INVERSE K-CHEVALLEY FORMULAS FOR SEMI-INFINITE FLAG MANIFOLDS, II: ARBITRARY WEIGHTS IN ADE TYPE

CRISTIAN LENART, SATOSHI NAITO, DANIEL ORR, AND DAISUKE SAGAKI

Abstract. We continue the study, begun in [KoNOS], of inverse Chevalley formulas for the equivariant K-group of semi-infinite flag manifolds. Using the language of alcove paths, we reformulate and extend our combinatorial inverse Chevalley formula to arbitrary weights in all simply-laced types (conjecturally also for E_8).

1. Introduction.

Let G be a simply-connected simple algebraic group over $\mathbb{C}$ with Borel subgroup $B = HN$, maximal torus H, unipotent radical N, Weyl group W, weight lattice $P = \sum_{i\in I} \mathbb{Z}\varpi_i$, root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$, and coroot lattice $Q^\vee = \sum_{i\in I} \mathbb{Z}\alpha_i^\vee$. The work [KoNOS] initiated the study of inverse Chevalley formulas in the equivariant K-group $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ of the semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$ associated with G, where the semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$ is a reduced ind-scheme whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}(\!(z)\!))/(H(\mathbb{C})\cdot N(\mathbb{C}(\!(z)\!)))$ (see [Kat2] for details), with the group $\mathbb{C}^*$ acting on $\mathbf{Q}_G^{\mathrm{rat}}$ by loop rotation; note that our K-group $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ is a variant of the Iwahori-equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$ introduced in [KaNS]. The K-group $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ has a topological $K_{H\times\mathbb{C}^*}(\mathrm{pt})$-basis consisting of Schubert classes $\{[\mathcal{O}_{\mathbf{Q}_G(x)}]\}_{x\in W_{\mathrm{af}}}$ indexed by the affine Weyl group $W_{\mathrm{af}} = W \ltimes Q^\vee$, where $K_{H\times\mathbb{C}^*}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm1}][P]$, with $K_H(\mathrm{pt}) = R(H) \cong \mathbb{Z}[P] = \mathbb{Z}[\mathbf{e}^\lambda \mid \lambda \in P]$ the character ring of H and $K_{\mathbb{C}^*}(\mathrm{pt}) = R(\mathbb{C}^*) \cong \mathbb{Z}[q^{\pm1}]$; here $q \in R(\mathbb{C}^*)$ denotes the character of loop rotation. By an *inverse* Chevalley formula, we mean a combinatorial formula for the product of an equivariant scalar $\mathbf{e}^\lambda \in K_H(\mathrm{pt}) = R(H)$ with a Schubert class $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, expressed as a $\mathbb{Z}[q^{\pm1}]$-linear combination of the twisted Schubert classes $\{[\mathcal{O}_{\mathbf{Q}_G(x)}(\mu)]\}_{x\in W_{\mathrm{af}},\mu\in P}$; here the twisted Schubert class $[\mathcal{O}_{\mathbf{Q}_G(x)}(\mu)]$ corresponds to the tensor product sheaf $[\mathcal{O}_{\mathbf{Q}_G(x)}] \otimes \mathcal{O}(\mu)$, with $\mathcal{O}(\mu)$ the equivariant line bundle over $\mathbf{Q}_G^{\mathrm{rat}}$ associated to $\mu \in P$. The results of [KoNOS] treat the case when λ is a (not necessarily dominant) minuscule weight and G is of simply-laced type.

This work is a sequel to [KoNOS]. Our main result is a combinatorial formula which generalizes the inverse Chevalley formula of [KoNOS] to arbitrary weights $\lambda \in P$. An important feature of our formula is that we formulate it using *alcove paths*, matching more closely existing results of [LP1] on equivariant Chevalley formulas for finite-dimensional flag manifolds and those of [LNS] regarding (non-inverse) Chevalley formulas in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$. (We recall that, while the ordinary and inverse Chevalley formulas for equivariant K-theory of finite-dimensional flag manifolds are essentially identical, the two types of Chevalley formulas are genuinely different for semi-infinite flag manifolds; see [KoNOS, Introduction].)

The ingredients in our combinatorial formula are roughly as follows (see §3.4 for details). Given any weight $\lambda \in P$, any alcove path Γ from the fundamental alcove $A_\circ$ to $A_\lambda = A_\circ + \lambda$, and any $w \in W$, we define a finite set $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ of combinatorial objects $(\mathbf{w}, \mathbf{b})$ (called decorated quantum walks) and various associated quantities: $(-1)^{(\mathbf{w},\mathbf{b})} \in \{\pm1\}$, $\deg(\mathbf{w}, \mathbf{b}) \in \mathbb{Z}$, $\mathrm{end}(\mathbf{w}) \in W$, $\mathrm{wt}(\mathbf{w}) \in P$, $\mathrm{qwt}(\mathbf{w}, \mathbf{b}) \in Q$ and $\mathrm{qwt}^\vee(\mathbf{w}, \mathbf{b}) \in Q^\vee$. Informally, an element $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ consists of a *walk* $\mathbf{w}$ in the quantum Bruhat graph $\mathrm{QBG}(W)$ (i.e., a directed path in $\mathrm{QBG}(W)$ with stationary steps allowed), which must begin at $w \in W$ and follow edges prescribed by Γ, together with some additional information recorded by $\mathbf{b}$ at special stationary steps in $\mathbf{w}$.

Our main result then reads as follows:

Theorem 1.1 (= Theorem 3.4)**.** *Assume that G is simply-laced, but not of type E_8. Let $\lambda \in P$ be an arbitrary weight, and let Γ be an arbitrary alcove path from $A_\circ$ to A_λ. For any $w \in W$, the following equality holds in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$:*

$$\mathbf{e}^{\lambda} \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \sum_{(\mathbf{w},\mathbf{b})\in\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)} (-1)^{(\mathbf{w},\mathbf{b})} q^{\deg(\mathbf{w},\mathbf{b})} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{w})t_{\mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})})}(-w_\circ\,\mathrm{wt}(\mathbf{w}) - w_\circ\,\mathrm{qwt}(\mathbf{w},\mathbf{b}))], \tag{1.1}$$

where $w_\circ$ denotes the longest element of W.

Here we mention that the sum on the right-hand side of (1.1) is an explicit finite sum, described in terms of decorated quantum walks, while the finiteness itself is proved in [O]. Also, we note that it suffices to consider only Schubert classes indexed by $x = w \in W$, rather than arbitrary $x \in W_{\mathrm{af}}$, by virtue of the right action of translations $\{t_\xi\}_{\xi\in Q^\vee} \subset W_{\mathrm{af}}$ on $\mathbf{Q}_G^{\mathrm{rat}}$.

We expect that (1.1) also holds, as stated, in type E_8. To prove this, however, further arguments will be necessary to handle the case of quasi-minuscule λ; we plan to provide the details of these arguments in a future work.

The first step in our proof of Theorem 1.1 is to reformulate the formula from [KoNOS], for minuscule λ, in terms of a particular (reduced) alcove path Γ (see Proposition 3.6). Then, our main efforts are devoted to establishing that the right-hand side of (1.1):

(1) is invariant under Yang-Baxter transformations (Theorem 3.12) and deletion procedures (Theorem 3.14) on alcove paths, and
(2) respects additivity in $\lambda \in P$ (Proposition 3.7).

In order to prove part (1), we establish the existence of a certain "sijection" (bijection between signed sets in the sense of [FK]) between decorated quantum walks associated to two alcove paths Γ_1, Γ_2 from $A_\circ$ to A_λ for $\lambda \in P$ such that Γ_2 is obtained from Γ_1 by a Yang-Baxter transformation or a deletion procedure. In [KoLN], it is shown that similar results hold for a certain generating function of some statistics including weights over admissible subsets in the quantum alcove model, which is closely related to the (non-inverse) Chevalley formula in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$. However, since decorated quantum walks are completely different combinatorial objects from admissible subsets, we need to construct the desired sijection for decorated quantum walks from scratch. In addition, our results above are much more difficult to prove than the corresponding results for admissible subsets in [KoLN] mainly because of the appearance of the rather delicate term $q^{\deg(\,\cdot\,)}$ on the right-hand side of (1.1). By a result of [S] (see also [LP2]), which asserts that an arbitrary $\lambda \in P$ can be written as a sum of (not necessarily dominant) minuscule weights in simply-laced types (except in type E_8), we are then able to deduce that (1.1) holds for all $\lambda \in P$ and all alcove paths Γ from $A_\circ$ to A_λ.

Here we should mention that in [Kat1], Kato established a $\mathbb{Z}[P]$-module isomorphism from the (small) H-equivariant quantum K-theory $QK_H(G/B)$ (see [ACT] for the finiteness result on $QK_H(G/B)$) of the finite-dimensional flag manifold G/B onto the $\mathbb{Z}[P]$-submodule (denoted by $K_H(\mathbf{Q}_G(e))$ in this paper) of the specialization of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ at $q = 1$ consisting of all finite linear combinations of the Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ for $x \in W_{\mathrm{af}}^{\geq 0} := W \times Q^{\vee,+}$ with coefficients in $\mathbb{Z}[P]$, where $Q^{\vee,+} := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee \subset Q^\vee = \sum_{i\in I} \mathbb{Z}\alpha_i^\vee$. This isomorphism sends each (opposite) Schubert class in $QK_H(G/B)$ to the corresponding Schubert class in $K_H(\mathbf{Q}_G(e))$; also, it respects the quantum multiplication with the line bundle class $[\mathcal{O}_{G/B}(-\varpi_i)]$ and the tensor product with the line bundle class $[\mathcal{O}_{\mathbf{Q}_G(e)}(w_\circ\varpi_i)]$ for all $i \in I$. Since the quantum multiplication in $QK_H(G/B)$ is uniquely determined by its $\mathbb{Z}[P]$-module structure and the quantum multiplication with the line bundle classes $[\mathcal{O}_{G/B}(-\varpi_i)]$ for $i \in I$ (see [BCMP]), it follows that under Kato's $\mathbb{Z}[P]$-module isomorphism above, the quantum multiplicative structure of $QK_H(G/B)$ can be described explicitly by means of (the specialization at $q = 1$ of) our inverse Chevalley formula (Theorem 1.1), together with (the specialization at $q = 1$ of) the (non-inverse) Chevalley formula in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ for anti-dominant weights $-\varpi_i$, obtained in [NOS]; recall that the (non-inverse) Chevalley formula in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ for anti-dominant weights expresses an arbitrary twisted

Schubert class $[\mathcal{O}_{\mathbf{Q}_G(x)}(-\varpi_i)]$ as an explicit finite linear combination of Schubert classes with coefficients in $\mathbb{Z}[q^{\pm 1}][P]$ in terms of the quantum alcove model.

In Appendix B, we give a detailed example in type A_2 of our inverse Chevalley formula (1.1).

Acknowledgements.

The authors would like to thank Takafumi Kouno for helpful discussions. C.L. was partially supported by the NSF grant DMS-1855592 and the Simons Foundation grant No. 584738. S.N. was supported in part by JSPS Grant-in-Aid for Scientific Research (B) 16H03920 D.O. was supported by a Collaboration Grant for Mathematicians from the Simons Foundation (Award ID: 638577). D.S. was supported in part by JSPS Grant-in-Aid for Scientific Research (C) 19K03415.

2. Basic notation.

2.1. **Root systems.** As above, let G be a simply-connected simple algebraic group over $\mathbb{C}$. We fix a maximal torus and Borel subgroup: $H \subset B \subset G$; let N denote the unipotent radical of B. We set $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$, and denote by $\langle \cdot\,,\,\cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ the canonical pairing, where $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$, $\Delta^+ \subset \Delta$ the positive roots (corresponding to B), and $\{\alpha_i\}_{i\in I} \subset \Delta^+$ the set of simple roots; we denote by $\alpha^\vee \in \mathfrak{h}$ the coroot corresponding to $\alpha \in \Delta$. For $\alpha \in \Delta$, we set $\mathrm{sgn}(\alpha) := 1$ (resp., $\mathrm{sgn}(\alpha) := -1$) if α is positive (resp., negative), and $|\alpha| := \mathrm{sgn}(\alpha)\alpha \in \Delta^+$. We denote by $\theta \in \Delta^+$ the highest root of Δ, and set $\rho := (1/2)\sum_{\alpha\in\Delta^+}\alpha$, $Q := \sum_{i\in I}\mathbb{Z}\alpha_i$, and $Q^\vee := \sum_{i\in I}\mathbb{Z}\alpha_i^\vee$; also, we set $Q^{\vee,+} := \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i^\vee \subset Q^\vee$.

For $i \in I$, let $\varpi_i \in \mathfrak{h}^*$ be the i-th fundamental weight determined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all $j \in I$, where $\delta_{i,j}$ denotes the Kronecker delta. The weight lattice P of $\mathfrak{g}$ is defined by $P := \sum_{i\in I}\mathbb{Z}\varpi_i$; note that $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Z}} P = \mathbb{R}\otimes_{\mathbb{Z}} Q$ is a real form of $\mathfrak{h}^*$. We denote by $\mathbb{Z}[P]$ the group algebra of P, that is, the associative algebra generated by the formal exponentials $\mathbf{e}^\lambda$, $\lambda \in P$, where $\mathbf{e}^\lambda\mathbf{e}^\mu := \mathbf{e}^{\lambda+\mu}$ for λ, $\mu \in P$.

For $\alpha \in \Delta$, the corresponding reflection $s_\alpha \in GL(\mathfrak{h}^*)$ is defined by $s_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee\rangle\alpha$ for $\lambda \in \mathfrak{h}^*$; we write $s_i := s_{\alpha_i}$ for $i \in I$. Then the Weyl group W of $\mathfrak{g}$ is the subgroup $\langle s_i \mid i \in I\rangle$ of $GL(\mathfrak{h}^*)$ generated by $\{s_i\}_{i\in I}$. We denote by $\ell(w)$ the length of $w \in W$ with respect to $\{s_i\}_{i\in I}$, and by $<$ the Bruhat order on W. The following fact is well-known.

Lemma 2.1. *Let $w \in W$ and $\alpha \in \Delta$. Then,*

$$\begin{aligned} ws_\alpha > w &\iff \ell(ws_\alpha) > \ell(w) \iff \mathrm{sgn}(w\alpha) = \mathrm{sgn}(\alpha), \\ ws_\alpha < w &\iff \ell(ws_\alpha) < \ell(w) \iff \mathrm{sgn}(w\alpha) = -\mathrm{sgn}(\alpha). \end{aligned}$$

Definition 2.2 (cf. [BFP, Definition 6.1]). The quantum Bruhat graph $\mathrm{QBG}(W)$ is the Δ^+-labeled directed graph whose vertices are the elements of W and whose (directed) edges are of the following form: $x \xrightarrow{\alpha} y$, with $x, y \in W$ and $\alpha \in \Delta^+$, such that $y = xs_\alpha$ and either of the following holds: (B) $\ell(y) = \ell(x) + 1$, (Q) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee\rangle + 1$. An edge satisfying (B) (resp. (Q)) is called a Bruhat edge (resp. a quantum edge).

An integral weight $\lambda \in P$ is said to be minuscule if $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Delta$. Remark that if a minuscule weight is dominant, then it is a fundamental weight; for the list of minuscule fundamental weights, see, e.g., [Hi, Chapter V, Section 2]. Also, if $\lambda \in P$ is minuscule, then every element in $W\lambda$ is also minuscule. Therefore, $W\lambda$ contains a unique fundamental minuscule weight.

Now, let $\mathfrak{g}_{\mathrm{af}} := (\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the (untwisted) affine Lie algebra over $\mathbb{C}$ associated to $\mathfrak{g}$, where c is the canonical central element and d is the scaling element, with Cartan subalgebra $\mathfrak{h}_{\mathrm{af}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We denote by $\langle \cdot\,,\,\cdot \rangle : \mathfrak{h}^*_{\mathrm{af}} \times \mathfrak{h}_{\mathrm{af}} \to \mathbb{C}$ the canonical pairing. Regarding $\lambda \in \mathfrak{h}^*$ as $\lambda \in \mathfrak{h}^*_{\mathrm{af}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{af}}, \mathbb{C})$ by setting $\langle \lambda, c\rangle = \langle \lambda, d\rangle = 0$, we have $\mathfrak{h}^* \subset \mathfrak{h}^*_{\mathrm{af}}$; under this identification, we see that the canonical pairing $\langle \cdot\,,\,\cdot \rangle$ on $\mathfrak{h}^*_{\mathrm{af}} \times \mathfrak{h}_{\mathrm{af}}$ extends that on $\mathfrak{h}^* \times \mathfrak{h}$.

We define δ to be the unique element of $\mathfrak{h}_{\mathrm{af}}^*$ which satisfies $\langle \delta, h \rangle = 0$ for all $h \in \mathfrak{h}$, $\langle \delta, c \rangle = 0$, and $\langle \delta, d \rangle = 1$, and set $\alpha_0 := -\theta + \delta \in \mathfrak{h}_{\mathrm{af}}^*$. Then the root system Δ_{af} of $\mathfrak{g}_{\mathrm{af}}$ has simple roots $\{\alpha_i\}_{i \in I_{\mathrm{af}}}$, where $I_{\mathrm{af}} := I \sqcup \{0\}$.

For $\alpha \in \Delta_{\mathrm{af}}$, we have the corresponding reflection $s_\alpha \in GL(\mathfrak{h}_{\mathrm{af}})$, defined as for $\mathfrak{g}$. Note that for $\alpha \in \Delta \subset \Delta_{\mathrm{af}}$, the restriction of the reflection s_α defined on $\mathfrak{h}_{\mathrm{af}}$ to $\mathfrak{h}$ coincides with the reflection s_α defined on $\mathfrak{h}$. We set $s_i := s_{\alpha_i}$ for $i \in I_{\mathrm{af}}$. Then, the Weyl group of $\mathfrak{g}_{\mathrm{af}}$ (called the affine Weyl group) W_{af} is defined to be the subgroup of $GL(\mathfrak{h}_{\mathrm{af}})$ generated by $\{s_i\}_{i \in I_{\mathrm{af}}}$, namely, $W_{\mathrm{af}} = \langle s_i \mid i \in I_{\mathrm{af}} \rangle$. In [Hu, Section 4.2], it is shown that $W_{\mathrm{af}} \simeq W \ltimes \{t_{\alpha^\vee} \mid \alpha^\vee \in Q^\vee\} \simeq W \ltimes Q^\vee$, where $t_{\alpha^\vee}$ is the translation element corresponding to $\alpha^\vee \in Q^\vee$; we set $W_{\mathrm{af}}^{\geq 0} := \{w t_{\alpha^\vee} \mid w \in W,\ \alpha^\vee \in Q^{\vee,+}\} \simeq W \times Q^{\vee,+} \subset W_{\mathrm{af}}$.

