})$, and let $\mathcal{O}_D$ be the unique quadratic order of discriminant D (lying in $\mathbb{Q}(\sqrt{D})$). For an integer r, define

$$\mathrm{O}^r := \{z \in \mathrm{O} : \mathrm{nrd}(z) = r\},$$

the set of elements of reduced norm r in O. Note that O^1 is a group under multiplication, and let

$$\Gamma = \Gamma_{\mathrm{O}} := \iota(\mathrm{O}^1)/\{\pm 1\}$$

be the image of O^1 in $\mathrm{PSL}(2, \mathbb{R})$, a discrete subgroup.

Definition 2. An *embedding* of $\mathcal{O}_D$ into O is a ring homomorphism $\phi : \mathcal{O}_D \to \mathrm{O}$. Call the embedding *optimal* if it does not extend to an embedding of a larger order into O. Note that if D is a fundamental discriminant, then all embeddings of $\mathcal{O}_D$ into O are optimal.

When $D < 0$, call the embedding ϕ *positive definite* if $\iota(\phi(\sqrt{D}))_{2,1} > 0$ (the lower left entry of the matrix is positive), and *negative definite* otherwise. This notion corresponds to whether the first root of $\iota(\phi(\sqrt{D}))$ (defined similarly to the hyperbolic case) lies in the upper half plane or not. While the individual definitenesses depend on the choice of ι, whether two optimal embeddings of negative discriminants have the same or opposite definiteness is independent of ι.

If ϕ, ϕ' are optimal embeddings, we define them to be equivalent if there exists $x \in \mathrm{O}^1$ such that

$$\phi' = \phi^x := x\phi x^{-1}.$$

Denote the equivalence class of ϕ by $[\phi]$. The notion of equivalence can be extended to pairs of optimal embeddings as follows:

$$(\phi_1, \phi_2) \sim (\phi_1', \phi_2') \text{ if there exists an } x \in \mathrm{O}^1 \text{ such that } x\phi_i x^{-1} = \phi_i' \text{ for } i = 1, 2.$$

For a fixed discriminant D, define $\mathrm{Emb}(\mathrm{O}, D)$ to be the set of equivalence classes of optimal embeddings of $\mathcal{O}_D$ into O, which is a finite set (see Proposition 23).

If D is a positive discriminant, let $\epsilon_D > 1$ be the fundamental unit with positive norm in $\mathcal{O}_D$. If ϕ is an optimal embedding of $\mathcal{O}_D$ into O, then $\iota(\phi(\epsilon_D))$ is a primitive hyperbolic element of Γ (in fact, all primitive hyperbolic elements of Γ arise in this fashion). Define ℓ_ϕ to be $\ell_{\iota(\phi(\epsilon_D))}$.

Definition 3. Let ϕ_1, ϕ_2 be optimal embeddings of positive discriminants D_1, D_2, and let $\gamma_i = \iota(\phi_i(\epsilon_{D_i}))$ for $i = 1, 2$. For any function f defined on transversal intersections, the weighted intersection number of ϕ_1, ϕ_2 is defined to be

$$\mathrm{Int}^f_{\mathrm{O}}(\phi_1, \phi_2) := \mathrm{Int}^f_{\Gamma_{\mathrm{O}}}(\gamma_1, \gamma_2).$$

Note that the intersection number only depends on the equivalence classes of ϕ_1, ϕ_2. In Proposition 1.8 of [Ric21a], an alternate interpretation of the intersection number is given. At each transversal intersection, we can lift the point and the local geodesic to the upper half plane. This corresponds to a pair of transversely intersecting root geodesics $\ell_{\sigma_1}, \ell_{\sigma_2}$, where $\sigma_i \sim \phi_i$ for $i = 1, 2$. The obstruction to uniqueness is the choice of lifted intersection point, which is only defined up to Γ-equivalence. Equivalently, the pair (σ_1, σ_2) is only defined up to simultaneous equivalence. This is formalized in the following proposition.

Proposition 4. *An alternate interpretation of the intersection number is*

$$\mathrm{Int}^f_{\mathrm{O}}(\phi_1, \phi_2) = \sum_{\substack{(\sigma_1,\sigma_2) \in ([\phi_1] \times [\phi_2])/\sim \\ |\ell_{\sigma_1} \pitchfork \ell_{\sigma_2}| = 1}} f(\sigma_1, \sigma_2).$$

Each intersection point z gives rise to a $\Gamma-$equivalence class of points in $\mathbb{H}$, as well as a unique intersection angle, measured from the tangent to ℓ_{σ_1} at z to the tangent to ℓ_{σ_2} at z.

2.3. **x-linking.** Proposition 4 still requires ι and ϵ_D to pass to the upper half plane. To make everything contained within the quaternion algebra, we introduce the notion of $x-$linking.

Definition 5. Let x be any integer such that $x^2 \neq D_1 D_2$. Call the pair (ϕ_1, ϕ_2) $x-$linked if

$$x = \frac{1}{2}\,\mathrm{trd}(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})).$$

In particular, if (ϕ_1, ϕ_2) is $x-$linked, then every pair in the equivalence class (of simultaneous equivalence) $[(\phi_1, \phi_2)]$ is $x-$linked.

The case $x^2 = D_1 D_2$ is a degenerate case, and will not be relevant here. If (ϕ_1, ϕ_2) are $x-$linked, then $x \equiv D_1 D_2 \pmod 2$. Whether two optimal embeddings intersect is completely determined by their $x-$linking, as demonstrated in the following proposition (proven in Section 4).

Proposition 6. *Assume that (ϕ_1, ϕ_2) are $x-$linked optimal embeddings of positive discriminants D_1, D_2. Then the root geodesics $\ell_{\phi_1}, \ell_{\phi_2}$ intersect transversely if and only if*

$$x^2 < D_1 D_2.$$

In this case,

(i) The intersection point is the upper half plane root of $\iota(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}))$, and so it corresponds to an (not necessarily optimal) embedding of the negative quadratic order $\mathcal{O}_{x^2-D_1D_2}$.

(ii) The angle of intersection θ satisfies

$$\cos(\theta) = \frac{x}{\sqrt{D_1 D_2}}.$$

Define

$$\operatorname{Emb}(\mathrm{O}, \phi_1, \phi_2, x) := \{(\sigma_1, \sigma_2) : \sigma_1 \sim \phi_1, \sigma_2 \sim \phi_2, (\sigma_1, \sigma_2) \text{ are } x\text{−linked}\}/\sim,$$

the equivalence classes of $x-$linked pairs of embeddings similar to ϕ_1, ϕ_2. Going further, write

$$\begin{aligned}\operatorname{Emb}(\mathrm{O}, D_1, D_2, x) :=& \{(\sigma_1, \sigma_2) : [\sigma_i] \in \operatorname{Emb}(\mathrm{O}, D_i), (\sigma_1, \sigma_2) \text{ are } x\text{−linked}\}/\sim \\ =& \bigcup_{[\phi_i]\in\operatorname{Emb}(\mathrm{O}, D_i)} \operatorname{Emb}(\mathrm{O}, \phi_1, \phi_2, x).\end{aligned}$$

In particular, the intersection number can be rephrased without reference to ι or the fundamental units as follows:

$$\operatorname{Int}^f_{\mathrm{O}}(\phi_1, \phi_2) = \sum_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} \sum_{[(\sigma_1,\sigma_2)]\in\operatorname{Emb}(\mathrm{O},\phi_1,\phi_2,x)} f(\sigma_1, \sigma_2).$$

Thus, an intersection of ϕ_1 with ϕ_2 can be thought of as an $x-$linked pair (σ_1, σ_2), with $|x| < \sqrt{D_1 D_2}$ and $\sigma_i \sim \phi_i$.

Definition 7. Let $\sigma_1 \times \sigma_2$ denote the unique optimal embedding which satisfies

$$\sigma_1 \times \sigma_2(x + \sqrt{x^2 - D_1 D_2}) = \sigma_1(\sqrt{D_1})\sigma_2(\sqrt{D_2}).$$

The *sign* of the intersection (σ_1, σ_2), denoted $\operatorname{sg}(\sigma_1, \sigma_2)$, is 1 if $\sigma_1 \times \sigma_2$ is positive definite, and -1 otherwise (it is left undefined if $x^2 > D_1 D_2$). The level of the intersection, denoted $\ell(\sigma_1, \sigma_2)$, is $\ell > 0$ where $\sigma_1 \times \sigma_2$ is an optimal embedding of discriminant $\frac{x^2 - D_1 D_2}{\ell^2}$.

Define $\operatorname{Emb}(\mathrm{O}, D_1, D_2, x, \ell)$ to be the set of pairs of intersections in $\operatorname{Emb}(\mathrm{O}, D_1, D_2, x)$ that have level ℓ, and if $x^2 < D_1 D_2$, define $\operatorname{Emb}^+(\mathrm{O}, D_1, D_2, x, \ell)$ to be the subset of pairs that also have positive sign.

Using the notion of sign and level, we can describe three different intersection functions f

(1) When $f(\sigma_1, \sigma_2) = 1$, $\operatorname{Int}^f_{\mathrm{O}}$ is called the unsigned intersection number, and is denoted $\operatorname{Int}_{\mathrm{O}}$.
(2) When $f(\sigma_1, \sigma_2) = \operatorname{sg}(\sigma_1, \sigma_2)$, $\operatorname{Int}^f_{\mathrm{O}}$ is called the signed intersection number, and is denoted $\operatorname{Int}^{\pm}_{\mathrm{O}}$.
(3) When q is a prime and $f(\sigma_1, \sigma_2) = \operatorname{sg}(\sigma_1, \sigma_2)(1 + v_q(\ell(\sigma_1, \sigma_2)))$, $\operatorname{Int}^f_{\mathrm{O}}$ is called the $q-$weighted intersection number, and is denoted $\operatorname{Int}^q_{\mathrm{O}}$.

Remark 8. The intersection sign can equivalently be defined as the topological intersection sign of the corresponding root geodesics.

2.4. **Main result.** The ϵ function defined in Gross and Zagier ([GZ85]) is very important for $x-$linking. We recall its definition here (see Definition 43 for a slight generalization).

Definition 9. Let D_1, D_2 be coprime fundamental discriminants, and let p be a prime for which $\left(\frac{D_1 D_2}{p}\right) \neq -1$. Define

$$\epsilon(p) := \begin{cases} \left(\dfrac{D_1}{p}\right) & \text{if } p \text{ and } D_1 \text{ are coprime;} \\ \\ \left(\dfrac{D_2}{p}\right) & \text{if } p \text{ and } D_2 \text{ are coprime.} \end{cases}$$

Note that ϵ is well defined if $p \nmid D_1 D_2$ (as $1 = \left(\frac{D_1 D_2}{p}\right)$), and it is defined on all prime factors of $\frac{D_1 D_2 - x^2}{4}$.

The "holy grail" of counting intersection numbers would be to identify the constituent terms in $\mathrm{Emb}^+(\mathrm{O}, \phi_1, \phi_2, x, \ell)$, though this seems nonviable (at least with the current approach). Thus, we settle for the more general term, where we replace the embedding ϕ_i with its discriminant D_i. A simplified version of our main result is the following theorem.

Theorem 10. *Let D_1, D_2 be positive coprime fundamental discriminants, and let x be any integer such that $x \equiv D_1 D_2 \pmod 2$. Then there is precisely one quaternion algebra B over $\mathbb{Q}$ which contains a maximal order O such that there exists $x-$linked optimal embeddings from $\mathcal{O}_{D_i}$ into O. Furthermore, factorize*

$$\frac{D_1 D_2 - x^2}{4} = \pm \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where the p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. If B has discriminant $\mathfrak{D}$, then:

(i) r is even and $\mathfrak{D} = \prod_{i=1}^{r} p_i$.

(ii) The size of $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ is $2^{r+1} \prod_{i=1}^{t} (g_i + 1)$.

(iii) The set $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x, \ell)$ is non-empty if and only if

$$\ell = \prod_{i=1}^{r} p_i^{e_i} \prod_{i=1}^{s} q_i^{f_i} \prod_{i=1}^{t} w_i^{g_i'},$$

where $2g_i' \leq g_i$.

(iv) Assume the above holds, and let n be the number of indices for which $2g_i' < g_i$. Then

$$|\mathrm{Emb}(\mathrm{O}, D_1, D_2, x, \ell)| = 2^{r+n+1}.$$

If $x^2 < D_1 D_2$, then exactly half of these embeddings have positive sign.

Theorem 63 is a generalization of this result, where we allow for Eichler orders, drop the requirement of fundamentalness, and weaken the coprimality condition. By adding an additional assumption, in Corollary 68 we also consider orientations (see Section 3.4) of the optimal embedding pairs in $\mathrm{Emb}^+(\mathrm{O}, D_1, D_2, x, \ell)$.

2.5. **Connection to other work.** In [GZ85], Gross and Zagier take D_1, D_2 to be negative fundamental coprime discriminants, and define the integral quantity $J(D_1, D_2)^2$, which is essentially the norm to $\mathbb{Q}$ of $j(\tau_1) - j(\tau_2)$. Their Theorem 1.3 says

$$J(D_1, D_2)^2 = \pm \prod_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} F_{\mathrm{GZ}}\left(\frac{D_1 D_2 - x^2}{4}\right),$$

for the function

$$F_{\mathrm{GZ}}(m) = \prod_{nn' = m, n > 0} n^{\epsilon(n')}.$$

Let D_1, D_2 be positive coprime fundamental discriminants, and let O be a maximal order of an indefinite quaternion algebra B of discriminant $\mathfrak{D}$ over $\mathbb{Q}$. A consequence of Proposition 6 and Theorem 10 is that the total unsigned intersection of discriminants D_1, D_2 into O is

$$\begin{aligned} \mathrm{Int}_{\mathrm{O}}(D_1, D_2) &:= \sum_{[\phi_1] \in \mathrm{Emb}(D_1, \mathrm{O})} \sum_{[\phi_2] \in \mathrm{Emb}(D_2, \mathrm{O})} \mathrm{Int}_{\mathrm{O}}(\phi_1, \phi_2) \\ &= \sum_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} F\left(\frac{D_1 D_2 - x^2}{4}\right). \end{aligned}$$

In particular, this naturally takes the exact same form (albeit with product replaced by sum). Taking the factorization as in Theorem 10, we have

- $F\left(\frac{D_1 D_2 - x^2}{4}\right) \neq 0$ if and only if $\mathfrak{D} = \prod_{i=1}^{r} p_i$;
- If this holds, then

$$F\left(\frac{D_1 D_2 - x^2}{4}\right) = 2^{r+1} \prod_{i=1}^{t} (g_i + 1) = 2^{r+1} \sum_{d \mid \frac{D_1 D_2 - x^2}{4\mathfrak{D}}} \epsilon(d).$$

On the Gross-Zagier side, take the same factorization as above, and take ℓ to be a prime. Then

- $v_\ell\left(F_{\mathrm{GZ}}\left(\frac{D_1 D_2 - x^2}{4}\right)\right) \neq 0$ if and only if $\ell = \prod_{i=1}^{r} p_i$ (i.e. $r = 1$ and $p_1 = \ell$);
- If this holds, then

$$v_\ell\left(F_{\mathrm{GZ}}\left(\frac{D_1 D_2 - x^2}{4}\right)\right) = (e_1 + 1) \prod_{i=1}^{t} (g_i + 1) = (e_1 + 1) \sum_{d \mid \frac{D_1 D_2 - x^2}{4\ell}} \epsilon(d).$$

In particular, this cements the analogy between the two situations.

Analogy 11. The total intersection number of positive discriminants, $\mathrm{Int}_{\mathrm{O}}(D_1, D_2)$, behaves like the exponents of primes in the factorization of $J(D_1, D_2)^2$ for negative discriminants.

The individual intersection numbers $\mathrm{Int}_{\mathrm{O}}(\phi_1, \phi_2)$ should then have an analogy involving the exponents of primes above ℓ in the factorization of $j(\tau_1) - j(\tau_2)$. To make such a connection concrete, we require a real quadratic analogue of $j(\tau_1) - j(\tau_2)$, and not just the exponents. This connection is the goal of Darmon-Vonk in [DV21].

In this work, given τ_1, τ_2 real quadratic points corresponding to coprime fundamental discriminants D_1, D_2 and a prime $p \leq 13$, they construct a p-adic quantity $J_p(D_1, D_2)$, which is conjecturally algebraic and belonging to the compositum of ring class fields associated to D_1, D_2.

Conjecture 12 (Conjecture 4.26 of [DV21]). *Let $\mathfrak{q}$ lie above the integer prime $q \neq p$. If q is split in $\mathbb{Q}(\sqrt{D_1})$ or $\mathbb{Q}(\sqrt{D_2})$, then $\operatorname{ord}_{\mathfrak{q}}(J_p(\tau_1, \tau_2)) = 0$. Otherwise, let O be a maximal order in the quaternion algebra ramified at p, q. Then there exist optimal embeddings ϕ_1, ϕ_2 of discriminants D_1, D_2 into O for which*

$$\operatorname{ord}_{\mathfrak{q}}(J_p(\tau_1, \tau_2)) = \operatorname{Int}_{\mathrm{O}}^q(\phi_1, \phi_2).$$

In other words, the exponents of primes above q in the factorizations of $J_p(\tau_1, \tau_2)$ are given by $q-$weighted intersection numbers associated to optimal embeddings of D_1, D_2 into a maximal order in the indefinite quaternion algebra ramified at p, q.

Besides the compelling analogy between Gross-Zagier, Darmon-Vonk, and this work, we have extensive computational evidence. I computed the intersection numbers $\operatorname{Int}_{\mathrm{O}}^q(\phi_1, \phi_2)$ for all pairs with $D_1 = 5, 13$ and $D_2 \leq 1000$, and compiled it into a 600 page document. On the other side, Jan Vonk computed the $q-$adic valuations of $J_p(\tau_1, \tau_2)$ for many of these examples, and the data matched perfectly.

2.6. **Computational aspects.** Everything described in this paper has been implemented by the author in PARI/GP ([PAR23]), and the corresponding package can be found on GitHub at [Ric21c]. In particular, this includes algorithms to:

- Initialize a quaternion algebra B over $\mathbb{Q}$ of a specified ramification, as well as an Eichler order O of a given level;
- Compute representatives of the equivalence classes in $\operatorname{Emb}(\mathrm{O}, D)$, divide them into classes by their orientation, and sort these classes by the action of $\operatorname{Cl}^+(D)$;
- Compute the sets $\operatorname{Emb}(\mathrm{O}, \phi_1, \phi_2, x)$, and the corresponding signs and levels.
- Compute all non-trivial unsigned, signed, and $q-$weighted intersection numbers of a given pair of discriminants D_1, D_2.

As mentioned in the last section, these computations were essential to establishing the connection to the work of Darmon and Vonk.

2.7. **Plan of attack.** Section 3 recalls and proves some basic results on quaternion algebras that will be useful later. Section 4 covers some basic results on intersection numbers. Section 5 studies the conditions on which there exist $x-$linked optimal embeddings of a given pair of discriminants. In Section 6, we count the Eichler orders containing a given pair of $x-$linked embeddings. Section 7 assembles all of the ingredients to prove the generalization of Theorem 10. The paper ends by providing some explicit examples demonstrating the main results.

3. Quaternionic background

In this section we recall properties of quaternion algebras and Eichler orders that are required in Sections 4 and beyond. The main focus will be on optimal embeddings. For a full exposition on quaternion algebras, see [Voi21].

