LOCAL DESCENT TO QUASI-SPLIT EVEN GENERAL SPIN GROUPS

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Abstract. Let $n > 1$ and τ be an irreducible unitary supercuspidal representation of GL_{2n} over a local non-archimedean field. Assuming the twisted symmetric square L-function of τ has a pole at $s = 0$, we construct the local descent of τ to the corresponding quasi-split even general spin group GSpin_{2n}. We prove this local descent is generic, unitary, supercuspidal and multiplicity free. Its irreducible quotients are "functorially related" to τ, in the analytic sense of a pole of a Rankin–Selberg type γ-function.

1. Introduction

Let G be a quasi-split classical group defined over a global number field with a ring of adeles $\mathbb{A}$. Cogdell *et al.* [CKPSS01, CKPSS04, CPSS11] proved that any globally generic cuspidal representation σ of $G(\mathbb{A})$ has a functorial lift to an automorphic representation of $\mathrm{GL}_N(\mathbb{A})$, for the proper N. Their work was extended to general spin groups by Asgari and Shahidi [AS06].

The descent method of Ginzburg *et al.* [GRS97a, GRS97b, GRS99a, GRS99b, GRS11] (see also [Sou06]) provides the global and local "inverse map" of the functorial lift, for certain automorphic representations. Given a globally generic automorphic representation τ of $\mathrm{GL}_N(\mathbb{A})$, which is an isobaric sum of cuspidal representations (and belongs to the image of the weak lift from cuspidal globally generic representations of $G(\mathbb{A})$), this method constructs a cuspidal representation of $G(\mathbb{A})$, whose irreducible subrepresentations σ are all globally generic and lift (functorially) to τ. The global and local descent were first developed for the metaplectic groups and the odd orthogonal groups (e.g., [GRS97a, GRS97b, GRS99a], the orthogonal case was obtained using the theta correspondence); the global descent for classical groups including orthogonal, symplectic, unitary or metaplectic groups was described in full detail in [GRS11]; and the local even unitary case was settled by Soudry and Tanai [ST15]. The global descent for general spin groups was developed by Hundley and Sayag [HS16], and for the exceptional group G_2 was obtained in the more recent work by Hundley and Liu [HL19].

The local descent constructs, for a supercuspidal (self-dual or self-dual up to a twist) representation τ of GL_N over a non-archimedean field, a generic supercuspidal representation of G whose irreducible quotients σ are all generic and lift to τ. This representation of G is irreducible in the metaplectic, odd orthogonal and even unitary cases ([GRS99a, JS03, ST15]), but is expected to be reducible for even special orthogonal or

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even general spin groups, because there is a representation of O_{2n} which lifts functorially to τ and its restriction to SO_{2n} can be of length 2. See Jiang and Soudry [PR12, Appendix].

The descent method has had numerous applications, including to local and global functoriality, local Langlands correspondence, rigidity theorems and globalization results ([JS03, CKPSS04, JS04, Sou06, JNQ10, PR12, JL14, ST15]). One of its strengths is that it provides an explicit realization of a Whittaker functional (locally, Whittaker model) for those representations σ. Recently Lapid and Mao [LM17] used the local realization as an ingredient in the proof of their conjecture on Whittaker–Fourier coefficients.

In this work we develop the local descent theory for quasi-split even general spin groups. Let F be a non-archimedean local field of characteristic 0, and fix a nontrivial additive character ψ of F. Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(F)$, $n > 1$, and ω be a unitary character of F^*. The Langlands–Shahidi L-function $L(s, \tau, \mathrm{Sym}^2 \otimes \omega)$ (defined in [Sha90]) is holomorphic in $\mathrm{Re}(s) > 0$, and has a pole at $s = 0$ precisely when $\tau \cong \omega^{-1} \otimes \tau^\vee$. In that case, by the Langlands functoriality principle τ should be the image of the lift of a representation of a unique split or quasi-split (split over a quadratic extension of F) even general spin group of absolute rank $n + 1$. Denote this group by $G(F)$, it is determined by the square-class of $\alpha \in F^*$. We define a unitary supercuspidal generic representation $\sigma_{\psi_\alpha}(\tau, \omega)$ of $G(F)$. Here by generic we mean generic with respect to some generic character of a maximal unipotent subgroup $N_G(F)$ of $G(F)$. Here is our main theorem.

Theorem 1.1. *(see Theorem 7.1) Assume $L(s, \tau, \mathrm{Sym}^2 \otimes \omega)$ has a pole at $s = 0$.*

(1) (Non-vanishing) There exists some $\alpha \in F^*$ such that $\sigma_{\psi_\alpha}(\tau, \omega) \neq 0$.
(2) The representation $\sigma_{\psi_\alpha}(\tau, \omega)$ is a supercuspidal, multiplicity free and admissible representation of $G(F)$. Its irreducible constituents are all unitary and generic.
(3) (Local Functorial Lift) Let σ be an irreducible supercuspidal $\psi_{N_G,\alpha}$-generic representation of $G(F)$. The Rankin-Selberg γ-factor $\gamma(s, \sigma \times (\tau \otimes \omega), \psi)$ defined by (3.7) has a pole at $s = 1$ if and only if $\sigma^\vee$ is a quotient of $\sigma_{\psi_\alpha}(\tau, \omega)$.

One may expect a description of α in terms of ω and the central character of τ. E.g., if the central character of τ is trivial, then ω should be a square. See [PR12, Appendix]. Such a result may require an approach different from the one presented here. Regarding the third property, the above condition on the γ-factor should hold when σ lifts to τ. We mention that Mœglin [Mœg14] obtained this lift, and thereby the Langlands parameter, for discrete series representations of classical groups (including the group G).

To prove our result we follow the paradigm introduced by Ginzburg *et al.* [GRS99a], but at certain points new ideas are needed. Let us briefly describe our method.

The representation $\sigma_{\psi_\alpha}(\tau, \omega)$ is defined using a certain twisted Jacquet module, which occurs naturally in a local Rankin–Selberg construction for $G \times \mathrm{GL}_k$ and the representations $\sigma \times \tau$. This construction was recently introduced by Cogdell *et. al.* [ACS17] in the global setting. We elaborate on the local aspects that we need here. In particular we prove a local "generic uniqueness" result, by which we can define a Rankin–Selberg γ-function $\gamma(s, \sigma \times (\tau \otimes \omega), \psi)$ as a proportionality factor between two integrals within a standard functional equation. One expects this γ-factor to coincide with the corresponding Langlands–Shahidi γ-factor, at least at the level of zeros and poles, but we have not pursued this here. Note that this work does not depend on the results of [ACS17]. See § 3 for more details.

In order to prove $\sigma_{\psi_\alpha}(\tau,\omega)$ is supercuspidal, we consider a "tower" of representations $\sigma_{\psi_\alpha}(l,\tau,\omega)$, $0<l\leq n$, where $\sigma_{\psi_\alpha}(\tau,\omega)=\sigma_{\psi_\alpha}(n,\tau,\omega)$, and show the vanishing of these representations for all $l<n$. The main ingredient we use for the proof is a class of exceptional representations. In our setting these are the representations of double coverings of general linear groups constructed by Kazhdan and Patterson [KP84], as well as the representations of double covers of general spin groups developed in [Kap17b] (following [BFG03], see also [LS10] for a more general construction). We mention that exceptional representations were constructed in greater generality in [Gao17]. We take the tensor product of two exceptional representations to form a representation of the linear group. Such a representation is typically quite large, and may be considered as a model (see [Kab01, Kap16a, Kap16b, Kap17a]). For example, one may prove multiplicity one results (e.g., [Kab01]), or analyze the structure of its irreducible quotients ([Kap17a]). In this spirit, we say that a representation of the linear group affords an exceptional model if it is a quotient of the tensor product of two exceptional representations of the double cover of the group.

Consider a supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ such that its symmetric square L-function has a pole at $s=0$. According to the results of [Kap16b] (see also [Yam17]), τ affords an exceptional model and so does the representation parabolically induced from $\tau\otimes 1$ to a general spin group. To prove the vanishing results we use the "smallness" of the exceptional representations, namely that a large class of their Jacquet modules vanishes ([BFG03, Kap17b]).

This technique is parallel to a method of Ginzburg *et. al.* [GRS99a, GRS99b]. They used the interplay between Shalika models, which are related to the pole of the exterior square L-function at $s=0$, linear models, and symplectic models (see § 5.1 for a more precise description). The presence of exceptional representations here is expected and understood, in light of the role these representations played in the (global) work of Bump and Ginzburg [BG92] on the integral representation of the symmetric square L-function, or even in the earlier low rank results [GJ78, PPS89].

To handle the twisted symmetric square L-function we use the recent construction of twisted exceptional representations for double coverings of general linear groups by Takeda [Tak14], who used them to develop an integral representation for the global partial L-function (extending [BG92]). We also rely on a result of Yamana [Yam17] who proved that if the twisted symmetric square L-function of τ has a pole at $s=0$, τ admits a (twisted) exceptional model. See § 5.

In § 6, we prove the non-vanishing of the descent $\sigma_{\psi_\alpha}(\tau,\omega)$. The main ingredients are the results of Jiang *et. al.* [JLS16] on the lifting of nilpotent orbits in the wave-front sets of representations and the results of Gomez *et. al.* [GGS17] on relations between degenerate Whittaker models and generalized Whittaker models of representations. More explicitly, let $\mathrm{LQ}(1,\tau\otimes\omega)$ be the image of the representation parabolically induced from $|\det|^{1/2}\tau\otimes\omega$ to $\mathrm{GSpin}_{4n+1}(F)$ under the standard intertwining operator (this image is the Langlands quotient). By [GGS17], the representation $\mathrm{LQ}(1,\tau\otimes\omega)$ has a nonzero generalized Whittaker model attached to the partition $(2n,2n,1)$. Then by [JLS16], $\mathrm{LQ}(1,\tau\otimes\omega)$ admits a similar nonzero model attached to the special expansion (in the sense of [CM93]) of $(2n,2n,1)$, and using [GGS17] we conclude that $\sigma_{\psi_\alpha}(\tau,\omega)\neq$

0. Our non-vanishing proof is the local counterpart of the global non-vanishing proof in [HS16], and is streamlined using the results of [JLS16] and [GGS17].

Let us also comment on the applicability of our results to non-archimedean local fields of characteristic $p > 0$. The results on the Rankin–Selberg integrals in § 3 and § 4 remain valid. Note that for § 3.2 the analytic properties of the intertwining operator follow from Waldspurger [Wal03], and the definition of the local coefficient follows from Lomeli [Lom15, Lom19, Lom]. The vanishing results of § 5 also hold with two reservations: one has to verify the applicability of [Yam17], and it is necessary to take $p > 2$ because double coverings are used. The non-vanishing arguments in § 6 rely on the work in [GGS17] which assumes that F has characteristic 0 (see [GGS17, Remark 5.1.4] for comments on the case of positive characteristic), and the work in [JLS16] which remains valid when the characteristic p is large enough. The non-vanishing results may still hold when the characteristic p is large enough with an argument by contradiction similar to [GRS11, Section 9.2].

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2. Groups and general notation

Let F be a non-archimedean local field of characteristic 0. Denote the residual cardinality of F by q. Let V be a finite-dimensional vector space over F, and ϕ be a quadratic form on V defined over F. Denote the special orthogonal group of ϕ by $\mathrm{SO}(\phi)$ and its simply connected cover by $\mathrm{Spin}(\phi)$. Let $\mathrm{pr} : \mathrm{Spin}(\phi) \twoheadrightarrow \mathrm{SO}(\phi)$ be the canonical isogeny and c be the nontrivial element in $\ker \mathrm{pr}$. Then $(-1, c)$ generates an order 2 subgroup $< (-1, c) >$ of $\mathrm{GL}_1 \times \mathrm{Spin}(\phi)$. Define $\mathrm{GSpin}(\phi) = (\mathrm{GL}_1 \times \mathrm{Spin}(\phi))/ < (-1, c) >$. Recall that the unipotent subgroups of $\mathrm{GSpin}(\phi)$ are isomorphic (as algebraic groups) to the unipotent subgroups of $\mathrm{SO}(\phi)$. The Weyl groups of $\mathrm{GSpin}(\phi)$ and $\mathrm{SO}(\phi)$ are also isomorphic. Throughout this work, when we say that $\mathrm{GSpin}(\phi)$ is quasi-split, we mean it is non-split over F, but split over a quadratic extension of F.

Let m be a positive integer. Let $H = \mathrm{GSpin}(\phi)$ where $\dim V = 2m + 1$ and ϕ is isotropic of index m. The group H is split over F. Fix maximal isotropic subspaces $V^{\pm}$ in duality with respect to the symmetric bilinear form $(\cdot, \cdot)$ associated with ϕ. Let $(e_1, \ldots, e_m, e_{m+1}, e_{-m}, \ldots, e_{-1})$ be a basis of V such that $V^+ = \mathrm{Span}\{e_1, \ldots, e_m\}$, $V^- = \mathrm{Span}\{e_{-1}, \ldots, e_{-m}\}$ and for all $1 \le i, j \le m$, $(e_i, e_{-j}) = \delta_{i,j}$ and $(e_i, e_{m+1}) = (e_{m+1}, e_{-j}) = 0$. The subspaces $V_l^+ = \mathrm{Span}\{e_1, \ldots, e_l\}$ form a maximal flag

$$0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_m^+ = V^+$$

in V. This choice then fixes the Borel subgroup $B' = T' \ltimes N'$ of $\mathrm{SO}(\phi)$, where T' is the torus. In general if $X' < \mathrm{SO}(\phi)$, we denote its preimage in H under pr by X. Now $B = \mathrm{pr}^{-1}(B')$ is a Borel subgroup of H, $B = T \ltimes N$ where T is a maximal torus.

For each $1 \le l \le m$, let $Q'_l = M'_l \ltimes U'_l$ denote the maximal parabolic subgroup of $\mathrm{SO}(\phi)$ which stabilizes V_l^+, and $P'_l = L'_l \ltimes N'_l < \mathrm{SO}(\phi)$ be the parabolic subgroup stabilizing the flag $V_1^+ \subset \ldots \subset V_l^+$. The unipotent radicals are U'_l and N'_l. Then, e.g., $Q_m = M_m \ltimes U_m$ is the standard Siegel parabolic subgroup of H. The center C_H of H is connected (because $\dim V$ is odd) and isomorphic to GL_1. Moreover, C_H is identified with the GL_1 component of $M_m \cong \mathrm{GL}_m \times \mathrm{GL}_1$.

For the group GL_l, we let $B_{\mathrm{GL}_l} = T_{\mathrm{GL}_l} \ltimes N_{\mathrm{GL}_l}$ denote its Borel subgroup of upper triangular invertible matrices, with the diagonal torus T_{GL_l}. If $\beta = (\beta_1, \ldots, \beta_r)$ is a composition of l, let $P_\beta = M_\beta \ltimes V_\beta$ denote the standard parabolic subgroup of GL_l corresponding to β ($V_\beta < N_{\mathrm{GL}_l}$). Also V_β^- denotes the unipotent subgroup opposite to V_β. Let $\mathcal{P}_l$ be the mirabolic subgroup of GL_l, i.e., the subgroup of invertible matrices whose last row is $(0, \ldots, 0, 1)$. For $a \in \mathrm{GL}_l$, denote its image in M_l by $\iota_l(a)$. For $l \le l'$, $\mathrm{GL}_l < \mathrm{GL}_{l'}$ via $a \mapsto \mathrm{diag}(a, I_{l'-l})$, then $\iota_l(a) = \iota_{l'}(a)$. Hence we simply denote $\iota = \iota_l$.

The algebraic groups in this work will be defined over F, and for any such group X, we identify $X = X(F)$ (which is an l-group). The center of any group G is denoted C_G, and if $x, y \in G$ and $Y < G$, ${}^x y = xyx^{-1}$ and ${}^x Y = \{{}^x y : y \in Y\}$. Representations are always complex and smooth. For a representation ρ of X, $\rho^\vee$ is the contragredient representation. We fix a nontrivial additive character ψ of F.

The induction Ind and compact induction ind functors are normalized as in [BZ77, 1.8]. For a representation ρ of $Y < H$ on a space V_ρ, a closed unipotent subgroup $U < Y$ and a character ψ_U of U, the Jacquet module $J_{U,\psi_U}(\rho)$ is the quotient of V_ρ by the subspace spanned by $\{\rho(u)\xi - \psi(u)\xi : u \in U, \xi \in V_\rho\}$. For any $Y_0 < Y$ let $N_{Y_0}(U, \psi_U)$ denote the subgroup of elements $y \in Y_0$ which normalize U and fix ψ_U. Then $J_{U,\psi_U}(\rho)$ is a representation of $N_Y(U, \psi_U)$. The action of $N_Y(U, \psi_U)$ is normalized as in [BZ77, 1.8], by the inverse square-root of the modulus character of U.

Throughout this work, all L-functions are the ones defined by Shahidi [Sha90]. The twisted symmetric square γ-factor is also the factor defined in [Sha90], but the standard γ-factor for a pair of representations of $\mathrm{GSpin}_{2n} \times \mathrm{GL}_m$ will be defined here using Rankin–Selberg integrals. We will not rely on the conjecture that this factor agrees with the similar factor of [Sha90].

