

SEQUENTIAL APPROXIMATIONS FOR TYPES AND KEISLER MEASURES

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ABSTRACT. This paper is a modified chapter of the author's Ph.D. thesis. We introduce the notions of sequentially approximated types and sequentially approximated Keisler measures. As the names imply, these are types which can be approximated by a sequence of realized types and measures which can be approximated by a sequence of "averaging measures" on tuples of realized types. We show that both generically stable types (in arbitrary theories) and Keisler measures which are finitely satisfiable over a countable model (in NIP theories) are sequentially approximated. We also introduce the notion of a smooth sequence in a measure over a model and give an equivalent characterization of generically stable measures (in NIP theories) via this definition. In the last section, we take the opportunity to generalize the main result of [8].

1. INTRODUCTION

One of the joys of working in a metric space is that the closure of a set coincides with its *sequential closure*. In particular, if X is a metric space, A is a subset of X , and b is in the closure of A , then there exists a sequence of elements in A which converges to b . In [17], Simon showed that global types which are finitely satisfiable over a countable model of a countable NIP theory admit a similar property. Let T be a complete, first-order theory, \mathcal{U} a monster model of T , and M a small submodel of \mathcal{U} . Simon proved the following ([17, Lemma 2.8]):

Theorem 1.1. *Let T be a countable NIP theory. Suppose p is a type in $S_x(\mathcal{U})$ and p is finitely satisfiable over M where $|M| = \aleph_0$. Then there exists a sequence of points $(a_i)_{i \in \omega}$ in M^x such that $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$.*

One of the goals of this paper is to *morally* generalize the proof of the above theorem in two different directions. By mimicking Simon's proof, we are able to prove the following,

- (T1) Let T be any countable theory. Suppose p is a type in $S_x(\mathcal{U})$ and p is generically stable over M . Then there exists a sequence of points $(a_i)_{i \in \omega}$ in M^x such that $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$.
- (T2) Let T be a countable NIP theory. Suppose μ is a Keisler measure in $\mathfrak{M}_x(\mathcal{U})$ and μ is finitely satisfiable over M where $|M| = \aleph_0$. Then there exists a sequence of points $(\bar{a}_i)_{i \in \omega}$ in $(M^x)^{<\omega}$ such that $\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i) = \mu$. More explicitly, for any formula $\varphi(x)$ in $\mathcal{L}_x(\mathcal{U})$, we have that

$$\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i)(\varphi(x)) = \mu(\varphi(x)).$$

The proofs of both of these theorems are slightly more *enjoyable* than one would anticipate. For example, we already know many diverse and useful approximation

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theorems for measures in NIP theories (and some for generically stable types in arbitrary theories) and so one might expect that our proofs rely on composing approximation techniques. However, stringing together different approximation methods can result in an array with some kind of *modes-of-convergence* problem.

As stated previously, the technique used to prove both these theorems mimics the argument used in [17, Lemma 2.8]. In the generically stable case, the set up is identical: Suppose p is in $S_x(\mathcal{U})$ where p is generically stable over M and I is a Morley sequence in p over M . As in Simon’s proof, we use both M and I to find an eventually indiscernible sequence of points in M^x which converge to $p|_{MI}$. The *eventual EM-type* of this sequence over M is precisely $p^{(\omega)}|_M$. Using generic stability and compactness, we conclude that this sequence must converge to p .

Our proof of the Keisler measure case is slightly more exotic since there is no standard notion of a “Morley sequence in a Keisler measure”. The proof we provide is *essentially* done in first order model theory (with an important exceptional lemma following from Ben Yaacov’s work on randomizations [2]). We expect that there exists other proofs using other methods such as continuous model theory¹. The proof we give here embraces the ideology first developed in [10] and shows that this can be resolved by replacing the Morley sequence (in Simon’s proof) by a *smooth sequence in μ over M* . This provides more evidence for the intuition that smooth measures can play the role of realized types, at least in the NIP context. After constructing a countable model N_ω “containing this sequence”, we find a sequence of points in $(M^x)^{<\omega}$ such that the corresponding average measures on these tuples converge to $\mu|_{N_\omega}$. After constructing an eventually indiscernible subsequence in this context, we are able to readapt most of Simon’s proof technique by making use of known approximation theorems, symmetry properties, and some basic integration techniques.

It is interesting to note that one can give another equivalent characterization of generically stable measures in NIP theories using smooth sequences. This characterization highlights the connection between generically stable types and generically stable measures. Recall that a type p is generically stable over a model M if for every Morley sequence $(a_i)_{i \in \omega}$ in p over M , $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$. We show that in an NIP theory, a measure μ is generically stable over a model M if and only if for every *smooth sequence in μ over M* , the limit of this sequence is precisely μ .

In addition to proving these theorems, we also introduce the classes of *sequentially approximated measures* and *sequentially approximated types*. These definitions can be seen as the *global analogue* to Khanaki’s definition of *Baire 1 definability* for local types (see [13]). Sequentially approximated measures should be thought of as a “halfway point” between finitely approximated measures and Keisler measures which are finitely satisfiable over a small model. For instance, we show that a Keisler measure is finitely approximated if and only if it is both definable and sequentially approximated (Proposition 3.4) and sequentially approximated measures commute with definable measures (Proposition 3.7). Sequentially approximated types remain a little more mysterious. We show that there exists a type such that its corresponding Keisler measure is sequentially approximated (even finitely approximated), but the type itself is not sequentially approximated (Proposition 4.14).

¹In fact, after this paper was posted to arXiv, another proof was discovered by Khanaki using BFT on an infinite product space [14].

In the last section, we consider connections to the local measure case and generalize the main result in [8] (Theorem 6.4). Explicitly, the main result in [8] demonstrates that if a formula φ is NIP and μ is a φ -measure which is φ -definable and finitely satisfiable over a *countable model*, then μ is φ -finitely approximated in said model. Here, we demonstrate that *countable* can be replaced by *small*.

This paper is structured as follows: In section 2, we discuss preliminaries. In section 3, we describe sequentially approximated measures and sequentially approximated types. In section 4, we show that if p is generically stable over M , then p is sequentially approximated over M . We also give some examples of types which are which are not sequentially approximated at the end of the section. In section 5, we show that if T is a countable NIP theory, and μ is finitely satisfiable over a countable model M , then μ is sequentially approximated over M . We then give an equivalent characterization of generically stable measures in NIP theories using smooth sequences. In section 6, we generalize the main theorem in [8].

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2. PRELIMINARIES

If r and s are real numbers and ϵ is a real number greater than 0, then we write $r \approx_\epsilon s$ to mean $|r - s| < \epsilon$.

Fix \mathcal{L} a countable language. Throughout this paper, we always have a countable, complete, first-order theory T and a monster model \mathcal{U} of T in the background. The letters M and N will be used to denote small elementary submodels of \mathcal{U} . The letters x, y, z will denote tuples of variables. If $A \subseteq \mathcal{U}$, we let $\mathcal{L}(A)$ be the collection of formulas with parameters from A (modulo logical equivalence). A formula in $\mathcal{L}(A)$ is called an “ $\mathcal{L}(A)$ -formula”. If x_0, \dots, x_k is a finite sequence of pairwise disjoint tuples of variables, we let $\mathcal{L}_{x_0, \dots, x_k}(A)$ be the collection of $\mathcal{L}(A)$ -formulas with free variables in these tuples. We write $\mathcal{L}_{x_0, \dots, x_k}(\emptyset)$ simply as $\mathcal{L}_{x_0, \dots, x_k}$. If $(x_i)_{i \in \omega}$ is a countable sequence of pairwise distinct tuples of variables, we let $\mathcal{L}_{(x_i)_{i \in \omega}}(A) = \bigcup_{k \in \omega} \mathcal{L}_{x_0, \dots, x_k}(A)$. For a tuple x , let $A^x = \{(a_0, \dots, a_{|x|-1}) : a_i \in A, i \leq |x|-1\}$. We let $(A^x)^{<\omega}$ be the collection of all finite sequences of points in A^x . If we call $\varphi(x, y)$ a *partitioned $\mathcal{L}_{x,y}(\mathcal{U})$ -formula*, we treat x as object variables and y as parameter variables. The formula $\varphi^*(y, x)$ denotes the exact same formula as $\varphi(x, y)$, but with the roles exchanged for parameters and object tuples. Generally speaking, in any instance where we have multiple tuples of variables (e.g. x and y , or (x_1, x_2, x_3, \dots)), we will always assume they are pairwise distinct without comment.

Unlike similar papers about Keisler measures, we do not identify a type and its corresponding Keisler measure. We let $S_x(A)$ denote the usual type space over A and $\mathfrak{M}_x(A)$ the space of Keisler measures over A . We let $\mathfrak{M}_{(x_i)_{i \in \omega}}(\mathcal{U})$ be the collection of finitely additive probability measures on $\mathcal{L}_{(x_i)_{i \in \omega}}(\mathcal{U})$. For any (tuple of) variable(s) x , and any subset $A \subseteq \mathcal{U}$, we have a map $\delta : S_x(A) \rightarrow \mathfrak{M}_x(A)$ via $\delta(p) = \delta_p$ where δ_p is the *Dirac measure at the type p* . We sometimes refer to δ_p as the *corresponding Keisler measure of p* . If $\bar{a} = (a_1, \dots, a_n)$ is a sequence

of points in \mathcal{U}^x , then we let $\text{Av}(\bar{a})$ be the associated average measure in $\mathfrak{M}_x(\mathcal{U})$. Explicitly, for any $\psi(x) \in \mathcal{L}_x(\mathcal{U})$, we define

$$\text{Av}(\bar{a})(\psi(x)) = \frac{|\{1 \leq i \leq n : \mathcal{U} \models \psi(a_i)\}|}{n}.$$

2.1. Basics of convergence. Recall that if $A \subseteq \mathcal{U}$, then both $S_x(A)$ and $\mathfrak{M}_x(A)$ carry a natural compact Hausdorff topology. For $S_x(A)$, we have the usual Stone space topology. Similarly, $\mathfrak{M}_x(A)$ admits a compact Hausdorff topology. There are two ways to describe this topology. First, this topology is the topology induced from the compact Hausdorff space $[0, 1]^{\mathcal{L}_x(A)}$ where we identify each measure with the obvious map from $\mathcal{L}_x(A)$ to $[0, 1]$. This topology on $\mathfrak{M}_x(A)$ can also be described as the coarsest topology such that for any continuous function $f : S_x(A) \rightarrow \mathbb{R}$, the map $\int f : \mathfrak{M}_x(A) \rightarrow \mathbb{R}$ is continuous.

We will routinely need to keep track of which sets of parameters our types and measures are converging over. Hence, we establish the following conventions.

Definition 2.1. Fix $A \subseteq \mathcal{U}$, $p \in S_x(A)$ and $\mu \in \mathfrak{M}_x(A)$.

- (i) We say that a sequence of types $(p_i)_{i \in \omega}$, where each p_i is in $S_x(A)$, **converges** to p if it converges in the Stone space topology on $S_x(A)$, which we write as “ $\lim_{i \rightarrow \infty} p_i = p$ in $S_x(A)$ ” or simply as “ $\lim_{i \rightarrow \infty} p_i = p$ ” when the underlying space is obvious. We recall that $\lim_{i \rightarrow \infty} p_i = p$ if for every $\psi(x) \in p$, there exists some natural number N_ψ such that for any $n > N_\psi$, $\psi(x) \in p_n$.
- (ii) We say that a sequence of measures $(\mu_i)_{i \in \omega}$, where each μ_i is in $\mathfrak{M}_x(A)$, **converges** to μ if this sequence converges in the compact Hausdorff topology on $\mathfrak{M}_x(A)$, which we write as “ $\lim_{i \rightarrow \infty} \mu_i = \mu$ in $\mathfrak{M}_x(A)$ ” or simply as “ $\lim_{i \rightarrow \infty} \mu_i = \mu$ ” when there is no possibility of confusion. Notice that $\lim_{i \rightarrow \infty} \mu_i = \mu$ if for every $\psi(x) \in \mathcal{L}_x(A)$ and $\epsilon > 0$, there exists some natural number $N_{\psi, \epsilon}$ such that for any $n > N_{\psi, \epsilon}$,

$$|\mu_n(\psi(x)) - \mu(\psi(x))| < \epsilon.$$

We now observe the relationship between finitely satisfiable types and measures and topological closure in their respective spaces.

Fact 2.2. Suppose $p \in S_x(\mathcal{U})$, $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$. Assume that p and μ are finitely satisfiable over M . Then the following are true.

- (i) The type p is in the closure of $\{tp(a/\mathcal{U}) : a \in M^x\}$ in $S_x(\mathcal{U})$.
- (ii) The associated Keisler measure δ_p is in the closure of $\{\delta_a : a \in M^x\}$ in $\mathfrak{M}_x(\mathcal{U})$.
- (iii) The measure μ is in the closure of

$$\left\{ \sum_{i=1}^n r_i \delta_{a_i} : n \in \mathbb{N}, r_i > 0, \sum_{i=1}^n r_i = 1, a_i \in M^x \right\}$$

in $\mathfrak{M}_x(\mathcal{U})$.

- (iv) The measure μ is in the closure of $\{\text{Av}(\bar{a}) : \bar{a} \in (M^x)^{<\omega}\}$ in $\mathfrak{M}_x(\mathcal{U})$.

We remark that the proof of (i) is a standard exercise and the proof of (ii) follows directly from (i). A proof of (iii) can be found at [4, Proposition 2.11] and (iv) follows directly from (iii).

2.2. Types. We recall some basic definitions and facts about special kinds of types (e.g. generically stable types). Our notion of an *EM-type* is not defined in complete generality since we are only concerned with countable sequences in this paper.