3.1. **Local and global quaternion algebras.** Let F be a field of characteristic 0, and $a, b \in F^\times$. Take $B = \left(\frac{a,b}{F}\right)$ to be the quaternion algebra associated to a, b, F. As an additive vector space, this is of dimension 4 over F, with basis $1, i, j, k$, and general element of the form

$$x = e + fi + gj + hk, \text{ where } e, f, g, h \in F.$$

The multiplicative structure is determined by the standard equations

$$i^2 = a, \qquad j^2 = b, \qquad k = ij = -ji.$$

The standard involution on B is denoted by an overline, and explicitly defined by

$$\overline{x} := e - fi - gj - hk.$$

The quaternion algebra also comes equipped with the reduced trace $\mathrm{trd} : B \to F$ and the reduced norm $\mathrm{nrd} : B \to F$, defined by

$$\begin{aligned}\mathrm{nrd}(x) :=& x\overline{x} = e^2 - af^2 - bg^2 + abh^2;\\ \mathrm{trd}(x) :=& x + \overline{x} = 2e.\end{aligned}$$

When $F = \mathbb{R}$, there are exactly two quaternion algebras up to isomorphism: $\mathrm{Mat}(2, \mathbb{R})$, and the Hamilton quaternions $\left(\frac{-1,-1}{\mathbb{R}}\right)$ (which is a division algebra). Similarly, over $\mathbb{Q}_p$, there are two quaternion algebras up to isomorphism: $\mathrm{Mat}(2, \mathbb{Q}_p)$, and a division algebra. The division algebra can be written as $\left(\frac{p,e}{\mathbb{Q}_p}\right)$, where e is any integer such that $\left(\frac{e}{p}\right) = -1$, and $\left(\frac{\cdot}{p}\right)$ is the Kronecker symbol.

Let $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a quaternion algebra over $\mathbb{Q}$. Much of the structure of B is determined by its local behaviour, i.e. the local quaternion algebras $B_v = B \otimes \mathbb{Q}_v = \left(\frac{a,b}{\mathbb{Q}_v}\right)$, where v is a place of $\mathbb{Q}$ and $\mathbb{Q}_\infty = \mathbb{R}$. Call v *ramified* in B if B_v is division, and call v *split* otherwise. Define the Hilbert symbol $(a, b)_v$ to be 1 if p is split in B, and -1 if B is ramified. The set of ramified places is both finite and of even size, and we say that B has discriminant $\mathfrak{D}$, where $\mathfrak{D}$ is the product of all ramifying places.

The quaternion algebra B over $\mathbb{Q}$ is uniquely determined (up to isomorphism) by the set of ramifying places, and furthermore, any finite even sized set of places corresponds to a quaternion algebra over $\mathbb{Q}$. We call B indefinite if ∞ is split, hence B is ramified at an even number of finite primes. We will generally be working with indefinite quaternion algebras over $\mathbb{Q}$, although some results work in more generality.

An order O of B is a lattice that is also a subring. A maximal order is an order which is not properly contained within another order. All maximal orders of $\mathrm{Mat}(2, \mathbb{Q}_p)$ are conjugate, whereas the division quaternion algebra over $\mathbb{Q}_p$ has a unique maximal order, consisting of all integral elements. Globally, all maximal orders in an indefinite quaternion algebra over $\mathbb{Q}$ are conjugate.

If $F = \mathbb{Q}, \mathbb{Q}_p$, then an order O is always a dimension four $\mathcal{O}_F = \mathbb{Z}, \mathbb{Z}_p-$module (respectively). Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be a basis of O, and define the discriminant of O to be

$$\mathrm{disc}(\mathrm{O}) = -d(\alpha_1, \alpha_2, \alpha_3, \alpha_4) := -\det(\mathrm{trd}(\alpha_i \alpha_j)_{i,j}).$$

This is always a square, and the reduced discriminant of O is defined by

$$\mathrm{discrd}(\mathrm{O})^2 = \mathrm{disc}(\mathrm{O}).$$

The reduced discriminant is only defined up to $\mathcal{O}_F^\times$, so over $\mathbb{Q}$ we take the convention that it is positive, and over $\mathbb{Q}_p$ we take it to be of the form p^e with $e \geq 0$. It follows that if $F = \mathbb{Q}$, then

$$\mathrm{discrd}(\mathrm{O}) = \prod_p \mathrm{discrd}(\mathrm{O}_p),$$

where the product is taken over all primes p and $\mathrm{O}_p = \mathrm{O} \otimes \mathbb{Z}_p$ is the corresponding local order in B_p.

Over $F = \mathbb{Q}$, an order is maximal if and only if its reduced discriminant is equal to the finite part of $\mathfrak{D}$, the discriminant of the quaternion algebra. A general order O will have $\operatorname{discrd}(\mathrm{O}) = \mathfrak{D}\mathfrak{M}$, where $\mathfrak{M}$ is called the level of the order.

Working locally will be essential, so we will state the local-global correspondence for lattices (which also holds for orders in a quaternion algebra).

Theorem 13 (Variant of Theorem 9.1.1 of [Voi21])**.** *Let V be a finite dimensional $\mathbb{Q}-$vector space, and let $M \subset V$ be a $\mathbb{Z}$-lattice. Then the map $N \to (N_p)_p$ gives a bijection between $\mathbb{Z}-$lattices $N \subset V$ and collections of $\mathbb{Z}_p$-lattices $(N_p)_p$ indexed by the primes which satisfy $M_p = N_p$ for all but finitely many primes p.*

An Eichler order O of B is an order that is the intersection of two (uniquely determined) maximal orders. Over $F = \mathbb{Q}_p$, if B is division there is exactly one maximal order, hence this is the only Eichler order. Otherwise, $B = \mathrm{Mat}(2, \mathbb{Q}_p)$, and there exist Eichler orders of levels p^e for all $e \geq 0$. They are all conjugate, and we define the standard Eichler order of level p^e to be

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^e\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Following the local-global principle of orders, when $F = \mathbb{Q}$, an order O is Eichler if and only if O_p is Eichler for all primes p.

Furthermore, if B is indefinite, a consequence of strong approximation is that all Eichler orders of the same level are conjugate over $B^\times$.

3.2. **Normalizer of an Eichler order.** Take B to be a quaternion algebra over $F = \mathbb{Q}$ or $F = \mathbb{Q}_p$.

Definition 14. Let O be an order in B, and define the subgroup of $x \in B^\times$ for which $x\mathrm{O}x^{-1} = \mathrm{O}$ to be $N_{B^\times}(\mathrm{O})$, the normalizer group of the order O.

Clearly $F^\times \mathrm{O}^\times \subseteq N_{B^\times}(\mathrm{O})$. As we will see in Proposition 17, this is a finite index subgroup.

Lemma 15. *Let $x \in B - F$. Then the set $C_B(x) := \{v \in B : vx = xv\}$ is an $F-$algebra and a two dimensional $F-$vector space spanned by $1, x$. We call it the centralizing algebra of x.*

Proof. This follows immediately from Proposition 7.7.8 of [Voi21]. ☐

Corollary 16. *Let $x_1, x_2 \in B^\times - F^\times$ have the same separable minimal polynomial. Then the set $C_B(x_1, x_2) := \{v \in B : vx_1 = x_2v\}$ is a two dimensional $F-$vector space.*

Proof. By Corollary 7.7.3 of [Voi21], the equality of the minimal polynomials of x_1, x_2 implies that there exists a $w \in B^\times$ with $wx_1w^{-1} = x_2$. Thus

$$v \in C_B(x_1, x_2) \Leftrightarrow vw^{-1}x_2 = x_2vw^{-1},$$

so the corollary follows from Lemma 15. ☐

We now describe the normalizer groups of Eichler orders over $\mathbb{Q}_p$.

Proposition 17. *Let B be a quaternion algebra over $\mathbb{Q}_p$ with Eichler order* O*. If B is division, we have*

$$N_{B^\times}(\mathrm{O}) = B^\times.$$

Otherwise, write $B = \mathrm{Mat}(2, \mathbb{Q}_p)$ *and take* $\mathrm{O} = \left(\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^e\mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$. *Let* $\omega := \left(\begin{smallmatrix} 0 & 1 \\ -p^e & 0 \end{smallmatrix}\right)$, *and then*

$$N_{B^\times}(\mathrm{O}) = \mathbb{Q}_p^\times \mathrm{O}^\times \langle \omega \rangle.$$

Proof. If B is division, then O is the unique maximal order. Since conjugates of O are also maximal orders, it is stabilized under conjugation by all of $B^\times$. When B is not division, this is Proposition 23.4.14 of [Voi21] (the definition of ω has been adjusted so that it has positive norm). $\square$

Note that if B is not division and O is maximal, then $N_{B^\times}(\mathrm{O}) = \mathbb{Q}_p^\times \mathrm{O}^\times$. Translating the above proposition into the global case yields the following proposition.

Proposition 18. *Let* B *be an indefinite quaternion algebra over* $\mathbb{Q}$ *with discriminant* $\mathfrak{D}$, *and let* O *be an Eichler order of* B *of level* $\mathfrak{M}$. *Then there exists a collection of elements* $\{\omega_p : p \mid \mathfrak{D}\mathfrak{M}\infty\}$ *with* $\mathrm{nrd}(\omega_p) = p^{v_p(\mathfrak{D}\mathfrak{M})}$ *for* $p < \infty$ *and* $\mathrm{nrd}(\omega_\infty) = -1$ *for which*

$$\frac{N_{B^\times}(\mathrm{O})}{\mathbb{Q}^\times \mathrm{O}^1} = \langle \omega_p \rangle_{p \mid \mathfrak{D}\mathfrak{M}\infty} \simeq \prod_{p \mid \mathfrak{D}\mathfrak{M}\infty} \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Proof. By combining Proposition 18.5.3 and Equation 23.4.20 of [Voi21] with the fact that O has class number one, we get the isomorphism

$$\frac{N_{B^\times}(\mathrm{O})}{\mathbb{Q}^\times \mathrm{O}^\times} \simeq \prod_{p \mid \mathfrak{D}\mathfrak{M}} \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

By taking a set of generators and looking locally, we can use Proposition 17 to show that we can find an equivalent set of generators $\{\omega_p\}_{p \mid \mathfrak{D}\mathfrak{M}}$ which satisfy $\mathrm{nrd}(\omega_p) = p^{v_p(\mathfrak{D}\mathfrak{M})}$ for $p < \infty$. Finally, we can pull out the ∞ by using $\mathrm{O}^\times = \mathrm{O}^1 \cup \mathrm{O}^{-1}$, and $\mathrm{O}^{-1} = \omega_\infty \mathrm{O}^1$ for any $\omega_\infty \in \mathrm{O}^{-1}$. $\square$

3.3. **Towers of Eichler orders.** The Bruhat-Tits tree provides a combinatorial aspect to the theory of maximal/Eichler orders of $B = \mathrm{Mat}(2, \mathbb{Q}_p)$. Vertices of the graph are maximal orders in B, and there exists an edge between O and O' if and only if $\mathrm{O} \cap \mathrm{O}'$ is an Eichler order of level p. A summary of the main facts of the graph (see Section 23.5 of [Voi21]) are:

- The graph is connected and has no cycles, hence it is a tree (as the name implies);
- Every vertex has degree $p + 1$;
- Let $\mathrm{O}_1, \mathrm{O}_2$ be maximal orders, and let $\mathrm{O} = \mathrm{O}_1 \cap \mathrm{O}_2$ be the corresponding Eichler order of level p^e. Then O corresponds to the unique path between O_1 and O_2. This path has length e, and the vertices on the path are precisely the $e + 1$ maximal orders which contain O.

Focusing on one Eichler order O of level p^e, we define the "inverted triangle" of superorders of O as follows:

- It is a graph consisting of all (necessarily Eichler) superorders $\mathrm{O}' \supseteq \mathrm{O}$ as vertices;
- The vertices are arranged into $e + 1$ rows, where the i^{th} row from the top (starting with row 0) consists of the Eichler orders of level p^i containing O.
- There is an edge between orders $\mathrm{O}_1, \mathrm{O}_2$ if and only if one order contains the other and they are in adjacent rows.

It follows directly from the Bruhat-Tits tree that there are $e + 1 - i$ vertices in the i^{th} row, and the graph can be drawn in the plane so that each vertex (besides those in row 0) is connected to the two closest vertices in the row above it. An Eichler order is the intersection of the two orders it is connected to in the above row. As an example, the inverted triangle for an Eichler order of level p^5 is displayed in Figure 1.

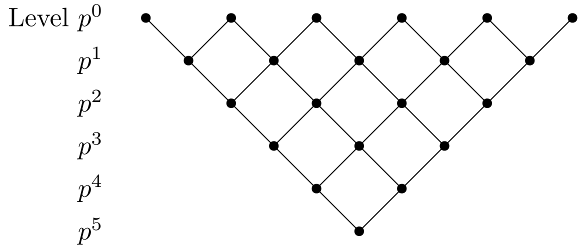

FIGURE 1. Inverted triangle of level p^5.

The inverted triangle of O allows one to count superorders of O of a specified level which do not contain certain given superorders (which is required in Section 6.2).

Remark 19. The inverted triangle of O is essentially the same concept as branches of orders, as found in [AC13] and [AACC18].

3.4. **Optimal embeddings.** Let B be a quaternion algebra over $F = \mathbb{Q}$ or $F = \mathbb{Q}_p$, and let O be an order in B. If $F = \mathbb{Q}_p$, we call an embedding of $\mathcal{O}_D$ into O optimal if it does not extend to an embedding of $\mathcal{O}_{D/p^2}$ (which is automatic if D/p^2 is not a discriminant). In particular, if $F = \mathbb{Q}$, an embedding ϕ into O is optimal if and only if the corresponding embeddings ϕ_p into O_p are optimal for all primes p.

Definition 20. For a discriminant D, define $p_D \in \{0, 1\}$ to be the parity of D. Let the field discriminant of $\mathbb{Q}(\sqrt{D})$ be D^{fund}.

Since $\mathcal{O}_D = \mathbb{Z}\left[\frac{p_D+\sqrt{D}}{2}\right]$, an embedding of $\mathcal{O}_D$ into O is equivalent to picking an element $x = \phi\left(\frac{p_D+\sqrt{D}}{2}\right) \in$ O which has the same characteristic polynomial as $\frac{p_D+\sqrt{D}}{2}$, i.e. an element x satisfying $x^2 - p_D x + \frac{p_D - D}{4} = 0$.

In certain proofs, it will be useful to assume that an optimal embedding takes a certain form. Corollary 22 allows us to do this.

Lemma 21 (Exercise 2.5 of [Voi21])**.** *Let B be a quaternion algebra over a field F of characteristic not equal to 2, and assume $x \in B \backslash F$ satisfies $x^2 = n \in F^\times$. Then there exists an $m \in F^\times$ and an isomorphism $\theta : B \to \left(\frac{n,m}{F}\right)$ satisfying $\theta(x) = i$.*

Proof. Consider the inner product defined as $\langle u, v\rangle = \frac{1}{2}\operatorname{trd}(u\overline{v})$. Pick any y such that B is generated as an $F-$algebra by x, y, and by applying the Gram-Schmidt orthogonalization process, we can assume that $0 = \langle 1, y\rangle = \langle x, y\rangle$. This implies that $y^2 = m \in F^\times$ and $xy = -yx$, whence we have the result. □

Corollary 22. *Let $\phi : \mathcal{O}_D \to \mathrm{O}$ be an (optimal) embedding into an order of the quaternion algebra B. Then there exists a quaternion algebra B' with order O' and an isomorphism $\theta : B \to B'$ taking O to O' such that $\theta \circ \phi : \mathcal{O}_D \to \mathrm{O}'$ is an (optimal) embedding with $\theta \circ \phi(\sqrt{D}) = i_{B'}$. In particular, given an (optimal) embedding, we can choose coordinates so that the image of $\sqrt{D}$ is i.*

Proof. Take $x = \phi(\sqrt{D})$ in Lemma 21, and consider the corresponding map θ. Let $\mathrm{O}' = \theta(\mathrm{O})$, and then O' is an isomorphic order for which $\theta \circ \phi$ is an (optimal) embedding into. □

We would like to count the set $\mathrm{Emb}(\mathrm{O}, D)$, and Chapter 30 of [Voi21] provides an excellent exposition of this in a more general context. We now restate the relevant results in our setting, and expand upon the notion of equivalence classes of the localized embeddings (which we refer to as orientation). If O is an Eichler order in a quaternion algebra B over $\mathbb{Q}_p$ or $\mathbb{R}$ ($\mathrm{O} = B$ if over $\mathbb{R}$), define $\mathrm{Emb}(\mathrm{O}, D)$ analogously to over $\mathbb{Q}$ (Section 2.2).

Proposition 23. *Let D be a discriminant, let B an indefinite quaternion algebra over $\mathbb{Q}$, and let O an Eichler order. Let $h^+(D)$ denote the narrow class number of discriminant D. Then,*

$$|\,\mathrm{Emb}(\mathrm{O}, D)| = h^+(D)\prod_v |\,\mathrm{Emb}(\mathrm{O}_v, D)|,$$

where the product is over all places of $\mathbb{Q}$.

Proof. The class number of any Eichler order over $\mathbb{Q}$ is one, and the result then follows from Theorem 30.7.3 of [Voi21] and the surrounding results. See also Sections 4.4 and 4.5 of [Ric21b] for an alternate presentation. □

In particular, $|\,\mathrm{Emb}(\mathrm{O}, D)|$ is $h^+(D)$ up to local factors. The local factors are as follows.

Proposition 24. *Let D be a discriminant, and let B a quaternion algebra over $\mathbb{R}$ or $\mathbb{Q}_p$.*

(i) If $B = \mathrm{O} = \mathrm{Mat}(2, \mathbb{R})$, then

$$|\,\mathrm{Emb}(\mathrm{O}, D)| = 1 + \mathbb{1}_{D<0}.$$

(ii) If B is division over $\mathbb{Q}_p$ with maximal order O, then

$$|\,\mathrm{Emb}(\mathrm{O}, D)| = \begin{cases} 0 & \text{if } p^2 \mid \frac{D}{D^{\mathrm{fund}}}; \\ 1 - \left(\dfrac{D}{p}\right) & \text{else.} \end{cases}$$

(iii) If $B = \mathrm{Mat}(2, \mathbb{Q}_p)$, O is an Eichler order of level p^e, and $\gcd(p^e, D) = 1$, then

$$|\,\mathrm{Emb}(\mathrm{O}, D)| = \begin{cases} 1 & \text{if } e = 0; \\ 1 + \left(\dfrac{D}{p}\right) & \text{if } e > 0. \end{cases}$$

Proof. The first part follows easily from the Skolem-Noether theorem. The second part follows from Proposition 30.5.3 of [Voi21] in the case of $p^2 \nmid \frac{D}{D^{\mathrm{fund}}}$. Otherwise, since O is the set of all integral elements in B, any embedding of $\mathcal{O}_D$ extends to an embedding of $\mathcal{O}_{D^{\mathrm{fund}}}$. The third part follows from Propositions 30.5.3 and 30.6.12 of [Voi21]. □

The above proposition omits the case of $B = \mathrm{Mat}(2, \mathbb{Q}_p)$, O is an Eichler order of level p^e with $e > 0$ and $p \mid D$. This case is much more complicated, and its description will not be of use to us. If desired, see Lemma 30.6.17 of [Voi21] for the details.

Definition 25. Assume B is indefinite over $\mathbb{Q}$, and let ϕ be an optimal embedding into an Eichler order O. For all places v, let $o_v(\phi)$ denote the local equivalence class of ϕ_v. The orientation of ϕ is defined to be

$$o(\phi) := (o_v(\phi))_v : v \text{ is a place},$$

the set of equivalence classes of the corresponding local embeddings.

