

A Converse Theorem for Split SO_{2l} over Finite Fields

Alexander HAZELTINE

*Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN 47907,
USA*
E-mail: ahazelti@purdue.edu

Baiying LIU

*Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN 47907,
USA*
E-mail: liu2053@purdue.edu

Abstract We prove a converse theorem for split even special orthogonal groups over finite fields. This is the only case left on converse theorems of classical groups and the difficulty is the existence of the outer automorphism. In this paper, we develop new ideas and overcome this difficulty.

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1 Introduction

Let G be a connected reductive group. In the representation theory of G over local fields F , converse theorems seek to uniquely identify a representation from its invariants. Among others, local gamma factors are important arithmetic invariants which play very important roles in the theory of Langlands program. More precisely, let F be a nonarchimedean local field, and let π be an irreducible generic representation of $\mathrm{GL}_n(F)$. The family of local twisted gamma factors $\gamma(s, \pi \times \tau, \psi)$, for τ any irreducible generic representation of $\mathrm{GL}_r(F)$, ψ an additive character of F and $s \in \mathbb{C}$, can be defined using Rankin–Selberg convolution [12] or the Langlands–Shahidi method [24]. The local converse problem is to determine which family of local twisted gamma factors will uniquely determine π . The following is the famous Jacquet’s conjecture on the local converse problem.

Conjecture 1.1 (Jacquet’s conjecture on the local converse problem) Let π_1, π_2 be irreducible generic representations of $\mathrm{GL}_n(F)$. Suppose that they have the same central character. If

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s , for all irreducible generic representations τ of $\mathrm{GL}_r(F)$ with $1 \leq r \leq [\frac{n}{2}]$, then $\pi_1 \cong \pi_2$.

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Conjecture 1.1 has recently been proved by Chai ([4]), and by Jacquet and the second-named author ([11]), independently, using different analytic methods. Hence we have a local converse theorem for GL_n . Local converse theorems for other classical groups also have been proved in recent years, mainly by Jiang-Soudry ([14], SO_{2n+1}), by Zhang ([25], Sp_{2n} , U_{2n} , [26], U_{2n+1}), and by Morimoto ([21], U_{2n}). For more references on local converse theorems, we refer to the introduction of [18].

Nien in [22] proved the finite fields analogue of Conjecture 1.1 for cuspidal representations of GL_n , using special properties of normalized Bessel functions and the twisted gamma factors defined by Roditty ([23]). In [19], the second-named author and Zhang defined the twisted gamma factors for generic cuspidal representations of Sp_{2n} , SO_{2n+1} , U_{2n} , and U_{2n+1} , by proving certain multiplicity one results, and proved the corresponding converse theorems.

The case left for the converse problems of classical groups over nonarchimedean local fields and finite fields is SO_{2n} . The difficulty is the existence of the outer automorphism. In this paper and in [10], we develop new ideas and overcome this difficulty, for split SO_{2n} over finite fields and over nonarchimedean local fields, respectively. More precisely, in this paper, we define the twisted gamma factors $\gamma(\pi \times \tau, \psi)$ (see Proposition 3.3) for irreducible generic cuspidal representation π of $\mathrm{SO}_{2l}(\mathbb{F}_q)$ and irreducible generic representation τ of $\mathrm{GL}_n(\mathbb{F}_q)$, and prove the following theorem.

Theorem 1.1 (The Converse Theorem for SO_{2l} , Theorem 8.2) *Let π and π' be irreducible cuspidal ψ -generic representations of split $\mathrm{SO}_{2l}(\mathbb{F}_q)$ with the same central character. If*

$$\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi),$$

for all irreducible generic representations τ of $\mathrm{GL}_n(\mathbb{F}_q)$ with $n \leq l$, then $\pi \cong \pi'$ or $\pi \cong c \cdot \pi'$, where c is the outer automorphism.

Theorem 1.1 implies that twisted gamma factors will not distinguish irreducible generic cuspidal representations π and $c \cdot \pi$ of $\mathrm{SO}_{2l}(\mathbb{F}_q)$ (see also Corollaries 6.4 and 7.8), which is a unique phenomenon for SO_{2l} among all the classical groups. This is consistent with the work of Arthur on the local Langlands correspondence and the local Langlands functoriality, and the work of Jiang and Soudry on local descent for SO_{2l} over nonarchimedean local fields (see [2] and [15]). The analogue of Theorem 1.1 for quasi-split non-split SO_{2l} has more subtleties and has been proved by the first named author for both finite and local fields [8, 9].

Now, we briefly introduce our new idea on proving Theorem 1.1. The following is the key result. As in other proven cases over finite fields, we make use of the normalized Bessel function $\mathcal{B}_{\pi, \psi}$ of π which is a particular Whittaker function in the Whittaker model of π (see Section 4 for the definition).

Theorem 1.2 (Theorem 8.1) *Let π and π' be irreducible cuspidal ψ -generic representations of split $\mathrm{SO}_{2l}(\mathbb{F}_q)$ with the same central character. If*

$$\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi),$$

for all irreducible generic representations τ of GL_n with $1 \leq n \leq l$, then we have that

$$(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(g) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(g)$$

for any $g \in \mathrm{SO}_{2l}(\mathbb{F}_q)$.

Our new idea is that for the GL_l twists, instead of considering the normalized Bessel function $\mathcal{B}_{\pi,\psi}$ and $\mathcal{B}_{c\cdot\pi,\psi}$ separately, we consider the summation of normalized Bessel functions $\mathcal{B}_{\pi,\psi} + \mathcal{B}_{c\cdot\pi,\psi}$. More precisely, to prove Theorem 1.2, as in [19], we study the support of $\mathcal{B}_{\pi,\psi}$ on $\mathrm{SO}_{2l}(\mathbb{F}_q)$ and partition it based on Bruhat cells: $B_n(\mathrm{SO}_{2l})$ for $n = 1, \dots, l$ and $B_l^c(\mathrm{SO}_{2l})$. Then we show that the twists by GL_n determine $\mathcal{B}_{\pi,\psi}$ and $\mathcal{B}_{c\cdot\pi,\psi}$ on Bruhat cells associated to Weyl elements in $B_n(\mathrm{SO}_{2l})$ (see Theorem 5.5), for $1 \leq n \leq l-2$, and the twists by GL_{l-1} determine $\mathcal{B}_{\pi,\psi}$ and $\mathcal{B}_{c\cdot\pi,\psi}$ on part of the Bruhat cells for those in $B_{l-1}(\mathrm{SO}_{2l})$ (see Theorem 6.2). However, the twists by GL_l determine the summation of the normalized Bessel functions $\mathcal{B}_{\pi,\psi} + \mathcal{B}_{c\cdot\pi,\psi}$ on the rest of the cells for $B_{l-1}(\mathrm{SO}_{2l})$ and the cells of $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$ (see Theorem 7.7). Comparing to the case of $\mathrm{Sp}_{2l}(\mathbb{F}_q)$ in [19] for example, where the authors show that the support of the normalized Bessel function $\mathcal{B}_{\pi,\psi}$ can be partitioned into l sets, $B_n(\mathrm{Sp}_{2l})$, $n = 1, \dots, l$, and for each n , the twists by GL_n determine exactly the normalized Bessel function on Bruhat cells associated to Weyl elements in $B_n(\mathrm{Sp}_{2l})$.

Following is the structure of this paper. In Section 2, we introduce the groups and representations considered in this paper. In Section 3, we prove the multiplicity one results (Propositions 3.1 and 3.2), and define the zeta integrals and gamma factors. In Section 4, we define Bessel functions and partition its support (Proposition 4.8). In Sections 5 - 7, we study the GL_n twists, $1 \leq n \leq l$, and show the relation between the GL_n twists and the support of Bessel functions (Theorems 5.5, 6.2, 7.7). In Section 8, we prove Theorem 1.2 and our main result Theorem 1.1.

2 The groups and representations

Let $n, l \in \mathbb{N}$ and $q = p^r$ for some prime number $p \neq 2$. Let \mathbb{F}_q be the finite field of q elements and fix a nontrivial additive character ψ on \mathbb{F}_q . Let GL_n to be the group of $n \times n$ matrices with entries in \mathbb{F}_q and non-zero determinant. Let I_n be the identity element and define J_n recursively by setting $J_1 = 1$ and

$$J_n = \begin{pmatrix} 0 & J_{n-1} \\ 1 & 0 \end{pmatrix}.$$

We set $\mathrm{SO}_n = \{g \in \mathrm{GL}_n \mid \det(g) = 1, {}^t g J_n g = J_n\}$ to be the split special orthogonal groups. Let U_{GL_n} and $U_{\mathrm{SO}_{2l}}$ be the subgroups of upper triangular unipotent matrices in GL_n and SO_{2l} , respectively. Fix $B_{\mathrm{SO}_{2l}} = T_{\mathrm{SO}_{2l}} U_{\mathrm{SO}_{2l}}$ to be the Borel subgroup of SO_{2l} with split torus $T_{\mathrm{SO}_{2l}}$.

Set

$$c = \mathrm{diag}(I_{l-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{l-1}).$$

We have $c \notin \mathrm{SO}_{2l}$; however, $c \mathrm{SO}_{2l} c^{-1} = c \mathrm{SO}_{2l} c = \mathrm{SO}_{2l}$. Given a representation π of SO_{2l} we define a new representation $c \cdot \pi$ of SO_{2l} by $c \cdot \pi(g) = \pi(cgc)$. Note that it is possible that $c \cdot \pi \cong \pi$.

We discuss the embeddings. These are the analogues for finite fields of the local cases found

in [16, 17]. If $n < l$ we embed SO_{2n+1} into SO_{2l} via

$$\begin{pmatrix} A & B & C \\ D & E & K \\ L & P & Q \end{pmatrix} \mapsto \mathrm{diag}(I_{l-n-1}, M^{-1} \begin{pmatrix} A & B & C \\ & 1 & \\ D & E & K \\ L & P & Q \end{pmatrix} M, I_{l-n-1}),$$

where A and Q are $n \times n$ matrices and

$$M = \mathrm{diag}(I_n, \begin{pmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{pmatrix}, I_n).$$

The embedding maps SO_{2n+1} into the standard Levi subgroup of SO_{2l} that is isomorphic to $\mathrm{GL}_{l-n-1} \times \mathrm{SO}_{2n+2}$.

If $n = l$, we embed SO_{2l} into SO_{2l+1} via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto M^{-1} \begin{pmatrix} A & B \\ & 1 \\ C & D \end{pmatrix} M, \quad (2.1)$$

where A, B, C , and D are $l \times l$ matrices and $M = \mathrm{diag}(I_{l-1}, \tilde{M}, I_{l-1})$, where

$$\tilde{M} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{-1}{2} & 1 & 1 \end{pmatrix}.$$

Note that the embedding takes the torus $T_{\mathrm{SO}_{2l}}$ to a torus in SO_{2l+1} , but not the standard one consisting of diagonal matrices. Indeed, the embedding sends $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$ to

$$\mathrm{diag}(s, \begin{pmatrix} \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) & \frac{1}{2}(t_l - t_l^{-1}) & 2(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) \\ \frac{1}{4}(t_l - t_l^{-1}) & \frac{1}{2}(t_l + t_l^{-1}) & \frac{-1}{2}(t_l - t_l^{-1}) \\ \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) & \frac{-1}{4}(t_l - t_l^{-1}) & \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) \end{pmatrix}, s^*),$$

where $s = \mathrm{diag}(t_1, t_2, \dots, t_{l-1})$ and $s^* = \mathrm{diag}(t_{l-1}^{-1}, \dots, t_2^{-1}, t_1^{-1})$.

Next, we define generic characters and generic representations. Recall that U_{GL_n} and $U_{\mathrm{SO}_{2l}}$ are the subgroups of upper triangular unipotent matrices in GL_n and SO_{2l} respectively and that we fixed an additive nontrivial character ψ of \mathbb{F}_q . We define a generic character, which by abuse of notation we still denote by ψ , on U_{GL_n} and $U_{\mathrm{SO}_{2l}}$. For $u = (u_{i,j})_{i,j=1}^n \in U_{\mathrm{GL}_n}$, we set $\psi(u) = \psi\left(\sum_{i=1}^{l-1} u_{i,i+1}\right)$. For $u = (u_{i,j})_{i,j=1}^l \in U_{\mathrm{SO}_{2l}}$ we set

$$\psi(u) = \psi\left(\sum_{i=1}^{l-2} u_{i,i+1} + \frac{1}{4}u_{l-1,l} - \frac{1}{2}u_{l-1,l+1}\right).$$

We say an irreducible representation π of SO_{2l} is ψ -generic if

$$\mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(\pi, \psi) \neq 0.$$

Similarly, we say an irreducible representation τ of GL_n is ψ -generic if

$$\mathrm{Hom}_{U_{\mathrm{GL}_n}}(\tau, \psi) \neq 0.$$

A nonzero intertwining operator in these spaces is called a Whittaker functional and it is well known that Whittaker functionals are unique up to scalars (by uniqueness of Whittaker models).

Fix a nonzero Whittaker functional $\Gamma \in \mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(\pi, \psi)$. For $v \in \pi$, let $W_v(g) = \Gamma(\pi(g)v)$ for any $g \in \mathrm{SO}_{2l}$ and set $\mathcal{W}(\pi, \psi) = \{W_v \mid v \in \pi\}$. $\mathcal{W}(\pi, \psi)$ is called the ψ -Whittaker model of π . By Frobenius reciprocity, $\mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(\pi, \psi) \cong \mathrm{Hom}_{\mathrm{SO}_{2l}}(\pi, \mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}}(\psi))$. Thus, π can be realized as a subrepresentation of $\mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}}(\psi)$ via the map $\pi \rightarrow \mathcal{W}(\pi, \psi)$ given by $v \mapsto W_v$. Moreover, by uniqueness of Whittaker models, this subrepresentation occurs with multiplicity one inside $\mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}}(\psi)$. We also note that the analogous results hold for ψ -generic representations τ of GL_n .

Let $Q_n = L_n V_n$ be the standard Siegel parabolic subgroup of SO_{2n+1} with Levi subgroup $L_n \cong \mathrm{GL}_n$. For $a \in \mathrm{GL}_n$ we let $l_n(a) = \mathrm{diag}(a, 1, a^*) \in L_n$ where $a^* = J_n^t a^{-1} J_n$. Let τ be an irreducible generic representation of GL_n and set $I(\tau) = \mathrm{Ind}_{Q_n}^{\mathrm{SO}_{2n+1}} \tau$. An element $\xi \in I(\tau)$ is a function $\xi : \mathrm{SO}_{2n+1} \rightarrow \tau$ satisfying

$$\xi(l_n(a)ug) = \tau(a)(\xi(g)), \forall a \in \mathrm{GL}_n, u \in V_n, g \in \mathrm{SO}_{2n+1}.$$

Let $\Lambda_\tau \in \mathrm{Hom}_{U_{\mathrm{GL}_n}}(\tau, \psi^{-1})$ be a fixed nonzero homomorphism. For $\xi \in I(\tau)$, we define the function $f_\xi : \mathrm{SO}_{2n+1} \times \mathrm{GL}_n \rightarrow \mathbb{C}$ by

$$f_\xi(g, a) = \Lambda_\tau(\tau(a)\xi(g)).$$

Let $I(\tau, \psi^{-1})$ be the space of functions generated by $f_\xi, \xi \in I(\tau)$. Note that for $f \in I(\tau, \psi^{-1})$, we have

$$f(g, ua) = \psi^{-1}(u)f(g, a), \forall g \in \mathrm{SO}_{2n+1}, u \in U_{\mathrm{GL}_n}, a \in \mathrm{GL}_n.$$

We also let τ^* be the contragredient representation of GL_n defined by $\tau^*(a) = \tau(a^*)$.

3 Multiplicity one theorems and the gamma factor

The goal of this section is to show that Bessel models for split even special orthogonal groups over finite fields are unique. Our primary reference for this is [17], but the setup there is local. Note that the notation of this section (introduced below) agrees with the notation of [6] with $H = N^{l-n} \times \mathrm{SO}_{2n+1}$, $\nu = \psi'$ in the case $n < l$ (ψ' is extended to be trivial on the special orthogonal group), and $H = \mathrm{SO}_{2l}$, $\nu = 1$ in the case $n = l$.

We begin by considering the case $n = l$. In this case, H is SO_{2l} embedded inside of SO_{2l+1} as in (2.1). Nonzero intertwining operators in $\mathrm{Hom}_{\mathrm{SO}_{2l}}(\pi, \tau)$ are called Bessel functionals which give Bessel models for π (similar to the Whittaker models). The following proposition shows that the Bessel models are unique when $n = l$.

Proposition 3.1 *Let $Q = LV$ be a parabolic subgroup of SO_{2l+1} and σ be an irreducible representation of L . Let π be an irreducible cuspidal representation of SO_{2l} and $\tau = \mathrm{Ind}_Q^{\mathrm{SO}_{2l+1}} \sigma$. Then*

$$\dim \mathrm{Hom}_{\mathrm{SO}_{2l}}(\pi, \tau) \leq 1.$$

Proof First, suppose that σ is cuspidal. Then, [5, Proposition 5.1] still holds in our case. Indeed, we can follow their proof closely, except we use the multiplicity one results of [1, 20]. This multiplicity one result was also known earlier in this case [7].

Suppose that σ is not cuspidal. Then there exists a parabolic subgroup $Q' = L'V'$ of L and cuspidal representation σ' of L' such that $\sigma \subseteq \text{Ind}_{Q'}^L \sigma'$. By transitivity of parabolic induction, $\tau \subseteq \text{Ind}_{L'V'V}^{\text{SO}_{2l+1}} \sigma' \otimes 1_V$ and hence the proposition follows from the case when σ was cuspidal.

Next, we consider the case $n < l$. We begin by defining the unipotent subgroup $N^{l-n} \subseteq \text{SO}_{2l}$. Consider the standard parabolic subgroup $P_{l-n-1} = M_{l-n-1}N_{l-n-1}$ of SO_{2l} with Levi subgroup $M_{l-n-1} \cong \text{GL}_{l-n-1} \times \text{SO}_{2n+2}$. Through this isomorphism, we embed $U_{\text{GL}_{l-n-1}}$ inside of SO_{2l} . We define $N^{l-n} = U_{\text{GL}_{l-n-1}}N_{l-n-1}$. That is,

$$N^{l-n} = \left\{ \begin{pmatrix} u_1 & v_1 & v_2 \\ & I_{2n+2} & v'_1 \\ & & u_1^* \end{pmatrix} \in \text{SO}_{2l} \mid u_1 \in U_{\text{GL}_{l-n-1}} \right\}.$$

For $v = (v_{i,j}) \in N^{l-n}$ we define a character ψ' of N^{l-n} by

$$\psi'(v) = \psi \left(\sum_{i=1}^{l-n-2} v_{i,i+1} + \frac{1}{4} v_{l-n-1,l} - \frac{1}{2} v_{l-n-1,l+1} \right).$$

Note that this character is trivial when $n = l - 1$. Let $H = \text{SO}_{2n+1}N^{l-n}$, where SO_{2n+1} is realized via the embedding into SO_{2n+2} inside M_{l-n-1} . Extend ψ' trivially across SO_{2n+1} so that ψ' is a character of H . Nonzero intertwining operators in $\text{Hom}_H(\pi, \tau \otimes \psi')$ are called Bessel functionals which give Bessel models for π . The below proposition gives the uniqueness of Bessel models for $n < l$.