Definition 2.3. Let $(a_i)_{i \in \omega}$ be a sequence of points in \mathcal{U}^x and let $B \subseteq \mathcal{U}$. Then the **Ehrenfeucht-Mostowski type** or **EM-type** of the sequence $(a_i)_{i \in \omega}$ over B , denoted $\text{EM}((a_i)_{i \in \omega}/B)$, is the following partial type:

$$\{\varphi(x_0, \dots, x_k) \in \mathcal{L}_{(x_i)_{i \in \omega}}(B) : \mathcal{U} \models \varphi(a_{i_0}, \dots, a_{i_k}) \text{ for any } i_0 < \dots < i_k\}.$$

We remark that this partial type corresponds to a subset of $S_{(x_i)_{i \in \omega}}(B)$.

Observation 2.4. *It is clear from the definition above that for any sequence of points $(a_i)_{i \in \omega}$ in \mathcal{U}^x and any $B \subseteq \mathcal{U}$, the type $\text{EM}((a_i)_{i \in \omega}/B)$ is complete if and only if the sequence $(a_i)_{i \in \omega}$ is indiscernible over B .*

The general notion of a *generically stable type* was introduced by Pillay and Tanović in [15]. The definition of a generically stable type provided below was proved to be equivalent in [6] (see Proposition 3.2). We also provide the definition of a dfs type which will be important throughout this paper. In general, the class of dfs types strictly contains the class of generically stable types.

Definition 2.5. Suppose that $p \in S_x(\mathcal{U})$.

- (i) We say that p is **dfs** if there exists a small model $M \prec \mathcal{U}$ such that p is both definable and finitely satisfiable over M . In this case, we say that p is **dfs over M** .
- (ii) We say that p is **generically stable** if there exists a small model $M \prec \mathcal{U}$ such that p is invariant over M and for any Morley sequence $(a_i)_{i \in \omega}$ in p over M , we have that $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$. In this case, we say that p is **generically stable over M** .

Finally, we provide a collection of standard facts about these classes of types.

Fact 2.6. *Let p be in $S_x(\mathcal{U})$ and $M \prec \mathcal{U}$.*

- (i) *If p is generically stable over M , then p is dfs over M ([15, Proposition 1]).*
- (ii) *If p is dfs over M , then any Morley sequence in p over M is totally indiscernible over M ([9, Proposition 3.2], proof does not use NIP).*
- (iii) *If p is generically stable/dfs over M and M_0 -invariant, then p is respectively generically stable/dfs over M_0 (generically stable case follows from (i) of [15, Proposition 1]; dfs case can be found in [16, Lemma 2.8]).*
- (iv) *(T is countable) If p is generically stable/dfs over M , there exists an elementary submodel M_0 such that $|M_0| = \aleph_0$ and p is generically stable/dfs over M_0 (Easy to check from (iii)).*
- (v) *(T is NIP) If p is dfs over M then p is generically stable over M (e.g. [16, Theorem 2.29]).*

2.3. Keisler measures. In this subsection, we will briefly recall some important definitions and facts about these measures. As with any paper about Keisler measures, we provide the following *standard atlas*.

Definition 2.7. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$.

- (i) μ is **invariant** if there exists a model $M \prec \mathcal{U}$ such that for every partitioned \mathcal{L} -formula $\varphi(x, y)$ and $b, b' \in \mathcal{U}^y$ such that $b \equiv_M b'$, $\mu(\varphi(x, b)) = \mu(\varphi(x, b'))$. In this case, we say that μ is **M -invariant** or **invariant over M** .

- (ii) If μ is invariant over M , then for every partitioned $\mathcal{L}(M)$ -formula $\varphi(x, y)$, we can define the map $F_{\mu, M}^\varphi : S_y(M) \rightarrow [0, 1]$ via $F_{\mu, M}^\varphi(q) = \mu(\varphi(x, b))$ where $b \models q$. When M is obvious we will simply write $F_{\mu, M}^\varphi$ as F_μ^φ .
- (iii) μ is **Borel-definable** if there exists a model $M \prec \mathcal{U}$ such that μ is M -invariant and for every partitioned \mathcal{L} -formula $\varphi(x, y)$, the map $F_{\mu, M}^\varphi$ is Borel. In this case, we say that μ is **Borel-definable over M** .
- (iv) μ is **definable** if there exists a model $M \prec \mathcal{U}$ such that μ is M -invariant and for every partitioned \mathcal{L} -formula $\varphi(x, y)$, the map $F_{\mu, M}^\varphi$ is continuous. In this case, we say that μ is **M -definable** or **definable over M** .
- (v) μ is **finitely satisfiable over a small model** if there exists $M \prec \mathcal{U}$ such that for every formula $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, if $\mu(\varphi(x)) > 0$ then there exists $a \in M^x$ such that $\mathcal{U} \models \varphi(a)$. In this case, we say that μ is **finitely satisfiable over M** .
- (vi) μ is **finitely approximated** if there exists a model $M \prec \mathcal{U}$ such that for every partitioned \mathcal{L} -formula $\varphi(x, y)$ and every $\epsilon > 0$, there exists $\bar{a} \in (M^x)^{<\omega}$ such that

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x, b)) - \text{Av}(\bar{a})(\varphi(x, b))| < \epsilon.$$

In this case, we say that μ is **finitely approximated over M** .

- (vii) μ is **smooth** if there exists a model $M \prec \mathcal{U}$ such that for any $\lambda \in \mathfrak{M}_x(\mathcal{U})$ if $\lambda|_M = \mu|_M$, then $\lambda = \mu$. If this is the case, we say that μ is **smooth over M** .

We now provide a collection of basic facts. Statements (i), (iii), (iv), and (v) in Fact 2.8 are relatively straightforward to prove and so we leave them as exercises.

Fact 2.8. *Assume that T is any theory and $\mu \in \mathfrak{M}_x(\mathcal{U})$ with $M \prec \mathcal{U}$.*

- (i) *If $\mu = \text{Av}(\bar{a})$ for some $\bar{a} \in (M^x)^{<\omega}$, then μ is smooth over M .*
- (ii) *If μ is smooth over M , then μ is finitely approximated over M , (e.g. [16, Proposition 7.10]).*
- (iii) *If μ is finitely approximated over M , then μ is both definable and finitely satisfiable over M .*
- (iv) *If μ is definable or finitely satisfiable over M , then μ is M -invariant.*
- (v) *The measure μ is definable over M if and only if for every partitioned $\mathcal{L}(M)$ -formula $\varphi(x, y)$ and for every $\epsilon > 0$, there exists formulas $\psi_1(y), \dots, \psi_n(y) \in \mathcal{L}_y(M)$ and real numbers $r_1, \dots, r_n \in [0, 1]$ such that*

$$\sup_{q \in S_y(M)} |F_{\mu, M}^\varphi(q) - \sum_{i=1}^n r_i \mathbf{1}_{\psi_i(y)}(q)| < \epsilon.$$

where $\mathbf{1}_{\psi_i(y)}$ is the characteristic function of the clopen set $[\psi_i(y)]$.

Moreover, if T is NIP then the following also hold.

- (vi) *If μ is invariant over M , then μ is Borel-definable over M (e.g. [16, Proposition 7.19]).*
- (vii) *Any measure μ is definable and finitely satisfiable over M if and only if μ is finitely approximated over M ([10, Proposition 3.2]).*
- (viii) *Every measure has a “smooth extension”. In particular, for any given $M \prec \mathcal{U}$ and $\mu \in \mathfrak{M}_x(\mathcal{U})$, there exists some N such that $M \prec N \prec \mathcal{U}$ and a measure $\lambda \in \mathfrak{M}_x(\mathcal{U})$ such that λ is smooth over N and $\lambda|_M = \mu|_M$ ([10, Lemma 2.2]).*

Proposition 2.9 (T is countable). *If μ is definable, finitely approximated, smooth or dfs, then there exists a countable model M_0 such that μ is definable, finitely approximated, smooth or dfs over M_0 (respectively).*

Proof. We notice that the properties of definability and smoothness only require the existence of \aleph_0 -many $\mathcal{L}(M)$ -formulas (by [10, Lemma 2.3] and (v) of Fact 2.8 respectively). If we choose an elementary submodel M_0 of M containing the parameters from these formulas, then μ will have the desired property over M_0 . Finitely approximated measures only require the existence of \aleph_0 -many elements of M . Choosing an elementary submodel M_0 of M with these elements demonstrates that μ is finitely approximated over M_0 .

Finally, if μ is dfs then μ is definable over a countable model M_0 . In particular, μ is invariant over M_0 and so μ is also finitely satisfiable over M_0 by the same argument as in [8, Proposition 4.13]. \square

Remark 2.10. Assuming T is countable, there are measures (even types) which are finitely satisfiable over a small submodel, but are not finitely satisfiable over a countable submodel. See Proposition 4.11 and Remark 4.12 for an explicit example.

Definition 2.11. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$ and assume that μ is Borel-definable over M . Then we define the **Morley product** of μ and ν (denoted $\mu \otimes \nu$) is the unique Keisler measure in $\mathfrak{M}_{x,y}(\mathcal{U})$ with the following property: for any formula $\varphi(x, y) \in \mathcal{L}_{x,y}(\mathcal{U})$,

$$\mu \otimes \nu(\varphi(x, y)) = \int_{S_y(N)} F_\mu^\varphi d(\nu|_N),$$

where N is any small elementary submodel of \mathcal{U} containing M and any parameters from φ and $\nu|_N$ is the associated regular Borel probability measure of the restriction of ν to N on the type space $S_y(N)$.

We remark that this this product is well-defined and the computation does not depend on our choice of N (assuming N contains M and all parameters in $\varphi(x, y)$) (see discussion after [16, Proposition 7.19]). This observation allows us to grow or shrink the space in which we are integrating over and we will make substantial use of this property in section 5. We end this section with a list of facts about measures and products.

Fact 2.12. *Assume that T is any theory and $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$. Assume that μ and ν are both M -invariant.*

- (i) *If μ is smooth and ν is Borel definable, then $\mu \otimes \nu = \nu \otimes \mu$ (see [10, Corollary 2.5]).*
- (ii) *If μ and ν are definable (over M), then $\mu \otimes \nu$ is definable (over M) and $\mu \otimes (\nu \otimes \lambda) = (\mu \otimes \nu) \otimes \lambda$ (see [6, Proposition 2.6]).*
- (iii) *If μ and ν are smooth (over M), then $\mu \otimes \nu$ is smooth (over M) (e.g. [5, Corollary 3.1]).*
- (iv) *If μ is Borel definable (over M) and ν is invariant (over M), then $\mu \otimes \nu$ is invariant (over M) (discussion before [16, Exercise 7.20]).*
- (v) *If μ and ν are dfs (over M), then $\mu \otimes \nu$ is dfs (over M) (e.g. [6, Proposition 2.10]).*

Moreover, if T is NIP then the following also hold.

- (a) *If μ, ν are invariant then $\mu \otimes (\nu \otimes \lambda) = (\mu \otimes \nu) \otimes \lambda$ (see [5]).*

(b) If μ is dfs and ν is invariant, then $\mu \otimes \nu = \nu \otimes \mu$ (see [10, Theorem 3.2]).

Definition 2.13 (T is NIP). Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and μ is invariant. Then, we define the following measures:

- (1) $\mu^{(0)}(x_0) = \mu(x_0)$.
- (2) $\mu^{(n)} = \mu(x_n) \otimes \mu^{(n-1)}(x_0, \dots, x_{n-1})$.
- (3) $\mu^{(\omega)} = \bigcup_{i \in \omega} \mu^{(n)}$ (where $\mu^{(\omega)} \in \mathfrak{M}_{(x_i)_{i \in \omega}}(\mathcal{U})$).

We note that $\mu^{(n)}$ and $\mu^{(\omega)}$ are well-defined by Fact 2.12, and moreover we do not need to worry about the ordering of the parentheses in the product.

3. SEQUENTIALLY APPROXIMATED TYPES AND MEASURES

We begin this section by isolating the property of *sequential approximability*. We again remark that these classes of objects are a global version of Khanaki's *Baire 1 definability* [13]. We assume that T is countable, but make no other global assumptions about T . As usual, \mathcal{U} is a fixed sufficiently saturated model of T . We now define sequentially approximated types and measures.

Definition 3.1. Let $p \in S_x(\mathcal{U})$ and $\mu \in \mathfrak{M}_x(\mathcal{U})$. We say that,

- (1) p is **sequentially approximated** if there exists $M \prec \mathcal{U}$ and a sequence of points $(a_i)_{i \in \omega}$ in M^x such that $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$ in $S_x(\mathcal{U})$. In this case, we say p is **sequentially approximated over M** .
- (2) μ is **sequentially approximated** if there exists $M \prec \mathcal{U}$ and a sequence of points $(\bar{a}_i)_{i \in \omega}$ in $(M^x)^{<\omega}$ such that $\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. In this case, we say μ is **sequentially approximated over M** .

We warn the reader that Definition 3.1 is only meaningful in the context of types and measures over large models. Indeed, if M is a countable model and T is a countable theory, then for every $p \in S_x(M)$, there exists a sequence of points in M^x such that $\lim_{i \rightarrow \infty} \text{tp}(a_i/M) = p$ in $S_x(M)$. The analogous statement also holds for measures.

We also emphasize to the reader that there is a real distinction between a type p being sequentially approximated over a model M and its associated Keisler measure δ_p being sequentially approximated over M . Proposition 4.14 gives an example of a type which is not sequentially approximated while its associated Keisler measure is sequentially approximated. However, the other implication holds almost trivially.