If $\gcd(D, \mathfrak{M}) = 1$, then all local embedding equivalence classes have size either 1 or 2. In particular, write $o_v(\phi) = 0$ or $o_v(\phi) = \pm 1$ for the one or two local equivalence classes (this is non-canonical and depends on an initial choice when there are two local classes).

Definition 26. For each orientation o of an optimal embedding of $\mathcal{O}_D$ into O, we denote by $\mathrm{Emb}_o(\mathrm{O}, D)$ the equivalence classes of optimal embeddings with orientation o.

Note that we can restrict the orientation to places $p \mid \mathfrak{D}\mathfrak{M}\infty$, since Proposition 24 implies that there is one local orientation at all other places. At those places, it will be useful to have a more explicit way to determine orientation.

Lemma 27. *Let B be a quaternion algebra over $\mathbb{Q}_p$ with Eichler order O of level $\mathfrak{M}$, let D be a discriminant, and let $\phi : \mathcal{O}_D \to \mathrm{O}$ be an optimal embedding.*

(i) If B is division, let $\mathfrak{p}$ be the maximal ideal of O. Then the orientation of ϕ is determined by $\phi\left(\frac{p_D+\sqrt{D}}{2}\right) \pmod{\mathfrak{p}}$.

(ii) If $B = \mathrm{Mat}(2, \mathbb{Q}_p)$, O is the standard Eichler order of level p^e with $e > 0$, and $p \nmid D$, then the orientation of ϕ is determined by $\phi\left(\frac{p_D+\sqrt{D}}{2}\right)_{1,1} \pmod{p^e}$.

Proof. If B is division, then $\mathrm{O}/\mathfrak{p} \simeq \mathbb{F}_{p^2}$ is commutative. Thus, equivalent embeddings give the same value of $\phi\left(\frac{p_D+\sqrt{D}}{2}\right) \pmod{\mathfrak{p}}$. If $p \mid D$ we are done, and otherwise, note that $\overline{\phi}$ (defined by $\overline{\phi}(x) := \overline{\phi(x)}$) is an optimal embedding with

$$\overline{\phi}\left(\frac{p_D+\sqrt{D}}{2}\right) \not\equiv \phi\left(\frac{p_D+\sqrt{D}}{2}\right) \pmod{\mathfrak{p}},$$

since this is equivalent to $\phi(\sqrt{D}) \not\equiv 0 \pmod{\mathfrak{p}}$. As there are two equivalence classes of optimal embeddings, it follows that the class is determined by $\phi\left(\frac{p_D+\sqrt{D}}{2}\right) \pmod{\mathfrak{p}}$.

If $B = \mathrm{Mat}(2, \mathbb{Q}_p)$, then a direct computation shows that $\phi\left(\frac{p_D+\sqrt{D}}{2}\right)_{1,1} \equiv \phi^u\left(\frac{p_D+\sqrt{D}}{2}\right)_{1,1} \pmod{p^e}$ for $u \in \mathrm{O}^1$ (see Equations (7.1) and (7.2) for this computation). As in the previous case, $\overline{\phi}$ is an optimal embedding with

$$\overline{\phi}\left(\frac{p_D+\sqrt{D}}{2}\right) \not\equiv \phi\left(\frac{p_D+\sqrt{D}}{2}\right) \pmod{p^e},$$

since $p \nmid D$. As there are two equivalence classes of optimal embeddings, it follows that the class is determined by $\phi\left(\frac{p_D+\sqrt{D}}{2}\right)_{1,1} \pmod{p^e}$. □

For $p \mid \mathfrak{D}\mathfrak{M}\infty$, we can use the elements $\omega_p \in N_B^\times(\mathrm{O})$ as described in Proposition 18 to pass between orientations.

Proposition 28. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$ with Eichler order O of level $\mathfrak{M}$, and let $\phi : \mathcal{O}_D \to \mathrm{O}$ be an optimal embedding. Then we have*

- *$o_v(\phi^{\omega_p}) = o_v(\phi)$ for all places $v \neq p$;*
- *$o_p(\phi^{\omega_p}) = -o_p(\phi)$ if $p \nmid \gcd(D, \mathfrak{M})$.*

In other words, the optimal embedding ϕ^{ω_p} only swaps orientation at p.

Proof. If $v \mid \mathfrak{D}$, let $\mathfrak{v}$ be the maximal order of O_v. Since $\mathrm{O}/\mathfrak{v} \simeq \mathbb{F}_{v^2}$ is commutative, the result follows for $p \neq v$ as $\mathrm{nrd}(\omega_p) \in \mathbb{Z}_v^\times$. If $p = v$, then we can assume that $p \nmid D$, as the result is trivial otherwise. By

Proposition 24, $\left(\frac{D}{p}\right) = -1$, whence we can write $B_p = \left(\frac{p, D}{\mathbb{Q}_p}\right)$. It suffices to prove the proposition for $\omega_p = i$ and $\phi_p(\sqrt{D}) = j$, and we indeed find that

$$\phi_p^i\left(\frac{p_D + \sqrt{D}}{2}\right) = \frac{p_D + iji^{-1}}{2} = \frac{p_D - j}{2} \not\equiv \frac{p_D + j}{2} = \phi_p\left(\frac{p_D + \sqrt{D}}{2}\right) \pmod{\mathfrak{p}}.$$

By Lemma 27, the embeddings have opposite orientation.

Next, take $v \mid \mathfrak{M}$, and assume that O_v is the standard Eichler order of level v^e. If $p \neq v$, then the computations in Equations (7.1) and (7.2) still hold true, and so we are done by Lemma 27. If $p = v$, then it suffices to take $\omega_p = \left(\begin{smallmatrix} 0 & 1 \\ -p^e & 0 \end{smallmatrix}\right)$. If $\phi_v\left(\frac{p_D+\sqrt{D}}{2}\right) = \left(\begin{smallmatrix} a & b \\ p^e c & p_D - a \end{smallmatrix}\right)$, then a direct computation shows that

$$\phi_v^{\omega_p} = \begin{pmatrix} p_D - a & -c \\ -p^e b & a \end{pmatrix},$$

whence by Lemma 27, the embeddings have the opposite orientation if and only if $a \not\equiv p_D - a \pmod{p^e}$. Assume otherwise, so that $2a - p_D \equiv 0 \pmod{p^e}$. If $p = 2$, then D is odd, and this is not possible. If p is odd, then by doubling the matrix expression for ϕ, we see that $-(2a - p_D)^2 \equiv D \pmod{p^e}$, hence this cannot be zero, as desired.

Finally, if $v = \infty$, then this follows directly by definition and an explicit computation. □

If $\gcd(D, \mathfrak{M}) = 1$, then by successively conjugating an embedding by the elements ω_p for $p \mid \mathfrak{D}\mathfrak{M}\infty$, we can pass between all possible orientations. In particular, this implies that for all orientations o,

$$|\operatorname{Emb}_o(\mathrm{O}, D)| = h^+(D).$$

In fact, more is true: there is a simply transitive action of the narrow class group $\mathrm{Cl}^+(D)$ on $\operatorname{Emb}_o(\mathrm{O}, D)$, valid for all discriminants D for which $\operatorname{Emb}(\mathrm{O}, D)$ is non-empty. See Section 4.5 of [Ric21b], or the discussion below Definition 4.22 of [DV21] for more details.

4. Basic results on intersection numbers

With the background out of the way, we turn our focus to intersection numbers. Proposition 1.10 of [Ric21a] gives nice descriptions of when root geodesics of hyperbolic matrices in $\mathrm{SL}(2, \mathbb{R})$ intersect. We state the relevant parts here (and change the expression for $\tan(\theta)$ into $\cos(\theta)$).

Proposition 29 (Proposition 1.10 of [Ric21a])**.** *Let $M_1, M_2 \in \mathrm{SL}(2, \mathbb{R})$ be hyperbolic matrices with respective upper half plane root geodesics ℓ_1, ℓ_2, and let $Z_i = M_i - \frac{\mathrm{Tr}(M_i)}{2}\mathrm{Id}$ for $i = 1, 2$. Then*

(i) ℓ_1, ℓ_2 intersect transversely if and only if

$$\det(M_1M_2 - M_2M_1) > 0.$$

(ii) In all cases,

$$\det(M_1M_2 - M_2M_1) = \det(Z_1Z_2 - Z_2Z_1) = 4\det(Z_1Z_2) - (\mathrm{Tr}(Z_1Z_2))^2.$$

(iii) If ℓ_1, ℓ_2 intersect transversely, then

(a) the intersection point is the fixed point of Z_1Z_2 that lies in $\mathbb{H}$.

(b) *the intersection angle θ (measured counterclockwise from the tangent to ℓ_1 to the tangent to ℓ_2) satisfies*

$$\cos(\theta) = \frac{\mathrm{Tr}(Z_1Z_2)}{2\sqrt{\det(Z_1Z_2)}}.$$

In particular, Proposition 6 is a corollary of this proposition.

Proof of Proposition 6. Assume that (ϕ_1, ϕ_2) are $x-$linked optimal embeddings of positive discriminants D_1, D_2. Let $M_i = \iota(\phi_i(\epsilon_{D_i}))$ for $i = 1, 2$, where the fundamental units can be written as $\epsilon_{D_i} = \frac{T_i+U_i\sqrt{D_i}}{2}$, with (T_i, U_i) being the smallest positive integer solution to $T^2 - D_iU^2 = 4$. In particular, $\ell_{\phi_i} = \ell_i$ for $i = 1, 2$. It follows that $Z_i = \frac{U_i}{2}\iota(\phi_i(\sqrt{D_i}))$, hence

$$\det(Z_i) = \frac{-U_i^2D_i}{4}, \qquad \mathrm{Tr}(Z_1Z_2) = \frac{U_1U_2x}{2}.$$

Therefore

$$4\det(Z_1Z_2) - (\mathrm{Tr}(Z_1Z_2))^2 = \frac{U_1^2U_2^2}{4}(D_1D_2 - x^2),$$

and the root geodesics intersect transversely if and only if $x^2 < D_1D_2$. This proves the first claim.

Assume the root geodesics intersect transversely, and let $T = \phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})$; the intersection point is the upper half plane fixed point of $\iota(T)$. Since T satisfies $T^2 - 2xT + D_1D_2 = 0$, T acts as $x + \sqrt{x^2 - D_1D_2}$. As $T \in (2\mathrm{O} + p_{D_1})(2\mathrm{O} + p_{D_2}) \subset 2\mathrm{O} + p_{D_1D_2}$, T corresponds to an embedding of $\mathcal{O}_{x^2-D_1D_2}$ into O, which is part i. This also implies that $x \equiv D_1D_2 \pmod 2$.

Finally, the angle of intersection satisfies

$$\cos(\theta) = \frac{U_1U_2x/2}{2\sqrt{U_1^2U_2^2D_1D_2/16}} = \frac{x}{\sqrt{D_1D_2}},$$

and the proof is finished. □

This implies that we can replace "study intersections of $\ell_{\phi_1}, \ell_{\phi_2}$" by "study $\mathrm{Emb}(\mathrm{O}, \phi_1, \phi_2, x)$ for $x^2 < D_1D_2$."

While the sets $\mathrm{Emb}(\mathrm{O}, \phi_1, \phi_2, x)$ can be computed in practice, it is a much harder task to access their theoretical properties. Instead, from now on we will focus on $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$, for positive discriminants D_1, D_2, which captures all possible $x-$linking of optimal embeddings of discriminants D_1, D_2 into O.

While we will eventually characterize and count $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$, we can already prove a strong necessary condition for this set to be non-empty.

Lemma 30. *Let $v_1, v_2 \in \mathrm{O}$. Then*

$$\mathfrak{D}\mathfrak{M} \mid \mathrm{nrd}(v_1v_2 - v_2v_1).$$

Proof. Let $p \mid \mathfrak{D}\mathfrak{M}$, and consider completing B at p. We can assume that the completion O_p is either the unique maximal order if B_p is division, or the standard Eichler order of level p^e otherwise. In the first case, let the unique maximal ideal of O_p be $\mathfrak{p}$, and then $\frac{\mathrm{O}_p}{\mathfrak{p}} \simeq \mathbb{F}_{p^2}$ is a field. Thus

$$v_1v_2 \equiv v_2v_1 \pmod{\mathfrak{p}},$$

which implies that $v_1v_2 - v_2v_1 \in \mathfrak{p}$, and so $p \mid \mathrm{nrd}(v_1v_2 - v_2v_1)$.

The second case follows from the fact that looking modulo p^e, we have upper triangular matrices. The diagonal of their product is unchanged when we swap the order of multiplication, and the result follows. □

Corollary 31. *If (ϕ_1, ϕ_2) are $x-$linked, then*

$$\mathfrak{D}\mathfrak{M} \mid \frac{D_1D_2 - x^2}{4}.$$

In particular, for a fixed pair of discriminants D_1, D_2, there is a finite set of non-isomorphic pairs (B, O) of an indefinite quaternion algebra B over $\mathbb{Q}$ with Eichler order O for which there exist optimal embeddings of D_1, D_2 into O giving a non-zero unweighted intersection number.

Proof. Let $v_i = \frac{p_{D_i}+\sqrt{D_i}}{2}$, and using Lemma 30 and a computation analogous to Proposition 29ii, we compute

$$\begin{aligned}\mathfrak{D}\mathfrak{M} &\mid \mathrm{nrd}(\phi_1(v_1)\phi_2(v_2) - \phi_2(v_2)\phi_1(v_1)) \\ &= \frac{\mathrm{nrd}(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}) - \phi_2(\sqrt{D_2})\phi_1(\sqrt{D_1}))}{16} = \frac{D_1D_2 - x^2}{4}.\end{aligned}$$

Intersections come from the finite set of x for which $x^2 < D_1D_2$, and this calculation shows that for each such x there are finitely many pairs $(\mathfrak{D}, \mathfrak{M})$ that satisfy the divisibility condition (in Theorem 44 we will show that $\mathfrak{D}$ is in fact uniquely determined from D_1, D_2, x). Therefore, there are finitely many Eichler orders for which there exist intersections of optimal embeddings of discriminants D_1, D_2. □

5. Existence of x-linked pairs

Rather than study the set $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ directly, we invert the setup. That is, we start with a pair of $x-$linked embeddings into B, and consider the possible Eichler orders which admit these (optimal) embeddings. We study this problem locally, and show how to lift the local results to global results. In this section, we we start this process by studying which quaternion algebras admit $x-$linked embeddings.

5.1. Simultaneous conjugation.

The fact that we are only allowing conjugation by elements of O^1 and not all of $B^\times$ is crucial to $x-$linking.

Lemma 32. *Let B be a quaternion algebra over a field F, and let (x_1, x_2) and (y_1, y_2) be pairs of elements of $B^\times$ for which:*

- *$x_i, y_i \notin F$ for $i = 1, 2$;*
- *x_i and y_i have the same irreducible minimal polynomial over F for $i = 1, 2$;*
- *x_1x_2 and y_1y_2 have the same minimal polynomial over F.*

Then the pairs are simultaneously conjugate over $B^\times$, i.e. there exists an $r \in B^\times$ for which $rx_1r^{-1} = x_2$ and $ry_1r^{-1} = y_2$.

Proof. The $F-$algebras $F[x_1, x_2]$ and $F[y_1, y_2]$ are $F-$subalgebras of B of (equal) dimension 2 or 4. If they have dimension 4, then they are equal to B, and are thus simple. Otherwise, they are equal to $F[x_1]$ and $F[y_1]$, which are again simple algebras since the minimal polynomials were irreducible.

Consider the map $\theta : F[x_1, x_2] \to F[y_1, y_2]$ defined by $\theta(x_i) = y_i$ for $i = 1, 2$. The equality of the minimal polynomials of x_i, y_i and x_1x_2, y_1y_2 implies that the map is indeed a well defined isomorphism. By the Skolem-Noether theorem, this map is inner in B (Corollary 7.7.2 of [Voi21]), and this implies the result. □

Applying Lemma 32 to optimal embeddings produces the following corollary.

Corollary 33. *Let B be a quaternion algebra over $F = \mathbb{Q}$ or $\mathbb{Q}_p$, and let $(\phi_1, \phi_2), (\phi_1', \phi_2')$ be pairs of $x-$linked embeddings from $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ respectively into B. Then $V = \{v \in B : v\phi_n = \phi_n' v \text{ for } n = 1, 2\}$ is a 1-dimensional F-vector space, generated by an element of B with non-zero norm. In particular, the pairs of embeddings are simultaneously conjugate over $B^\times$.*

Proof. Let $V_n = \{v \in B : v\phi_n = \phi_n' v\}$ for $n = 1, 2$; by Corollary 16, this is a two dimensional F-vector space. Furthermore, we have $V_n = r_n(F + \phi_n(\sqrt{D_n})F)$ for $n = 1, 2$ for some $r_1, r_2 \in B^\times$. We claim that V_1 and V_2 are distinct: otherwise, right multiplication by $\phi_1(\sqrt{D_1})$ on V_1 remains in V_1, hence it is true for V_2 as well. This implies that $\phi_1(\sqrt{D_1}) \in F + \phi_2(\sqrt{D_2})F$, and therefore $\phi_1(\sqrt{D_1})$ is a scalar multiple of $\phi_2(\sqrt{D_2})$ (by taking traces). Writing $\phi_1(\sqrt{D_1}) = f\phi_2(\sqrt{D_2})$ for $f \in F^\times$, squaring gives us $D_1 = f^2 D_2$ and $x = \frac{1}{2}\operatorname{trd}\left(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})\right) = fD_2$. Thus $x^2 = f^2 D_2^2 = D_1 D_2$, which is a contradiction by definition of $x-$linkage.

Since $V = V_1 \cap V_2$, V has dimension 0 or 1 as V_1, V_2 are distinct. We apply Lemma 32 to the images of $\sqrt{D_1}, \sqrt{D_2}$ under (ϕ_1, ϕ_2) and (ϕ_1', ϕ_2') respectively. The minimal polynomials satisfy the requirements, whence the lemma implies that V has an invertible element. Thus V has dimension 1, as desired. ☐

5.2. **Orders containing x-linked pairs.** Given a pair of embeddings $\phi_i : \mathcal{O}_{D_i} \to B$ $(i = 1, 2)$, there does not need to be an order that contains the images of both $\mathcal{O}_{D_i}$. The following definition and lemma describe when there is such an order.

Definition 34. Let (D_1, D_2, x) be a triple of integers. We call the triple *admissible* if the following hold:

- D_1 and D_2 are positive discriminants;
- $x \equiv D_1 D_2 \pmod 2$ and $x^2 \neq D_1 D_2$.

A consequence of the following lemma is that there exists an order containing given $x-$linked embeddings of discriminants D_1, D_2 if (D_1, D_2, x) is admissible.