Proposition 3.2 *Let $Q = LV$ be a parabolic subgroup of SO_{2n+1} and σ be an irreducible representation of the Levi subgroup L . Let π be an irreducible cuspidal representation of SO_{2l} and $\tau = \text{Ind}_Q^{\text{SO}_{2n+1}} \sigma$. Then,*

$$\dim \text{Hom}_H(\pi, \tau \otimes \psi') \leq 1.$$

Proof The proof is the same as Proposition 3.1, except instead of using the special orthogonal analogue of [5, Proposition 5.1], we use the special orthogonal analogue of [5, Proposition 5.3]. Again, we also must substitute the appropriate multiplicity one results which follow from [1, 20]. \square

3.1 The Zeta Integrals

Let π be an irreducible ψ -generic cuspidal representation of SO_{2l} and τ be a generic representation of GL_n . Let $W \in \mathcal{W}(\pi, \psi)$ and $f \in I(\tau, \psi^{-1})$. Next, we shall define the zeta “integrals” $\Psi(W, f)$ analogous to the local integrals of [17]. Note that [17] defines integrals for any n and l ; however, we only need the case of $n \leq l$ for the converse theorem and so we do not consider the case of $n > l$. These integrals lie in the Bessel models and the uniqueness of these models give rise to the definition of the γ -factors.

First, suppose that $n = l$. Then we define

$$\Psi(W, f) = \sum_{g \in U_{\text{SO}_{2l}} \backslash \text{SO}_{2l}} W(g) f(w_{l,l} g, I_l),$$

where

$$w_{l,l} = \begin{pmatrix} \frac{1}{2} \cdot I_l & & \\ & 1 & \\ & & 2 \cdot I_l \end{pmatrix} \in \mathrm{SO}_{2l+1}.$$

The integral satisfies the property $\Psi(g \cdot W, g \cdot f) = \Psi(W, f)$ for any $g \in \mathrm{SO}_{2l}$.

Next, suppose that $n < l$. Then we define

$$\Psi(W, f) = \sum_{g \in U_{\mathrm{SO}_{2n+1}} \setminus \mathrm{SO}_{2n+1}} \left(\sum_{r \in R^{l,n}} W(rw^{l,n}g(w^{l,n})^{-1}) \right) f(g, I_l),$$

where

$$w^{l,n} = \begin{pmatrix} I_n & & & \\ & I_{l-n-1} & & \\ & & I_2 & \\ & & & I_{l-n-1} \\ & & & & I_n \end{pmatrix} \in \mathrm{SO}_{2l},$$

and

$$R^{l,n} = \left\{ \begin{pmatrix} I_n & & & \\ & I_{l-n-1} & & \\ & & I_2 & \\ & & & I_{l-n-1} \\ & & & & x' & I_n \end{pmatrix} \in \mathrm{SO}_{2l} \right\}.$$

The integral satisfies the property $\Psi((gn) \cdot W, g \cdot f) = (\psi')^{-1}(n)\Psi(W, f)$ for any $g \in \mathrm{SO}_{2n+1}$ and $n \in N^{l-n}$. Note that in the case $n < l$, our integral differs from [17] slightly. The difference is a right translation of the Whittaker function by $(w^{l,n})^{-1}$.

Let $n \leq l$. We define an intertwining operator $M(\tau, \psi^{-1}) : I(\tau, \psi^{-1}) \rightarrow I(\tau^*, \psi^{-1})$ by

$$M(\tau, \psi^{-1})f(h, a) = \sum_{u \in V_n} f(w_n u h, d_n a^*),$$

where $w_n = \begin{pmatrix} I_n & & \\ & (-1)^n & & \\ & & I_n \end{pmatrix}$, $d_n = \mathrm{diag}(-1, 1, -1, \dots, (-1)^n) \in \mathrm{GL}_n$, $a^* = J_n^t a^{-1} J_n$, and

V_n is the unipotent radical of the standard Siegel parabolic subgroup $Q_n = L_n V_n$ in SO_{2n+1} with $L_n \cong \mathrm{GL}_n$.

The following proposition gives the definition of the γ -factor.

Proposition 3.3 *Let π be an irreducible ψ -generic cuspidal representation of SO_{2l} , τ be a generic representation of GL_n , $W \in \mathcal{W}(\pi, \psi)$, and $f \in I(\tau, \psi^{-1})$. Then there exists a complex constant $\gamma(\pi \times \tau, \psi)$, called the γ -factor of π and τ , such that*

$$\gamma(\pi \times \tau, \psi) \Psi(W, f) = \Psi(W, M(\tau, \psi^{-1})f).$$

Proof This is immediate from Propositions 3.1 and 3.2. \square

We refer to these integrals and γ -factors as the twists by GL_n .

4 Bessel Functions

In this section, we define Bessel functions and study their properties. The Bessel functions are particular functions in the Whittaker model of a ψ -generic representation. The analysis of the zeta integrals evaluated on Bessel functions is crucial in our proof of the converse theorem.

Consider π as a representation of $B_{\mathrm{SO}_{2l}}$. Let $\pi(U_{\mathrm{SO}_{2l}}, \psi)$ be the subspace of π generated by

$$\{\pi(u)v - \psi(u)v \mid u \in U_{\mathrm{SO}_{2l}}, v \in \pi\},$$

and let $\pi_{U_{\mathrm{SO}_{2l}}, \psi} = \pi/\pi(U_{\mathrm{SO}_{2l}}, \psi)$ be the twisted Jacquet module. Since π is an irreducible ψ -generic representation, $\dim(\pi_{U_{\mathrm{SO}_{2l}}, \psi}) = 1$. Let $v \in \pi$ with $v \notin \pi(U_{\mathrm{SO}_{2l}}, \psi)$ and set

$$v_0 = \frac{1}{|U_{\mathrm{SO}_{2l}}|} \sum_{u \in U_{\mathrm{SO}_{2l}}} \psi^{-1}(u) \pi(u)v.$$

By the Jacquet-Langlands lemma [3, Lemma 2.33], $v_0 \neq 0$. Recall that we fixed a nonzero Whittaker functional $\Gamma \in \mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(\pi, \psi)$. Then we have $\Gamma(v_0) \neq 0$ and by construction $\pi(u)v_0 = \psi(u)v_0$ for any $u \in U_{\mathrm{SO}_{2l}}$. Any such vector is called a Whittaker vector of π . For $g \in \mathrm{SO}_{2l}$, set

$$\mathcal{B}_{\pi, \psi}(g) = \frac{\Gamma(\pi(g)v_0)}{\Gamma(v_0)}.$$

$\mathcal{B}_{\pi, \psi}$ is called the normalized Bessel function for π and it is simple to verify that $\mathcal{B}_{\pi, \psi} \in \mathcal{W}(\pi, \psi)$.

The following proposition follows from the definitions.

Proposition 4.1 $\mathcal{B}_{\pi, \psi}(I_{2l}) = 1$ and $\mathcal{B}_{\pi, \psi}(u_1 g u_2) = \psi(u_1 u_2) \mathcal{B}_{\pi, \psi}(g)$ for any $g \in \mathrm{SO}_{2l}$ and any $u_1, u_2 \in U_{\mathrm{SO}_{2l}}$.

Let $W(\mathrm{SO}_{2l})$ be the Weyl group of SO_{2l} and let $\Delta(\mathrm{SO}_{2l})$ be the set of simple roots. We say that a Weyl element $w \in W(\mathrm{SO}_{2l})$ supports Bessel functions if for any $\alpha \in \Delta(\mathrm{SO}_{2l})$, $w\alpha$ is either negative or simple. We let $B(\mathrm{SO}_{2l})$ denote the set of Weyl elements which support Bessel functions. We call $B(\mathrm{SO}_{2l})$ the Bessel support. Recall $B_{\mathrm{SO}_{2l}} = T_{\mathrm{SO}_{2l}} U_{\mathrm{SO}_{2l}}$ is a fixed Borel subgroup of SO_{2l} . The following lemma justifies our terminology.

Lemma 4.2 Let π be an irreducible ψ -generic representation of SO_{2l} with Bessel function $\mathcal{B}_{\pi, \psi}$. Then, for any $w \in W(\mathrm{SO}_{2l}) \setminus B(\mathrm{SO}_{2l})$, we have $\mathcal{B}_{\pi, \psi}(g) = 0$, for any $g \in B_{\mathrm{SO}_{2l}} w B_{\mathrm{SO}_{2l}}$.

Proof Since $B_{\mathrm{SO}_{2l}} = T_{\mathrm{SO}_{2l}} U_{\mathrm{SO}_{2l}} = U_{\mathrm{SO}_{2l}} T_{\mathrm{SO}_{2l}}$ as a set and by Proposition 4.1,

$$\mathcal{B}_{\pi, \psi}(g) = \mathcal{B}_{\pi, \psi}(t_1 u_1 w t_2 u_2) = \mathcal{B}_{\pi, \psi}(u'_1 t'_1 w t_2 u_2) = \psi(u'_1 u_2) \mathcal{B}_{\pi, \psi}(t'_1 w t_2),$$

for some $u_1, u'_1, u_2 \in U_{\mathrm{SO}_{2l}}$ and $t_1, t'_1, t_2 \in T_{\mathrm{SO}_{2l}}$. Thus it is enough to show that $\mathcal{B}_{\pi, \psi}(t_1 w t_2) = 0$ for any $t_1, t_2 \in T_{\mathrm{SO}_{2l}}$. By definition of the Weyl group, $w t_2 = t'_2 w$ for some $t'_2 \in T_{\mathrm{SO}_{2l}}$. Hence, $\mathcal{B}_{\pi, \psi}(t_1 w t_2) = \mathcal{B}_{\pi, \psi}(t_1 t'_2 w)$ and so it is enough to show that $\mathcal{B}_{\pi, \psi}(t w) = 0$ for any $t \in T_{\mathrm{SO}_{2l}}$.

Since $w \notin B(\mathrm{SO}_{2l})$, there exists $\alpha \in \Delta(\mathrm{SO}_{2l})$ such that $w\alpha$ is positive but not simple. Let $x \in \mathbb{F}_q$ and let $\mathbf{x}_\alpha(x)$ be an element in the root space of α (note that the root space of α lies in $U_{\mathrm{SO}_{2l}}$ and is isomorphic to \mathbb{F}_q). Then $t w \mathbf{x}_\alpha(x) = \mathbf{x}_{w\alpha}(x') t w$ for some $x' \in \mathbb{F}_q$ (because $T_{\mathrm{SO}_{2l}}$ normalizes $U_{\mathrm{SO}_{2l}}$). Since the simple root spaces are exactly the support of ψ , $\psi(\mathbf{x}_\alpha(x))$ is a

nonzero constant multiple of $\psi(x)$ and this constant is independent of x . Since $w\alpha$ is positive but not simple, $\psi(\mathbf{x}_{w\alpha}(x')) = \psi(0) = 1$. Thus, by Proposition 4.1, $\psi(x)\mathcal{B}_{\pi,\psi}(tw) = \mathcal{B}_{\pi,\psi}(tw)$ for any $x \in \mathbb{F}_q$. Since ψ is nontrivial, $\mathcal{B}_{\pi,\psi}(tw) = 0$. This proves the lemma. \square

Next, we determine the support of a Bessel function on the torus.

Lemma 4.3 *If $t \in T_{\mathrm{SO}_{2l}}$ and $\mathcal{B}_{\pi,\psi}(t) \neq 0$, then t is in the center of SO_{2l} .*

Proof Let $t \in T_{\mathrm{SO}_{2l}}$ such that $\mathcal{B}_{\pi,\psi}(t) \neq 0$ and let δ be any simple root of SO_{2l} . For $x \in \mathbb{F}_q$, let $\mathbf{x}_\delta(x)$ be in the root subgroup of δ . Then $t\mathbf{x}_\delta(x) = \mathbf{x}_\delta(\delta(t)x)t$. Hence, by Proposition 4.1,

$$\psi(\mathbf{x}_\delta(x))\mathcal{B}_{\pi,\psi}(t) = \psi(\mathbf{x}_\delta(\delta(t)x))\mathcal{B}_{\pi,\psi}(t).$$

Thus, $\psi(\mathbf{x}_\delta(x)) = \psi(\mathbf{x}_\delta(\delta(t)x))$ for any $x \in \mathbb{F}_q$ and simple root $\delta \in \Delta(\mathrm{SO}_{2l})$. Since ψ is nontrivial and x is arbitrary, we must have $\delta(t) = 1$ for all $\delta \in \Delta(\mathrm{SO}_{2l})$. Thus, t is in the center of SO_{2l} which proves the lemma. \square

4.1 Bessel Function of the Conjugate representation

In this section, we determine the normalized Bessel function in the ψ -Whittaker model of the conjugate representation $c \cdot \pi$.

Let ψ_c be the character on $U_{\mathrm{SO}_{2l}}$ defined by $\psi_c(u) = \psi(cuc)$. Recall that we fixed a nonzero Whittaker functional $\Gamma \in \mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(\pi, \psi)$. It follows that $\Gamma \in \mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(c \cdot \pi, \psi_c)$ and

$$\mathcal{B}_{c \cdot \pi, \psi_c}(g) = \frac{\Gamma(c \cdot \pi(g)v_0)}{\Gamma(v_0)} = \mathcal{B}_{\pi, \psi}(cgc)$$

defines the normalized Bessel function for $c \cdot \pi$ in $\mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}} \psi_c$. On the other hand, $c \cdot \pi$ is also ψ -generic, since $\psi_c(\tilde{t}^{-1}u\tilde{t}) = \psi(u)$ for any $u \in U_{\mathrm{SO}_{2l}}$, where

$$\tilde{t} = \mathrm{diag}(I_{l-1}, \frac{-1}{2}, -2, I_{l-1}) \in T_{\mathrm{SO}_{2l}}. \quad (4.1)$$

Let

$$\mathcal{B}_{c \cdot \pi, \psi}(g) := \mathcal{B}_{c \cdot \pi, \psi_c}(\tilde{t}^{-1}g\tilde{t}) = \mathcal{B}_{\pi, \psi}(c\tilde{t}^{-1}g\tilde{t}c).$$

Then we have the following.

Proposition 4.4 *$\mathcal{B}_{c \cdot \pi, \psi}$ is the normalized Bessel function for $c \cdot \pi$ in $\mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}} \psi$.*

Proof Let Γ' be defined by $\Gamma'(v) = \Gamma(c \cdot \pi(\tilde{t}^{-1}v))$. Then, $\Gamma' \in \mathrm{Hom}_{U_{\mathrm{SO}_{2l}}}(c \cdot \pi, \psi)$. Since $\tilde{t}c = c\tilde{t}^{-1}$, $\mathcal{B}_{c \cdot \pi, \psi}$ is the right translation by \tilde{t}^{-1} of a Whittaker function of $c \cdot \pi$ in $\mathrm{Ind}_{U_{\mathrm{SO}_{2l}}}^{\mathrm{SO}_{2l}} \psi$. Thus, $\mathcal{B}_{c \cdot \pi, \psi}$ is a Whittaker function of $c \cdot \pi$.

Next, $\mathcal{B}_{c \cdot \pi, \psi}(I_{2l}) = \mathcal{B}_{\pi, \psi}(I_{2l}) = 1$ and for $u_1, u_2 \in U_{\mathrm{SO}_{2l}}$ and $g \in \mathrm{SO}_{2l}$

$$\begin{aligned} \mathcal{B}_{c \cdot \pi, \psi}(u_1 g u_2) &= \mathcal{B}_{c \cdot \pi, \psi_c}(\tilde{t}^{-1}u_1\tilde{t}\tilde{t}^{-1}g\tilde{t}\tilde{t}^{-1}u_2\tilde{t}) \\ &= \psi_c(\tilde{t}^{-1}u_1\tilde{t})\psi_c(\tilde{t}^{-1}u_2\tilde{t})\mathcal{B}_{c \cdot \pi, \psi_c}(\tilde{t}^{-1}g\tilde{t}) \\ &= \psi(u_1)\psi(u_2)\mathcal{B}_{c \cdot \pi, \psi}(g). \end{aligned}$$

By Proposition 4.1, $\mathcal{B}_{c \cdot \pi, \psi}$ is the normalized Bessel function of $c \cdot \pi$ which proves the proposition. \square

4.2 Partition of the Bessel Support

In this section we aim to partition the Bessel support and provide some preparation for future calculations. We define the Weyl elements that appear in the computations of the zeta integrals and then show that these can be used to partition the Bessel support. When $n < l$, we embed w_n into SO_{2l} and use the image to define the Weyl elements we are interested in. When $n = l$, we take a different approach since the embedding is $\mathrm{SO}_{2l} \hookrightarrow \mathrm{SO}_{2l+1}$ and $w_l \in \mathrm{SO}_{2l+1}$.

Let $n < l$ and recall that $w_n = \begin{pmatrix} & & I_n \\ & (-1)^n & \\ I_n & & \end{pmatrix} \in \mathrm{SO}_{2n+1}$. Let

$$\hat{w}_n = \begin{cases} \begin{pmatrix} I_{l-n-1} & & & I_n \\ & \ddots & & \\ & & 1 & \\ & & & I_n \\ & & & & I_{l-n-1} \\ & & & & & \ddots \\ & & & & & & I_n \\ & & & & & & & I_{l-n-1} \end{pmatrix} & \text{if } n \text{ is odd,} \\ \begin{pmatrix} I_{l-n-1} & & & I_n \\ & \ddots & & \\ & & 1 & \\ & & & I_n \\ & & & & I_{l-n-1} \\ & & & & & \ddots \\ & & & & & & I_n \\ & & & & & & & I_{l-n-1} \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Then the image of w_n in SO_{2l} is $t'_n \hat{w}_n$, where $t'_n = \tilde{t}$ as in (4.1) if n is odd, and $t'_n = I_{2l}$ if n is even. Note that if n is odd, then $cw_n c = \tilde{t}^{-1} \hat{w}_n$, and if n is even then $cw_n c = \hat{w}_n$ (here we realize $cw_n c \in \mathrm{SO}_{2l}$ via the embedding mentioned before). Recall that

$$w^{l,n} = \begin{pmatrix} & & I_n & & \\ & I_{l-n-1} & & & \\ & & I_2 & & \\ & & & I_{l-n-1} & \\ & & & & I_n \end{pmatrix} \in \mathrm{SO}_{2l}.$$

Let

$$\tilde{w}_n = w^{l,n} \hat{w}_n (w^{l,n})^{-1} = \begin{pmatrix} & & & & I_n \\ & I_{l-n-1} & & & \\ & & X & & \\ & & & I_{l-n-1} & \\ I_n & & & & \end{pmatrix},$$

where

$$X = \begin{cases} J_2 & \text{if } n \text{ is odd,} \\ I_2 & \text{if } n \text{ is even.} \end{cases}$$

This Weyl element \tilde{w}_n occurs in the computations of the zeta integrals of the twists by GL_n and is also used to partition the Bessel support. \tilde{w}_n also shows up naturally in the computations of the zeta integrals after applying the intertwining operator.

Let α_i be the simple roots of SO_{2l} given by $\alpha_i(t) = \frac{t_i}{t_{i+1}}$ for $i \leq l-1$ and $\alpha_l(t) = t_{l-1}t_l$, where $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$. Let $\Phi(\mathrm{SO}_{2l})$, resp. $\Phi^+(\mathrm{SO}_{2l})$, be the set of roots, resp. positive roots, of SO_{2l} . Then, for $n \leq l-2$, the action of \tilde{w}_n on the simple roots is given by

$$\begin{aligned} \tilde{w}_n \alpha_i &= \alpha_{n-i} \text{ for } 1 \leq i \leq n-1, \\ \tilde{w}_n \alpha_n(t) &= t_1^{-1} t_{n+1}^{-1}, \\ \tilde{w}_n \alpha_i &= \alpha_i \text{ for } n+1 \leq i \leq l-2, \\ \tilde{w}_n \alpha_{l-1} &= \begin{cases} \alpha_l & \text{if } n \text{ is odd,} \\ \alpha_{l-1} & \text{if } n \text{ is even,} \end{cases} \\ \tilde{w}_n \alpha_l &= \begin{cases} \alpha_{l-1} & \text{if } n \text{ is odd,} \\ \alpha_l & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Note that \tilde{w}_{l-1} sends α_{l-1} and α_l to negative roots and the rest to simple roots. Specifically,

$$\begin{aligned} \tilde{w}_{l-1} \alpha_i &= \alpha_{l-1-i} \text{ for } 1 \leq i \leq l-2, \\ \tilde{w}_{l-1} \alpha_{l-1}(t) &= \begin{cases} t_1^{-1} t_l^{-1} & \text{if } l-1 \text{ is odd,} \\ t_1^{-1} t_l & \text{if } l-1 \text{ is even,} \end{cases} \\ \tilde{w}_{l-1} \alpha_l(t) &= \begin{cases} t_1^{-1} t_l & \text{if } l-1 \text{ is odd,} \\ t_1^{-1} t_l^{-1} & \text{if } l-1 \text{ is even.} \end{cases} \end{aligned}$$

Next, we consider the case $n = l$. Note that the construction of \tilde{w}_n for $n < l$ arose from the embedding of the Weyl elements involved in the intertwining operator and the Weyl elements in the zeta integrals. Reversing this for $n = l$, would require knowing what Weyl elements in SO_{2l} embed into certain subsets involving the Weyl element $w_l \in \mathrm{SO}_{2l+1}$. We answer this question later in Propositions 7.2, 7.3, and 7.4. For now, we motivate our choices by using the following bijection between $\mathcal{P}(\Delta(\mathrm{SO}_{2l}))$ and the Bessel support.