Remark 2.1. *Irreducible supercuspidal representations are injective and projective in the category of (smooth) representations of G on which C_G° acts by a character. Indeed since C_G is abelian and the index of C_G° in C_G is finite, any representation π of G decomposes into a direct sum of eigenspaces under the action of the quotient group $C_G^\circ \backslash C_G$. The center C_G acts by a fixed character on each of these eigenspaces, and the statement then follows immediately from the the injectivity and projectivity of irreducible supercuspidal representations in the category of representations with a fixed central character.*

3. The local Rankin–Selberg integral

3.1. **The integral and γ-factor.** Let τ be an irreducible generic representation of GL_m and ω be a quasi-character of F^*. Let $1 \le n \le m$ and $l = m - n$. We define the family of local Rankin–Selberg integrals for the groups $\mathrm{GSpin}_{2n} \times \mathrm{GL}_m$. While the integrals can also be defined for $n > m$, this case will not appear in this work. Note that in § 3.2 and § 4 we will allow n (and thereby l) to vary, in § 5 we shall take $l > m/2$, then in § 7 we specialize to $m = 2n$.

Let $\alpha \in F^*$ and fix a vector $y_\alpha = e_m + \frac{\alpha}{2} e_{-m}$ of length α (i.e., $(y_\alpha, y_\alpha) = \alpha$). Define the character $\psi_{l,\alpha}$ of N_l by

$$\psi_{l,\alpha}(u) = \psi(\sum_{i=2}^{l} (u \cdot e_i, e_{-(i-1)}) + (u \cdot y_\alpha, e_{-l})), \tag{3.1}$$

where on the r.h.s. (right hand side) u is identified with its projection into N'_l. The stabilizer of $\psi_{l,\alpha}$ in P_l is isomorphic to the group $G = \mathrm{GSpin}_{2n}$, which is split over F if α is a square in F^*, otherwise it is quasi-split. To see this note that the pointwise stabilizer of $\{e_i, e_{-i}\}_{1\leq i\leq l}$ in $\mathrm{SO}(\phi)$ is SO_{2n+1} and inside that group, the stabilizer of y_α is a form of SO_{2n} (see [Kap13c, § 2.1.1]). In this way we construct an embedding of G in H. Since $P_l \cong (B_{\mathrm{GL}_l} \times \mathrm{GSpin}_{2n+1}) \ltimes U_l$, this embedding already determines a Borel subgroup $B_G = T_G \ltimes N_G$ of G by the requirement $(N_{\mathrm{GL}_l} \times N_G) \ltimes U_l = N$. We identify N_G with

$$\left\{ \mathrm{diag}(I_l, \begin{pmatrix} z & v & x & -\frac{2}{\alpha}v & u \\ & 1 & 0 & 0 & -\frac{2}{\alpha}v' \\ & & 1 & 0 & x' \\ & & & 1 & v' \\ & & & & z^* \end{pmatrix}, I_l) \in N' : z \in N_{\mathrm{GL}_{n-1}},\, v, x \in F^{n-1} \right\}. \tag{3.2}$$

For $n > 1$, C_G is disconnected, its identity component C_G° is isomorphic to GL_1 and under the embedding $G < H$, C_G° is identified with C_H.

Denote the Whittaker model of τ with respect to $\psi_{N_{\mathrm{GL}_m}}(z) = \psi(\sum_{i=1}^{m-1} z_{i,i+1})$ ($z \in N_{\mathrm{GL}_m}$) by $W(\tau, \psi_{N_{\mathrm{GL}_m}})$. Set $Q = Q_m$. For a complex parameter s, let $V(s, \tau\otimes\omega)$ denote the space of the (normalized) induced representation

$$\mathrm{Ind}_Q^H(|\det|^{s-1/2} W(\tau, \psi_{N_{\mathrm{GL}_m}}) \otimes \omega).$$

There are standard notions of holomorphic or meromorphic sections of $V(\tau\otimes\omega)$. Briefly, a holomorphic section of $V(\tau\otimes\omega)$ is a function f on $\mathbb{C}\times H$ such that for each $h \in H$, $s \mapsto f(s,h)$ is holomorphic, and for each s, $h \mapsto f(s,h)$ belongs to $V(s,\tau\otimes\omega)$. Denote $f_s(h) = f(s,h)$ and $h_0 \cdot f_s(h) = f_s(hh_0)$. We regard f_s as a complex-valued function by evaluating at the identity.

Define a generic character $\psi_{N_G,\alpha}$ of N_G by $\psi_{N_G,\alpha}(u) = \psi^{-1}_{N_{\mathrm{GL}_{n-1}}}(z)\psi^{-1}(v_n)$, where u is given by (3.2) and v_n is the last row of v. Note that if one writes the root subgroup of G corresponding to the simple root $\epsilon_{n-1} + \epsilon_n$ by $(x, y) \in F^2$, the restriction of $\psi_{N_G,\alpha}$ to (x, y) is given by $\psi^{-1}(\frac{1}{4}x - \frac{\alpha}{2}y)$ in the split case and $\psi^{-1}(\frac{1}{2}y)$ in the quasisplit case ($\psi^{-1}_{N_G,\alpha}$ is the character [Kap13c, (3.1)]). In particular $\psi_{N_G,\alpha}$ depends on α. Let σ be an irreducible $\psi_{N_G,\alpha}$-generic representation of G and $W(\sigma, \psi_{N_G,\alpha})$ denote the corresponding $\psi_{N_G,\alpha}$-Whittaker model.

For $W \in W(\sigma, \psi_{N_G,\alpha})$ and a holomorphic section f of $V(\tau\otimes\omega)$, define the integral

$$I(s, W, f) = \int\limits_{C_G^\circ N_G\backslash G} W(g) \int\limits_{N_l^{\beta_{l,\alpha}}} f_s(\beta_{l,\alpha} u g)\psi^{-1}_{l,\alpha}(u)\, du\, dg\,.$$

Here $N_l^{\beta_{l,\alpha}} = \beta^{-1}_{l,\alpha} Q \beta_{l,\alpha} \cap N_l$ where $\beta_{l,\alpha}$ is a representative for the unique open orbit of the right-action of $G \ltimes N_l$ on $Q\backslash H$, which we take to be a fixed element in

$$\mathrm{pr}^{-1}\left(\iota(\begin{pmatrix} 0 & \frac{2}{\alpha}I_{m-l} \\ I_l & 0 \end{pmatrix}) \begin{pmatrix} 0 & 0 & I_l \\ 0 & I_{2m-2l+1} & 0 \\ I_l & 0 & 0 \end{pmatrix} (-1)^l I_m \right).$$

(The embedding ι was defined in § 2.) This is the local analog of the global Eulerian integral introduced in [ACS17].

Proposition 3.1. *There exists $s_0 \in \mathbb{R}$ such that $I(s, W, f)$ is absolutely convergent for all s with $\mathrm{Re}(s) > s_0$, for all $W \in W(\sigma, \psi_{N_G,\alpha})$ and holomorphic sections f.*

Proof. Since we mod out by C_G° in the domain of integration for the outer integral, the proof is a straightforward adaptation of the proof of convergence for the integrals for odd orthogonal groups, see [Sou93, § 4.4–4.6]. ☐

Lemma 3.2. *In the domain* $\mathrm{Re}(s) > s_0$*, for all* $g_0 \in G$ *and* $u_0 \in N_l$*,*

$$I(s, \sigma(g_0)W, (g_0u_0)\cdot f) = \psi_{l,\alpha}(u_0) I(s, W, f)\,. \tag{3.3}$$

Proof. This follows from the fact that $I(s, W, f)$ is invariant with respect to G, G normalizes N_l and stabilizes $\psi_{l,\alpha}$, and from

$$\int\limits_{N_l^{\beta_{l,\alpha}}} f_s(\beta_{l,\alpha}uu_0)\psi_{l,\alpha}^{-1}(u)\,du = \psi_{l,\alpha}(u_0)\int\limits_{N_l^{\beta_{l,\alpha}}} f_s(\beta_{l,\alpha}u)\psi_{l,\alpha}^{-1}(u)\,du,$$

in $\mathrm{Re}(s) > s_0$. ☐

Theorem 3.3. *Let* σ *be an irreducible generic representation of* G *(with respect to* $\psi_{N_G,\alpha}$ *or a different generic character of* N_G*). Except for a finite number of values of* q^{-s}*, the space of bilinear forms satisfying* (3.3) *is at most one-dimensional. If* σ *is supercuspidal, the dimension is at most one for all* s*. If however,* σ *is irreducible supercuspidal and non-generic, the dimension is* 0 *for all* s*.*

Proof. The proof is the adaptation of the local uniqueness result for the Rankin–Selberg integrals for $\mathrm{SO}_{2n}\times\mathrm{GL}_m$ ([Kap13c, § 4.1], for the global counterpart see [Kap12, § 3.2], see also the local uniqueness proofs in [Sou93, § 8.2] and [GRS99a, § 6.2]). Briefly, according to [BZ77, 1.9],

$$\begin{aligned}&\mathrm{Bil}_G(\sigma, J_{N_l,\psi_{l,\alpha}}(\mathrm{Ind}_Q^H(|\det|^{s-1/2}\tau\otimes\omega)))\\ &\cong \mathrm{Bil}_H(\mathrm{ind}_R^H(\sigma\otimes\psi_{l,\alpha}^{-1}), \mathrm{Ind}_Q^H(|\det|^{s-1/2}\tau\otimes\omega)).\end{aligned} \tag{3.4}$$

Here $\mathrm{Bil}_G(\cdot,\cdot)$ denotes the space of G-equivariant bilinear forms and $R = G \ltimes N_l$. For $h \in H$, let

$$\mathrm{Hom}(h) = \mathrm{Hom}_{({}^hR)\cap Q}({}^h(\sigma\otimes\psi_{l,\alpha}^{-1})\otimes(|\det|^{s-1/2}\tau\otimes\omega), \delta).$$

Here for a representation ρ, ${}^h\rho(x) = \rho({}^{h^{-1}}x)$, and $\delta = \delta_Q^{-1/2}\cdot{}^h(\delta_R^{-1}\delta_{R\cap(h^{-1}Qh)}^{-1})$. The group R acts on the right on $Q\backslash H$ with finitely many orbits. Write $H = \coprod_{h\in\Omega} QhR$ for a finite set Ω. According to the Bruhat theory (see e.g., [Sil79, Theorems 1.9.4, 1.9.5]), the space (3.4) injects into the semi-simplification $\bigoplus_{h\in\Omega}\mathrm{Hom}(h)$.

According to the proof of [Kap12, Claim 3.1], $\mathrm{Hom}(h) = 0$ unless $QhR = Qh_0R$ for some $h_0 \in H$ such that ${}^{h_0}N_l$ is the opposite subgroup N_l^- (see also [GRS11, Proposition 5.1]). In the quasi-split case this already determines h uniquely; in the split case there are three such representatives, for two of which ${}^{h^{-1}}Q\cap G$ is a parabolic subgroup of G and any morphism in $\mathrm{Hom}(h)$ defines a morphism in

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_{{}^{h^{-1}}Q\cap G}^G(\delta_{{}^{h^{-1}}Q\cap G}^{-1/2}\cdot{}^{h^{-1}}(\delta|\det|^{1/2-s}\tau^\vee\otimes\omega))). \tag{3.5}$$

The latter space is zero outside finitely many values of q^{-s} by [GPSR87, Lemma 10.1.2], which was stated for SO_{2n} but immediately implies the similar result for G. Thus there is only one more representative which we denote by h_0 in both the split and quasi-split cases. Note that $Qh_0R = Q\beta_{l,\alpha}R$.

The above arguments (including [GPSR87, Lemma 10.1.2]) apply whether σ is generic or not.

Let $V < N_G$ be the subgroup with $z = I_{n-1}$ and $v = 0$, using (3.2) (V is R'_l of [Kap12, § 3.2, p. 154]). Then $N_G = N_{\mathrm{GL}_n} \ltimes V$ and ${}^{h_0}V < {}^{h_0}G \cap U_m$. Since U_m acts trivially on the space of $|\det|^{s-1/2}\tau \otimes \omega$, each morphism in $\mathrm{Hom}(h_0)$ factors through $J_V(\sigma)$. (Note that V is not a unipotent radical of a parabolic subgroup of G.) In addition, each morphism in $\mathrm{Hom}(h_0)$ factors through $J_{{}^{h_0}N_l \cap M_m, {}^{h_0}\psi_{l,\alpha}}(\delta' |\det|^{1/2-s}\tau^\vee \otimes \omega)$, where δ' is a suitable modulus character. Also note that $(C_G V)\backslash({}^{h_0^{-1}}Q \cap G) = \mathcal{P}_n$. It follows that

$$\mathrm{Hom}(h_0) \subset \mathrm{Bil}_{\mathcal{P}_n}(J_V(\sigma), J_{{}^{h_0}N_l \cap M_m, {}^{h_0}\psi_{l,\alpha}}(\delta' |\det|^{1/2-s}\tau^\vee \otimes \omega)). \tag{3.6}$$

Now by [GPSR87, Proposition 8.2] (stated for representations of SO_{2n+1} but the proof is applicable to G as well), $J_V(\sigma)$ admits a finite Jordan–Hölder composition series as a $\mathcal{P}_n$-module. This series contains the irreducible representation $\mathrm{ind}_{N_{\mathrm{GL}_n}}^{\mathcal{P}_n}(\psi_{N_{\mathrm{GL}_n}})$ with multiplicity at most 1. In fact the multiplicity is 1 if and only if σ is generic (a Whittaker functional on $J_V(\sigma)$ lifts to a Whittaker functional on σ). The similar assertion applies to $J_{{}^{h_0}N_l \cap M_m, {}^{h_0}\psi_{l,\alpha}}(\delta'(|\det|^{1/2-s}\tau^\vee \otimes \omega))$ by [BZ76, 5.15]. The result then follows as in [JPSS83, 2.10], using the structure of irreducible representations of $\mathcal{P}_n$ ([BZ76, 5.13]).

Now assume σ is supercuspidal. Then (3.5) vanishes for all s (see the remark on [GPSR87, p. 117], again for SO_{2n}). Regarding the r.h.s. of (3.6), since the i-th derivatives of $J_V(\sigma)$ (in the sense of [BZ77]) for $0 < i < n$ all factor through Jacquet modules (non-twisted — without a character) of standard unipotent radicals of G, they all vanish. By [BZ77, Proposition 3.7b, c], the contribution to (3.6) comes from the n-th derivatives of both $J_V(\sigma)$ and $J_{{}^{h_0}N_l \cap M_m, {}^{h_0}\psi_{l,\alpha}}(\delta' |\det|^{1/2-s}\tau^\vee \otimes \omega)$, which correspond to the Whittaker characters on $N_{\mathrm{GL}_n} < \mathcal{P}_n$. This contribution is of dimension one if σ is generic, and 0 otherwise. □

Proposition 3.4. *The integral $I(s, W, f)$ can be made a nonzero constant (independent of s) for some choice of data (W, f) where f is holomorphic.*

Proof. See [Kap13a, Lemma 3.1, Lemma 4.1]. □

Corollary 3.5. *$I(s, W, f)$ is a rational function of q^{-s}.*

Proof. By Proposition 3.1 and Lemma 3.2, in a right half plane $I(s, W, f)$ can be regarded as a G-equivariant bilinear form on $\sigma \times J_{N_l, \psi_{l,\alpha}}(V(s, \tau \otimes \omega))$. The result now follows from Theorem 3.3 and Proposition 3.4 together with Bernstein's continuation principle (in [Ban98]). □

Let $A(s, w) : V(s, \tau \otimes \omega) \to V(1 - s, (\omega^{-1} \circ \det)\tau^\vee \otimes \omega)$ denote the standard intertwining operator, defined for $\mathrm{Re}(s) \gg 0$ by an absolutely convergent integral, where w is a representative for the long Weyl group element modulo the Weyl group of the Levi part of Q. In this case the local coefficient attached to $A(s, w)$ is given by $\gamma(2s - 1, \tau, \mathrm{Sym}^2 \otimes \omega)$ ([Sha81, Sha90]). By the results above and in particular, Theorem 3.3, there is a well defined and not identically zero function $\gamma(s, \sigma \times (\tau \otimes \omega), \psi) \in \mathbb{C}(q^{-s})$ such that for all W and f,

$$\gamma(s, \sigma \times (\tau \otimes \omega), \psi) I(s, W, f) = I(1 - s, W, \gamma(2s - 1, \tau, \mathrm{Sym}^2 \otimes \omega) A(s, w) f). \tag{3.7}$$

3.2. **The descent map.** Assume $n > 1$, σ and τ are irreducible supercuspidal, τ and ω are unitary, and σ is $\psi_{N_G,\alpha}$-generic. Then $A(s, w)$ is holomorphic for $\mathrm{Re}(s) > 1/2$ ([Sha81, Lemma 2.2.5]). Its image $A(1, w)V(1, \tau \otimes \omega)$ (at $s = 1$) is isomorphic to the

Langlands quotient $\mathrm{LQ}(1,\tau\otimes\omega)$ of $V(1,\tau\otimes\omega)$. For any $1\le l\le m$, define a descent map by $\sigma_{\psi_\alpha}(l,\tau,\omega)=J_{N_l,\psi_{l,\alpha}}(\mathrm{LQ}(1,\tau\otimes\omega))$.