2.2. **Alcove paths.** For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we set

$$H_{\alpha,k} := \{\nu \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \nu, \alpha^\vee \rangle = k\}. \tag{2.1}$$

Let $\widehat{r}_{\alpha,k}$ denote the affine reflection with respect to the affine hyperplane $H_{\alpha,k}$; we have

$$\widehat{r}_{\alpha,k}(\nu) = \nu - (\langle \nu, \alpha^\vee \rangle - k)\alpha = s_\alpha \nu + k\alpha \quad \text{for } \nu \in \mathfrak{h}_{\mathbb{R}}^*. \tag{2.2}$$

The affine reflections $\widehat{r}_{\alpha,k}$ generate the affine Weyl group $W'_{\mathrm{af}} = W \ltimes \{t_\alpha \mid \alpha \in Q\} \simeq W \ltimes Q$.

The hyperplanes $H_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbb{Z}$, divide the real vector space $\mathfrak{h}_{\mathbb{R}}^*$ into open regions, called alcoves; the fundamental alcove is defined as

$$A_\circ := \{\nu \in \mathfrak{h}_{\mathbb{R}}^* \mid 0 < \langle \nu, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}.$$

We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\alpha} B$ for $\alpha \in \Delta$ if the common wall is orthogonal to α, and α points in the direction from A to B.

Definition 2.3 ([LP1])**.** An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that A_{j-1} and A_j are adjacent for $j = 1, 2, \ldots, m$. We say that an alcove path $(A_0, A_1, \ldots, A_m)$ is reduced if it has minimal length among all alcove paths from A_0 to A_m.

Given an element x in W'_{af}, there is a well-known bijection between alcove paths $(A_0, A_1, \ldots, A_m)$ from the fundamental alcove $A_0 = A_\circ$ to $A_m = x(A_\circ)$ and the decompositions $x = s_{i_1} \ldots s_{i_m}$ of x (reduced or not), as products of the Coxeter generators of W'_{af}; see [Hu, Section 4.5], while more details are given in [LP1, Lemma 5.3]. (Note that the element $x \in W'_{\mathrm{af}}$ corresponding to A_m is unique, by the simple transitivity of the action of W'_{af} on alcoves.) The mentioned bijection is explicitly given by $A_j = s_{i_1} \ldots s_{i_j}(A_\circ)$, for $j = 0, \ldots, m$. Moreover, the above alcove path is reduced if and only if the corresponding decomposition is reduced. On another hand, it is well-known that any two decompositions of an element of W'_{af} are related by successively applying the Coxeter relations in W'_{af}; the corresponding elementary transformations relating the respective alcove paths are indicated in [LNS, Remark 40].

Let $\lambda \in P$, and let $\Gamma = (A_0, A_1, \ldots, A_m)$ be an alcove path from the fundamental alcove $A_\circ$ to $A_\lambda := A_\circ + \lambda$. If $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Delta$ are such that

$$A_\circ = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda \ (= A_\circ + \lambda), \tag{2.3}$$

then we write $\Gamma = (\gamma_1, \ldots, \gamma_m)$; this notation makes sense, as the above sequence of alcoves can be recovered from the corresponding sequence of roots. For each $1 \leq t \leq m$, let H_{γ_t, l_t} be the affine hyperplane between A_{t-1} and A_t, and set

$$l'_t := \langle \lambda, \gamma_t^\vee \rangle - l_t. \tag{2.4}$$

Let $\mathsf{AP}(\lambda)$ (resp., $\mathsf{AP}_{\mathrm{red}}(\lambda)$) denote the set of all alcove paths (resp., all reduced alcove paths) from $A_\circ$ to A_λ.

Let $\lambda, \mu \in P$, and let $\Gamma = (\gamma_1, \ldots, \gamma_m) \in \mathsf{AP}(\lambda)$, $\Xi = (\xi_1, \ldots, \xi_p) \in \mathsf{AP}(\mu)$. The concatenation

$$\Gamma * \Xi := (\gamma_1, \ldots, \gamma_m, \xi_1, \ldots, \xi_p) \tag{2.5}$$

of Γ with Ξ is an alcove path from $A_\circ$ to $A_{\lambda+\mu}$; namely, $\Gamma * \Xi \in \mathsf{AP}(\lambda + \mu)$. Let us briefly explain this fact. We first consider the extended affine Weyl group $\widehat{W'_{\mathrm{af}}} := W \ltimes \{t_\nu \mid \nu \in P\} \simeq W \ltimes P$, where t_ν is the translation element corresponding to $\nu \in P$. It is clear (since affine transformations act as homeomorphisms) that $\widehat{W}'_{\mathrm{af}}$ permutes the set of alcoves (transitively, albeit not simply transitively); see [Hu, Sections 4.2 and 4.3]. Moreover, by the same reasoning, given adjacent alcoves $A \xrightarrow{\alpha} B$ and the weight λ considered above, we have $t_\lambda(A) \xrightarrow{\alpha} t_\lambda(B)$. Therefore, we can concatenate the sequence of alcoves corresponding to Γ with the translation by λ of the sequence of alcoves corresponding to Ξ, and we obtain the sequence of alcoves corresponding to $\Gamma * \Xi$.

Remark 2.4. Keep the notation and setting above. For each $1 \le q \le p$, let H_{ξ_q,k_q} be the affine hyperplane between the $(q-1)$-th alcove and the q-th alcove in Ξ. Then the affine hyperplane between the $(t-1)$-th alcove and the t-th alcove in $\Gamma * \Xi$ is H_{γ_t,l_t} (resp., $H_{\xi_{t-m},\langle\lambda,\,\xi_{t-m}^\vee\rangle+k_{t-m}}$) for $1 \le t \le m$ (resp., $m+1 \le t \le m+p$).

2.3. **Simply-laced assumption.** In this paper, we assume throughout that G is *simply-laced*. As a result, by means of the non-degenerate W-invariant symmetric bilinear form $(\cdot\,,\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$, normalized so that $(\alpha,\alpha) = 2$ for all $\alpha \in \Delta$, we can identify roots with coroots; note that $\langle \nu,\, \alpha^\vee\rangle = (\nu,\alpha)$ for $\nu \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Under this identification, if $\alpha \in \Delta$ is of the form $\alpha = \sum_{i\in I} c_i\alpha_i$, with $c_i \in \mathbb{Z}$ for $i \in I$, then we can write $\alpha^\vee = \sum_{i\in I} c_i\alpha_i^\vee$.

3. Main results.

3.1. **Semi-infinite flag manifolds.** Recall (see [KaNS] or [Kat2]) that the semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$ associated to G is a pure ind-scheme of infinite type whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}(\!(z)\!))/(H(\mathbb{C}) \cdot N(\mathbb{C}(\!(z)\!)))$, defined as an inductive limit of copies of a (reduced) closed subscheme $\mathbf{Q}_G \subset \prod_{i\in I} \mathbb{P}(V(\varpi_i)[\![z]\!])$ of infinite type, where $V(\varpi_i)$ denotes the irreducible G-module with highest weight ϖ_i. A Schubert variety $\mathbf{Q}_G(x) \subset \mathbf{Q}_G^{\mathrm{rat}}$ is by definition the closure of the orbit under the Iwahori subgroup $\mathbf{I} = \mathrm{ev}_{z=0}^{-1}(B) \subset G(\mathbb{C}[\![z]\!])$ (where $\mathrm{ev}_{z=0}$ is the evaluation map) through the $(H \times \mathbb{C}^*)$-fixed point labeled by $x \in W_{\mathrm{af}}$. The union of these $\mathbf{I}$-orbits over $x \in W_{\mathrm{af}}$ exhausts $\mathbf{Q}_G^{\mathrm{rat}}$, and the labeling of fixed points is determined as follows: for $x = wt_\xi \in W \ltimes Q^{\vee,+}$, the corresponding fixed point is the collection of lines $(z^{\langle\varpi_i,\,-w_\circ\xi\rangle}V(\varpi_i)_{ww_\circ(\varpi_i)})_{i\in I}$, where $V(\varpi_i)_\mu \subset V(\varpi_i)$ denotes the μ-weight space for $\mu \in P$; note that $\mathbf{Q}_G(e) = \mathbf{Q}_G$, where e is the identity element of W_{af}. For each $\lambda = \sum_{i\in I} m_i\varpi_i \in P$, we also have a $G(\mathbb{C}[\![z]\!]) \rtimes \mathbb{C}^*$-equivariant (with $\mathbb{C}^*$ acting on $G(\mathbb{C}[\![z]\!])$ by loop rotation) line bundle $\mathcal{O}(\lambda)$ on $\mathbf{Q}_G$ given by the restriction of the line bundle $\boxtimes_{i\in I}\mathcal{O}(m_i)$ on $\prod_{i\in I} \mathbb{P}(V(\varpi_i)[\![z]\!])$. This extends to a $G(\mathbb{C}(\!(z)\!)) \rtimes \mathbb{C}^*$-equivariant line bundle on $\mathbf{Q}_G^{\mathrm{rat}}$. We note that we are following the conventions of [KaNS] for indexing equivariant line bundles and Schubert varieties in $\mathbf{Q}_G^{\mathrm{rat}}$.

3.2. **K-groups.** Let $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ denote the equivariant K-group of the semi-infinite flag manifold $\mathbf{Q}_G^{\mathrm{rat}}$ introduced in [KaNS, Section 6], where $\widetilde{\mathbf{I}} = \mathbf{I} \rtimes \mathbb{C}^*$ is the semi-direct product of the Iwahori subgroup $\mathbf{I}$ and loop rotation $\mathbb{C}^*$. Correspondingly, $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ is a module over $\mathbb{Z}[P](\!(q^{-1})\!)$, which acts by equivariant scalar multiplication.

One has the following important classes in $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$, for each $x \in W_{\mathrm{af}}$ and $\lambda \in P$:

- Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$,
- equivariant line bundle classes $[\mathcal{O}(\lambda)]$,
- classes $[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)]$ corresponding to the tensor product sheaves $\mathcal{O}_{\mathbf{Q}_G(x)} \otimes \mathcal{O}(\lambda)$.

Definition 3.1 ($(H \times \mathbb{C}^*)$-equivariant K-groups of $\mathbf{Q}_G^{\mathrm{rat}}$ and $\mathbf{Q}_G$)**.** Let $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ denote the $\mathbb{Z}[q^{\pm1}][P]$-submodule of $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ consisting of all convergent infinite $\mathbb{Z}[q^{\pm1}][P]$-linear combinations of Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ for $x \in W_{\mathrm{af}}$, where convergence holds in the sense of [KaNS, Proposition 5.11].

Similarly, we define $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ to be the $\mathbb{Z}[q^{\pm1}][P]$-submodule of $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ consisting of all convergent infinite $\mathbb{Z}[q^{\pm1}][P]$-linear combinations of Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ for $x \in W_{\mathrm{af}}^{\ge 0}$.

The classes $\{[\mathcal{O}_{\mathbf{Q}_G(x)}]\}_{x\in W_{\mathrm{af}}}$ satisfy a notion of topological linear independence in $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ given precisely by [KaNS, Proposition 5.11]. In particular, a convergent sum $\sum_{x\in W_{\mathrm{af}}} c_x \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}]$ with $c_x \in \mathbb{Z}[q^{\pm1}][P]$ can equal 0 if and only if all $c_x = 0$. Thus, the $\{[\mathcal{O}_{\mathbf{Q}_G(x)}]\}_{x\in W_{\mathrm{af}}}$ form a topological $\mathbb{Z}[q^{\pm 1}][P]$-basis of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ since they are linearly independent in this way and each element of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ is, by definition, a convergent $\mathbb{Z}[q^{\pm 1}][P]$-linear combination of them. Also, one has $[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)] \in K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ for any $x \in W_{\mathrm{af}}$ and $\lambda \in P$, thanks to the Chevalley formulas for dominant weights [KaNS] and anti-dominant weights [NOS]. Similar (in fact, equivalent) assertions hold for $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$.

Definition 3.2. Define $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ to be the $\mathbb{Z}[q^{\pm 1}]$-submodule consisting of all finite $\mathbb{Z}[q^{\pm 1}]$-linear combinations of the classes $\{[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)]\}_{x\in W_{\mathrm{af}},\,\lambda\in P}$.

By definition, $\mathbb{K}$ is only a $\mathbb{Z}[q^{\pm 1}]$-submodule of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$. But, as shown in [O, Theorem 5.1], $\mathbb{K}$ is indeed a $\mathbb{Z}[q^{\pm 1}][P]$-submodule of $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$; here we recall the identification $K_{H\times\mathbb{C}^*}(\mathrm{pt}) \simeq \mathbb{Z}[q^{\pm 1}][P]$. We note that the classes $\{[\mathcal{O}_{\mathbf{Q}_G(x)}(\lambda)]\}_{x\in W_{\mathrm{af}},\,\lambda\in P}$ are linearly independent over $\mathbb{Z}[q^{\pm 1}]$, again by the Chevalley formula of [KaNS].

To summarize, we have the following chain of $\mathbb{Z}[q^{\pm 1}][P]$-modules (and $K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}})$ is in fact a $\mathbb{Z}[P](\!(q^{-1})\!)$-module):

$$\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}}) \subset K_{\widetilde{\mathbf{I}}}(\mathbf{Q}_G^{\mathrm{rat}}).$$

3.3. **Inverse K-Chevalley formula for minuscule weights.** In this subsection, we review the inverse Chevalley formula for minuscule weights, obtained in [KoNOS, Theorem 3.14]. Assume that $\mathfrak{g}$ is simply-laced, but not of type E_8; in this subsection, we identify $\sum_{i\in I} c_i\alpha_i \in Q$ with $\sum_{i\in I} c_i\alpha_i^\vee \in Q^\vee$, as mentioned in Section 2.3. Let $\lambda \in P$ be a (not necessarily dominant) minuscule weight, with ϖ_k the unique minuscule fundamental weight contained in $W\lambda$. Let $x \in W$ be the unique minimal-length element of W such that $\lambda = x\varpi_k$, and let $y \in W$ be the (unique) element such that yx is the unique minimal-length element in $\{w \in W \mid w\varpi_k = w_\circ\varpi_k\}$; it is easy to verify that $\ell(yx) = \ell(y) + \ell(x)$. We fix reduced expressions $x = s_{j_a}\cdots s_{j_1}$ and $y = s_{i_1}\cdots s_{i_b}$, and define

$$\beta_r := s_{j_a}s_{j_{a-1}}\cdots s_{j_{r+1}}\alpha_{j_r} \in \Delta^+ \qquad \text{for } 1 \le r \le a,$$
$$\gamma_s := s_{i_b}s_{i_{b-1}}\cdots s_{i_{s+1}}\alpha_{i_s} \in \Delta^+ \qquad \text{for } 1 \le s \le b.$$

We set

$$\vec{\eta} := (\eta_1, \dots, \eta_m) = (\beta_a,\, \dots,\, \beta_1,\, \gamma_1,\, \dots,\, \gamma_b), \tag{3.1}$$

where $m = a + b$. For $w \in W$, let $\mathbf{QW}^{\mathrm{I}}_{\lambda,w}$ denote the set of sequences $(w_0, w_1, \dots, w_m)$ such that

- $w_0 = w$;
- $w_t \in \{w_{t-1},\, s_{\eta_t}w_{t-1}\}$ for all $1 \le t \le m$;
- for $1 \le t \le m$ such that $w_t = s_{\gamma_t}w_{t-1}$, $w_{t-1} \to w_t = s_{\gamma_t}w_{t-1}$ is an edge (labeled by $|w_{t-1}^{-1}\gamma_t| = |w_t^{-1}\gamma_t|$) in the quantum Bruhat graph $\mathrm{QBG}(W)$.