Observation 3.2. If a type p in $S_x(\mathcal{U})$ is sequentially approximated over a model M , then the associated Keisler measure δ_p is sequentially approximated over M .

Proof. If $\lim_{i \rightarrow \infty} \text{tp}(a_i/\mathcal{U}) = p$ in $S_x(\mathcal{U})$, then $\lim_{i \rightarrow \infty} \delta_{a_i} = \delta_p$ in $\mathfrak{M}_x(\mathcal{U})$ since $\delta : S_x(\mathcal{U}) \rightarrow \mathfrak{M}_x(\mathcal{U})$ is a topological embedding. \square

3.1. Basic properties. We now connect sequentially approximated types and measures to standard model-theoretic properties. For the reader's intuition, sequential approximability (at least in the case of measures) should be thought of as a strong version of finite satisfiability over a small model or a weak version of finite approximability. Sequentially approximated types remain a little more mysterious.

Proposition 3.3. Assume that $p \in S_x(\mathcal{U})$ and $\mu \in \mathfrak{M}_x(\mathcal{U})$.

- (i) If p and μ are sequentially approximated over M , then p and μ are finitely satisfiable over M . Even more, p and μ are finitely satisfiable over a countable elementary submodel of M .

- (ii) If p and μ are sequentially approximated over M , then p and μ are Borel-definable over M .
- (iii) If μ is finitely approximated over M , then μ is sequentially approximated over M . (Warning: In general, this fails for types.)
- (iv) If T is NIP, then p is sequentially approximated over M if and only if δ_p is sequentially approximated over M .
- (v) Assume that $k \subseteq \{1, 2, \dots, n\}$ and let $\pi_k : S_n(\mathcal{U}) \rightarrow S_k(\mathcal{U})$ and $\rho_k : \mathfrak{M}_n(\mathcal{U}) \rightarrow \mathfrak{M}_k(\mathcal{U})$ be the obvious projection maps. If $p \in S_n(\mathcal{U})$ and p is sequentially approximated over M , then $\pi_k(p)$ is sequentially approximated over M . Similarly, if $\mu \in \mathfrak{M}_n(\mathcal{U})$ is sequentially approximated over M then so is $\rho_k(\mu)$.

Proof. We prove the claims.

- (i) The first part of (i) is obvious. For the second part, we only need to choose a submodel containing a sequence which sequentially approximates our type or measure. Since T is countable, we can choose a countable model.
- (ii) The proofs for both the type and measure cases are similar, so we prove the measure case. Assume that $(\bar{a}_i)_{i \in \omega}$ is a sequence of points in $(M^x)^{<\omega}$ such that $\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. By part (i), μ is finitely satisfiable over M and hence M -invariant. So, for any partitioned formula $\varphi(x, y)$ in \mathcal{L} , the map $F_\mu^\varphi : S_y(M) \rightarrow [0, 1]$ is well-defined. By sequential approximability, the sequence of continuous functions $(F_{\text{Av}(\bar{a}_i)}^\varphi)_{i \in \omega}$ converges pointwise to F_μ^φ . Hence, F_μ^φ is Baire-1 (and therefore Borel).
- (iii) This follows from an encoding argument. Let $(\varphi_n(x, y_n))_{n \in \omega}$ be an enumeration of the partitioned \mathcal{L} -formulas. For each $n \in \mathbb{N}$, consider the partitioned formula $\theta_n(x; y_0, \dots, y_n, z_*, z_0, \dots, z_n)$ where $|z_*| = |z_i| = 1$ and

$$\theta_n(x; \bar{y}, \bar{z}) := \bigwedge_{i \leq n} \left(\left(z_* = z_i \wedge \bigwedge_{\substack{j \leq n \\ j \neq i}} z_j \neq z_* \right) \rightarrow \varphi_i(x, y_i) \right).$$

Since μ is finitely approximated over M , for $\epsilon = \frac{1}{n}$, there exists some \bar{a}_n in $(M^x)^{<\omega}$ such that for every $(\bar{b}, \bar{c}) \in \mathcal{U}^{\bar{y}\bar{z}}$,

$$|\text{Av}(\bar{a}_n)(\theta_n(x, \bar{b}, \bar{c})) - \mu((\theta_n(x, \bar{b}, \bar{c})))| < \epsilon.$$

Notice that $\theta_n(x; \bar{y}, \bar{z})$ encodes the definable sets which are obtained by the formulas $\varphi_0(x, y_0), \dots, \varphi_n(x, y_n)$. In particular, for every $b \in \mathcal{U}^{y_j}$ where $j \leq n$, consider then tuple $(\bar{d}_b, \bar{c}_j) = (d_0, \dots, d_{j-1}, b, d_{j+1}, \dots, d_n, c_*, c_0, \dots, c_n)$ where the d_i 's are arbitrary and $c_* = c_l$ if and only if $l = j$. Then

$$|\text{Av}(\bar{a}_n)(\varphi_j(x, b)) - \mu(\varphi_j(x, b))| = |\text{Av}(\theta_n(x, \bar{d}_b, \bar{c}_j)) - \mu((\theta_n(x, \bar{d}_b, \bar{c}_j)))|.$$

So for any $j \leq n$ and $b \in \mathcal{U}^{y_j}$,

$$|\text{Av}(\bar{a}_n)(\varphi_j(x, b)) - \mu(\varphi_j(x, b))| < \frac{1}{n}.$$

It is clear that $\lim_{n \rightarrow \infty} \text{Av}(\bar{a}_n) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$.

- (iv) The forward direction is Observation 3.2. We consider the converse. If δ_p is sequentially approximated over M then δ_p is finitely satisfiable over a countable submodel M_0 by (i) above. Then p is finitely satisfiable over M_0 and so by Theorem 1.1, p is sequentially approximated over M_0 (and also over M).

- (v) Simply consider the approximating sequence restricted to the appropriate coordinates. \square

Proposition 3.4. *A measure μ is sequentially approximated and definable over M if and only if μ is finitely approximated over M .*

Proof. We first prove the forward direction. The proof is similar to the proof of [8, Theorem 4.8]. Fix $\epsilon > 0$. For any partitioned \mathcal{L} -formula $\varphi(x, y)$, consider the map $F_\mu^\varphi : S_y(M) \rightarrow [0, 1]$. Let $(\bar{a}_i)_{i \in \omega}$ be a sequence of points in $(M^x)^{<\omega}$ such that $\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. Observe that each map $F_{\text{Av}(\bar{a})}^\varphi : S_y(M) \rightarrow [0, 1]$ is continuous and the sequence $(F_{\text{Av}(\bar{a}_i)}^\varphi)_{i \in \omega}$ converge pointwise to F_μ^φ . Since μ is definable, the map F_μ^φ is continuous. By the Riesz representation theorem and dominated convergence theorem, we have that $(F_{\text{Av}(\bar{a}_i)}^\varphi)_{i \in \omega}$ converges weakly to F_μ^φ in $C(S_y(M))$. By a standard application of Mazur's lemma, there exists a sequence of functions $(g_j)_{j \in \omega}$ such that each g_j is a rational convex combination of $\{F_{\text{Av}(\bar{a}_i)}^\varphi : i \leq n_j\}$ for some natural number n_j and the sequence $(g_j)_{j \in \omega}$ converges uniformly to F_μ^φ . Choose $m \in \mathbb{N}$ so that

$$\sup_{p \in S_y(M)} |F_\mu^\varphi(p) - g_m(p)| < \epsilon.$$

By construction, $g_m = F_{\text{Av}(\bar{c})}^\varphi$ for some $\bar{c} \in (M^x)^{<\omega}$. Notice that

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x, b)) - \text{Av}(\bar{c})(\varphi(x, b))| < \epsilon.$$

For the converse, μ is definable over M by (iii) of Fact 2.8. Moreover, μ is sequentially approximated over M by (iii) of Proposition 3.3. \square

We now show that sequentially approximated measures commute with definable measures. It is well-known that in the context of NIP theories definable measures commute with measures which are finitely satisfiable over a small model (see [10, Lemma 3.1] or [16, Proposition 7.22]). Recently, it was shown that in general, measures which are finitely satisfiable over a small model (even dfs measures) do not always commute with definable measures (see [7, Proposition 7.14]). We first present a topological proof (in NIP theories) which shows that measures which are finitely satisfiable over a small model commute with definable measures. We will then modify this proof (by replacing an instance of continuity by the dominated convergence theorem) to show that sequentially approximated measures commute with definable ones in any theory. Recall the following facts.

Fact 3.5. *Let $\nu \in \mathfrak{M}_y(\mathcal{U})$, $N \prec \mathcal{U}$, and $\varphi(x, y)$ be an $\mathcal{L}_{x, y}(N)$ formula. Let $\mathfrak{M}_x(\mathcal{U}, N)$ denote the collection of measures in $\mathfrak{M}_x(\mathcal{U})$ which are finitely satisfiable over N .*

- (i) *If ν is definable over N , then the map from $\mathfrak{M}_x(\mathcal{U})$ to $[0, 1]$ defined via $\mu \rightarrow \nu \otimes \mu(\varphi(x, y))$ is continuous ([7, Lemma 5.4]).*
- (ii) *(T is NIP) If ν is any measure, then the map from $\mathfrak{M}_x(\mathcal{U}, N)$ to $[0, 1]$ defined via $\mu \rightarrow \mu \otimes \nu(\varphi(x, y))$ is well-defined and continuous ([4, Proposition 6.3]).*

We remark that statement (ii) of Fact 3.5 requires NIP for two reasons. First, it is not true in general that measures which are finitely satisfiable over a small model are Borel definable. In NIP theories, this is true ((vi) of Fact 2.8). Secondly, the proof that this map is continuous relies on the existence of a smooth extension

of $\nu|_N$. Without NIP, this map need not be continuous. The first proof of the following proposition can be found in [10].

Proposition 3.6 (T is NIP). *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. If μ is finitely satisfiable over a small model and ν is definable, then $\mu \otimes \nu = \nu \otimes \mu$.*

Proof. Fix a formula $\varphi(x, y) \in \mathcal{L}_{x,y}(\mathcal{U})$. Choose N such that μ is finitely satisfiable over N , ν is definable over N , and N contains all the parameters from φ . Since μ is finitely satisfiable over N , there exists a net of measures $(\text{Av}(\bar{a}_i))_{i \in I}$ such that each $\bar{a}_i \in (N^x)^{<\omega}$ and $\lim_{i \in I} \text{Av}(\bar{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$ ((iv) of Fact 2.2). By Fact 3.5

$$\begin{aligned} \mu \otimes \nu(\varphi(x, y)) &= \int_{S_y(N)} F_\mu^\varphi d(\nu|_N) \stackrel{(a)}{=} \lim_{i \in I} \int_{S_y(N)} F_{\text{Av}(\bar{a}_i)}^\varphi d(\nu|_N) \\ &\stackrel{(b)}{=} \lim_{i \in I} \int_{S_x(N)} F_\nu^{\varphi^*} d(\text{Av}(\bar{a}_i)|_N) \\ &\stackrel{(c)}{=} \int_{S_x(N)} F_\nu^{\varphi^*} d(\mu|_N) = \nu \otimes \mu(\varphi(x, y)). \end{aligned}$$

Where the equalities (a) and (c) follow from the fact that continuous functions commute with nets. The equality (b) is simple to check and is also justified by statement (i) of Fact 2.12. \square

Proposition 3.7. *Sequentially approximated and definable measures commute. Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. If μ is sequentially approximated and ν is definable, then $\mu \otimes \nu = \nu \otimes \mu$.*

Proof. Fix a formula $\varphi(x, y) \in \mathcal{L}_{x,y}(\mathcal{U})$. Choose N such that μ is sequentially approximated over N , ν is definable over N , and N contains all the parameters from φ . Let $(\bar{a}_i)_{i \in \omega}$ be a sequence of points in $(N^x)^{<\omega}$ such that $\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$. Now we consider the following computation.

$$\begin{aligned} \mu \otimes \nu(\varphi(x, y)) &= \int_{S_y(N)} F_\mu^\varphi d(\nu|_N) \stackrel{(a)}{=} \lim_{i \rightarrow \infty} \int_{S_y(N)} F_{\text{Av}(\bar{a}_i)}^\varphi d(\nu|_N) \\ &\stackrel{(b)}{=} \lim_{i \rightarrow \infty} \int_{S_x(N)} F_\nu^{\varphi^*} d(\text{Av}(\bar{a}_i)|_N) \\ &\stackrel{(c)}{=} \int_{S_x(N)} F_\nu^{\varphi^*} d(\mu|_N) = \nu \otimes \mu(\varphi(x, y)). \end{aligned}$$

Where the equality (a) now holds from the dominated convergence theorem, equality (c) holds from (i) of Fact 3.5 and the observation that continuous functions commute with nets, and equality (b) is easy to check (also (i) of Fact 2.12). \square

Corollary 3.8. *Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$. If μ is finitely approximated and ν is definable, then $\mu \otimes \nu = \nu \otimes \mu$.*

Proof. By (iii) of Proposition 3.3, μ is sequentially approximated. Apply Proposition 3.7. \square

3.2. Egorov's theorem. It is interesting to note that sequentially approximated measures are not too far away from finitely approximated measures. In particular, if we fix some measure on the parameter space, any sequentially approximated measure is *almost* finitely approximated. This result is in a similar vein as Khanaki's *almost definable* coheirs in the local setting ([12]). A direct application of Egorov's theorem gives our result.