Proposition 3.6. *Let f be a holomorphic section. The function $I(s,W,f)$ is holomorphic, and $I(1-s,W,A(s,w)f)$ is holomorphic except perhaps at the poles of $A(s,w)$.*

Proof. Since σ is supercuspidal, any W is compactly supported modulo C_GN_G. This together with the fact that the inner integral $\int f_s(\beta_{l,\alpha}ug)\psi_{l,\alpha}^{-1}(u)du$ stabilizes for large compact open subgroups of $N_l^{\beta_{l,\alpha}}$ (see [Kap13b, § 4.2]) implies that $I(s,W,f)$ is holomorphic. The statement regarding $I(1-s,W,A(s,w)f)$ follows immediately. □

Theorem 3.7. *If $\gamma(s,\sigma\times(\tau\otimes\omega),\psi)$ has a pole at $s=1$, then $L(s,\tau^\vee,\mathrm{Sym}^2\otimes\omega^{-1})$ has a pole at $s=0$ and σ pairs nontrivially with $\sigma_{\psi_\alpha}(l,\tau,\omega)$. The converse is also true, under the additional assumption that $\sigma_{\psi_\alpha}(l,\tau,\omega)$ is semi-simple.*

Proof. By Proposition 3.4, we may choose data (W,f) where f is holomorphic, such that $I(s,W,f)=1$ for all $s\in\mathbb{C}$. According to the definitions and [Sha90, Proposition 7.3], $\gamma(2s-1,\tau,\mathrm{Sym}^2\otimes\omega,\psi)$ has a pole at $s=1$ if and only if $L(2-2s,\tau^\vee,\mathrm{Sym}^2\otimes\omega^{-1})$ does. Since $A(1,w)$ is holomorphic, it follows from (3.7) and Proposition 3.6 that $\gamma(s,\sigma\times(\tau\otimes\omega),\psi)$ has a pole at $s=1$ if and only if $L(2-2s,\tau^\vee,\mathrm{Sym}^2\otimes\omega^{-1})$ does and $I(1-s,\cdot,A(s,w)\cdot)|_{s=1}\neq 0$, which means that $I(1-s,W,A(s,w)f)$ is nonzero at $s=1$ for some W and holomorphic section f. The latter condition, i.e., that $I(1-s,\cdot,A(s,w)\cdot)|_{s=1}\neq 0$, implies that $\mathrm{Bil}_G(\sigma,\sigma_{\psi_\alpha}(l,\tau,\omega))\neq 0$.

Note that if the r.h.s. of (3.7) has a pole at $s=1$, then so does $\gamma(s,\sigma\times(\tau\otimes\omega),\psi)$, even if $I(s,W,f)|_{s=1}=0$.

For the converse direction we assume $\sigma_{\psi_\alpha}(l,\tau,\omega)$ is semi-simple, $L(s,\tau^\vee,\mathrm{Sym}^2\otimes\omega^{-1})$ has a pole at $s=0$ and $\mathrm{Bil}_G(\sigma,\sigma_{\psi_\alpha}(l,\tau,\omega))\neq 0$. Since

$$J_{N_l,\psi_{l,\alpha}}(A(1,w)V(1,\tau\otimes\omega))\cong\sigma_{\psi_\alpha}(l,\tau,\omega)$$

which is semi-simple, $\sigma^\vee$ is also a subrepresentation of $J_{N_l,\psi_{l,\alpha}}(A(1,w)V(1,\tau\otimes\omega))$ and thereby of $J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)$.

Because the embedding $G<H$ identifies C_G° with C_H, C_G° acts on the space of $J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)$ by a fixed character. Hence Remark 2.1 implies that $\sigma^\vee$ is a direct summand of $J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)$ and we write

$$J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)=E_{\sigma^\vee}\oplus E,$$

where $E_{\sigma^\vee}$ is the space of $\sigma^\vee$. This gives rise to a projection $\Lambda: J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)\to\sigma^\vee$ which vanishes on E.

Next, denote

$$\mathscr{H}=\mathrm{Bil}_G(\sigma,J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)).$$

By Theorem 3.3 and Proposition 3.4, $\dim\mathscr{H}=1$. Hence $\sigma^\vee$ appears as a quotient of $J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)$ with multiplicity 1. It follows that there is a unique projection $J_{N_l,\psi_{l,\alpha}}V(0,\omega^{-1}\tau^\vee\otimes\omega)\to\sigma^\vee$, up to scaling.

Since the integral regarded as a bilinear form in $\mathscr{H}$ is nonzero, it is a (nonzero) scalar multiple of Λ. As such, the integral does not vanish upon restriction to $A(1,w)V(1,\tau\otimes\omega)$, since the latter contains $E_{\sigma^\vee}$. Therefore $I(1-s,\cdot,A(s,w)\cdot)|_{s=1}\neq 0$. As explained above, together with the assumption that $L(2-2s,\tau^\vee,\mathrm{Sym}^2\otimes\omega^{-1})$ has a pole at $s=1$, we infer $\gamma(s,\sigma\times(\tau\otimes\omega),\psi)$ has a pole at $s=1$. □

4. The tower Property

We proceed with the notation of § 3.1. In this section we prove that the Jacquet module $J_{N_l,\psi_{l,\alpha}}(\mathrm{Ind}_Q^H(|\det|^{1/2}\tau\otimes\omega))$, for a supercuspidal τ, satisfies the so-called tower property. Namely, for the maximal l such that this module is nonzero, it is supercuspidal.

Recall $l=m-n$. Let $U_{G,p}$ be the unipotent radical of a standard parabolic subgroup $Q_{G,p}$ of G whose Levi part is isomorphic to $\mathrm{GL}_p\times\mathrm{GSpin}_{2(n-p)}$ ($1\leq p\leq n$ if G is split, otherwise $1\leq p\leq n-1$). Denote $\mathcal{U}=V_{(p+1,1^{l-1})}\ltimes U_{p+l}<N_{p+l}$, where $V_{(p+1,1^{l-1})}$ is identified with its images in H under ι.

Proposition 4.1. *Let π be a representation of H. Fix $w=w_{p,l}$ such that $\mathrm{pr}(w)=\mathrm{diag}(\begin{pmatrix} & I_p\\ I_l & \end{pmatrix},I_{2(m-l-p)+1},\begin{pmatrix} & I_l\\ I_p & \end{pmatrix})$. Then $J_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi))\cong J_{\mathcal{U},\psi_{l+p,\alpha}}(\pi)$ as vector spaces.*

Proof. Assume $p<n$. Put $\mathcal{N}={}^{w}(N_l\rtimes U_{G,p})$, and write the elements of $\mathcal{N}$ in the form

$$[z;u,a,d,e;x_1,x_2]=\begin{pmatrix} I_p & 0 & x_1 & d & x_2\\ u & z & a & e & d'\\ 0 & 0 & I_{2(n-p)+1} & a' & x_1'\\ 0 & 0 & 0 & z^* & 0\\ 0 & 0 & 0 & u' & I_p\end{pmatrix},\quad z\in N_{\mathrm{GL}_l},\quad \begin{pmatrix} I_p & x_1 & x_2\\ & I_{2(n-p)+1} & x_1'\\ & & I_p\end{pmatrix}\in N_G$$

(see (3.2)). Put $\mathcal{L}=V_{(p,l)}^-$. Throughout the proof we identify $N_{\mathrm{GL}_l}=\{[z;0,0,0,0;0,0]:z\in N_{\mathrm{GL}_l}\}$ and $\mathcal{L}<\{[I_l;u,0,0,0;0,0]\}$. Extending $\psi_{l,\alpha}$ trivially on $U_{G,p}$ to a character of $N_lU_{G,p}$ gives a character of $\mathcal{N}$, $\psi_{\mathcal{N}}={}^{w}\psi_{l,\alpha}$. In particular $\psi_{\mathcal{N}}|_{N_{\mathrm{GL}_l}}=\psi_{N_{\mathrm{GL}_l}}$ and because $p<n$, $\psi_{\mathcal{N}}|_{\mathcal{L}}=1$. Then $J_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi))\cong J_{\mathcal{N},\psi_{\mathcal{N}}}(\pi)$ (as vector spaces). Note that $J_{\mathcal{N},\psi_{\mathcal{N}}}(\pi)$ is a representation of ${}^{w}G\ltimes\mathcal{N}$.

For $0\leq i\leq l-1$, let $\mathcal{L}^i$ be the subgroup of $\mathcal{L}$ consisting of matrices $\begin{pmatrix} I_p & \\ z & I_l\end{pmatrix}$ such that all rows of z except the $(l-i)$-th row are 0, and $\mathcal{L}_i$ denote the subgroup of elements $\begin{pmatrix} I_p & \\ z & I_l\end{pmatrix}$ where the last $i+1$ rows, i.e., rows $l-i,\dots,l$, are 0.

We argue by repeatedly applying the local "exchange of roots" [GRS99a, Lemma 2.2]. Let $C^0=\{[I_l;0,a,d,e;x_1,x_2]\}$ and $Y^0=\mathcal{L}^0$. Let X^0 be the unipotent subgroup of H consisting of the coordinates in the blocks x_1,x_2,x_1' depicted in $[I_l;0,0,0,0;x_1,x_2]$ above, without the prescribed condition with respect to N_G. The group $X^0\cap C^0$ is normal in X^0 and the quotient is abelian, because it is isomorphic to F^p (note that if $x_1=0$, the conditions imposed on x_2 by the definitions of G or H coincide). Also $X^0C^0=U_{l+p}<H$. In addition, $X^0\cap C^0$ acts trivially on the space of $J_{\mathcal{N},\psi_{\mathcal{N}}}(\pi)$ because $\psi_{\mathcal{N}}|_{X^0\cap C^0}=1$. Put $\psi_{C^0}=\psi_{\mathcal{N}}|_{C^0}$. By the local analog of [GRS11, Lemma 7.1], namely the extension of [GRS99a, Lemma 2.2] to the case where the intersections $X\cap C$ or $Y\cap C$ are nontrivial, where X,Y and C are the subgroups defined in [GRS11, § 7.1] and [GRS99a, Lemma 2.2] (we take $C=C^0$, $X=X^0$, $Y=Y^0$; $C^0\cap X^0$ is nontrivial), $J_{Y^0\ltimes C^0,\psi_{Y^0\ltimes C^0}}(\pi)\cong J_{X^0C^0,\psi_{X^0C^0}}(\pi)$ (as vector spaces), where the characters $\psi_{\dots}$ are extended trivially on Y^0 and X^0. Since $\mathcal{L}=\mathcal{L}_0\cdot\mathcal{L}^0$ (direct product), $\mathcal{N}=N_{\mathrm{GL}_l}\ltimes(\mathcal{L}_0\ltimes(Y^0\ltimes C^0))$ and $\psi_{X^0C^0}=\psi_{l+p,\alpha}|_{U_{l+p}}$,

$$\begin{aligned}J_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi))&\cong J_{N_{\mathrm{GL}_l},\psi_{N_{\mathrm{GL}_l}}}(J_{\mathcal{L}_0}(J_{Y^0\ltimes C^0,\psi_{Y^0\ltimes C^0}}(\pi)))\\&\cong J_{N_{\mathrm{GL}_l},\psi_{N_{\mathrm{GL}_l}}}(J_{\mathcal{L}_0}(J_{U_{l+p},\psi_{l+p,\alpha}}(\pi))).\end{aligned}$$

Next we handle the group $\mathcal{L}_0$. For $1\leq i\leq l-1$, denote $\mathcal{U}^i=V_{(p+l-i,1^i)}\ltimes U_{p+l}<N_{p+l}$. Embed $N_{\mathrm{GL}_{i-1}}<N_{\mathrm{GL}_i}$ via $z\mapsto\mathrm{diag}(z,1)$, then $N_{\mathrm{GL}_i}=N_{\mathrm{GL}_{i-1}}\ltimes V_{(i-1,1)}$. Also note

that $\mathcal{L}_i = \mathcal{L}^{i+1} \cdot \mathcal{L}_{i+1}$. Let $C^1 = V_{(l-1,1)} \ltimes U_{l+p}$ where $V_{(l-1,1)} < N_{\mathrm{GL}_l}$; $Y^1 = \mathcal{L}^1$; and $X^1 = V_{(p+l-1,1)} \cap V_{(p,l)} < N_{\mathrm{GL}_{l+p}}$ (identified with its image under ι). Then $X^1 \ltimes C^1 = \mathcal{U}^1$, $\psi_{C^1} = \psi_{l+p,\alpha}|_{C^1}$ and by [GRS99a, Lemma 2.2] ($X^1 \cap C^1$ is trivial), $J_{Y^1 \ltimes C^1, \psi_{Y^1 \ltimes C^1}}(\pi) \cong J_{X^1 \ltimes C^1, \psi_{X^1 \ltimes C^1}}(\pi)$ where again the characters $\psi_{\dots}$ are extended trivially on Y^1 and X^1. Hence

$$J_{N_{\mathrm{GL}_l}, \psi_{N_{\mathrm{GL}_l}}}(J_{\mathcal{L}_0}(J_{X^0 C^0, \psi_{X^0 C^0}}(\pi))) \cong J_{N_{\mathrm{GL}_{l-1}}, \psi_{N_{\mathrm{GL}_{l-1}}}}(J_{\mathcal{L}_1}(J_{\mathcal{U}^1, \psi_{l+p,\alpha}}(\pi))).$$

We repeat this argument for $i = 1, \dots, l-2$, with $C^{i+1} = V_{(l-i-1,1)} \ltimes \mathcal{U}^i$, $Y^{i+1} = \mathcal{L}^{i+1}$ and $X^{i+1} = V_{(p+l-i-1,1)} \cap V_{(p,l-i)}$, each time [GRS99a, Lemma 2.2] implies

$$J_{N_{\mathrm{GL}_{l-i}}, \psi_{N_{\mathrm{GL}_{l-i}}}}(J_{\mathcal{L}_i}(J_{X^i \ltimes C^i, \psi_{X^i \ltimes C^i}}(\pi))) \cong J_{N_{\mathrm{GL}_{l-i-1}}, \psi_{N_{\mathrm{GL}_{l-i-1}}}}(J_{\mathcal{L}_{i+1}}(J_{\mathcal{U}^{i+1}, \psi_{l+p,\alpha}}(\pi))).$$

Thus $J_{U_{G,p}}(J_{N_l, \psi_{l,\alpha}}(\pi)) \cong J_{\mathcal{U}^{l-1}, \psi_{l+p,\alpha}}(\pi)$ as claimed ($\mathcal{U}^{l-1} = \mathcal{U}$).

It remains to consider $p = n$ (then G is split). There are two standard maximal parabolic subgroups $Q_{G,p}$, but they are conjugate, hence it suffices to prove the statement for one. Write $\alpha = \beta^2$ and consider $h \in \mathrm{pr}^{-1}(h_0)$ where $h_0 \in \mathrm{SO}_{2m+1}$ is defined by

$$h_0 = \mathrm{diag}(I_{m-1}, \begin{pmatrix} -\beta/4 & 1/2 & 1/(2\beta) \\ \beta/2 & 0 & 1/\beta \\ \beta/2 & 1 & -1/\beta \end{pmatrix}, I_{m-1}) \quad (h_0^{-1} = \mathrm{diag}(I_{m-1}, \begin{pmatrix} -1/\beta & 1/\beta & 1/(2\beta) \\ 1 & 0 & 1/2 \\ \beta/2 & \beta/2 & -\beta/4 \end{pmatrix}, I_{m-1})).$$

Then $h_0 y_\alpha = e_{m+1}$ and the only difference between ${}^h\psi_{l,\alpha}$ and (3.1) is that y_α in (3.1) is replaced with e_{m+1}. The group ${}^hG$ is the stabilizer of ${}^h\psi_{l,\alpha}$, and ${}^hJ_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi)) \cong J_{{}^hU_{G,p}}(J_{N_l, {}^h\psi_{l,\alpha}}(\pi))$. We choose $Q_{G,p}$ such that (3.2) for ${}^hU_{G,p}$ takes the form $\{\left(\begin{smallmatrix} I_n & u \\ 0 & I_n \end{smallmatrix}\right)\}$ (note that for u in the form (3.2), the column x in ${}^hu$ becomes 0). With this "patch" we proceed as above, now with $p = n$: denote $\mathcal{N} = {}^w(N_l \rtimes {}^hU_{G,p})$, the elements $[z; u, a, d, e; x_1, x_2]$ take the same form except $x_1 = 0$; $\mathcal{L} = V^-_{(p,l)}$; $\psi_{\mathcal{N}} = {}^{wh}\psi_{l,\alpha}$, $\psi_{\mathcal{N}}|_{N_{\mathrm{GL}_l}} = \psi_{N_{\mathrm{GL}_l}}$ and because of the conjugation by h, $\psi_{\mathcal{N}}|_{\mathcal{L}} = 1$. ☐

Regard $\mathcal{P}_{p+1}$ as a subgroup of H via ι.