Lemma 35. *Let $F = \mathbb{Q}$ or $\mathbb{Q}_p$, and let B be a quaternion algebra over F. Let $\phi_i : \mathcal{O}_{D_i} \to B$ be embeddings of the orders of discriminants D_1, D_2 into B, and take $v_i = \phi_i\left(\frac{p_{D_i} + \sqrt{D_i}}{2}\right)$ for $i = 1, 2$. Assume that $x = \frac{1}{2}\operatorname{trd}(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})) \in p_{D_1 D_2} + 2\mathcal{O}_F$ and $x^2 \neq D_1 D_2$. Then*

$$\mathrm{O}_{\phi_1,\phi_2} := \langle 1, v_1, v_2, v_1 v_2 \rangle_{\mathcal{O}_F}$$

is an order of B, necessarily the smallest order of B for which both ϕ_1 and ϕ_2 embed into. Furthermore,

$$\operatorname{discrd}(\mathrm{O}_{\phi_1,\phi_2}) = \frac{D_1 D_2 - x^2}{4}.$$

Proof. For ease of notation write $\mathrm{O} = \mathrm{O}_{\phi_1,\phi_2}$. First,

$$\operatorname{trd}(v_1 v_2) = \frac{p_{D_1} p_{D_2} + x}{2} = p_{D_1 D_2} + \frac{x - p_{D_1 D_2}}{2} \in \mathcal{O}_F,$$

and $\operatorname{nrd}(v_1 v_2) = \operatorname{nrd}(v_1)\operatorname{nrd}(v_2) \in \mathcal{O}_F$, whence $v_1 v_2$ is integral. We will demonstrate that $v_2 v_1 \in \mathrm{O}$, and the rest of the equations to prove that $\mathrm{O}_{\phi_1,\phi_2}$ is closed under multiplication can be deduced from this and the minimal polynomials for v_1, v_2. We compute

$$\begin{aligned} v_1v_2 + v_2v_1 =& \frac{p_{D_1D_2} + p_{D_1}\phi_2(\sqrt{D_2}) + p_{D_2}\phi_1(\sqrt{D_1})}{2} \\ &+ \frac{\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}) + \phi_2(\sqrt{D_2})\phi_1(\sqrt{D_1})}{4} \\ =& p_{D_1}v_2 + p_{D_2}v_1 + \frac{x - p_{D_1D_2}}{2}, \end{aligned}$$

whence v_2v_1 lies in O, as claimed.

The fact that O is an order will follow from computing its reduced discriminant, and seeing that it is non-zero. To ease our calculations, write

$$\begin{pmatrix} 1 \\ \phi_1(\sqrt{D_1}) \\ \phi_2(\sqrt{D_2}) \\ \phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -p_{D_1} & 2 & 0 & 0 \\ -p_{D_2} & 0 & 2 & 0 \\ p_{D_1}p_{D_2} & -2p_{D_2} & -2p_{D_1} & 4 \end{pmatrix} \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_1v_2 \end{pmatrix},$$

and we have the equation

$$d(1, \phi_1(\sqrt{D_1}), \phi_2(\sqrt{D_2}), \phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})) = \det(M)^2 d(1, v_1, v_2, v_1v_2),$$

where M is the transition matrix above. We compute $\det(M) = 16$ and

$$\begin{aligned} d(1, \phi_1(\sqrt{D_1}), \phi_2(\sqrt{D_2}), \phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})) &= \det \begin{pmatrix} 2 & 0 & 0 & 2x \\ 0 & 2D_1 & 2x & 0 \\ 0 & 2x & 2D_2 & 0 \\ 2x & 0 & 0 & 4x^2 - 2D_1D_2 \end{pmatrix} \\ &= -16(D_1D_2 - x^2)^2. \end{aligned}$$

Since $\mathrm{discrd}(\mathrm{O})^2 = -d(1, v_1, v_2, v_1v_2)$, the reduced discriminant is as claimed (and is non-zero by the assumption of $x^2 \neq D_1D_2$).

It is immediate that O is the smallest order for which ϕ_1, ϕ_2 embed into, as such an order must contain $\{1, v_1, v_2\}$, and O is generated as an $\mathcal{O}_F$ algebra by these elements. $\square$

Lemma 35 has some historical connections. The proof of Theorem 2' in [Kan89] details a similar computation in a definite quaternion algebra. Furthermore, as noted by Gross, this definite computation leads to a simple argument that a prime p dividing $\mathrm{Nm}(j(\tau_1) - j(\tau_2))$ must satisfy $p \mid \frac{D_1D_2 - x^2}{4}$ for $x^2 < D_1D_2$ (see Theorem 65 and Proposition 66 of [Gis20] for the full argument).

Our first application of Lemma 35 is to show that $x-$linked pairs of embeddings can be detected locally.

Lemma 36. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$, let O be an Eichler order in B, and let (D_1, D_2, x) be an admissible triple. Then the set $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ is non-empty if and only if $\mathrm{Emb}(\mathrm{O}_p, D_1, D_2, x)$ is non-empty for all finite primes p.*

Proof. If such a pair $(\phi_1, \phi_2) \in \mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ exists, then the corresponding maps to the completions gives elements of $\mathrm{Emb}(\mathrm{O}_p, D_1, D_2, x)$ for all p.

To prove the opposite direction, assume that $(\alpha_p, \beta_p) \in \mathrm{Emb}(\mathrm{O}_p, D_1, D_2, x)$ for all p. A consequence of Proposition 23 is that there exists an embedding ϕ_1 of $\mathcal{O}_{D_1}$ into B. By Corollary 22, we can assign coordinates so that $\phi_1(\sqrt{D_1}) = i$. In this case, we are considering the existence of a map ϕ_2 such that $\phi_2(\sqrt{D_2}) = fi + gj + hk$, where

$$\operatorname{nrd}(fi + gj + hk) = -D_2 \qquad \text{and} \qquad 2x = \operatorname{trd}(i(fi + gj + hk)) = 2fD_1.$$

With the substitution of $f = \frac{x}{D_1}$, the equation $\operatorname{nrd}\left(\frac{x}{D_1}i + gj + hk\right) + D_2 = 0$ is a quadratic form in g, h. This will have a solution in $\mathbb{R}$ since B is indefinite, and it will have a solution in $\mathbb{Q}_p$ for all p since $\operatorname{Emb}(\mathrm{O}_p, D_1, D_2, x)$ is non-empty. By Hasse's principle, it has a solution over $\mathbb{Q}$; let the corresponding map be ϕ_2.

Following Lemma 35, let $\mathrm{O}' = \mathrm{O}_{\phi_1,\phi_2}$ be the smallest order for which ϕ_1, ϕ_2 embed into. By Corollary 33, for all finite primes p there exists an $r_p \in B_p^\times$ for which $r_p(\alpha_p, \beta_p)r_p^{-1} = (\phi_{1,p}, \phi_{2,p})$. By the definition of O', it follows that $\mathrm{O}'_p \subseteq r_p\mathrm{O}_p r_p^{-1}$. For all primes p,

- let $s_p = r_p$ if $\mathrm{O}'_p \neq \mathrm{O}_p$ or $p \mid D_1D_2$;
- let $s_p = 1$ otherwise.

Consider the sequence of local orders $\{s_p\mathrm{O}_p s_p^{-1}\}_p$. Since $\mathrm{O}'_p = \mathrm{O}_p$ holds for all but finitely many primes, by Theorem 13 there exists an order O'' of B which completes to $s_p\mathrm{O}_p s_p^{-1}$ for all primes p. In particular, we note that O'' is an Eichler order of level $\mathfrak{M}$, and ϕ_1, ϕ_2 give embeddings into O''. When $p \mid D_1D_2$ the local embeddings are optimal since (α_p, β_p) were optimal, hence ϕ_1, ϕ_2 are optimal embeddings into O''. Since all Eichler orders of the same level are conjugate, let $r\mathrm{O}''r^{-1} = \mathrm{O}$, and then $[r(\phi_1, \phi_2)r^{-1}] \in \operatorname{Emb}(\mathrm{O}, D_1, D_2, x)$, as required. ☐

In particular, the non-emptyness of $\operatorname{Emb}(\mathrm{O}, D_1, D_2, x)$ can be studied locally.

5.3. **Local x-linking.** While we were concerned with orders in Lemma 36, we will drop this for now and instead consider embeddings into the entire quaternion algebra.

Definition 37. Let (D_1, D_2, x) be an admissible triple, and define $\operatorname{Emb}(B, D_1, D_2, x)$ to be the set of all pairs (ϕ_1, ϕ_2) of $x-$linked embeddings of discriminants D_1, D_2 into B.

Note that Lemma 36 also applies to the sets $\operatorname{Emb}(B, D_1, D_2, x)$ and $\operatorname{Emb}(B_p, D_1, D_2, x)$. Our next goal is to determine when $\operatorname{Emb}(B_p, D_1, D_2, x)$ is non-empty. Before getting into these local computations, we require a lemma about the solutions to Pell's equation over $\mathbb{Z}_p$.

Lemma 38. *Let p be a prime, let A be a non-zero integer, and let D be a positive discriminant coprime to p. Then the equation*

$$X^2 - DY^2 = A \tag{5.1}$$

has a solution $(X, Y) \in \mathbb{Z}_p^2$ if and only if one of the following conditions hold:

- $\left(\frac{D}{p}\right) = 1$, *and if $p = 2$ we additionally have $v_2(A) \neq 1$;*
- $\left(\frac{D}{p}\right) = -1$ *and $v_p(A)$ is even.*

Proof. If $\left(\frac{D}{p}\right) = 1$, then $\sqrt{D} \in \mathbb{Z}_p$ (noting that if $p = 2$ then $D \equiv 1 \pmod 8$ as it is a discriminant). By factoring $(X - \sqrt{D}Y)(X + \sqrt{D}Y) = A = uv$, this will always have a solution if p is odd. If $p = 2$, then we require u and v to have opposite parity, which gives $v_2(A) \neq 1$.

Otherwise, $\mathbb{Q}_p(\sqrt{D})$ is the unramified degree 2 extension of $\mathbb{Q}_p$, and $X^2 - DY^2$ is the norm form from $\mathbb{Z}_p[\sqrt{D}]$ to $\mathbb{Z}_p$. The result follows, e.g. by Chapter 2, Section 4 of [Lan94]. $\square$

We start the local calculations by considering the division algebra case. Recall the Hilbert symbol $(a, b)_p$, which has an alternate characterization via Hilbert's criterion: $(a, b)_p = 1$ if and only if $ax^2 + by^2 = 1$ has solutions with $x, y \in \mathbb{Q}_p$ (see Section 12.4 of [Voi21]).

Lemma 39. *Let (D_1, D_2, x) be an admissible triple, and let B be the division algebra over $\mathbb{Q}_p$. Then $\mathrm{Emb}(B, D_1, D_2, x)$ is non-empty if and only if*

$$(D_1, x^2 - D_1D_2)_p = -1.$$

If $p \nmid D_1$, this is equivalent to

$$\left(\frac{D_1}{p}\right) = -1 \text{ and } v_p\left(\frac{D_1D_2 - x^2}{4}\right) \text{ is odd.}$$

Proof. If there does not exist an embedding of $\mathcal{O}_{D_1}$ into B, then the same is true for $\mathcal{O}_{D_1^{\mathrm{fund}}}$. By Proposition 24ii, $\left(\frac{D_1^{\mathrm{fund}}}{p}\right) = 1$, and therefore by Hilbert's criterion, $(D_1, N)_p = 1$ for all $N \neq 0$. In particular, $(D_1, x^2 - D_1D_2)_p \neq -1$, as desired.

Otherwise, by Corollary 22 we can write $B = \left(\frac{D_1, e}{\mathbb{Q}_p}\right)$ for some non-zero $e \in \mathbb{Z}_p$, where $\phi_1(\sqrt{D_1}) = i$ and $(D_1, e)_p = -1$. Writing $\phi_2(\sqrt{D_2}) = fi + gj + hk$, it suffices to solve the equations

$$D_1f^2 + eg^2 - D_1eh^2 = D_2, \qquad x = fD_1.$$

Therefore $f = \frac{x}{D_1}$, and the first equation rearranges to

$$g^2 - D_1h^2 = \frac{D_1D_2 - x^2}{eD_1}. \tag{5.2}$$

If this has a solution with $h = h_1$, then by Hensel's lemma there will be a solution with $h = h_1 + p^k$ for large enough k. In particular, they correspond to distinct g's, so we can solve the equation with the assumption that $g \neq 0$. Equation (5.2) then rearranges to

$$D_1(h/g)^2 + \frac{D_1D_2 - x^2}{eD_1}(1/g)^2 = 1,$$

which is in the format of Hilbert's criterion. The properties of the Hilbert symbol imply that

$$1 = \left(D_1, \frac{D_1D_2 - x^2}{eD_1}\right)_p = (D_1, (x^2 - D_1D_2)e)_p = -(D_1, x^2 - D_1D_2)_p,$$

from which the first result follows.

If $p \nmid D_1$, then $\left(\frac{D_1}{p}\right) = -1$, and we claim that $v_p(e)$ is odd. If p is odd this follows immediately, since $(a, b)_p = 1$ if $p \nmid ab$. If $p = 2$, then $D_1 \equiv 5 \pmod 8$, and this follows by computing $(D_1, b)_2$ for all $b \in \mathbb{Q}_2^\times/Q_2^{\times 2}$, and seeing that $(D_1, b)_2 = 1$ if $v_2(b)$ is even (see Table 12.4.16 of [Voi21] for this computation).

Scaling Equation (5.2) by powers of p, it is equivalent to solve

$$g^2 - D_1h^2 = \frac{D_1D_2 - x^2}{eD_1}p^{2r}, \tag{5.3}$$

for $r \geq 0$ and $g, h \in \mathbb{Z}_p$. Lemma 38 implies that Equation (5.3) has a solution if and only if

$$v_p\left(\frac{D_1D_2 - x^2}{eD_1}p^{2r}\right)$$

is even, which is equivalent to our condition. $\square$

Now we consider non-division algebras.

Lemma 40. *Let (D_1, D_2, x) be an admissible triple. Then $\mathrm{Emb}(\mathrm{Mat}(2, \mathbb{Q}_p), D_1, D_2, x)$ is non-empty if and only if*

$$(D_1, x^2 - D_1D_2)_p = 1.$$

If $p \nmid D_1$ this is equivalent to either

$$\left(\frac{D_1}{p}\right) = 1 \qquad \text{or} \qquad \left(\frac{D_1}{p}\right) = -1 \text{ and } v_p\left(\frac{D_1D_2 - x^2}{4}\right) \text{ is even.}$$

Proof. Since embeddings of a fixed discriminant are all conjugate over $B^\times$, we can fix the first embedding to be $\phi_1(\sqrt{D_1}) = \left(\begin{smallmatrix} 0 & D_1 \\ 1 & 0 \end{smallmatrix}\right)$, and write $\phi_2(\sqrt{D_2}) = \left(\begin{smallmatrix} e & f \\ g & -e \end{smallmatrix}\right) \in \mathrm{Mat}(2, \mathbb{Q}_p)$. We will have a solution if and only if

$$e^2 + fg = D_2, \qquad D_1g + f = 2x.$$

This implies that $f = 2x - D_1g$, and plugging this into the first equation and rearranging gives

$$e^2 - D_1\left(g - \frac{x}{D_1}\right)^2 = \frac{D_1D_2 - x^2}{D_1}.$$

Let $X = e$ and $Y = g - \frac{x}{D_1}$, and then the equation is

$$X^2 - D_1Y^2 = \frac{D_1D_2 - x^2}{D_1}.$$

The rest of the proof is analogous to Lemma 39, where Lemma 38 completes the characterization of the solubility when $p \nmid D_1$. $\square$

Lemmas 39 and 40 immediately imply the following corollary

Corollary 41. *Let (D_1, D_2, x) be an admissible triple, and let B be the division algebra over $\mathbb{Q}_p$. Then exactly one of B and $\mathrm{Mat}(2, \mathbb{Q}_p)$ admits $x-$linked embeddings of $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$, and which one is determined by if $(D_1, x^2 - D_1D_2)_p$ is -1 or 1, respectively.*

Remark 42. The first half of Lemmas 39, 40 and Corollary 41 still holds when $p = \infty$, where $\mathbb{Q}_\infty = \mathbb{R}$.

5.4. **Global x-linking.** Fix an admissible triple (D_1, D_2, x). Corollary 41 combined with Lemma 36 implies that there is precisely one quaternion algebra B over $\mathbb{Q}$ for which there exist embeddings ϕ_i of $\mathcal{O}_{D_i}$ into B that are $x-$linked, and it can be given by

$$\left(\frac{D_1, x^2 - D_1D_2}{\mathbb{Q}}\right).$$

We describe the ramification of this quaternion algebra by using a generalization of the ϵ function (Definition 9).

Definition 43. Let D_1, D_2 be discriminants, and let p be any prime such that

$$p \nmid \gcd(D_1^{\mathrm{fund}}, D_2^{\mathrm{fund}}) \qquad \text{and} \qquad \left(\frac{D_1^{\mathrm{fund}}D_2^{\mathrm{fund}}}{p}\right) \neq -1.$$

Define

$$\epsilon(p) := \begin{cases} \left(\dfrac{D_1^{\text{fund}}}{p}\right) & \text{if } p \text{ and } D_1^{\text{fund}} \text{ are coprime;} \\ \\ \left(\dfrac{D_2^{\text{fund}}}{p}\right) & \text{if } p \text{ and } D_2^{\text{fund}} \text{ are coprime.} \end{cases}$$

Theorem 44. *Let (D_1, D_2, x) be an admissible triple. Then the only quaternion algebra over $\mathbb{Q}$ that admits $x-$linked embeddings from $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ is*

$$B = \left(\frac{D_1, x^2 - D_1D_2}{\mathbb{Q}}\right).$$

Furthermore, let $N = \gcd(D_1^{\text{fund}}, D_2^{\text{fund}})$, and factorize

$$\frac{D_1D_2 - x^2}{4} = \pm N' \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where N' is minimal so that $\frac{D_1D_2-x^2}{4N'}$ is coprime to N, p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Then B is ramified at

$$\{p_1, p_2, \ldots, p_r\} \cup \{p : p \mid N', (D_1, x^2 - D_1D_2)_p = -1\}.$$

Proof. It suffices to compute $(D_1, x^2 - D_1D_2)_p$ for $p \mid \frac{D_1D_2-x^2}{4}$ satisfying $p \nmid N'$. If $p \nmid D_1$, Lemmas 39 and 40 imply that the Hilbert symbol is -1 if and only $\epsilon(p) = -1$ and $v_p\left(\frac{D_1D_2-x^2}{4}\right)$ is odd, i.e. $p = p_i$ for some i. Since

$$(D_1, x^2 - D_1D_2)_p = (D_2, x^2 - D_1D_2)_p,$$

the same holds for $p \nmid D_2$. As we assume that $p \nmid N'$, the final case is (without loss of generality) $p \nmid D_1^{\text{fund}}$ and $p \mid D_1, D_2$. By Lemma 51, we can replace (D_1, D_2, x) by $(D_1/p^2, D_2, x/p)$, and repeat. $\square$

In particular, if $\gcd(D_1^{\text{fund}}, D_2^{\text{fund}}) = 1$, then B is ramified at exactly $\{p_1, p_2, \ldots, p_r\}$.

Remark 45. The value of $(D_1, x^2 - D_1D_2)_p$ for $p \mid N'$ is full of technical casework, and there is little benefit in listing the cases out.

Remark 46. To work with an explicit $x-$linked pair, take $B = \left(\frac{D_1, x^2-D_1D_2}{\mathbb{Q}}\right)$, and define

$$\phi_1(\sqrt{D_1}) = i, \qquad \phi_2(\sqrt{D_2}) = \frac{xi + k}{D_1}.$$

This pair is $x-$linked and corresponds to $\phi_1 \times \phi_2(\sqrt{x^2 - D_1D_2}) = j$.

6. Counting Eichler orders containing x-linked pairs

Thanks to Theorem 44, we have a good description of quaternion algebras that exhibit $x-$linking. We now turn our focus to describing Eichler orders that admit $x-$linked pairs, i.e. Eichler superorders of $\mathrm{O}_{\phi_1,\phi_2}$. Once again, it suffices to do this locally.