Theorem 3.9 (Egorov's Theorem). *Let (X, B, μ) be a finite measure space. Assume that $(f_i)_{i \in \omega}$ is a sequence of measurable functions from $X \rightarrow \mathbb{R}$ such that $(f_i)_{i \in \omega}$ converges to a function f pointwise. Then for every $\epsilon > 0$ there exists a $Y_\epsilon \in B$ such that $f_i|_{Y_\epsilon}$ converges to $f|_{Y_\epsilon}$ uniformly on Y_ϵ and $\mu(X \setminus Y_\epsilon) < \epsilon$.*

A proof of Egorov's theorem can be found in [19, Theorem 3.2.4.1]. Restating this theorem in our context gives the following result.

Corollary 3.10. *Assume that p and μ are sequentially approximated over M . Let $\nu \in \mathfrak{M}_y(M)$. Then, for every $\epsilon > 0$, there exists a Borel set $Y_\epsilon \subset S_y(M)$ such that*

- (1) $\nu(Y_\epsilon) > 1 - \epsilon$.
- (2) *For every $\delta > 0$ and every partitioned \mathcal{L} -formula $\varphi(x, y)$, there exists \bar{a}_δ in $(M^x)^{<\omega}$ such that for every $b \in \mathcal{U}^y$ so that $\text{tp}(b/M) \in Y_\epsilon$, we have*

$$|\mu(\varphi(x, b)) - \text{Av}(\bar{a}_\delta)(\varphi(x, b))| < \delta.$$

- (3) *For every partitioned \mathcal{L} -formula $\varphi(x, y)$, there exists a in M^x such that for every $b \in \mathcal{U}^y$ so that $\text{tp}(b/M) \in Y_\epsilon$, we have*

$$\varphi(x, b) \in p \iff \models \varphi(a, b).$$

4. GENERICALLY STABLE TYPES

Throughout this section, we let T be a countable theory and \mathcal{U} be a monster model of T . We show that if a type p is generically stable over a small submodel M of \mathcal{U} , then p is sequentially approximated over M . Toward proving this result, we actually prove a slightly stronger lemma than what is necessary. Namely, let p be a dfs type and let M be a countable model such that p is dfs over M (for any dfs type, these models always exist by (iv) of Fact 2.6). We show that there exists a special sequence of points in M such that the *limiting behavior* of this sequence *resembles* a Morley sequence in p over M . In the case where p is generically stable over M , we show that this special sequence converges to p . This is enough to show the result since every generically stable type is generically stable over some countable model. We now begin with a discussion on eventually indiscernible sequences, which were introduced in [17].

Definition 4.1. Let $(c_i)_{i \in \omega}$ be a sequence of points in \mathcal{U}^x and $A \subset \mathcal{U}$. We say that $(c_i)_{i \in \omega}$ is an **eventually indiscernible sequence over A** if for any formula $\varphi(x_0, \dots, x_k)$ in $\mathcal{L}_{(x_i)_{i \in \omega}}(A)$, there exists some natural number N_φ such that for any indices $n_k > \dots > n_0 > N_\varphi$ and $m_k > \dots > m_0 > N_\varphi$, we have that

$$\mathcal{U} \models \varphi(c_{n_0}, \dots, c_{n_k}) \leftrightarrow \varphi(c_{m_0}, \dots, c_{m_k}).$$

Fact 4.2. *Let $(b_i)_{i \in \omega}$ be a sequence of points in \mathcal{U}^x and $A \subset \mathcal{U}$ such that $|A| = \aleph_0$. Then there exists a subsequence $(c_i)_{i \in \omega}$ of $(b_i)_{i \in \omega}$ such that $(c_i)_{i \in \omega}$ is eventually indiscernible over A .*

The proof is a standard application of Ramsey's theorem and taking the diagonal (as mentioned in [17]). We prove a “continuous” version of this fact in the next section and the proof is analogous (see Proposition 5.3 for details).

For any eventually indiscernible sequence $(c_i)_{i \in \omega}$ over a set of parameters A , we can associate to this sequence a unique type in $S_{(x_i)_{i \in \omega}}(A)$. We call this the *eventual Ehrenfeucht-Mostowski type* (or EEM-type) of $(c_i)_{i \in \omega}$ over A . We now give the formal definition.

Definition 4.3. Let $(b_i)_{i \in \omega}$ be a sequence of points in \mathcal{U}^x and $A \subset \mathcal{U}$. Then the **eventual Ehrenfeucht-Mostowski type** (or **EEM-type**) of $(b_i)_{i \in \omega}$ over A , which is written as $\text{EEM}((b_i)_{i \in \omega}/A)$, is a subset of $\mathcal{L}_{(x_i)_{i \in \omega}}(A)$ defined as follows:

Let $\varphi(x_{i_0}, \dots, x_{i_k})$ be a formula in $\mathcal{L}_{(x_i)_{i \in \omega}}(A)$ where the indices are ordered $i_0 < \dots < i_k$. Then $\varphi(x_{i_0}, \dots, x_{i_k}) \in \text{EEM}((b_i)_{i \in \omega}/A)$ if and only if there exists an N_φ such that for any $n_k > \dots > n_0 > N_\varphi$, we have that $\mathcal{U} \models \varphi(b_{n_0}, \dots, b_{n_k})$.

Notice that an EEM-type of a sequence is always indiscernible in the following sense: If we have indices i_0, \dots, i_k and j_0, \dots, j_k where $i_0 < \dots < i_k$ and $j_0 < \dots < j_k$, then $\varphi(x_{i_0}, \dots, x_{i_k})$ is in the EEM-type of $(b_i)_{i \in \omega}$ over A if and only if $\varphi(x_{j_0}, \dots, x_{j_k})$ is. This follows directly from the definition. We have some basic observations.

Observation 4.4. Let $(c_i)_{i \in \omega}$ be an eventually indiscernible sequence over A .

- (1) Then $\text{EEM}((c_i)_{i \in \omega}/A)$ is a complete type in $S_{(x_i)_{i \in \omega}}(A)$.
- (2) If $(c_i)_{i \in \omega}$ is A -indiscernible, then $\text{EEM}((c_i)_{i \in \omega}/A) = \text{EM}((c_i)_{i \in \omega}/A)$.
- (3) If $\text{tp}((b_i)_{i \in \omega}/A) = \text{EEM}((c_i)_{i \in \omega}/A)$, then $(b_i)_{i \in \omega}$ is A -indiscernible.

Proof. Clear from the definitions and discussion above. \square

We warn the reader that an eventually indiscernible sequence need not “realize” its own EEM-type. Consider the following example:

Example 4.5. Let $T_{<}$ be the theory of $(\mathbb{R}; <)$. Let \mathcal{U} be a monster model of T_{real} and $\mathbb{R} \prec \mathcal{U}$. Then the sequence $(a_i)_{i \in \omega}$ where $a_i = i$ is eventually indiscernible over \mathbb{R} while the sequence $(b_i)_{i \in \omega}$ where $b_i = i(-1)^i$ is not. Clearly, $(a_i)_{i \in \omega}$ is not \mathbb{R} -indiscernible. Moreover, for each $r \in \mathbb{R}$, the formula $x_0 > r$ is in $\text{EEM}((a_i)_{i \in \omega}/\mathbb{R})$ while $a_1 > 2$ clearly does not hold. So if $\text{tp}((c_i)_{i \in \omega}/\mathbb{R}) = \text{EEM}((a_i)_{i \in \omega}/\mathbb{R})$, then $c_i > \mathbb{R}$ for each $i \in \omega$.

The next two lemmas prove the bulk of this section's main theorem and their proofs are similar to the proof of Theorem 1.1. The proof strategy for this theorem is the following: If p is in $S_x(\mathcal{U})$ and p is dfs, then we can find a countable model M such that p is dfs over M . Let I be a Morley sequence in p over M . Using the fact that p is finitely satisfiable over M , we can find a sequence of points in M^x which converge to $p|_{MI}$ in $S_x(MI)$. After moving to an eventually indiscernible subsequence, we show that the EEM-type of this eventually indiscernible sequence is precisely $p^\omega|_M$. With the stronger assumption that our type p is generically stable (instead of just dfs), we show that this eventually indiscernible subsequence must converge to p in $S_x(\mathcal{U})$.

Lemma 4.6. Suppose p is in $S_x(\mathcal{U})$ and p is dfs over M where $|M| = \aleph_0$. Then there exists a sequence $(c_i)_{i \in \omega}$ in M^x such that $\text{EEM}((c_i)_{i \in \omega}/M) = p^\omega|_M$.

Proof. Let $I = (a_i)_{i \in \omega}$ be a Morley sequence in p over M . Since T , M , and I are countable, $\mathcal{L}_x(MI)$ is countable. It follows that $p|_{MI}$ is countable and we may enumerate this collection of formulas as $(\varphi_i(x))_{i \in \omega}$. Since p is dfs over M , in particular

p is finitely satisfiable over M . For each natural number n , we choose b_n in M^x such that $\mathcal{U} \models \bigwedge_{j \leq n} \varphi_j(b_n)$. By construction, we have that $\lim_{i \rightarrow \infty} \text{tp}(b_i/MI) = p|_{MI}$ in $S_x(MI)$. By Fact 4.2, we may choose a subsequence $(c_i)_{i \in \omega}$ of $(b_i)_{i \in \omega}$ such that $(c_i)_{i \in \omega}$ is eventually indiscernible over MI . For ease of notation, we write $(c_i)_{i \in \omega}$ as J .

We now show that $\text{EEM}(J/M) = \text{EM}(I/M) = p^\omega|_M$. We remind the reader that $\text{EM}(I/M) = p^\omega|_M$ follows directly from the definition of a Morley sequence. We prove the first equality by induction on the number of free variables occurring in a formula. We begin with the base case. It suffices to show that for every $\varphi(x_0) \in \mathcal{L}_{x_0}(M)$, if $\varphi(x_0) \in \text{EM}(I/M)$, then $\varphi(x_0) \in \text{EEM}(J/M)$. Notice that $\lim_{n \rightarrow \infty} \text{tp}(b_n/MI) = p|_{MI}$, and $(c_i)_{i \in \omega}$ is a subsequence of $(b_n)_{n \in \omega}$, $\lim_{i \rightarrow \infty} \text{tp}(c_i/MI) = p|_{MI}$. This clearly implies the base case.

Fix k and suppose that for any formula $\theta(x_0, \dots, x_k)$ in $\mathcal{L}_{x_0, \dots, x_k}(M)$, we have that $\theta(x_0, \dots, x_k) \in \text{EM}(I/M)$ if and only if $\theta(x_0, \dots, x_k) \in \text{EEM}(J/M)$.

Towards a contradiction, we assume that $\neg\theta(x_0, \dots, x_{k+1}) \in \text{EEM}(J/M)$ and $\theta(x_0, \dots, x_{k+1}) \in \text{EM}(I/M)$. Since $\neg\theta(\bar{x}) \in \text{EEM}(J/M)$, there exists some natural number N_{θ_1} such that for any $n_{k+1} > \dots > n_0 > N_{\theta_1}$, we have that $\mathcal{U} \models \neg\theta(c_{n_0}, \dots, c_{n_{k+1}})$. Since $\theta(\bar{x}) \in \text{EM}(I/M)$, we conclude that $\mathcal{U} \models \theta(a_0, \dots, a_{k+1})$. Since p is dfs over M , I is totally indiscernible over M by Fact 2.6. Therefore, $\mathcal{U} \models \theta(a_{k+1}, a_0, \dots, a_k)$ and so $\theta(x, a_0, \dots, a_k) \in p|_{Ma_0, \dots, a_k}$. Since $\lim_{i \rightarrow \infty} \text{tp}(c_i/MI) = p|_{MI}$, there exists some N_{θ_2} such that for every $n > N_{\theta_2}$, we have that $\mathcal{U} \models \theta(c_n, a_0, \dots, a_k)$. Choose $n_* > \max\{N_{\theta_1}, N_{\theta_2}\}$. Then the formula $\theta(c_{n_*}, x_0, \dots, x_k) \in \text{tp}(a_0, \dots, a_k/M)$. By our induction hypothesis, we have that $\theta(c_{n_*}, \bar{x}) \in \text{EEM}(J/M)$ and so there exists N_{θ_3} such that for any $m_k > \dots > m_0 > N_{\theta_3}$, we have that $\mathcal{U} \models \theta(c_{n_*}, c_{m_0}, \dots, c_{m_k})$. Now consider what happens when $m_0 > \max\{N_{\theta_3}, n_*\}$. Then $m_k > \dots > m_0 > n_* > N_{\theta_1}$ and so $\mathcal{U} \models \neg\theta(c_{n_*}, c_{m_0}, \dots, c_{m_k})$ by our assumption. However, $m_k > \dots > m_0 > N_{\theta_3}$ and therefore $\mathcal{U} \models \theta(c_{n_*}, c_{m_0}, \dots, c_{m_k})$. This is a contradiction. \square

Lemma 4.7. *Suppose p is in $S_x(\mathcal{U})$ and $M \prec \mathcal{U}$. Assume that p is generically stable over M . If $(c_i)_{i \in \omega}$ is a sequence in M^x such that $\text{EEM}((c_i)_{i \in \omega}/M) = p^\omega|_M$, then $\lim_{i \rightarrow \infty} \text{tp}(c_i/\mathcal{U}) = p$.*

Proof. Let p , $(c_i)_{i \in \omega}$ and M be as in the statement of the lemma. Let $J = (c_i)_{i \in \omega}$. We first argue that the sequence of global types $(\text{tp}(c_i/\mathcal{U}))_{i \in \omega}$ converges and then argue that this sequence converges to p .

Claim 1: The sequence $(\text{tp}(c_i/\mathcal{U}))_{i \in \omega}$ converges to a some type in $S_x(\mathcal{U})$.