Theorem 4.2. *Let τ be a supercuspidal representation of GL_m and π be a subquotient of $\mathrm{Ind}_Q^H(|\det|^{1/2}\tau \otimes \omega)$. As vector spaces,*

$$J_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi)) \cong \mathrm{ind}_{N_{\mathrm{GL}_{p+1}}}^{\mathcal{P}_{p+1}}(J_{N_{l+p},\psi_{l+p,\alpha}}(\pi)). \tag{4.1}$$

In particular $J_{N_l,\psi_{l,\alpha}}(\pi)$ enjoys the following "tower property": For the largest l such that $J_{N_l,\psi_{l,\alpha}}(\pi) \neq 0$, $J_{N_l,\psi_{l,\alpha}}(\pi)$ is supercuspidal.

Proof. By Proposition 4.1 applied to π, $J_{U_{G,p}}(J_{N_l,\psi_{l,\alpha}}(\pi)) \cong J_{\mathcal{U},\psi_{l+p,\alpha}}(\pi)$. For brevity, denote the latter representation by ρ. Observe that ρ is a representation of $\mathcal{P}_{p+1}$. By [BZ77, 3.5] (see also [BZ76, 5.15]), there is a filtration of $\mathcal{P}_{p+1}$-modules

$$0 \subset \rho_{p+1} \subset \ldots \subset \rho_1 = \rho, \quad \rho_i = (\Phi^+)^{i-1}(\Phi^-)^{i-1}(\rho), \quad \rho_{i+1} \backslash \rho_i = (\Phi^+)^{i-1}\Psi^+(\rho^{(i)}).$$

Here $\rho^{(i)} = \Psi^-(\Phi^-)^{i-1}(\rho)$ is the i-th derivative of ρ, and $\Phi^\mp, \Psi^\mp$ are the functors from [BZ77]. Since τ is supercuspidal and π is a constituent of $\mathrm{Ind}_Q^H(|\det|^{1/2}\tau \otimes \omega)$, $\rho^{(i)} = 0$ for all $1 \leq i \leq p$ by [BZ77, Corollary 2.13a]. Hence $\rho = \rho_{p+1} = (\Phi^+)^p(\Phi^-)^p(\rho)$ and (4.1) follows because $(\Phi^-)^p(\rho) = J_{N_{l+p},\psi_{l+p,\alpha}}(\pi)$.

The second assertion is clear because by (4.1) and the maximality of l, the Jacquet modules of π along the standard maximal parabolic subgroups of G all vanish. ☐

5. The vanishing results

5.1. **Outline of the method.** In this section we prove the vanishing of the descent map in the relevant range (see Theorem 5.5 below). We briefly recall the analogous result for SO_{2n+1} (for details see [GRS99a, GRS99b]). Let τ be an irreducible unitary supercuspidal representation of GL_m such that $L(s,\tau,\wedge^2)$ has a pole at $s=0$ (in particular, m is even). Then τ affords a Shalika model, hence also a linear model, i.e., τ embeds into $C^\infty(\mathrm{GL}_{m/2}\times\mathrm{GL}_{m/2}\setminus\mathrm{GL}_m)$. It then follows that the representation parabolically induced from $|\det|^{1/2}\tau$ to Sp_{2m} injects into $C^\infty(\mathrm{Sp}_m\times\mathrm{Sp}_m\setminus\mathrm{Sp}_{2m})$. The vanishing properties of the tower of local descent maps are proved by showing that the corresponding Jacquet modules vanish on $C^\infty(\mathrm{Sp}_m\times\mathrm{Sp}_m\setminus\mathrm{Sp}_{2m})$ ([GRS99b, Theorem 17]).

We consider a representation τ such that the symmetric or twisted symmetric square L-function has a pole at $s=0$. Let us begin with the case of $L(s,\tau,\mathrm{Sym}^2)$. One can find a pair (θ,θ') of exceptional representations (in the sense of [KP84]) of a double cover $\widetilde{\mathrm{GL}}_m$ of GL_m, such that τ is a quotient of $\theta\otimes\theta'$ ([Yam17, Theorem 3.19], see also [Kap16b, Theorem 1.3]). Note that the tensor product $\theta\otimes\theta'$ is a well defined representation of GL_m. Then in [Kap16b, Proposition 4.1] it was proved that $\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}\tau\otimes 1)$ is a quotient of $\Theta\otimes\Theta'$, where Θ and Θ' are exceptional representations of a double cover $\widetilde{H}$ of $H(=\mathrm{GSpin}_{2m+1})$, defined in [Kap17b]. The remaining step, proving the vanishing of the Jacquet modules of $\Theta\otimes\Theta'$, has already been established in [Kap16b, Theorem 1.1] but only for the "ground level", i.e., the generic case ($l=m$). Here we complete the proof for all $l>m/2$.

Now assume ω is a unitary character of F^* and $L(s,\tau,\mathrm{Sym}^2\otimes\omega)$ has a pole at $s=0$. According to Yamana [Yam17, Theorem 3.19], τ (or a twist of τ for odd m) is a quotient of a tensor product of two "extended exceptional representations" (see § 5.2 below for the definition). To proceed, we need to define an extended version of the exceptional representations of $\widetilde{H}$. This can be done along the line of arguments of [Tak14, Kap17b], then [Kap16b, Proposition 4.1] becomes applicable in this setup as well. In this manner we treat Sym^2 and $\mathrm{Sym}^2\otimes\omega$ simultaneously.

We further note, for the interested reader, that the case where $L(s,\tau,\mathrm{Sym}^2)$ has a pole at $s=0$ is also relevant for the descent construction from GL_{2n} to SO_{2n}, which uses the Rankin–Selberg integrals of [Kap13c]. The details of that case have not yet appeared in print, but the expected vanishing result will follow immediately from the proof here, under a certain mild assumption on the field, by replacing H with SO_{2m+1} and using non-extended exceptional representations. See [Kap16b, § 5].

5.2. **Exceptional representations.** Recall the fixed Borel subgroup $B=T\ltimes N$ of H, the maximal parabolic subgroups $Q_k=M_k\ltimes U_k$, $T<M_k\cong\mathrm{GL}_k\times\mathrm{GSpin}_{2(m-k)+1}$ and $U_k<N$, the embedding $\iota:\mathrm{GL}_m\to M_m$, and the fixed character ψ of F (see § 2). Let $\Upsilon=\Upsilon_m$ denote the "canonical" character of H constructed in [Kap17b, § 1.2]. Fixing an identification $[,]:\mathrm{GL}_k\times\mathrm{GSpin}_{2(m-k)+1}\to M_k$ for each k, this character satisfies $\Upsilon([a,b])=\det(a)\cdot\Upsilon_{m-k}(b)$ ($a\in\mathrm{GL}_k$, $b\in\mathrm{GSpin}_{2(m-k)+1}$) and in particular for $k=m$ (then $b\in\mathrm{GL}_1$), $\Upsilon([a,b])=\det(a)\cdot b^{-2}$. Let $\widetilde{H}$ be the double cover of H, constructed in [Kap17b] by restricting the double cover of Spin_{2m+3} of Matsumoto [Mat69] and using the cocycle σ of Banks *et. al.* [BLS99] (in [Kap17b] we showed that σ is block-compatible). We fix a section $\mathfrak{s}:H\to\widetilde{H}$ such that $\sigma(h,h')=\mathfrak{s}(h)\mathfrak{s}(h')\mathfrak{s}(hh')^{-1}$. This

section restricts to a homomorphism of N. For any $X \subset H$ let $\widetilde{X}$ be its preimage in $\widetilde{H}$. In particular $\widetilde{\mathrm{GL}}_m$ is defined by restriction from $\widetilde{H}$, and it is a nontrivial double cover of the class studied in [KP84, Tak14]. We have $C_{\widetilde{H}} = \widetilde{C}_H$, in contrast with double coverings of GL_{2m} ($\widetilde{C}_H$ denotes the preimage of C_H in $\widetilde{H}$).

The exceptional representations of $\widetilde{H}$ were developed (locally and globally) in [Kap17b], by adapting the construction of Bump *et. al.* [BFG03, BFG06] for a covering of SO_{2m+1}. For a convenient summary see [Kap16b, § 2.8]. Let ξ be a genuine character of $C_{\widetilde{T}}$, whose restriction to $C_{\widetilde{T}_{\mathrm{GL}_m}}$ and $C_{\widetilde{H}}$ is a genuine lift of $\delta_{B_{\mathrm{GL}_m}}^{1/4} \cdot |\det|^{(m+1)/4}$ and the trivial character, respectively (note that $C_{\widetilde{T}_{\mathrm{GL}_m}} < C_{\widetilde{T}}$). This determines ξ uniquely when m is even, in the odd case there is an additional choice of a Weil factor. Let $\rho(\xi)$ denote the corresponding genuine irreducible representation of $\widetilde{T}$ (see e.g., [KP84, McN12]). Then $\mathrm{ind}_{\widetilde{B}}^{\widetilde{H}}(\rho(\xi))$ has a unique irreducible quotient Θ_0. An exceptional representation Θ of $\widetilde{H}$ is then any twist of Θ_0 by a non-genuine character of H, i.e., $\Theta = (\chi \circ \Upsilon) \cdot \Theta_0$ where χ is a quasi-character of F^*.

The main property of Θ is that its Jacquet module along a unipotent radical of a parabolic subgroup is, essentially, an exceptional representation of the stabilizer. See [Kap17b, Proposition 2.19] for a more precise statement (see also [BFG03, Theorem 2.3]). This result and the fact that exceptional representations of $\widetilde{\mathrm{GL}}_m$ do not afford a Whittaker functional for $m > 2$ ([KP84, Kap17a], see also [Yam17]), imply through a series of intermediate results, that Θ is "small" in the sense that it is attached to one of the unipotent orbits next to the minimal one (see [BFG03]). The following theorem encapsulates all the vanishing properties of Θ.

Regarding the elements of U_1 as row vectors, any character λ of U_1 takes the form $\lambda(u) = \psi\left(\sum_{i=1}^{2m-1} \beta_i u_i\right)$, with $\beta_i \in F$. The length of λ is defined by $2\sum_{i=1}^{m-1} \beta_i\beta_{2m-i} + \beta_m^2$. While the length depends on ψ, it is zero or not independently of ψ.

Theorem 5.1. *(*[BFG03, Theorem 2.6], [BFG06, Proposition 3], [Kap17b, Lemma 2.25]*) For any λ with nonzero length, $J_{U_1,\lambda}(\Theta) = 0$.*

Corollary 5.2. *Let $V < U_1$ (as algebraic groups) and λ be a character of V, such that the action of $N_{M_1}(V, \lambda)$ (see § 2 for this notation) on the set of characters of $V\backslash U_1$ has finitely many orbits. Assume any extension of λ to a character of U_1 has a nonzero length. Then $J_{V,\lambda}(\Theta) = 0$.*

Proof. The quotient $V\backslash U_1$ is abelian. By [BZ76, 5.9–5.12], if $J_{V\backslash U_1,\lambda'}(J_{V,\lambda}(\Theta)) = 0$ when λ' varies over a complete set of representatives for the orbits, $J_{V,\lambda}(\Theta) = 0$ (by our assumptions, λ' varies over a finite set). Since $J_{V\backslash U_1,\lambda'}(J_{V,\lambda}(\Theta)) = J_{U_1,\lambda_1}(\Theta)$ where λ_1 extends λ, the length of λ_1 is nonzero and $J_{U_1,\lambda_1}(\Theta) = 0$ by Theorem 5.1. □

For example when $m = 2$, if V is defined by $u_1 = 0$, the character $\lambda(v) = \psi(\beta u_2)$ satisfies the requirement of the corollary for any $\beta \neq 0$.

Corollary 5.3. *Let λ be a character of U_1 defined by the vector $(\beta, 0, \dots, 0)$, where $\beta \neq 0$. The subgroup U_2 acts trivially on $J_{U_1,\lambda}(\Theta)$.*

Proof. We argue exactly as in the proof of [BFG06, Proposition 4]. □

As mentioned above, the Jacquet functor takes exceptional representations into exceptional representations. We describe the particular case of J_{U_m}. Let θ_0 be the unique irreducible quotient of $\mathrm{ind}_{\widetilde{B}_{\mathrm{GL}_m}}^{\widetilde{\mathrm{GL}}_m}(\rho(\xi_0))$, where ξ_0 is a lift of $\delta_{B_{\mathrm{GL}_m}}^{1/4}$ to a genuine character of $C_{\widetilde{T}_{\mathrm{GL}_m}}$. This lift is unique when m is even and depends on a Weil factor in the odd case; still, if we fix one Weil factor, the representations θ_0 corresponding to the different lifts are twists of one another by a square-trivial character (see [Kap17a, Claim 2.6]). The exceptional representations of $\widetilde{\mathrm{GL}}_m$ are thus θ_0 and its twists $\theta = (\chi \circ \det) \cdot \theta_0$ (see [KP84, BG92, Kab01]). Then

$$\delta_{Q_m}^{1/2} J_{U_m}(\Theta_0) = |\det|^{(m-1)/4}\theta_0 \otimes 1 \tag{5.1}$$

([Kap16b, (2.8)], the Jacquet functor there was not normalized; see also [Kap17b, Claim 2.21]). Note that the direct factors of M_m commute in the cover, but this is a special phenomenon, which does not hold for M_k with $k < m$. Equality (5.1) implies (almost formally) that when we take a unitary quotient τ of $\theta \otimes \theta'$, there is a suitable unitary character ω of F^* (depending on θ and θ') and a pair of exceptional representations (Θ, Θ') of $\widetilde{H}$, such that $\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}\tau \otimes \omega)$ is a quotient of $\Theta \otimes \Theta'$ ([Kap16b, Proposition 4.1]).

As explained in § 5.1, to handle the twisted symmetric square L-function we need to consider a wider class of exceptional representations, which we call extended exceptional representations (exceptional ones are included in the definition). For $\widetilde{\mathrm{GL}}_2$ these representations are formed by extending the Weil representation of $\widetilde{\mathrm{Sp}}_2$ to a subgroup of $\widetilde{\mathrm{GL}}_2$ of finite index, then inducing to $\widetilde{\mathrm{GL}}_2$ ([Gel76, GPS80]). For $\widetilde{\mathrm{GL}}_m$ they were constructed by Banks [Ban94] under the assumption that F is p-adic of odd residual characteristic. The general case of $\widetilde{\mathrm{GL}}_m$ is due to Takeda [Tak14, § 2.2–2.4].

We begin with a brief description of his construction. Let χ be a unitary character of F^* such that $\chi(-1) = -1$. Denote by ω_ψ^- the irreducible summand of the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_2$ consisting of odd functions. One can extend ω_ψ^- to a representation $\omega_{\psi,\chi}^-$ of the subgroup $\widetilde{\mathrm{GL}}_2^{(2)}$ of $\widetilde{\mathrm{GL}}_2$, where $\mathrm{GL}_2^{(2)} = \{g \in \mathrm{GL}_2 : \det g \in (F^*)^2\}$, by letting C_{GL_2} act by χ. More precisely if $\mathfrak{s} : \mathrm{GL}_2 \to \widetilde{\mathrm{GL}}_2$ is the chosen section, the action is given by $\mathfrak{s}(aI_2) \mapsto \chi(a)\gamma_{\psi'}(a)$, where $\gamma_{\psi'}$ is the Weil factor corresponding to an additive character ψ' of F. The extended exceptional representation $\theta_2^\chi = \mathrm{Ind}_{\widetilde{\mathrm{GL}}_2^{(2)}}^{\widetilde{\mathrm{GL}}_2}(\omega_{\psi,\chi}^-)$ is irreducible supercuspidal and unitary, and independent of the choice of ψ'. (In contrast, the exceptional representations of [KP84] are not supercuspidal.)

Now assume m is even and let $\beta = (2^{m/2})$ be a composition of m. Consider the representation

$$\mathrm{Ind}_{\widetilde{P}_\beta}^{\widetilde{\mathrm{GL}}_m}((\theta_2^\chi \widetilde{\otimes} \ldots \widetilde{\otimes} \theta_2^\chi)\delta_{P_\beta}^{1/4}).$$

Here $\widetilde{\otimes}$ is the metaplectic tensor product ([Kab01, Mez04]), which in this case is canonical (see [Tak16, Remark 4.3]). Since the inducing data is tempered, the Langlands quotient theorem — proved for metaplectic groups by Ban and Jantzen [BJ13] implies that it has a unique irreducible quotient θ_m^χ, which is defined to be an extended exceptional representation of $\widetilde{\mathrm{GL}}_m$. The representation θ_m^χ is also the image of the intertwining operator with respect to the longest Weyl element relative to P_β. We also have the following the "periodicity result" ([Tak14, Proposition 2.36])

$$J_{V_\beta}(\theta_m^\chi) = (\theta_2^\chi \widetilde{\otimes} \ldots \widetilde{\otimes} \theta_2^\chi)\delta_{P_\beta}^{-1/4}$$

(in *loc. cit.* J_{V_β} was not normalized). See [KP84, Theorem I.2.9] for this statement on exceptional representations. As above, we can twist θ_m^χ by $\chi_1 \circ \det$ for a quasi-character χ_1.