6.1. **Local Eichler orders containing x-linked pairs.** Up until now, we have mostly worked in full generality. However, as evidenced by the end of Theorem 44, this generality can (and will) start to make results rather unwieldy. As such, we would like to find a middle ground between a pleasant exposition and full generality. The following definition is our choice for such a middle ground.

Definition 47. Given an an admissible triple (D_1, D_2, x), we call it *nice* if

$$\gcd\left(D_1, D_2, D_1D_2 - x^2\right) = 1.$$

Note that a nice triple has at least one of D_1, D_2 being odd.

More generally, if p is a prime, we call p *nice* (with respect to (D_1, D_2, x)) if

$$p \nmid \gcd\left(D_1, D_2, D_1D_2 - x^2\right).$$

From now on, we will mostly be working with nice triples/nice primes.

In order to determine if an order is Eichler or not, we consider the Eichler symbol (see Section 24.3 of [Voi21]). Working in $B = \left(\frac{a,b}{\mathbb{Q}_p}\right)$, for $\alpha \in B$, define

$$\Delta(\alpha) = \operatorname{trd}(\alpha)^2 - 4\operatorname{nrd}(\alpha) = 4(af^2 + bg^2 - abh^2),$$

where $\alpha = e + fi + gj + hk$. For an order O of B, define (O, p) to be the set of values that $\left(\frac{\Delta(\alpha)}{p}\right)$ takes as α ranges over O, where $\left(\frac{\cdot}{p}\right)$ is the Kronecker symbol.

Lemma 48. *The set (O, p) determines the possible Eichler superorders of O as follows:*

- *The order O is Eichler and non-maximal if and only if $(\mathrm{O}, p) = \{0, 1\}$ (i.e. O is "residually split").*
- *If $-1 \in (\mathrm{O}, p)$, then O is contained in precisely one maximal order.*

Proof. The first point is a direct consequence of Lemma 24.3.6 of [Voi21]. For the second point, if O' is a superorder of O, then $(\mathrm{O}, p) \subseteq (\mathrm{O}', p)$. In particular, no superset has $(\mathrm{O}, p) = \{0, 1\}$, whence O is not contained in a non-maximal Eichler order. If O were contained in two maximal orders, it would be contained in their intersection, a non-maximal Eichler order, contradiction. ◻

Lemma 48 allows us to compute the Eichler orders containing $\mathrm{O}_{\phi_1,\phi_2}$.

Lemma 49. *Let (D_1, D_2, x) be admissible, and let ϕ_1, ϕ_2 be $x-$linked embeddings of discriminants D_1, D_2 into B, a quaternion algebra over $\mathbb{Q}_p$, where p is nice. Let $\mathrm{O} = \mathrm{O}_{\phi_1,\phi_2}$, and then:*

(i) If $p \nmid \frac{D_1D_2 - x^2}{4}$, then O is maximal;
(ii) If $\epsilon(p) = -1$, then O is contained in a unique maximal order;
(iii) If $\epsilon(p) = 1$, then O is Eichler.

Proof. By Lemma 35, the reduced discriminant of O is $\frac{D_1D_2 - x^2}{4}$. Thus if $p \nmid \frac{D_1D_2 - x^2}{4}$, O is maximal.

Now, assume that $p \mid \frac{D_1D_2 - x^2}{4}$, which implies that $p^{1+2v_2(p)} \mid x^2 - D_1D_2$. As p is nice, it follows that $p \nmid \gcd(D_1, D_2)$, so without loss of generality assume that $p \nmid D_1$. Take B, ϕ_i as in Remark 46, and then by Lemma 35, a general element of O is of the form

$$\begin{aligned}\alpha = A_0 + A\frac{p_{D_1} + i}{2} &+ B\frac{p_{D_2} + (xi + k)/D_1}{2} + C\frac{j}{2} \\ &= \left(A_0 + A\frac{p_{D_1}}{2} + B\frac{p_{D_2}}{2}\right) + \left(\frac{A}{2} + \frac{Bx}{2D_1}\right)i + \frac{C}{2}j + \frac{B}{2D_1}k,\end{aligned}$$

for $A_0, A, B, C \in \mathbb{Z}_p$. Therefore

$$\Delta(\alpha) = D_1 \left(A + \frac{Bx}{D_1}\right)^2 + (x^2 - D_1D_2)C^2 - D_1(x^2 - D_1D_2)\frac{B^2}{D_1^2},$$

whence

$$\Delta(\alpha) \equiv D_1 \left(A + \frac{Bx}{D_1}\right)^2 \pmod{p^{1+2v_2(p)}}.$$

Thus $(\mathrm{O}, p) = \{0, \epsilon(p)\}$, which by Lemma 48 completes the second and third points. $\square$

Lemma 49 implies that locally, there is a minimal Eichler order containing $\mathrm{O}_{\phi_1,\phi_2,p}$, which is either the order itself, or the unique maximal order it is contained within. Therefore the result is true globally, and we make this a definition.

Definition 50. Let ϕ_1, ϕ_2 be $x-$linked embeddings of discriminants D_1, D_2 into B, an indefinite quaternion algebra over $\mathbb{Q}$ or $\mathbb{Q}_p$, where (D_1, D_2, x) is nice. Then there exists a minimal Eichler order containing $\mathrm{O}_{\phi_1,\phi_2}$, denoted $\mathrm{O}_{\phi_1,\phi_2}^{\mathrm{Eich}}$.

Since we are concerned with the optimality of embeddings, we need to determine which orders containing $\mathrm{O}_{\phi_1,\phi_2}^{\mathrm{Eich}}$ admit ϕ_1, ϕ_2 as optimal embeddings.

Lemma 51. *Let (D_1, D_2, x) be admissible, and let ϕ_1, ϕ_2 be $x-$linked embeddings of D_1, D_2 into B, an indefinite quaternion algebra over $\mathbb{Q}$. Let p be a prime for which $p \mid \frac{D_1}{D_1^{\mathrm{fund}}}$, and let ϕ_1' be the corresponding embedding of $\mathcal{O}_{D_1/p^2}$ into B that agrees with ϕ_1 on $\mathcal{O}_{D_1}$. Then (ϕ_1', ϕ_2) are $\frac{x}{p}-$linked embeddings into B if and only if $p \mid \frac{D_1D_2 - x^2}{4}$.*

Proof. Since

$$\frac{1}{2}\operatorname{trd}\left(\phi_1'\left(\sqrt{D_1/p^2}\right)\phi_2\left(\sqrt{D_2}\right)\right) = \frac{x}{p},$$

(ϕ_1', ϕ_2) are $\frac{x}{p}-$linked embeddings if and only if $\frac{x}{p}$ is an integer congruent to $\frac{D_1D_2}{p^2}$ modulo 2.

If this is the case, then by Lemma 35 the reduced discriminant of $\mathrm{O}_{\phi_1',\phi_2}$ is $\frac{D_1D_2 - x^2}{4p^2}$, which implies that $p \mid \frac{D_1D_2 - x^2}{4}$, as required.

If $p \mid \frac{D_1D_2 - x^2}{4}$, first assume that p is odd. Then $p \mid x^2$, whence $p \mid x$, and $\frac{x}{p}$ is an integer with the same parity as $\frac{D_1D_2}{p^2}$, as required.

If $p = 2$, then $8 \mid D_1D_2 - x^2$. If D_2 is even or $8 \mid D_1$, then $8 \mid D_1D_2$, so $8 \mid x^2$, and hence $4 \mid x$. Therefore $\frac{x}{2} \equiv 0 \equiv \frac{D_1D_2}{2^2} \pmod 2$, as required. Otherwise, $4 \,||\, D_1$ and D_2 is odd. As $D_1/4$ is a discriminant, it is equivalent to $1 \pmod 4$, and so $D_1D_2 \equiv 4 \pmod{16}$. This implies that $x^2 \equiv 4 \pmod 8$, and so $x \equiv 2 \pmod 4$. Then $\frac{x}{2} \equiv 1 \equiv \frac{D_1D_2}{2^2} \pmod 2$, which completes the proof. $\square$

We are now able to study the optimality of embeddings in $\mathrm{O}_{\phi_1,\phi_2}^{\mathrm{Eich}}$, as well as the level of this order.

Definition 52. Let D_1, D_2 be discriminants. Define a prime p to be *potentially bad* (with respect to D_1, D_2) if

$$p \mid \frac{D_1D_2}{D_1^{\mathrm{fund}}D_2^{\mathrm{fund}}}.$$

Define $\mathrm{PB}(D_1, D_2)$ to be the product of all potentially bad primes. In particular, D_1 and D_2 are both fundamental if and only if $\mathrm{PB}(D_1, D_2) = 1$.

It suffices to consider the optimality of (ϕ_1, ϕ_2) at potentially bad primes.

Proposition 53. *Let ϕ_1, ϕ_2 be $x-$linked embeddings of discriminants D_1, D_2 into B, an indefinite quaternion algebra over $\mathbb{Q}$, where (D_1, D_2, x) is nice. Factorize*

$$\frac{D_1D_2 - x^2}{4} = \pm \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Then

(i) The order $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$ is Eichler of level $\prod_{i=1}^{t} w_i^{g_i}$;

(ii) The embeddings ϕ_1, ϕ_2 are optimal embeddings into $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$ if and only if none of primes p_i and q_i are potentially bad.

Proof. Let $\mathrm{O} = \mathrm{O}_{\phi_1,\phi_2}$ and $\mathrm{O}^{\mathrm{Eich}} = \mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$. Lemma 35 computes the reduced discriminant of O to be $\frac{D_1D_2-x^2}{4}$, so it suffices to compute the change in reduced discriminant between O and $\mathrm{O}^{\mathrm{Eich}}$, which can be done locally. Lemma 49 implies that $\mathrm{O}_p = \mathrm{O}^{\mathrm{Eich}}_p$ for $p = w_i$, hence those prime factors remain in the level. For $p = p_i, q_i$, O_p is contained in a unique maximal order, hence those prime factors disappear. This completes the first point.

For optimality, assume that ϕ_1 is not optimal with respect to $\mathrm{O}^{\mathrm{Eich}}$. Thus there exists a $p \mid \frac{D_1}{D_1^{\mathrm{fund}}}$ for which $\phi(\mathcal{O}_{D_1/p^2})$ lands inside $\mathrm{O}^{\mathrm{Eich}}$. Let ϕ_1' denote this embedding (which agrees with ϕ on $\mathcal{O}_D$), and then (ϕ_1', ϕ_2) are $\frac{x}{p}-$linked. By definition, we have

$$\mathrm{O} \subseteq \mathrm{O}_{\phi_1',\phi_2} \subseteq \mathrm{O}^{\mathrm{Eich}},$$

and Lemma 35 says that the reduced discriminant of $\mathrm{O}_{\phi_1',\phi_2}$ is $\frac{D_1D_2-x^2}{4p^2}$. Therefore $p = p_i, q_i, w_i$, so assume that $p = w_i$. By Lemma 49, $\mathrm{O}_p = \mathrm{O}^{\mathrm{Eich}}_p$, hence this is equal to $\mathrm{O}_{\phi_1',\phi_2,p}$ as well, which contradicts the fact that the level of $\mathrm{O}_{\phi_1',\phi_2}$ differs from the level of O by the factor p^2. Therefore $p = p_i$ or $p = q_i$, as claimed.

To finish, it suffices to show that if $p \mid \frac{D_1}{D_1^{\mathrm{fund}}}, \frac{D_1D_2-x^2}{4}$ satisfies $\epsilon(p) = -1$, then the embedding ϕ_1 is not optimal into $\mathrm{O}^{\mathrm{Eich}}$. As above, let ϕ_1' denote the embedding of $\mathcal{O}_{D_1/p^2}$ corresponding to ϕ. By Lemma 51, (ϕ_1', ϕ_2) are $\frac{x}{p}-$linked, so by Lemma 35, $\mathrm{O}_{\phi_1',\phi_2}$ is an order of reduced discriminant $\frac{D_1D_2-x^2}{4p^2}$. Since $\mathrm{O} \subseteq \mathrm{O}_{\phi_1',\phi_2}$ and O_p is contained in a unique maximal order, this must be the same maximal order that contains $\mathrm{O}_{\phi_1',\phi_2,p}$. Therefore $\mathrm{O}^{\mathrm{Eich}}_p = \mathrm{O}^{\mathrm{Eich}}_{\phi_1',\phi_2,p}$, and so ϕ_1' embeds into $\mathrm{O}^{\mathrm{Eich}}_p$, hence it embeds into $\mathrm{O}^{\mathrm{Eich}}$, which proves that ϕ_1 is not optimal. $\square$

To finish off with optimality, we need to consider the optimality of ϕ_1, ϕ_2 into superorders O' of $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$. Assume that none of the p_i, q_i are potentially bad, so that ϕ_1, ϕ_2 are optimal in $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$. The only way that ϕ_1 would fail optimality in O' is if O' admitted the embedding ϕ_1' of discriminant $\mathcal{O}_{D_1/w_j^2}$ (some $1 \leq j \leq t$) that agrees with ϕ on $\mathcal{O}_{D_1}$. The pair (ϕ_1', ϕ_2) is $\frac{x}{w_j}-$linked by Lemma 51, and $\mathrm{O}^{\mathrm{Eich}}_{\phi_1',\phi_2}$ is an Eichler order of level $\frac{1}{w_j^2}\prod_{i=1}^{t} w_i^{g_i}$ by Proposition 53i. Therefore O' admits ϕ_1 as an optimal embedding if and only if $\mathrm{O}' \not\supseteq \mathrm{O}^{\mathrm{Eich}}_{\phi_1',\phi_2}$.

Definition 54. With notation and assumptions as above, let S_{ϕ_1,ϕ_2} be the (possibly empty) set of orders $\mathrm{O}^{\mathrm{Eich}}_{\phi_1',\phi_2}$ and $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2'}$, each of which corresponds to a $1 \leq j \leq t$ for which $w_j \mid \frac{D_1}{D_1^{\mathrm{fund}}}$ or $w_j \mid \frac{D_2}{D_2^{\mathrm{fund}}}$ respectively.

The above discussion is the proof of the following proposition.

Proposition 55. *Take the notation as in Proposition 53, and assume that none of p_i, q_i are potentially bad. Then a superorder O' of $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}$ admits ϕ_1, ϕ_2 as optimal embeddings if and only if it does not contain any order in S_{ϕ_1,ϕ_2}.*

6.2. **Local x-linking with level.** Given an admissible triple (D_1, D_2, x), Theorem 44 determines the unique quaternion algebra for which there exists $x-$linked optimal embeddings. Under the additional restriction of niceness, Propositions 53 and 55 determine the possible Eichler orders that an $x-$linked pair of embeddings becomes optimal in. In this section, we study the possible levels of such embeddings.

Lemma 56. *Let B be a quaternion algebra over $F = \mathbb{Q}$ or $\mathbb{Q}_p$. Let $v_1, v_2, v_3 \in B$ be such that $\langle 1, v_i, v_j, v_iv_j\rangle_{\mathcal{O}_F}$ is an order for $(i, j) = (1, 2), (1, 3), (2, 3)$. Then*

$$\mathrm{O} = \langle 1, v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3, v_1v_2v_3\rangle_{\mathcal{O}_F}$$

is an order.

Proof. It suffices to show that any product $v = v_{i_1} \cdots v_{i_k}$ lands in O for any sequence $i_1, \ldots, i_k$ with $i_j \in \{1, 2, 3\}$ for all j. This is accomplished via induction: the base case of $k = 0$ is trivial. For the inductive step, assume it is true up to $k - 1 \geq 0$. If $i_j \neq 1$ for all j, then $v \in \langle 1, v_2, v_3, v_2v_3\rangle_{\mathcal{O}_F}$ (as this is an order), and we are done. Otherwise, take the last occurrence of 1, say i_m. If $i_{m-1} = 1$, then $v_{i_{m-1}}v_{i_m} = v_1^2 \in \langle 1, v_1\rangle_{\mathcal{O}_F}$, and by replacing it we are done by induction. Otherwise, if $m > 1$, then $i_{m-1} = j \neq 1$, hence $v_{i_{m-1}}v_{i_m} = v_jv_1 \in \langle 1, v_1, v_j, v_1v_j\rangle_{\mathcal{O}_F}$. By writing $v_{i_{m-1}}v_{i_m}$ in this basis and using induction, we see that it suffices to prove the claim when we swap v_{i_m} and $v_{i_{m-1}}$. By successively repeating this process, we can assume that v starts with a v_1 and has no other terms v_1. But then $v_{i_2} \cdots v_{i_k}$ lies in $\langle 1, v_2, v_2, v_2v_3\rangle_{\mathcal{O}_F}$, and a left multiplication by v_1 still lands us in O, as desired. □

The generalization of $\mathrm{O}_{\phi_1,\phi_2}$ is the following.

Definition 57. Let ϕ_1, ϕ_2 be $x-$linked embeddings from $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ to B. Let $\ell \in \mathbb{Z}^+$ be such that $\frac{x^2-D_1D_2}{\ell^2}$ is a discriminant, and define $\mathrm{O}_{\phi_1,\phi_2}(\ell)$ to be the smallest order for which $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}, \mathcal{O}_{(x^2-D_1D_2)/\ell^2}$ embed into via $\phi_1, \phi_2, \phi_1 \times \phi_2$ respectively, if it exists.

Lemma 58. *Let $F = \mathbb{Q}$, and assume (D_1, D_2, x) is nice. Then $\mathrm{O}_{\phi_1,\phi_2}(\ell)$ exists if and only if $\ell^2 \mid \frac{D_1D_2-x^2}{4}$, and when it does, it has reduced discriminant $\frac{D_1D_2-x^2}{4\ell^2}$.*

Proof. Let $D_3 = \frac{x^2-D_1D_2}{\ell^2}$, and let $\phi_3 : \mathcal{O}_{D_3} \to B$ be the embedding induced by $\phi_1 \times \phi_2$. Let $w_i = \phi_i(\sqrt{D_i})$ and $v_i = \phi_i\left(\frac{p_{D_i}+\sqrt{D_i}}{2}\right)$ for $i = 1, 2, 3$, and let $x = \frac{1}{2}\operatorname{trd}(w_1w_2) \equiv D_1D_2 \pmod 2$ by assumption. We have $w_3 = \frac{w_1w_2-x}{\ell}$, whence

$$\frac{1}{2}\operatorname{trd}(w_1w_3) = \frac{1}{2}\operatorname{trd}\left(\frac{D_1w_2 - xw_1}{\ell}\right) = 0.$$

Similarly, $\frac{1}{2}\operatorname{trd}(w_2w_3) = 0$. If D_3 is odd, then since D_1 or D_2 is odd, $p_{D_iD_3} = 1 \not\equiv 0 \pmod 2$ for $i = 1$ or 2, whence $\langle 1, v_i, v_3, v_iv_3\rangle_{\mathbb{Z}}$ is not an order, and $\mathrm{O}_{\phi_1,\phi_2}(\ell)$ does not exist. Since D_3 is a discriminant, if it is not odd it must be a multiple of 4. In particular, we have that $\ell^2 \mid \frac{D_1D_2-x^2}{4}$. In this case, $0 \equiv D_iD_3 \pmod 2$ for $i = 1, 2$, and so by Lemma 35, $\langle 1, v_i, v_j, v_iv_j\rangle_{\mathbb{Z}}$ is an order for $(i, j) = (1, 2), (1, 3), (2, 3)$. Thus by Lemma 56, $\mathrm{O} = \langle 1, v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3, v_1v_2v_3\rangle_{\mathbb{Z}}$ is an order, necessarily the smallest order for which ϕ_i embeds into for all $i = 1, 2, 3$.