Given $\mathbf{w} = (w_0, w_1, \dots, w_m) \in \mathbf{QW}^{\mathrm{I}}_{\lambda,w}$, let $S^-(\mathbf{w})^{\mathrm{I}}$ denote the set of steps t, for $1 \le t \le a$, such that $w_t = w_{t-1}$ and $(\rho, w_{t-1}^{-1}\eta_t) = 1$ (or equivalently, $w_{t-1}^{-1}\eta_t$ is a simple root). Similarly, let $S^+(\mathbf{w})^{\mathrm{I}}$ denote the set of steps t, for $a < t \le m$, such that $w_t = w_{t-1}$ and $(\rho, w_{t-1}^{-1}\eta_t) = -1$ (or equivalently, $-w_{t-1}\eta_t$ is a simple root). Let $S(\mathbf{w})^{\mathrm{I}} = S^-(\mathbf{w})^{\mathrm{I}} \cup S^+(\mathbf{w})^{\mathrm{I}}$, and define $\widetilde{\mathbf{QW}}^{\mathrm{I}}_{\lambda,w}$ to consist of all pairs $(\mathbf{w}, \mathbf{b})$ where $\mathbf{w} \in \mathbf{QW}^{\mathrm{I}}_{\lambda,w}$ and $\mathbf{b}$ is a $\{0,1\}$-valued function on $S(\mathbf{w})^{\mathrm{I}}$. For $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}^{\mathrm{I}}_{\lambda,w}$, we define

$$(-1)^{(\mathbf{w},\mathbf{b})} := \prod_{\substack{1\le t\le a\\ w_t<w_{t-1}}} (-1) \prod_{\substack{a<t\le m\\ w_t>w_{t-1}}} (-1) \prod_{t\in S(\mathbf{w})^{\mathrm{I}}} (-1)^{\mathbf{b}(t)},$$

and

$$\mathrm{wt}_0(\mathbf{w}, \mathbf{b})^{\mathrm{I}} := 0,$$

$$\mathrm{wt}_t(\mathbf{w}, \mathbf{b})^{\mathrm{I}} := \mathrm{wt}_{t-1}(\mathbf{w}, \mathbf{b})^{\mathrm{I}} + \begin{cases} -\mathbf{b}(t)w_\circ w_{t-1}^{-1}\eta_t & \text{if } t \in S^-(\mathbf{w})^{\mathrm{I}}, \\ w_\circ w_{t-1}^{-1}\eta_t & \text{if } w_t < w_{t-1}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 \le t \le a$$

$$\mathrm{wt}_t(\mathbf{w},\mathbf{b})^{\mathrm{I}} := \mathrm{wt}_{t-1}(\mathbf{w},\mathbf{b})^{\mathrm{I}} + \begin{cases} \mathbf{b}(t) w_\circ w_{t-1}^{-1}\eta_t & \text{if } t \in S^+(\mathbf{w})^{\mathrm{I}}, \\ w_\circ w_{t-1}^{-1}\eta_t & \text{if } w_t < w_{t-1}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } a < t \le m;$$

define $\mathrm{wt}(\mathbf{w},\mathbf{b})^{\mathrm{I}} := \mathrm{wt}_n(\mathbf{w},\mathbf{b})^{\mathrm{I}}$, and set $d_t(\mathbf{w},\mathbf{b}) := \mathrm{wt}_t(\mathbf{w},\mathbf{b})^{\mathrm{I}} - \mathrm{wt}_{t-1}(\mathbf{w},\mathbf{b})^{\mathrm{I}}$ for $1 \le t \le m$. Then we define

$$\deg_0^-(\mathbf{w},\mathbf{b})^{\mathrm{I}} = 0,$$

$$\deg_t^-(\mathbf{w},\mathbf{b})^{\mathrm{I}} = \deg_{t-1}^-(\mathbf{w},\mathbf{b})^{\mathrm{I}} + \frac{(d_t(\mathbf{w},\mathbf{b}), d_t(\mathbf{w},\mathbf{b}))}{2} + (d_t(\mathbf{w},\mathbf{b}), \mathrm{wt}_{t-1}(\mathbf{w},\mathbf{b})^{\mathrm{I}}) \quad \text{for } 1 \le t \le a$$

$$\deg_a^+(\mathbf{w},\mathbf{b})^{\mathrm{I}} = \deg_a^-(\mathbf{w},\mathbf{b})^{\mathrm{I}} + (-w_\circ w_a^{-1}\lambda,\, \mathrm{wt}_a(\mathbf{w},\mathbf{b})^{\mathrm{I}})$$

$$\deg_t^+(\mathbf{w},\mathbf{b})^{\mathrm{I}} = \deg_{t-1}^+(\mathbf{w},\mathbf{b})^{\mathrm{I}} + \frac{(d_t(\mathbf{w},\mathbf{b}), d_t(\mathbf{w},\mathbf{b}))}{2} + (d_t(\mathbf{w},\mathbf{b}), \mathrm{wt}_{t-1}(\mathbf{w},\mathbf{b})^{\mathrm{I}}) \qquad \text{for } a < t \le m;$$

define $\deg(\mathbf{w},\mathbf{b})^{\mathrm{I}} = \deg_m^+(\mathbf{w},\mathbf{b})^{\mathrm{I}} \in \mathbb{Z}$.

Theorem 3.3 ([KoNOS, Theorem 3.14])**.** *For any minuscule $\lambda \in P$ and $w \in W$, we have in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$,*

$$\mathbf{e}^\lambda \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \sum_{(\mathbf{w},\mathbf{b})\in\widetilde{\mathbf{QW}}^{\mathrm{I}}_{\lambda,w}} (-1)^{(\mathbf{w},\mathbf{b})} q^{\deg(\mathbf{w},\mathbf{b})^{\mathrm{I}}} \cdot [\mathcal{O}_{\mathbf{Q}_G(w_m t_{-w_\circ \mathrm{wt}(\mathbf{w},\mathbf{b})^{\mathrm{I}}})}(-w_\circ w_a^{-1}\lambda + \mathrm{wt}(\mathbf{w},\mathbf{b})^{\mathrm{I}})]. \tag{3.2}$$

3.4. **Inverse K-Chevalley formula for arbitrary weights.** Let $\lambda \in P$, and

$$\Gamma : A_\circ = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda$$

be an alcove path from the fundamental alcove $A_\circ$ to A_λ; for $1 \le t \le m$, let H_{γ_t,l_t} be the affine hyperplane between A_{t-1} and A_t. For $w \in W$, let $\mathbf{QW}_{\lambda,w} = \mathbf{QW}_{\lambda,w}(\Gamma)$ denote the set of sequences $(w_0, w_1, \dots, w_m)$ such that

- $w_0 = w$;
- $w_t \in \{w_{t-1},\, s_{\gamma_t} w_{t-1}\}$ for all $1 \le t \le m$;
- for $1 \le t \le m$ such that $w_t = s_{\gamma_t} w_{t-1}$, $w_{t-1} \to w_t = s_{\gamma_t} w_{t-1}$ is an edge in the quantum Bruhat graph $\mathrm{QBG}(W)$; note that the label of this edge is $|w_{t-1}^{-1}\gamma_t| = |w_t^{-1}\gamma_t|$.

For $\mathbf{w} = (w_0, w_1, \dots, w_m) \in \mathbf{QW}_{\lambda,w}$, we define

$$\mathrm{end}(\mathbf{w}) := w_m, \tag{3.3}$$

$$T(\mathbf{w}) := \{1 \le t \le m \mid w_t = s_{\gamma_t} w_{t-1}\}, \tag{3.4}$$

$$T^-(\mathbf{w}) := \{1 \le t \le m \mid w_{t-1} > w_t\} \subset T(\mathbf{w}), \tag{3.5}$$

$$S(\mathbf{w}) := \{1 \le t \le m \mid w_t = w_{t-1} \text{ and } -w_t^{-1}\gamma_t \text{ is a simple root}\}. \tag{3.6}$$

We set

$$\widetilde{\mathbf{QW}}_{\lambda,w} = \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) := \{(\mathbf{w},\mathbf{b}) \mid \mathbf{w} \in \mathbf{QW}_{\lambda,w}(\Gamma) \text{ and } \mathbf{b} : S(\mathbf{w}) \to \{0,1\}\}. \tag{3.7}$$

For $(\mathbf{w},\mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}$, we define

$$\mathrm{end}(\mathbf{w},\mathbf{b}) := \mathrm{end}(\mathbf{w}), \tag{3.8}$$

$$\begin{aligned}(-1)^{(\mathbf{w},\mathbf{b})} &:= \prod_{\substack{1\le t\le m \\ \gamma_t \in \Delta^- \\ w_{t-1} > w_t}} (-1) \prod_{\substack{1\le t\le m \\ \gamma_t \in \Delta^+ \\ w_{t-1} < w_t}} (-1) \prod_{t \in S(\mathbf{w})} (-1)^{\mathbf{b}(t)} \\ &= \prod_{\substack{t \in T(\mathbf{w}) \\ w_{t-1}^{-1}\gamma_t \in \Delta^+}} (-1) \prod_{t\in S(\mathbf{w})} (-1)^{\mathbf{b}(t)} = \prod_{\substack{t \in T(\mathbf{w}) \\ w_t^{-1}\gamma_t \in \Delta^-}} (-1) \prod_{t\in S(\mathbf{w})} (-1)^{\mathbf{b}(t)},\end{aligned} \tag{3.9}$$

$$\begin{cases} \mathrm{qwt}_t(\mathbf{w},\mathbf{b}) := \displaystyle\sum_{\substack{1\le u\le t\\ u\in T^-(\mathbf{w})}} |w_u^{-1}\gamma_u| + \sum_{\substack{1\le u\le t\\ u\in S(\mathbf{w})}} (-\mathbf{b}(u)w_u^{-1}\gamma_u) \quad \text{for } 0\le t\le m, \\ \mathrm{qwt}(\mathbf{w},\mathbf{b}) := \mathrm{qwt}_m(\mathbf{w},\mathbf{b}), \end{cases} \tag{3.10}$$

$$\begin{cases} \mathrm{qwt}^\vee_t(\mathbf{w},\mathbf{b}) := \displaystyle\sum_{\substack{1\le u\le t\\ u\in T^-(\mathbf{w})}} |w_u^{-1}\gamma_u|^\vee + \sum_{\substack{1\le u\le t\\ u\in S(\mathbf{w})}} (-\mathbf{b}(u)w_u^{-1}\gamma_u^\vee) \quad \text{for } 0\le t\le m, \\ \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}) := \mathrm{qwt}^\vee_m(\mathbf{w},\mathbf{b}), \end{cases} \tag{3.11}$$

$$\begin{cases} \widehat{r}_0(\mathbf{w}) := w^{-1}, \\ \widehat{r}_t(\mathbf{w}) := \begin{cases} \widehat{r}_{t-1}(\mathbf{w})\widehat{r}_{\gamma_t,l_t} & \text{if } t\in T(\mathbf{w}), \\ \widehat{r}_{t-1}(\mathbf{w}) & \text{otherwise}, \end{cases} \quad \text{for } 1\le t\le m, \end{cases} \tag{3.12}$$

$$\begin{cases} \mathrm{wt}_0(\mathbf{w}) := \widehat{r}_0(\mathbf{w})\lambda = w^{-1}\lambda, \\ \mathrm{wt}_t(\mathbf{w}) := \widehat{r}_t(\mathbf{w})\lambda - \widehat{r}_{t-1}(\mathbf{w})\lambda \quad \text{for } 1\le t\le m, \\ \mathrm{wt}(\mathbf{w}) := \widehat{r}_m(\mathbf{w})\lambda = \displaystyle\sum_{t=0}^m \mathrm{wt}_t(\mathbf{w}), \qquad \mathrm{wt}(\mathbf{w},\mathbf{b}) := \mathrm{wt}(\mathbf{w}), \end{cases} \tag{3.13}$$

$$\deg(\mathbf{w},\mathbf{b}) := \frac{1}{2}\langle \mathrm{qwt}(\mathbf{w},\mathbf{b}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle + \deg'(\mathbf{w},\mathbf{b}), \tag{3.14}$$

where

$$\deg'(\mathbf{w},\mathbf{b}) := \sum_{t=1}^m \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}^\vee_t(\mathbf{w},\mathbf{b})\rangle - \sum_{t\in T^-(\mathbf{w})} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t\in S(\mathbf{w})} \mathbf{b}(t) l'_t. \tag{3.15}$$

The main result of this paper is the following inverse K-Chevalley formula in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$, which generalizes Theorem 3.3 to the case that $\lambda \in P$ is an arbitrary weight (see also Section 4 below).

Theorem 3.4. *Assume that $\mathfrak{g}$ is simply-laced, but not of type E_8. Let $\lambda \in P$ be an arbitrary weight, and $\Gamma \in \mathsf{AP}(\lambda)$. For $w \in W$, the following equality holds in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$:*

$$\mathbf{e}^\lambda \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] =$$

$$\underbrace{\sum_{(\mathbf{w},\mathbf{b})\in\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)} \overbrace{(-1)^{(\mathbf{w},\mathbf{b})} q^{\deg(\mathbf{w},\mathbf{b})} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{w})t_{\mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})})}(-w_\circ\,\mathrm{wt}(\mathbf{w}) - w_\circ\,\mathrm{qwt}(\mathbf{w},\mathbf{b}))]}^{=:\mathbf{G}(\mathbf{w},\mathbf{b})}}_{=:\mathbf{F}_{\lambda,w}(\Gamma)}. \tag{3.16}$$

An example in type A_2 is given in Appendix B.

Remark 3.5. The degree function deg defined in (3.14) and (3.15) may seem ad hoc to the reader. In fact, it arises naturally from commutation relations in the q-Heisenberg algebra used in [KoNOS].

3.5. **Outline of the proof of Theorem 3.4.** Keep the setting of Theorem 3.4. Using Theorem 3.3, we first prove the following.

Proposition 3.6 (to be proved in Section 4)**.** *Assume that $\lambda \in P$ is a minuscule weight. Then, there exists $\Gamma \in \mathsf{AP}_{\mathrm{red}}(\lambda)$ for which Theorem 3.4 holds.*

For a minuscule weight $\lambda \in P$, let $\mathsf{AP}^\circ_{\mathrm{red}}(\lambda)$ denote the subset of $\mathsf{AP}_{\mathrm{red}}(\lambda)$ consisting of those elements for which Theorem 3.4 holds.

Next we prove the following.

Proposition 3.7 (to be proved in Section 5). *Let $\lambda, \mu \in P$, and $\Gamma \in \mathsf{AP}(\lambda)$, $\Xi \in \mathsf{AP}(\mu)$. Assume that both of the equalities $\mathbf{e}^{\lambda} \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \mathbf{F}_{\lambda,w}(\Gamma)$ and $\mathbf{e}^{\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \mathbf{F}_{\mu,w}(\Xi)$ hold in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ for all $w \in W$. Then we have $\mathbf{e}^{\lambda+\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \mathbf{F}_{\lambda+\mu,w}(\Gamma * \Xi)$ in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ for all $w \in W$.*

Here we know the following fact from [S] (see also [L, Theorem 2.1]); recall that $\mathfrak{g}$ is simply-laced, but not of type E_8.

Proposition 3.8. *For each $\lambda \in P$, there exist minuscule weights $\nu_1, \nu_2, \dots, \nu_s \in P$ such that $\lambda = \nu_1 + \nu_2 + \cdots + \nu_s$.*

For $\lambda \in P$, we define $\mathbf{m}(\lambda)$ to be the set of all finite sequences $(\nu_1, \nu_2, \dots, \nu_s)$, $s \geq 0$, of minuscule weights in P such that $\lambda = \nu_1 + \nu_2 + \cdots + \nu_s$, and set

$$\mathsf{AP}^*(\lambda) := \bigcup_{(\nu_1,\nu_2,\dots,\nu_s)\in\mathbf{m}(\lambda)} \bigl\{\Gamma_1 * \cdots * \Gamma_s \mid \Gamma_u \in \mathsf{AP}^{\circ}_{\mathrm{red}}(\nu_u),\ 1 \leq u \leq s\bigr\}; \tag{3.17}$$

notice that $\mathsf{AP}^*(\lambda) \neq \emptyset$ by Propositions 3.6 and 3.8. Combining Propositions 3.6 and 3.7, we obtain the following.

Corollary 3.9. *Theorem 3.4 holds for arbitrary $\lambda \in P$ and $\Gamma \in \mathsf{AP}_0(\lambda)$.*

Also, we know the following fact from the proof of [LP2, Lemma 9.3].

Proposition 3.10. *Let $\lambda \in P$. Let $\Gamma \in \mathsf{AP}_{\mathrm{red}}(\lambda)$ and $\Gamma' \in \mathsf{AP}(\lambda)$. Then, Γ can be obtained from Γ' by repeated application of the following procedures* (YB) *and* (D):

(YB) *for $\alpha, \beta \in \Delta$ such that $\langle \alpha, \beta^\vee \rangle \leq 0$, or equivalently, $\langle \beta, \alpha^\vee \rangle \leq 0$, one replaces a segment $\alpha,\ s_\alpha\beta,\ s_\alpha s_\beta\alpha,\ \dots,\ s_\beta\alpha,\ \beta$ by $\beta,\ s_\beta\alpha,\ \dots,\ s_\alpha s_\beta\alpha,\ s_\alpha\beta,\ \alpha$;*

(D) *one deletes a segment of the form $\alpha, -\alpha$ for $\alpha \in \Delta$.*

Remark 3.11. Since $\mathfrak{g}$ is simply-laced, if $\alpha, \beta \in \Delta$ satisfy $\langle \alpha, \beta^\vee \rangle \leq 0$, or equivalently $\langle \beta, \alpha^\vee \rangle \leq 0$, then either of the following holds:

$$\text{(a) } \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0, \qquad \text{(b) } \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1.$$

If (a) (resp., (b)) holds, then (YB) is just the following replacement:

$$\alpha,\ \beta \mapsto \beta,\ \alpha \qquad (\text{resp., } \alpha,\ \alpha+\beta,\ \beta \mapsto \beta,\ \alpha+\beta,\ \alpha).$$

Theorem 3.12 (to be proved in Section 7). *Let $\lambda \in P$ and $w \in W$. Let $\Gamma, \Xi \in \mathsf{AP}(\lambda)$, and assume that Ξ is obtained from Γ by* (YB) *in Proposition 3.10. Then, there exist a subset $\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma)$ of $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ and a subset $\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Xi)$ of $\widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ such that the following hold :*

(1) *There exists a bijection $\mathsf{YB} : \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma) \to \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Xi)$ satisfying the conditions that for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma)$,*

$$\begin{cases} (-1)^{\mathsf{YB}(\mathbf{w},\mathbf{b})} = (-1)^{(\mathbf{w},\mathbf{b})}, & \mathrm{end}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{end}(\mathbf{w},\mathbf{b}), \\ \mathrm{qwt}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}(\mathbf{w},\mathbf{b}), & \mathrm{qwt}^\vee(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}), \\ \mathrm{wt}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{wt}(\mathbf{w},\mathbf{b}), & \deg(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \deg(\mathbf{w},\mathbf{b}). \end{cases} \tag{3.18}$$

(2) *There exists an involution YB on $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \setminus \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma)$ satisfying the conditions that for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \setminus \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma)$,*

$$\begin{cases} (-1)^{\mathsf{YB}(\mathbf{w},\mathbf{b})} = -(-1)^{(\mathbf{w},\mathbf{b})}, & \mathrm{end}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{end}(\mathbf{w},\mathbf{b}), \\ \mathrm{qwt}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}(\mathbf{w},\mathbf{b}), & \mathrm{qwt}^\vee(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}), \\ \mathrm{wt}(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \mathrm{wt}(\mathbf{w},\mathbf{b}), & \deg(\mathsf{YB}(\mathbf{w},\mathbf{b})) = \deg(\mathbf{w},\mathbf{b}). \end{cases} \tag{3.19}$$

(3) *There exists an involution YB on $\widetilde{\mathbf{QW}}_{\lambda,w}(\Xi) \setminus \widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Xi)$ satisfying the same conditions as in* (3.19).