It suffices to argue that for any formula $\psi(x) \in \mathcal{L}_x(\mathcal{U})$, $\lim_{i \rightarrow \infty} \mathbf{1}_\psi(c_i)$ exists (recall that $\mathbf{1}_{\psi(x)}$ is the characteristic function of the definable set $\psi(x)$). Assume not. Then we may choose a subsequence $(c'_i)_{i \in \omega}$ of $(c_i)_{i \in \omega}$ such that $\mathcal{U} \models \psi(c'_i) \leftrightarrow \neg\psi(c'_{i+1})$. For notational purposes, we also denote $(c'_i)_{i \in \omega}$ as J' . It is clear that $(c'_i)_{i \in \omega}$ is also eventually indiscernible over M and $\text{EEM}((c'_i)_{i \in \omega}/M) = \text{EEM}((c_i)_{i \in \omega}/M)$. By using J' , one can show that the following type is finitely consistent:

$$\Theta_1 = \text{EEM}(J'/M) \cup \bigcup_{i \text{ is even}} \{\psi(x_i) \wedge \neg\psi(x_{i+1})\}.$$

Let $(d_i)_{i \in \omega}$ realize this type. Then $(d_i)_{i \in \omega}$ is a Morley sequence in p over M because

$$\text{EM}((d_i)_{i \in \omega}/M) = \text{EEM}(J'/M) = \text{EEM}(J/M) = p^\omega|_M.$$

Then $\mathcal{U} \models \psi(d_i)$ if and only if i is even. This contradicts generic stability since $\lim_{i \rightarrow \infty} \text{tp}(d_i/M)$ does not converge.

Claim 2: The sequence $(\text{tp}(c_i/\mathcal{U}))_{i \in \omega}$ converges to p .

Again, assume not. By claim 1, $\lim_{i \rightarrow \infty} \text{tp}(c_i/\mathcal{U}) = q$ for some $q \in S_x(\mathcal{U})$. By assumption, $q \neq p$ and so there exists a formula $\psi(x)$ such that $\psi(x) \in p$ and $\neg\psi(x) \in q$. Since $(\text{tp}(c_i/\mathcal{U}))_{i \in \omega}$ converges to q , there is an N such that for every $n > N$, we have that $\mathcal{U} \models \neg\theta(c_n)$. By a similar argument as the previous claim, one can show the following type is finitely consistent:

$$\Theta_2 = \text{EEM}(J/M) \cup \bigcup_{i \in \omega} \neg\theta(x_i).$$

Again, we let $(d_i)_{i \in \omega}$ realize this type. Then $(d_i)_{i \in \omega}$ is a Morley sequence in p over M and we have that $\lim_{i \rightarrow \infty} \text{tp}(d_i/\mathcal{U}) \neq p$ in $S_x(\mathcal{U})$. This again contradicts the definition of generic stability. \square

Theorem 4.8. *Suppose p is in $S_x(\mathcal{U})$ and p is generically stable (over M). Then p is sequentially approximated (over M).*

Proof. If p is generically stable, then p is generically stable over a countable submodel M_0 contained in M by Fact 2.6. Then p is dfs over M_0 and so by Lemma 4.6, one can choose $(c_i)_{i \in \omega}$ where each $c_i \in M_0^x$ and $\text{EEM}((c_i)_{i \in \omega}/M_0) = p|_{M_0}$. By Lemma 4.7, $\lim_{i \rightarrow \infty} \text{tp}(c_i/\mathcal{U}) = p$. \square

Corollary 4.9. *Assume that T' is countable or uncountable in the language \mathcal{L}' , $\mathcal{U}' \models T'$, and M' a submodel of \mathcal{U}' . Assume that p is generically stable over M' . Then for any countable collection of formulas $\Delta = \{\psi_i(x, y_i)\}_{i \in \omega}$ in \mathcal{L}' , there exists a sequence of points $(c_i)_{i \in \omega}$ each in $(M')^x$ such that $\lim_{i \rightarrow \infty} \text{tp}_\Delta(c_i/\mathcal{U}) = p|_\Delta$.*

Proof. Let \mathcal{L} be a countable sublanguage of \mathcal{L}' containing all the formulas in Δ . The corresponding type $p|_\mathcal{L}$ is generically stable over the model M where $M = M'|_\mathcal{L}$ (see [6, Remark 3.3]). Hence we may apply Theorem 4.8. \square

4.1. Examples and non-examples. We begin this subsection by collecting the known examples of sequentially approximated types. We then go on to give two examples of types which are not sequentially approximated (over any model).

Observation 4.10. *Assume that $p \in S_x(\mathcal{U})$ and let M be a small elementary submodel. Then, p is sequentially approximated over M if*

- (i) T is stable, and p is invariant over M ,
- (ii) T is NIP, $|M| = \aleph_0$, and p is finitely satisfiable over M , or
- (iii) p is generically stable over M .

We just proved (iii). Clearly, (i) follows from (iii) (we remark that it also follows from (ii)). As noted previously, the proof of (ii) is precisely [17, Lemma 2.8].

We now exhibit some concrete examples of types which are not sequentially approximated. We begin by describing a type in an NIP theory which is finitely satisfiable over a small model but not sequentially approximated (and its associated Keisler measure is not sequentially approximated either). We then discuss a finitely approximated type which is not sequentially approximated.

Proposition 4.11. *Let ω_1 be the first uncountable ordinal, $M = (\omega_1; <)$ with the usual ordering, and let $T_{<}$ be the theory of M in the language $\{<\}$. Recall that $T_{<}$ is NIP. Let $p \in S_x(\omega_1)$ be the complete type extending $\{\alpha < x : \alpha < \omega_1\}$. Let \mathcal{U} be*

a monster model of $T_{<}$ such that $M \prec \mathcal{U}$ and let $p_* \in S_x(\mathcal{U})$ be the unique global coheir of p . Then, p_* is not sequentially approximated over any model.

Proof. Assume for the sake of contradiction that p_* is sequentially approximated over some model N . Then there exists a sequence of points $(b_i)_{i \in \omega}$ in N such that $\lim_{i \rightarrow \infty} \text{tp}(b_i/\mathcal{U}) = p_*$ in $S_x(\mathcal{U})$. There is either an infinite subsequence which is strictly increasing or strictly decreasing and so without loss of generality, $(b_i)_{i \in \omega}$ has one of these two properties. First assume that $(b_i)_{i \in \omega}$ is strictly increasing. Notice that $b_i < x \in p_*$. Since p_* is a coheir of p , p_* is finitely satisfiable over ω_1 . So, for each b_i there exists α in ω_1 such that $b_i < \alpha$. Now, for each b_i , we define $\alpha_i := \min\{\alpha \in \omega_1 : \mathcal{U} \models b_i < \alpha\}$. Since ω_1 is well-ordered, α_i is well-defined. We let β be the supremum (in ω_1) of $\{\alpha_i : i \in \omega\}$. Then $\mathcal{U} \models b_i < \beta$ for each $i \in \omega$, and so but $x < \beta \in p_*$, contradiction.

Now we assume that $(b_i)_{i \in \omega}$ is a strictly decreasing subsequence. Notice that for each $i \in \omega$, $b_i > x \in p_*$. Let $\Theta(x) = \{\alpha < x : \alpha \in \omega_1\} \cup \{x < b_i : i \in \omega\}$. By compactness, choose c_∞ in \mathcal{U} satisfying $\Theta(x)$. Since p_* is finitely satisfiable over ω_1 , we have $c_\infty > x \in p_*$. But since $\mathcal{U} \models b_i > c_\infty$ for each $i \in \omega$, we have that $x > c_\infty \in p$, contradiction. \square

Remark 4.12. The type p_* in Proposition 4.11 is finitely satisfiable over a small model, but not finitely satisfiable over any countable submodel by Theorem 1.1.

Proposition 4.13. Let p_* be as in Proposition 4.11. Then the associated Keisler measure δ_{p_*} is not sequentially approximated.

Proof. Clear from (iv) of Proposition 3.3. \square

Proposition 4.14. Let T_s^2 be the theory of the random K_s -free graph in the language $\mathcal{L} = \{E(x, y)\}$. Let p_* be the unique global complete type extending the formulas $\{\neg E(x, b) : b \in \mathcal{U}\}$. Then, δ_{p_*} is sequentially approximated (even finitely approximated over any submodel) but p_* is not sequentially approximated. Moreover, T_s^2 admits no (non-realized) sequentially approximated types.

Proof. The proof that δ_{p_*} is finitely approximated can be found in [6, Theorem 5.8]. By (iii) of Proposition 3.3, δ_{p_*} is sequentially approximated. By (v) of Proposition 3.3, it suffices to show that there are no non-realized types in one variable which are sequentially approximated. Let p be any non-realized type in $S_1(\mathcal{U})$ and assume that $(b_i)_{i \in \omega}$ is a sequence of points in \mathcal{U}^x such that $\lim_{i \rightarrow \infty} \text{tp}(b_i/\mathcal{U}) = p$. Since p is non-realized, we may assume that the points in $(b_i)_{i \in \omega}$ are distinct. Then, by Ramsey's theorem, there is a subsequence which is either independent or complete. It cannot be complete, because that would violate K_s -freeness. Therefore, $(b_i)_{i \in \omega}$ contains an independent subsequence, call it $(c_i)_{i \in \omega}$. By compactness, there exists an a in \mathcal{U} such that $\mathcal{U} \models E(c_i, a)$ if and only if i is even. Then, $(\text{tp}(c_i/\mathcal{U}))_{i \in \omega}$ does not converge in $S_x(\mathcal{U})$ and so $(\text{tp}(b_i/\mathcal{U}))_{i \in \omega}$ does not converge in $S_x(\mathcal{U})$. \square

Question 4.15. We say a global type p in $S_x(\mathcal{U})$ is **sad**² if it is both sequentially approximated and definable. Does there exist a global type p which is sad over a model M but is not generically stable over M ? It is clear that if T is NIP, then all sad types are generically stable. Therefore an example of such a type must come from *the wild*.

²Credit to James Hanson for the terminology.

5. SEQUENTIAL APPROXIMATIONS OF MEASURES IN NIP THEORIES

Throughout this section, we assume that T is a countable NIP theory and \mathcal{U} is a monster model of T . We show that measures which are finitely satisfiable over a countable model of T are sequentially approximated (Theorem T2). To do this, we introduce the notion of a *smooth sequence*. These are sequences of global measures which are intended to play the role of a Morley sequence for a measure. Unfortunately, these sequences only exist (a priori) in the NIP context and it is currently not known how to expand this idea to IP theories. At the end of this section, we give a characterization of generic stability using smooth sequences (again, only in the NIP context).

To motivate the machinery introduced in this section, we explain why Theorem T2 does not follow directly from some approximation results currently in the literature. One might assume that one could prove Theorem T2 from Theorem 1.1 in tandem with the following fact [16, Proposition 7.11],

Fact 5.1 (T is NIP). *Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and μ is finitely satisfiable over M . Then, for any formula $\varphi(x, y) \in \mathcal{L}$ and every $\epsilon > 0$, there exists types $p_1, \dots, p_n \in S_x(\mathcal{U})$, where for each $i \leq n$ the type p_i is finitely satisfiable over M , and*

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x, b)) - \text{Av}(\bar{p})(\varphi(x, b))| < \epsilon.$$

If μ is in $\mathfrak{M}_x(\mathcal{U})$ and is finitely satisfiable over a countable model M , then one can use Theorem 1.1 and Fact 5.1 together to produce:

- (1) a sequence of global measures $(\text{Av}(\bar{p}_i))_{i \in \mathbb{N}}$ such that each $\bar{p}_i = (p_{i_1}, \dots, p_{i_k})$, each $p_{i_k} \in S_x(\mathcal{U})$ is finitely satisfiable over M , and $\lim_{i \rightarrow \infty} \text{Av}(\bar{p}_i) = \mu$ in $\mathfrak{M}_x(\mathcal{U})$,
- (2) for each $i \in \mathbb{N}$, a sequence of points $(\bar{a}_{i_j})_{j \in \mathbb{N}}$ each in $(M^x)^{<\omega}$ so that $\lim_{j \rightarrow \infty} \text{Av}(\bar{a}_{i_j}) = \text{Av}(\bar{p}_i)$.

This construction gives an *array* of points $(\bar{a}_{i_j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ in $(M^x)^{<\omega}$ so that

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} (\text{Av}(\bar{a}_{i_j})) = \mu \text{ in } \mathfrak{M}_x(\mathcal{U}).$$

A priori, the convergence of an array *does not imply* that there exists a subsequence of that array which converges to the array's limit³. A similar situation arises by trying to iterate Theorem 5.13. So, we must work slightly harder. As previously stated, our proof essentially mimics the proof of Theorem 1.1 but with Morley sequences replaced by *smooth sequences*. Finally we remark that if there were an *elementary proof* using an array to show this result, then we would have a moderately simple proof that dfs measures are finitely approximated in NIP theories. In particular, this proof would bypass the implicit use of randomizations (i.e. (i) of Fact 5.5).

We formally begin this section by discussing a “continuous” analogue of eventually indiscernible sequences.

5.1. Eventually indiscernible sequences revisited. We fix some notation. Fix distinct tuples of variables x and x_0, \dots, x_n such that $|x| = |x_i|$ for $i \leq n$. If

³For example, any Baire-2 function which is not Baire-1 can be written as the limit of an array of continuous functions, but cannot be written as the sequential limit of continuous functions.