Let $p_i = p_{D_i}$, and compute

$$\begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ v_1v_2 \\ v_1v_3 \\ v_2v_3 \\ v_1v_2v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{p_1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{p_2}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{p_1p_2+x}{4} & \frac{p_2}{4} & \frac{p_1}{4} & \frac{\ell}{4} \\ 0 & \frac{-x}{4\ell} & \frac{D_1}{4\ell} & \frac{p_1}{4} \\ 0 & \frac{-D_2}{4\ell} & \frac{x}{4\ell} & \frac{p_2}{4} \\ \frac{x^2-D_1D_2}{8\ell} & \frac{-p_2x-p_1D_2}{8\ell} & \frac{p_1x+p_2D_1}{8\ell} & \frac{p_1p_2+x}{8} \end{pmatrix} \begin{pmatrix} 1 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

Let this transition matrix be M. From the calculation in Lemma 35, we can compute that $d(1, w_1, w_2, w_3) = \left(\frac{1}{\ell}\right)^2 d(1, w_1, w_2, w_1w_2) = -\frac{16(D_1D_2-x^2)^2}{\ell^2}$. It suffices to show that the rows of M generate a $\mathbb{Z}-$lattice with determinant $\frac{1}{16\ell}$, as then we have the discriminant of O being $\frac{(D_1D_2-x^2)^2}{16\ell^4}$, whence the reduced discriminant is $\frac{D_1D_2-x^2}{4\ell^2}$, as desired. The calculation of the rowspace is done by hand in Appendix A. $\square$

Remark 59. The statement $\ell^2 \mid \frac{D_1D_2-x^2}{4}$ only requires (D_1, D_2, x) to be nice at $p = 2$. If it is not nice at $p = 2$, then this does not need to hold. For example, take $D_1 = 20$, $D_2 = 68$, $x = 2$, and B to be ramified at $3, 113$. Then $\mathrm{O}_{\phi_1,\phi_2}(2)$ exists, but $2^2 \nmid \frac{D_1D_2-x^2}{4} = 339$.

Since $\mathrm{O}_{\phi_1,\phi_2} \subseteq \mathrm{O}_{\phi_1,\phi_2}(\ell)$, the inclusion holds when we complete at p. Considering Lemma 49, we find that

- If $p \nmid \frac{D_1D_2-x^2}{4}$, then $\mathrm{O}_{\phi_1,\phi_2,p}(\ell)$ is maximal;
- If $\epsilon(p) = -1$, then $\mathrm{O}_{\phi_1,\phi_2,p}(\ell)$ is contained in a unique maximal order, necessarily the same maximal order as the one containing $\mathrm{O}_{\phi_1,\phi_2,p}$;
- If $\epsilon(p) = 1$, then $\mathrm{O}_{\phi_1,\phi_2,p}(\ell)$ is Eichler.

In particular, this implies that there exists a minimal Eichler order containing $\mathrm{O}_{\phi_1,\phi_2}(\ell)$, denoted $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}(\ell)$. Factorize

$$\frac{D_1D_2-x^2}{4} = \pm \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. The local conditions imply that

$$\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2} = \mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}\left(\prod_{i=1}^{r} p_i^{e_i} \prod_{i=1}^{s} q_i^{f_i}\right),$$

i.e. that the maximum possible level always occurs at the prime factors p of $\frac{D_1D_2-x^2}{4}$ for which $\epsilon(p) = -1$. The analogous assessment of the prime factors p for which $\epsilon(p) = 1$ leads to the following proposition.

Proposition 60. *Let (D_1, D_2, x) be nice, and let $\ell = \prod_{i=1}^{r} p_i^{e_i'} \prod_{i=1}^{s} q_i^{f_i'} \prod_{i=1}^{t} w_i^{g_i'}$, where $e_i' \le e_i$, $f_i' \le f_i$, and $2g_i' \le g_i$. Then the Eichler order $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}(\ell)$ has level $\prod_{i=1}^{t} w_i^{g_i-2g_i'}$. Furthermore, assume that all the p_i, q_i are not potentially bad. Let*

$$S = \{w_i : w_i \mid \mathrm{PB}(D_1, D_2)\}$$

be the set of potentially bad primes among the w_i. Then a superorder O' of $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}(\ell)$ admits ϕ_1, ϕ_2 as optimal embeddings if and only if O' does not contain $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}(w_i)$ for all $w_i \in S$. This implies $g_i' = 0$ for all i such that $w_i \in S$.

Proof. The first half of the proposition has been proven in the above discussion. For the second half, the optimality of ϕ_1, ϕ_2 can only fail if we have a w_j for which ϕ_1 (without loss of generality) descends to an embedding of $\mathcal{O}_{D_1/w_j^2}$. Call this embedding ϕ_1', and as in Proposition 53 the order $\mathrm{O}^{\mathrm{Eich}}_{\phi_1',\phi_2}$ has level $\frac{1}{w_j^2}\prod_{i=1}^t w_i^{g_i}$. It suffices to show that $\mathrm{O}_{\phi_1',\phi_2,w_j} = \mathrm{O}_{\phi_1,\phi_2,w_j}(w_j)$, as this means that picking up a factor of w_j in the level is equivalent to killing optimality.

These Eichler orders have the same level, so it suffices to show inclusion only. However this is immediate, as the embedding $\phi_1' \times \phi_2$ corresponds to an embedding of discriminant $\frac{x^2-D_1D_2}{p^2}$ induced from

$$\phi_1'(\sqrt{D_1/p^2})\phi_2(\sqrt{D_2}) = \frac{1}{p}\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}).$$

$\square$

An embedding pair having level exactly ℓ in O' is equivalent to O' containing $\mathrm{O}_{\phi_1,\phi_2}(\ell)$ but not containing $\mathrm{O}_{\phi_1,\phi_2}(p\ell)$ for any prime p. At long last, we can describe the levels and counts of Eichler orders admitting ϕ_1, ϕ_2 as optimal embeddings.

Theorem 61. *Let ϕ_1, ϕ_2 be $x-$linked embeddings of discriminants D_1, D_2 into B, an indefinite quaternion algebra over $\mathbb{Q}$, let ℓ be a positive integer, and assume that (D_1, D_2, x) is nice. Factorize*

$$\frac{D_1D_2 - x^2}{4} = \pm\prod_{i=1}^r p_i^{2e_i+1}\prod_{i=1}^s q_i^{2f_i}\prod_{i=1}^t w_i^{g_i},$$

where p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Then,

(i) This setup is possible if and only if B is ramified at exactly $p_1, p_2, \ldots p_r$;

(ii) There exists an Eichler order of level $\mathfrak{M}$ for which ϕ_1, ϕ_2 are optimal embeddings into if and only if both of the following are satisfied:

- *None of the p_i, q_i are potentially bad;*
- *$\mathfrak{M} = \prod_{i=1}^t w_i^{g_i'}$ with $g_i' \le g_i$.*

(iii) Let $\mathfrak{M}$ satisfy the above. The number of Eichler orders of level $\mathfrak{M}$ for which ϕ_1, ϕ_2 are optimal embeddings into is

$$\prod_{i=1}^t \begin{cases} g_i + 1 - g_i' & \text{if } w_i \nmid \mathrm{PB}(D_1, D_2); \\ 2 & \text{if } w_i \mid \mathrm{PB}(D_1, D_2) \text{ and } g_i' < g_i; \\ 1 & \text{if } w_i \mid \mathrm{PB}(D_1, D_2) \text{ and } g_i' = g_i. \end{cases}$$

(iv) There exists an Eichler order of level $\mathfrak{M}$ for which ϕ_1, ϕ_2 are optimal embeddings of into of level exactly ℓ if and only we have

$$\ell = \prod_{i=1}^r p_i^{e_i}\prod_{i=1}^s q_i^{f_i}\prod_{i=1}^t w_i^{g_i''},$$

where $2g_i'' \le g_i - g_i'$ and $g_i'' = 0$ if $w_i \mid \mathrm{PB}(D_1, D_2)$.

(v) Let $\mathfrak{M}, \ell$ satisfy the above. Let n be the number of indices i for which $2g_i'' < g_i - g_i'$. Then the number of Eichler orders of level $\mathfrak{M}$ for which ϕ_1, ϕ_2 are optimal embeddings into of level exactly ℓ is 2^n.

Proof. Part i is the content of Theorem 44, and the necessity of the conditions in part ii follows from Proposition 53. To complete part ii, it suffices to prove it locally, and Proposition 55 implies that there is an Eichler order of level $w_i^{g_i-2}$ whose containment must be avoided for each i such that $w_i \mid \mathrm{PB}(D_1, D_2)$ (and no other orders need be avoided).

Recall the inverted triangle of local Eichler orders, as described in Section 3.3. The local Eichler orders containing $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2,w_i}$ form an inverted triangle with $g_i + 1$ rows. There are $g_i + 1 - n$ Eichler orders of level w_i^n in the n^{th} row of the triangle, starting at $n = 0$ and ending at $n = g_i$. Therefore if $w_i \nmid \mathrm{PB}(D_1, D_2)$, there are $g_i + 1 - g_i'$ possible Eichler orders of level $w_i^{g_i'}$. If $w_i \mid \mathrm{PB}(D_1, D_2)$, then there is one when $g_i' = g_i$, and on all rows above it there are two, as the order that we cannot contain has level $w_i^{g_i-2}$. In particular, this implies part ii as this is a non-zero number.

By the local-global principle for orders (Theorem 13), the total count for global orders is the product of the local counts. The count in part iii follows from this and the previous paragraph.

For parts iv, v, Proposition 60 and the discussion surrounding it imply that ℓ has the prime factorization as claimed. The necessity of $2g_i'' \leq g_i - g_i'$ comes from the level of $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2}(w_i^{g_i''})$ having valuation $g_i - 2g_i''$ at w_i. Proposition 60 also implies that if $w_i \mid \mathrm{PB}(D_1, D_2)$, then the valuation of ℓ at w_i must be 0, i.e. $g_i'' = 0$.

To count this, we again work locally and use the local-global principle. The local count is unchanged at the primes w_i for which $w_i \mid \mathrm{PB}(D_1, D_2)$. For primes w_i not satisfying this, we no longer have to worry about optimality. For ease of notation, if the level of the embedding pair is w_i^k, we say it has intersection level k. The Eichler order $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2,w_i}(w_i^n)$ has level $w_i^{g_i-2n}$, and an intersection level is at least n if and only if the order contains $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2,w_i}(w_i^n)$. Drawing the inverted triangle as before, it follows by induction that (noting that all of the orders $\mathrm{O}^{\mathrm{Eich}}_{\phi_1,\phi_2,w_i}(w_i^n)$ are contained inside each other)

- In level $w_i^{g_i-2n}$, there are $2n+1$ orders, of which there are 2 of each intersection level $0, 1, \ldots, n-1$, and one of intersection level n;
- In level $w_i^{g_i-2n+1}$, there are $2n$ orders, of which there are 2 of each intersection level $0, 1, \ldots, n-1$.

In particular, there are 2 orders of intersection level g_i'' when $2g_i'' < g_i - g_i'$, and one when $2g_i'' = g_i - g_i'$. The condition coming from $w_i \mid \mathrm{PB}(D_1, D_2)$ was there are two if $g_i' < g_i$, and one if we had equality. Since $g_i'' = 0$, this condition is absorbed by $2g_i'' < g_i - g_i'$. This completes parts iv, v. $\square$

7. Proof of the main theorem

We are now ready to study $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$.

7.1. **Total x-linking into a given Eichler order.**

As alluded to at the start of Section 5, we need to pass between Eichler orders containing a fixed pair of $x-$linked embeddings, and elements of $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$. This is accomplished in the "inversion theorem", which we now set up for.

Let F be $\mathbb{Q}$ or $\mathbb{Q}_p$, and let B be a quaternion algebra over F of discriminant $\mathfrak{D}$, which is indefinite if $F = \mathbb{Q}$. Let O be an Eichler order of level $\mathfrak{M}$ in B. Assume that D_1, D_2 are positive discriminants for which $\mathrm{Emb}(B, D_1, D_2, x)$ is non-empty, fix $[(\phi_1, \phi_2)] \in \mathrm{Emb}(B, D_1, D_2, x)$, let $\ell^2 \mid \frac{D_1D_2-x^2}{4}$, and define

$$\begin{aligned} T_{\phi_1,\phi_2}(\mathfrak{M}) :=& \{\mathrm{E} : \ \mathrm{E} \text{ is an Eichler order of } B \text{ of level } \mathfrak{M} \\ & \qquad \text{for which } \phi_1, \phi_2 \text{ give optimal embeddings into}\}; \\ T_{\phi_1,\phi_2}(\mathfrak{M}, \ell) :=& \{\mathrm{E} \in T_{\phi_1,\phi_2}(\mathfrak{M}) \text{ such that } (\phi_1, \phi_2) \text{ has level } \ell \text{ in E}\}. \end{aligned}$$

Proposition 62. *We have*

$$|\operatorname{Emb}(\mathrm{O}, D_1, D_2, x, \ell)| = \left|\frac{N_{B^\times}(\mathrm{O})}{F^\times \mathrm{O}^1}\right| |T_{\phi_1,\phi_2}(\mathfrak{M}, \ell)|,$$

and the analogous result without the ℓ. If $F = \mathbb{Q}$, then

$$\left|\frac{N_{B^\times}(\mathrm{O})}{F^\times \mathrm{O}^1}\right| = 2^{\omega(\mathfrak{D}\mathfrak{M})+1}.$$

Proof. By Corollary 33 and Eichler orders of the same level being conjugate, we have that $T_{\phi_1,\phi_2}(\mathfrak{M})$ is non-empty if and only if $S = \operatorname{Emb}(\mathrm{O}, D_1, D_2, x)$ is non-empty. In particular, we can assume that (ϕ_1, ϕ_2) give a class in S, and we will use this pair to define a map $\theta : S \to T_{\phi_1,\phi_2}(\mathfrak{M})$. Given optimal embeddings (ϕ_1', ϕ_2') representing a class in S, by Corollary 33, there exists an $r \in B^\times$ for which $r\phi_i' r^{-1} = \phi_i$ for $i = 1, 2$. Define

$$\theta((\phi_1', \phi_2')) = r\mathrm{O}r^{-1}.$$

It is clear that $r\mathrm{O}r^{-1} \in T_{\phi_1,\phi_2}(\mathfrak{M})$, but we need to check that all choices were well defined. By Corollary 33, the element r is defined up to multiplication by $F^\times$, which does not change $r\mathrm{O}r^{-1}$. If $(\phi_1', \phi_2') \sim (\phi_1'', \phi_2'')$ in S, then there exists an $s \in \mathrm{O}^1$ for which $\phi_i' = s\phi_i'' s^{-1}$ for $i = 1, 2$. The corresponding element r can then be taken to be $r' = rs$, and then $r'\mathrm{O}r'^{-1} = rs\mathrm{O}s^{-1}r^{-1} = r\mathrm{O}r^{-1}$, as desired. Therefore the map θ is well defined.

Next, it is clear that θ is surjective. Indeed, if $\mathrm{E} \in T_{\phi_1,\phi_2}(\mathfrak{M})$, then there exists a $b \in B^\times$ for which $b\mathrm{E}b^{-1} = \mathrm{O}$. Then $(\phi_1^b, \phi_2^b) \in S$, and this pair maps via θ to E, as desired.

Therefore, it suffices to show that θ is a $\left|\frac{N_{B^\times}(\mathrm{O})}{F^\times \mathrm{O}^1}\right|$-to-one map. Assume that $\theta((\phi_1', \phi_2')) = \theta((\phi_1'', \phi_2''))$, and that the pairs correspond to r, s respectively. Then $r\mathrm{O}r^{-1} = s\mathrm{O}s^{-1}$, hence $t = r^{-1}s \in N_{B^\times}(\mathrm{O})$. Writing $s = rt$, it follows that $t^{-1}\phi_i' t = \phi_i''$, so it suffices to determine how $t^{-1}(\phi_1', \phi_2')t$ varies as t ranges over $N_{B^\times}(\mathrm{O})$. For a fixed t, by Corollary 33, the set of elements conjugating (ϕ_1', ϕ_2') to any form in the class of $t^{-1}(\phi_1', \phi_2')t$ is $\mathrm{O}^1 t^{-1} F^\times = t^{-1} F^\times \mathrm{O}^1$. Thus, for distinct t_1, t_2, they correspond to the same image if and only if

$$t_1^{-1} F^\times \mathrm{O}^1 = t_2^{-1} F^\times \mathrm{O}^1,$$

which is equivalent to $t_2 t_1^{-1} \in F^\times \mathrm{O}^1$. This proves the first claim without the ℓ. It is clear that the level of intersection remains constant under θ, hence the statements remain true when we add in the level ℓ.

When $F = \mathbb{Q}$, Proposition 18 yields

$$\frac{N_{B^\times}(\mathrm{O})}{\mathbb{Q}^\times \mathrm{O}^1} \simeq \prod_{p \mid \mathfrak{D}\mathfrak{M}\infty} \frac{\mathbb{Z}}{2\mathbb{Z}},$$

which implies the final result. □

Combining Proposition 62 with Theorem 61 produces the count of $x-$linking.

Theorem 63. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$, let O be an Eichler order of level $\mathfrak{M}$, let (D_1, D_2, x) be nice, and let ℓ be a positive integer. Factorize*

$$\frac{D_1 D_2 - x^2}{4} = \pm \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where the p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Then *(i) The set* $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ *is non-empty if and only if all of the following hold:*

- $\mathfrak{D} = \prod_{i=1}^{r} p_i$*;*
- *None of the* p_i, q_i *are potentially bad;*
- $\mathfrak{M} = \prod_{i=1}^{t} w_i^{g_i'}$ *with* $g_i' \leq g_i$*.*

(ii) Assume the above holds. Then

$$|\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)| = 2^{\omega(\mathfrak{D}\mathfrak{M})+1} \prod_{i=1}^{t} \begin{cases} g_i + 1 - g_i' & \text{if } w_i \nmid \mathrm{PB}(D_1, D_2); \\ 2 & \text{if } w_i \mid \mathrm{PB}(D_1, D_2) \text{ and } g_i' < g_i; \\ 1 & \text{if } w_i \mid \mathrm{PB}(D_1, D_2) \text{ and } g_i' = g_i. \end{cases}$$

(iii) The set $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x, \ell)$ *is non-empty if and only if* ℓ *takes the form*

$$\ell = \prod_{i=1}^{r} p_i^{e_i} \prod_{i=1}^{s} q_i^{f_i} \prod_{i=1}^{t} w_i^{g_i''},$$

where $2g_i'' \leq g_i - g_i'$ *and* $g_i'' = 0$ *if* $w_i \mid \mathrm{PB}(D_1, D_2)$*.*

(iv) Assume the above holds. Let n *be the number of indices* i *for which* $2g_i'' < g_i - g_i'$*. Then*

$$|\mathrm{Emb}(\mathrm{O}, D_1, D_2, x, \ell)| = 2^{\omega(\mathfrak{D}\mathfrak{M})+n+1}.$$

Most of Theorem 10 now follows by specializing Theorem 63 to the case of D_1, D_2 being coprime, fundamental, and O being maximal. The only unproven claim is the final one about the signs of intersections, and this is considered in the next section.

7.2. **Orientations and sign of intersection.** Up until now, the orientations of optimal embeddings and the sign of intersection has been completely ignored; we now address this issue.