Corollary 3.13. *Let $\lambda \in P$ and $w \in W$. Let $\Gamma, \Xi \in \mathsf{AP}(\lambda)$. If Ξ is obtained from Γ by* (YB), *then $\mathbf{F}_{\lambda,w}(\Gamma) = \mathbf{F}_{\lambda,w}(\Xi)$ holds in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$.*

Theorem 3.14 (to be proved in Section 6)**.** *Let $\lambda \in P$ and $w \in W$. Let $\Gamma, \Xi \in \mathsf{AP}(\lambda)$, and assume that Ξ is obtained from Γ by* (D) *in Proposition 3.10. Then, there exists a subset $\widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$ of $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ such that the following hold :*

(1) *There exists a bijection $\mathsf{D} : \widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma) \to \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfying the conditions that for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$,*

$$\begin{cases} (-1)^{\mathsf{D}(\mathbf{w},\mathbf{b})} = (-1)^{(\mathbf{w},\mathbf{b})}, & \mathrm{end}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{end}(\mathbf{w},\mathbf{b}), \\ \mathrm{qwt}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}(\mathbf{w},\mathbf{b}), & \mathrm{qwt}^\vee(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}), \\ \mathrm{wt}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{wt}(\mathbf{w},\mathbf{b}), & \deg(\mathsf{D}(\mathbf{w},\mathbf{b})) = \deg(\mathbf{w},\mathbf{b}). \end{cases} \tag{3.20}$$

(2) *There exists an involution D on $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \setminus \widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$ satisfying the conditions that for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \setminus \widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$,*

$$\begin{cases} (-1)^{\mathsf{D}(\mathbf{w},\mathbf{b})} = -(-1)^{(\mathbf{w},\mathbf{b})}, & \mathrm{end}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{end}(\mathbf{w},\mathbf{b}), \\ \mathrm{qwt}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}(\mathbf{w},\mathbf{b}), & \mathrm{qwt}^\vee(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}), \\ \mathrm{wt}(\mathsf{D}(\mathbf{w},\mathbf{b})) = \mathrm{wt}(\mathbf{w},\mathbf{b}), & \deg(\mathsf{D}(\mathbf{w},\mathbf{b})) = \deg(\mathbf{w},\mathbf{b}). \end{cases} \tag{3.21}$$

Corollary 3.15. *Let $\lambda \in P$ and $w \in W$. Let $\Gamma, \Xi \in \mathsf{AP}(\lambda)$. If Ξ is obtained from Γ by* (D), *then $\mathbf{F}_{\lambda,w}(\Gamma) = \mathbf{F}_{\lambda,w}(\Xi)$ holds in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$.*

Let $\Gamma \in \mathsf{AP}^*(\lambda)$ and $\Xi \in \mathsf{AP}_{\mathrm{red}}(\lambda)$. By Proposition 3.10, there exists a sequence $\Gamma = \Gamma_0, \Gamma_1, \dots, \Gamma_p = \Xi$ of elements in $\mathsf{AP}(\lambda)$ such that Γ_q is obtained from Γ_{q-1} by (YB) or (D) for each $1 \le q \le p$. By making use of Corollaries 3.13 and 3.15 repeatedly, we deduce that $\mathbf{F}_{\lambda,w}(\Gamma) = \mathbf{F}_{\lambda,w}(\Xi)$ holds in $\mathbb{K}$. Since $\mathbf{e}^\lambda \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \mathbf{F}_{\lambda,w}(\Gamma)$ in $\mathbb{K}$ by Corollary 3.9, we obtain the following.

Corollary 3.16. *Theorem 3.4 holds for arbitrary $\lambda \in P$ and $\Gamma \in \mathsf{AP}_{\mathrm{red}}(\lambda)$.*

Finally, let $\Gamma \in \mathsf{AP}(\lambda)$ and $\Xi \in \mathsf{AP}_{\mathrm{red}}(\lambda)$. By the same argument as above, we deduce that $\mathbf{F}_{\lambda,w}(\Gamma) = \mathbf{F}_{\lambda,w}(\Xi)$ holds in $\mathbb{K}$. Since $\mathbf{e}^\lambda \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] = \mathbf{F}_{\lambda,w}(\Xi)$ in $\mathbb{K}$ by Corollary 3.16, we conclude that Theorem 3.4 holds for arbitrary $\lambda \in P$ and $\Gamma \in \mathsf{AP}(\lambda)$.

3.6. **A few technical remarks about the proof.** Keep the notation and setting of Section 3.4. Let $\mathbf{w} \in \mathbf{QW}_{\lambda,w}(\Gamma)$. Let $0 \le t \le m$, and let $t_1 < t_2 < \cdots < t_c$ be the elements of $T(\mathbf{w})$ (see (3.4)) less than or equal to t. Then we have

$$\begin{aligned} \widehat{r}_t(\mathbf{w})\lambda &= w^{-1}\widehat{r}_{\gamma_{t_1},l_{t_1}} \cdots \widehat{r}_{\gamma_{t_c},l_{t_c}}(\lambda) \\ &= w^{-1}\widehat{r}_{\gamma_{t_1},l_{t_1}} \cdots \widehat{r}_{\gamma_{t_{c-1}},l_{t_{c-1}}}(s_{\gamma_{t_c}}\lambda + l_{t_c}\gamma_{t_c}) \\ &= w^{-1}\widehat{r}_{\gamma_{t_1},l_{t_1}} \cdots \widehat{r}_{\gamma_{t_{c-2}},l_{t_{c-2}}}(s_{\gamma_{t_{c-1}}}s_{\gamma_{t_c}}\lambda + l_{t_c}s_{\gamma_{t_{c-1}}}\gamma_{t_c} + l_{t_{c-1}}\gamma_{t_{c-1}}) \\ &= \cdots\cdots\cdots\cdots \\ &= \underbrace{w^{-1}s_{\gamma_{t_1}} \cdots s_{\gamma_{t_{c-1}}}s_{\gamma_{t_c}}}_{=w_t^{-1}}\lambda + \sum_{a=1}^{c} l_{t_a}w^{-1}s_{\gamma_{t_1}} \cdots s_{\gamma_{t_{a-1}}}\gamma_{t_a}. \end{aligned} \tag{3.22}$$

Using this formula, we can show that for $1 \le t \le m$,

$$\mathrm{wt}_t(\mathbf{w}) = \widehat{r}_t(\mathbf{w})\lambda - \widehat{r}_{t-1}(\mathbf{w})\lambda = \begin{cases} -l'_t w_{t-1}^{-1}\gamma_t & \text{if } t \in T(\mathbf{w}), \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

Indeed, if $t \notin T(\mathbf{w})$, then it is obvious that $\mathrm{wt}_t(\mathbf{w}) = 0$ since $\widehat{r}_t(\mathbf{w}) = \widehat{r}_{t-1}(\mathbf{w})$. Assume that $t \in T(\mathbf{w})$; note that in this case, $t_c = t$, and $w^{-1}s_{\gamma_{t_1}} \cdots s_{\gamma_{t_{c-1}}}\gamma_{t_c} = w_{t-1}^{-1}\gamma_t$. We see that

$$\begin{aligned}\mathrm{wt}_t(\mathbf{w}) &= \widehat{r}_t(\mathbf{w})\lambda - \widehat{r}_{t-1}(\mathbf{w})\lambda = w_t^{-1}\lambda - w_{t-1}^{-1}\lambda + l_t w_{t-1}^{-1}\gamma_t \\ &= w_{t-1}^{-1}s_{\gamma_t}\lambda - w_{t-1}^{-1}\lambda + l_t w_{t-1}^{-1}\gamma_t = -\langle \lambda, \gamma_t^\vee\rangle w_{t-1}^{-1}\gamma_t + l_t w_{t-1}^{-1}\gamma_t \\ &= -l_t' w_{t-1}^{-1}\gamma_t.\end{aligned}$$

This proves (3.23), as desired.

4. Proof of Proposition 3.6.

In this section, we assume that $\lambda \in P$ is a (not necessarily dominant) minuscule weight; as in Section 3.3, let ϖ_k be the unique minuscule fundamental weight contained in $W\lambda$, and let $x \in W$ be the unique minimal-length element of W such that $\lambda = x\varpi_k$. Also, let $y \in W$ be the (unique) element such that yx is the unique minimal-length element in $\{w \in W \mid w\varpi_k = w_\circ\varpi_k\}$; recall that $\ell(yx) = \ell(y) + \ell(x)$. Let $x = s_{j_a} \cdots s_{j_1}$ and $y = s_{i_1} \cdots s_{i_b}$ be reduced expressions for x and y, respectively, and set

$$\begin{aligned}\beta_c &:= s_{j_a} \cdots s_{j_{c+1}}\alpha_{j_c} \in \Delta^+ \qquad \text{for } 1 \le c \le a, \\ \zeta_d &:= s_{i_b} \cdots s_{i_{d+1}}\alpha_{i_d} \in \Delta^+ \qquad \text{for } 1 \le d \le b.\end{aligned}$$

Lemma 4.1 (cf. (3.1))**.** *In the notation and setting above,*

$$\Gamma = (-\beta_a, \dots, -\beta_1, \zeta_1, \dots, \zeta_b) \in \mathsf{AP}_{\mathrm{red}}(\lambda). \tag{4.1}$$

Moreover, for $1 \le c \le a$, the affine hyperplane between the $(c-1)$-th alcove and the c-th alcove in Γ is $H_{-\beta_{a-c+1},0}$, and for $1 \le d \le b$, the affine hyperplane between the $(a+d-1)$-th alcove and the $(a+d)$-th alcove in Γ is $H_{\zeta_d,1}$.

Proof. Consider a dominant weight μ, and an element w in the set W^μ of minimal(-length) coset representatives modulo the stabilizer of μ, denoted W_μ (as a parabolic subgroup). In the extended affine Weyl group, we have the following length-additive factorization of the translation element $t_{w\mu}$:

$$t_{w\mu} = w(t_\mu w^{-1}). \tag{4.2}$$

Indeed, this follows from the well-known equality $\ell(t_{w\mu}) = \ell(t_\mu)$ (see, for example, [M, (2.4.1)]) and the fact that

$$\ell(t_\mu w^{-1}) = \ell(wt_{-\mu}) = \ell(t_{-\mu}) - \ell(w) = \ell(t_\mu) - \ell(w).$$

The first and last equalities above are obvious, while the second one is the straightforward extension to the extended affine Weyl group of the corresponding result in [LS, Lemma 3.3]; indeed, the hypothesis of the mentioned lemma is satisfied, as $-\mu$ is anti-dominant and $w \in W^\mu$.

Now consider the following reduced alcove paths: Δ from $A_\circ$ to $wA_\circ$, and Ξ from $A_\circ$ to $w^{-1}A_\circ + \mu$. The reduced alcove paths Δ and $w\Xi$ can be concatenated (as sequences of alcoves), and we obtain in this way the alcove path $\Delta * w\Xi$ in $\mathsf{AP}(w\mu)$. In fact, $\Delta * w\Xi$ is in $\mathsf{AP}_{\mathrm{red}}(w\mu)$, due to the length-additive factorization (4.2).

We now specialize $\mu = \varpi_k$ and $w = x$, so we obtain $\Delta * x\Xi$ in $\mathsf{AP}_{\mathrm{red}}(\lambda)$. The reduced alcove path Δ, written as a sequence of roots, can be obtained from the reduced decomposition $x = s_{j_a} \dots s_{j_1}$ as needed; see [Hu, Theorem 4.5].

It remains to analyze the reduced alcove path $x\Xi$. Upon translation by $-\lambda = -x\varpi_k$ and reversal, we obtain a reduced alcove path from $A_\circ$ to $x(A_\circ - \varpi_k)$. We claim that $A_\circ - \varpi_k = \lfloor w_\circ \rfloor^{-1}A_\circ$, where $\lfloor u \rfloor \in W^{\varpi_k}$ denotes the minimal(-length) coset representative of uW_{ϖ_k}. Therefore, we have $x(A_\circ - \varpi_k) = x\lfloor w_\circ \rfloor^{-1}A_\circ = y^{-1}A_\circ$. We conclude that, as a sequence of roots, the alcove path $x\Xi$ is obtained from the reduced decomposition $y^{-1} = s_{i_b} \dots s_{i_1}$ by reversing the sequence of roots given by [Hu, Theorem 4.5].

Finally, we address the claim that $A_\circ - \varpi_k = \lfloor w_\circ \rfloor^{-1}A_\circ$. Using the assumption that ϖ_k is a minuscule fundamental weight, we have $A_\circ - \varpi_k = vA_\circ$ for some $v \in W$. Indeed, for any positive root β, we have precisely $\langle \varpi_k, \beta^\vee \rangle$ hyperplanes separating $A_\circ$ and $A_\circ - \varpi_k$, that is, either 0 or 1; in the second case, the respective hyperplane is $H_{\beta,0}$. On another hand, we have

$$\langle \varpi_k, \beta^\vee \rangle \neq 0 \iff \beta \in \Delta^+ \setminus \Delta^+_{\varpi_k} \iff \beta \in \mathrm{Inv}(w_\circ w_\circ^{\varpi_k});$$

here, $\Delta^+_{\varpi_k}$ denotes the positive roots corresponding to the parabolic subgroup W_{ϖ_k}, $w_\circ^{\varpi_k}$ denotes the longest element of W_{ϖ_k}, and $\mathrm{Inv}(u)$ denotes the inversion set of u, namely $\Delta^+ \cap u^{-1}(-\Delta^+)$. We conclude that the hyperplanes separating $A_\circ$ from $vA_\circ$ are precisely those $H_{\beta,0}$ for which $\beta \in \mathrm{Inv}(\lfloor w_\circ \rfloor)$. But it is well-known that these hyperplanes correspond to $\mathrm{Inv}(v^{-1})$. As Weyl group elements are uniquely determined by their inversion sets (see, for example, [HL, Proposition 2.1] or [M, (2.2.6)]), we deduce $v = \lfloor w_\circ \rfloor^{-1}$. This concludes the proof. $\square$

We write the Γ in Lemma 4.1 as $\Gamma = (\gamma_1, \dots, \gamma_{a+b})$, i.e.,

$$\Gamma = (\gamma_1, \dots, \gamma_{a+b}) = (-\beta_a, \dots, -\beta_1, \zeta_1, \dots, \zeta_b).$$

For $1 \le t \le a+b$, let H_{γ_t, l_t} be the affine hyperplane between the $(t-1)$-th alcove and the t-th alcove in Γ; recall that $l'_t = \langle \lambda, \gamma_t^\vee \rangle - l_t$ for $1 \le t \le a+b$. Then we see that

$$(l_1, \dots, l_a, l_{a+1}, \dots, l_{a+b}) = (0, \dots, 0, 1, \dots, 1),$$
$$(l'_1, \dots, l'_a, l'_{a+1}, \dots, l'_{a+b}) = (1, \dots, 1, 0, \dots, 0).$$

Let $w \in W$. We see that the sets $\mathbf{QW}_{\lambda,w} = \mathbf{QW}_{\lambda,w}(\Gamma)$ and $\widetilde{\mathbf{QW}}_{\lambda,w} = \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ agree with $\mathbf{QW}^{\mathrm{I}}_{\lambda,w}$ and $\widetilde{\mathbf{QW}}^{\mathrm{I}}_{\lambda,w}$ (defined by using $\vec{\eta} = (\beta_a, \dots, \beta_1, \zeta_1, \dots, \zeta_b)$), respectively. Let $\mathbf{w} = (w_0, w_1, \dots, w_a, w_{a+1}, \dots, w_{a+b}) \in \mathbf{QW}_{\lambda,w}$. Then we deduce from Lemma 4.1 and (3.22) that $\mathrm{wt}(\mathbf{w}) = w_a^{-1}\lambda$. Also, we see that the set $S(\mathbf{w})$ agrees with $S(\mathbf{w})^{\mathrm{I}}$, and that $\mathrm{qwt}(\mathbf{w}, \mathbf{b})$ (and $\mathrm{qwt}^\vee(\mathbf{w}, \mathbf{b})$) is identical to $-w_\circ \, \mathrm{wt}(\mathbf{w}, \mathbf{b})^{\mathrm{I}}$ for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}$; remark that $\mathrm{qwt}(\mathbf{w}, \mathbf{b}) = \mathrm{qwt}^\vee(\mathbf{w}, \mathbf{b})$ under the identification of roots and coroots, mentioned in Section 2.3.