$\varphi(x_0, \dots, x_n)$ is a formula in $\mathcal{L}_{x_0, \dots, x_n}(\mathcal{U})$ and $\bar{a}_0, \dots, \bar{a}_n$ is a finite sequence of elements where each $\bar{a}_i \in (\mathcal{U}^x)^{<\omega}$ and $\bar{a}_i = (a_{i,0}, \dots, a_{i,m_i})$ for $i \leq n$, then we write $\varphi_c(\bar{a}_0, \dots, \bar{a}_n)$ to mean,

$$\bigotimes_{i=0}^n \text{Av}(\bar{a}_i)_{x_i}(\varphi(x_0, \dots, x_n)).$$

Notice that $\varphi_c(\bar{a}_0, \dots, \bar{a}_n)$ is a real number. We observe that by unpacking the definition of the product measure, our formula can be computed as follows:

$$\varphi_c(\bar{a}_0, \dots, \bar{a}_n) = \frac{1}{\prod_{i=0}^n (m_i + 1)} \sum_{j_0=0}^{m_0} \dots \sum_{j_n=0}^{m_n} \mathbf{1}_\varphi(a_{0,j_0}, \dots, a_{n,j_n}).$$

Definition 5.2. Let $(\bar{a}_i)_{i \in \omega}$ be a sequence of elements in $(\mathcal{U}^x)^{<\omega}$ and let $A \subset \mathcal{U}$ be a collection of parameters. Then we say that the sequence $(\bar{a}_i)_{i \in \omega}$ is **eventually indiscernible over** A if for any formula $\varphi(x_0, \dots, x_n)$ in $\mathcal{L}_{(x_i)_{i \in \omega}}(A)$ and any $\epsilon > 0$, there exists $N_{\epsilon, \varphi}$ such that for any $n_k > \dots > n_0 > N_{\epsilon, \varphi}$ and $m_k > \dots > m_0 > N_{\epsilon, \varphi}$,

$$|\varphi_c(\bar{a}_{n_0}, \dots, \bar{a}_{n_k}) - \varphi_c(\bar{a}_{m_0}, \dots, \bar{a}_{m_k})| < \epsilon.$$

Proposition 5.3. Let $(\bar{a}_i)_{i \in \omega}$ be a sequence of tuples in $(\mathcal{U}^x)^{<\omega}$. If A is a countable set of parameters, then there exists some subsequence $(\bar{c}_i)_{i \in \omega}$ of $(\bar{a}_i)_{i \in \omega}$ such that $(\bar{c}_i)_{i \in \omega}$ is eventually indiscernible over A .

Proof. This proof is a standard application of Ramsey's theorem applied to the "continuous" setting. Enumerate all pairs in $\mathcal{L}_{(x_i)_{i \in \omega}}(A) \times \mathbb{N}_{>0}$. Let $(\bar{a}_i^0)_{i \in \omega} := (\bar{a}_i)_{i \in \omega}$ and set $B_0 = \{\bar{a}_i^0 : i \in \omega\}$. Now, assume we have constructed the subsequence $(\bar{a}_i^l)_{i \in \omega}$ and B_l (where $B_l = \{\bar{a}_i^l : i \in \omega\}$). We now construct $(\bar{a}_i^{l+1})_{i \in \omega}$ and B_{l+1} . Assume that $(\varphi(x_0, \dots, x_k), n)$ is the $l+1$ indexed pair in our enumeration. Then we define the coloring $r_{l+1} : [B_l]^{k+1} \rightarrow \{0, \dots, n\}$ via

$$r(\{\bar{a}_{i_0}^l, \dots, \bar{a}_{i_k}^l\}) = \lfloor n \cdot \varphi_c(\bar{a}_{i_0}^l, \dots, \bar{a}_{i_k}^l) \rfloor.$$

where $i_0 < i_1 < \dots < i_k$. By Ramsey's theorem, there is an infinite monochromatic subset B'_l of B_l . Let $(\bar{a}_i^{l+1})_{i \in \omega}$ be the obvious reindexed subsequence of $(\bar{a}_i^l)_{i \in \omega}$ with the elements only from the monochromatic set B'_l . We let $B_{l+1} = \{\bar{a}_i^{l+1} : i \in \omega\}$. By construction, the sequence $(\bar{a}_i^i)_{i \in \omega}$ is eventually indiscernible. \square

We now present a collection of facts which will help us prove that the associated average measures along eventually indiscernible sequences always converge to a measure in $\mathfrak{M}_x(\mathcal{U})$ when the underlying theory is NIP. The first fact is elementary and left to the reader as an exercise.

Fact 5.4. Assume that $(\mu_i)_{i \in \omega}$ is a sequence of Keisler measures in $\mathfrak{M}_x(\mathcal{U})$. If for every formula $\varphi(x) \in \mathcal{L}_x(\mathcal{U})$, $\lim_{i \rightarrow \infty} \mu_i(\varphi(x))$ converges, then $(\mu_i)_{i \in \omega}$ converges to a measure in $\mathfrak{M}_x(\mathcal{U})$.

The next collection of facts can be found in [10]. In particular, (i) follows immediately from Lemma 2.10 while (ii) and (iii) are from Corollary 2.14. The proof of Lemma 2.10 is non-trivial and is an interpretation of results in [2]. Implicitly, our proof uses the fact that the randomization of an NIP theory is NIP.

Fact 5.5 (T is NIP). Suppose that $\lambda \in \mathfrak{M}_{(x_i)_{i \in \omega}}$ where $|x_i| = |x_j|$ for each $i, j < \omega$. λ is said to be **M-indiscernible** if for every increasing sequence of indices i_0, \dots, i_n and any formula $\varphi(x_{i_0}, \dots, x_{i_n})$ in $\mathcal{L}_{(x_i)_{i \in \omega}}(M)$, we have that

$$\lambda(\varphi(x_{i_0}, \dots, x_{i_n})) = \lambda(\varphi(x_0, \dots, x_n)).$$

Let $\mu, \nu \in \mathfrak{M}_x(\mathcal{U})$ such that μ, ν are invariant over M . The following statements are true.

- (i) If λ is M -indiscernible, then for any formula $\varphi(x, b) \in \mathcal{L}_x(\mathcal{U})$, we have that $\lim_{i \rightarrow \infty} \lambda(\varphi(x_i, b))$ exists.
- (ii) The measures $\mu^{(\omega)}$ and $\nu^{(\omega)}$ are M -indiscernible.
- (iii) If $\mu^{(\omega)}|_M = \nu^{(\omega)}|_M$, then $\mu = \nu$.

We now establish a formal connection between eventually indiscernible sequences of tuples and indiscernible measures. We use this connection to show that the eventually indiscernible sequences converges to a measure in $\mathfrak{M}_x(\mathcal{U})$.

Proposition 5.6. *Let $(\bar{c}_i)_{i \in \omega}$ be a sequence of points in $(\mathcal{U}^x)^{<\omega}$. If $(\bar{c}_i)_{i \in \omega}$ is an eventually indiscernible sequence over some model M , then the sequence $(\text{Av}(\bar{c}_i))_{i \in \omega}$ converges in $\mathfrak{M}_x(\mathcal{U})$.*

Proof. Assume not. Then there exists some formula $\psi(x, b)$ in $\mathcal{L}_x(\mathcal{U})$, some $\epsilon_0 > 0$, and some subsequence $(\bar{c}'_i)_{i \in \omega}$ of $(\bar{c}_i)_{i \in \omega}$ such that for each natural number i ,

$$|\text{Av}(\bar{c}'_i)(\psi(x; b)) - \text{Av}(\bar{c}'_{i+1})(\psi(x; b))| > \epsilon_0.$$

It is clear that $(\bar{c}'_i)_{i \in \omega}$ is also eventually indiscernible over M . We now aim to contradict (i) of Fact 5.5 via (topological) compactness of the space $\mathfrak{M}_\omega(\mathcal{U}) := \mathfrak{M}_{(x_i)_{i \in \omega}}(\mathcal{U})$. For any formula $\varphi(x_{i_0}, \dots, x_{i_k}) \in \mathcal{L}_{(x_i)_{i \in \omega}}(M)$, we let r_φ be the unique real number such that for every $\epsilon > 0$, there exists an $N_{\epsilon, \varphi}$ so that for any $n_k > \dots > n_0 > N_{\epsilon, \varphi}$ we have

$$|\varphi_c(\bar{c}'_{n_0}, \dots, \bar{c}'_{n_k}) - r_\varphi| < \epsilon.$$

Since the sequence $(\bar{c}'_i)_{i \in \omega}$ is eventually indiscernible over M , r_φ exists for each $\varphi(\bar{x}) \in \mathcal{L}_{(x_i)_{i \in \omega}}(M)$. Now, for every $\varphi(\bar{x}) \in \mathcal{L}_{(x_i)_{i \in \omega}}(M)$ and $\epsilon > 0$, we define the following family of closed subsets of $\mathfrak{M}_\omega(\mathcal{U})$;

$$C_{\epsilon, \varphi} = \left\{ \lambda \in \mathfrak{M}_\omega(\mathcal{U}) : r_\varphi - \epsilon \leq \lambda(\varphi(\bar{x})) \leq r_\varphi + \epsilon \right\}.$$

We also define another family of sets and argue that they are closed; let

$$D_i = \left\{ \lambda \in \mathfrak{M}_\omega(\mathcal{U}) : |\lambda(\psi(x_i, b)) - \lambda(\psi(x_{i+1}, b))| \geq \frac{\epsilon_0}{2} \right\}.$$

Notice that D_i is closed since for every natural number i , the evaluation map $E_i : \mathfrak{M}_\omega(\mathcal{U}) \rightarrow [0, 1]$ via $E_i(\lambda) = \lambda(\varphi(x_i, b))$ is continuous. Indeed, define $F_i = E_i - E_{i+1}$ and $H_i = E_{i+1} - E_i$. Then we have $D_i = F_i^{-1}([\frac{\epsilon_0}{2}, 1]) \cup H_i^{-1}([\frac{\epsilon_0}{2}, 1])$ and so D_i is a union of two closed sets and therefore closed. Using $(\bar{c}'_i)_{i \in \omega}$, the collection $\Phi = \{C_{\epsilon, \varphi} : \epsilon > 0, \varphi(\bar{x}) \in \mathcal{L}_\omega(M)\} \cup \{D_i : i \in \omega\}$ has the finite intersection property. Therefore, there exists some $\lambda \in \mathfrak{M}_\omega(\mathcal{U})$ in the intersection of all the sets in Φ . Moreover, λ is M -indiscernible by construction. Since λ is in D_i for each i , its existence contradicts (i) of Fact 5.5. \square

5.2. Smooth sequences. In this subsection, we define the notion of a smooth sequence and prove the main theorem. If μ is a global M -invariant measure, then a smooth sequence is a collection of models and measures meant to replicate a Morley sequence. The ideology is the following: A Morley sequence in p over M is to the infinite type $p^\omega|_M$ as a smooth sequence in μ over M is to the measure $\mu^{(\omega)}|_M$. We now provide the formal definition.

Definition 5.7. Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ and assume that μ is invariant over some small model M . Then, a **smooth sequence in μ over M** is a sequence of pairs of measures and small models, $(\mu_i, N_i)_{i \in \omega}$, such that:

- (i) $M \prec N_0$ and $N_i \prec N_{i+1}$ and each N_i is small.
- (ii) μ_i is smooth over N_i .
- (iii) $\mu_0|_M = \mu|_M$ and for $i > 0$, $\mu_i|_{N_{i-1}} = \mu|_{N_{i-1}}$.

Furthermore, we define $\bigotimes_{i=0}^{\omega} \mu_i = \bigcup_{i=0}^{\omega} \bigotimes_{i=0}^n \mu_i$ which is an element of $\mathfrak{M}_{(x_i)_{i \in \omega}}(\mathcal{U})$. We let $N_{\omega} = \bigcup_{i \in \omega} N_i$. Notice that for each $i \in \omega$, the measure μ_i is smooth over N_{ω} .