Lemma 64. *Let* B *be an indefinite quaternion algebra over* $\mathbb{Q}$*, let* O *be an Eichler order of level* $\mathfrak{M}$*, let* (ϕ_1, ϕ_2) *be* $x-$*linked optimal embeddings of positive discriminants* D_1, D_2 *respectively where* (D_1, D_2, x) *is admissible, let* $v \mid \mathfrak{D}\mathfrak{M}\infty$*, and let* $\omega_v \in N_B^{\times}(\mathrm{O})$ *be as in Proposition 18. Then* $(\phi_1^{w_v}, \phi_2^{w_v})$ *is an* $x-$*linked pair of optimal embeddings into* O *with the same level as* (ϕ_1, ϕ_2)*. Furthermore, if* $x^2 < D_1D_2$*, then*

- *If* $v = \infty$ *then the orientations are the same, but the sign of intersection is opposite.*
- *If* $v < \infty$*, then the orientations are negated at* v *only, and the sign of intersection is the same.*

Proof. It is clear that $(\phi_1^{w_v}, \phi_2^{w_v})$ remains $x-$linked, optimal, has the same intersection level, and the orientation follows from Proposition 28. Having opposite sign of intersection is equivalent to $\phi_1 \times \phi_2$ swapping orientation at ∞ when conjugating by w_v, and this also follows from Proposition 28. □

In particular, any element of O^{-1} (reduced norm -1) acts as an involution on $\mathrm{Emb}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$, dividing it into equal sized sets of intersection sign being 1 and -1. This completes the final claim of Theorem 10.

Definition 65. If o_1, o_2 are orientations of optimal embeddings, then attaching the subscript o_1, o_2 to any of the sets defined as $\mathrm{Emb}(\mathrm{O}, D_1, D_2, \ldots)$ means we only take the pairs of optimal embeddings of the specified orientations. Thus, $\mathrm{Emb}^{+}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$ counts the equivalence classes of pairs $[(\phi_1, \phi_2)]$ of optimal embeddings of discriminants D_1, D_2 and orientations o_1, o_2 that are $x-$linked of level ℓ with positive sign.

Lemma 66. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$, let O be an Eichler order of level $\mathfrak{M}$, and let (D_1, D_2, x) be admissible. Assume that $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ is non-empty, let o_1 be a possible orientation of an optimal embedding of $\mathcal{O}_{D_1}$ into O, and assume that $\gcd(D_1D_2, \mathfrak{M}) = 1$. Then there exists a $[(\phi_1, \phi_2)] \in \mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ for which ϕ_1 has orientation o_1. For each $p \mid \mathfrak{D}\mathfrak{M}$, we also have:*

- *If $p \nmid D_1$, then $o_p(\phi_2)$ is uniquely determined;*
- *If $p \mid D_1$ but $p \nmid D_2$, then $o_p(\phi_2)$ can be both 1 and -1.*

Finally, there is a positive integer N such that for all orientations (o_1, o_2), we have

$$|\mathrm{Emb}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)| \in \{0, N\},$$

and the same result holds with $N/2$ for $\mathrm{Emb}^+_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$ if $x^2 < D_1D_2$.

Proof. Start with $[(\phi_1', \phi_2')] \in \mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$, and from Lemma 64 we can conjugate the pair by ω_p for $p \mid \mathfrak{D}\mathfrak{M}$ to get (ϕ_1, ϕ_2) with ϕ_1 having orientation o_1.

If $p \mid D_1$ but $p \nmid D_2$, the local orientation result follows from from conjugating the embeddings by ω_p, as $o_p(\phi_1) = 0$.

Next, assume $p \nmid D_1$. It suffices to prove this lemma locally, so first assume we have $p \mid \mathfrak{D}$, i.e. B_p is division. As in the proof of Lemma 39, write $B_p = \left(\frac{D_1, e}{\mathbb{Q}_p}\right)$, with $\phi_{1,p}(\sqrt{D_1}) = i$, $(D_1, e)_p = -1$, and $v_p(e)$ being necessarily odd. Assume p is odd, let $\phi_{2,p}(\sqrt{D_2}) = fi + gj + hk$ for $f, g, h \in \mathbb{Z}_p$, and the trace condition gives that $f = \frac{x}{D_1}$. Let $\mathfrak{p}$ be the maximal order in O_p, and since $p \mid \mathrm{nrd}(j), \mathrm{nrd}(k)$,

$$\phi_{2,p}(\sqrt{D_2}) \equiv \frac{x}{D_1} i \pmod{\mathfrak{p}},$$

which only depends on x, D_1. Therefore by Lemma 27, the local orientation of ϕ_2 at p is fixed. If $p = 2$, then the analogous computations involving $\phi_{2,p}\left(\frac{p_{D_2} + \sqrt{D_2}}{2}\right)$ imply the result.

Otherwise, assume that $B_p = \mathrm{Mat}(2, \mathbb{Q}_p)$, and O_p is the standard Eichler level of order p^e with $e > 0$. Let $e_1 = e + v_2(p)$, and then working modulo p^{e_1} we write

$$\phi_1(\sqrt{D_1}) \equiv \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \pmod{p^{e_1}}, \qquad \phi_2(\sqrt{D_2}) \equiv \begin{pmatrix} c & d \\ 0 & -c \end{pmatrix} \pmod{p^{e_1}}.$$

Therefore $x \equiv ac \pmod{p^{e_1}}$, and since $p \nmid a$ (else $p \mid D_1$), we have $c \equiv \frac{x}{a} \pmod{p^{e_1}}$. By Lemma 27, the local orientation of ϕ_2 at p is fixed.

Finally, the above shows that we can pass between all pairs (o_1, o_2) for which $\mathrm{Emb}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$ is non-empty via conjugation by ω_p for $p \mid \mathfrak{D}\mathfrak{M}$, hence these sets all have the same size. If $x^2 < D_1D_2$, then exactly half of the pairs in a given set have positive intersection sign, which completes the lemma. □

We can say even more about how the possible x's divide across a pair of orientations.

Proposition 67. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$, let O be an Eichler order of level $\mathfrak{M}$, let D_1, D_2 be positive discriminants, and let o_1, o_2 be possible orientations of optimal embeddings of discriminants D_1, D_2 into O. Then there exists an integer x_{o_1,o_2} such that for all optimal embeddings $\phi_i \in \mathrm{Emb}_{o_i}(\mathrm{O}, D_i)$ $(i = 1, 2)$, we have*

$$x_{o_1,o_2} \equiv \frac{1}{2}\,\mathrm{trd}\left(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2})\right) \pmod{2\mathfrak{D}\mathfrak{M}}.$$

In particular, the possible x−linkings across an orientation pair are all equivalent modulo $2\mathfrak{D}\mathfrak{M}$.

Proof. Fix another pair $\phi_i' \in \mathrm{Emb}_{o_i}(\mathrm{O}, D_i)$, and say that ϕ_1, ϕ_2 are $x-$linked and ϕ_1', ϕ_2' are $x'-$linked. It suffices to show that $x \equiv x' \pmod{2\mathfrak{D}\mathfrak{M}}$. We can work locally, so start with $p \mid \mathfrak{D}$, and assume that ϕ_i, ϕ_i' now land in O_p. Let $\mathfrak{p}$ be the unique maximal order of O_p, and as the embeddings have the same orientation, there exists $u_1, u_2 \in \mathrm{O}_p^1$ for which $\phi_i' = \phi_i^{u_i}$ for $i = 1, 2$. Since $\mathrm{O}_p/\mathfrak{p} \simeq \mathbb{F}_{p^2}$ is commutative, when working modulo $\mathfrak{p}$ we can rearrange terms freely. Thus

$$\phi_1^{u_1}\left(\frac{p_{D_1} + \sqrt{D_1}}{2}\right)\phi_2^{u_2}\left(\frac{p_{D_2} + \sqrt{D_2}}{2}\right) \equiv \phi_1\left(\frac{p_{D_1} + \sqrt{D_1}}{2}\right)\phi_2\left(\frac{p_{D_2} + \sqrt{D_2}}{2}\right) \pmod{\mathfrak{p}}.$$

Taking reduced traces implies that

$$\frac{p_{D_1}p_{D_2} + x'}{2} \equiv \frac{p_{D_1}p_{D_2} + x}{2} \pmod{\mathfrak{p}}.$$

If $p \neq 2$, it follows that $x' \equiv x \pmod{\mathfrak{p}}$, whence $x' \equiv x \pmod{p}$ by subtracting and taking the norm. If $p = 2$, then $x' \equiv x \pmod{2\mathfrak{p}}$, and so subtracting and taking norms gives $8 \mid (x' - x)^2$, hence $x \equiv x' \pmod{4}$.

Next, assume that $p^e \mid\mid \mathfrak{M}$ with $e > 0$, and assume that O_p is the standard Eichler order of level p^e. As the embeddings have the same orientation, there exists $u_1, u_2 \in \mathrm{O}_p^1$ for which $\phi_i' = \phi_i^{u_i}$ for $i = 1, 2$. Explicitly write

$$\phi_i\left(\frac{p_{D_i} + \sqrt{D_i}}{2}\right) = \begin{pmatrix} a_i & b_i \\ p^e c_i & p_{D_i} - a_i \end{pmatrix}, \qquad u_i = \begin{pmatrix} f_i & g_i \\ p^e h_i & k_i \end{pmatrix}. \tag{7.1}$$

It follows that $f_i k_i \equiv 1 \pmod{p^e}$. Modulo p^e, we compute

$$\phi_i^{u_i}\left(\frac{p_{D_i} + \sqrt{D_i}}{2}\right) \equiv \begin{pmatrix} a_i & f_i(p_{D_i}g_i - 2g_i a_i + f_i b_i) \\ 0 & p_{D_i} - a_i \end{pmatrix} \pmod{p^e}. \tag{7.2}$$

By taking the explicit expressions for $\phi_i\left(\frac{p_{D_i}+\sqrt{D_i}}{2}\right)$, doubling and subtracting p_{D_i}, and multiplying together, we find that

$$x \equiv (2a_1 - p_{D_1})(2a_2 - p_{D_2}) \equiv x' \pmod{p^{e+v_2(p)}},$$

as claimed.

Combining the above shows that $x \equiv x' \pmod{2\mathfrak{D}\mathfrak{M}}$ if $2 \mid \mathfrak{D}\mathfrak{M}$, and $x \equiv x' \pmod{\mathfrak{D}\mathfrak{M}}$ otherwise. In this case, $x \equiv p_{D_1}p_{D_2} \equiv x' \pmod{2}$, so the same conclusion follows. $\square$

If D_1 is coprime to $\mathfrak{D}\mathfrak{M}$, then Lemma 66 and Proposition 67 can be used to show that for o_1 fixed, the integers x_{o_1,o_2} are all distinct modulo $2\mathfrak{D}\mathfrak{M}$ across all orientations o_2. If D_1 has factors in common with $\mathfrak{D}\mathfrak{M}$, this no longer needs to be true at those primes. Furthermore, not all x's satisfying the congruence condition will necessarily appear as $x-$linkings, as this depends on the actual factorization of $\frac{D_1D_2-x^2}{4}$, and not just on congruences. For example, this number will always be divisible by $\mathfrak{D}\mathfrak{M}$, but prime factors of $\mathfrak{D}$ could appear to even powers.

Lemma 66 allows us to count the sizes of $\mathrm{Emb}^+_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$, by dividing $|\mathrm{Emb}^+(\mathrm{O}, D_1, D_2, x, \ell)|$ across the total number of orientations. We record this in the final corollary.

Corollary 68. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\mathfrak{D}$, let O be an Eichler order of level $\mathfrak{M}$, let (D_1, D_2, x) be nice, and let ℓ be a positive integer. Factorize*

$$\frac{D_1D_2 - x^2}{4} = \pm\prod_{i=1}^{r} p_i^{2e_i+1}\prod_{i=1}^{s} q_i^{2f_i}\prod_{i=1}^{t} w_i^{g_i},$$

where the p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Assume that

- $\mathfrak{D} = \prod_{i=1}^{r} p_i$;
- *None of the p_i or q_i are potentially bad;*
- $\mathfrak{M} = \prod_{i=1}^{t} w_i^{g_i'}$ *with* $g_i' \le g_i$ *and* $\gcd(\mathfrak{M}, D_1D_2) = 1$;
- $\ell = \prod_{i=1}^{r} p_i^{e_i} \prod_{i=1}^{s} q_i^{f_i} \prod_{i=1}^{t} w_i^{g_i''}$, *where* $2g_i'' \le g_i - g_i'$ *and* $g_i'' = 0$ *if* $w_i \mid \mathrm{PB}(D_1, D_2)$.

Let n be the number of indices i for which $2g_i'' < g_i - g_i'$. If $x^2 < D_1D_2$, then for every pair of orientations (o_1, o_2), we have

$$|\operatorname{Emb}^{+}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)| = 2^n \text{ or } 0.$$

If $x^2 > D_1D_2$, then the same result holds without the $+$ and n replaced by $n+1$.

Proof. By Theorem 63, the count without the orientations or $+$ is $2^{\omega(\mathfrak{D}\mathfrak{M})+n+1}$. If $p \mid \mathfrak{D}\mathfrak{M}$, then since the triple is nice and $\gcd(\mathfrak{M}, D_1D_2) = 1$, Lemma 66 implies that there are precisely 2 pairs $(o_p(\phi_1), o_p(\phi_2))$ which admit $x-$linking. Hence we divide by 2 for all such p, eliminating the factor of $2^{\omega(\mathfrak{D}\mathfrak{M})}$. Finally, if $x^2 < D_1D_2$, exactly half of the embeddings have positive sign, which implies the result. □

Corollary 68 approaches the limits of what we can do with this approach. When non-empty, the set $\operatorname{Emb}^{+}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$ has size 2^n, and distributes itself across the $h^+(D_1)h^+(D_2)$ pairs of equivalence classes of the specified orientations. A rough description of what we can say about this distribution is as follows:

- Fix $[(\phi_1, \phi_2)] \in \operatorname{Emb}^{+}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$. Then the map θ found in Proposition 62 combined with the work on $T_{\phi_1,\phi_2}(\mathfrak{M})$ allows us to describe possible values of $\mathrm{nrd}(r)$ for $r \in \mathrm{O}$ such that $[(\phi_1^r, \phi_2^r)] \in \operatorname{Emb}^{+}_{o_1,o_2}(\mathrm{O}, D_1, D_2, x, \ell)$ and $[(\phi_1^r, \phi_2^r)] \neq [(\phi_1, \phi_2)]$; they are essentially products of powers of prime divisors p of $\frac{D_1D_2 - x^2}{4}$ with $\epsilon(p) = 1$.
- The integers represented by the element of the class group $\mathrm{Cl}^+(D_i)$ taking ϕ_i to ϕ_i^r correspond to the norms of elements in O conjugating ϕ_i to ϕ_i^r (see Sections 4.4 and 4.5 of [Ric21b]).
- In particular, the distribution relates to the representations of products of primes $p \mid \frac{D_1D_2 - x^2}{4}$ with $\epsilon(p) = 1$ by binary quadratic forms of discriminants D_1, D_2.

Of course, even if we could make this more formal and explicit, it does not tell us how the distinct x values interact, which is important for intersection numbers.

8. Examples

We present a few examples that illustrate the results of Theorem 63 and Corollary 68. All computations were done in PARI/GP ([PAR23]), and the code to replicate these examples can be found in [Ric21c].

Example 69. Let $D_1 = 5$ and $D_2 = 381$, so that D_1, D_2 are coprime and fundamental. Since $43 < \sqrt{5 \cdot 381} < 44$, to compute which algebras admit non-trivial intersections of D_1, D_2, it suffices to compute $\frac{5 \cdot 381 - x^2}{4}$ for odd $|x| \le 43$, and find $\epsilon(p)$ for all prime divisors. The values of $\epsilon(p)$ with $p \le 80$ are in Table 1.

Table 2 displays the possible ramifications of the quaternion algebras, along with the corresponding positive x's (since x and $-x$ correspond to the same algebra).

Let's focus on $B = \left(\frac{3,-1}{\mathbb{Q}}\right)$, which is ramified at 2, 3. Let $\mathrm{O} = \langle 1, i, j, \frac{1+i+j+k}{2} \rangle_{\mathbb{Z}}$, which is maximal. There are four orientations and $h^+(5) = 1$, hence by Proposition 23 there are 4 embedding classes of discriminant

TABLE 1. $\epsilon(p)$ for $D_1 = 5$, $D_2 = 381$, $p \leq 80$.

p	2	3	5	7	17	19	29	31	43	47	59	61	67	79
$\epsilon(p)$	-1	-1	1	-1	-1	1	1	1	-1	-1	1	1	-1	1

TABLE 2. Quaternion algebras admitting non-trivial intersections in a maximal order for discriminants 5 and 381.

Ramifying primes	$\emptyset$	$2, 3$	$2, 7$	$2, 17$	$2, 43$	$2, 47$
Positive x's	$7, 17, 25, 31$	$3, 9, 21, 27, 39$	$13, 29, 41, 43$	35	23	5
Ramifying primes	$2, 67$	$2, 193$	$2, 223$	$3, 7$	$3, 17$	$7, 17$
Positive x's	37	19	11	15	33	1

5. Since $h^+(381) = 2$ and $3 \mid 381$, there are two orientations, and 4 total embedding classes of discriminant 381. Representative embeddings are given in Table 3.

TABLE 3. Optimal embedding classes for $D = 5, 381$.

D	$o_2(\phi)$	$o_3(\phi)$	$\phi\left(\frac{p_D+\sqrt{D}}{2}\right)$
5	1	1	$\frac{1-i-j+k}{2}$
5	-1	1	$\frac{1-i-j-k}{2}$
5	1	-1	$\frac{1+i+j+k}{2}$
5	-1	-1	$\frac{1+i+j-k}{2}$
381	1	0	$\frac{1-11i-3j+3k}{2}$
381	1	0	$\frac{1+9i-3j+7k}{2}$
381	-1	0	$\frac{1-11i-3j-3k}{2}$
381	-1	0	$\frac{1+9i-3j-7k}{2}$

The possible x's have $|x| = \{3, 9, 21, 27, 39\}$. For each x, we factor $\frac{5\cdot 381 - x^2}{4}$ in Table 4, and determine the possible levels.

It turns out that each x corresponds to a unique level, though this need not be the case in general. This data says that $|\operatorname{Emb}^+_{o_1,o_2}(\mathrm{O}, 5, 381, x, \ell)|$ should be 0 or 2 for $|x| \in \{3, 9, 21\}$, and 0 or 1 for the $|x| \in \{27, 39\}$. Let ϕ_1 be the first embedding of discriminant 5 as given in Table 3, and let σ_1, σ_2 be the first two embeddings of discriminant 381 as given in the same table. For each intersection of ϕ_1 with σ_i, we take a pair (ϕ_1', σ_i) representing the intersection, and record the data in Table 5 (the signed level is the product of the sign and the level).

TABLE 4. Factorization of $\frac{5\cdot 381-x^2}{4}$ for $|x| = \{3, 9, 21, 27, 39\}$.

$\lvert x\rvert$	$\frac{5\cdot 381-x^2}{4}$	$\prod p_i^{e_i}$	$\prod q_i^{f_i}$	$\prod w_i^{g_i}$	Possible levels	n
3	474	$2^1 3^1$		79^1	1	1
9	456	$2^3 3^1$		19^1	2	1
21	366	$2^1 3^1$		61^1	1	1
27	294	$2^1 3^1$	7^2		7	0
39	96	$2^5 3^1$			4	0

TABLE 5. Intersection of ϕ_1 with σ_1, σ_2.