Let $\theta_w = \{\alpha \in \Delta(\mathrm{SO}_{2l}) \mid w\alpha \in \Phi^+(\mathrm{SO}_{2l})\}$. The assignment $w \mapsto \theta_w$ gives a bijection from $\mathrm{B}(\mathrm{SO}_{2l})$ to the power set of $\Delta(\mathrm{SO}_{2l})$, which we denote by $\mathcal{P}(\Delta(\mathrm{SO}_{2l}))$. Then, for $n < l-1$, $\theta_{\tilde{w}_n} = \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_n\}$. Also, we have $\theta_{\tilde{w}_{l-1}} = \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}, \alpha_l\}$. In the cases of [19], the Weyl elements for which $\theta_w = \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_i\}$ for some i are used to partition the Bessel support. This suggests that we need at least 2 more Weyl elements which we define below. These correspond to the sets $\Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}\}$ and $\Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}$.

We extend the definition of \tilde{w}_n to the case of $n = l$. We begin by defining \tilde{w}'_l and define \tilde{w}_l

as either this Weyl element or its conjugation by c depending on the parity of l . If l is even, let

$$\tilde{w}'_l = \begin{pmatrix} & I_l \\ I_l & \end{pmatrix},$$

while if l is odd, let

$$\tilde{w}'_l = \begin{pmatrix} & & I_{l-1} \\ 1 & & & \\ & & & 1 \\ & I_{l-1} & & \end{pmatrix}.$$

We let w_{long} be the long Weyl element of SO_{2l} . Then,

$$w_{\text{long}} = \begin{cases} \begin{pmatrix} & J_{l-1} \\ & I_2 \\ J_{l-1} & \end{pmatrix} & \text{if } l \text{ is odd,} \\ J_{2l} & \text{if } l \text{ is even.} \end{cases}$$

We have that

$$w_{\text{long}} \tilde{w}'_l = \begin{pmatrix} J_l & \\ & J_l \end{pmatrix}$$

which is the long Weyl element of a standard Levi subgroup that is isomorphic to GL_l . This suggests that $\theta_{\tilde{w}'_l}$ should be $\Delta(\text{SO}_{2l}) \setminus \{\alpha_{l-1}\}$ or $\Delta(\text{SO}_{2l}) \setminus \{\alpha_l\}$. We verify this explicitly below. Also, its conjugation by c is the remaining set since c acts on the simple roots by swapping α_{l-1} and α_l and fixing the rest. Set $\tilde{w}'_l^c = cw'_l c$.

We have $\tilde{w}'_l t (\tilde{w}'_l)^{-1} = \text{diag}(t_l^{-1}, \dots, t_2^{-1}, t_1^{(-1)^{l+1}}, t_1^{(-1)^l}, t_2, \dots, t_l)$. Hence,

$$\tilde{w}'_l \alpha_i = \alpha_{l-i} \text{ for } 1 \leq i \leq l-2,$$

$$\begin{aligned} \tilde{w}'_l \alpha_{l-1}(t) &= \begin{cases} t_2^{-1} t_1^{-1} & \text{if } l \text{ is odd,} \\ \alpha_1(t) & \text{if } l \text{ is even,} \end{cases} \\ \tilde{w}'_l \alpha_l(t) &= \begin{cases} \alpha_1(t) & \text{if } l \text{ is odd,} \\ t_2^{-1} t_1^{-1} & \text{if } l \text{ is even.} \end{cases} \end{aligned}$$

Thus, $\theta_{\tilde{w}'_l} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_{l-1}\}$ if l is odd and $\theta_{\tilde{w}'_l} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_l\}$ if l is even. Similarly, we have $\theta_{\tilde{w}'_l^c} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_l\}$ if l is odd and $\theta_{\tilde{w}'_l^c} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_{l-1}\}$ if l is even. Hence, we set

$$\tilde{w}_l = \begin{cases} \tilde{w}'_l^c & \text{if } l \text{ is odd,} \\ \tilde{w}'_l & \text{if } l \text{ is even,} \end{cases}$$

and $\tilde{w}_l^c = c \tilde{w}_l c$. Then, $\theta_{\tilde{w}_l} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_l\}$ and $\theta_{\tilde{w}_l^c} = \Delta(\text{SO}_{2l}) \setminus \{\alpha_{l-1}\}$.

For $x \in \text{GL}_n$, we set $t_n(x) = \text{diag}(x, I_{2l-2n}, x^*) \in \text{SO}_{2l}$. For $n \leq l-2$, let $B_n(\text{SO}_{2l})$ be the set of $w \in B(\text{SO}_{2l})$ such that there exists $w' \in W(\text{GL}_n)$ such that $w = t_n(w') \tilde{w}_n$. We also let $B_l(\text{SO}_{2l})$ be the set of $w \in B(\text{SO}_{2l})$ such that there exists $w' \in W(\text{GL}_l)$ such that $w = t_l(w') \tilde{w}_l$

and $cwc \neq w$. Also, let $B_l^c(\mathrm{SO}_{2l})$ be the set of cwc where $w \in B_l(\mathrm{SO}_{2l})$. We let $B_{l-1}(\mathrm{SO}_{2l})$ be the set of $w \in B(\mathrm{SO}_{2l})$ such that there exists $w' \in W(\mathrm{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$ and $cwc = w$. By convention, we also define $B_0(\mathrm{SO}_{2l}) = \{I_{2l}\}$. Note that $\theta_{I_{2l}} = \Delta(\mathrm{SO}_{2l})$. From the definition, we have that $B_l^c(\mathrm{SO}_{2l})$ is the set of $w' \in W(\mathrm{GL}_l)$ such that $w = t_l^c(w')\tilde{w}_l^c$ and $cwc \neq w$ where $t_l^c(w') = ct_l(w')c$. From Proposition 4.6 below, a similar definition holds for $B_{l-1}(\mathrm{SO}_{2l})$. That is, $B_{l-1}(\mathrm{SO}_{2l})$ is the set of $w \in B(\mathrm{SO}_{2l})$ such that there exists $w' \in W(\mathrm{GL}_{l-1})$ such that $w = t_{l-1}(w')\tilde{w}_{l-1}$.

We remark on the significance of the condition $cwc \neq w$ in $B_l(\mathrm{SO}_{2l})$. Let $\widetilde{B}_l(\mathrm{SO}_{2l})$ be the set of $w \in B(\mathrm{SO}_{2l})$ such that there exists $w' \in W(\mathrm{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$ and $\widetilde{B}_l^c(\mathrm{SO}_{2l})$ be the set of cwc such that $w \in \widetilde{B}_l(\mathrm{SO}_{2l})$. Then $\widetilde{B}_l(\mathrm{SO}_{2l}) \cap \widetilde{B}_l^c(\mathrm{SO}_{2l}) = B_{l-1}(\mathrm{SO}_{2l})$ is a nonzero intersection and hence we would not be able to partition the Bessel support with these sets. Thus it is necessary to include the condition $cwc \neq w$ in the definitions of $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$. This does not happen for the classical groups considered in [19].

Later, we see that the twists by GL_n determine the normalized Bessel function on Bruhat cells associated to Weyl elements in $B_n(\mathrm{SO}_{2l})$, for $n \leq l-2$, the twists by GL_{l-1} determine only a part of the Bruhat cells for those in $B_{l-1}(\mathrm{SO}_{2l})$, and the twists by GL_l determine the rest of the Bruhat cells for $B_{l-1}(\mathrm{SO}_{2l})$ and the cells of $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$. The goal for the rest of this section is to show that these sets partition the Bessel support. Before we show this, we present some computational results for $B_{l-1}(\mathrm{SO}_{2l})$.

Proposition 4.5 *Let $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $w' \in W(\mathrm{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$. Then*

$$w' = \begin{pmatrix} & w'' \\ 1 & \end{pmatrix},$$

where $w'' \in W(\mathrm{GL}_{l-1})$.

Conversely, if $w \in B(\mathrm{SO}_{2l})$ is such that $w = t_l(w')\tilde{w}_l$ where

$$w' = \begin{pmatrix} & w'' \\ 1 & \end{pmatrix},$$

for some $w'' \in W(\mathrm{GL}_{l-1})$, then $w \in B_{l-1}(\mathrm{SO}_{2l})$.

Proof We only prove the first claim as the second claim is easy to verify directly. Since $w \in B_{l-1}(\mathrm{SO}_{2l})$, we have $cwc = w$. Thus, $t_l(w')\tilde{w}_l = ct_l(w')cc\tilde{w}_l c$. First, assume that l is odd, then $c\tilde{w}_l c = \tilde{w}'_l$. Also $ct_l(w') = t_l(w')\tilde{w}_l \tilde{w}'_l c$ and

$$\tilde{w}_l \tilde{w}'_l = \begin{pmatrix} & & & 1 \\ & I_{l-2} & & \\ & & J_2 & \\ & & & I_{l-2} \\ 1 & & & \end{pmatrix}.$$

Let $w' = (w'_{i,j})_{i,j=1}^l$. Then,

$$t_l(w')\tilde{w}_l\tilde{w}'_l c = \begin{pmatrix} 0 & (w'_{1,j})_{j=2}^l & 0 & w'_{1,1} \\ 0 & (w'_{i,j})_{i,j=2}^l & 0 & (w'_{i,1})_{i=2}^l \\ * & 0 & * & 0 \\ * & 0 & * & 0 \end{pmatrix},$$

where the $*$'s represent coordinates in $(w')^*$. On the other hand,

$$ct_l(w') = \begin{pmatrix} (w'_{i,j})_{i,j=1}^{l-1} & (w'_{i,l})_{i=1}^l & 0 & 0 \\ 0 & 0 & * & * \\ w'_{l,1} & (w'_{l,j})_{j=2}^l & 0 & 0 \\ 0 & 0 & * & * \end{pmatrix}.$$

Hence, we must have $w'_{i,1} = 0$ for any $i = 1, \dots, l-1$. Since w' is a Weyl element of GL_l , $w'_{l,1} = 1$ and hence the claim follows.

Next, we consider the case that l is even. Again, we have $t_l(w')\tilde{w}_l = ct_l(w')cc\tilde{w}_l c$ and hence $t_l(w')\tilde{w}_l c\tilde{w}_l cc = ct_l(w')$. Again,

$$\tilde{w}_l c\tilde{w}_l c = \begin{pmatrix} & & & 1 \\ & I_{l-2} & & \\ & & J_2 & \\ 1 & & & I_{l-2} \end{pmatrix}.$$

The rest of this case follows exactly as above. This completes the proof of the proposition. \square

The following proposition relates $B_{l-1}(\mathrm{SO}_{2l})$ with \tilde{w}_{l-1} .

Proposition 4.6 *Suppose that $w \in B_{l-1}(\mathrm{SO}_{2l})$. Then there exists $\omega \in W(\mathrm{GL}_{l-1})$ such that $w = t_{l-1}(\omega)\tilde{w}_{l-1}$. Conversely, if $w \in B(\mathrm{SO}_{2l})$ is such that $w = t_{l-1}(\omega)\tilde{w}_{l-1}$ for some $\omega \in W(\mathrm{GL}_{l-1})$, then $w \in B_{l-1}(\mathrm{SO}_{2l})$.*

Proof By Proposition 4.5, $w = t_l(w')\tilde{w}_l$ where

$$w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix},$$

for some $w'' \in W(\mathrm{GL}_{l-1})$. We show that the claim follows with $\omega = w''$.

If l is even, then

$$w = t_l(w')\tilde{w}_l = \begin{pmatrix} w'' \\ J_2 \\ (w'')^* \end{pmatrix}.$$

It is immediate to check then $t_{l-1}(w'')\tilde{w}_{l-1} = w$. Hence $\omega = w''$ gives the claim.

If l is odd, then

$$w = t_l(w')\tilde{w}_l = \begin{pmatrix} & w'' \\ & I_2 \\ (w'')^* & \end{pmatrix}.$$

Again, it is immediate to check then $t_{l-1}(w'')\tilde{w}_{l-1} = w$. Hence $\omega = w''$ gives the claim. This completes the proof of the proposition. \square

For $0 \leq n \leq l$, let

$$P_n = \{\theta \subseteq \Delta(\mathrm{SO}_{2l}) \mid w_\theta \in \mathrm{B}_n(\mathrm{SO}_{2l})\},$$

and

$$P_l^c = \{\theta \subseteq \Delta(\mathrm{SO}_{2l}) \mid w_\theta \in \mathrm{B}_l^c(\mathrm{SO}_{2l})\}.$$

The next proposition is crucial to show that the sets $\mathrm{B}_n(\mathrm{SO}_{2l})$ for $0 \leq n \leq l$ and $\mathrm{B}_l^c(\mathrm{SO}_{2l})$ form a partition of the Bessel support.

Proposition 4.7 *For $n < l - 1$,*

$$P_n = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \{\alpha_{n+1}, \dots, \alpha_l\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_n\}\}.$$

We also have $P_0 = \Delta(\mathrm{SO}_{2l})$,

$$P_l = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}\},$$

$$P_l^c = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \{\alpha_l\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}\}\},$$

and

$$P_{l-1} = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}, \alpha_l\}\}.$$

Proof First, assume that $n \leq l - 2$ and suppose $\theta \in P_n$. Then, $w_\theta \in \mathrm{B}_n(\mathrm{SO}_{2l})$. Thus, there exists $w' \in W(\mathrm{GL}_n)$ such that $w = t_n(w')\tilde{w}_n$. $t_n(w')$ acts on $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$ by permuting the t_i 's for $1 \leq i \leq n$. By direct computation, $\{\alpha_{n+1}, \dots, \alpha_l\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_n\}$.

Suppose that $n \leq l - 2$ and $\{\alpha_{n+1}, \dots, \alpha_l\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_n\}$. First we show that w_θ takes the set $\{\alpha_{l-1}, \alpha_l\}$ to itself. Suppose that $w_\theta \alpha_l = \alpha_j$ for some $j \leq l - 2$ for contradiction. Let $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$ and $s = w_\theta t w_\theta^{-1} = \mathrm{diag}(s_1, \dots, s_l, s_l^{-1}, \dots, s_1^{-1})$. Then, $w_\theta \alpha_l(t) = t_j t_{j+1}^{-1}$. So, $w_\theta \alpha_{l-1}(t) = t_j t_{j+1}$. That is, $w_\theta \alpha_{l-1}$ is a positive, nonsimple root. But w_θ should be in the Bessel support and hence we have a contradiction. Therefore, w_θ takes the set $\{\alpha_{l-1}, \alpha_l\}$ to itself. So, $s_{l-1} = t_{l-1}$ and $s_l = t_l$ or $s_l = t_l^{-1}$.

Next, we show that $w_\theta \alpha_i = \alpha_i$ for $n + 1 \leq i \leq l - 2$. We do this by considering α_{l-2} , then α_{l-3} , and so on until we reach α_{n+1} . We have $w_\theta \alpha_{l-2}(t) = t_j^\varepsilon t_{l-1}^{-1}$ for some $j \leq l - 2$ and $\varepsilon = \pm 1$. If $\varepsilon = -1$, this is a negative root, but it should be simple by assumption. Thus $\varepsilon = 1$ and $w_\theta \alpha_{l-2}$ is a positive root. In this case, $w_\theta \alpha_{l-2}$ is simple if and only if $j = l - 2$. Hence, $s_{l-2} = t_{l-2}$. We repeat this argument for α_{l-3} and then α_{l-4} and so on until we obtain α_{n+1} . In summary, we have $w_\theta \alpha_i = \alpha_i$ for $n + 1 \leq i \leq l - 2$. Hence, $s_i = t_i$ for $n + 1 \leq i \leq l - 1$ and $s_l \in \{t_l, t_l^{-1}\}$.

Then, $w_\theta \alpha_n(t) = t_j^\varepsilon t_{n+1}^{-1}$ for some $j \leq n$ and $\varepsilon = \pm 1$. Since $w_\theta \alpha_n$ is negative, $\varepsilon = -1$. Finally, since $w_\theta \alpha_i$ is either simple or negative for $i \leq n - 1$, it follows that $s_i \in \{t_1^{-1}, \dots, t_n^{-1}\}$ for any $i \leq n$. Let $w' \in W(\mathrm{GL}_n)$ be the Weyl element which takes $\mathrm{diag}(t_1, \dots, t_n)$ to $\mathrm{diag}(s_1^{-1}, \dots, s_n^{-1})$.

Then, $w_\theta = t_n(w')\tilde{w}_n$ for some $w' \in W(\mathrm{GL}_n)$ (note that the X in the definition of \tilde{w}_n comes from needing $\det(w_\theta) = 1$ and this also determines the action of w_θ on α_{l-1} and α_l). Therefore, for $n < l-1$, we have

$$P_n = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \{\alpha_{n+1}, \dots, \alpha_l\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_n\}\}.$$

Next, we examine P_{l-1} . Suppose $\theta \in P_{l-1}$. Then, $w_\theta \in B_{l-1}(\mathrm{SO}_{2l})$. Thus, there exists $w' \in W(\mathrm{GL}_l)$ such that $w_\theta = t_l(w')\tilde{w}_l$. Also, $cw_\theta c = w_\theta$. By Proposition 4.5,

$$w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix},$$

for some $w'' \in W(\mathrm{GL}_{l-1})$. Since $\tilde{w}_l t_l \tilde{w}_l^{-1} = \mathrm{diag}(t_l^{\pm 1}, \dots, t_1^{-1}, t_1, \dots, t_l^{\mp 1})$, it follows that the (l, l) -entry of $w_\theta t_l w_\theta^{-1}$ is $t_l^{\pm 1}$ (the sign depends on the parity of l). Hence $w_\theta \alpha_l(t) = t_j^{-1} t_l^{\pm 1}$ for some $j \leq l-1$ and hence $w_\theta \alpha_l$ is a negative root. So, $\alpha_l \notin \theta$. $w_\theta \alpha_{l-1}(t) = t_j^{-1} t_l^{\mp 1}$ is also a negative root. So $\alpha_{l-1} \notin \theta$ and hence $\theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}, \alpha_l\}$.

Now we suppose that $\theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}, \alpha_l\}$. Then $w_\theta \alpha_{l-1}$ and $w_\theta \alpha_l$ are negative. Let $s = w_\theta t_l w_\theta^{-1} = \mathrm{diag}(s_1, \dots, s_l, s_l^{-1}, \dots, s_1^{-1})$. Then $w_\theta \alpha_{l-1}(t) = s_{l-1} s_l^{-1}$ and $w_\theta \alpha_l(t) = s_{l-1} s_l$. We show that $s_{l-1} = t_j^{-1}$ for some j by contradiction. Suppose $s_{l-1} = t_j$. If $s_l = t_k$, then $w_\theta \alpha_l(t) = t_j t_k$ is a positive root which is a contradiction. But, if $s_l = t_k^{-1}$, then $w_\theta \alpha_{l-1}(t) = t_j t_k$ is positive and hence we have another contradiction. Therefore, we must have $s_{l-1} = t_j^{-1}$.