Proposition 5.8. *If T is a countable NIP theory, $\mu \in \mathfrak{M}_x(\mathcal{U})$, and μ is invariant over M where $|M| = \aleph_0$, then there exists a smooth sequence $(\mu_i, N_i)_{i \in \omega}$ in μ over M such that each N_i is countable.*

Proof. This follows directly from Proposition 2.9. \square

Proposition 5.9 (T is NIP). *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and μ is M -invariant. Let $(\mu_i, N_i)_{i \in \omega}$ be a smooth sequence in μ over M . Then, $\bigotimes_{i=0}^{\omega} \mu_i|_M = \mu^{(\omega)}|_M$. Hence, $\bigotimes_{i=0}^{\omega} \mu_i$ is M -indiscernible.*

Proof. We prove this via induction on formulas in $\mathcal{L}_{(x_i)_{i \in \omega}}(\mathcal{U})$. For our base case, it is true by construction that $\mu_0|_M = \mu|_M$. For our induction hypothesis, we assume that $\mu^{(k-1)}|_M = \bigotimes_{i=0}^{k-1} \mu_i|_M$. For ease of notation, we set $\lambda = \bigotimes_{i=0}^{k-1} \mu_i$ and show the induction step: Let $\varphi(x_0, \dots, x_k)$ be any formula in $\mathcal{L}_{x_0, \dots, x_k}(M)$. Since the product of smooth measures is smooth (by (iii) of Fact 2.12), we have that λ is smooth over N_{k-1} . In particular, λ is invariant over N_{k-1} . We let $\bar{x} = (x_0, \dots, x_{k-1})$ and $\theta(x_k; \bar{x}) = \varphi(x_0, \dots, x_k)$. We consider the following computation followed by a list of justifications.

$$\begin{aligned}
\mu_k \otimes \lambda(\varphi(x_0, \dots, x_k)) &= \int_{S_{\bar{x}}(N_k)} F_{\mu_k}^{\theta} d(\lambda|_{N_k}) \stackrel{(a)}{=} \int_{S_{x_k}(N_k)} F_{\lambda}^{\theta*} d(\mu_k|_{N_k}) \\
&\stackrel{(b)}{=} \int_{S_{x_k}(N_{k-1})} F_{\lambda}^{\theta*} d(\mu_k|_{N_{k-1}}) \stackrel{(c)}{=} \int_{S_{x_k}(N_{k-1})} F_{\lambda}^{\theta*} d(\mu|_{N_{k-1}}) \stackrel{(a)}{=} \int_{S_{\bar{x}}(N_{k-1})} F_{\mu}^{\theta} d(\lambda|_{N_{k-1}}) \\
&\stackrel{(d)}{=} \int_{S_{\bar{x}}(M)} F_{\mu}^{\theta} d(\lambda|_M) \stackrel{(e)}{=} \int_{S_{\bar{x}}(M)} F_{\mu}^{\theta} d(\mu^{(k-1)}|_M) = \mu \otimes \mu^{(k-1)}(\varphi(x_0, \dots, x_k)).
\end{aligned}$$

We provide the following justifications:

- (a) Smooth measures commute with invariant measures.
- (b) Changing space of integration since λ is invariant over N_{k-1} .
- (c) By construction of smooth sequences, we have that $\mu_k|_{N_{k-1}} = \mu|_{N_{k-1}}$.
- (d) Changing space of integration since μ is invariant over M .
- (e) By our induction hypothesis. \square

We now begin the proof of our main theorem. Again, the proof is similar to both the generically stable case in the previous section and even more so to the proof of Lemma 2.8 in [17]. Here, the major difference is that we replace the Morley sequence in that proof with a countable model, N_{ω} , which “contains” a smooth sequence in μ over M . Then we find a sequence of elements in $(M^x)^{<\omega}$ such that the associated average measures converge to $\mu|_{N_{\omega}}$ in $\mathfrak{M}_x(N_{\omega})$. After choosing an eventually indiscernible subsequence, we know from our NIP assumption that this

new sequence converges to a global measure ν in $\mathfrak{M}_x(\mathcal{U})$. Finally, we demonstrate that $\nu^{(\omega)}|_M = \mu^{(\omega)}|_M$ which completes the proof.

Theorem 5.10 (T is NIP). *Let μ be finitely satisfiable over a countable model M . Then there exists a sequence $(\bar{a}_i)_{i \in \omega}$ of elements, each in $(M^x)^{<\omega}$, such that for any $\theta(x) \in \mathcal{L}_x(\mathcal{U})$, we have that,*

$$\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i)(\theta(x)) = \mu(\theta(x)).$$

Proof. Choose a smooth sequence $(\mu_i, N_i)_{i \in \omega}$ in μ over M . By Proposition 5.8 we may choose this sequence so that for each $i \in \omega$, N_i is countable. In particular, this implies that N_ω is a countable model. We begin by constructing a sequence of elements $(\bar{a}_i)_{i \in \omega}$ in $(M^x)^{<\omega}$ such that $(\text{Av}(\bar{a}_i)|_{N_\omega})_{i \in \omega}$ converges to $\mu|_{N_\omega}$ in $\mathfrak{M}_x(N_\omega)$. Since N_ω is countable, we let $(\theta_i(x))_{i \in \omega}$ be an enumeration of the formulas in $\mathcal{L}_x(N_\omega)$. Since μ is finitely satisfiable over M , we can find we find $\bar{a}_k \in (M^x)^{<\omega}$ such that for any $j \leq k$, we have that,

$$|\mu(\theta_j(x)) - \text{Av}(\bar{a}_k)(\theta_j(x))| < \frac{1}{k}.$$

By construction, it is clear that the sequence $(\text{Av}(\bar{a}_i)|_{N_\omega})_{i \in \omega}$ converges to $\mu|_{N_\omega}$ in $\mathfrak{M}_x(N_\omega)$. Now, we let $(\bar{c}_i)_{i \in \omega}$ be a subsequence of $(\bar{a}_i)_{i \in \omega}$ so that $(\bar{c}_i)_{i \in \omega}$ is eventually indiscernible over N_ω . Then the sequence $(\text{Av}(\bar{c}_i))_{i \in \omega}$ converges in $\mathfrak{M}_x(\mathcal{U})$ by Proposition 5.6. Assume that $(\text{Av}(\bar{c}_i))_{i \in \omega}$ converges to some measure $\nu \in \mathfrak{M}_x(\mathcal{U})$. Hence, ν is finitely satisfiable over M by (i) of Proposition 3.3 and therefore ν is invariant over M . We show that $\nu^{(\omega)}|_M = \mu^{(\omega)}|_M$. This will conclude the proof by (iii) of Fact 5.5.

Since $(\bar{c}_i)_{i \in \omega}$ is a subsequence of $(\bar{a}_i)_{i \in \omega}$, it follows that $\nu|_{N_\omega} = \mu|_{N_\omega}$ and therefore $\nu|_M = \mu|_M$. We now proceed by induction. Assume that $\nu^{(k-1)}|_M = \mu^{(k-1)}|_M$. Fix $\varphi(x_0, \dots, x_k)$ in $\mathcal{L}_{x_0, \dots, x_k}(M)$. For ease of notation, set $\lambda = \bigotimes_{i=0}^{k-1} \mu_i$. We recall that λ is smooth over N_ω (see Fact 2.12). By Proposition 5.9, $\mu^{(k-1)}|_M = \lambda|_M$. We let $\bar{x} = (x_0, \dots, x_{k-1})$ and let $\theta(x_k; \bar{x}) = \varphi(x_0, \dots, x_k)$. We now consider the critical computation followed a small glossary of justifications.

$$\begin{aligned} \nu^{(k)}(\varphi(x_0, \dots, x_k)) &= \int_{S_{\bar{x}}(M)} F_\nu^\theta d(\nu^{(k-1)}|_M) \stackrel{(a)}{=} \int_{S_{\bar{x}}(M)} F_\nu^\theta d(\mu^{(k-1)}|_M) \\ &\stackrel{(b)}{=} \int_{S_{\bar{x}}(M)} F_\nu^\theta d(\lambda|_M) \stackrel{(c)}{=} \int_{S_{\bar{x}}(N_\omega)} F_\nu^\theta d(\lambda|_{N_\omega}) \stackrel{(d)}{=} \int_{S_{x_k}(N_\omega)} F_\lambda^{\theta^*} d(\nu|_{N_\omega}) \\ &\stackrel{(e)}{=} \int_{S_{x_k}(N_\omega)} F_\lambda^{\theta^*} d(\mu|_{N_\omega}) \stackrel{(d)}{=} \int_{S_{\bar{x}}(N_\omega)} F_\mu^\theta d(\lambda|_{N_\omega}) \stackrel{(c)}{=} \int_{S_{\bar{x}}(M)} F_\mu^\theta d(\lambda|_M) \\ &\stackrel{(b)}{=} \int_{S_{\bar{x}}(M)} F_\mu^\theta d(\mu^{(k-1)}|_M) = \mu^{(k)}(\varphi(x_0, \dots, x_k)). \end{aligned}$$

We provide the following justifications:

- (a) Induction hypothesis.
- (b) $\mu^{(k-1)}|_M = \lambda|_M$.
- (c) Changing the space of integration.
- (d) Smooth measures commute with invariant measures.
- (e) $\nu|_{N_\omega} = \mu|_{N_\omega}$

□

We now observe that we have another proof of the theorem that global measures in NIP theories which are definable and finitely satisfiable are also finitely approximated.

Corollary 5.11. *If T' is a countable or uncountable NIP theory and μ is dfs over M , then μ is finitely approximated over M .*

Proof. After restricting to a countable language, we still have a dfs measures (by [6, Proposition 2.9]). By Proposition 2.9, μ restricted to this language is dfs over a countable model, M_0 . By the previous result, μ is sequentially approximated over M_0 . Since μ is also definable, an application of Proposition 3.4 yields the result. \square

Observation 5.12. *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$ and let M be a small elementary submodel. Then, μ is sequentially approximated over M if*

- (1) T is stable, and μ is invariant over M ,
- (2) T is NIP, $|M| = \aleph_0$, and μ is finitely satisfiable over M , or
- (3) μ is finitely approximated over M .

Finally, one may ask what happens in the local context. We remark that there exists two proofs for a local version of Theorem T2 which both rely on an important result of Bourgain, Fremlin, and Talagrand whose connection to model theory is (by now) well-known (e.g. [11, 18, 12, 8]). Chronologically, the first proof of the following theorem is implicit in the work of Khanaki (see [12, Remark 3.21, Theorem 3.26]) (through the observation that measures are types over models of the randomization in continuous model theory and [1, Proposition 1.1]),

Theorem 5.13. *Suppose μ is a Keisler measure in $\mathfrak{M}_\varphi(\mathcal{U})$, μ is finitely satisfiable over M where $|M| = \aleph_0$, and $\varphi(x, y)$ is an NIP formula. Then there exists a sequence of points $(\bar{a}_i)_{i \in \omega}$ in $(M^x)^{<\omega}$ such that for each $b \in \mathcal{U}^y$,*

$$\lim_{i \rightarrow \infty} \text{Av}(\bar{a}_i)(\varphi(x, b)) = \mu(\varphi(x, b)).$$

There is another proof for the case of just Keisler measures via the VC theorem (see [8, Lemma 4.7]) which came later.

5.3. Smooth sequences and generically stable measures in NIP theories.

We now give an equivalent characterization for generically stable measures in NIP theories. We invite the reader to review the definition of a generically stable type prior to reading this section. Recall the following theorem due to Hrushovski, Pillay, and Simon [10, Theorem 3.2].

Theorem 5.14 (T is NIP). *Assume that $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then the following are equivalent.*

- (i) μ is dfs.
- (ii) μ is finitely approximated.
- (iii) μ is fim (see [10, Definition 2.7]).
- (iv) μ is invariant and $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.

*Moreover, a Keisler measure (in an NIP theory) is called **generically stable** if it satisfies any/all of (i) – (iv).*

We will now show that smooth sequences can also give a characterization of generically stable measures in NIP theories.

Lemma 5.15 (T is NIP). *Let $\mu \in \mathfrak{M}_x(\mathcal{U})$. Suppose that μ is generically stable over M . For any smooth sequence $(\mu_i, N_i)_{i \in \omega}$ in μ over M , we have that $\lim_{i \rightarrow \infty} \mu_i = \mu$ in $\mathfrak{M}_x(\mathcal{U})$.*

Proof. Since $(\mu_i, N_i)_{i \in \omega}$ is a smooth sequence in μ over M , the measure $\bigotimes_{i=0}^{\omega} \mu_i$ is indiscernible over M by Proposition 5.9. By (i) of Fact 5.5, we know that $\lim_{i \rightarrow \infty} \mu_i = \nu$ for some $\nu \in \mathfrak{M}_x(\mathcal{U})$. Since each μ_i is finitely satisfiable over N_i , it follows that ν is finitely satisfiable over N_ω . By (iii) of Fact 5.5, it is enough to show that $\nu^{(\omega)}|_{N_\omega} = \mu^{(\omega)}|_{N_\omega}$. The base case is trivial. Assume that $\nu^{(k-1)}|_{N_\omega} = \mu^{(k-1)}|_{N_\omega}$. Fix $\varphi(x_0, \dots, x_k) \in \mathcal{L}_{x_0, \dots, x_k}(N_\omega)$ and $\epsilon > 0$. Let $\bar{x} = (x_0, \dots, x_{k-1})$ and $\theta(x_k; \bar{x}) = \varphi(x_0, \dots, x_k)$. Since μ is generically stable over M , $\mu^{(k-1)}$ is generically stable over M ((v) of Fact 2.12) and so also definable over N_ω . Therefore by (v) of Fact 2.8, there exists formulas $\psi_1(x_k), \dots, \psi_n(x_k) \in \mathcal{L}_{x_k}(N_\omega)$ and real numbers $r_1, \dots, r_n \in [0, 1]$ so that

$$\sup_{q \in S_{x_k}(N_\omega)} |F_{\mu^{(k-1)}}^{\theta^*}(q) - \sum_{i=1}^n r_i \mathbf{1}_{\psi_i(x_k)}(q)| < \epsilon.$$

Consider the following sequence of equations followed by a short list of justifications.

$$\begin{aligned} \nu^{(k)}(\varphi(x_0, \dots, x_k)) &= \int_{S_{\bar{x}}(N_\omega)} F_{\nu}^{\theta} d(\nu^{(k-1)}|_{N_\omega}) \stackrel{(a)}{=} \int_{S_{\bar{x}}(N_\omega)} F_{\nu}^{\theta} d(\mu^{(k-1)}|_{N_\omega}) \\ &\stackrel{(b)}{=} \int_{S_{x_k}(N_\omega)} F_{\mu^{(k-1)}}^{\theta^*} d(\nu|_{N_\omega}) \approx_{\epsilon} \int_{S_{x_k}(N_\omega)} \sum_{i=1}^n r_i \mathbf{1}_{\psi_i(x_k)} d(\nu|_{N_\omega}) \\ &= \sum_{i=1}^n r_i \nu(\psi_i(x_k)) \stackrel{(c)}{=} \sum_{i=1}^n r_i \mu(\psi_i(x_k)) = \int_{S_{x_k}(N_\omega)} \sum_{i=1}^n r_i \mathbf{1}_{\psi_i(x_k)} d(\mu|_{N_\omega}) \\ &\approx_{\epsilon} \int_{S_{x_k}(N_\omega)} F_{\mu^{(k-1)}}^{\theta^*} d(\mu|_{N_\omega}) \stackrel{(b)}{=} \int_{S_{x_k}(N_\omega)} F_{\mu}^{\theta} d(\mu^{(k-1)}|_{N_\omega}) = \mu^{(k)}(\varphi(x_0, \dots, x_k)). \end{aligned}$$

(a) Induction hypothesis.