We follow a similar paradigm to construct extended exceptional representations of $\widetilde{H}$. Let $R = A \ltimes V$ be the standard parabolic subgroup of H with $A \cong M_\beta \times \mathrm{GL}_1$. Consider the representation

$$\Pi^\chi = \mathrm{Ind}_{\widetilde{R}}^{\widetilde{H}}(((\theta_2^\chi \widetilde{\otimes} \ldots \widetilde{\otimes} \theta_2^\chi) \otimes 1)\delta_R^{1/4}).$$

Again, according to the Langlands quotient theorem [BJ13] this representation has a unique irreducible quotient Θ^χ. Since the inducing data is supercuspidal, according to [BZ77, Corollary 2.13c] $J_V(\Pi^\chi)$ is glued from (that is, admits a filtration whose subquotients are)

$$^w(((\theta_2^\chi \widetilde{\otimes} \ldots \widetilde{\otimes} \theta_2^\chi) \otimes 1)\delta_R^{1/4}),$$

where w varies over the Weyl elements of H which satisfy ${}^wA = A$ and are reduced modulo the Weyl group of A. The periodicity result becomes

$$J_V(\Theta^\chi) = ((\theta_2^\chi \widetilde{\otimes} \ldots \widetilde{\otimes} \theta_2^\chi) \otimes 1)\delta_R^{-1/4}. \tag{5.2}$$

See [Kap17b, Proposition 2.16] for this statement for Θ. A family of extended exceptional representations can be obtained by varying χ, and twisting using $\chi_1 \circ \Upsilon$.

Let $E = B \ltimes Z$ be a standard parabolic subgroup of H. By [BZ77, Corollary 2.13], if $Z \cap A$ is nontrivial, $J_Z(\Theta^\chi) = 0$, and if B contains A, $J_Z(\Theta^\chi)$ is irreducible (combine *loc. cit.* with the transitivity of the Jacquet functor and (5.2)).

By virtue of the above observations, the results of [Kap17b, § 2.3.1] for Θ are applicable to Θ^χ as well. Also note that Yamana [Yam17] proved θ_m^χ does not afford a Whittaker functional when $m \geq 3$. Thus the arguments of [Kap17b, § 2.3.2] are valid as well, in particular [Kap17b, Lemma 2.25], and we deduce that Theorem 5.1 and its corollaries are applicable also to Θ^χ.

The analog of (5.1) holds as well (see [Kap17b, Claim 2.21]), with (Θ_0, θ_0) replaced by $(\Theta^\chi, \theta_m^\chi)$, whence the proof of [Kap16b, Proposition 4.1] extends to Θ^χ. By that proposition, if τ is a quotient of $\theta_m^{\chi^{-1}} \otimes \theta_m$, $\mathrm{Ind}_{Q_m}^H(|\det|^{1/2}\tau \otimes \chi)$ is a quotient of two extended exceptional representations of $\widetilde{H}$.

Since Θ^χ enjoys the same properties of Θ relevant to this work, namely Theorem 5.1, Corollary 5.2 and Corollary 5.3, we omit χ and simply write Θ in all cases.

5.3. **The generic Jacquet modules of Θ.** Recall the Weil representation of the metaplectic group. In this section we prove that the Jacquet module of an extended exceptional representation Θ with respect to C_{U_l} (the center of U_l) and a "generic character" is a direct sum of Weil representations. For $l = m$ this was proved in [Kap16b, Theorem 1.4].

Assume $0 < l \leq m$ is even and set $l = 2j$ and $C = C_{U_l} \cong F^{l \times l}$. Let ψ_j denote a character of C whose stabilizer in Q_l is $\mathrm{Sp}_l \ltimes U_l$; such a character is called generic and for convenience we may assume it is defined by $\psi_j(c) = \psi(\sum_{i=1}^j c_{i,i})$.

Put $r = l(2(m-l)+1)$. Let $\mathcal{H}$ be a generalized Heisenberg group of rank $r+1$. Identify $\mathcal{H}$ with the set of elements $(a, b; c)$, where a and b are rows in $F^{r/2}$, $c \in F$, and the product is given by

$$(a, b; c) \cdot (a', b'; c') = \left(a + a', b + b', c + c' + \tfrac{1}{2}(a, b)\begin{pmatrix} & J_{r/2} \\ -J_{r/2} & \end{pmatrix} {}^{\mathrm{t}}(a', b')\right).$$

Here $J_{r/2}$ is the $r/2\times r/2$ permutation matrix having 1 on its anti-diagonal and ${}^{t}(a',b')$ is the transpose of (a',b'). We have the epimorphism $\ell : U_l \to \mathcal{H}$ defined by

$$\ell(u) = (a_1,\ldots,a_j,b_1,\ldots,b_j,\tfrac{1}{2}(\sum_{i=1}^{j} c_{i,i} - c_{l+i,l+i})),$$

where $b_1,\ldots,b_j$ are the first j rows of v and $a_1,\ldots,a_j$ are the last and we recall that u is written using the notation of (5.4) (with $z = I_l$). Also let $R < \mathcal{H}$ be the subgroup consisting of elements $(0, b; 0)$.

Since ψ_j is trivial on the kernel of ℓ, we may regard $J_{C,\psi_j}(\Theta)$ as a smooth representation of $\mathcal{H}$, and as such it is the direct sum of irreducible Weil representations ω_ψ, where ψ is our fixed character of F.

The representation ω_ψ extends to a representation of $\widetilde{\mathrm{Sp}}_r \ltimes \mathcal{H}$, where $\widetilde{\mathrm{Sp}}_r$ is the metaplectic double cover of Sp_r. Using the action of Sp_l on each of the $2(m-l)+1$ columns of v we construct an embedding of Sp_l in Sp_r. Moreover, the covering $\widetilde{\mathrm{Sp}}_l$ obtained by restricting $\widetilde{H}$ does not split over Sp_l, hence it is the metaplectic double cover, therefore the embedding extends to an embedding of the coverings, also denoted ℓ (one may also apply the strong block compatibility property of the cocycle [BLS99, Theorem 2.7] to deduce this).

As a smooth representation of a generalized Jacobi group, $J_{C,\psi_j}(\Theta)$ is isomorphic to a representation $\kappa\otimes\omega_\psi$, where $(\kappa\otimes\omega_\psi)(\ell(g)h) = \kappa(g)\otimes\omega_\psi(\ell(g)h)$ for $g\in\widetilde{\mathrm{Sp}}_l$ and $h\in\mathcal{H}$, and κ is a non-genuine representation (see e.g., [Ike94, § 1]). The following claim proves κ is trivial.

Theorem 5.4. *As a representation of $\widetilde{\mathrm{Sp}}_l \ltimes \mathcal{H}$, $J_{C,\psi_j}(\Theta)$ is isomorphic to a (possibly infinite) direct sum of copies of the representation ω_ψ.*

Proof. The proof of [Kap16b, Theorem 1.4] (when $l = m$) carries over to $l < m$, we describe the argument briefly. We need to show κ is a trivial representation. Let

$$Y = \left\{ \begin{pmatrix} 1 & & y \\ & I_{l-2} & \\ & & 1 \end{pmatrix} \right\} < \mathrm{Sp}_l\,.$$

It is enough to show $J_{\ell(Y),\psi_\beta}(\kappa) = 0$, where $\psi_\beta(y) = \psi(\beta y)$, for all $\beta\neq 0$. Consider

$$V = \left\{ \begin{pmatrix} 1 & 0 & y & b & 0 & 0 & * \\ & I_{l-2} & & & & & 0 \\ & & 1 & & & & 0 \\ & & & I_{2(m-l)+1} & & & b' \\ & & & & 1 & & y' \\ & & & & & I_{l-2} & 0 \\ & & & & & & 1 \end{pmatrix} \right\} < U_1.$$

The mapping ℓ is an isomorphism of V onto the direct product $\ell(Y)\cdot R_1$, where $R_1 < R$ consists of elements $(0,(b,0,\ldots,0);0)$ with $b\in F^{2(m-l)+1}$. First observe that

$$J_{V,\psi_\beta\circ\ell}(J_{C,\psi_j}(\Theta)) = 0. \tag{5.3}$$

This follows because this space is a quotient of $J_{V\cdot(C\cap U_1),(\psi_\beta\circ\ell)\psi_j}(\Theta)$ which vanishes by Corollary 5.2. Indeed for $c\in C\cap U_1$, $\psi_j(c) = \psi(c_{1,1})$, thus any extension of $(\psi_\beta\circ\ell)\psi_j$ to a character of U_1 is a character of nonzero length (the extension is defined by a character of $V_{(1,l-2)}$ taken from one of 2 orbits under the action of $\mathrm{GL}_{l-2} < P_{(1,l-2)}$).

Since $J_R(\omega_\psi)$ is one-dimensional ([Kap16b, Claim 2.4]), there is a vector φ in the space of ω_ψ such that the Jacquet integrals

$$\varphi^{\mathcal{Y},\mathcal{R}} = \int_{\mathcal{Y}} \int_{\mathcal{R}} \omega_\psi(yr)\varphi \, dr \, dy$$

do not vanish for all compact subgroups $\mathcal{Y} < Y$ and $\mathcal{R} < R$ (see [BZ76, 2.33]). Then given ξ in the space of κ, arguing as in [Kap16b, p. 922] using the fact that $J_R(\omega_\psi)$ is one-dimensional and (5.3), one shows that for sufficiently large $\mathcal{Y}$ and $\mathcal{R}$,

$$\int_{\mathcal{Y}} \kappa(y)\xi\psi_\beta^{-1}(y) \, dy \otimes \varphi^{\mathcal{Y},\mathcal{R}} = 0.$$

This implies that ξ vanishes in $J_{\ell(Y),\psi_\beta}(\kappa)$ ([BZ76, 2.33]). $\square$

5.4. **Vanishing results.** Let τ be an irreducible unitary supercuspidal representation of GL_m, and ω be a unitary character of F^*. Assume $L(s,\tau,\mathrm{Sym}^2\otimes\omega)$ has a pole at $s=0$. We prove $\sigma_{\psi_\alpha}(l,\tau,\omega)=0$ for $m/2<l\leq m$.

If m is even, by [Yam17, Theorem 3.19(1)] τ is a quotient of $\theta_m^\omega \otimes \theta_m$ where the extended exceptional representations are determined (non-uniquely) by τ and ω. Then according to [Kap16b, Proposition 4.1] there is a pair of extended exceptional representations (Θ,Θ') of $\widetilde{H}$ such that $\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}\tau\otimes\omega)$ is a quotient of $\Theta\otimes\Theta'$.

When m is odd, let ω_τ be the central character of τ and $\eta=\omega_\tau^{-1}\omega^{-(m-1)/2}$. By [Yam17, Theorem 3.19(2)], $(\eta\circ\det)\tau$ is a quotient of $\theta\otimes\theta'$ and moreover $\eta^2=\omega$, then [Kap16b, Proposition 4.1] implies $\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}(\eta\circ\det)\tau\otimes 1)$ is a quotient of some $\Theta\otimes\Theta'$. Since

$$\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}(\eta\circ\det)\tau\otimes 1) = (\eta\circ\Upsilon)\,\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}\tau\otimes\eta^2),$$

$\mathrm{Ind}_{Q_m}^{H}(|\det|^{1/2}\tau\otimes\omega)$ is a quotient of (Θ'',Θ') with $\Theta''=(\eta^{-1}\circ\Upsilon)\Theta$.

Theorem 5.5. *For any Θ, Θ' and $m/2<l\leq m$, $J_{N_l,\psi_{l,\alpha}}(\Theta\otimes\Theta')=0$. In particular, $\sigma_{\psi_\alpha}(l,\tau,\omega)=0$ for $m/2<l\leq m$.*

Remark 5.6. *We do not assume anything on α (except that $\alpha\neq 0$). This is reasonable because Theorem 5.1 applies to any character of nonzero length.*

Proof. For $l=m$ this is [Kap16b, Theorem 1.1]. The framework of the proof for $l\leq m$ is similar, but the proof involves several new difficulties (e.g., see Lemma 5.7 below).

Recall the unipotent radical N_l of P_l, which we write in the form

$$N_l = \left\{ \begin{pmatrix} z & v & c \\ & I_{2(m-l)+1} & v' \\ & & z^* \end{pmatrix} : z \in N_{\mathrm{GL}_l} \right\}. \tag{5.4}$$

Let $C=C_{U_l}$. The group M_l acts on the characters of C with $\lfloor l/2 \rfloor$ orbits. We choose representatives for these orbits: $\psi_0(c)=1$ and

$$\psi_j(c) = \psi(\sum_{i=1}^{j} c_{l-2j+i,i}), \qquad 0<j\leq\lfloor l/2\rfloor.$$

Here c is regarded as an $l\times l$ matrix. Denote the stabilizer of ψ_j in Q_l by St_j,

$$\mathrm{St}_j = (\mathrm{GL}_{l-2j}\times\mathrm{Sp}_{2j}\times\mathrm{GSpin}_{2(m-l)+1})\ltimes V_{(l-2j,2j)})\ltimes U_l.$$

By virtue of the Geometric Lemma of Bernstein and Zelevinsky ([BZ77, Theorem 5.2] and [BZ76, 5.9–5.12]), as a representation of $\widetilde{Q}_l$, Θ is glued from

$$\mathrm{ind}_{\widetilde{\mathrm{St}}_j}^{\widetilde{Q}_l}(J_{C,\psi_j}(\Theta)), \qquad 0 \le j \le \lfloor l/2 \rfloor.$$

A similar result applies to Θ', where we use ψ^{-1} for the representatives of the orbits. In turn $\Theta \otimes \Theta'$ is glued from tensor products of such representations, with indices j, j'. According to [Kap17a, Lemma 2.3], when we apply the Jacquet functor with respect to N_l and $\psi_{l,\alpha}$ only those with $j = j'$ remain. Since the tensor product of two genuine representations is a non-genuine representation, we need to show that for all j,

$$J_{N_l,\psi_{l,\alpha}}(\mathrm{ind}_{\mathrm{St}_j}^{Q_l}(J_{C,\psi_j}(\Theta) \otimes J_{C,\psi_j^{-1}}(\Theta'))) = 0. \tag{5.5}$$

Since ψ_j is trivial on $U_{l-2j} \cap C$ (e.g., on C if $j = 0$), $J_{C,\psi_j}(\Theta)$ is a representation of $E = U_{l-2j}/(U_{l-2j} \cap C)$ and thereby of U_{l-2j} (which is the trivial group when $j = l/2$). We apply a filtration argument to $J_{C,\psi_j}(\Theta)$ as a U_{l-2j}-module. Since there are infinitely many orbits of characters of U_{l-2j} with respect to the action of St_j, we will carry out the argument in stages. Let $U^1_{l-2j} = U_{l-2j} \cap U_1$ and define inductively $U^i_{l-2j} = (U_{l-2j} \cap U_i)/(\prod_{k=1}^{i-1} U^k_{l-2j})$ for $1 < i \le l-2j$ ($U^i_{l-2j} \ltimes \prod_{k=1}^{i-1} U^k_{l-2j}$). Set $E^i = U^i_{l-2j}/(C \cap U^i_{l-2j})$, namely the i-th row of E.

Lemma 5.7. *The representation $J_{C,\psi_j}(\Theta)$ is glued from the representations*

$$J_{U_{l-2j}C,\lambda_k\psi_j}(\Theta), \qquad 0 \le k \le \min(m-l, l-2j),$$

where λ_k is a character of U_{l-2j}, $\lambda_0 = 1$ and for $k > 0$, $\lambda_k(u) = \prod_{i=1}^k \psi(u_{l-2j-i+1,l-i+1})$.

The proof is given below, after the proof of the theorem.

The lemma holds trivially when $j = l/2$; when $l = m$, the lemma (whose proof is now shorter) shows that U_{l-2j} acts trivially on $J_{U_{l-2j}C,\lambda_k\psi_j}(\Theta)$, simplifying the following arguments; the case $l < m$ involves zero length characters defined on $U_i \cap U_{l-2j}$.

If $0 < k < l-2j$, $V_{(l-2j-k,k)}$ is nontrivial. The group $V_{(l-2j-k,k)}$ normalizes U_{l-2j} and stabilizes λ_k, we prove that its action on $J_{U_{l-2j}C,\lambda_k\psi_j}(\Theta)$ is trivial. The set of characters of $V_{(l-2j-k,k)}$ is partitioned into finitely many orbits, under the action of $M_{(l-2j-k,k)}$ embedded in the stabilizer of $\lambda_k\psi_j$ via $\mathrm{diag}(x,y) \mapsto \iota(\mathrm{diag}(x,y,I_{l-k},y))$ ($x \in \mathrm{GL}_{l-2j-k}$, $y \in \mathrm{GL}_k$). Thus it suffices to show that for any nontrivial character μ of $V_{(l-2j-k,k)}$,

$$J_{V_{(l-2j-k,k)} \ltimes U_{l-2j},\mu\,\lambda_k\psi_j}(\Theta) = 0. \tag{5.6}$$

Indeed, as in [Kap16b, Claim 3.3] applying [GRS99a, Lemma 2.2] and another conjugation we see that $J_{V_{(l-2j-k,k)}U_{l-2j}C,\mu\,\lambda_k\psi_j}(\Theta)$ is a quotient of $J_{U_1,\nu}(\Theta)$, where $\nu(u) = \psi(u_1)$, and the action of the $(l-2j-k+1)$-th row of U_l given by the restriction of λ_k to this row transforms into a nontrivial action of U_2. This contradicts Corollary 5.3 unless (5.6) holds.