We will show that the degree function $\deg(\mathbf{w}, \mathbf{b})$ defined by (3.14) agrees with $\deg(\mathbf{w}, \mathbf{b})^{\mathrm{I}}$. Fix $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}$. We set

$$A := \sum_{u \in T^-(\mathbf{w})} A_u, \qquad B := \sum_{u \in S(\mathbf{w})} B_u,$$

where

$$A_u := \sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}), |w_u^{-1}\gamma_u|^\vee \rangle \qquad \text{for } u \in T^-(\mathbf{w}),$$

$$B_u := -\mathbf{b}(u) \sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}), w_u^{-1}\gamma_u^\vee \rangle \qquad \text{for } u \in S(\mathbf{w}),$$

so that

$$\begin{aligned}
&\sum_{t=1}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}), \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b}) \rangle \\
&\quad = \sum_{t=1}^{a+b} \sum_{\substack{1 \le u \le t \\ u \in T^-(\mathbf{w})}} \langle \mathrm{wt}_t(\mathbf{w}), |w_u^{-1}\gamma_u|^\vee \rangle + \sum_{t=1}^{a+b} \sum_{\substack{1 \le u \le t \\ u \in S(\mathbf{w})}} (-\mathbf{b}(t)) \langle \mathrm{wt}_t(\mathbf{w}), w_t^{-1}\gamma_t^\vee \rangle \\
&\quad = \underbrace{\sum_{u \in T^-(\mathbf{w})} \overbrace{\sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}), |w_u^{-1}\gamma_u|^\vee \rangle}^{=A_u}}_{=A} + \underbrace{\sum_{u \in S(\mathbf{w})} \overbrace{(-\mathbf{b}(u)) \sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}), w_t^{-1}\gamma_t^\vee \rangle}^{=B_u}}_{=B}.
\end{aligned}$$

By (3.23), we observe that $\mathrm{wt}_t(\mathbf{w}) = 0$ for all $a < t \le a+b$ since $l'_t = 0$ for such a t. Hence we have

$$\begin{aligned} A_u &= 0 \quad \text{for } u \in T^-(\mathbf{w}) \text{ such that } a < u \le a+b, \\ B_u &= 0 \quad \text{for } u \in S(\mathbf{w}) \text{ such that } a < u \le a+b. \end{aligned}$$

Since $l_t = 0$ for all $1 \le t \le a$, we see by (3.22) that $\widehat{r}_t(\mathbf{w})\lambda = w_t^{-1}\lambda$ for $0 \le t \le a$. Hence, for $u \in T^-(\mathbf{w})$ with $1 \le u \le a$, we have

$$\begin{aligned} A_u &= \sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}),\, |w_u^{-1}\gamma_u|^\vee \rangle = \sum_{t=u}^{a} \langle \mathrm{wt}_t(\mathbf{w}),\, |w_u^{-1}\gamma_u|^\vee \rangle \\ &= \langle \widehat{r}_a(\mathbf{w})\lambda - \widehat{r}_{u-1}(\mathbf{w})\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle \\ &= \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle - \langle w_{u-1}^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle. \end{aligned}$$

Here we note that $\gamma_u \in \Delta^-$ for all $1 \le u \le a$, which implies that $w_u^{-1}\gamma_u \in \Delta^-$ for all $u \in T^-(\mathbf{w})$ with $1 \le u \le a$. Therefore,

$$\begin{aligned} A_u &= \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle + \langle w_{u-1}^{-1}\lambda,\, w_u^{-1}\gamma_u^\vee \rangle \\ &= \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle - \langle \lambda,\, \gamma_u^\vee \rangle = \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle - 1 \\ &= \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle + \mathrm{sgn}(\gamma_u), \end{aligned}$$

and hence

$$\begin{aligned} A = \sum_{u \in T^-(\mathbf{w})} A_u &= \sum_{\substack{u \in T^-(\mathbf{w}) \\ 1 \le u \le a}} A_u = \sum_{\substack{u \in T^-(\mathbf{w}) \\ 1 \le u \le a}} \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle + \sum_{\substack{u \in T^-(\mathbf{w}) \\ 1 \le u \le a}} \mathrm{sgn}(\gamma_u) \\ &= \sum_{\substack{u \in T^-(\mathbf{w}) \\ 1 \le u \le a}} \langle w_a^{-1}\lambda,\, |w_u^{-1}\gamma_u|^\vee \rangle + \sum_{u \in T^-(\mathbf{w})} \mathrm{sgn}(\gamma_u) l'_u. \end{aligned}$$

Similarly, for $u \in S(\mathbf{w})$ with $1 \le u \le a$, we have

$$\begin{aligned} B_u &= -\mathbf{b}(u) \sum_{t=u}^{a+b} \langle \mathrm{wt}_t(\mathbf{w}),\, w_u^{-1}\gamma_u^\vee \rangle = -\mathbf{b}(u) \sum_{t=u}^{a} \langle \mathrm{wt}_t(\mathbf{w}),\, w_u^{-1}\gamma_u^\vee \rangle \\ &= -\mathbf{b}(u) \langle \widehat{r}_a(\mathbf{w})\lambda - \widehat{r}_{u-1}(\mathbf{w})\lambda,\, w_u^{-1}\gamma_u^\vee \rangle = -\mathbf{b}(u)(\langle w_a^{-1}\lambda,\, w_u^{-1}\gamma_u^\vee \rangle - \langle w_{u-1}^{-1}\lambda,\, w_u^{-1}\gamma_u^\vee \rangle) \\ &= -\mathbf{b}(u)(\langle w_a^{-1}\lambda,\, w_u^{-1}\gamma_u^\vee \rangle - \langle \lambda,\, \gamma_u^\vee \rangle) \quad (\text{note that } w_{u-1} = w_u \text{ since } u \in S(\mathbf{w})) \\ &= -\mathbf{b}(u)(\langle w_a^{-1}\lambda,\, w_u^{-1}\gamma_u^\vee \rangle - 1) \quad (\text{since } \langle \lambda,\, \gamma_u^\vee \rangle = \langle \lambda,\, -\beta_{a-u+1}^\vee \rangle = 1), \end{aligned}$$

and hence

$$\begin{aligned} B = \sum_{u \in S(\mathbf{w})} B_u &= \sum_{\substack{u \in S(\mathbf{w}) \\ 1 \le u \le a}} B_u = \sum_{\substack{u \in S(\mathbf{w}) \\ 1 \le u \le a}} \langle w_a^{-1}\lambda,\, -\mathbf{b}(u) w_u^{-1}\gamma_u^\vee \rangle + \sum_{\substack{u \in S(\mathbf{w}) \\ 1 \le u \le a}} \mathbf{b}(u) \\ &= \sum_{\substack{u \in S(\mathbf{w}) \\ 1 \le u \le a}} \langle w_a^{-1}\lambda,\, -\mathbf{b}(u) w_u^{-1}\gamma_u^\vee \rangle + \sum_{\substack{u \in S(\mathbf{w}) \\ 1 \le u \le a}} \mathbf{b}(u) l'_u. \end{aligned}$$

Putting all this together, we deduce that $\deg'(\mathbf{w}, \mathbf{b}) = \langle w_a^{-1}\lambda,\, \mathrm{qwt}_a^\vee(\mathbf{w}, \mathbf{b}) \rangle$, and hence that

$$\begin{aligned} \deg(\mathbf{w}, \mathbf{b}) &= \frac{1}{2} \langle \mathrm{qwt}(\mathbf{w}, \mathbf{b}),\, \mathrm{qwt}^\vee(\mathbf{w}, \mathbf{b}) \rangle + \langle w_a^{-1}\lambda,\, \mathrm{qwt}_a^\vee(\mathbf{w}, \mathbf{b}) \rangle \\ &= \frac{1}{2} (\mathrm{qwt}(\mathbf{w}, \mathbf{b}), \mathrm{qwt}(\mathbf{w}, \mathbf{b})) + (w_a^{-1}\lambda, \mathrm{qwt}_a(\mathbf{w}, \mathbf{b})), \end{aligned}$$

which agrees with $\deg(\mathbf{w}, \mathbf{b})^{\mathrm{I}}$, as desired. Therefore, by Theorem 3.3, we conclude that Theorem 3.4 holds for $\Gamma \in \mathsf{AP}_{\mathrm{red}}(\lambda)$ of the form (4.1). This completes the proof of Proposition 3.6.

5. Proof of Proposition 3.7.

Recall the notation and setting of Proposition 3.7, equation (2.5), and Remark 2.4. We write $\Gamma \in \mathsf{AP}(\lambda)$ and $\Xi \in \mathsf{AP}(\mu)$ as: $\Gamma = (\gamma_1, \dots, \gamma_m)$ and $\Xi = (\xi_1, \dots, \xi_p)$, respectively. If we set $\gamma_t := \xi_{t-m}$ for $m+1 \le t \le m+p$, then

$$\Gamma * \Xi = (\gamma_1, \dots, \gamma_m, \xi_1, \dots, \xi_p) = (\gamma_1, \dots, \gamma_{m+p}).$$

For $1 \le t \le m$, let H_{γ_t, l_t} be the affine hyperplane between the $(t-1)$-th alcove and the t-th alcove in Γ. Similarly, for $1 \le q \le p$, let H_{ξ_q, k_q} be the affine hyperplane between the $(q-1)$-th alcove and the q-th alcove in Ξ. Then the affine hyperplane between the $(t-1)$-th alcove and the t-th alcove in $\Gamma * \Xi$ is H_{γ_t, l_t} (resp., $H_{\xi_{t-m}, \langle \lambda, \xi_{t-m}^\vee \rangle + k_{t-m}}$) for $1 \le t \le m$ (resp., $m+1 \le t \le m+p$); we set $l_t := \langle \lambda, \xi_{t-m}^\vee \rangle + k_{t-m}$ for $m+1 \le t \le m+p$. Also, we set

$$\begin{cases} l_t'' := \langle \lambda + \mu, \gamma_t^\vee \rangle - l_t & \text{for } 1 \le t \le m+p, \\ l_t' := \langle \lambda, \gamma_t^\vee \rangle - l_t & \text{for } 1 \le t \le m, \\ k_q' := \langle \mu, \xi_q^\vee \rangle - k_q = \langle \mu, \gamma_{m+q}^\vee \rangle - l_{m+q} & \text{for } 1 \le q \le p. \end{cases} \tag{5.1}$$

For $\mathbf{w} = (w_0, \dots, w_m) \in \mathbf{QW}_{\lambda, w}(\Gamma)$ and $\mathbf{v} = (v_0, v_1, \dots, v_p) \in \mathbf{QW}_{\mu, \mathrm{end}(\mathbf{w})}(\Xi)$ (note that $\mathrm{end}(\mathbf{w}) = w_m$), we set

$$\mathbf{w} * \mathbf{v} := (w_0, \dots, w_m, v_1, \dots, v_p) \in \mathbf{QW}_{\lambda+\mu, w}(\Gamma * \Xi); \tag{5.2}$$

if we set $w_q := v_{q-m}$ for $m+1 \le q \le m+p$, then we can write $\mathbf{w} * \mathbf{v}$ as:

$$\mathbf{w} * \mathbf{v} = (w_0, \dots, w_m, v_1, \dots, v_p) = (w_0, \dots, w_m, w_{m+1}, \dots, w_{m+p});$$

note that $S(\mathbf{w} * \mathbf{v}) \subset [1, m+p]$. We see that $S(\mathbf{w} * \mathbf{v}) = S(\mathbf{w}) \sqcup \{m + q \mid q \in S(\mathbf{v})\}$. For $\mathbf{b} : S(\mathbf{w}) \to \{0,1\}$ and $\mathbf{c} : S(\mathbf{v}) \to \{0,1\}$, we define $\mathbf{b} * \mathbf{c} : S(\mathbf{w} * \mathbf{v}) \to \{0,1\}$ by

$$(\mathbf{b} * \mathbf{c})(t) := \begin{cases} b(t) & \text{if } t \in S(\mathbf{w}), \\ c(t-m) & \text{if } t \in \{m+q \mid q \in S(\mathbf{v})\}. \end{cases} \tag{5.3}$$

Then it follows that

$$\widetilde{\mathbf{QW}}_{\lambda+\mu, w}(\Gamma * \Xi) = \left\{ (\mathbf{w} * \mathbf{v}, \mathbf{b} * \mathbf{c}) \;\middle|\; \begin{array}{l} (\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda, w}(\Gamma), \\ (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\mu, \mathrm{end}(\mathbf{w})}(\Xi) \end{array} \right\}. \tag{5.4}$$

By the assumption, we have in $\mathbb{K} \subset K_{H \times \mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$,

$$\begin{aligned} &\mathbf{e}^{\lambda+\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(w)}] \\ &= \sum_{(\mathbf{w},\mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)} (-1)^{(\mathbf{w},\mathbf{b})} q^{\deg(\mathbf{w},\mathbf{b})} \mathbf{e}^{\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{w}) t_{\mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})})}(-w_\circ \,\mathrm{wt}(\mathbf{w}) - w_\circ \,\mathrm{qwt}(\mathbf{w},\mathbf{b}))] \\ &= \sum_{(\mathbf{w},\mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)} \; \sum_{(\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\mu,\mathrm{end}(\mathbf{w})}(\Xi)} (-1)^{(\mathbf{w},\mathbf{b})} (-1)^{(\mathbf{v},\mathbf{c})} \times \\ &\quad q^{\deg(\mathbf{w},\mathbf{b}) + \deg(\mathbf{v},\mathbf{c}) + \langle \mathrm{wt}(\mathbf{v}) + \mathrm{qwt}(\mathbf{v},\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}) \rangle} \times \\ &\quad [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{v}) t_{\mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}) + \mathrm{qwt}^\vee(\mathbf{v},\mathbf{c})})}(-w_\circ \,\mathrm{wt}(\mathbf{w}) - w_\circ \,\mathrm{qwt}(\mathbf{w},\mathbf{b}) - w_\circ \,\mathrm{wt}(\mathbf{v}) - w_\circ \,\mathrm{qwt}(\mathbf{v},\mathbf{c}))] \\ &=: (\spadesuit) \end{aligned}$$

We can easily verify that

$$\begin{cases} (-1)^{(\mathbf{w},\mathbf{b})}(-1)^{(\mathbf{v},\mathbf{c})} = (-1)^{(\mathbf{w}*\mathbf{c},\mathbf{b}*\mathbf{c})}, \\ \mathrm{qwt}(\mathbf{w},\mathbf{b}) + \mathrm{qwt}(\mathbf{v},\mathbf{c}) = \mathrm{qwt}(\mathbf{w}*\mathbf{c},\mathbf{b}*\mathbf{c}), \\ \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}) + \mathrm{qwt}^\vee(\mathbf{v},\mathbf{c}) = \mathrm{qwt}^\vee(\mathbf{w}*\mathbf{c},\mathbf{b}*\mathbf{c}), \\ \mathrm{end}(\mathbf{v}) = \mathrm{end}(\mathbf{w}*\mathbf{v}). \end{cases} \tag{5.5}$$

First we claim that

$$\mathrm{wt}(\mathbf{w}) + \mathrm{wt}(\mathbf{v}) = \mathrm{wt}(\mathbf{w}*\mathbf{v}). \tag{5.6}$$

Let $\mathbf{w} = (w_0, w_1, \ldots, w_m) \in \mathbf{QW}_{\lambda,w}$ and $\mathbf{v} = (v_0, v_1, \ldots, v_p) \in \mathbf{QW}_{\mu,\mathrm{end}(\mathbf{w})}$. If we write

$$T(\mathbf{w}) = \{1 \le t \le m \mid w_{t-1} \ne w_t\} = \{t_1 < t_2 < \cdots < t_c\},$$
$$T(\mathbf{v}) = \{1 \le q \le p \mid v_{q-1} \ne v_q\} = \{q_1 < q_2 < \cdots < q_d\},$$

then we see by (3.22) that

$$\mathrm{wt}(\mathbf{w}) = \underbrace{w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_{c-1}}}s_{\gamma_{t_c}}}_{=w_m^{-1}=\mathrm{end}(\mathbf{w})^{-1}}\lambda + \sum_{a=1}^{c} l_{t_a} w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_{a-1}}}\gamma_{t_a},$$

and that

$$\begin{aligned} \mathrm{wt}(\mathbf{v}) &= w_m^{-1}s_{\xi_{q_1}}\cdots s_{\xi_{q_{d-1}}}s_{\xi_{q_d}}\mu + \sum_{b=1}^{d} k_{q_b} w_m^{-1}s_{\xi_{q_1}}\cdots s_{\xi_{q_{b-1}}}\xi_{q_b} \\ &= w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_c}}s_{\xi_{q_1}}\cdots s_{\xi_{q_d}}\mu + \sum_{b=1}^{d} k_{q_b} w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_c}}s_{\xi_{q_1}}\cdots s_{\xi_{q_{b-1}}}\xi_{q_b}. \end{aligned}$$

Also, we deduce from Remark 2.4 and (3.22) that

$$\begin{aligned} &\mathrm{wt}(\mathbf{w}*\mathbf{v}) = w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_c}}s_{\xi_{q_1}}\cdots s_{\xi_{q_d}}(\lambda+\mu) \\ &\quad + \sum_{a=1}^{c} l_{t_a} w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_{a-1}}}\gamma_{t_a} + \sum_{b=1}^{d}(\langle\lambda,\, \xi_{q_b}^\vee\rangle + k_{q_b}) w^{-1}s_{\gamma_{t_1}}\cdots s_{\gamma_{t_c}}s_{\xi_{q_1}}\cdots s_{\xi_{q_{b-1}}}\xi_{q_b}. \end{aligned}$$

Since

$$s_{\xi_{q_1}}\cdots s_{\xi_{q_d}}\lambda = \lambda - \sum_{b=1}^{d}\langle\lambda,\, \xi_{q_b}^\vee\rangle s_{\xi_{q_1}}\cdots s_{\xi_{q_{b-1}}}\xi_{q_b},$$

we obtain (5.6), as desired.