Our next step is to show that $s_l = t_l$ or $s_l = t_l^{-1}$ depending on the parity of l . Since $s_{l-1} = t_j^{-1}$, $w_\theta \alpha_{l-2}(t) = s_{l-2} t_j$. Since $w_\theta \in B(\mathrm{SO}_{2l})$, this root must be simple or negative. It is simple if $s_{l-2} = t_{j+1}^{-1}$. It is negative if $s_{l-2} = t_r^{-1}$ for some $r < j$. In either case we have that $s_{l-2} = t_r^{-1}$ for some r . By continuing this process for all the roots, we find that for each $n < l$ there exists a unique j_n such that $s_n = t_{j_n}^{-1}$. Suppose that $s_l \neq t_l$ and $s_l \neq t_l^{-1}$ for contradiction. Then, there exists n such that $s_n = t_l^{-1}$. Then $w_\theta \alpha_n(t) = t_l^{-1} t_{j_{n+1}}$. This root is positive for any index j_{n+1} . It is simple when $j_{n+1} = l-1$. Thus we must have $s_{n+1} = t_{l-1}^{-1}$. Next, $w_\theta \alpha_{n+1}(t) = t_{l-1}^{-1} t_{j_{n+2}}$. This root is positive for any index $j_{n+2} < l-1$. It is simple when $j_{n+1} = l-2$. Thus we must have $s_{n+2} = t_{l-2}^{-1}$. Continuing in this fashion we find that $s_i = t_{l+n-i}^{-1}$ for $n \leq i \leq l-2$. Thus $w_\theta \alpha_{l-1}(t) = t_{l+n-(l-2)}^{-1} t_j$ where $j \leq l+n-(l-2)$. This is always a positive root. However, by assumption, it is negative. Hence we have a contradiction. Therefore, we must have $s_l = t_l$ or $s_l = t_l^{-1}$. Thus, $w_\theta t_l w_\theta^{-1} = \mathrm{diag}(t_{j_1}^{-1}, \dots, t_{j_{l-1}}^{-1}, t_l^\epsilon, t_l^{-\epsilon}, t_{j_{l-1}}, \dots, t_{j_1})$ for some $\epsilon \in \{\pm 1\}$. Let $w'' \in W(\mathrm{GL}_{l-1})$ be such that $w'' \mathrm{diag}(t_1, \dots, t_{l-1})(w'')^{-1} = \mathrm{diag}(t_{j_{l-1}}, \dots, t_{j_1})$. Then, $t_{l-1}(w'')\tilde{w}_{l-1} = w_\theta$ (note that the parity of l determines ϵ since we need the determinant to be 1) and hence $w_\theta \in B(\mathrm{SO}_{2l})$. Therefore, we have

$$P_{l-1} = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_{l-1}, \alpha_l\}\}.$$

Next, we show the claim for P_l . Suppose that $w_\theta = t_l(w')\tilde{w}_l$ for some $w' \in W(\mathrm{GL}_l)$ with $cw_\theta c \neq w_\theta$. For l odd, we have

$$\tilde{w}_l t_l \tilde{w}_l^{-1} = \mathrm{diag}(t_l, t_{l-1}^{-1}, \dots, t_2^{-1}, t_1^{-1}, t_1, t_2, \dots, t_{l-1}, t_l^{-1}),$$

and for l even

$$\tilde{w}_l t_l \tilde{w}_l^{-1} = \mathrm{diag}(t_l^{-1}, t_{l-1}^{-1}, \dots, t_2^{-1}, t_1^{-1}, t_1, t_2, \dots, t_{l-1}, t_l).$$

Let $w' = (w'_{i,j})_{i,j=1}^l$. Since $cw_\theta c \neq w_\theta$, by Proposition 4.5, $w'_{l,1} = 0$. Let

$$w_\theta tw_\theta^{-1} = \mathrm{diag}(s_1, \dots, s_l, s_l^{-1}, \dots, s_1^{-1}).$$

For l odd, $s_i \in \{t_1^{-1}, \dots, t_{l-1}^{-1}, t_l\}$ and for l even $s_i \in \{t_1^{-1}, \dots, t_{l-1}^{-1}, t_l^{-1}\}$. Since $w'_{l,1} = 0$, $s_l \neq t_l^{\pm 1}$. Thus, $w_\theta \alpha_{l-1} = s_{l-1} s_l^{-1} = s_{l-1} t_r$ and $w_\theta \alpha_{l-1} = s_{l-1} s_l = s_{l-1} t_r^{-1}$ for some $r < l$. If $s_{l-1} = t_l$, then $w_\theta \alpha_{l-1}$ is positive and we must have $r = l - 1$ so that it is simple. Then, $w_\theta \alpha_l = t_l t_{l-1}^{-1}$ is negative and hence we have $\{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}$. If $s_{l-1} = t_l^{-1}$, then $w_\theta \alpha_{l-1}$ is again positive since $r < l$. We must have $r = l - 1$ to ensure it is simple. But, then $w_\theta \alpha_{l-2}(t) = t_j t_l$ for some $j < l - 1$. This root is positive and never simple. Hence we have a contradiction. Thus $s_{l-1} \neq t_l^{-1}$. Thus we have shown the claim for $s_{l-1} = t_l$ (and t_l^{-1} for which we had a contradiction) so far. Suppose $s_{l-1} = t_j^{-1}$ for some $j < l$. Then $w_\theta \alpha_{l-1}(t) = t_j^{-1} t_r$ and $w_\theta \alpha_l(t) = t_j^{-1} t_r^{-1}$. So $w_\theta \alpha_l$ is always negative. $w_\theta \alpha_{l-1}$ is negative if $j < r$ and simple if $j = r + 1$. We show that we must have $j = r + 1$. Suppose $j < r$ for contradiction. Then there exists n such that $s_n = t_l$ or t_l^{-1} . Suppose $s_n = t_l$. Then $w_\theta \alpha_n(t) = s_n s_{n+1}^{-1} = t_l t_{j_{n+1}}$. This root is positive and only simple if $j_{n+1} = l - 1$. Then $w_\theta \alpha_{n+1}(t) = t_{l-1}^{-1} t_{j_{n+2}}$ for some $j_{n+2} < l - 1$. Again this root is always positive and is simple only if $j_{n+2} = l - 2$. Continuing this, we find $s_i = t_{l+n-i}^{-1}$ for $n + 1 \leq i \leq l$. Thus we have $k = n$ and $w_\theta \alpha_{l-1}(t) = t_{n+1}^{-1} t_n$ is simple and $w_\theta \alpha_l(t) = t_{n+1}^{-1} t_n^{-1}$ is negative. Therefore, $\{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}$. We address the final case. Suppose that $s_n = t_l^{-1}$. Then $w_\theta \alpha_n(t) = t_l^{-1} t_{j_{n+1}}$ with $j_{n+1} < l$. This root is always positive. It is simple only if $j_{n+1} = l - 1$. The rest of this case follows exactly as in the previous case. Again we find $\{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}$ and $\theta \in P_l$.

Next, suppose that $\{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}$. Let

$$w_\theta tw_\theta^{-1} = \mathrm{diag}(s_1, \dots, s_l, s_l^{-1}, \dots, s_1^{-1}).$$

Then, $w_\theta \alpha_{l-1}(t) = s_{l-1} s_l^{-1}$ is simple and $w_\theta \alpha_{l-1}(t) = s_{l-1} s_l$ is negative. Suppose first that the simple root $w_\theta \alpha_{l-1}(t)$ is $s_{l-1} s_l^{-1} = t_r t_{r+1}^{-1}$ for some $r < l$. If $s_l = t_{r+1}$, then $w_\theta \alpha_l(t) = t_r t_{r+1}$ is positive which is a contradiction. Therefore, we must have $s_{l-1} = t_{r+1}^{-1}$ and $s_l = t_r^{-1}$. Hence, $w_\theta \alpha_{l-2}(t) = s_{l-2} s_{l-1}^{-1} = t_{j_{l-2}}^\epsilon t_{r+1}$ where $\epsilon \in \{\pm 1\}$. This is simple if $\epsilon = -1$ and $j_{l-2} = r + 2$. It is negative if $\epsilon = -1$ and $j_{l-2} < r + 1$. In either case, we have $s_{l-2} = t_{j_{l-2}}^{-1}$. We can continue this argument until we arrive at $s_{n+1} = t_{l-1}^{-1}$. That is, $s_i = t_{j_i}^{-1}$ for $n + 1 \leq i \leq l$. The next root is $w_\theta \alpha_n(t) = s_n s_{n+1}^{-1} = t_{j_n}^\epsilon t_{l-1}$ where $\epsilon \in \{\pm 1\}$. This is simple for two cases: when $j_n = l$ and $\epsilon = \pm 1$. ϵ is determined by the parity of l . Specifically, if $\epsilon = 1$, then l must be odd and if $\epsilon = -1$, then l must be even. Suppose that $s_n = t_l$. Then $w_\theta \alpha_{n-1}(t) = s_{n-1} s_n^{-1} = t_{j_{n-1}}^\epsilon t_l^{-1}$ where $j_{n-1} < r$ and $\epsilon \in \{\pm 1\}$. Since $j_{n-1} < r < l - 1$, this root can never be simple. Thus it must be negative and so $\epsilon = -1$. That is, $s_{n-1} = t_{j_{n-1}}^{-1}$. Continuing in the above manner, we again find that $s_i = t_{j_i}^{-1}$ for $i < n$. Let $w'' \in W(\mathrm{GL}_l)$ be such that $w'' \mathrm{diag}(t_l, t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^{-1} = \mathrm{diag}(s_1, \dots, s_l)$. Then, $w_\theta = t_l(w'') \tilde{w}_l \in B_l(\mathrm{SO}_{2l})$. Next, we must address the case $s_n = t_l^{-1}$. Then $w_\theta \alpha_{n-1}(t) = s_{n-1} s_n^{-1} = t_{j_{n-1}}^\epsilon t_l$ where $j_{n-1} < r$ and $\epsilon \in \{\pm 1\}$. Since $j_{n-1} < r < l - 1$, this root can never be simple. Thus it must be negative and so $\epsilon = -1$. That is, $s_{n-1} = t_{j_{n-1}}^{-1}$. Continuing in the above manner, we again find that $s_i = t_{j_i}^{-1}$ for $i < n$. Let $w'' \in W(\mathrm{GL}_l)$ be such that $w'' \mathrm{diag}(t_l^{-1}, t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^{-1} = \mathrm{diag}(s_1, \dots, s_l)$. Then, $w_\theta = t_l(w'') \tilde{w}_l \in B_l(\mathrm{SO}_{2l})$. Thus, if $w_\theta \alpha_{l-1}(t) = t_r t_{r+1}^{-1}$ for some $r < l$, then

$$P_l = \{\theta \in \Delta(\mathrm{SO}_{2l}) \mid \{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\mathrm{SO}_{2l}) \setminus \{\alpha_l\}\},$$

Suppose that $w_\theta\alpha_{l-1}(t)$ is the remaining simple root $s_{l-1}s_l^{-1} = t_{l-1}t_l$. If $s_{l-1} = t_{l-1}$ then $w_\theta\alpha_l$ is simple which is a contradiction. Thus, we must have $s_{l-1} = t_l$ and $s_l = t_{l-1}^{-1}$.

Thus, $w_\theta\alpha_{l-2}(t) = s_{l-2}t_l^{-1}$. This root cannot be simple and hence we must have $s_{l-2} = t_{j_{l-2}}^{-1}$. Next, $w_\theta\alpha_{l-3}(t) = s_{l-3}t_{j_{l-2}} = t_{j_{l-3}}^\epsilon t_{j_{l-2}}$. This root is always positive and not simple if $\epsilon = 1$. Thus we must have $s_{l-3} = t_{j_{l-3}}^{-1}$. Continuing in this manner for all the remaining roots shows that for any $i \leq l-2$ we have $s_i \in \{t_1^{-1}, \dots, t_{l-2}^{-1}\}$. Thus,

$$w_\theta = \begin{pmatrix} & & w_1 \\ & w_2 \\ w'_1 \end{pmatrix},$$

where $w_1 \text{diag}(t_{l-2}^{-1}, \dots, t_1^{-1})w_1^{-1} = \text{diag}(s_1, \dots, s_{l-2})$, and

$$w_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\det(w_2) = -1$ and hence this is only possible if l is odd (for l even, $\det w_\theta = -1$). Let $w'' \in W(\text{GL}_l)$ be such that $w'' \text{diag}(t_l, t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^{-1} = \text{diag}(s_1, \dots, s_l)$. Then, we have $w_\theta = t_l(w'')\tilde{w}_l \in B_l(\text{SO}_{2l})$. Thus, in both the cases of $w_\theta\alpha_{l-1}(t)$, we have

$$P_l = \{\theta \in \Delta(\text{SO}_{2l}) \mid \{\alpha_{l-1}\} \subseteq \theta \subseteq \Delta(\text{SO}_{2l}) \setminus \{\alpha_l\}\}.$$

Finally, conjugation by c fixes the roots α_i for $i \leq l-2$ and maps $\alpha_{l-1} \mapsto \alpha_l$ and $\alpha_l \mapsto \alpha_{l-1}$. Using the result for P_l , it is straightforward to check that

$$P_l^c = \{\theta \in \Delta(\text{SO}_{2l}) \mid \{\alpha_l\} \subseteq \theta \subseteq \Delta(\text{SO}_{2l}) \setminus \{\alpha_{l-1}\}\}.$$

This finishes the proof of the proposition. \square

Next, we interpret the previous proposition in terms of the Bessel support.

Proposition 4.8 *The sets $B_n(\text{SO}_{2l})$ for $n = 0, 1, \dots, l$ and $B_l^c(\text{SO}_{2l})$ form a partition of $B(\text{SO}_{2l})$.*

Proof We have that

$$\bigsqcup_{n=0}^{l-2} P_n \sqcup P_l \sqcup P_l^c \sqcup P_{l-1} = \mathcal{P}(\Delta(\text{SO}_{2l})),$$

the power set of $\Delta(\text{SO}_{2l})$. The claim then follows from Proposition 4.7. \square

We need the following proposition to show that the zeta integrals are nonzero in later sections.

Proposition 4.9 *If $w \in W(\text{GL}_l)$ and $w \neq I_l$, then $t_l(w) \notin B(\text{SO}_{2l})$. In particular, if $a \in \text{GL}_l$ and a is not upper triangular, then $\mathcal{B}_{\pi, \psi}(t_l(a)) = 0$.*

Proof We use the partition of the Bessel support in Proposition 4.8. First, suppose that $t_l(w) = t_n(w')\tilde{w}_n$ for some n with $1 \leq n \leq l$ and $w' \in W(\text{GL}_n)$. Then, $\tilde{w}_n \in W(\text{GL}_l)$.

However, this is not the case for any n . Next, suppose $t_l(w) = ct_l(w')\tilde{w}_l c$ (this is the case that $t_l(w) \in \mathrm{B}_l^c(\mathrm{SO}_{2l})$). Thus, $ct_l(w)c = t_l(w')\tilde{w}_l$. Let $w = (w_{i,j})_{i,j=1}^l$. Then,

$$ct_l(w)c = \begin{pmatrix} (w_{i,j})_{i,j=1}^{l-1} & 0 & (w_{i,l})_{i=1}^{l-1} & 0 \\ 0 & * & 0 & * \\ (w_{l,j})_{j=2}^{l-1} & 0 & w_{l,l} & 0 \\ 0 & * & 0 & * \end{pmatrix}.$$

If l is even then,

$$t_l(w')\tilde{w}_l = \begin{pmatrix} w' \\ (w')^* \end{pmatrix}.$$

Hence, $w_{l,l} = 0$ and $w_{i,j} = 0$ for any $1 \leq i, j \leq l-1$. Thus,

$$w = \begin{pmatrix} 0_{(l-1) \times (l-1)} & * \\ * & 0 \end{pmatrix}.$$

But, $w \in W(\mathrm{GL}_l)$ and hence this is not possible.

If l is odd, we have

$$t_l(w')\tilde{w}_l = \begin{pmatrix} 0_{l \times (l-1)} & (w'_{i,1})_{i=1}^l & 0_{l \times 1} & (w'_{i,j})_{i=1,j=2}^l \\ * & 0 & * & 0 \end{pmatrix}.$$

Again, $w_{i,j} = 0$ for any $1 \leq i, j \leq l-1$, but since $w \in W(\mathrm{GL}_l)$, this is not possible. Therefore, $w \notin \mathrm{B}_l^c(\mathrm{SO}_{2l})$ and $w \notin \mathrm{B}_n(\mathrm{SO}_{2l})$ for any n . By Proposition 4.8, $w \notin \mathrm{B}(\mathrm{SO}_{2l})$.

For the second part of the claim, suppose $a \in \mathrm{GL}_l$ is not upper triangular. Then, by the Bruhat decomposition of GL_l , there exists $w \in W(\mathrm{GL}_l)$ such that $w \neq I_l$ and $a = u_1 t w u_2$ where $t \in T_{\mathrm{GL}_l}$ and $u_1, u_2 \in U_{\mathrm{GL}_l}$. We obtain $\mathcal{B}_{\pi, \psi}(t_l(a)) = \mathcal{B}_{\pi, \psi}(t_l(t)t_l(w))\psi(t_l(u_1))\psi(t_l(u_2))$. However, $t_l(w) \notin \mathrm{B}(\mathrm{SO}_{2l})$ and hence $\mathcal{B}_{\pi, \psi}(t_l(t)t_l(w)) = 0$. Thus, we have proven the proposition. \square

5 Twists by GL_n for $n \leq l-2$

In this section, we calculate the zeta integrals for twists up to GL_{l-2} . Let $v \in \tau$ be a fixed vector and define $\xi_v \in I(\tau)$ by $\mathrm{supp}(\xi_v) = Q_n$ and

$$\xi_v(l_n(a)u) = \tau(a)v, \forall a \in \mathrm{GL}_n, u \in V_n,$$

where $l_n(a) = \mathrm{diag}(a, 1, a^*)$ and $Q_n = L_n V_n$ is the standard Siegel parabolic of SO_{2n+1} . Let $f_v = f_{\xi_v} \in I(\tau, \psi^{-1})$. That is, $f_v(g, a) = \Lambda_\tau(\tau(a)\xi_v(g))$, where $\Lambda_\tau \in \mathrm{Hom}_{U_{\mathrm{GL}_n}}(\tau, \psi^{-1})$, $a \in \mathrm{GL}_n$, and $g \in \mathrm{SO}_{2n+1}$. We also fix the Whittaker function $W_v(a) = \Lambda_\tau(\tau(a)v)$. We show that the zeta integrals in this case are nonzero.

Proposition 5.1 $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = W_v(I_n)$. In particular, we may choose $v \in \tau$ such that $\Psi(\mathcal{B}_{\pi, \psi}, f_v) \neq 0$.

Proof By definition,

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \sum_{g \in U_{\mathrm{SO}_{2n+1}} \setminus \mathrm{SO}_{2n+1}} \left(\sum_{r \in R^{l,n}} \mathcal{B}_{\pi, \psi}(rw^{l,n}g(w^{l,n})^{-1}) \right) f_v(g, I_n).$$

The support of $f_v(g, I_n)$ is contained in $V_n L_n$. Since $V_n \subseteq U_{\mathrm{SO}_{2n+1}}$, we have

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \sum_{a \in U_{\mathrm{GL}_n} \setminus \mathrm{GL}_n} \left(\sum_{r \in R^{l, n}} \mathcal{B}_{\pi, \psi}(rw^{l, n}l_n(a)(w^{l, n})^{-1}) \right) W_v(a).$$

The embedding of SO_{2n+1} into SO_{2l} takes $l_n(a)$ to $q_n(a) = \mathrm{diag}(I_{l-n-1}, a, I_2, a^*, I_{l-n-1})$. We have $w^{l, n}q_n(a)(w^{l, n})^{-1} = \mathrm{diag}(a, I_{2l-2n}, a^*)$. Let

$$r_x = \begin{pmatrix} I_n & & & \\ x & I_{l-n-1} & & \\ & & I_2 & \\ & & & I_{l-n-1} \\ & & & x' & I_n \end{pmatrix} \in R^{l, n}.$$

Then,

$$r_x w^{l, n} q_n(a)(w^{l, n})^{-1} = \begin{pmatrix} a & & & \\ xa & I_{l-n-1} & & \\ & & I_2 & \\ & & & I_{l-n-1} \\ & & & x'a^* & a^* \end{pmatrix}.$$

By Proposition 4.9, $\mathcal{B}_{\pi, \psi}(r_x w^{l, n} q_n(a)(w^{l, n})^{-1}) = 0$ unless a is upper triangular and $x = 0$. Hence,

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \sum_{a \in T_{\mathrm{GL}_n}} \mathcal{B}_{\pi, \psi}(t_n(a)) W_v(a).$$

Finally, by Lemma 4.3, $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = W_v(I_n)$. If we choose v to be a Whittaker vector for τ , then $W_v(I_n) \neq 0$. This completes the proof of the proposition. \square

Let $\tilde{f}_v = M(\tau, \psi^{-1})f_v$ and $W_v^*(a) = \Lambda_\tau(\tau(d_n a^*)v)$. The following lemma is the analogue of [19, Lemma 3.6] and holds for any n . The proof follows similarly.