Let $\mathrm{St}_{j,k} = N_{M_{l-2j} \cap Q_l}(U_{l-2j}C, \lambda_k\psi_j)$. As a representation of $\widetilde{\mathrm{St}}_j$, $J_{C,\psi_j}(\Theta)$ is filtered by $\mathrm{ind}_{\widetilde{\mathrm{St}}_{j,k}}^{\widetilde{\mathrm{St}}_j} J_{U_{l-2j}C,\lambda_k\psi_j}(\Theta)$ with k varying as in Lemma 5.7. Applying the same argument to $J_{C,\psi_j^{-1}}(\Theta')$, the l.h.s. of (5.5) is filtered by $J_{N_l,\psi_{l,\alpha}} \mathrm{ind}_{\mathrm{St}_j}^{Q_l}(\Pi_{j,k,k'})$ where $0 \le k, k' \le \min(m-l, l-2j)$ and

$$\Pi_{j,k,k'} = \mathrm{ind}_{\widetilde{\mathrm{St}}_{j,k}}^{\widetilde{\mathrm{St}}_j} J_{U_{l-2j}C,\lambda_k\psi_j}(\Theta) \otimes \mathrm{ind}_{\widetilde{\mathrm{St}}_{j,k'}}^{\widetilde{\mathrm{St}}_j} J_{U_{l-2j}C,\lambda_{k'}^{-1}\psi_j^{-1}}(\Theta').$$

We prove $J_{N_l,\psi_{l,\alpha}} \operatorname{ind}_{\mathrm{St}_j}^{Q_l}(\Pi_{j,k,k'})$ vanishes by analyzing distributions on the orbits of the right action of N_l on the (infinite) homogenous space $\mathrm{St}_j\backslash Q_l$. By the Frobenius reciprocity, the space of distributions on the orbit $\mathrm{St}_j hN_l$ is isomorphic to

$$\mathscr{H}(h) = \mathrm{Hom}_{S_h}({}^{h^{-1}}\Pi_{j,k,k'}, \psi_{l,\alpha}).$$

Here $S_h = {}^{h^{-1}}\mathrm{St}_j \cap N_l$. We show $\mathscr{H}(h) = 0$ for all h, then $J_{N_l,\psi_{l,\alpha}} \operatorname{ind}_{\mathrm{St}_j}^{Q_l}(\Pi_{j,k,k'}) = 0$ by [BZ76, Theorem 6.9] concluding the proof of (5.5).

Let $Q = Q_l \cap Q_{l-2j}$. Since $\mathrm{St}_j\backslash Q = \mathrm{Sp}_{2j}\backslash \mathrm{GL}_{2j}$, and $Q\backslash Q_l/N_l = P_{(l-2j,2j)}\backslash \mathrm{GL}_l / N_{\mathrm{GL}_l}$ can be parameterized using the Weyl group of GL_l, we can assume $h = tw$, where $t \in \mathrm{GL}_{2j}$ is a representative of $\mathrm{Sp}_{2j}\backslash \mathrm{GL}_{2j}$ and w is a permutation matrix in GL_l.

We note that one can take $t \in \mathcal{P}_{2j}$. To see this consider $b = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \in \mathrm{GL}_{2j}$ where b_1 is a block of size $(2j-1)\times(2j-1)$. If $b_1 \in \mathrm{GL}_{2j-1}$, we can find $x_1 \in \mathrm{Sp}_{2j}$ such that $x_1 b \in P_{(2j-1,1)}$ (the last row of x_1 will be $\left(\begin{smallmatrix} -b_3b_1^{-1} & 1 \end{smallmatrix}\right)$), then take a torus element $x_2 \in \mathrm{Sp}_{2j}$ for which $x_2x_1b \in \mathcal{P}_{2j}$. If $b_1 \notin \mathrm{GL}_{2j-1}$, then because the first $2j-1$ columns of b contain a nonzero minor of order $2j-1$, we can find a permutation matrix $x_0 \in \mathrm{Sp}_{2j}$ such that $x_0 b = \left(\begin{smallmatrix} b_1' & b_2' \\ b_3' & b_4' \end{smallmatrix}\right)$ and $b_1' \in \mathrm{GL}_{2j-1}$.

The action of U_{l-2j} on the space of $\Pi_{j,k,k'}$ is given by $\lambda_{j,k,k'} = \lambda_k\lambda_{k'}^{-1}$. Note that GL_{2j} stabilizes $\psi_{j,k,k'}$.

First we claim $\mathscr{H}(h) = 0$ unless $\mathrm{St}_j hN_l = \mathrm{St}_j twN_l$ for $w = \left(\begin{smallmatrix} & I_{l-2j} \\ I_{2j} & \end{smallmatrix}\right)$. This is trivial when $j = 0$ or $l = 2j$, assume $0 < j < l/2$. If w is not of this form, then because $\psi_{l,\alpha}|_{N_{\mathrm{GL}_l}} = \psi_{N_{\mathrm{GL}_l}}$, $\psi_{l,\alpha}$ and ${}^h\lambda_{j,k,k'}$ do not coincide on $N_{\mathrm{GL}_l} \cap {}^{h^{-1}}U_{l-2j}$ ($N_l = N_{\mathrm{GL}_l} \ltimes U_l$).

Assume w takes this form and $0 \le j < l/2$. Since $\psi_{l,\alpha}$ is trivial on the first $l-1$ rows of U_l/C and restricts to a character of nonzero length on the last row, now $\psi_{l,\alpha}$ and ${}^h\lambda_{j,k,k'}$ do not agree on $U_l \cap {}^{h^{-1}}U_{l-2j}$ unless $k = k' > 0$. Note that the case $k + k' = 1$ is ruled out because the lengths of both $\lambda_k|_{E^{l-2j}}$ and $\lambda_{k'}|_{E^{l-2j}}$ are zero.

We can thus assume $k = k' > 0$, in which case $\lambda_{j,k,k'} = \lambda_{j,k,k} = 1$. Consider $k < l-2j$. Because $V_{(l-2j-k,k)} < \mathrm{St}_j$ and ${}^{h^{-1}}V_{(l-2j-k,k)} < N_l$, we have ${}^{h^{-1}}V_{(l-2j-k,k)} < S_h$ then by (5.6), ${}^{h^{-1}}\Pi_{j,k,k} = J_{{}^{h^{-1}}V_{(l-2j-k,k)}}({}^{h^{-1}}\Pi_{j,k,k})$. Since $\psi_{l,\alpha}|_{{}^{h^{-1}}V_{(l-2j-k,k)}} \ne 1$ (${}^{h^{-1}}V_{(l-2j-k,k)} = {}^{w^{-1}}V_{(l-2j-k,k)}$), again $\mathscr{H}(h) = 0$ unless $V_{(l-2j-k,k)}$ is trivial which is only possible if $k = l-2j$ (because $k > 0$). In particular if $j = 0$, our assumption $m/2 < l$ implies $m - l < l = l - 2j$ whence $k < l - 2j$, so that $\mathscr{H}(h) = 0$ for all k in this case.

It remains to consider $0 < j \le \lfloor l/2 \rfloor$ and $k = l - 2j$. Since U_{l-2j} is a normal subgroup of St_j which acts trivially on $\Pi_j = \Pi_{j,l-2j,l-2j}$, $\Pi_j = J_{U_{l-2j}}(\Pi_j)$ then by [Kap17a, Lemma 2.3],

$$\Pi_j = \operatorname{ind}_{\mathrm{St}_{j,l-2j}}^{\mathrm{St}_j}(\Xi_j(\Theta,\Theta')), \qquad \Xi_j(\Theta,\Theta') = J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta) \otimes J_{U_{l-2j}C,\lambda_{l-2j}^{-1}\psi_j^{-1}}(\Theta').$$

Note that ${}^{h^{-1}}\Pi_j = \operatorname{ind}_{{}^{h^{-1}}\mathrm{St}_{j,l-2j}}^{{}^{h^{-1}}\mathrm{St}_j}({}^{h^{-1}}\Xi_j(\Theta,\Theta'))$.

To describe $\mathrm{St}_{j,l-2j}$ we introduce the following notation. For $0 \le a \le m$, identify GSpin_{2a+1} with the natural direct factor of M_{m-a}, and for $a \ge l$ let $Q_l^a = M_l^a \ltimes U_l^a$ denote the standard maximal parabolic subgroup of GSpin_{2a+1} with $M_l^a = \mathrm{GL}_l \times \mathrm{GSpin}_{2(a-l)+1}$. Also let $\mathrm{GL}_{l-2j}^{\triangle}$ denote the embedding of GL_{l-2j} in M_{2l-2j} given by $b \mapsto \iota(\mathrm{diag}(b, I_{2j}, b))$. With this notation

$$\mathrm{St}_{j,l-2j} = ((\mathrm{GL}_{l-2j}^{\triangle} \times \mathrm{Sp}_{2j} \times \mathrm{GSpin}_{2(m-2l+2j)+1}) \ltimes U_l^{m-l+2j}) \ltimes U_{l-2j}.$$

In order to prove $\mathscr{H}(h)=0$ we study distributions on the orbits of the right action of S_h on ${}^{h^{-1}}\mathrm{St}_{j,l-2j}\backslash{}^{h^{-1}}\mathrm{St}_j$. By the Frobenius reciprocity, the space of distributions on $({}^{h^{-1}}\mathrm{St}_{j,l-2j})({}^{h^{-1}}g)S_h$, where $g\in\mathrm{St}_j$, is isomorphic to

$$\mathscr{H}(h,g)=\mathrm{Hom}_{S_{h,g}}({}^{(gh)^{-1}}\Xi_j(\Theta,\Theta'),\psi_{l,\alpha}).$$

Here $S_{h,g}={}^{(gh)^{-1}}\mathrm{St}_{j,l-2j}\cap S_h$.

To parameterize the representatives ${}^{h^{-1}}g$ ($g\in\mathrm{St}_j$), note that ${}^{h^{-1}}\iota(\mathrm{diag}(I_{l-2j},V_{(2j,l-2j)}))$ is contained in S_h because $U_l<\mathrm{St}_j\cap N_l$ is normalized by h. Thus we can assume $g=\iota(\mathrm{diag}(I_l,d))g_0$ where $d\in\mathrm{GL}_{l-2j}$ and g_0 is a representative of $Q_{l-2j}^{m-l}\backslash\mathrm{GSpin}_{2(m-l)+1}$.

Let $\mathcal{H}={}^{\iota(\mathrm{diag}(I_{l-2j},w^{-1}))}(U_{2j}^{m-2l+4j})$ which is the generalized Heisenberg group of rank $r+1=2j(2(m-2l+2j)+1)+1$ (see § 5.3). Since $\mathcal{H}<\mathrm{St}_j\cap\mathrm{St}_{j,l-2j}$, $J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta)$ is a representation of $\mathcal{H}$. In addition $\mathrm{Sp}_{2j}<\mathrm{St}_j\cap\mathrm{St}_{j,l-2j}$, thus $J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta)$ is a representation of $\widetilde{\mathrm{Sp}}_{2j}\ltimes\mathcal{H}$.

Lemma 5.8. *As a representation of $\widetilde{\mathrm{Sp}}_{2j}\ltimes\mathcal{H}$, $J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta)$ is isomorphic to a (possibly infinite) direct sum of copies of the representation ω_ψ.*

We proceed to prove $\mathscr{H}(h)=0$. According to Lemma 5.8, as a representation of $\mathrm{Sp}_{2j}\ltimes\mathcal{H}$ the representation $\Xi_j(\Theta,\Theta')$ is a direct sum of representations $\omega_\psi\otimes\omega_{\psi^{-1}}$.

Since g and h normalize U_l, and $\mathcal{H}<U_l<\mathrm{St}_j$, ${}^{(gh)^{-1}}\mathcal{H}<{}^{h^{-1}}\mathrm{St}_j\cap N_l=S_h$. Also ${}^{(gh)^{-1}}\mathcal{H}<{}^{(gh)^{-1}}\mathrm{St}_{j,l-2j}$, because $\mathcal{H}<\mathrm{St}_{j,l-2j}$. Therefore ${}^{(gh)^{-1}}\mathcal{H}<S_{h,g}$. Note that $\psi_{l,\alpha}|_{{}^{(gh)^{-1}}\mathcal{H}}$ is nontrivial if and only if $l=2j$ (because of the conjugation by w^{-1}, note that $j>0$ and that w is trivial when $l=2j$). By the definitions, any morphism in $\mathscr{H}(h,g)$ factors through

$$J_{{}^{(gh)^{-1}}\mathcal{H},\psi_{l,\alpha}}({}^{(gh)^{-1}}\Xi_j(\Theta,\Theta'))\cong{}^{(gh)^{-1}}J_{\mathcal{H},{}^{(gh)}\psi_{l,\alpha}}(\Xi_j(\Theta,\Theta')).$$

These are representations of ${}^{(gh)^{-1}}\mathcal{H}$ and of the stabilizer of $\psi_{l,\alpha}|_{{}^{(gh)^{-1}}\mathcal{H}}$ in ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}$. This stabilizer is ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}$ unless $l<2j$, in which case it is ${}^{(gh)^{-1}}(\mathrm{Sp}_l\cap\mathcal{P}_l)$ because $\psi_{l,\alpha}$ is nontrivial only on the last row of $C_{\mathcal{H}}\backslash\mathcal{H}$, and the action of g on the characters of each row of $C_{\mathcal{H}}\backslash\mathcal{H}$ carries nontrivial characters to nontrivial ones as well as preserves their length.

Since g commutes with $\iota(\mathrm{GL}_l)$, ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}={}^{h^{-1}}\mathrm{Sp}_{2j}=\iota({}^{t^{-1}}\mathrm{Sp}_{2j})$ (recall $h=tw$). Hence ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}<{}^{h^{-1}}\mathrm{St}_j$, and clearly ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}<{}^{(gh)^{-1}}\mathrm{St}_{j,l-2j}$. Identify $N_{\mathrm{GL}_{2j}}$ with $\iota(\mathrm{diag}(I_{l-2j},N_{\mathrm{GL}_{2j}}))$, ${}^{w^{-1}}N_{\mathrm{GL}_{2j}}=\iota(N_{\mathrm{GL}_{2j}})$. Put $\mathrm{Sp}_{2j}^t={}^{t^{-1}}\mathrm{Sp}_{2j}\cap N_{\mathrm{GL}_{2j}}$. Then ${}^{(gh)^{-1}}\mathrm{Sp}_{2j}\cap{}^{w^{-1}}N_{\mathrm{GL}_{2j}}=\iota(\mathrm{Sp}_{2j}^t)$. Also $\iota(\mathrm{Sp}_{2j}^t)<\iota(N_{\mathrm{GL}_{2j}})<N_l$. Thus $\iota(\mathrm{Sp}_{2j}^t)<S_{h,g}$. Moreover $\iota(\mathrm{Sp}_{2j}^t)$ belongs to the stabilizer of $\psi_{l,\alpha}|_{{}^{(gh)^{-1}}\mathcal{H}}$: This is clear for $l<2j$, and holds when $l=2j$ because as explained above we can take $t\in\mathcal{P}_l$, then $\iota(\mathrm{Sp}_l^t)={}^{t^{-1}}(\mathrm{Sp}_l\cap{}^tN_{\mathrm{GL}_l})<{}^{(gh)^{-1}}(\mathrm{Sp}_l\cap\mathcal{P}_l)$.

Therefore it suffices to show

$$\mathrm{Hom}_{\iota(\mathrm{Sp}_{2j}^t)}({}^{(gh)^{-1}}J_{\mathcal{H},{}^{(gh)}\psi_{l,\alpha}}(\omega_\psi\otimes\omega_{\psi^{-1}}),\psi_{l,\alpha})=0. \tag{5.7}$$

On the one hand, by [Kap16b, Claim 2.5] if ν is any character of $C_{\mathcal{H}}\backslash\mathcal{H}$ (trivial or not), $J_{\mathcal{H},\nu}(\omega_\psi\otimes\omega_{\psi^{-1}})$ is the trivial one-dimensional representation of the stabilizer of ν in Sp_r (note that Sp_r acts transitively on the nontrivial characters of $C_{\mathcal{H}}\backslash\mathcal{H}$). Thus ${}^{(gh)^{-1}}J_{\mathcal{H},{}^{(gh)}\psi_{l,\alpha}}(\omega_\psi\otimes\omega_{\psi^{-1}})$ is trivial. On the other hand by Offen and Sayag [OS08, Proposition 2], for any generic character ψ of $N_{\mathrm{GL}_{2j}}$, $\psi|_{\mathrm{Sp}_{2j}^t}\neq1$ for all $t\in\mathrm{GL}_{2j}$ (use

$\mathcal{H}^{r,r'}$ with $r = 0$ and $r' = 2j$, in their notation). Since $\psi_{l,\alpha}|_{\iota(N_{\mathrm{GL}_{2j}})}$ is generic we conclude (5.7) whence $\mathscr{H}(h,g) = 0$ and thereby $\mathscr{H}(h) = 0$, in all cases. □

Proof of Lemma 5.7. We apply a filtration argument to $J_{C,\psi_j}(\Theta)$ according to the orbits of characters of E^{l-2j} with respect to the action of St_j. If μ is such a character and is nontrivial, we can assume it takes the form

$$b \mapsto \psi(\epsilon b_1 + (1-\epsilon)b_{2j+1} + (1-\epsilon)\beta b_{2(m-l+j)+1}), \qquad b \in F^{2(m-l+j)+1} \cong E^{l-2j},$$

$\epsilon \in \{0, 1\}$ and $\beta \in F$ is either 0 or a representative of a coset of $(F^*)^2\backslash F^*$. In particular the number of orbits is finite. Here if $j = 0$, we can already take $\epsilon = 0$.