Next we claim that

$$\deg(\mathbf{w},\mathbf{b}) + \deg(\mathbf{v},\mathbf{c}) + \langle \mathrm{wt}(\mathbf{v}) + \mathrm{qwt}(\mathbf{v},\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle = \deg(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}). \tag{5.7}$$

We use $\mathrm{qwt}(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}) = \mathrm{qwt}(\mathbf{w},\mathbf{b}) + \mathrm{qwt}(\mathbf{v},\mathbf{c})$ and $\mathrm{qwt}^\vee(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}) = \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b}) + \mathrm{qwt}^\vee(\mathbf{v},\mathbf{c})$ to obtain

$$\begin{aligned} &\frac{1}{2}\langle \mathrm{qwt}(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})\rangle \\ &= \frac{1}{2}\langle \mathrm{qwt}(\mathbf{w},\mathbf{b}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle + \frac{1}{2}\langle \mathrm{qwt}(\mathbf{v},\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{v},\mathbf{c})\rangle, \\ &\qquad + \underbrace{\frac{1}{2}\langle \mathrm{qwt}(\mathbf{v},\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle + \frac{1}{2}\langle \mathrm{qwt}(\mathbf{w},\mathbf{b}),\, \mathrm{qwt}^\vee(\mathbf{v},\mathbf{c})\rangle}_{=\langle \mathrm{qwt}(\mathbf{v},\mathbf{c}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle \text{ since } \mathfrak{g} \text{ is simply-laced}}, \end{aligned}$$

which enables us to simplify (5.7) as:

$$\deg'(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}) = \deg'(\mathbf{w},\mathbf{b}) + \deg'(\mathbf{v},\mathbf{c}) + \langle \mathrm{wt}(\mathbf{v}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle. \tag{5.8}$$

Thus, what we need to show is:

$$\begin{aligned}
&\sum_{t=1}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})\rangle - \sum_{t\in T^-(\mathbf{w}*\mathbf{v})}\mathrm{sgn}(\gamma_t)l''_t - \sum_{t\in S(\mathbf{w}*\mathbf{v})}(\mathbf{b}*\mathbf{c})(t)l''_t \\
&= \sum_{t=1}^{m}\langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w},\mathbf{b})\rangle - \sum_{t\in T^-(\mathbf{w})}\mathrm{sgn}(\gamma_t)l'_t - \sum_{t\in S(\mathbf{w})}\mathbf{b}(t)l'_t \\
&+ \sum_{q=1}^{p}\langle \mathrm{wt}_q(\mathbf{v}),\, \mathrm{qwt}_q^\vee(\mathbf{v},\mathbf{c})\rangle - \sum_{q\in T^-(\mathbf{v})}\mathrm{sgn}(\xi_q)k'_q - \sum_{q\in S(\mathbf{v})}\mathbf{c}(q)k'_q \\
&+ \langle \mathrm{wt}(\mathbf{v}),\, \mathrm{qwt}^\vee(\mathbf{w},\mathbf{b})\rangle.
\end{aligned} \tag{5.9}$$

Let us write a part (i.e., the first sum) of $\deg'(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})$ as follows:

$$\sum_{t=1}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})\rangle = \sum_{u\in T^-(\mathbf{w}*\mathbf{v})}A_u + \sum_{u\in S(\mathbf{w}*\mathbf{v})}B_u,$$

where

$$\begin{aligned}
A_u &:= \sum_{t=u}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, |w_u^{-1}\gamma_u|^\vee\rangle \qquad \text{for } u\in T^-(\mathbf{w}*\mathbf{v}),\\
B_u &:= -(\mathbf{b}*\mathbf{c})(u)\sum_{t=u}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, w_u^{-1}\gamma_u^\vee\rangle \qquad \text{for } u\in S(\mathbf{w}*\mathbf{v}).
\end{aligned}$$

For $m<t\le m+p$, we have

$$l''_t = \langle \lambda+\mu,\, \xi_{t-m}^\vee\rangle - (k_{t-m}+\langle\lambda,\, \xi_{t-m}^\vee\rangle) = \langle \mu,\, \xi_{t-m}^\vee\rangle - k_{t-m} = k'_{t-m}, \tag{5.10}$$

and hence

$$\begin{aligned}
\mathrm{wt}_t(\mathbf{w}*\mathbf{v}) &= \begin{cases} -l''_t w_{t-1}^{-1}\gamma_t & \text{if } t\in T(\mathbf{w}*\mathbf{v}),\\ 0 & \text{otherwise}\end{cases} \qquad \text{by (3.23)}\\
&= \begin{cases} -k'_{t-m} v_{t-m-1}^{-1}\xi_{t-m} & \text{if } t-m\in T(\mathbf{v}),\\ 0 & \text{otherwise}\end{cases}\\
&= \mathrm{wt}_{t-m}(\mathbf{v}).
\end{aligned}$$

Therefore, we deduce that

$$A_u = \sum_{t=u}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, |w_u^{-1}\gamma_u|^\vee\rangle = \sum_{t=u-m}^{p}\langle \mathrm{wt}_t(\mathbf{v}),\, |v_{u-m}^{-1}\xi_{u-m}|^\vee\rangle$$

for $u\in T^-(\mathbf{w}*\mathbf{v})$ with $m<u\le m+p$, and that

$$B_u = -(\mathbf{b}*\mathbf{c})(u)\sum_{t=u}^{m+p}\langle \mathrm{wt}_t(\mathbf{w}*\mathbf{v}),\, w_u^{-1}\gamma_u^\vee\rangle = -\mathbf{c}(u-m)\sum_{t=u-m}^{p}\langle \mathrm{wt}_t(\mathbf{v}),\, v_{u-m}^{-1}\xi_{u-m}^\vee\rangle$$

for $u\in S(\mathbf{w}*\mathbf{v})$ with $m<u\le m+p$. Here we remark that

$$\sum_{q=1}^{p}\langle \mathrm{wt}_q(\mathbf{v}),\, \mathrm{qwt}_q^\vee(\mathbf{v},\mathbf{c})\rangle = \sum_{u\in T^-(\mathbf{v})}A'_u + \sum_{u\in S(\mathbf{v})}B'_u,$$

where

$$A'_u := \sum_{t=u}^{p} \langle \mathrm{wt}_t(\mathbf{v}),\, |v_u^{-1}\xi_u|^\vee \rangle \qquad \text{for } u \in T^-(\mathbf{v}),$$

$$B'_u := -\mathbf{c}(u) \sum_{t=u}^{p} \langle \mathrm{wt}_t(\mathbf{v}),\, v_u^{-1}\xi_u^\vee \rangle \qquad \text{for } u \in S(\mathbf{v}).$$

Since $A_u = A'_{u-m}$ for $u \in T^-(\mathbf{w} * \mathbf{v}) \cap \{m+1, m+2, \dots, m+p\} = \{m+u' \mid u' \in T^-(\mathbf{v})\}$, and since $B_u = B'_{u-m}$ for $u \in S(\mathbf{w} * \mathbf{v}) \cap \{m+1, m+2, \dots, m+p\} = \{m+u' \mid u' \in S(\mathbf{v})\}$, it follows that

$$\sum_{\substack{u \in T^-(\mathbf{w}*\mathbf{v}) \\ m<u\leq m+p}} A_u + \sum_{\substack{u \in S(\mathbf{w}*\mathbf{v}) \\ m<u\leq m+p}} B_u = \sum_{q=1}^{p} \langle \mathrm{wt}_q(\mathbf{v}),\, \mathrm{qwt}_q^\vee(\mathbf{v}, \mathbf{c}) \rangle.$$

Thus, equation (5.9) (which we need to show) is equivalent to:

$$\begin{aligned}
&\sum_{\substack{u \in T^-(\mathbf{w}*\mathbf{v}) \\ 1\leq u\leq m}} A_u + \sum_{\substack{u \in S(\mathbf{w}*\mathbf{v}) \\ 1\leq u\leq m}} B_u - \sum_{t \in T^-(\mathbf{w}*\mathbf{v})} \mathrm{sgn}(\gamma_t) l''_t - \sum_{t \in S(\mathbf{w}*\mathbf{v})} (\mathbf{b} * \mathbf{c})(t) l''_t \\
&= \sum_{t=1}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b}) \rangle - \sum_{t \in T^-(\mathbf{w})} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t \in S(\mathbf{w})} \mathbf{b}(t) l'_t \qquad (5.11) \\
&\quad - \sum_{q \in T^-(\mathbf{v})} \mathrm{sgn}(\xi_q) k'_q - \sum_{q \in S(\mathbf{v})} \mathbf{c}(q) k'_q + \langle \mathrm{wt}(\mathbf{v}),\, \mathrm{qwt}^\vee(\mathbf{w}, \mathbf{b}) \rangle.
\end{aligned}$$

If $1 \leq t \leq m$, then we have

$$l''_t = \langle \lambda + \mu,\, \gamma_t^\vee \rangle - l_t = \langle \lambda,\, \gamma_t^\vee \rangle - l_t + \langle \mu,\, \gamma_t^\vee \rangle = l'_t + \langle \mu,\, \gamma_t^\vee \rangle, \qquad (5.12)$$

and hence

$$\begin{aligned}
\mathrm{wt}_t(\mathbf{w} * \mathbf{v}) &= \begin{cases} -l''_t w_{t-1}^{-1}\gamma_t & \text{if } t \in T(\mathbf{w} * \mathbf{v}), \\ 0 & \text{otherwise} \end{cases} \qquad \text{by (3.23)} \\
&= \begin{cases} -(l'_t + \langle \mu,\, \gamma_t^\vee \rangle) w_{t-1}^{-1}\gamma_t & \text{if } t \in T(\mathbf{w}), \\ 0 & \text{otherwise} \end{cases} \\
&= \mathrm{wt}_t(\mathbf{w}) + w_t^{-1}\mu - w_{t-1}^{-1}\mu.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{t=u}^{m+p} \mathrm{wt}_t(\mathbf{w} * \mathbf{v}) &= \sum_{t=u}^{m} \mathrm{wt}_t(\mathbf{w} * \mathbf{v}) + \sum_{t=m+1}^{m+p} \underbrace{\mathrm{wt}_t(\mathbf{w} * \mathbf{v})}_{=\mathrm{wt}_{t-m}(\mathbf{v})} \\
&= w_m^{-1}\mu - w_{u-1}^{-1}\mu + \sum_{t=u}^{m} \mathrm{wt}_t(\mathbf{w}) + \mathrm{wt}(\mathbf{v}) - w_m^{-1}\mu.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
A_u &= \langle -w_{u-1}^{-1}\mu,\, |w_u^{-1}\gamma_u|^\vee \rangle + \sum_{t=u}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, |w_u^{-1}\gamma_u|^\vee \rangle + \langle \mathrm{wt}(\mathbf{v}),\, |w_u^{-1}\gamma_u|^\vee \rangle \\
&= \mathrm{sgn}(\gamma_u) \langle \mu,\, \gamma_u^\vee \rangle + \sum_{t=u}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, |w_u^{-1}\gamma_u|^\vee \rangle + \langle \mathrm{wt}(\mathbf{v}),\, |w_u^{-1}\gamma_u|^\vee \rangle
\end{aligned}$$

for $u \in T^{-}(\mathbf{w} * \mathbf{v})$ with $1 \le u \le m$, and that

$$B_u = \mathbf{b}(u)\langle \mu,\, \gamma_u^{\vee} \rangle - \mathbf{b}(u) \sum_{t=u}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, w_u^{-1}\gamma_u^{\vee} \rangle - \mathbf{b}(u)\langle \mathrm{wt}(\mathbf{v}),\, w_u^{-1}\gamma_u^{\vee} \rangle$$

for $u \in S(\mathbf{w} * \mathbf{v})$ with $1 \le u \le m$. Here we remark that

$$\sum_{t=1}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^{\vee}(\mathbf{w}, \mathbf{b}) \rangle = \sum_{u \in T^{-}(\mathbf{w})} A''_u + \sum_{u \in S(\mathbf{w})} B''_u,$$

where

$$A''_u := \sum_{t=u}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, |w_u^{-1}\gamma_u|^{\vee} \rangle \qquad \text{for } u \in T^{-}(\mathbf{w}),$$

$$B''_u := -\mathbf{b}(u) \sum_{t=u}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, w_u^{-1}\gamma_u^{\vee} \rangle \qquad \text{for } u \in S(\mathbf{w}).$$

Since $A_u = A''_u + \mathrm{sgn}(\gamma_u)\langle \mu,\, \gamma_u^{\vee} \rangle + \langle \mathrm{wt}(\mathbf{v}),\, |w_u^{-1}\gamma_u|^{\vee} \rangle$ for $u \in T^{-}(\mathbf{w} * \mathbf{v}) \cap \{1, 2, \dots, m\} = T^{-}(\mathbf{w})$, and since $B_u = B''_u + \mathbf{b}(u)\langle \mu,\, \gamma_u^{\vee} \rangle - \mathbf{b}(u)\langle \mathrm{wt}(\mathbf{v}),\, w_u^{-1}\gamma_u^{\vee} \rangle$ for $u \in S(\mathbf{w} * \mathbf{v}) \cap \{1, 2, \dots, m\} = S(\mathbf{w})$, it follows that

$$\begin{aligned}
\sum_{\substack{u \in T^{-}(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} A_u + \sum_{\substack{u \in S(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} B_u &= \sum_{t=1}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^{\vee}(\mathbf{w}, \mathbf{b}) \rangle \\
&\quad + \sum_{\substack{u \in T^{-}(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \big(\mathrm{sgn}(\gamma_u)\langle \mu,\, \gamma_u^{\vee} \rangle + \langle \mathrm{wt}(\mathbf{v}),\, |w_u^{-1}\gamma_u|^{\vee} \rangle \big) \\
&\quad + \sum_{\substack{u \in S(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \big(\mathbf{b}(u)\langle \mu,\, \gamma_u^{\vee} \rangle - \mathbf{b}(u)\langle \mathrm{wt}(\mathbf{v}),\, w_u^{-1}\gamma_u^{\vee} \rangle \big) \\
&= \sum_{t=1}^{m} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^{\vee}(\mathbf{w}, \mathbf{b}) \rangle + \langle \mathrm{wt}(\mathbf{v}),\, \mathrm{qwt}^{\vee}(\mathbf{w}, \mathbf{b}) \rangle \\
&\quad + \sum_{\substack{u \in T^{-}(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \mathrm{sgn}(\gamma_u)\langle \mu,\, \gamma_u^{\vee} \rangle + \sum_{\substack{u \in S(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \mathbf{b}(u)\langle \mu,\, \gamma_u^{\vee} \rangle.
\end{aligned}$$

Thus, equation (5.11) (which we need to show) is equivalent to:

$$\begin{aligned}
&\sum_{\substack{u \in T^{-}(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \mathrm{sgn}(\gamma_u)\langle \mu,\, \gamma_u^{\vee} \rangle + \sum_{\substack{u \in S(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \mathbf{b}(u)\langle \mu,\, \gamma_u^{\vee} \rangle \\
&- \sum_{t \in T^{-}(\mathbf{w} * \mathbf{v})} \mathrm{sgn}(\gamma_t) l''_t - \sum_{t \in S(\mathbf{w} * \mathbf{v})} (\mathbf{b} * \mathbf{c})(t) l''_t \qquad (5.13) \\
&= - \sum_{t \in T^{-}(\mathbf{w})} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t \in S(\mathbf{w})} \mathbf{b}(t) l'_t - \sum_{q \in T^{-}(\mathbf{v})} \mathrm{sgn}(\xi_q) k'_q - \sum_{q \in S(\mathbf{v})} \mathbf{c}(q) k'_q.
\end{aligned}$$

We see by (5.1) and (5.10) that

$$\sum_{\substack{u \in T^{-}(\mathbf{w} * \mathbf{v}) \\ 1 \le u \le m}} \mathrm{sgn}(\gamma_u)\langle \mu,\, \gamma_u^{\vee} \rangle - \sum_{t \in T^{-}(\mathbf{w} * \mathbf{v})} \mathrm{sgn}(\gamma_t) l''_t = - \sum_{t \in T^{-}(\mathbf{w})} \mathrm{sgn}(\gamma_t) l'_t - \sum_{q \in T^{-}(\mathbf{v})} \mathrm{sgn}(\xi_q) k'_q,$$

$$\sum_{\substack{u\in S(\mathbf{w}*\mathbf{v})\\ 1\le u\le m}} \mathbf{b}(u)\langle \mu,\, \gamma_u^\vee\rangle - \sum_{t\in S(\mathbf{w}*\mathbf{v})} (\mathbf{b}*\mathbf{c})(t) l''_t = - \sum_{t\in S(\mathbf{w})} \mathbf{b}(t) l'_t - \sum_{q\in S(\mathbf{v})} \mathbf{c}(q) k'_q.$$

This proves (5.13), and hence (5.7), as desired.

Substituting (5.5), (5.6), and (5.7) into (♠), we conclude that

$$\begin{aligned}(\spadesuit) = & \sum_{(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})\in \widetilde{\mathbf{QW}}_{\lambda+\mu,w}(\Gamma*\Xi)} (-1)^{(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})} q^{\deg(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})} \times \\ & \qquad [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{w}*\mathbf{v}) t_{\mathrm{qwt}^\vee(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c})})}(-w_\circ \,\mathrm{wt}(\mathbf{w}*\mathbf{v}) - w_\circ \,\mathrm{qwt}(\mathbf{w}*\mathbf{v},\mathbf{b}*\mathbf{c}))] \\ = & \; \mathbf{F}_{\lambda+\mu,w}(\Gamma*\Xi).\end{aligned}$$

This completes the proof of Proposition 3.7.

6. Proof of Theorem 3.14.

Let $\lambda \in P$ be an arbitrary weight, and let

$$\Gamma : A_\circ = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda \tag{6.1}$$

be an alcove path from the fundamental alcove $A_\circ$ to $A_\lambda = A_\circ + \lambda$. Assume that $\gamma_s = \alpha$, $\gamma_{s+1} = -\alpha$ for some $\alpha \in \Delta$ and $1 \le s \le m-1$, i.e.,

$$\cdots \xrightarrow{\gamma_{s-1}} A_{s-1} \xrightarrow{\alpha} A_s \xrightarrow{-\alpha} A_{s+1} \xrightarrow{\gamma_{s+2}} A_{s+2} \cdots ;$$

note that $A_{s-1} = A_{s+1}$. We define $\Xi = (\beta_1, \dots, \beta_{s-1}, \beta_{s+2}, \dots, \beta_m)$, where $\beta_k := \gamma_k$ for $1 \le k \le m$ with $k \ne s, s+1$. Then, Ξ is an alcove path from $A_\circ$ to A_λ of the form:

$$\Xi : A_\circ = A_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{s-1}} A_{s-1} = A_{s+1} \xrightarrow{\gamma_{s+2}} A_{s+2} \xrightarrow{\gamma_{s+3}} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda. \tag{6.2}$$

Now, let $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$, with $\mathbf{w} = (w_0, w_1, \dots, w_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$. In the following, we will define $\mathsf{D}(\mathbf{w}, \mathbf{b})$.

Case 1. Assume that $(\mathbf{w}, \mathbf{b})$ satisfies $w_{s-1} = w_s = w_{s+1}$; observe that $S(\mathbf{w}) \cap \{s, s+1\} = \emptyset$, $\{s\}$, or $\{s+1\}$.

Subcase 1.1. Assume that $S(\mathbf{w}) \cap \{s, s+1\} = \emptyset$, or $S(\mathbf{w}) \cap \{s, s+1\} \ne \emptyset$ and $\mathbf{b}(t) = 0$ for $t \in S(\mathbf{w}) \cap \{s, s+1\}$. In this case, we set

$$v_k := w_k \qquad \text{for } 0 \le k \le m \text{ with } k \ne s, s+1.$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_{s-1}, v_{s+2}, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) = S(\mathbf{w}) \setminus \{s, s+1\}$. We set $\mathbf{c}(t) := \mathbf{b}(t)$ for $t \in S(\mathbf{v})$. Then, $\mathsf{D}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.20). Indeed, we can easily show the equalities in (3.20), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that

$$X := \sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b})\rangle - \sum_{t\in T^-(\mathbf{w})\cap\{s,s+1\}} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t\in S(\mathbf{w})\cap\{s,s+1\}} \mathbf{b}(t) l'_t \tag{6.3}$$

is equal to 0. Indeed, since $\mathrm{wt}_s(\mathbf{w}) = \mathrm{wt}_{s+1}(\mathbf{w}) = 0$, $T^-(\mathbf{w}) \cap \{s, s+1\} = \emptyset$, and $\mathbf{b}(t) = 0$ for $t \in S(\mathbf{w}) \cap \{s, s+1\}$, we obtain $X = 0$, as desired.