(b) (T is NIP) Generically stable measures commute with invariant measures (see (b) of Fact 2.12).

(c) Base case.

As ϵ was arbitrary, this proves the result. \square

Lemma 5.16 (T is NIP). *Assume that μ is M -invariant. If for every smooth sequence $(\mu_i, N_i)_{i \in \mathbb{N}}$ in μ over M , we have that $\lim_{i \rightarrow \infty} \mu_i = \mu$, then μ is generically stable over M .*

Proof. Since T is NIP, all invariant measures are Borel definable. By Theorem 5.14, it suffices to show that μ commutes with itself, i.e. $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$. Fix $\varphi(x, y) \in \mathcal{L}_{x, y}(\mathcal{U})$. Let M_1 be a small model such that $M \prec M_1$ and M_1 contains all the parameters from $\varphi(x, y)$. We choose a smooth sequence $(\mu_{i, x}, N_i)_{i \in \omega}$ in μ_x over M_1 and let $N_\omega = \bigcup_{i \in \omega} N_i$. By construction, the sequence $(\mu_{i, x}, N_i)_{i \in \omega}$ is a smooth sequence in μ_x over M . Consider the following computation.

$$\mu_x \otimes \mu_y(\varphi(x, y)) = \int_{S_y(M_1)} F_{\mu_x}^{\varphi} d(\mu_y|_{M_1}) \stackrel{(a)}{=} \int_{S_y(N_\omega)} F_{\mu_x}^{\varphi} d(\mu_y|_{N_\omega})$$

$$\begin{aligned}
& \stackrel{(b)}{=} \lim_{i \rightarrow \infty} \int_{S_y(N_\omega)} F_{\mu_{i,x}}^\varphi d(\mu_y|_{N_\omega}) \stackrel{(c)}{=} \lim_{i \rightarrow \infty} \int_{S_x(N_\omega)} F_{\mu_y}^{\varphi*} d(\mu_{i,x}|_{N_\omega}) \\
& \stackrel{(d)}{=} \lim_{i \rightarrow \infty} \int_{S_x(M_1)} F_{\mu_y}^{\varphi*} d(\mu_{i,x}|_{M_1}) \stackrel{(e)}{=} \lim_{i \rightarrow \infty} \int_{S_x(M_1)} F_{\mu_y}^{\varphi*} d(\mu_x|_{M_1}) \\
& = \int_{S_x(M_1)} F_{\mu_y}^{\varphi*} d(\mu_x|_{M_1}) = \mu_y \otimes \mu_x(\varphi(x, y)).
\end{aligned}$$

We provide a list of the following justifications:

- (a) Changing the space of integration.
- (b) Dominated convergence theorem.
- (c) Smooth measures commute with Borel definable measures.
- (d) Since μ_y is M_1 invariant.
- (e) Since $\mu_{i,x}|_{M_1} = \mu_x|_{M_1}$ for any $i \in \omega$. □

Theorem 5.17 (T is NIP). *Let $\mu \in \mathfrak{M}_x(\mathcal{U})$. Then the following are equivalent:*

- (1) μ is generically stable over M .
- (2) For any smooth sequence $(\mu_i, N_i)_{i \in \omega}$ in μ over M ,

$$\lim_{i \rightarrow \infty} \mu_i = \mu \text{ in } \mathfrak{M}_x(\mathcal{U}).$$

Proof. Follows directly from the previous two lemmas. □

6. LOCAL MEASURES REVISITED

We generalize the main theorem of [8]. Fix a partitioned NIP formula $\varphi(x, y)$ and let μ be a φ -measure. In [8], we proved two main theorems. We showed that if $\varphi(x, y)$ is an NIP formula and μ is φ -definable and finitely satisfiable over a **countable** model M , then μ is φ -finitely approximated. We then proved that if μ is definable and finitely satisfiable over any small model M , then μ is finitely approximated in M by reducing to the previous theorem. But this was somewhat unsatisfactory and the following question was left open: if μ is φ -definable and finitely satisfiable over a **small** model, then is μ φ -finitely approximated? We give a positive answer to this question by modifying one of the important technical lemmas in the proof. Let us first recall some definitions.

Definition 6.1. Fix \mathcal{U} and a formula $\varphi(x, y)$ in $\mathcal{L}(\mathcal{U})$.

- (1) $\mathcal{L}_\varphi(\mathcal{U})$ denotes the Boolean algebra of definable sets of \mathcal{U}^x generated by the collection $\{\varphi(x, b) : b \in \mathcal{U}\}$.
- (2) A φ -measure is a finitely additive measure on the Boolean algebra $\mathcal{L}_\varphi(\mathcal{U})$.
- (3) The collection of all φ -measures is denoted $\mathfrak{M}_\varphi(\mathcal{U})$.
- (4) Let $M \prec \mathcal{U}$ and assume that M contains all the parameters from $\varphi(x, y)$. For any $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$, we say that μ is (M, φ) -invariant if for any $b, c \in \mathcal{U}^y$ such that $\text{tp}(b/M) = \text{tp}(c/M)$, we have that $\mu(\varphi(x, b)) = \mu(\varphi(x, c))$.
- (5) Let $\mu \in \mathfrak{M}_\varphi(M)$. If μ is (M, φ) -invariant, then we define the fiber map $F_\mu^\varphi : S_y(M) \rightarrow [0, 1]$ via $F_{\mu, M}^\varphi(q) = \mu(\varphi(x, b))$ where $b \models q|_M$. When M is clear from context, we write $F_{\mu, M}^\varphi$ simply as F_μ^φ .
- (6) Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Then μ is said to be φ -definable if the map $F_{\mu, M}^\varphi : S_y(M) \rightarrow [0, 1]$ is continuous.
- (7) Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. Then μ is said to be definable if for any formula $\theta(x, \bar{y})$ in the algebra generated by $\{\varphi(x, y_i) : i \in \mathbb{N}\}$, μ is (M, θ) -invariant and the map $F_\mu^\varphi : S_y(M) \rightarrow [0, 1]$ is continuous.

- (8) For any $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$, μ is said to be finitely satisfiable in M if for every $\theta(x) \in \mathcal{L}_\varphi(\mathcal{U})$ such that $\mu(\theta(x)) > 0$, there exists some $a \in M$ so that $\mathcal{U} \models \theta(a)$.
- (9) For each $a \in M$ we let $F_a^\varphi : S_y(M) \rightarrow [0, 1]$ via $F_a^\varphi = \mathbf{1}_{\varphi(a, y)}$. We denote the collection of such functions as \mathbb{F}_M . We let $\text{conv}(\mathbb{F}_M)$ be the collection of convex combinations of elements in \mathbb{F}_M . We let $F = [0, 1]^{S_y(M)}$ endowed with the Tychonoff topology and if $A \subset F$, we let $\text{cl}(A)$ denote its closure in this space and so the set $\text{cl}(\text{conv}(A))$ is well-defined.

Recall the following facts about φ -measures which can be found in [8].

Fact 6.2. *Let $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$ and $M \prec \mathcal{U}$.*

- (i) *If μ is finitely satisfiable or φ -definable over M then μ is (M, φ) -invariant.*
- (ii) *If μ is φ -definable over M then μ is (M_0, φ) -invariant for some $M_0 \prec M$ such that $|M_0| = \aleph_0$.*
- (iii) *If μ is finitely satisfiable over M then $F_{\mu, M}^\varphi$ is in $\text{cl}(\text{conv}(\mathbb{F}_M))$.*
- (iv) *If $|M| = \aleph_0$ and $\varphi(x, y)$ is NIP, there exists a sequence of elements $(g_i)_{i \in \omega}$ with each $g_i \in \text{conv}(\mathbb{F}_M)$ so that $\lim_{i \rightarrow \infty} g_i = F_{\mu, M}^\varphi$.*

The following lemma is essentially the *missing lemma* from [8]. The missed observation is that one can consider finitely many parameters at once (instead of a single parameter).

Lemma 6.3. *Suppose that $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$ and μ is finitely satisfiable in a small submodel N and (M, φ) -invariant. Then the map $F_{\mu, M}^\varphi \in \text{cl}(\text{conv}(\mathbb{F}_M))$.*

Proof. The proof is similar to the proof for types [16, Lemma 2.18] as well as the proof for measures [8, Proposition 4.13] (which has both a stronger assumption and conclusion). It suffices to show that for any finite collection of types $p_1, \dots, p_n \in S_y(M)$ and $\epsilon > 0$ there exists $\bar{a} \in (M^x)^{<\omega}$ such that $F_{\text{Av}(\bar{a}), M}^\varphi(p_i) \approx_\epsilon F_{\mu, M}^\varphi(p_i)$ for each $i \leq n$.

Fix $p_1, \dots, p_n \in S_y(M)$ and $\epsilon > 0$. Choose $b_i \models p_i$ for $i \leq n$. Let $q = \text{tp}(N/M) \in S_{|N|}(M)$. Let $\hat{q} \in S_{|N|}(\mathcal{U})$ such that $\hat{q} \supset q$ and \hat{q} is finitely satisfiable in M , i.e. \hat{q} is a global coheir of q . Let $N_1 \models \hat{q}|_{Mb_1, \dots, b_n}$.

By compactness, there exists elements $b'_1, \dots, b'_n \in \mathcal{U}$ such that $\text{tp}(N_1 b_1, \dots, b_n/M) = \text{tp}(N b'_1, \dots, b'_n/M)$. Since μ is (M, φ) -invariant, we have that

$$F_{\mu, M}^\varphi(p_i) = \mu(\varphi(x, b_i)) = \mu(\varphi(x, b'_i)),$$

for each $i \leq n$. Since μ is finitely satisfiable in N , there exists some m and $\bar{c} \in (N^x)^m$ such that $\text{Av}(\bar{c})(\varphi(x, b'_i)) \approx_\epsilon \mu(\varphi(x, b'_i))$ for $i \leq n$. Let $B_i = \{j \leq m : \models \varphi(c_j, b'_i)\}$. Now consider the formula

$$\theta(x_1, \dots, x_m, y_1, \dots, y_n) = \bigwedge_{i \leq n} \left(\bigwedge_{j \in B_i} \varphi(x_j, y_i) \wedge \bigwedge_{j \notin B_i} \neg \varphi(x_j, y_i) \right).$$

By construction $\theta(\bar{x}, \bar{y}) \in \text{tp}(\bar{c}, \bar{b}'/M)$ and so for an appropriate choice of indices, $\theta(\bar{x}, \bar{y}) \in \text{tp}(N b'_1, \dots, b'_n/M)$. Hence $\theta(\bar{x}, \bar{y}) \in \text{tp}(N_1 b_1, \dots, b_n/M)$ and so $\theta(\bar{x}, \bar{b}) \in \text{tp}(N_1/M b_1, \dots, b_n) \subset \hat{q}$. Since \hat{q} is finitely satisfiable in M , there exists $\bar{a} \in (M^x)^m$ such that $\models \theta(\bar{a}, \bar{b})$. By construction, we have that for any $i \leq n$,

$$F_{\text{Av}(\bar{a}), M}^\varphi(p_i) = \text{Av}(\bar{a})(\varphi(x, b_i)) = \text{Av}(\bar{c})(\varphi(x, b'_i)) \approx_\epsilon \mu(\varphi(x, b'_i)) = F_{\mu, M}^\varphi(p_i).$$

This concludes the proof. \square

Theorem 6.4. *Fix a formula $\varphi(x, y)$ and a small model M containing all the parameters from $\varphi(x, y)$. Assume that $\mu \in \mathfrak{M}_\varphi(\mathcal{U})$. If*

- (1) $\varphi(x; y)$ is NIP,
- (2) μ is φ -definable over M ,
- (3) and μ is finitely satisfiable in M ,

Then for every $\epsilon > 0$, there exists $a_1, \dots, a_n \in M^x$ such that,

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x, b)) - \text{Av}(\bar{a})(\varphi(x, b))| < \epsilon.$$

Proof. We remark that the proof is similar to that of Proposition 3.4. Since μ is φ -definable over M , μ is (M_0, φ) -invariant where M_0 is a countable submodel of M . By Lemma 6.3, the map $F_{\mu, M_0}^\varphi \in \text{cl}(\text{conv}(\mathbb{F}_{M_0}))$. By Fact 6.2, there exists a sequence $(g_i)_{i \in I}$ so that $\lim_{i \rightarrow \infty} g_i = F_{\mu, M_0}^\varphi$. By Mazur's lemma, for every $\epsilon > 0$, there exists a finite set $I \subset \mathbb{N}$ and positive real numbers $\{r_i : i \in I\}$ such that $\sum_{i \in I} r_i = 1$ and

$$\sup_{q \in S_y(M_0)} |F_{\mu, M_0}^\varphi(q) - \sum_{i \in I} r_i g_i(q)| < \epsilon.$$

The map $\sum_{i \in I} r_i g_i$ can clearly be uniformly approximated by an average function. More explicitly, there exists $\bar{d} \in (M^x)^{<\omega}$ such that

$$\sup_{q \in S_y(M)} |\sum_{i \in I} r_i g_i(q) - F_{\text{Av}(\bar{d}), M}^\varphi(q)| < \epsilon.$$

Hence

$$\sup_{b \in \mathcal{U}^y} |\mu(\varphi(x, b)) - \text{Av}(\bar{d})(\varphi(x, b))| = \sup_{q \in S_y(M)} |F_{\mu, M}^\varphi(q) - F_{\text{Av}(\bar{d}), M}^\varphi(q)| < 2\epsilon.$$

which completes the proof. \square

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