Lemma 5.2 1. If $\tilde{f}_v(g, I_n) \neq 0$, then $g \in Q_n w_n Q_n = Q_n w_n V_n$.

2. If $x \in V_n$, then $\tilde{f}_v(w_n x, I_n) = W_v^*(I_n)$.

3. For $a \in \mathrm{GL}_n$ and $x \in V_n$, $\tilde{f}_v(l_n(a)w_n x, I_n) = W_v^*(a)$.

In [19], each twist utilizes [22, Lemma 3.1]. For twists up to GL_{l-2} , [22, Lemma 3.1] would be sufficient. However, for twists by GL_{l-1} and GL_l , it is expedient to have the below lemma. Note that [22, Lemma 3.1] is precisely the case when $X = \mathrm{GL}_n$ in the below lemma.

Lemma 5.3 Suppose X is a subset of GL_n for which $ux \in X$ for any $u \in U_{\mathrm{GL}_n}$ and $x \in X$. Let H be a function on X such that $H(ug) = \psi(u)H(g)$ for any $u \in U_{\mathrm{GL}_n}$ and $g \in X$. Suppose that

$$\sum_{x \in X} H(x) W_v(x) = 0$$

for any $v \in \tau$ and with τ running through all generic irreducible representations of GL_n . Then, $H(x) = 0$ for any $x \in X$.

Proof Let H' be the function on GL_n defined by $H'(g) = H(g)$ if $g \in X$ and $H'(g) = 0$ if $g \notin X$. Then, $H'(ug) = \psi(u)H'(g)$ for any $u \in U_{\mathrm{GL}_n}$ and $g \in \mathrm{GL}_n$. By [22, Lemma 3.1], $H' \equiv 0$ on GL_n and hence the lemma follows. \square

The following theorem shows that, for $n \leq l-2$, the twists by GL_n determine that the Bessel functions are equal on $B_n(\mathrm{SO}_{2l})$. Note that \tilde{w}_n fixes α_{l-1} and α_l when n is even and switches them when n is odd. Recall that $t'_n = \tilde{t} = \mathrm{diag}(I_{l-1}, \frac{-1}{2}, -2, I_{l-1})$ if n is odd and $t'_n = I_{2l}$ if n is even. We explain the inclusion of the term t'_n in the below theorem. Let $x \in \mathbb{F}_q$ and $\mathbf{x}_{\alpha_l}(x)$ be an element of the root space of α_l . By Proposition 4.1, we have

$$\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)(-\psi(x)/2).$$

When n is odd,

$$\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(\mathbf{x}_{\alpha_{l-1}}(x)t_n(a)\tilde{w}_n) = \mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)(\psi(x)/4).$$

Since ψ is nontrivial, it follows that $\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n) = 0$. However, if we include $t'_n = \tilde{t}$, we have

$$\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)(-\psi(x)/2)$$

and

$$\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(\mathbf{x}_{\alpha_{l-1}}(-2x)t_n(a)\tilde{w}_n) = \mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)(-\psi(x)/2).$$

Thus, $\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)$ may be nonzero.

When n is even, we have

$$\mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(\mathbf{x}_{\alpha_l}(x)t_n(a)\tilde{w}_n) = \mathcal{B}_{\pi, \psi}(t_n(a)\tilde{w}_n)(-\psi(x)/2)$$

and so $t'_n = I_{2l}$ suffices. Thus, we see that the inclusion of the term t'_n is necessary in the below theorem.

Theorem 5.4 *Let π and π' be irreducible cuspidal ψ -generic representations of SO_{2l} which share the same central character. If $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n , then $\mathcal{B}_{\pi, \psi}(t_n(a)t'_n \tilde{w}_n) = \mathcal{B}_{\pi', \psi}(t_n(a)t'_n \tilde{w}_n)$ for any $a \in \mathrm{GL}_n$.*

Proof By Proposition 5.1, we have $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \Psi(\mathcal{B}_{\pi', \psi}, f_v) = W_v(I_n)$. From our assumption, $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n , and hence it follows that $\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{\pi', \psi}, \tilde{f}_v)$. By definition,

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \sum_{g \in U_{\mathrm{SO}_{2n+1}} \setminus \mathrm{SO}_{2n+1}} \left(\sum_{r \in R^{l, n}} \mathcal{B}_{\pi, \psi}(rw^{l, n}g(w^{l, n})^{-1}) \right) \tilde{f}_v(g, I_n).$$

By Lemma 5.2,

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \sum_{\substack{a \in U_{\mathrm{GL}_n} \setminus \mathrm{GL}_n \\ x \in V_n}} \left(\sum_{r \in R^{l, n}} \mathcal{B}_{\pi, \psi}(rw^{l, n}l_n(a)w_n x(w^{l, n})^{-1}) \right) \tilde{f}_v(l_n(a)w_n x, I_n).$$

The embedding of SO_{2n+1} into SO_{2l} takes w_n to $t'_n \tilde{w}_n$ (see §4.2) and $l_n(a)$ to the element $q_n(a) = \mathrm{diag}(I_{l-n-1}, a, I_2, a^*, I_{l-n-1})$. The embedding also takes the unipotent element $x =$

$\begin{pmatrix} I_n & * & * \\ & 1 & * \\ & & I_n \end{pmatrix}$ to $\tilde{x} = \text{diag}(I_{l-n-1}, \begin{pmatrix} I_n & * & * \\ & I_2 & * \\ & & I_n \end{pmatrix}, I_{l-n-1})$, and we have

$$\tilde{x}(w^{l,n})^{-1} = (w^{l,n})^{-1} \begin{pmatrix} I_n & 0 & * & 0 & * \\ & I_{l-n-1} & 0 & 0 & 0 \\ & & I_2 & 0 & * \\ & & & I_{l-n-1} & 0 \\ & & & & I_n \end{pmatrix}.$$

The character is trivial on the last unipotent matrix and hence, by Lemma 5.2,

$$\Psi(\mathcal{B}_{\pi,\psi}, \tilde{f}_v) = |V_n| \sum_{a \in U_{GL_n} \setminus \text{GL}_n} \left(\sum_{r \in R^{l,n}} \mathcal{B}_{\pi,\psi}(rw^{l,n}q_n(a)t'_n \hat{w}_n(w^{l,n})^{-1}) \right) W_v^*(a).$$

Next, $w^{l,n}q_n(a) = t_n(a)w^{l,n}$ where $t_n(a) = \text{diag}(a, I_{2l-2n}, a^*)$. Also, $w^{l,n}t'_n = t'_n w^{l,n}$. Let

$$r_x = \begin{pmatrix} I_n & & & & \\ x & I_{l-n-1} & & & \\ & & I_2 & & \\ & & & I_{l-n-1} & \\ & & & x' & I_n \end{pmatrix} \in R^{l,n}.$$

Then, $r_x t_n(a) t'_n = t_n(a) t'_n r_{ax}$. Recall that $\tilde{w}_n = w^{l,n} \hat{w}_n (w^{l,n})^{-1}$. Then,

$$\Psi(\mathcal{B}_{\pi,\psi}, \tilde{f}_v) = |V_n| \sum_{a \in U_{GL_n} \setminus \text{GL}_n} \left(\sum_{r_x \in R^{l,n}} \mathcal{B}_{\pi,\psi}(t_n(a) t'_n r_{ax} \tilde{w}_n) \right) W_v^*(a).$$

Note that

$$r_{ax} \tilde{w}_n = \tilde{w}_n \begin{pmatrix} I_n & & & & \\ & I_{l-n-1} & & & \\ & & I_2 & & \\ & & & I_{l-n-1} & \\ & & & & I_n \end{pmatrix}.$$

The character is trivial on the last unipotent element and hence

$$\Psi(\mathcal{B}_{\pi,\psi}, \tilde{f}_v) = |V_n| |R^{l,n}| \sum_{a \in U_{GL_n} \setminus \text{GL}_n} \mathcal{B}_{\pi,\psi}(t_n(a) t'_n \tilde{w}_n) W_v^*(a).$$

Since $\Psi(\mathcal{B}_{\pi,\psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{\pi',\psi}, \tilde{f}_v)$, it follows that

$$0 = \sum_{a \in U_{GL_n} \setminus \text{GL}_n} (\mathcal{B}_{\pi,\psi} - \mathcal{B}_{\pi',\psi})(t_n(a) t'_n \tilde{w}_n) W_v^*(a).$$

Let f be the function on GL_n defined by $f(a) = (\mathcal{B}_{\pi,\psi} - \mathcal{B}_{\pi',\psi})(t_n(a) t'_n \tilde{w}_n)$. Then, $f(ua) = \psi(u)f(a)$ and

$$0 = \sum_{a \in U_{GL_n} \setminus \text{GL}_n} f(a) W_v^*(a).$$

By Lemma 5.3, $f(a) = 0$ for any $a \in \mathrm{GL}_n$ which proves the theorem. \square

As a consequence, we see that the twists by GL_n determine the Bessel functions on the Bruhat cells for $w \in \mathrm{B}_n(\mathrm{SO}_{2l})$.

Theorem 5.5 *Let π and π' be irreducible cuspidal ψ -generic representations of SO_{2l} which share the same central character. If $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n , then $\mathcal{B}_{\pi, \psi}(b_1 w b_2) = \mathcal{B}_{\pi', \psi}(b_1 w b_2)$ for any $b_1, b_2 \in B_{\mathrm{SO}_{2l}}$ and $w \in \mathrm{B}_n(\mathrm{SO}_{2l})$.*

Proof We have $b_1 w b_2 = u_1 t w u_2$ for some $t \in T_{\mathrm{SO}_{2l}}$, and $u_1, u_2 \in U_{\mathrm{SO}_{2l}}$. By Proposition 4.1, it suffices to show that $\mathcal{B}_{\pi, \psi}(tw) = \mathcal{B}_{\pi', \psi}(tw)$. We shall see that the support of these functions on $T_{\mathrm{SO}_{2l}} \mathrm{B}_n(\mathrm{SO}_{2l})$ is contained in the set $\{t_n(a) t'_n \tilde{w}_n, |, a \in \mathrm{GL}_n\}$. Suppose that $\mathcal{B}_{\pi, \psi}(tw) \neq 0$. By definition of $\mathrm{B}_n(\mathrm{SO}_{2l})$, there exists $w' \in W(\mathrm{GL}_n)$ such that $w = t_n(w') \tilde{w}_n$. Recall that

$$\begin{aligned} \tilde{w}_n \alpha_i &= \alpha_{n-i} \text{ for } 1 \leq i \leq n-1, \\ \tilde{w}_n \alpha_n(t) &= t_1^{-1} t_{n+1}^{-1}, \\ \tilde{w}_n \alpha_i &= \alpha_i \text{ for } n+1 \leq i \leq l-2, \\ \tilde{w}_n \alpha_{l-1} &= \begin{cases} \alpha_l & \text{if } n \text{ is odd,} \\ \alpha_{l-1} & \text{if } n \text{ is even,} \end{cases} \\ \tilde{w}_n \alpha_l &= \begin{cases} \alpha_{l-1} & \text{if } n \text{ is odd,} \\ \alpha_l & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Also $t_n(w') \alpha_i = \alpha_i$ for any $n+1 \leq i \leq l$. Let $x \in \mathbb{F}_q$ and $\mathbf{x}_{\alpha_i}(x)$ be an element of the root space of α_i . By Proposition 4.1, for $n+1 \neq i \leq l-2$ we have

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_i}(x)) = \mathcal{B}_{\pi, \psi}(tw) \psi(x).$$

On the other hand,

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_i}(x)) = \mathcal{B}_{\pi, \psi}(tw) \psi(\alpha_i(t)x).$$

Since ψ is nontrivial, we must have $\alpha_i(t) = 1$ for any $n+1 \leq i \leq l-2$. That is, $t_i = t_{i+1}$ for any $n+1 \leq i \leq l-2$.

Similarly,

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(tw)(-\psi(x)/2)$$

and

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_{l-1}}(x)) = \mathcal{B}_{\pi, \psi}(tw)(\psi(x)/4).$$

When n is odd,

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi, \psi}(\mathbf{x}_{\alpha_{l-1}}(\alpha_{l-1}(t)x) tw) = \mathcal{B}_{\pi, \psi}(tw)(\psi(\alpha_{l-1}(t)x)/4),$$

and

$$\mathcal{B}_{\pi, \psi}(tw \mathbf{x}_{\alpha_{l-1}}(x)) = \mathcal{B}_{\pi, \psi}(\mathbf{x}_{\alpha_l}(\alpha_l(t)x) tw) = \mathcal{B}_{\pi, \psi}(tw)(-\psi(\alpha_l(t)x)/2),$$

Since ψ is nontrivial, we have $\frac{\alpha_{l-1}(t)}{4} = \frac{-1}{2}$ and $\frac{-\alpha_l(t)}{2} = \frac{1}{4}$. Write $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$. We find that $t_{l-1} = \pm 1$ and $t_l = \mp \frac{1}{2}$. Furthermore, $t_i = t_{i+1} = t_{l-1} = \pm 1$ for any $n+1 \leq i \leq l-2$. Let $a = \mathrm{diag}(t_1, \dots, t_n) w'$. Then $tw = t_n(a) \tilde{t} \tilde{w}_n$ if $t_{l-1} = 1$ and by Theorem 5.4, we

have $\mathcal{B}_{\pi,\psi}(tw) = \mathcal{B}_{\pi',\psi}(tw)$. If $t_{l-1} = -1$, then $tw = -t_n(a)\tilde{t}\tilde{w}_n$. Since π and π' have the same central character, say χ , by Theorem 5.4,

$$\begin{aligned}\mathcal{B}_{\pi,\psi}(tw) &= \mathcal{B}_{\pi,\psi}(-t_n(a)\tilde{t}\tilde{w}_n) \\ &= \chi(-1)\mathcal{B}_{\pi,\psi}(t_n(a)\tilde{t}\tilde{w}_n) \\ &= \chi(-1)\mathcal{B}_{\pi',\psi}(t_n(a)\tilde{t}\tilde{w}_n) \\ &= \mathcal{B}_{\pi',\psi}(-t_n(a)\tilde{t}\tilde{w}_n) \\ &= \mathcal{B}_{\pi',\psi}(tw).\end{aligned}$$

This proves the corollary when n is odd.

Suppose now that n is even. We have

$$\mathcal{B}_{\pi,\psi}(tw\mathbf{x}_{\alpha_l}(x)) = \mathcal{B}_{\pi,\psi}(\mathbf{x}_{\alpha_l}(\alpha_l(t)x)tw) = \mathcal{B}_{\pi,\psi}(tw)(\psi(\alpha_l(t)x)/4),$$

and

$$\mathcal{B}_{\pi,\psi}(tw\mathbf{x}_{\alpha_{l-1}}(x)) = \mathcal{B}_{\pi,\psi}(\mathbf{x}_{\alpha_{l-1}}(\alpha_{l-1}(t)x)tw) = \mathcal{B}_{\pi,\psi}(tw)(-\psi(\alpha_{l-1}(t)x)/2),$$

Since ψ is nontrivial, we have $\alpha_{l-1}(t) = 1$ and $\alpha_l(t) = 1$. Write $t = \text{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$. We find that $t_{l-1} = t_l = \pm 1$. Furthermore, $t_i = t_{i+1} = t_{l-1} = \pm 1$ for any $n+1 \leq i \leq l-2$. Let $a = \text{diag}(t_1, \dots, t_n)w'$. Then $tw = t_n(a)\tilde{w}_n$ if $t_{l-1} = 1$ and by Theorem 5.4, we have $\mathcal{B}_{\pi,\psi}(tw) = \mathcal{B}_{\pi',\psi}(tw)$. If $t_{l-1} = -1$, then $tw = -t_n(a)\tilde{w}_n$. Since π and π' have the same central character, say χ , by Theorem 5.4,

$$\begin{aligned}\mathcal{B}_{\pi,\psi}(tw) &= \mathcal{B}_{\pi,\psi}(-t_n(a)\tilde{w}_n) \\ &= \chi(-1)\mathcal{B}_{\pi,\psi}(t_n(a)\tilde{w}_n) \\ &= \chi(-1)\mathcal{B}_{\pi',\psi}(t_n(a)\tilde{w}_n) \\ &= \mathcal{B}_{\pi',\psi}(-t_n(a)\tilde{w}_n) \\ &= \mathcal{B}_{\pi',\psi}(tw).\end{aligned}$$

This proves the corollary when n is even. \square

6 Twists by GL_{l-1}

In this section, we consider the twists of GL_{l-1} . The arguments of this section are similar to the previous section, except that $R^{l,n}$ and $w^{l,n}$ are trivial which leads to minor differences.

Let $v \in \tau$ be a fixed vector and define $\xi_v \in I(\tau)$ by $\text{supp}(\xi_v) = L_{l-1}V_{l-1} = Q_{l-1}$ and

$$\xi_v(l_{l-1}(a)u) = \tau(a)v, \forall a \in \text{GL}_{l-1}, u \in V_{l-1}.$$

Let $f_v = f_{\xi_v} \in I(\tau, \psi^{-1})$. That is, for $\Lambda_\tau \in \text{Hom}_{U_{\text{GL}_{l-1}}}(\tau, \psi^{-1})$, $a \in \text{GL}_{l-1}$, and $g \in \text{SO}_{2l-1}$, we have $f_v(g, a) = \Lambda_\tau(\tau(a)\xi_v(g))$. We also let $W_v(a) = \Lambda_\tau(\tau(a)v)$. We show that the zeta integrals in this case are nonzero.

Proposition 6.1 *Suppose $\mathcal{B}_{\pi,\psi}$ be the normalized Bessel function of π . Then, we have $\Psi(\mathcal{B}_{\pi,\psi}, f_v) = W_v(I_{l-1})$. In particular, we may choose $v \in \tau$ such that $\Psi(\mathcal{B}_{\pi,\psi}, f_v) \neq 0$.*

Proof By definition,

$$\Psi(\mathcal{B}_{\pi,\psi}, f_v) = \sum_{g \in U_{\text{SO}_{2l-1}} \backslash \text{SO}_{2l-1}} \mathcal{B}_{\pi,\psi}(g) f_v(g, I_{l-1}).$$

The support of $f_v(\cdot, I_{l-1})$ is Q_{l-1} . Thus,

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \sum_{a \in U_{\mathrm{GL}_{l-1}} \backslash \mathrm{GL}_{l-1}} \mathcal{B}_{\pi, \psi}(l_{l-1}(a)) f_v(l_{l-1}(a), I_{l-1}).$$

Under the embedding of SO_{2l-1} into SO_{2l} , $l_{l-1}(a)$ maps to $t_{l-1}(a)$. Hence,

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \sum_{a \in U_{\mathrm{GL}_{l-1}} \backslash \mathrm{GL}_{l-1}} \mathcal{B}_{\pi, \psi}(t_{l-1}(a)) W_v(a).$$

By Proposition 4.9, $\mathcal{B}_{\pi, \psi}(t_{l-1}(a)) = 0$ unless a is upper triangular. Thus, by Lemma 4.3,

$$\Psi(\mathcal{B}_{\pi, \psi}, f_v) = W_v(I_{l-1}).$$

If we choose v to be a Whittaker vector for τ , then $W_v(I_{l-1}) \neq 0$. This proves the proposition. \square

The next theorem shows that the twists by GL_{l-1} determine that the Bessel functions are equal on a subset of the Bruhat cells in $B_{l-1}(\mathrm{SO}_{2l})$. More specifically, by Proposition 4.1, to determine the normalized Bessel functions on a Bruhat cell of $w \in B_{l-1}(\mathrm{SO}_{2l})$, it is enough to determine it on $T_{\mathrm{SO}_{2l}} w$. The below theorem determines the subset of $T_{\mathrm{SO}_{2l}} w$ for $t \in T_{\mathrm{SO}_{2l}}$ such that the l -th coordinate is $t_l = 1$ if n is odd and $t_l = \frac{1}{2}$ if n is even. Using the central character we can determine $t_l = \pm 1$ if n is odd and $t_l = \pm \frac{1}{2}$ if n is even. Later, Theorem 7.7 shows that the twists by GL_l determine the remaining part where $t_l \neq \pm 1$ if n is odd and $t_l \neq \pm \frac{1}{2}$ if n is even.