Subcase 1.2 (to be paired with Case 4 below)**.** Assume that $S(\mathbf{w}) \cap \{s, s+1\} \ne \emptyset$, and $\mathbf{b}(t) = 1$ for $t \in S(\mathbf{w}) \cap \{s, s+1\}$. We deduce that $|w_{s-1}^{-1}\alpha|$ is a simple root. Hence it follows that

$$w_{s-1} \xrightarrow{|w_{s-1}^{-1}\alpha|} s_\alpha w_{s-1} \xrightarrow{|w_{s-1}^{-1}\alpha|} w_{s-1};$$

notice that one of these edges is a Bruhat edge, and the other is a quantum edge. We set

$$\begin{cases} v_k := w_k & \text{for } 0 \le k \le m \text{ with } k \ne s, s+1, \\ v_s := s_\alpha w_{s-1}, & v_{s+1} := w_{s-1}. \end{cases}$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_{s-1}, v_s, v_{s+1}, v_{s+2}, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \cap \{s, s+1\} = \emptyset$, and $S(\mathbf{v}) = S(\mathbf{w}) \setminus \{s, s+1\}$. We set $\mathbf{c}(t) := \mathbf{b}(t)$ for $t \in S(\mathbf{v})$. Then, $\mathsf{D}(\mathbf{w}, \mathbf{b}) = (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.21). Indeed, we can easily show the equalities in (3.21), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show that $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that X in (6.3) is equal to

$$Y := \sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{v}, \mathbf{c}) \rangle - \sum_{t \in T^-(\mathbf{v}) \cap \{s,s+1\}} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t \in S(\mathbf{v}) \cap \{s,s+1\}} \mathbf{c}(t) l'_t. \tag{6.4}$$

Since $l'_{s+1} = -l'_s$, and $\mathrm{wt}_s(\mathbf{v}) = -l'_s w_{s-1}^{-1}\alpha$, $\mathrm{wt}_{s+1}(\mathbf{v}) = l'_s w_{s-1}^{-1}\alpha = -\mathrm{wt}_s(\mathbf{v})$, we deduce that

$$\sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{v}, \mathbf{c}) \rangle = \langle l'_s w_{s-1}^{-1}\alpha,\, \delta_{s+1 \in T^-(\mathbf{v})} |w_{s-1}^{-1}\alpha|^\vee \rangle,$$

where for a statement P, we define $\delta_{\mathsf{P}} := 1$ (resp., $:= 0$) if P is true (resp., false). We see that $\delta_{s+1 \in T^-(\mathbf{v})} |w_{s-1}^{-1}\alpha|^\vee = \delta_{s+1 \in T^-(\mathbf{v})} \mathrm{sgn}(\alpha) w_{s-1}^{-1}\alpha^\vee$. Hence it follows that

$$\sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{v}, \mathbf{c}) \rangle = 2\delta_{s+1 \in T^-(\mathbf{v})} l'_s \,\mathrm{sgn}(\alpha).$$

Also, we see that

$$\sum_{t \in T^-(\mathbf{v}) \cap \{s,s+1\}} \mathrm{sgn}(\gamma_t) l'_t = \mathrm{sgn}(\alpha) l'_s \quad \text{and} \quad \sum_{t \in S(\mathbf{v}) \cap \{s,s+1\}} \mathbf{c}(t) l'_t = 0.$$

Hence it follows that $Y = (2\delta_{s+1 \in T^-(\mathbf{v})} - 1) l'_s \,\mathrm{sgn}(\alpha)$. As for X, we have

$$\sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b}) \rangle = \sum_{t \in T^-(\mathbf{w}) \cap \{s,s+1\}} \mathrm{sgn}(\gamma_t) l'_t = 0.$$

Also, we deduce that

$$\begin{aligned} &s \in S(\mathbf{w}) \iff s+1 \in T^-(\mathbf{v}) \text{ and } \alpha \in \Delta^-, \text{ or } s+1 \notin T^-(\mathbf{v}) \text{ and } \alpha \in \Delta^+, \\ &s+1 \in S(\mathbf{w}) \iff s+1 \in T^-(\mathbf{v}) \text{ and } \alpha \in \Delta^+, \text{ or } s+1 \notin T^-(\mathbf{v}) \text{ and } \alpha \in \Delta^-, \end{aligned}$$

which implies that

$$X = -\sum_{t \in S(\mathbf{w}) \cap \{s,s+1\}} \mathbf{b}(t) l'_t = (2\delta_{s+1 \in T^-(\mathbf{v})} - 1) l'_s \,\mathrm{sgn}(\alpha).$$

This proves $X = Y$, as desired.

Case 2 (to be paired with Case 3 below)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies $w_{s-1} \xrightarrow{|w_{s-1}^{-1}\alpha|} w_s = w_{s+1}$. We set

$$\begin{cases} v_k := w_k & \text{for } 0 \le k \le m \text{ with } k \ne s, s+1, \\ \underbrace{w_{s-1}}_{=v_{s-1}=:v_s} \xrightarrow{|w_{s-1}^{-1}\alpha|} \underbrace{s_\alpha w_{s-1} = w_s = w_{s+1}}_{=:v_{s+1}}. \end{cases}$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_{s-1}, v_s, v_{s+1}, v_{s+2}, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that

$$S(\mathbf{w}) \cap \{s, s+1\} = \emptyset \text{ (resp., } = \{s+1\}) \text{ if and only if } S(\mathbf{v}) \cap \{s, s+1\} = \emptyset \text{ (resp., } = \{s\}).$$

We set $\mathbf{c}(t) := \mathbf{b}(t)$ for $t \in S(\mathbf{v}) \setminus \{s, s+1\} = S(\mathbf{w}) \setminus \{s, s+1\}$, and $\mathbf{c}(s) := \mathbf{b}(s+1)$ if $S(\mathbf{v}) \cap \{s, s+1\} = \{s\}$. Then, $\mathsf{D}(\mathbf{w}, \mathbf{b}) = (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.21). For the equality $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, let us show that X in (6.3) is equal to Y in (6.4). We compute

$$\begin{aligned} X &= \sum_{t=s,s+1} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b}) \rangle - \sum_{t \in T^-(\mathbf{w}) \cap \{s,s+1\}} \mathrm{sgn}(\gamma_t) l'_t - \sum_{t \in S(\mathbf{w}) \cap \{s,s+1\}} \mathbf{b}(t) l'_t \\ &= \langle \mathrm{wt}_s(\mathbf{w}),\, \mathrm{qwt}_s^\vee(\mathbf{w}, \mathbf{b}) \rangle - \delta_{s \in T^-(\mathbf{w})} \mathrm{sgn}(\gamma_s) l'_s - \delta_{s+1 \in S(\mathbf{w})} \mathbf{b}(s+1) l'_{s+1} \\ &= \langle \mathrm{wt}_s(\mathbf{w}),\, \mathrm{qwt}_s^\vee(\mathbf{w}, \mathbf{b}) \rangle - \delta_{s+1 \in T^-(\mathbf{v})} \mathrm{sgn}(\gamma_{s+1}) l'_{s+1} + \delta_{s \in S(\mathbf{v})} \mathbf{c}(s) l'_s, \end{aligned}$$

$$Y = \langle \mathrm{wt}_{s+1}(\mathbf{v}),\, \mathrm{qwt}_{s+1}^\vee(\mathbf{v}, \mathbf{c}) \rangle - \delta_{s+1 \in T^-(\mathbf{v})} \mathrm{sgn}(\gamma_{s+1}) l'_{s+1} - \delta_{s \in S(\mathbf{v})} \mathbf{c}(s) l'_s.$$

Here,

$$\langle \mathrm{wt}_s(\mathbf{w}),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle = \langle -l'_s w_{s-1}^{-1}\gamma_s,\, \mathrm{qwt}_{s-1}^\vee(\mathbf{w},\mathbf{b}) + \delta_{s\in T^-(\mathbf{w})}|w_s^{-1}\gamma_s|^\vee\rangle,$$

$$\begin{aligned}&\langle \mathrm{wt}_{s+1}(\mathbf{v}),\, \mathrm{qwt}_{s+1}^\vee(\mathbf{v},\mathbf{c})\rangle \\ &= \langle -\underbrace{l'_{s+1}v_s^{-1}\gamma_{s+1}}_{=l'_s w_{s-1}^{-1}\gamma_s},\, \underbrace{\mathrm{qwt}_{s-1}^\vee(\mathbf{v},\mathbf{c})}_{=\mathrm{qwt}_{s-1}^\vee(\mathbf{w},\mathbf{b})} + \underbrace{\delta_{s+1\in T^-(\mathbf{v})}|v_{s+1}^{-1}\gamma_{s+1}|^\vee}_{=\delta_{s\in T^-(\mathbf{w})}|w_s^{-1}\gamma_s|^\vee} - \delta_{s\in S(\mathbf{v})}\mathbf{c}(s)\ \underbrace{v_s^{-1}\gamma_s^\vee}_{=w_{s-1}^{-1}\gamma_s^\vee}\ \rangle \\ &= \langle \mathrm{wt}_s(\mathbf{w}),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + 2\delta_{s\in S(\mathbf{v})}\mathbf{c}(s)l'_s.\end{aligned}$$

Combining these equalities, we obtain $X = Y$, as desired.

Case 3 (to be paired with Case 2)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies $w_{s-1} = w_s \xrightarrow{|w_{s-1}^{-1}\alpha|} s_\alpha w_s = w_{s+1}$. We set

$$\begin{cases} v_k := w_k & \text{for } 0 \le k \le m \text{ with } k \ne s, s+1, \\ \underbrace{w_{s-1} = w_s}_{=v_{s-1}} \xrightarrow{|w_{s-1}^{-1}\alpha|} \underbrace{s_\alpha w_s = w_{s+1}}_{=:v_s = v_{s+1}}. \end{cases}$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_{s-1}, v_s, v_{s+1}, v_{s+2}, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that

$$S(\mathbf{w}) \cap \{s, s+1\} = \emptyset \text{ (resp., } = \{s\}) \text{ if and only if } S(\mathbf{v}) \cap \{s, s+1\} = \emptyset \text{ (resp., } = \{s+1\}).$$

We set $\mathbf{c}(t) := \mathbf{b}(t)$ for $t \in S(\mathbf{v}) \setminus \{s, s+1\} = S(\mathbf{w}) \setminus \{s, s+1\}$, and $\mathbf{c}(s+1) := \mathbf{w}(s)$ if $S(\mathbf{v}) \cap \{s, s+1\} = \{s+1\}$. Then, $\mathsf{D}(\mathbf{w},\mathbf{b}) = (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.21); we can show the equalities in (3.21) in exactly the same way as in Case 2.

Case 4 (to be paired with Subcase 1.2)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies $w_{s-1} \xrightarrow{|w_{s-1}^{-1}\alpha|} w_s \xrightarrow{|w_{s-1}^{-1}\alpha|} w_{s+1}$; note that $w_{s+1} = w_{s-1}$. Also, notice that $S(\mathbf{w}) \cap \{s, s+1\} = \emptyset$, and $|w_{s-1}^{-1}\alpha|$ is a simple root. We set

$$\begin{cases} v_k := w_k & \text{for } 0 \le k \le m \text{ with } k \ne s, s+1, \\ w_{s-1} = v_{s-1} = v_s = v_{s+1}. \end{cases}$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_{s-1}, v_s, v_{s+1}, v_{s+2}, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \setminus \{s, s+1\} = S(\mathbf{w}) \setminus \{s, s+1\}$, and that $S(\mathbf{v}) \cap \{s, s+1\}$ is either $\{s\}$ or $\{s+1\}$ since $|w_{s-1}^{-1}\alpha|$ is a simple root. We set

$$\mathbf{c}(t) := \begin{cases} \mathbf{b}(t) & \text{for } t \in S(\mathbf{v}) \setminus \{s, s+1\}, \\ 1 & \text{for } t \in S(\mathbf{v}) \cap \{s, s+1\}. \end{cases}$$

Then, $\mathsf{D}(\mathbf{w},\mathbf{b}) = (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.21); we can show the equalities in (3.21) in exactly the same way as in Subcase 1.2.

Now, we set

$$\widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma) := \bigl\{(\mathbf{w},\mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \mid \mathsf{D}(\mathbf{w},\mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)\bigr\}.$$

We can easily verify that the map $(\mathbf{w},\mathbf{b}) \mapsto \mathsf{D}(\mathbf{w},\mathbf{b})$ gives a bijection from $\widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \setminus \widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$ onto $\widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfying (3.20), and also an involution on $\widetilde{\mathbf{QW}}^{(1)}_{\lambda,w}(\Gamma)$ satisfying (3.21). This completes the proof of Theorem 3.14.

7. Proof of Theorem 3.12.

In what follows, we indicate that an edge $w \xrightarrow{\alpha} ws_\alpha$ in $\mathrm{QBG}(W)$ is a quantum edge (resp., Bruhat edge) by writing $w \xrightarrow[\mathsf{Q}]{\alpha} ws_\alpha$ (resp., $w \xrightarrow[\mathsf{B}]{\alpha} ws_\alpha$).

7.1. **In type A_2.** We give a proof of Theorem 3.12 in the case that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$ (see Remark 3.11). Assume that $\Gamma \in \mathsf{AP}(\lambda)$ is of the form

$$\Gamma : A_\circ = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda, \tag{7.1}$$

with $\gamma_{s+1} = \alpha$, $\gamma_{s+2} = \alpha + \beta$, $\gamma_{s+3} = \beta$, i.e.,

$$\cdots \xrightarrow{\gamma_s} A_s \xrightarrow{\alpha} A_{s+1} \xrightarrow{\alpha+\beta} A_{s+2} \xrightarrow{\beta} A_{s+3} \xrightarrow{\gamma_{s+4}} \cdots \qquad (\text{in } \Gamma).$$

Then, $\Xi = (\beta_1, \dots, \beta_m)$, where $\beta_k := \gamma_k$ for $1 \le k \le m$ with $k \ne s+1, s+2, s+3$, and $\beta_{s+1} = \beta$, $\beta_{s+2} = \alpha + \beta$, $\beta_{s+3} = \alpha$; note that Ξ is an alcove path from $A_\circ$ to A_λ of the form:

$$\Xi : A_\circ = A_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} A_s \xrightarrow{\beta} B_{s+1} \xrightarrow{\alpha+\beta} B_{s+2} \xrightarrow{\alpha} A_{s+3} \xrightarrow{\gamma_{s+4}} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda \tag{7.2}$$

for some alcoves B_{s+1} and B_{s+2}; observe that (the closure of) $A_s \sqcup A_{s+1} \sqcup A_{s+2} \sqcup A_{s+3} \sqcup B_{s+1} \sqcup B_{s+2}$ forms a "hexagon" lying in $\mathbb{R}\alpha \oplus \mathbb{R}\beta \subset \mathfrak{h}^*_{\mathbb{R}}$. For $1 \le t \le m$, let H_{γ_t, l_t} (resp., H_{β_t, k_t}) be the affine hyperplane between the $(t-1)$-th alcove and the t-th alcove in Γ (resp., Ξ). Then we see that $l_t = k_t$ for all $1 \le t \le m$ with $t \ne s+1, s+2, s+3$, and that

$$k_{s+3} = l_{s+1}, \qquad k_{s+1} = l_{s+3}, \qquad l_{s+2} = l_{s+1} + l_{s+3} = k_{s+1} + k_{s+3} = k_{s+2}. \tag{7.3}$$

Recall that $l'_t = \langle \lambda, \gamma_t^\vee \rangle - l_t$ for $1 \le t \le m$. Set $k'_t := \langle \lambda, \beta_t^\vee \rangle - k_t$ for $1 \le t \le m$. We see that $l'_t = k'_t$ for all $1 \le t \le m$ with $t \ne s+1, s+2, s+3$, and that

$$k'_{s+3} = l'_{s+1}, \qquad k'_{s+1} = l'_{s+3}, \qquad l'_{s+2} = l'_{s+1} + l'_{s+3} = k'_{s+1} + k'_{s+3} = k'_{s+2}. \tag{7.4}$$

Now, for $\mathbf{w} = (w_0, w_1, \dots, w_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, we set

$$v_k := w_k \qquad \text{for } 0 \le k \le m \text{ with } k \ne s+1, s+2. \tag{7.5}$$

In the following, we will define v_{s+1} and v_{s+2} in such a way that $\mathbf{v} := (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma) \sqcup \mathbf{QW}_{\lambda,w}(\Xi)$. Note that $S(\mathbf{w}) \setminus \{s+1, s+2, s+3\} = S(\mathbf{v}) \setminus \{s+1, s+2, s+3\}$. Also, for $\mathbf{b} : S(\mathbf{w}) \to \{0,1\}$, we will define $\mathbf{c} : S(\mathbf{v}) \to \{0,1\}$ in such a way that

$$\mathbf{c}|_{S(\mathbf{v}) \setminus \{s+1,s+2,s+3\}} = \mathbf{b}|_{S(\mathbf{w}) \setminus \{s+1,s+2,s+3\}}, \tag{7.6}$$

and $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \sqcup \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$. Then we set

$$\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma) := \bigl\{ (\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \mid \mathsf{YB}(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi) \bigr\}. \tag{7.7}$$

Case 1. Assume that $w_s = w_{s+1} = w_{s+2} = w_{s+3}$. Then we set $w_s = v_s = v_{s+1} = v_{s+2} = v_{s+3} = w_{s+3}$. It is obvious that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$.