If $\epsilon = 0$ and $\beta \neq 0$, we can use a conjugation of E^{l-2j} such that $J_{E^{l-2j},\mu}(J_{C,\psi_j}(\Theta))$ becomes a quotient of $J_{V,\lambda}(\Theta)$, where λ is a character of $V < U_1$ and any extension of λ to a character of U_1 is of nonzero length. But by Corollary 5.2 (or directly Theorem 5.1 when $l - 2j = 1$) with $V\backslash U_1 = V_{(1,l-2j-1)}$ and the action of $\mathrm{GL}_{l-2j-1} < P_{(1,l-2j-1)}$ on the characters of $V_{(1,l-2j-1)}$, under which there are only 2 orbits, the Jacquet module vanishes. Hence $\beta = 0$ in all cases (for all ϵ).

If $\epsilon = 1$, we argue as in [Kap16b, pp. 924–925]: Consider another filtration, now of $J_{E^{l-2j},\mu}(J_{C,\psi_j}(\Theta))$ along $V_{(l-2j,1)} \cap U_{l-2j-1} \cong F^{l-2j-1}$. Using the action of the transpose of $\mathcal{P}_{l-2j}$ we see there is only one orbit of characters to consider, the trivial (because μ is nontrivial on b_1). Applying [GRS99a, Lemma 2.2] we can replace $V_{(l-2j,1)} \cap U_{l-2j-1}$ with $V^-_{(l-2j-1,1)}$, and we also conjugate by o_1 where

$$o_i = \iota(\mathrm{diag}(1, \begin{pmatrix} & I_{l-2j-2} & 1 \\ 1 & & \end{pmatrix}))\iota(\begin{pmatrix} & I_{l-2j-i} & 1 \end{pmatrix}), \qquad 1 \leq i \leq l-2j.$$

Note that if $l - 2j = 1$, $V^-_{(l-2j-1,1)}$ is trivial and the conjugation is not needed. Then $J_{V^-_{(l-2j-1,1)}}(J_{E^{l-2j},\mu}(J_{C,\psi_j}(\Theta)))$ becomes a quotient of $J_{U_1,\lambda}(\Theta)$, where $\lambda(u) = \psi(u_1)$, and the action of the top left coordinate of C under ψ_j becomes a nontrivial action of U_2, contradicting Corollary 5.3 unless $J_{E^{l-2j},\mu}(J_{C,\psi_j}(\Theta)) = 0$. Thus we can assume $\epsilon = 0$.

We deduce $J_{C,\psi_j}(\Theta)$ is filtered by $J_{E^{l-2j}}(J_{C,\psi_j}(\Theta))$ and $J_{E^{l-2j},\lambda}(J_{C,\psi_j}(\Theta))$, where $\lambda(b) = \psi(b_{2j+1})$. To unify the notation let λ_i denote the character of E which is trivial on $E^{i'}$ for $i' \neq i$, and is given by $b \mapsto \psi(b_{l-i+1})$ on E^i $(1 \leq i \leq l-2j)$. Then $J_{C,\psi_j}(\Theta)$ is filtered by $J_{E^{l-2j},\lambda_{l-2j}^{d_{l-2j}}}(J_{C,\psi_j}(\Theta))$ with $d_{l-2j} \in \{0, 1\}$.

We proceed with E^{l-2j-1} and consider $J_{E^{l-2j},\lambda_{l-2j}^{d_{l-2j}}}(J_{C,\psi_j}(\Theta))$. Assume $d_{l-2j} = 0$. The above parametrization of μ applies here to the nontrivial characters of E^{l-2j-1}. Then for $\epsilon = 0$ we argue exactly as before to deduce $\beta = 0$.

If $\epsilon = 1$, we argue as above with $V_{(l-2j,1)} \cap U_{l-2j-2} \cong F^{l-2j-2}$, and using [GRS99a, Lemma 2.2] we replace $V_{(l-2j,1)} \cap U_{l-2j-2}$ with $V^-_{(l-2j-2,1)}$. Since the action of the leftmost coordinate of E^{l-2j} on $J_{E^{l-2j}}(J_{C,\psi_j}(\Theta))$ is now trivial and $\epsilon = 1$, we apply [GRS99a, Lemma 2.2] once more to replace this coordinate with $\mathrm{diag}(I_{l-2j-2}, V_{(1,1)})$, then we conjugate by o_2 to deduce $J_{V^-_{(l-2j-1,1)}}(J_{E^{l-2j-1}E^{l-2j},\mu}(J_{C,\psi_j}(\Theta)))$ is a quotient of $J_{U_1,\lambda}(\Theta)$ on which U_2 acts nontrivially, because of the conjugation of the top left coordinate of C. Hence $J_{E^{l-2j-1}E^{l-2j},\mu}(J_{C,\psi_j}(\Theta)) = 0$ by Corollary 5.3. Thus $J_{E^{l-2j}}(J_{C,\psi_j}(\Theta))$ is filtered by $J_{E^{l-2j-1}E^{l-2j},\lambda_{l-2j-1}^{d_{l-2j-1}}}(J_{C,\psi_j}(\Theta))$ with $d_{l-2j-1} \in \{0, 1\}$.

For the case $d_{l-2j} = 1$, the nontrivial characters μ of E^{l-2j-1} under the action of the stabilizer of λ_{l-2j} inside St_j take the form

$$b \mapsto \psi(\epsilon b_1 + (1-\epsilon)b_{2j+2} + (1-\epsilon)\beta b_{2(m-l+j)} + \epsilon_1 b_{2(m-l+j)+1}),$$

where ϵ and β are as above (if $j = 0$, $\epsilon = 0$) and $\epsilon_1 \in \{0, 1\}$.

If $\epsilon = 0$ and $\beta \neq 0$, again $J_{E^{l-2j-1}E^{l-2j},\mu\lambda_{l-2j}}(J_{C,\psi_j}(\Theta)) = 0$ by Corollary 5.2: A conjugation of μ will take it to a character of a subgroup of U_1 whose extensions to U_1 all have nonzero lengths. Hence we can assume $\beta = 0$ (for all ϵ).

If $\epsilon_1 = 1$, we can conjugate by $\iota(\mathrm{diag}(I_{l-2j-2}, \left(\begin{smallmatrix}1 & \\ 1 & 1\end{smallmatrix}\right)))$ to assume μ is nontrivial on b_{2j+1}, and again apply Corollary 5.2 to deduce the Jacquet module vanishes. Thus $\epsilon_1 = 0$ in all cases.

Now for $\epsilon = 1$ we argue as in the case $d_{l-2j} = 0$ and $\epsilon = 1$, applying [GRS99a, Lemma 2.2] twice and deducing the vanishing by Corollary 5.3. Note that λ_{l-2j} does not interfere with the argument and we still use the top left coordinate of C.

We deduce $J_{E^{l-2j},\lambda_{l-2j}^{d_{l-2j}}}(J_{C,\psi_j}(\Theta))$ is filtered by $J_{E^{l-2j-1}E^{l-2j},\lambda_{l-2j-1}^{d_{l-2j-1}}\lambda_{l-2j}^{d_{l-2j}}}(J_{C,\psi_j}(\Theta))$, where $d_{l-2j-1} \in \{0, 1\}$.

For E_{l-2j-2} the reasoning is similar. Briefly, the nontrivial characters μ of E_{l-2j-2} take the form

$$b \mapsto \psi(\epsilon b_1 + (1-\epsilon)b_{2j+3} + (1-\epsilon)\beta b_{2(m-l+j)-1} + \epsilon_2 b_{2(m-l+j)} + \epsilon_1 b_{2(m-l+j)+1}).$$

Here when $d_{l-2j} = 0$, we can already take $\epsilon_1 = 0$ and if $d_{l-2j-1} = 0$, $\epsilon_2 = 0$. Corollary 5.2 implies that for $\epsilon = 0$, one must have $\beta = 0$. If $\epsilon_1 = 1$ or $\epsilon_2 = 1$, we use a conjugation by either $\iota(\mathrm{diag}(I_{l-2j-3}, \begin{pmatrix}1 & & \\ & 1 & \\ 1 & & 1\end{pmatrix}))$ or $\iota(\mathrm{diag}(I_{l-2j-3}, \left(\begin{smallmatrix}1 & \\ 1 & 1\end{smallmatrix}\right)))$, then apply Corollary 5.2 to deduce the vanishing. Thus $\epsilon_1 = \epsilon_2 = 0$. For the case $\epsilon = 1$ the auxiliary unipotent subgroups are $V_{(l-2j,1)} \cap U_{l-2j-3} \cong F^{l-2j-3}$, $V^-_{(l-2j-3,1)}$ and $\mathrm{diag}(I_{l-2j-3}, V_{(1,2)})$, one conjugates by o_3 then applies Corollary 5.3.

Repeating these arguments for $i = l-2j-3, \ldots, 1$ we obtain a filtration of $J_{C,\psi_j}(\Theta)$ as a representation of E by the modules $J_{E,\lambda_d}(J_{C,\psi_j}(\Theta)) = J_{U_{l-2j}C,\lambda_d\psi_j}(\Theta)$, where d varies over $\{0, 1\}^{l-2j}$ and $\lambda_d = \prod_{i=1}^{l-2j} \lambda_i^{d_i}$. Moreover $|\{i : d_i = 1\}| \leq m - l$, otherwise λ_d can be conjugated into a character whose restriction to some E^i is $b \mapsto \psi(b_{2j+m-l+1})$, then $J_{U_{l-2j}C,\lambda_d\psi_j}(\Theta) = 0$ by Corollary 5.2.

The statement of the lemma now follows using another conjugation, to arrange the characters λ_d in the requested form. □

Proof of Lemma 5.8. The first step is to show $J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta)$ is isomorphic to a Jacquet module of Θ which factors through U_{2l-4j}. We can assume $j < l/2$, otherwise this is trivially true. Using conjugations by elements in $N_{Q_{l-2j}}(U_{l-2j}C, \psi_j)$, we can replace λ_{l-2j} with the character $u \mapsto \prod_{i=1}^{l-2j} \psi(u_{i,2j+i})$ (of E) which we re-denote by λ_{l-2j}. Write a general element $u \in U_{l-2j}C$ in the form

$$\begin{pmatrix}
1 & & * & x_1 & * & * & * & * & * & * & * \\
 & I_{l-2j-1} & * & y & x_2 & * & * & * & * & * & * \\
 & & I_{2j} & & & & & & c^j & * & * \\
 & & & 1 & & & & & & * & * \\
 & & & & I_{l-2j-1} & & & & & * & * \\
 & & & & & I_{2(m-2l+2j)+1} & & & & * & * \\
 & & & & & & I_{l-2j-1} & & & x'_2 & * \\
 & & & & & & & 1 & & y' & x'_1 \\
 & & & & & & & & I_{2j} & * & * \\
 & & & & & & & & & I_{l-2j-1} & \\
 & & & & & & & & & & 1
\end{pmatrix}.$$

Here the coordinates marked by asterisks are either arbitrary or already determined by the other coordinates and the quadratic form defining SO_{2m+1}, and similarly x_i' and y' are uniquely determined by x_i and y and the form. With this notation $\lambda_{l-2j}(u) = \psi(x_1 + \sum_{i=1}^{l-2j-1}(x_2)_{i,i})$ and $\psi_j(u) = \psi(\sum_{i=1}^{j}(c^j)_{i,i})$. Also note that if $l-2j-1=0$, the rows and columns in the matrix above corresponding to y and x_2 are omitted. Conjugating u by $o = \iota(\mathrm{diag}(1, \left(\begin{smallmatrix} & 1 \\ I_{l-1} & \end{smallmatrix}\right)))$, we obtain

$$\begin{pmatrix} 1 & x_1 & & * & * & * & * & * & * & * & * \\ & 1 & & & & & & & * & & * \\ & y & I_{l-2j-1} & * & x_2 & * & * & * & * & * & * \\ & & & I_{2j} & & & & c^j & * & & * \\ & & & & I_{l-2j-1} & & & & * & & * \\ & & & & & I_{2(m-2l+2j)+1} & & & * & & * \\ & & & & & & I_{l-2j-1} & & x_2' & & * \\ & & & & & & & I_{2j} & * & & * \\ & & & & & & & & I_{l-2j-1} & & \\ & & & & & & & & y' & 1 & x_1' \\ & & & & & & & & & & 1 \end{pmatrix}.$$

Apply [GRS99a, Lemma 2.2] to exchange y with the missing roots of U_1 (to the right of x_1). Then ${}^{o}J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta) \cong J_{U,\lambda_{l-2j}\psi_j}(\Theta)$ where U is the subgroup of elements

$$\begin{pmatrix} 1 & x_1 & * & * & * & * & * & * & * & * & * \\ & 1 & & & & & & & * & & * \\ & & I_{l-2j-1} & * & x_2 & * & * & * & * & * & * \\ & & & I_{2j} & & & & c^j & * & & * \\ & & & & I_{l-2j-1} & & & & * & & * \\ & & & & & I_{2(m-2l+2j)+1} & & & * & & * \\ & & & & & & I_{l-2j-1} & & x_2' & & * \\ & & & & & & & I_{2j} & * & & * \\ & & & & & & & & I_{l-2j-1} & & * \\ & & & & & & & & & 1 & x_1' \\ & & & & & & & & & & 1 \end{pmatrix}.$$

Note that λ_{l-2j} and ψ_j (defined as above) are also characters of U. Since $J_{U,\lambda_{l-2j}\psi_j}(\Theta)$ factors through $J_{U_1,\nu}(\Theta)$ where $\nu(u) = \psi(u_1)$, and UU_2 is a group, by Corollary 5.3 $J_{U,\lambda_{l-2j}\psi_j}(\Theta) = J_{UU_2,\lambda_{l-2j}\psi_j}(\Theta)$ with $\lambda_{l-2j}\psi_j$ extended to UU_2 trivially on U_2.

Looking at the inner $(2(m-2)+1)\times(2(m-2)+1)$ block of UU_2 we see the subgroup $U_{l-1-2j}^{m-2}C_{U_{l-1}^{m-2}}$, $\lambda_{l-2j}|_{U_{l-1-2j}^{m-2}} = \lambda_{l-1-2j}$ (the character λ_{l-1-2j} of U_{l-1-2j}^{m-2}) and $\psi_j|_{C_{U_{l-1}^{m-2}}}$ which is given by the restriction of ψ_j to c^j is a generic character of $C_{U_{l-1}^{m-2}}$, in fact the generic character of the form chosen after (5.4). We are thus in a position to apply the above procedure again, to the inner $(2(m-2)+1)\times(2(m-2)+1)$ block, with (l,j,m) replaced by $(l-1,j,m-2)$.

Repeating this $l-2j$ times we deduce $J_{U_{l-2j}C,\lambda_{l-2j}\psi_j}(\Theta) \cong J_{N_{2l-4j}\rtimes C^j,\psi^\circ\psi_j}(\Theta)$, where $C^j = C_{U_{2j}^{m-2l+4j}}$, ψ_j is the generic character of C^j and ψ° is the character of $N_{\mathrm{GL}_{2l-4j}}$ defined by $\psi^\circ(z) = \psi(\sum_{i=1}^{l-2j} z_{2i-1,2i})$, i.e., the character of the so called semi-Whittaker functional (see [BG92, Tak14]). This Jacquet module factors through $J_{U_{2l-4j}}$ as promised.

Next, by [Kap17b, Proposition 2.19] and [Kap17b, Lemma 2.13] and because $J_{U_{2l-4j}}(\Theta)$ is irreducible (see [Kap17b, p. 641]),

$$J_{U_{2l-4j}}(\Theta) = \mathrm{Ind}_{\widetilde{\mathrm{GL}}_{2l-4j}^{(2)} \times \widetilde{\mathrm{GSpin}}_{2(m-2l+4j)+1}}^{\widetilde{M}_{2l-4j}}(\theta^{(2)} \otimes \Theta).$$

Here $\mathrm{GL}_{2l-4j}^{(2)} = \{g \in \mathrm{GL}_{2l-4j} : \det g \in (F^*)^2\}$; θ is an extended exceptional representation of $\widetilde{\mathrm{GL}}_{l-2j}$ and $\theta^{(2)}$ denotes its restriction to $\mathrm{GL}_{2l-4j}^{(2)}$; and Θ is an extended exceptional representation of $\widetilde{\mathrm{GSpin}}_{2(m-2l+4j)+1}$. Note that by [Kap16b, (2.1)], $\mathrm{GL}_{2l-4j}^{(2)}$

and $\mathrm{GSpin}_{2(m-21+4j)+1}$ commute in $\widetilde{M}_{2l-4j}$ whence the tensor product here is the standard one.