Theorem 6.2 *Let π and π' be irreducible cuspidal ψ -generic representations of SO_{2l} which share the same central character. If $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_{l-1} , then we have $\mathcal{B}_{\pi, \psi}(t_{l-1}(a) t'_{l-1} \tilde{w}_{l-1}) = \mathcal{B}_{\pi', \psi}(t_{l-1}(a) t'_{l-1} \tilde{w}_{l-1})$ for any $a \in \mathrm{GL}_{l-1}$.*

Proof By Proposition 6.1, $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \Psi(\mathcal{B}_{\pi', \psi}, f_v) = W_v(I_{l-1})$. From our assumption, $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_{l-1} , and hence it follows that $\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{\pi', \psi}, \tilde{f}_v)$. By definition,

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \sum_{g \in U_{\mathrm{SO}_{2l-1}} \backslash \mathrm{SO}_{2l-1}} \mathcal{B}_{\pi, \psi}(g) \tilde{f}_v(g, I_{l-1}).$$

From Lemma 5.2, we get

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \sum_{\substack{a \in U_{\mathrm{GL}_{l-1}} \backslash \mathrm{GL}_{l-1} \\ u \in V_{l-1}}} \mathcal{B}_{\pi, \psi}(l_{l-1}(a) w_{l-1} u) W_v^*(a).$$

The embedding of SO_{2l-1} into SO_{2l} takes $l_{l-1}(a)$ to $t_{l-1}(a)$ and w_{l-1} to $t'_{l-1} \tilde{w}_{l-1}$. Let

$$u = \begin{pmatrix} I_{l-1} & x & y \\ & 1 & x' \\ & & I_{l-1} \end{pmatrix}.$$

Then the embedding takes u to

$$\tilde{u} = \begin{pmatrix} I_{l-1} & x & \frac{x}{2} & y \\ & 1 & \frac{x'}{2} & \\ & & 1 & x' \\ & & & I_{l-1} \end{pmatrix}.$$

Since $\psi(\tilde{u}) = 1$, we have

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = |V_{l-1}| \sum_{a \in U_{\mathrm{GL}_{l-1}} \backslash \mathrm{GL}_{l-1}} \mathcal{B}_{\pi, \psi}(t_{l-1}(a)t'_{l-1}\tilde{w}_{l-1}) W_v^*(a).$$

From $\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{\pi', \psi}, \tilde{f}_v)$, it follows that

$$0 = \sum_{a \in U_{\mathrm{GL}_{l-1}} \backslash \mathrm{GL}_{l-1}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(t_{l-1}(a)t'_{l-1}\tilde{w}_{l-1}) W_v^*(a).$$

Let f be the function on GL_{l-1} defined by $f(a) = (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(t_{l-1}(a)t'_{l-1}\tilde{w}_{l-1})$. Then, $f(ua) = \psi(u)f(a)$ and

$$0 = \sum_{a \in \mathrm{GL}_{l-1}} f(a) W_v^*(a).$$

By Lemma 5.3, $f(a) = 0$ for any $a \in \mathrm{GL}_{l-1}$ which gives the theorem. \square

The following corollary shows that Theorems 5.5 and 6.2 hold for the Bessel functions of $c \cdot \pi$ and $c \cdot \pi'$ as well.

Corollary 6.3 *Let π and π' be irreducible cuspidal ψ -generic representations of SO_{2l} which share the same central character. If $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n for any $n \leq l-1$, then $\mathcal{B}_{c \cdot \pi, \psi}(t_n(a)t'_n\tilde{w}_n) = \mathcal{B}_{c \cdot \pi', \psi}(t_n(a)t'_n\tilde{w}_n)$ for any $a \in \mathrm{GL}_n$.*

Proof By Theorems 5.5 and 6.2, $\mathcal{B}_{\pi, \psi}(t_n(a)t'_n\tilde{w}_n) = \mathcal{B}_{\pi', \psi}(t_n(a)t'_n\tilde{w}_n)$ for any $a \in \mathrm{GL}_n$. Also,

$$\begin{aligned} \mathcal{B}_{\pi, \psi}(t_n(a)t'_n\tilde{w}_n) &= \mathcal{B}_{\pi, \psi}(c\tilde{t}^{-1}\tilde{t}ct_n(a)t'_n\tilde{w}_n\tilde{t}^{-1}\tilde{t}c) = \mathcal{B}_{c \cdot \pi, \psi}(\tilde{t}ct_n(a)t'_n\tilde{w}_n\tilde{t}^{-1}) \\ &= \mathcal{B}_{c \cdot \pi, \psi}(t_n(a)t'_n\tilde{w}_n). \end{aligned}$$

for any $n \leq l-1$. This proves the corollary. \square

The following corollary shows the equivalence of γ -factors of conjugate representations for $n \leq l-1$. This is not necessary to show the converse theorem, but when later paired with its analogue for $n = l$, Corollary 7.8, it shows that the γ -factor cannot distinguish between a representation and its conjugate. In the setting of local fields, this is known [2, 13].

Corollary 6.4 *Let π be an irreducible cuspidal ψ -generic representation of SO_{2l} . Then we have $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$ for all irreducible generic representations τ of GL_n for any $n \leq l-1$.*

Proof We abuse notation and consider the zeta integrals simultaneously for each $n \leq l-1$. By Propositions 5.1 and 7.1 we can choose $v \in \tau$ such that $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \Psi(\mathcal{B}_{c \cdot \pi, \psi}, f_v) \neq 0$. By the proofs of theorems 5.5 and 6.2,

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = |V_n| |R^{l,n}| \sum_{a \in U_{\mathrm{GL}_n} \backslash \mathrm{GL}_n} \mathcal{B}_{\pi, \psi}(t_n(a)t'_n\tilde{w}_n) W_v^*(a).$$

Since for any $n \leq l-1$, we have

$$\begin{aligned}\mathcal{B}_{\pi,\psi}(t_n(a)t'_n\tilde{w}_n) &= \mathcal{B}_{\pi,\psi}(c\tilde{t}^{-1}\tilde{t}ct_n(a)t'_n\tilde{w}_n\tilde{c}\tilde{t}^{-1}\tilde{t}c) \\ &= \mathcal{B}_{c\cdot\pi,\psi}(\tilde{t}ct_n(a)t'_n\tilde{w}_n\tilde{c}\tilde{t}^{-1}) \\ &= \mathcal{B}_{c\cdot\pi,\psi}(t_n(a)t'_n\tilde{w}_n),\end{aligned}$$

it follows that $\Psi(\mathcal{B}_{\pi,\psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{c\cdot\pi,\psi}, \tilde{f}_v)$. Finally, since $\Psi(\mathcal{B}_{\pi,\psi}, f_v) = \Psi(\mathcal{B}_{c\cdot\pi,\psi}, f_v)$ is nonzero, we have $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$ for all irreducible generic representations τ of GL_n for any $n \leq l-1$. This concludes the proof of the corollary. \square

7 Twists by GL_l

In this section, we consider the twists by GL_l . These twists are significantly different from the previous twists. Note that we still need to determine the Bessel support on the Bruhat cells corresponding to Weyl elements in $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$, and the Bessel support on the rest of Bruhat cells corresponding to Weyl elements in $B_{l-1}(\mathrm{SO}_{2l})$ that are not determined by Theorem 6.2. The difference from previous cases is that c acts nontrivially on these sets.

Let $v \in \tau$ be a fixed vector and define $\xi_v \in I(\tau)$ by $\mathrm{supp}(\xi_v) = L_l V_l = Q_l$ and

$$\xi_v(l_l(a)u) = \tau(a)v,$$

for any $a \in \mathrm{GL}_l, u \in V_l$. Let $f_v = f_{\xi_v} \in I(\tau, \psi^{-1})$. That is $f_v(g, a) = \Lambda_\tau(\tau(a)\xi_v(g))$, where $\Lambda_\tau \in \mathrm{Hom}_{U_{\mathrm{GL}_l}}(\tau, \psi^{-1}), a \in \mathrm{GL}_l$, and $g \in \mathrm{SO}_{2l+1}$. Let $W_v(a) = \Lambda_\tau(\tau(a)v)$. The following proposition shows that the zeta integrals in this case are nonzero.

Proposition 7.1 $\Psi(\mathcal{B}_{\pi,\psi}, f_v) = W_v(w_{l,l})$. In particular, we may choose $v \in \tau$ such that $\Psi(\mathcal{B}_{\pi,\psi}, f_v) \neq 0$.

Proof By definition.

$$\Psi(\mathcal{B}_{\pi,\psi}, f_v) = \sum_{g \in U_{\mathrm{SO}_{2l}} \backslash \mathrm{SO}_{2l}} \mathcal{B}_{\pi,\psi}(g) f_v(w_{l,l}g, I_l).$$

By definition, the support of $f_v(\cdot, I_l)$ is Q_l . Thus,

$$\Psi(\mathcal{B}_{\pi,\psi}, f_v) = \sum_{a \in U_{\mathrm{GL}_l} \backslash \mathrm{GL}_l} \mathcal{B}_{\pi,\psi}(t_l(a)) f_v(w_{l,l}l_l(a), I_l).$$

By Proposition 4.9, $\mathcal{B}_{\pi,\psi}(t_l(a)) = 0$ for any a which isn't upper triangular. Thus, by Lemma 4.3, $\Psi(\mathcal{B}_{\pi,\psi}, f_v) = W_v(w_{l,l})$. If we let \tilde{v} be a Whittaker vector for τ and $v = \tau(w_{l,l}^{-1})\tilde{v}$, then $\Psi(\mathcal{B}_{\pi,\psi}, f_v) = W_v(w_{l,l}) = W_{\tilde{v}}(I_l) \neq 0$. This proves the proposition. \square

The support of $\tilde{f}_v = M(\tau, \psi^{-1})f_v$ is $Q_l w_l V_l$. Recall that the twists by GL_n for $n \leq l-2$ determine the Bessel functions on the Bruhat cells for $B_n(\mathrm{SO}_{2l})$, while the twists by GL_{l-1} only determine a portion of the Bruhat cells for $B_{l-1}(\mathrm{SO}_{2l})$. We first show that the rest of the Bruhat cells, along with the Bruhat cells for $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$, embed into $Q_l w_l V_l$. This follows from the next three propositions.

Proposition 7.2 Let $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \in T_{\mathrm{SO}_{2l}}$. Then $tw \in Q_l w_l V_l$ (via the embedding of SO_{2l} into SO_{2l+1}) if and only if $t_l \neq 1$ if l is odd, or $t_l \neq \frac{-1}{2}$ if l is even.

Proof We begin with a sketch of the proof. We first compute the embedding of the torus in coordinates and the embedding of the Weyl element w , then compute $Q_l w_l V_l$ in terms of block matrices. We see that the lower left corner is determined by the torus. In particular, this lower left corner must be invertible if $tw \in Q_l w_l V_l$ and the determinant of this block gives us the condition we need. We proceed with the details of the proof.

Recall that the embedding sends $t = \text{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$ to

$$\text{diag}(s, \begin{pmatrix} \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) & \frac{1}{2}(t_l - t_l^{-1}) & 2(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) \\ \frac{1}{4}(t_l - t_l^{-1}) & \frac{1}{2}(t_l + t_l^{-1}) & \frac{-1}{2}(t_l - t_l^{-1}) \\ \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) & \frac{-1}{4}(t_l - t_l^{-1}) & \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) \end{pmatrix}, s^*),$$

where $s = \text{diag}(t_1, t_2, \dots, t_{l-1})$. By Proposition 4.5, there exists $w' \in W(\text{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$ and

$$w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix},$$

for some $w'' \in W(\text{GL}_{l-1})$. Then,

$$w = \begin{pmatrix} & & w'' \\ & J_2^{l-1} & \\ (w'')^* & & \end{pmatrix}.$$

Thus, the embedding into SO_{2l+1} of w is

$$\begin{pmatrix} & & w'' \\ & A^{l-1} & \\ (w'')^* & & \end{pmatrix},$$

where

$$A = \begin{pmatrix} \frac{-1}{8} & \frac{3}{4} & \frac{9}{4} \\ \frac{-3}{8} & \frac{5}{4} & \frac{3}{4} \\ \frac{9}{16} & \frac{-3}{8} & \frac{-1}{8} \end{pmatrix}.$$

Note that $A^{l-1} = I_3$ when l is odd.

Thus, we can compute the embedding of tw in SO_{2l+1} . We obtain

$$\begin{pmatrix} & & \text{diag}(t_1, \dots, t_{l-1})w'' \\ & Z & \\ \text{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* & & \end{pmatrix},$$

where

$$Z = \begin{pmatrix} \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) & \frac{1}{2}(t_l - t_l^{-1}) & 2(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) \\ \frac{1}{4}(t_l - t_l^{-1}) & \frac{1}{2}(t_l + t_l^{-1}) & \frac{-1}{2}(t_l - t_l^{-1}) \\ \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) & \frac{-1}{4}(t_l - t_l^{-1}) & \frac{1}{2} + \frac{1}{4}(t_l + t_l^{-1}) \end{pmatrix} A^{l-1}.$$

Next, we turn towards computing $Q_l w_l V_l$. Let $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$. We wish to compute $l_l(a) n_1 w_l n_2$. Write

$$n_1 = \begin{pmatrix} I_l & X & Y \\ & 1 & X' \\ & & I_l \end{pmatrix},$$

where $X \in \mathbb{F}_q^l$, $Y \in \mathrm{Mat}_{l \times l}(\mathbb{F}_q)$, $X' = -{}^t X J_l$, and $J_l Y + {}^t X' \cdot X' + {}^t Y J_l = 0$. Similarly, we write

$$n_2 = \begin{pmatrix} I_l & M & N \\ & 1 & M' \\ & & I_l \end{pmatrix}.$$

Then,

$$l_l(a) n_1 w_l n_2 = \begin{pmatrix} aY & aX + aYM & a + aXM' + aYN \\ X' & 1 + X'M & M' + X'N \\ a^* & a^*M & a^*N \end{pmatrix}.$$

Thus, $tw = l_l(a) n_1 w_l n_2$ has a solution as long as we can find a suitable a^* . We see that $a^* = \begin{pmatrix} Z_{3,1} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* \end{pmatrix}$ where $Z_{3,1}$ is the $(3, 1)$ -entry of Z . Such a^* exists as long as this matrix is invertible. Hence we need $Z_{3,1} \neq 0$.

If l is odd, then $Z_{3,1} = \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1}))$. This is nonzero as long as $t_l \neq 1$ and hence the proposition follows.

If l is even then $Z_{3,1} = \frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1}$. This is nonzero as long as $t_l \neq -\frac{1}{2}$ and hence the proposition follows. This finishes the proof of the proposition. \square

Next we examine the Bruhat cells for $B_l(\mathrm{SO}_{2l})$ and $B_l^c(\mathrm{SO}_{2l})$.

Proposition 7.3 *Let $w \in B_l(\mathrm{SO}_{2l})$. Then, for any $t \in T_{\mathrm{SO}_{2l}}$, we have $tw \in Q_l w_l V_l$ under the embedding of SO_{2l} into SO_{2l+1} .*

Proof Since $w \in B_l(\mathrm{SO}_{2l})$, $w \notin B_{l-1}(\mathrm{SO}_{2l})$. By Proposition 4.5, there exists $w' \in W(\mathrm{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$ where $w' = (w'_{i,j})_{i,j=1}^l \in W(\mathrm{GL}_l)$ with $w'_{l,1} = 0$. Similarly, we also write $(w')^* = (w'^*_{i,j})_{i,j=1}^l$.

As in the previous proof, we compute $Q_l w_l V_l$. Let $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$. We wish to compute $l_l(a) n_1 w_l n_2$. Write

$$n_1 = \begin{pmatrix} I_l & X & Y \\ & 1 & X' \\ & & I_l \end{pmatrix},$$

where $X \in \mathbb{F}_q^l$, $Y \in \mathrm{Mat}_{l \times l}(\mathbb{F}_q)$, $X' = -{}^t X J_l$, and $J_l Y + {}^t X' \cdot X' + {}^t Y J_l = 0$. Similarly, we write

$$n_2 = \begin{pmatrix} I_l & M & N \\ & 1 & M' \\ & & I_l \end{pmatrix}.$$

Then,

$$l_l(a)n_1w_ln_2 = \begin{pmatrix} aY & aX + aYM & a + aXM' + aYN \\ X' & 1 + X'M & M' + X'N \\ a^* & a^*M & a^*N \end{pmatrix}.$$

We proceed by computing the image of w under the embedding given by Equation 2.1. Recall that we defined the matrix

$$\tilde{M} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{-1}{2} & 1 & 1 \end{pmatrix}$$

which was used in the embedding. The image of the embedding depends on the parity of l .

Let l be even. Then the embedding of SO_{2l} into SO_{2l+1} takes w to

$$\begin{pmatrix} 0_{(l-1) \times (l-1)} & \begin{pmatrix} 0 & 0 & (w'_{i,1})_{i=1}^{l-1} \end{pmatrix} \tilde{M} & (w'_{i,j})_{i=1, j=2}^{i=l-1, j=l} \\ \tilde{M}^{-1} \begin{pmatrix} 0 \\ 0 \\ (w'^*_{1,j})_{j=1}^{l-1} \end{pmatrix} & \tilde{M}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{M} & \tilde{M}^{-1} \begin{pmatrix} (w'_{l,j})_{j=2}^l \\ 0 \\ 0 \end{pmatrix} \\ (w'^*_{i,j})_{i=2, j=1}^{i=l, j=l-1} & \begin{pmatrix} (w'^*_{i,l})_{i=2}^l & 0 & 0 \end{pmatrix} \tilde{M} & 0_{(l-1) \times (l-1)} \end{pmatrix}.$$

Thus, $tw = l_l(a)n_1w_ln_2$ as long as we can find a such that

$$a^* = \begin{pmatrix} \frac{t_l^{-1}(w'^*_{1,j})_{j=1}^{l-1}}{4} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'^*_{i,j})_{i=2, j=1}^{i=l, j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{(w'^*_{i,l})_{i=2}^l}{4} \end{pmatrix}.$$

This is possible as the determinant of this matrix is nonzero. Indeed, the determinant of this matrix is (up to a sign) $\frac{1}{16}t_l^{-1}t_{l-1}^{-1} \cdots t_1^{-1}$ since $(w')^* \in W(\mathrm{GL}_l)$ and $(w'^*_{i,j})_{i=2, j=1}^{i=l, j=l-1}$ contains a row of zeroes. Therefore, $tw \in Q_lw_lV_l$ for any $t \in T_{\mathrm{SO}_{2l}}$.