Intersections with σ_1			Intersections with σ_2		
$\phi_1'\left(\frac{1+\sqrt{5}}{2}\right)$	x	Signed level	$\phi_1'\left(\frac{1+\sqrt{5}}{2}\right)$	x	Signed level
$\frac{1-13i-55j-29k}{2}$	3	-1	$\frac{1+i-j-k}{2}$	3	1
$\frac{1-13i+197j-113k}{2}$	3	-1	$\frac{1+101i+359j-181k}{2}$	3	1
$\frac{1+31i+131j+69k}{2}$	-9	2	$\frac{1-i-j+k}{2}$	-9	-2
$\frac{1+31i-469j+269k}{2}$	-9	2	$\frac{1-41i-145j+73k}{2}$	-9	-2
$\frac{1-87i-373j-197k}{2}$	-21	-1	$\frac{1+i+5j-3k}{2}$	-21	1
$\frac{1-711i-3031j-1599k}{2}$	-21	-1	$\frac{1+11i+41j-21k}{2}$	-21	1
$\frac{1+223i+953j+503k}{2}$	27	7	$\frac{1-3i-13j+7k}{2}$	27	-7
$\frac{1-i-j+k}{2}$	39	4	$\frac{1-29i+71j+29k}{2}$	39	-4

This data agrees with the theoretical claim. It also satisfies Proposition 67, since the $x-$values are all equivalent modulo $2\mathfrak{D}\mathfrak{M} = 12$. For the other orientation of 381, we have essentially the same data, except the x's are all negated.

For another interesting example, we consider a non-maximal Eichler order, and compare it to the results for the maximal order.

Example 70. Let $D_1 = 73$, $D_2 = 937$, and $x = 89$. Then D_1, D_2 are coprime, fundamental, and have class number 1 each. Let $B = \left(\frac{7,5}{\mathbb{Q}}\right)$, which is ramified at $5, 7$. Let O be a maximal order and O' an Eichler order of level 3, given by

$$\mathrm{O} = \left\langle 1, \frac{1+j}{2}, i, \frac{1+i+j+k}{2} \right\rangle_{\mathbb{Z}}, \qquad \mathrm{O}' = \left\langle 1, i, \frac{1+3j}{2}, \frac{1+i+j+k}{2} \right\rangle_{\mathbb{Z}}.$$

There are 4 embedding classes into O and 8 embedding classes into O′ of each discriminant, each corresponding to a distinct orientation. Since

$$\frac{73 \cdot 937 - 89^2}{4} = (5^1 7^1)()(2^4 3^3),$$

with $\epsilon(5) = \epsilon(7) = -1$ and $\epsilon(2) = \epsilon(3) = 1$ (the empty parentheses indicate the absence of q_i's), the sets $\mathrm{Emb}(X, 73, 937, 89)$ should be non-empty for $X = \mathrm{O}, \mathrm{O}'$. Fix the optimal embeddings

$$\phi_1\left(\frac{1+\sqrt{73}}{2}\right) = \frac{1-2i+3j}{2}, \qquad \phi_2\left(\frac{1+\sqrt{937}}{2}\right) = \frac{1+14i+5j-4k}{2},$$

which land in and are optimal with respect to both O and O′. Since

$$\frac{1}{2}\operatorname{trd}(\phi_1(\sqrt{73})\phi_2(\sqrt{937})) = -121 \equiv 89 \pmod{2 \cdot 3 \cdot 5 \cdot 7},$$

$\mathrm{Int}_{\mathrm{O}}(\phi_1, \phi_2)$ and $\mathrm{Int}_{\mathrm{O}'}(\phi_1, \phi_2)$ should have 89−linkage. As the class numbers are both one, this is all of the 89−linkage for the given orientations. Corollary 68 predicts the levels and counts, which is recorded in Table 6.

Table 6. Theoretical prediction for counts of levels.

ℓ	$\lvert \mathrm{Emb}^+_{o_1,o_2}(\mathrm{O}, 73, 937, 89, \ell)\rvert$	$\lvert \mathrm{Emb}^+_{o_1,o_2}(\mathrm{O}', 73, 937, 89, \ell)\rvert$
1	4	4
2	4	4
3	4	2
4	2	2
6	4	2
12	2	1

The difference in counts comes only at $w_i = 3$, where $2g_i'' < g_i - g_i' = 3 - g_i'$ is true for $g_i'' = 0, 1$ when the level is maximal, but is only true for $g_i'' = 0$ when $g_i' = 1$, the Eichler order of level 3.

We compute the 89−linkage of ϕ_1, ϕ_2. For each intersection with positive sign, we take a representative pair (ϕ_1, ϕ_2'), and record ϕ_2' and the level in Tables 7 and 8.

This data agrees with Table 6.

For a final example, we introduce a non-fundamental discriminant.

Example 71. Let $D_1 = 241$ and $D_2 = 2736$, which are coprime, and let $x = 324$. Note that D_1 is fundamental, but $D_2 = 2^2 3^2 76$, where 76 is fundamental. Take $B = \left(\frac{77,-1}{\mathbb{Q}}\right)$, which is ramified at 7, 11. Let $\mathrm{O} = \langle 1, \frac{1+i}{2}, j, \frac{j+k}{2}\rangle_{\mathbb{Z}}$, which is maximal. We have $h^+(241) = 1$ and $h^+(2736) = 4$, and consider the 5 optimal embeddings in Table 9 (one being of discriminant 241, and the other 4 being one entire orientation of discriminant 2736).

Factorize

$$\frac{241 \cdot 2736 - 324^2}{4} = (7^1 11^1)()(2^3 3^2 5^2),$$

Table 7. Positive 89−linking of ϕ_1 with ϕ_2 in O.

$\phi_2'\left(\frac{1+\sqrt{937}}{2}\right)$	ℓ	$\phi_2'\left(\frac{1+\sqrt{937}}{2}\right)$	ℓ
$\frac{1+22559i+21061j-12851k}{2}$	1	$\frac{1+119i+117j-69k}{2}$	3
$\frac{1+1769i+1657j-1009k}{2}$	1	$\frac{1+1428689i+1333449j-813783k}{2}$	3
$\frac{1+1769i+1657j+1009k}{2}$	1	$\frac{1+14i+19j-8k}{2}$	4
$\frac{1+22559i+21061j+12851k}{2}$	1	$\frac{1+14i+19j+8k}{2}$	4
$\frac{1+584i+551j+334k}{2}$	2	$\frac{1+6907484i+6446991j-3934506k}{2}$	6
$\frac{1+584i+551j-334k}{2}$	2	$\frac{1+4664i+4359j+2658k}{2}$	6
$\frac{1+44i+47j+26k}{2}$	2	$\frac{1+6907484i+6446991j+3934506k}{2}$	6
$\frac{1+44i+47j-26k}{2}$	2	$\frac{1+4664i+4359j-2658k}{2}$	6
$\frac{1+119i+117j+69k}{2}$	3	$\frac{1+179534i+167571j-102264k}{2}$	12
$\frac{1+1428689i+1333449j+813783k}{2}$	3	$\frac{1+179534i+167571j+102264k}{2}$	12

Table 8. Positive 89−linking of ϕ_1 with ϕ_2 in O′.

$\phi_2'\left(\frac{1+\sqrt{937}}{2}\right)$	ℓ	$\phi_2'\left(\frac{1+\sqrt{937}}{2}\right)$	ℓ
$\frac{1+1769i+1657j+1009k}{2}$	1	$\frac{1+1428689i+1333449j+813783k}{2}$	3
$\frac{1+119i+117j+69k}{2}$	1	$\frac{1+119i+117j-69k}{2}$	3
$\frac{1+1428689i+1333449j-813783k}{2}$	1	$\frac{1+14i+19j-8k}{2}$	4
$\frac{1+22559i+21061j-12851k}{2}$	1	$\frac{1+179534i+167571j+102264k}{2}$	4
$\frac{1+44i+47j+26k}{2}$	2	$\frac{1+6907484i+6446991j-3934506k}{2}$	6
$\frac{1+584i+551j-334k}{2}$	2	$\frac{1+4664i+4359j-2658k}{2}$	6
$\frac{1+6907484i+6446991j+3934506k}{2}$	2	$\frac{1+179534i+167571j-102264k}{2}$	12
$\frac{1+4664i+4359j+2658k}{2}$	2		

where $\epsilon(7) = \epsilon(11) = -1$ and $\epsilon(2) = \epsilon(3) = \epsilon(5) = 1$. As $\mathrm{PB}(241, 2736) = 2 \cdot 3$, the primes $2, 3$ are potentially bad and therefore cannot occur in the intersection level. In particular, for 324−linking, the only valid intersection levels are $1, 5$ (whereas if D_1, D_2 were fundamental, we could get all divisors of 30). The table of predicted levels and counts is found in Table 10.

Since

$$\frac{1}{2}\,\mathrm{trd}(\phi(\sqrt{241})\sigma_1(\sqrt{2736})) = 786 \equiv 324 \pmod{2 \cdot 7 \cdot 11},$$

intersections of ϕ with σ_i should exhibit the above $324-$linking behaviour. We compute the possible positive $324-$linking between ϕ and σ_i for $i = 1, 2, 3, 4$, and represent each intersection by a pair (ϕ', σ_i). The corresponding data is found in Table 11.

This data agrees with the theoretical claim.

TABLE 9. Optimal embedding classes for $D = 241, 2736$.

Label	D	$o_7(\phi)$	$o_{11}(\phi)$	$\phi\left(\frac{p_D+\sqrt{D}}{2}\right)$
ϕ	241	1	1	$\frac{1+i-12j+2k}{2}$
σ_1	2736	1	1	$\frac{2i-50j+8k}{2}$
σ_2	2736	1	1	$\frac{10i-281j+31k}{2}$
σ_3	2736	1	1	$\frac{2i-50j-8k}{2}$
σ_4	2736	1	1	$\frac{10i-281j-31k}{2}$

TABLE 10. Theoretical prediction for counts of levels.

ℓ	$\lvert \mathrm{Emb}^+_{o_1,o_2}(\mathrm{O}, 241, 2736, 324, \ell)\rvert$
1	8
5	4

TABLE 11. Positive $324-$linking of ϕ_1 with σ_i.

i	$\phi_1'\left(\frac{1+\sqrt{241}}{2}\right)$	ℓ	i	$\phi_1'\left(\frac{1+\sqrt{241}}{2}\right)$	ℓ
1	$\frac{1+51079i+839827j-80937k}{2}$	1	3	$\frac{1-5i-89j-9k}{2}$	1
1	$\frac{1+39i-397j+23k}{2}$	1	3	$\frac{1-449i+4531j+255k}{2}$	1
1	$\frac{1+2433i-24575j+1387k}{2}$	5	3	$\frac{1-7i+65j+3k}{2}$	5
2	$\frac{1-17i+1220j-138k}{2}$	1	4	$\frac{1-87657i+1615987j+161959k}{2}$	1
2	$\frac{1+259i-4786j+480k}{2}$	1	4	$\frac{1-21i+373j+37k}{2}$	1
2	$\frac{1+5i-89j+9k}{2}$	5	4	$\frac{1-1395i+25706j+2576k}{2}$	5

Appendix A. Hermite normal form calculation

We calculate the determinant of the row-space of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{p_1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{p_2}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{p_1p_2+x}{4} & \frac{p_2}{4} & \frac{p_1}{4} & \frac{\ell}{4} \\ 0 & \frac{-x}{4\ell} & \frac{D_1}{4\ell} & \frac{p_1}{4} \\ 0 & \frac{-D_2}{4\ell} & \frac{x}{4\ell} & \frac{p_2}{4} \\ \frac{x^2-D_1D_2}{8\ell} & \frac{-p_2x-p_1D_2}{8\ell} & \frac{p_1x+p_2D_1}{8\ell} & \frac{p_1p_2+x}{8} \end{pmatrix},$$

where:

- D_1, D_2 are discriminants with parities p_1, p_2 respectively;
- $\gcd\left(D_1, D_2, D_1D_2 - x^2\right) = 1$
- $4\ell^2 \mid D_1D_2 - x^2$.

Let L be this rowspace, and label the rows $r_1, \ldots, r_8$. Since $L \supseteq \mathbb{Z}^4$ and $\mathbb{Z}^4$ has determinant 1, we see that the determinant of L is $\frac{1}{N}$ for some positive integer N. Our aim is to show that $N = 16\ell$. We can compute N by tensoring our space with $\mathbb{Z}_p$ for all primes p, and determining the power of p dividing the determinant of the corresponding $\mathbb{Z}_p$ lattice.

Note that all denominators of M divide 8ℓ. Hence $p \nmid 2\ell$ implies that $L_p = \mathbb{Z}_p^4$, and so $v_p(N) = 0$, as desired.

Next, assume that $p \mid 2\ell$ is odd. Thus $p \mid \ell \mid D_1D_2 - x^2$, which implies that D_1 and D_2 are not both divisible by p. The first four rows of M_p span $\mathbb{Z}_p^4$, and the fifth row is already in this span. Since $\ell \mid \ell^2 \mid D_1D_2 - x^2$, by removing the powers of 2 and applying row operations, the last three rows (labeled r_6', r_7', r_8' in order) become

$$\begin{pmatrix} 0 & \frac{-x}{\ell} & \frac{D_1}{\ell} & 0 \\ 0 & \frac{-D_2}{\ell} & \frac{x}{\ell} & 0 \\ 0 & \frac{-p_2x-p_1D_2}{\ell} & \frac{p_1x+p_2D_1}{\ell} & 0 \end{pmatrix}.$$

First, $r_8' = p_2r_6' + p_1r_7'$, so we can ignore r_8'. Next, we have

$$xr_6' - D_1r_7', D_2r_6' - xr_7' \in \langle r_1, r_2, r_3, r_4 \rangle_{\mathbb{Z}_p}.$$

Without loss of generality assume that $p \nmid D_1$, whence $r_7' \in \langle r_1, r_2, r_3, r_4, r_6' \rangle_{\mathbb{Z}_p}$. Then $r_3 \in \langle r_1, r_2, r_4, r_6' \rangle_{\mathbb{Z}_p}$, and thus our basis is spanned by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-x}{\ell} & \frac{D_1}{\ell} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The power of p dividing the denominator of this determinant is $v_p(\ell) = v_p(16\ell)$, as desired.

The remaining case is $p = 2$. Let $v_2(\ell) = k \geq 0$, write $\ell = 2^k \ell'$ with ℓ' odd, and without loss of generality, assume that D_1 is odd. Working over $\mathbb{Z}_2$, we multiply out by odd factors to obtain the row-space

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{p_2}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{p_2+x}{4} & \frac{p_2}{4} & \frac{1}{4} & \ell' 2^{k-2} \\ 0 & \frac{-x}{2^{k+2}} & \frac{D_1}{2^{k+2}} & \frac{\ell'}{4} \\ 0 & \frac{-D_2}{2^{k+2}} & \frac{x}{2^{k+2}} & \frac{p_2 \ell'}{4} \\ \frac{x^2 - D_1 D_2}{2^{k+3}} & \frac{-p_2 x - D_2}{2^{k+3}} & \frac{x + p_2 D_1}{2^{k+3}} & \frac{(p_2+x)\ell'}{8} \end{pmatrix}.$$

We now find the span of the first 5 rows, and successively add in rows 6 through 8 in the various cases.

- If D_2 is even,
  - If $k = 0$,
    - ∗ If $2 \mid\mid x$, rows 1 to 5 give

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

      Rows 6 and 7 already lie in this span, and row 8 shifts to $\left(\frac{x^2 - D_1 D_2}{8} \; \frac{-D_2}{8} \; \frac{1}{4} \; \frac{1}{4} \right)$. If $4 \mid\mid D_2$, it follows that $8 \mid D_1 D_2 - x^2$, and after a $\mathbb{Z}_2^4$ shift, row 8 becomes $\left(0 \; \frac{1}{2} \; \frac{1}{4} \; \frac{1}{4} \right)$, which is already in the span. Otherwise, $8 \mid D_2$, and by a $\mathbb{Z}_2^4$ shift we arrive at $\left(\frac{1}{2} \; 0 \; \frac{1}{4} \; \frac{1}{4} \right)$. Thus rows 1 through 5 sufficed, we get the determinant 2^{-4}, so the power of two dividing the denominator is $4 = k + 4$, as desired.
    - ∗ If $4 \mid x$, rows 1 to 5 give

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

      and the last three rows already lie in this span. The determinant is again 2^{-4}, as desired.
  - If $k = 1$, then $16 \mid D_1 D_2 - x^2$.
    - ∗ If $2 \mid\mid x$, then $4 \mid\mid D_2$ necessarily. The first 5 rows give

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Shifting the sixth row gives $\left(0, \pm\frac{1}{4}, \frac{1}{8}, \frac{1}{4}\right)$ (using $D_1 \equiv 1 \pmod 4$), which can replace row two, giving

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \pm\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The seventh and eighth rows lie in this span, and the determinant is 2^{-5}, as desired.

∗ If $4 \mid x$, then $16 \mid D_2$ necessarily. The first 5 rows give

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Rows 7 and 8 already lie in this span, and row 6 shifts to $\left(0, \frac{-x}{8}, \frac{1}{8}, \frac{1}{4}\right)$. If $4 \mid\mid x$ we can replace the second row, and if $8 \mid x$ we can replace the third row, giving

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \pm\frac{1}{2} & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

respectively. This gives determinant 2^{-5}, as desired.

– If $k \geq 2$, then $64 \mid 2^{2k+2} \mid D_1D_2 - x^2$. The last three rows shift to

$$\begin{pmatrix} 0 & \frac{-x}{2^{k+2}} & \frac{D_1}{2^{k+2}} & \frac{\ell'}{4} \\ 0 & \frac{-D_2}{2^{k+2}} & \frac{x}{2^{k+2}} & 0 \\ 0 & \frac{-D_2}{2^{k+3}} & \frac{x}{2^{k+3}} & \frac{x\ell'}{8} \end{pmatrix}.$$

Since $xr_6 - D_1r_7$ lies in the span of the first four rows, so we can eliminate r_7 from consideration. Similarly, $\frac{x}{2}r_6 - D_1r_8$ also lies in this span, so we can eliminate r_8 from consideration too; only the first 6 rows are left.

∗ If $2 \mid\mid x$, rows 1 to 5 give us

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

We can replace r_2 with r_6 giving

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{-x}{2^{k+2}} & \frac{D_1}{2^{k+2}} & \frac{\ell'}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

which has determinant $\frac{-x}{2^{k+5}}$, as desired (since $v_2(x) = 1$).

* If $4 \mid x$, rows 1 to 5 give us

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

In this case we can replace r_3 with r_6, giving

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-x}{2^{k+2}} & \frac{D_1}{2^{k+2}} & \frac{\ell'}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

which has determinant $\frac{D_1}{2^{k+4}}$, as desired (since D_1 is odd).

- If D_2 is odd,
  - If $k = 0$, then the first 5 rows give us

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{-x}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The last three rows lie in this span, so we get determinant 2^{-4}, as desired.

  - If $k \geq 1$, the first five rows give

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{-x}{4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The second row can be replaced by the seventh, giving

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{-D_2}{2^{k+2}} & \frac{x}{2^{k+2}} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

This span also contains the sixth and eighth rows, hence is a valid basis. The $2-$adic valuation of this determinant is $-(k+4)$, as desired.

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University of Colorado Boulder, Boulder, Colorado, USA

Email address: `james.rickards@colorado.edu`

URL: `https://math.colorado.edu/~jari2770/`