We compute $J_{N_{2l-4j}C^j,\psi^\circ\psi_j}(\Theta)$ using [BZ77, Theorem 5.2]. Let $\mathrm{St}'_j=\mathrm{St}_j^{m-2l+4j}$ denote the stabilizer of ψ_j in $Q_{2j}^{m-2l+4j}$ ($\mathrm{St}'_j=\mathrm{Sp}_{2j}\ltimes U_{2j}^{m-2l+4j}$). Because the space

$$\mathrm{GL}_{2l-4j}^{(2)}\,\mathrm{GSpin}_{2(m-2l+4j)+1}\backslash M_{2l-4j}/\,\mathrm{GL}_{2l-4j}\,\mathrm{St}'_j$$

is trivial,

$$J_{U_{2l-4j}C^j,\psi_j}(\Theta)=J_{C^j,\psi_j}(J_{U_{2l-4j}}(\Theta))=\mathrm{Ind}_{\widetilde{\mathrm{GL}}_{l-2j}^{(2)}\times\widetilde{\mathrm{St}}'_j}^{\widetilde{\mathrm{GL}_{l-2j}\,\mathrm{St}'_j}}(\theta^{(2)}\otimes J_{C^j,\psi_j}(\Theta)).$$

Since $\mathrm{GL}_{2l-4j}^{(2)}\backslash\mathrm{GL}_{2l-4j}$ is a finite abelian group, $J_{U_{2l-4j}C^j,\psi_j}(\Theta)|_{\widetilde{\mathrm{GL}}_{2l-4j}^{(2)}\times\widetilde{\mathrm{St}}'_j}$ is isomorphic to $\bigoplus_a(\theta^{(2)}\otimes\varrho_aJ_{C^j,\psi_j}(\Theta))$, where ϱ_a is the character of $\mathrm{GSpin}_{2(m-2l+2j)+1}$ given by $\varrho_a(g)=(\Upsilon(g),a)_2$, $(,)_2$ is the quadratic Hilbert symbol and a varies over $(F^*)^2\backslash F^*$ (see [Kap16b, (2.1)] and [Kap17b, p. 634]).

Now $J_{N_{2l-4j}C^j,\psi^\circ\psi_j}(\Theta)=J_{N_{\mathrm{GL}_{2l-4j}},\psi^\circ}J_{U_{2l-4j}C^j,\psi_j}(\Theta)$ and

$$\begin{aligned}J_{N_{\mathrm{GL}_{2l-4j}},\psi^\circ}(\theta^{(2)}\otimes\varrho_aJ_{C^j,\psi_j}(\Theta))&=J_{N_{\mathrm{GL}_{2l-4j}},\psi^\circ}(\theta^{(2)})\otimes\varrho_aJ_{C^j,\psi_j}(\Theta)\\&=J_{N_{\mathrm{GL}_{2l-4j}},\psi^\circ}(\theta)\otimes\varrho_aJ_{C^j,\psi_j}(\Theta).\end{aligned}$$

By [Tak14, Proposition 2.51], $\dim J_{N_{\mathrm{GL}_{2l-4j}},\psi^\circ}(\theta)=1$. Finally, under the above conjugations $\mathrm{Sp}_{2j}\ltimes\mathcal{H}$ is bijected into St'_j, then the result follows from Theorem 5.4 applied to $\mathrm{GSpin}_{2(m-2l+4j)+1}$ and $J_{C^j,\psi_j}(\Theta)$. □

6. The non-vanishing results

We prove a non-vanishing result for the descent map (see Theorem 6.5 below).

6.1. **Generalized and degenerate Whittaker models.** We recall the generalized and degenerate Whittaker models attached to nilpotent orbits, following the formulation of [GGS17]. Let A be an algebraic reductive group (defined over F, $A=A(F)$, see § 2), $\mathfrak{a}$ denote the Lie algebra of A and κ denote the Killing form on $\mathfrak{a}$. For $a\in A$, let A_a denote the centralizer of a in A, and similarly $\mathfrak{a}_x$ denotes the centralizer of $x\in\mathfrak{a}$ in $\mathfrak{a}$.

Any nilpotent $u\in\mathfrak{a}$ defines a function $\psi_u:\mathfrak{a}\to\mathbb{C}^*$ by $\psi_u(x)=\psi(\kappa(u,x))$. In this sense we regard u as an element of $\mathfrak{a}^*$.

If $y\in\mathfrak{a}$ is semisimple, $\mathfrak{a}$ decomposes (under the adjoint action) into a direct sum of eigenspaces $\mathfrak{a}^y_{\lambda_i}$ of y corresponding to eigenvalues λ_i. The element y is called rational semisimple if the eigenvalues of $\mathrm{ad}y$ are all rational. In this case define for $r\in\mathbb{Q}$, $\mathfrak{a}^y_{\geq r}=\oplus_{\lambda_i\geq r}\mathfrak{a}^y_{\lambda_i}$ and $\mathfrak{u}_y=\mathfrak{a}^y_{\geq 1}$.

A Whittaker pair is a pair (y,u) where y is a rational semisimple element and $u\in\mathfrak{a}^y_{-2}$. If (y,u) is a Whittaker pair, we say that y is a neutral element for u if there exists $v\in\mathfrak{a}^y_2$ such that (v,y,u) is an $\mathfrak{sl}_2$-triple. In this case we also call (y,u) a neutral pair.

For a Whittaker pair (y,u), $b_u(X,Y)=\kappa(u,[X,Y])$ is an anti-symmetric form on $\mathfrak{a}$. Let $\mathfrak{n}_{y,u}$ be the radical of the restriction of b_u to $\mathfrak{u}_y$. Then $[\mathfrak{u}_y,\mathfrak{u}_y]\subset\mathfrak{a}^y_{\geq 2}\subset\mathfrak{n}_{y,u}$. By [GGS17, Lemma 3.2.6], $\mathfrak{n}_{y,u}=\mathfrak{a}^y_{\geq 2}+\mathfrak{a}^y_1\cap\mathfrak{a}_u$. If (s,u) is a neutral pair, $\mathfrak{n}_{y,u}=\mathfrak{a}^y_{\geq 2}$. Let $U_y=\exp(\mathfrak{u}_y)$ and $N_{y,u}=\exp(\mathfrak{n}_{y,u})$ be the corresponding unipotent subgroups of H. Define a character of $N_{y,u}$ by $\psi_u(x)=\psi(\kappa(u,\log(x)))$. Let $N^\psi_{y,u}=N_{y,u}\cap\ker(\psi_u)$. If $U_y\neq N_{y,u}$, then the quotient $U_y/N^\psi_{y,u}$ is a Heisenberg group and its center is $N_{y,u}/N^\psi_{y,u}$.

Let ρ be an irreducible representation of A (ρ is then admissible because F is p-adic) and (y, u) be a Whittaker pair. The Jacquet module $J_{N_{y,u},\psi_u}(\rho)$ is called a degenerate Whittaker model of ρ. If (y, u) is neutral, $J_{N_{y,u},\psi_u}(\rho)$ is also called a generalized Whittaker model of ρ. The wave-front set $\mathfrak{n}(\rho)$ of ρ is defined to the set of nilpotent orbits $\mathcal{O}$ for which $J_{N_{y,u},\psi_u}(\rho) \neq 0$, for some neutral pair (y, u) with $u \in \mathcal{O}$. (This is well defined, i.e., independent of the choice of a neutral pair). Let $\mathfrak{n}^{\max}(\rho)$ be the set of maximal elements in $\mathfrak{n}(\rho)$ under the natural order of nilpotent orbits. We will use the following theorem, which follows immediately from [GGS17, Theorem A]:

Theorem 6.1. *Let (y, u) be a neutral pair and (y', u') be a Whittaker pair. If u belongs to the closure $\overline{A_{y'}u'}$ of the orbit of u' in $\mathfrak{a}^*$ under the coadjoint action of $A_{y'}$, $J_{N_{y',u'},\psi_{u'}}(\rho) \neq 0$ implies $J_{N_{y,u},\psi_u}(\rho) \neq 0$.*

Let A be either G or H. In this case, since the projection $\mathrm{GSpin}_{2k} \to \mathrm{SO}_{2k}$ splits over unipotent subgroups, as in the corresponding orthogonal cases (SO_{2n} and SO_{2m+1}), the nilpotent orbits of A are parameterized by pairs (α, Φ), where α is an orthogonal partition (even parts occur with even multiplicities) and Φ is a set of non-degenerate quadratic forms (see e.g., [Wp01, § I.6]). With ρ as above, we say that ρ admits an α generalized model if there is u in the nilpotent orbit attached to (α, Φ) for some Φ and a semisimple $y \in \mathfrak{a}$ such that (y, u) is a neutral pair and $J_{N_{y,u},\psi_u}(\rho) \neq 0$. When the pair (y, u) is not important we simply denote this Jacquet module by ρ_α. In addition let $\mathfrak{p}^{\max}(\rho)$ be the set of partitions corresponding to the nilpotent orbits in $\mathfrak{n}^{\max}(\rho)$.

An orthogonal partition α is called special if the number of odd parts smaller than every even number occurring in the partition is even ([JLS16, Definition 10.1], see also [CM93, § 6.3]). For example, if m is even, then $(m, m, 1)$ is not special. In this case, $(m + 1, m - 1, 1)$ is special, called the special expansion of $(m, m, 1)$, i.e., the smallest orthogonal special partition of $2m + 1$ which is greater than $(m, m, 1)$. By the main results of [JLS16] (generalizing [Mœg96]), any $\alpha \in \mathfrak{p}^{\max}(\rho)$ is special.

6.2. **Non-vanishing results.** Assume m is even. Let τ and ω be as in § 3.2 and put $\rho = \mathrm{LQ}(1, \tau \otimes \omega)$.

Proposition 6.2. *ρ admits an $(m, m, 1)$ generalized model.*

Proof. By [Wp01, § I.6], there is only one nilpotent orbit $\mathcal{O}$ corresponding to the partition $(m, m, 1)$. By Theorem 6.1, it suffices to show that the Jacquet module $J_{N,\psi}(\rho)$ is nonzero, where ψ restricts to a nontrivial character on each of the long simple roots of H and is trivial on the unique short simple root. Thus $J_{N,\psi}(\rho) = J_{N_{\mathrm{GL}_m},\psi_{N_{\mathrm{GL}_m}}}(J_{U_m}(\rho))$. Since ρ is a quotient of $V(1, \tau \otimes \omega)$ and τ is supercuspidal, $J_{U_m}(\rho)$ is nonzero and its irreducible constituents are isomorphic to unramified twists of $\tau \otimes \omega$ ([BZ77, Corollary 2.13]), each of which is automatically generic (in fact $J_{U_m}(\rho)$ is of length 1), whence $J_{N,\psi}(\rho) \neq 0$. □

Remark 6.3. *See* [LM15, Appendix 3] *for the similar assertion for symplectic groups but when τ is tempered; see also* [GRS02, § 5.7].

Proposition 6.4. *ρ admits an $(m + 1, m - 1, 1)$ generalized model.*

Proof. By Proposition 6.2 and [JLS16, Theorem 11.1], $\mathfrak{p}^{\max}(\rho)$ contains the special expansion of $(m, m, 1)$, i.e., the smallest orthogonal special partition of $2m + 1$ which is greater than $(m, m, 1)$. This partition is $(m + 1, m - 1, 1)$. Note that m is even. □

Theorem 6.5. *Let $m = 2n$. There exists $\alpha \in F^*$ such that $\sigma_{\psi_\alpha}(n, \tau, \omega) \neq 0$.*

Proof. According to [Wp01, § I.6], the nilpotent orbits corresponding to $(m+1, m-1, 1)$ are parameterized by certain one-dimensional quadratic forms, i.e., square-classes $\{\alpha_{m+1}, \alpha_{m-1}, \alpha_1\}$, corresponding to the parts $(m+1), (m-1)$ and 1. In fact by [JLS16, Proposition 8.1] we can take the square-classes to be $\alpha_{m+1} = \alpha$, $\alpha_{m-1} = -\alpha$ and $\alpha_1 = 1$ for some $\alpha \in F^*$.

By Proposition 6.4, there is a neutral pair (y, u) such that $J_{N_{y,u},\psi_u}(\rho) \neq 0$, where u belongs to an orbit corresponding to $(m+1, m-1, 1)$ and $\{\alpha, -\alpha, 1\}$. One can assume $u = u_1 + u_2$ where $u_1 = \sum_{i=1}^{n-1} x_{-e_i+e_{i+1}}(1) + x_{-e_n+e_{2n}}(1) + x_{-e_n-e_{2n}}(\alpha/2)$, u_2 is any representative of the nilpotent orbit in the Levi part of the stabilizer of u_1 which is $\mathrm{GSpin}_m(\phi)$ for a certain quadratic form ϕ, corresponding to the partition $((m-1), 1)$ and square-classes $\{-\alpha, 1\}$. Let y_1 be such that (y_1, u_1) is a neutral pair.

Now, by letting certain torus element in the Levi part of the stabilizer of u_1 go to zero, it is easy to see $u_1 \in \overline{H_y u}$. Then by Theorem 6.1, $J_{N_{y_1,u_1},\psi_{u_1}}(\rho) \neq 0$ and the result follows because $J_{N_{y_1,u_1},\psi_{u_1}}(\rho) \cong \sigma_{\psi_\alpha}(n, \tau, \omega)$. □

7. The local descent

Theorem 7.1. *Let τ be an irreducible unitary supercuspidal representation of GL_{2n} ($n > 1$) and ω be a unitary character of F^* such that $L(s, \tau, \mathrm{Sym}^2 \otimes \omega)$ has a pole at $s = 0$. Let $m = 2n$ and denote $\sigma_{\psi_\alpha}(\tau, \omega) = \sigma_{\psi_\alpha}(n, \tau, \omega)$.*

(1) There exists some $\alpha \in F^*$ such that $\sigma_{\psi_\alpha}(\tau, \omega) \neq 0$.
(2) The representation $\sigma_{\psi_\alpha}(\tau, \omega)$ is a supercuspidal, multiplicity free and admissible representation of G. Its irreducible constituents are all unitary and generic (for some generic characters of N_G).
(3) Let σ be an irreducible supercuspidal $\psi_{N_G,\alpha}$-generic representation of G. The Rankin-Selberg γ-factor $\gamma(s, \sigma \times (\tau \otimes \omega), \psi)$ has a pole at $s = 1$ if and only if $\sigma^\vee$ is a quotient of $\sigma_{\psi_\alpha}(\tau, \omega)$.

Proof. Part (1) follows immediately from Theorem 6.5. Then Theorems 4.2 and 5.5 imply $\sigma_{\psi_\alpha}(\tau, \omega)$ is supercuspidal.

We show that $\sigma_{\psi_\alpha}(\tau, \omega)$ is semi-simple. Because $C_G^\circ \cong C_H$ under the embedding $G < H$, C_G° acts on the space of $\sigma_{\psi_\alpha}(\tau, \omega)$ by a fixed character. This character is unitary because $\mathrm{LQ}(1, \tau \otimes \omega)$ is unitary, by [Sha90, Theorem 8.1b]. Thus by Remark 2.1, any irreducible subrepresentation σ of $\sigma_{\psi_\alpha}(\tau, \omega)$ (which is necessarily supercuspidal) is a direct summand of $\sigma_{\psi_\alpha}(\tau, \omega)$. Therefore $\sigma_{\psi_\alpha}(\tau, \omega)$ is semi-simple.

The representation $\sigma_{\psi_\alpha}(\tau, \omega)$ is now multiplicity free by Theorem 3.3 (for $s = 1$), since it is semi-simple and its quotients are supercuspidal. The semi-simplicity of $\sigma_{\psi_\alpha}(\tau, \omega)$ and Theorem 3.3 also imply that the irreducible constituents of $\sigma_{\psi_\alpha}(\tau, \omega)$ are generic. Furthermore, because $C_G^\circ \backslash C_G$ is finite and as mentioned above, C_G° acts on $\sigma_{\psi_\alpha}(\tau, \omega)$ by a unitary character, the irreducible constituents are unitary (an irreducible supercuspidal representation with a unitary central character is unitary).

Because $\sigma_{\psi_\alpha}(\tau, \omega)$ is semi-simple and multiplicity free, and the action of C_G° on its space is given by a fixed character, we deduce that $\sigma_{\psi_\alpha}(\tau, \omega)$ is admissible (for any compact open $K < G$, there are only finitely many irreducible supercuspidal representations of G with a nonzero K-fixed vector and a fixed C_G°-action).

Finally assume $L(s, \tau, \mathrm{Sym}^2 \otimes \omega)$ has a pole at $s = 0$. Using the identity

$$L(s, \tau \times \omega\tau) = L(s, \tau, \mathrm{Sym}^2 \otimes \omega) L(s, \tau, \wedge^2 \otimes \omega)$$

(see [Sha92, Lemma 3.6], [Yam17, Theorem 3.19] and [Hen10]) and by [JPSS83, § 8], we deduce $\tau \cong \omega^{-1} \otimes \tau^\vee$. Hence $L(s, \tau, \mathrm{Sym}^2 \otimes \omega) = L(s, \tau^\vee, \mathrm{Sym}^2 \otimes \omega^{-1})$ (by the results of [Hen10]). Then the last assertion follows from Theorem 3.7. □

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