The proof for l odd is similar. We obtain that $tw = l_l(a)n_1w_ln_2$ as long as we can find a such that

$$a^* = \begin{pmatrix} \frac{t_l^{-1}(w'^*_{1,j})_{j=1}^{l-1}}{4} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'^*_{i,j})_{i=2, j=1}^{i=l, j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{-(w'^*_{i,l})_{i=2}^l}{2} \end{pmatrix}.$$

Again, this is always possible as the determinant of this matrix is (up to a sign) $\frac{1}{8}t_l^{-1} \cdots t_1^{-1}$. Therefore, $tw \in Q_lw_lV_l$ for any $t \in T_{\mathrm{SO}_{2l}}$ which concludes the proof of the proposition. \square

Proposition 7.4 *Let $w \in \mathrm{B}_l^c(\mathrm{SO}_{2l})$. Then for any $t \in T_{\mathrm{SO}_{2l}}$ $tw \in Q_lw_lV_l$ under the embedding of SO_{2l} into SO_{2l+1} .*

Proof The proof is similar to that of the previous proposition. We record the a^* 's for future reference.

For l even, we have $tw = l_l(a)n_1w_ln_2$ where $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$ if

$$a^* = \begin{pmatrix} \frac{-t_l(w'^*_{1,j})_{j=1}^{l-1}}{2} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'^*_{i,j})_{i=2, j=1}^{i=l, j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{-(w'^*_{i,l})_{i=2}^l}{2} \end{pmatrix}.$$

For l odd, we have $tw = l_l(a)n_1w_ln_2$ where $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$ if

$$a^* = \begin{pmatrix} \frac{-t_l(w'_{1,j})_{j=1}^{l-1}}{2} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'_{i,j})_{i=2,j=1}^{i=l,j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{(w'_{i,l})_{i=2}^l}{4} \end{pmatrix}.$$

□

The next 2 lemmas relate Bessel function for π and $c \cdot \pi$. In particular, the below lemma shows that the contribution of the zeta integral from $B_l^c(\mathrm{SO}_{2l})$ for π can be described using an integral over $B_l(\mathrm{SO}_{2l})$ and the Bessel function for $c \cdot \pi$.

Lemma 7.5 *Let π be an irreducible cuspidal ψ -generic representations of SO_{2l} . Then we have*

$$\sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l^c(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l}twu, I_l) = \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{c \cdot \pi, \psi}(twu) \tilde{f}_v(w_{l,l}twu, I_l).$$

Proof We write Let $t = \mathrm{diag}(t_1, \dots, t_{l-1}, t_l, t_l^{-1}, t_{l-1}^{-1}, \dots, t_1^{-1})$. By Proposition 4.1, we have $\mathcal{B}_{\pi, \psi}(twu) = \mathcal{B}_{\pi, \psi}(tw)\psi(u)$ for any $t \in T_{\mathrm{SO}_{2l}}$, $u \in U_{\mathrm{SO}_{2l}}$, and $w \in W(\mathrm{SO}_{2l})$. Also,

$$\mathcal{B}_{\pi, \psi}(tw) = \mathcal{B}_{\pi, \psi}(c\tilde{t}ctwct\tilde{t}^{-1}c) = \mathcal{B}_{c \cdot \pi, \psi}(tctwct\tilde{t}^{-1}) = \mathcal{B}_{c \cdot \pi, \psi}(t'cwc),$$

where $t' = \tilde{t}ctc(cwc\tilde{t}^{-1}(cwc)^{-1})$. For $w \in B_l^c(\mathrm{SO}_{2l})$, we have $cwc \in B_l(\mathrm{SO}_{2l})$. By Proposition 4.5, there exists $w' \in W(\mathrm{GL}_l)$ such that $cwc = t_l(w')\tilde{w}_l$ where $w' = (w'_{i,j})_{i,j=1}^l \in W(\mathrm{GL}_l)$ with $w'_{l,1} = 0$. We also let $(w')^* = (w'_{i,j})_{i,j=1}^l$ and r be such that $w'_{r,1} = 1$.

For l even, $(cwc\tilde{t}^{-1}(cwc)^{-1}) = t_l(\mathrm{diag}(1, \dots, 1, -2, 1, \dots, 1))$ where -2 is the r -th entry. Thus, $t' = t_l(\mathrm{diag}(t_1, \dots, t_{r-1}, -2t_r, t_{r+1}, \dots, t_{l-1}, \frac{-t_l^{-1}}{2}))$. Let $a_1, a_2 \in \mathrm{GL}_l$ and $n_1, n_2, n_3, n_4 \in V_l$ be such that $tw = l_l(a_1)n_1w_ln_2$ and $t'cwc = l_l(a_2)n_3w_ln_4$. By the proofs of Proposition 7.3 and 7.4, we have

$$a_1^* = \begin{pmatrix} \frac{-t_l(w'_{1,j})_{j=1}^{l-1}}{2} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'_{i,j})_{i=2,j=1}^{i=l,j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{-(w'_{i,l})_{i=2}^l}{2} \end{pmatrix},$$

and

$$a_2^* = \begin{pmatrix} \frac{t_l^{-1}(w'_{1,j})_{j=1}^{l-1}}{4} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'_{i,j})_{i=2,j=1}^{i=l,j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{(w'_{i,l})_{i=2}^l}{4} \end{pmatrix}.$$

Performing the change of variables $t_r \mapsto \frac{-t_r}{2}$ and $t_l \mapsto \frac{-t_l^{-1}}{2}$ takes $t' \mapsto t$ and $a_1^* \mapsto a_2^*$. Note that $w'_{l+1-r,l} = 1$ and this is the coordinate for t_r^{-1} . Thus we obtain the claim for l even:

$$\begin{aligned} & \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l^c(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l}twu, I_l) \\ &= \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l^c(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{c \cdot \pi, \psi}(t'cwc) \tilde{f}_v(w_{l,l}twu, I_l) \\ &= \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{c \cdot \pi, \psi}(twu) \tilde{f}_v(w_{l,l}twu, I_l). \end{aligned}$$

For l is odd, we obtain a different change of variables which matches the a^* 's in the odd cases of Propositions 7.3 and 7.4. We omit the details. Again, we find that

$$\sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l^c(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l}twu, I_l)$$

$$= \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in B_l(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{c \cdot \pi, \psi}(twu) \tilde{f}_v(w_{l,l} twu, I_l).$$

□

The contribution of $B_{l-1}(\mathrm{SO}_{2l})$ to the zeta integral is more nuanced. Since $cwc = w$ for $w \in B_{l-1}(\mathrm{SO}_{2l})$, one has to split the integral in half and then conjugate one half in order to obtain an integral involving a sum of Bessel functions for π and $c \cdot \pi$. Note that from Theorem 6.2 it follows that we need to only consider the subset of the torus defined by

$$T_l = \{t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \in T_{\mathrm{SO}_{2l}} \mid t_l \neq \pm c_l\},$$

where $c_l = 1$ if l is odd or $\frac{1}{2}$ if l is even.

Lemma 7.6 *Let π be an irreducible cuspidal ψ -generic representation of SO_{2l} . There exists a subset $A_l \subseteq T_l$, specified in the proof, such that*

$$\sum_{t \in T_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l} twu, I_l) = \sum_{t \in A_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l).$$

Proof Let $t = \mathrm{diag}(t_1, \dots, t_{l-1}, t_l, t_l^{-1}, t_{l-1}^{-1}, \dots, t_1^{-1})$. We have

$$\mathcal{B}_{\pi, \psi}(tw) = \mathcal{B}_{\pi, \psi}(c\tilde{t}^{-1}\tilde{t}ctw\tilde{c}\tilde{t}^{-1}\tilde{t}c) = \mathcal{B}_{c \cdot \pi, \psi}(\tilde{t}ctw\tilde{c}\tilde{t}^{-1}) = \mathcal{B}_{c \cdot \pi, \psi}(t'cwc),$$

where $t' = \tilde{t}ctw\tilde{c}\tilde{t}^{-1}(cwc)^{-1}$. For $w \in B_{l-1}(\mathrm{SO}_{2l})$, $cwc = w$. Hence $t' = \tilde{t}ctcw\tilde{t}^{-1}w^{-1}$ and $\mathcal{B}_{c \cdot \pi, \psi}(t'cwc) = \mathcal{B}_{c \cdot \pi, \psi}(t'w)$.

If l is even, then $w\tilde{t}^{-1}w^{-1} = \tilde{t}$ and hence $t' = \mathrm{diag}(t_1, \dots, t_{l-1}, \frac{t_l^{-1}}{4}, 4t_l, t_{l-1}^{-1}, \dots, t_1^{-1})$. The map $t_l \mapsto \frac{t_l^{-1}}{4}$ sends t' to t and $\frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1}$ to itself. The fixed points of the map $t_l \mapsto \frac{t_l^{-1}}{4}$ are $t_l = \pm \frac{1}{2}$. Furthermore, the mapping is an involution. Therefore, we can partition $\mathbb{F}_q^\times \setminus \{\pm \frac{1}{2}\}$ into two disjoint sets A and B such that if $t_l \in A$ then $\frac{t_l^{-1}}{4} \in B$. Hence, we partition T_l into two sets A_l and B_l such that $t \in A_l$ if and only if $t_l \in A$. Note that in this case $tw = l_l(a)n_1w_ln_2$ for $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$ with

$$a^* = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* \end{pmatrix}.$$

The map $t_l \mapsto \frac{t_l^{-1}}{4}$ takes $a^* \mapsto a^*$. Therefore,

$$\begin{aligned} & \sum_{t \in T_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &= \sum_{t \in A_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ & \quad + \sum_{t \in B_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &= \sum_{t \in A_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

The case for l odd is similar. We have $w\tilde{t}^{-1}w^{-1} = \tilde{t}^{-1}$ and hence $t' = ctc$. The map $t_l \mapsto t_l^{-1}$ sends t' to t and $\frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1}))$ to itself. The fixed points of the map $t_l \mapsto t_l^{-1}$ are $t_l = \pm 1$. Furthermore, the mapping is an involution. Therefore, we can partition $\mathbb{F}_q^\times \setminus \{\pm 1\}$ into two

disjoint sets A and B such that if $t_l \in A$ then $t_l^{-1} \in B$. Furthermore we partition T_l into two sets A_l and B_l such that $t \in A_l$ if and only if $t_l \in A$. Note that in this case $tw = l_l(a)n_1w_ln_2$ for $a \in \mathrm{GL}_l$ and $n_1, n_2 \in V_l$ with

$$a^* = \begin{pmatrix} \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* \end{pmatrix}.$$

The map $t_l \mapsto t_l^{-1}$ takes $a^* \mapsto a^*$. As in the previous case, we find

$$\begin{aligned} & \sum_{t \in T_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} \mathcal{B}_{\pi, \psi}(twu) \tilde{f}_v(w_l, t w u, I_l) \\ &= \sum_{t \in A_l, w \in B_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_l, t w u, I_l). \end{aligned}$$

Hence we have proven the lemma for any l . \square

We are ready to compute the sets determined by the twists by GL_n , $n \leq l$. We remark that it is possible to prove the below theorem assuming that the γ -factors are equal only for twists by GL_l . Indeed, computations of the embedding of SO_{2l} into SO_{2l+1} show that any element of a Bruhat cell corresponding to $B_n(\mathrm{SO}_{2l})$ for $n \leq l-2$ does not embed into the support of \tilde{f}_v (and we obtain a similar result for the set determined by Theorem 6.2). However, this is not necessary for the converse theorem. Instead, we use the results of the previous sections to eliminate the contributions from Bruhat cells corresponding to $B_n(\mathrm{SO}_{2l})$ for $n \leq l-2$ (and the set determined by Theorem 6.2).

Theorem 7.7 *Let π and π' be irreducible cuspidal ψ -generic representations of SO_{2l} with the same central character. If $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n where $n \leq l$, then*

(1)

$$(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw),$$

for any $t \in T_{\mathrm{SO}_{2l}}$ and $w \in B_l(\mathrm{SO}_{2l}) \cup B_l^c(\mathrm{SO}_{2l})$;

(2)

$$(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw),$$

for any $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t \in T_{\mathrm{SO}_{2l}}$ with $t_l \neq \pm 1$ if l is odd or $t_l \neq \pm \frac{1}{2}$ if l is even.

Proof By Proposition 7.1, we have $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \Psi(\mathcal{B}_{\pi', \psi}, f_v) = W_v(w_l, l)$. By assumption, $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_l , and hence we have that $\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \Psi(\mathcal{B}_{\pi', \psi}, \tilde{f}_v)$. By definition,

$$\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) = \sum_{g \in U_{\mathrm{SO}_{2l}} \backslash \mathrm{SO}_{2l}} \mathcal{B}_{\pi, \psi}(g) \tilde{f}_v(w_l, l g, I_l),$$

and hence

$$0 = \sum_{g \in U_{\mathrm{SO}_{2l}} \backslash \mathrm{SO}_{2l}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(g) \tilde{f}_v(w_l, l g, I_l).$$

Let $T_l = \{t = \text{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \in T_{\text{SO}_{2l}} \mid t_l \neq \pm c_l\}$ where $c_l = 1$ if l is odd or $\frac{1}{2}$ if l is even. Since $\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi)$ for all irreducible generic representations τ of GL_n with $n \leq l-1$, by Propositions 7.2, 7.3, 7.4, and Theorems 5.5 and 6.2,

$$\begin{aligned} 0 &= \sum_{t \in T_{\text{SO}_{2l}}, w \in B_l(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in T_{\text{SO}_{2l}}, w \in B_l^c(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in T_l, w \in B_{l-1}(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

By Lemma 7.5,

$$\begin{aligned} &\sum_{t \in T_{\text{SO}_{2l}}, w \in B_l(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in T_{\text{SO}_{2l}}, w \in B_l^c(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &= \sum_{t \in T_{\text{SO}_{2l}}, w \in B_l(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

By Lemma 7.6,

$$\begin{aligned} &\sum_{t \in T_l, w \in B_{l-1}(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &= \sum_{t \in A_l, w \in B_{l-1}(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} 0 &= \sum_{t \in T_{\text{SO}_{2l}}, w \in B_l(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in A_l, w \in B_{l-1}(\text{SO}_{2l}), u \in U_{\text{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

Next, we define a function f on a subset of GL_l so that we may apply Lemma 5.3. The first step in this is to describe the arguments of the W_v^* 's. That is, we describe a_i , where the image of tw under the embedding into SO_{2l+1} is $t_l(a_i)n_1w_ln_2$ where $a_i \in \text{GL}_l$ and $n_1, n_2 \in V_l$ and $i = 1, 2$ if $w \in B_l(\text{SO}_{2l})$ or $w \in B_{l-1}(\text{SO}_{2l})$ respectively. We recall the setup. For $w \in B_l^c(\text{SO}_{2l})$, by Proposition 4.5, there exists $w' \in W(\text{GL}_l)$ such that $w = t_l(w')\tilde{w}_l$ where $w' = (w'_{i,j})_{i,j=1}^l \in W(\text{GL}_l)$ with $w'_{l,1} = 0$. We also let $(w')^* = (w'^*_{i,j})_{i,j=1}^l$ and r be such that $w'_{r,1} = 1$.

First, suppose that l is even. Let

$$a_1^* = \begin{pmatrix} \frac{t_l^{-1}(w'^*_{1,j})_{j=1}^{l-1}}{4} & \frac{1}{4} \\ \text{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'^*_{i,j})_{i=2,j=1}^{l-1} & \text{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{(w'^*_{i,l})_{i=2}^l}{4} \end{pmatrix}.$$

Then,

$$a_1^* = \begin{pmatrix} 1 & & t_r & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \mathrm{diag} \left(\frac{t_l^{-1}}{4}, t_{l-1}^{-1}, \dots, t_{r+1}^{-1}, \frac{t_r^{-1}}{4}, t_{r-1}^{-1}, \dots, t_1^{-1} \right) w'^*.$$

where t_r is the $(1, l-r+1)$ entry of the unipotent matrix. Thus, we obtain

$$a_1 = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & -t_r & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \mathrm{diag} (t_1, \dots, t_{r-1}, 4t_r, t_{r+1}, \dots, t_{l-1}, 4t_l) w',$$

where $-t_r$ is the (r, l) entry of the unipotent matrix. This determines a on the $B_l(\mathrm{SO}_{2l})$ sum.

Let $t_{w'} \in T_{\mathrm{GL}_l}$ be such that

$$t_{w'} \mathrm{diag} (t_1, \dots, t_{r-1}, 4t_r, t_{r+1}, \dots, t_{l-1}, 4t_l) = \mathrm{diag} (t_1, \dots, t_{r-1}, t_r, t_{r+1}, \dots, t_{l-1}, t_l).$$

That is, $t_{w'}$ is a diagonal matrix consisting of 1's on the diagonal, except in the (r, r) and (l, l) coordinates where it is $\frac{1}{4}$.

Next, we consider the a_2 in the $B_{l-1}(\mathrm{SO}_{2l})$ sum. Let $w' = \begin{pmatrix} & w'' \\ 1 & \end{pmatrix}$ and

$$a_2^* = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* \end{pmatrix}.$$

Then,

$$a_2^* = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1} & & & \\ & t_{l-1}^{-1} & & \\ & & \ddots & \\ & & & t_1^{-1} \end{pmatrix} (w')^*.$$

Hence

$$a_2 = \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1})^{-1} \end{pmatrix} w'.$$

Recall we partitioned $\mathbb{F}_q^\times \setminus \{\pm \frac{1}{2}\}$ into two disjoint sets A and B such that if $t_l \in A$ then $\frac{t_l^{-1}}{4} \in B$. Suppose $t_l, s_l \in A$ with $\frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1} = \frac{1}{4} + \frac{1}{4}s_l + \frac{1}{16}s_l^{-1}$. This gives a quadratic equation in

s_l whose roots are $s_l = t_l$ and $s_l = \frac{t_l^{-1}}{4}$. Since $s, t \in A_l$ it follows that we must have $t_l = s_l$. Let

$$A_{GL_l} = \left\{ \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1})^{-1} \end{pmatrix} \mid t_1, \dots, t_{l-1} \in \mathbb{F}_q^\times, t_l \in A \right\}.$$

The following map is well defined on A_{GL_l} :

$$\xi \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{4} + \frac{1}{4}t_l + \frac{1}{16}t_l^{-1})^{-1} \end{pmatrix} = \text{diag}(t_1, \dots, t_l).$$

Let $u = (u_{i,j})_{i,j=1}^l$. Note that $u_{l,l+1} = 0$. Then, the embedding of u in SO_{2l+1} is

$$\dot{u} = \begin{pmatrix} (u_{i,j})_{i,j=1}^{i,j=l-1} & \left(\frac{(u_{i,l})_{i=1}^{l-1}}{4} - \frac{(u_{i,l+1})_{i=1}^{l-1}}{2} \right) & * & * \\ & I_3 & & * \\ & & & * \end{pmatrix}.$$

Let $\tilde{u} = \begin{pmatrix} (u_{i,j})_{i,j=1}^{i,j=l-1} & \frac{(u_{i,l})_{i=1}^{l-1}}{4} - \frac{(u_{i,l+1})_{i=1}^{l-1}}{2} \\ 0 & 1 \end{pmatrix}$. Then $\dot{u} = l_l(\tilde{u})n_3$, where $n_3 \in V_l$. The embedding takes twu to $l_l(a_i)n_1w_ln_2l_l(\tilde{u})n_3 = n_4l_l(a_i\tilde{u}^*)w_ln_5$ where $n_4, n_5 \in V_l$ and $i = 1, 2$ if $w \in B_l(\text{SO}_{2l})$ or $w \in B_{l-1}(\text{SO}_{2l})$ respectively. Thus, by Proposition 5.2, $\tilde{f}_v(w_l, twu, I_l) = W_v^*(\text{diag}(\frac{1}{2}, \dots, \frac{1}{2})a_i\tilde{u}^*)$.

Next, we define a function on a subset of GL_l using its Bruhat decomposition. Specifically, we partition the Weyl group of $W(\text{GL}_l)$ into $W_1(\text{GL}_l)$ and $W_2(\text{GL}_l)$ where $w' \in W_1(\text{GL}_l)$ if $w' \neq \begin{pmatrix} w'' \\ 1 \end{pmatrix}$ for any $w'' \in W(\text{GL}_{l-1})$, and $w' \in W_2(\text{GL}_l)$ if $w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix}$ for some $w'' \in W(\text{GL}_{l-1})$. By Proposition 4.5, we have $w = t_l(w')\tilde{w}_l \in B_l(\text{SO}_{2l})$ if $w' \in W_1(\text{GL}_l)$ and $w = t_l(w')\tilde{w}_l \in B_{l-1}(\text{SO}_{2l})$ if $w' \in W_2(\text{GL}_l)$. Let

$$X_l = \left(\bigsqcup_{w' \in W_1(\text{GL}_l)} U_{\text{GL}_l} T_{\text{GL}_l} w' U_{\text{GL}_l} \right) \bigsqcup \left(\bigsqcup_{w' \in W_2(\text{GL}_l)} U_{\text{GL}_l} A_{\text{GL}_l} w' U_{\text{GL}_l} \right).$$

Recall the definition of $t_{w'}$ above. For $g = u_1tw'u_2 \in U_{\text{GL}_l} T_{\text{GL}_l} w' U_{\text{GL}_l}$ such that $w' \in W_1(\text{GL}_l)$ we define

$$f(g) = (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(t_l(\text{diag}(2, \dots, 2))t_l(u_1t_{w'}tw'u_2)\tilde{w}_l).$$

For $g = u_1tw'u_2 \in U_{\text{GL}_l} A_{\text{GL}_l} w' U_{\text{GL}_l}$ such that $w' \in W_2(\text{GL}_l)$ we define

$$f(g) = (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(t_l(\text{diag}(2, \dots, 2))t_l(u_1\xi(t)w'u_2)\tilde{w}_l).$$

We have $f(ug) = \psi(u)f(g)$ for any $u \in U_{\mathrm{GL}_l}$ and $g \in X_l$. Also, $t_l(g)\tilde{w}_l = \tilde{w}_l t_l(g^*)$ for any $g \in \mathrm{GL}_l$ (we are still assuming that l is even). Therefore,

$$0 = \sum_{t \in T_{\mathrm{GL}_l}, w \in W_1(\mathrm{GL}_l), u \in U_{\mathrm{SO}_{2l}}} f(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2})tw\tilde{u}^*)W_v^*(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2})tw\tilde{u}^*) \\ + \sum_{t \in A_{\mathrm{GL}_l}, w \in W_2(\mathrm{GL}_l), u \in U_{\mathrm{SO}_{2l}}} f(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2})tw\tilde{u}^*)W_v^*(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2})tw\tilde{u}^*).$$

Thus, by Lemma 5.3, f must identically vanish on X_l . Therefore, for l even, we have that $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $t \in \mathrm{SO}_{2l}$ and any $w \in B_l(\mathrm{SO}_{2l}) \cup B_l^c(\mathrm{SO}_{2l})$. Also, $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t \in A_l$. Also, conjugating the previous equation by $\tilde{t}c$ gives $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t \in B_l$ hence the theorem for l even.

Next, suppose that l is odd. Let

$$a_1^* = \begin{pmatrix} \frac{t_l^{-1}(w'_{1,j})_{j=1}^{l-1}}{4} & \frac{1}{4} \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'_{i,j})_{i=2, j=1}^{i=l, j=l-1} & \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})\frac{-(w'_{i,l})_{i=2}^l}{2} \end{pmatrix}.$$

Then,

$$a_1^* = \begin{pmatrix} 1 & \frac{-t_r}{2} \\ \ddots & \\ & 1 & \ddots \\ & & & 1 \end{pmatrix} \mathrm{diag}\left(\frac{t_l^{-1}}{4}, t_{l-1}^{-1}, \dots, t_{r+1}^{-1}, \frac{-t_r^{-1}}{2}, t_{r-1}^{-1}, \dots, t_1^{-1}\right) w'^*,$$

where t_r is the $(1, l-r+1)$ entry of the unipotent matrix. Thus, we obtain

$$a_1 = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \frac{t_r}{2} \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \mathrm{diag}(t_1, \dots, t_{r-1}, -2t_r, t_{r+1}, \dots, t_{l-1}, 4t_l) w',$$

where $-t_r$ is the (r, l) entry of the unipotent matrix. This determines a on the $B_l(\mathrm{SO}_{2l})$ sum. Let $t_{w'} \in T_{\mathrm{GL}_l}$ be such that

$$t_{w'} \mathrm{diag}(t_1, \dots, t_{r-1}, -2t_r, t_{r+1}, \dots, t_{l-1}, 4t_l) = \mathrm{diag}(t_1, \dots, t_{r-1}, t_r, t_{r+1}, \dots, t_{l-1}, t_l).$$

That is, $t_{w'}$ is a diagonal matrix consisting of 1's on the diagonal, except in the (r, r) and (l, l) coordinates where it is $\frac{-1}{2}$ and $\frac{1}{4}$ respectively.

Next, we consider the a in the $B_{l-1}(\mathrm{SO}_{2l})$ sum. Let $w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix}$ and

$$a_2^* = \begin{pmatrix} \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) \\ \mathrm{diag}(t_{l-1}^{-1}, \dots, t_1^{-1})(w'')^* \end{pmatrix}.$$

Then,

$$a_2^* = \begin{pmatrix} \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) & & & \\ & t_{l-1}^{-1} & & \\ & & \ddots & \\ & & & t_1^{-1} \end{pmatrix} (w')^*.$$

Hence

$$a_2 = \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})))^{-1} \end{pmatrix} w'.$$

Recall we partitioned $\mathbb{F}_q^\times \setminus \{\pm \frac{1}{2}\}$ into two disjoint sets A and B such that if $t_l \in A$ then $t_l^{-1} \in B$. Suppose $t_l, s_l \in A$ with $\frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})) = \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(s_l + s_l^{-1}))$. This gives a quadratic equation in s_l whose roots are $s_l = t_l$ and $s_l = t_l^{-1}$. Since $s, t \in A_l$ it follows that we must have $t_l = s_l$.

Let

$$A_{GL_l} = \left\{ \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})))^{-1} \end{pmatrix} \mid t_1, \dots, t_{l-1} \in \mathbb{F}_q^\times, t_l \in A \right\}.$$

The following map is well defined on A_{GL_l} :

$$\xi \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{l-1} & \\ & & & (\frac{1}{2}(\frac{1}{2} - \frac{1}{4}(t_l + t_l^{-1})))^{-1} \end{pmatrix} = \text{diag}(t_1, \dots, t_l).$$

Let $u = (u_{i,j})_{i,j=1}^l$. Note that $u_{l,l+1} = 0$. Then, the embedding of u in SO_{2l+1} is

$$\dot{u} = \begin{pmatrix} (u_{i,j})_{i,j=1}^{i,j=l-1} & \left(\frac{(u_{i,l})_{i=1}^{l-1}}{4} - \frac{(u_{i,l+1})_{i=1}^{l-1}}{2} \right) * & * \\ & I_3 & * \\ & & * \end{pmatrix}.$$

Let $\tilde{u} = \begin{pmatrix} (u_{i,j})_{i,j=1}^{i,j=l-1} & \frac{(u_{i,l})_{i=1}^{l-1}}{4} - \frac{(u_{i,l+1})_{i=1}^{l-1}}{2} \\ 0 & 1 \end{pmatrix}$. Then $\dot{u} = l_l(\tilde{u})n_3$, where $n_3 \in V_l$. The embedding takes twu to $l_l(a_i)n_1w_l n_2 l_l(\tilde{u})n_3 = n_4 l_l(a_i \tilde{u}^*)w_l n_5$ where $n_4, n_5 \in V_l$ and $i = 1, 2$ if $w \in B_l(\text{SO}_{2l})$ or $w \in B_{l-1}(\text{SO}_{2l})$ respectively. Thus, by Proposition 5.2, $\tilde{f}_v(w_l, twu, I_l) = W_v^*(\text{diag}(\frac{1}{2}, \dots, \frac{1}{2})a_i \tilde{u}^*)$.

Next, we define a function on a subset of GL_l using its Bruhat decomposition. Specifically, we partition the Weyl group of $W(\text{GL}_l)$ into two sets, $W_1(\text{GL}_l)$ and $W_2(\text{GL}_l)$, by $w' \in W_1(\text{GL}_l)$

if $w' \neq \begin{pmatrix} w'' \\ 1 \end{pmatrix}$ for any $w'' \in W(\mathrm{GL}_{l-1})$ and $w' \in W_2(\mathrm{GL}_l)$ if $w' = \begin{pmatrix} w'' \\ 1 \end{pmatrix}$ for some $w'' \in W(\mathrm{GL}_{l-1})$. By Proposition 4.5, we have $w = t_l(w')\tilde{w}_l \in B_l(\mathrm{SO}_{2l})$ if $w' \in W_1(\mathrm{GL}_l)$ and $w = t_l(w')\tilde{w}_l \in B_{l-1}(\mathrm{SO}_{2l})$ if $w' \in W_2(\mathrm{GL}_l)$. Let

$$X_l = \left(\bigsqcup_{w' \in W_1(\mathrm{GL}_l)} U_{\mathrm{GL}_l} T_{\mathrm{GL}_l} w' U_{\mathrm{GL}_l} \right) \bigsqcup \left(\bigsqcup_{w' \in W_2(\mathrm{GL}_l)} U_{\mathrm{GL}_l} A_{\mathrm{GL}_l} w' U_{\mathrm{GL}_l} \right).$$

Recall the definition of $t_{w'}$ above. For $g = u_1 t w' u_2 \in U_{\mathrm{GL}_l} T_{\mathrm{GL}_l} w' U_{\mathrm{GL}_l}$ such that $w' \in W_1(\mathrm{GL}_l)$ we define

$$f(g) = (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(t_l(\mathrm{diag}(2, \dots, 2)) t_l(u_1 t w' t w' u_2) \tilde{w}_l).$$

For $g = u_1 t w' u_2 \in U_{\mathrm{GL}_l} A_{\mathrm{GL}_l} w' U_{\mathrm{GL}_l}$ with $w' \in W_2(\mathrm{GL}_l)$ we define

$$f(g) = (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi, \psi} - \mathcal{B}_{c \cdot \pi', \psi})(t_l(\mathrm{diag}(2, \dots, 2)) t_l(u_1 \xi(t) w' u_2) \tilde{w}_l).$$

We have $f(ug) = \psi(u)f(g)$ for any $u \in U_{\mathrm{GL}_l}$ and $g \in X_l$. Let $g = (g_{i,j})_{i,j=1}^l$ and $g^* = (g_{i,j}^*)_{i,j=1}^l$.

Then

$$t_l(g)\tilde{w}_l = \tilde{w}_l \begin{pmatrix} (g_{i,j})_{i,j=1}^{l-1} & 0 & (g_{i,l})_{i=1}^{l-1} & 0 \\ 0 & g_{1,1}^* & 0 & (g_{1,j}^*)_{j=2}^l \\ (g_{l,j})_{j=1}^{l-1} & 0 & g_{l,l} & 0 \\ 0 & (g_{i,1}^*)_{i=2}^l & 0 & (g_{i,j}^*)_{i,j=2}^l \end{pmatrix}.$$

In particular, if $u \in U_{\mathrm{GL}_l}$, then

$$t_l(u)\tilde{w}_l = \tilde{w}_l \begin{pmatrix} (g_{i,j})_{i,j=1}^{l-1} & 0 & (g_{i,l})_{i=1}^{l-1} & 0 \\ & 1 & 0 & (g_{1,j}^*)_{j=2}^l \\ & & 1 & 0 \\ & & & (g_{i,j}^*)_{i,j=2}^l \end{pmatrix},$$

and the last matrix is upper triangular. Therefore,

$$\begin{aligned} 0 &= \sum_{t \in T_{\mathrm{GL}_l}, w \in W_1(\mathrm{GL}_l), u \in U_{\mathrm{SO}_{2l}}} f(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2}) t w \tilde{w}^*) W_v^*(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2}) t w \tilde{w}^*) \\ &+ \sum_{t \in A_{\mathrm{GL}_l}, w \in W_2(\mathrm{GL}_l), u \in U_{\mathrm{SO}_{2l}}} f(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2}) t w \tilde{w}^*) W_v^*(\mathrm{diag}(\frac{1}{2}, \dots, \frac{1}{2}) t w \tilde{w}^*). \end{aligned}$$

Thus, by Lemma 5.3, f must identically vanish on X_l . Therefore, for l odd, we have that $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $t \in \mathrm{SO}_{2l}$ and $w \in B_l(\mathrm{SO}_{2l})$. Also, we have $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t \in A_l$. Conjugation by tc gives $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $w \in B_{l-1}(\mathrm{SO}_{2l})$ and $t \in B_l$ for l odd.

Finally, for any l , we have $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $t \in \mathrm{SO}_{2l}$ and $w \in B_l(\mathrm{SO}_{2l})$. Also,

$$\begin{aligned} \mathcal{B}_{\pi, \psi}(tw) &= \mathcal{B}_{\pi, \psi}(c \tilde{t}^{-1} \tilde{t} c t w c \tilde{t}^{-1} \tilde{t} c) \\ &= \mathcal{B}_{c \cdot \pi, \psi}(\tilde{t} c t w c \tilde{t}^{-1}) \end{aligned}$$

$$= \mathcal{B}_{c \cdot \pi, \psi}(t' c w c),$$

where $t' = \tilde{t} c t c (c w c \tilde{t}^{-1} (c w c)^{-1})$. Hence for any $t \in \mathrm{SO}_{2l}$ and $w \in \mathrm{B}_l^c(\mathrm{SO}_{2l})$, we have that $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ and thus we have proved the theorem. \square

The following corollary, when combined with Corollary 6.4, shows that γ -factor is unable to distinguish between a representation and its conjugate. That is, $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$ for all irreducible generic representations τ of GL_n with $n \leq l$.

Corollary 7.8 *Let π be an irreducible cuspidal ψ -generic representation of SO_{2l} . Then we have $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$ for all irreducible generic representations τ of GL_l .*

Proof By Corollary 6.4, $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$ for all irreducible generic representations τ of GL_n with $n \leq l-1$. So, by Theorems 5.5 and 6.2,

$$\begin{aligned} & \Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) - \Psi(\mathcal{B}_{c \cdot \pi, \psi}, \tilde{f}_v) \\ &= \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in \mathrm{B}_l(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in T_{\mathrm{SO}_{2l}}, w \in \mathrm{B}_l^c(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l) \\ &+ \sum_{t \in T_l, w \in \mathrm{B}_{l-1}(\mathrm{SO}_{2l}), u \in U_{\mathrm{SO}_{2l}}} (\mathcal{B}_{\pi, \psi} - \mathcal{B}_{c \cdot \pi, \psi})(twu) \tilde{f}_v(w_{l,l} twu, I_l). \end{aligned}$$

Performing the change of variables as in the proof of Theorem 7.7 on the $\mathcal{B}_{\pi, \psi}$ sums gives $\Psi(\mathcal{B}_{\pi, \psi}, \tilde{f}_v) - \Psi(\mathcal{B}_{c \cdot \pi, \psi}, \tilde{f}_v) = 0$. By Proposition 7.1, we may choose a nonzero $v \in \tau$ such that $\Psi(\mathcal{B}_{\pi, \psi}, f_v) = \Psi(\mathcal{B}_{c \cdot \pi, \psi}, f_v) = W_v(w_{l,l}) \neq 0$. Thus, we have $\gamma(\pi \times \tau, \psi) = \gamma(c \cdot \pi \times \tau, \psi)$. This proves the corollary. \square

8 The converse theorem

In this section, we prove the converse theorem. First, we combine the results of the previous sections to obtain the following theorem.

Theorem 8.1 *Let π and π' be irreducible cuspidal ψ -generic representations of split $\mathrm{SO}_{2l}(\mathbb{F}_q)$ with the same central character. If*

$$\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi),$$

for all irreducible generic representations τ of GL_n with $n \leq l$, then we have that

$$(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(g) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(g)$$

for any $g \in \mathrm{SO}_{2l}(\mathbb{F}_q)$.

Proof By the Bruhat decomposition, we may assume $g = u_1 twu_2 \in B_{\mathrm{SO}_{2l}} w B_{\mathrm{SO}_{2l}}$ for some $w \in W(\mathrm{SO}_{2l})$. By Proposition 4.1 and the definition of the Bessel support, it is enough to show that $(\mathcal{B}_{\pi, \psi} + \mathcal{B}_{c \cdot \pi, \psi})(tw) = (\mathcal{B}_{\pi', \psi} + \mathcal{B}_{c \cdot \pi', \psi})(tw)$ for any $t \in T_{\mathrm{SO}_{2l}}$ and $w \in B(\mathrm{SO}_{2l})$. This follows from Lemma 4.3 (we use that π and π' have the same central character here), Proposition 4.8, Theorems 5.5, 6.2, 7.7, and Corollary 6.3. This concludes the proof of the theorem. \square

Theorem 8.2 (The Converse Theorem for SO_{2l}) *Let π and π' be irreducible cuspidal ψ -generic representations of split $\mathrm{SO}_{2l}(\mathbb{F}_q)$ with the same central character. If*

$$\gamma(\pi \times \tau, \psi) = \gamma(\pi' \times \tau, \psi),$$

for all irreducible generic representations τ of $GL_n(\mathbb{F}_q)$ with $n \leq l$, then $\pi \cong \pi'$ or $\pi \cong c \cdot \pi'$.

Proof By Theorem 8.1, $(\mathcal{B}_{\pi,\psi} + \mathcal{B}_{c \cdot \pi,\psi}) = (\mathcal{B}_{\pi',\psi} + \mathcal{B}_{c \cdot \pi',\psi})$ on all of SO_{2l} . We let $W_\pi, W_{c \cdot \pi}, W_{\pi'}$, and $W_{c \cdot \pi'}$ be the Whittaker models of $\pi, c \cdot \pi, \pi'$, and $c \cdot \pi'$ respectively.

First, suppose that π is isomorphic to $c \cdot \pi$. By uniqueness of Whittaker models, we have $\mathcal{B}_{\pi,\psi} + \mathcal{B}_{c \cdot \pi,\psi} = 2\mathcal{B}_{\pi,\psi} \in W_\pi$. Then, $2\mathcal{B}_{\pi,\psi} = \mathcal{B}_{\pi',\psi} + \mathcal{B}_{c \cdot \pi',\psi}$ and hence $W_\pi \cap W_{\pi'} \oplus W_{c \cdot \pi'} \neq 0$. Since W_π is isomorphic to π and is hence irreducible, we have that $W_\pi \cap W_{\pi'} \oplus W_{c \cdot \pi'} = W_\pi$ and therefore, by uniqueness of Whittaker models, W_π must be isomorphic to one of $W_{\pi'}$ or $W_{c \cdot \pi'}$.

Second, suppose π is not isomorphic to $c \cdot \pi$. We have that $\mathcal{B}_{\pi,\psi} \in W_\pi$. We also have that $\mathcal{B}_{\pi',\psi} + \mathcal{B}_{c \cdot \pi',\psi} - \mathcal{B}_{c \cdot \pi,\psi} \in W_{\pi'} \oplus W_{c \cdot \pi'} \oplus W_{c \cdot \pi}$. Thus, $W_\pi \cap W_{\pi'} \oplus W_{c \cdot \pi'} \oplus W_{c \cdot \pi} \neq 0$ and hence the intersection is a nonzero subrepresentation of W_π . Since W_π is isomorphic to π and is hence irreducible, we have that $W_\pi \cap W_{\pi'} \oplus W_{c \cdot \pi'} \oplus W_{c \cdot \pi} = W_\pi$ and therefore, by uniqueness of Whittaker models, W_π must be isomorphic to one of $W_{c \cdot \pi}, W_{\pi'}$, or $W_{c \cdot \pi'}$. By assumption, W_π is not isomorphic to $W_{c \cdot \pi}$ and hence must be isomorphic to $W_{\pi'}$ or $W_{c \cdot \pi'}$.

Therefore, in either case, we have shown the converse theorem. \square

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