We see that $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is one of $\emptyset$, $\{s+1\}$, $\{s+2\}$, $\{s+3\}$, and $\{s+1, s+3\}$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \ (\text{resp., } \{s+1\}, \{s+2\}, \{s+3\}, \{s+1, s+3\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \ (\text{resp., } \{s+3\}, \{s+2\}, \{s+1\}, \{s+1, s+3\}). \end{aligned}$$

Hence we can define $\mathbf{c}(t) := \mathbf{b}(2s+4-t)$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) = (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18). Indeed, we can easily show the equalities in (3.18), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show that $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that

$$\begin{aligned} X := &\overbrace{\sum_{t=s+1,s+2,s+3} \langle \mathrm{wt}_t(\mathbf{w}),\, \mathrm{qwt}_t^\vee(\mathbf{w}, \mathbf{b}) \rangle}^{=:X_1} \\ &- \underbrace{\sum_{t \in T^-(\mathbf{w}) \cap \{s+1,s+2,s+3\}} \mathrm{sgn}(\gamma_t) l'_t}_{=:X_2} - \underbrace{\sum_{t \in S(\mathbf{w}) \cap \{s+1,s+2,s+3\}} \mathbf{b}(t) l'_t}_{=:X_3} \end{aligned} \tag{7.8}$$

is equal to

$$
\begin{aligned}
Y := & \overbrace{\sum_{t=s+1,s+2,s+3} \langle \mathrm{wt}_t(\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{v},\mathbf{c})\rangle}^{=:Y_1} \\
& - \underbrace{\sum_{t\in T^-(\mathbf{v})\cap\{s+1,s+2,s+3\}} \mathrm{sgn}(\beta_t)k'_t}_{=:Y_2} - \underbrace{\sum_{t\in S(\mathbf{v})\cap\{s+1,s+2,s+3\}} \mathbf{c}(t)k'_t}_{=:Y_3}.
\end{aligned}
\tag{7.9}
$$

We see that $X_1 = Y_1 = 0$. Also, it is easily verified by (5.12) that $X_2 = Y_2$ and $X_3 = Y_3$. This proves $X = Y$, as desired.

Case 2. Assume that $w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s = w_{s+1} = w_{s+2} = w_{s+3}$. Then, we define v_{s+1} and v_{s+2} by the following directed path in $\mathrm{QBG}(W)$:

$$(w_s =) \quad v_s = v_{s+1} = v_{s+2} \xrightarrow{|v_{s+2}^{-1}\alpha|} s_\alpha v_{s+2} = v_{s+3} \quad (= w_{s+3}).$$

It is easily checked that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$.

We see that $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is one of $\emptyset$, $\{s+2\}$, and $\{s+3\}$, and that

$$
\begin{aligned}
& S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\},\ \{s+3\}) \\
\iff & S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\},\ \{s+2\}).
\end{aligned}
$$

Hence we can define $\mathbf{c}(t) := \mathbf{b}(t+1)$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) = (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18). Indeed, we can easily show the equalities in (3.18), except for $\deg(\mathbf{v},\mathbf{c}) = \deg(\mathbf{w},\mathbf{b})$. In order to show that $\deg(\mathbf{v},\mathbf{c}) = \deg(\mathbf{w},\mathbf{b})$, it suffices to show that $\deg'(\mathbf{v},\mathbf{c}) = \deg'(\mathbf{w},\mathbf{b})$. For this equality, we claim that X in (7.8) is equal to Y in (7.9). We have

$$
\begin{aligned}
X_1 &= \langle \mathrm{wt}_{s+1}(\mathbf{w}),\, \mathrm{qwt}_{s+1}^\vee(\mathbf{w},\mathbf{b})\rangle = \langle -l'_{s+1} w_s^{-1}\gamma_{s+1},\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) + \delta_{s+1\in T^-(\mathbf{w})}|w_{s+1}^{-1}\gamma_{s+1}|^\vee\rangle \\
&= \langle -l'_{s+1} w_s^{-1}\alpha,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + \langle -l'_{s+1} w_s^{-1}\alpha,\, \delta_{s+1\in T^-(\mathbf{w})}|w_s^{-1}\alpha|^\vee\rangle,
\end{aligned}
$$

$$
\begin{aligned}
Y_1 &= \langle \mathrm{wt}_{s+3}(\mathbf{v}),\, \mathrm{qwt}_{s+3}^\vee(\mathbf{v},\mathbf{c})\rangle \\
&= \langle -k'_{s+3} v_{s+2}^{-1}\beta_{s+3},\, \mathrm{qwt}_s^\vee(\mathbf{v},\mathbf{c})\rangle + \langle -k'_{s+3} v_{s+2}^{-1}\beta_{s+3},\, \delta_{s+3\in T^-(\mathbf{v})}|v_{s+3}^{-1}\beta_{s+3}|^\vee\rangle \\
&\quad - \langle -k'_{s+3} v_{s+2}^{-1}\beta_{s+3},\, \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)v_{s+1}^{-1}\beta_{s+1}^\vee + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)v_{s+2}^{-1}\beta_{s+2}^\vee\rangle \\
&= \langle -l'_{s+1} w_s^{-1}\alpha,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + \langle -l'_{s+1} w_s^{-1}\alpha,\, \delta_{s+1\in T^-(\mathbf{w})}|w_s^{-1}\alpha|^\vee\rangle \\
&\quad - \langle -l'_{s+1} w_s^{-1}\alpha,\, \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)w_s^{-1}\beta^\vee + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)w_s^{-1}(\alpha+\beta)^\vee\rangle \\
&= \langle -l'_{s+1} w_s^{-1}\alpha,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + \langle -l'_{s+1} w_s^{-1}\alpha,\, \delta_{s+1\in T^-(\mathbf{w})}|w_s^{-1}\alpha|^\vee\rangle \\
&\quad - \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)l'_{s+1} + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)l'_{s+1},
\end{aligned}
$$

and hence

$$Y_1 = X_1 - \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)l'_{s+1} + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)l'_{s+1}.$$

Also, we have

$$X_2 = \delta_{s+1\in T^-(\mathbf{w})}\,\mathrm{sgn}(\gamma_{s+1})l'_{s+1} = \delta_{s+3\in T^-(\mathbf{v})}\,\mathrm{sgn}(\beta_{s+3})k'_{s+3} = Y_2,$$

and

$$
\begin{aligned}
X_3 &= \delta_{s+2\in S(\mathbf{w})}\mathbf{b}(s+2)l'_{s+2} + \delta_{s+3\in S(\mathbf{w})}\mathbf{b}(s+3)l'_{s+3}, \\
Y_3 &= \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)k'_{s+1} + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)k'_{s+2}.
\end{aligned}
$$

Therefore, we deduce that

$$
\begin{aligned}
Y &= Y_1 - Y_2 - Y_3 \\
&= X_1 - \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)l'_{s+1} + \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)l'_{s+1} - X_2
\end{aligned}
$$

$$
\begin{aligned}
&- \delta_{s+1\in S(\mathbf{v})}\mathbf{c}(s+1)k'_{s+1} - \delta_{s+2\in S(\mathbf{v})}\mathbf{c}(s+2)k'_{s+2} \\
&= X_1 - \delta_{s+2\in S(\mathbf{w})}\mathbf{b}(s+2)l'_{s+1} + \delta_{s+3\in S(\mathbf{w})}\mathbf{b}(s+3)l'_{s+1} - X_2 \\
&\quad - \delta_{s+2\in S(\mathbf{w})}\mathbf{b}(s+2)l'_{s+3} - \delta_{s+3\in S(\mathbf{w})}\mathbf{b}(s+3)l'_{s+2} \\
&= X_1 - X_2 - X_3 = X,
\end{aligned}
$$

as desired.

Case 3. Assume that $w_s = w_{s+1} = w_{s+2} \xrightarrow{|w_s^{-1}\beta|} s_\beta w_{s+2} = w_{s+3}$. Then, we define v_{s+1} and v_{s+2} by the following directed path in $\mathrm{QBG}(W)$:

$$(w_s =) \quad v_s \xrightarrow{|v_s^{-1}\beta|} s_\alpha v_s = v_{s+1} = v_{s+2} = v_{s+3} \quad (= w_{s+3}).$$

It is easily checked that $\mathbf{v} = (v_0, v_1, \dots, v_m) = \mathbf{v} \in \mathbf{QW}_{\lambda,w}(\Xi)$.

We see that $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is one of $\emptyset$, $\{s+1\}$, and $\{s+2\}$, and that

$$
\begin{aligned}
&S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\},\ \{s+2\}) \\
&\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\},\ \{s+3\}).
\end{aligned}
$$

Hence we can define $\mathbf{c}(t) := \mathbf{b}(t-1)$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. As in Case 2, we can show that $\mathsf{YB}(\mathbf{w}, \mathbf{b}) = (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Case 4. Assume that $w_s = w_{s+1} \xrightarrow{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2} = w_{s+3}$; note that $w_{s+1}^{-1}(\alpha+\beta) = w_s^{-1}\alpha + w_s^{-1}\beta$, and $\langle w_s^{-1}\alpha,\ w_s^{-1}\beta^\vee\rangle = \langle w_s^{-1}\beta,\ w_s^{-1}\alpha^\vee\rangle = -1$. Also, since $w_{s+1}^{-1}\gamma_{s+1} = w_s^{-1}\alpha$ and $w_{s+3}^{-1}\gamma_{s+3} = w_s^{-1}s_{\alpha+\beta}\beta = -w_s^{-1}\alpha$, we see that $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is one of $\emptyset$, $\{s+1\}$, and $\{s+3\}$.

Subcase 4.1 (to be paired with Subcase 6.1 below)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies either

$$
\begin{cases}
S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root)}, \\[2ex]
\mathbf{b}(s+1) = 1, \quad w_s^{-1}\beta \in \Delta^-, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2},
\end{cases}
\tag{7.10}
$$

or

$$
\begin{cases}
S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root)}, \\[2ex]
\mathbf{b}(s+3) = 1, \quad w_s^{-1}\beta \in \Delta^+, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}.
\end{cases}
\tag{7.11}
$$

It follows from Lemma A.1 (applied to w_s, $w_s^{-1}\alpha$, and $w_s^{-1}\beta$) that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} \underbrace{s_\alpha w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} \underbrace{s_{\alpha+\beta} s_\alpha w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} \underbrace{s_\beta s_{\alpha+\beta} s_\alpha w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in } \mathrm{QBG}(W),$$

and that

$$\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)) = -\mathrm{sgn}(\alpha) = -\mathrm{sgn}(\beta) = -\mathrm{sgn}(\alpha+\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). Since $\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta))$, we have $(-1)^{(\mathbf{w},\mathbf{b})} = -(-1)^{(\mathbf{v},\mathbf{c})}$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19). Indeed, we can easily show the equalities in (3.19), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show that $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that X of (7.8) is equal to

$$
\begin{aligned}
Z :=& \overbrace{\sum_{t=s+1,s+2,s+3} \langle \mathrm{wt}_t(\mathbf{v}),\, \mathrm{qwt}_t^\vee(\mathbf{v},\mathbf{c})\rangle}^{=:Z_1} \\
&- \underbrace{\sum_{t\in T^-(\mathbf{v})\cap\{s+1,s+2,s+3\}} \mathrm{sgn}(\gamma_t) l'_t}_{=:Z_2} - \underbrace{\sum_{t\in S(\mathbf{v})\cap\{s+1,s+2,s+3\}} \mathbf{c}(t) l'_t}_{=:Z_3}.
\end{aligned}
\tag{7.12}
$$

We give a proof only in the case that (7.10) holds; the proof in the case that (7.11) holds is similar. We have

$$
\begin{aligned}
X_1 &= \langle \mathrm{wt}_{s+2}(\mathbf{w}),\, \mathrm{qwt}_{s+2}^\vee(\mathbf{w},\mathbf{b})\rangle \\
&= \langle -l'_{s+2} w_{s+1}^{-1}\gamma_{s+2},\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) + |w_{s+2}^{-1}\gamma_{s+2}|^\vee - w_{s+1}^{-1}\gamma_{s+1}^\vee\rangle \\
&= \langle -l'_{s+2} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) + |w_s^{-1}(\alpha+\beta)|^\vee - w_s^{-1}\alpha^\vee\rangle \\
&= \langle -l'_{s+2} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + 2l'_{s+2} + l'_{s+2},
\end{aligned}
$$

$$
X_2 = l'_{s+2}, \qquad X_3 = l'_{s+1},
$$

and hence

$$
X = X_1 - X_2 - X_3 = \langle -l'_{s+2} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + 2l'_{s+2} - l'_{s+1}.
$$

Also, we have

$$
\begin{aligned}
Z_1 &= \langle \mathrm{wt}_{s+1}(\mathbf{v}),\, \mathrm{qwt}_{s+1}^\vee(\mathbf{v},\mathbf{c})\rangle + \langle \mathrm{wt}_{s+2}(\mathbf{v}),\, \mathrm{qwt}_{s+2}^\vee(\mathbf{v},\mathbf{c})\rangle + \langle \mathrm{wt}_{s+3}(\mathbf{v}),\, \mathrm{qwt}_{s+3}^\vee(\mathbf{v},\mathbf{c})\rangle \\
&= \langle -l'_{s+1} v_s^{-1}\gamma_{s+1},\, \mathrm{qwt}_s^\vee(\mathbf{v},\mathbf{c}) + |v_{s+1}^{-1}\gamma_{s+1}|^\vee\rangle \\
&\quad + \langle -l'_{s+2} v_{s+1}^{-1}\gamma_{s+2},\, \mathrm{qwt}_s^\vee(\mathbf{v},\mathbf{c}) + |v_{s+1}^{-1}\gamma_{s+1}|^\vee + |v_{s+2}^{-1}\gamma_{s+2}|^\vee\rangle \\
&\quad + \langle -l'_{s+3} v_{s+2}^{-1}\gamma_{s+3},\, \mathrm{qwt}_s^\vee(\mathbf{v},\mathbf{c}) + |v_{s+1}^{-1}\gamma_{s+1}|^\vee + |v_{s+2}^{-1}\gamma_{s+2}|^\vee + |v_{s+3}^{-1}\gamma_{s+3}|^\vee\rangle \\
&= \langle -l'_{s+1} w_s^{-1}\alpha,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) - w_s^{-1}\alpha^\vee\rangle \\
&\quad + \langle -l'_{s+2} w_s^{-1}\beta,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) - w_s^{-1}\alpha^\vee - w_s^{-1}\beta^\vee\rangle \\
&\quad + \langle -l'_{s+3} w_s^{-1}\alpha,\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b}) - w_s^{-1}\alpha^\vee - w_s^{-1}\beta^\vee - w_s^{-1}\alpha^\vee\rangle \\
&= \langle -l'_{s+2} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + 2l'_{s+1} + l'_{s+2} + 3l'_{s+3},
\end{aligned}
$$

$$
Z_2 = l'_{s+1} + l'_{s+2} + l'_{s+3}, \qquad Z_3 = 0,
$$

and hence

$$
Z = Z_1 - Z_2 - Z_3 = \langle -l'_{s+2} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}_s^\vee(\mathbf{w},\mathbf{b})\rangle + l'_{s+1} + 2l'_{s+3}.
$$

Since $2l'_{s+2} - l'_{s+1} = 2(l'_{s+1} + l'_{s+3}) - l'_{s+1} = l'_{s+1} + 2l'_{s+3}$, we obtain $X = Z$, as desired.

Subcase 4.2 (to be paired with Subcase 6.2 below)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies either

$$
\begin{cases}
S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root)}, \\
\mathbf{b}(s+1) = 1, \quad w_s^{-1}\beta \in \Delta^+, \quad \alpha,\beta \in \Delta^-, \quad \text{and} \\
w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2},
\end{cases}
\tag{7.13}
$$

or

$$
\begin{cases}
S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root)}, \\
\mathbf{b}(s+3) = 1, \quad w_s^{-1}\beta \in \Delta^-, \quad \alpha,\beta \in \Delta^+, \quad \text{and} \\
w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}.
\end{cases}
\tag{7.14}
$$

It follows from Lemma A.2 (applied to w_s, $w_s^{-1}\alpha$, and $w_s^{-1}\beta$) that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_\alpha w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} \underbrace{s_{\alpha+\beta}s_\alpha w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_\beta s_{\alpha+\beta}s_\alpha w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(w_s^{-1}\beta) = -\operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = \operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta) = \operatorname{sgn}(\alpha+\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \ldots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19), which we can verify in exactly the same way as in Subcase 4.1.

Subcase 4.3 (to be paired with Subcase 6.3 below)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies either

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+1) = 1, \quad w_s^{-1}\beta \in \Delta^+, \quad \alpha \in \Delta^+, \quad \text{and} \\ w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_{s+1} = w_{s+2}, \end{cases} \tag{7.15}$$

or

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+3) = 1, \quad w_s^{-1}\beta \in \Delta^-, \quad \alpha \in \Delta^-, \quad \text{and} \\ w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_{s+1} = w_{s+2}. \end{cases} \tag{7.16}$$

It follows from Lemma A.4 that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} \underbrace{s_\beta w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_{\alpha+\beta}s_\beta w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_\alpha s_{\alpha+\beta}s_\beta w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(w_s^{-1}\beta) = -\operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = -\operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta) = \operatorname{sgn}(\alpha+\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \ldots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18), which we can verify in exactly the same way as in Cases 1 and 2.

Subcase 4.4 (to be paired with Subcase 6.4 below)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies either

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e.,, } -w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+1) = 1, \quad w_s^{-1}\beta \in \Delta^+, \quad \alpha \in \Delta^-, \quad \beta \in \Delta^+, \quad \text{and} \\ w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_{s+1} = w_{s+2}, \end{cases} \tag{7.17}$$

or

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+3) = 1, \quad w_s^{-1}\beta \in \Delta^-, \quad \alpha \in \Delta^+, \quad \beta \in \Delta^-, \quad \text{and} \\ w_{s+1} \xrightarrow[\mathsf{Q}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_{s+1} = w_{s+2}. \end{cases} \tag{7.18}$$

It follows from Lemma A.5 that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_\beta w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_{\alpha+\beta}s_\beta w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} \underbrace{s_\alpha s_{\alpha+\beta}s_\beta w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(w_s^{-1}\beta) = -\operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = \operatorname{sgn}(\alpha) = -\operatorname{sgn}(\beta) = \operatorname{sgn}(\alpha+\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18), which we can verify in exactly the same way as in Cases 1 and 2.

Subcase 4.5 (to be paired with Subcase 6.5 below)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+1) = 1, \quad \alpha \in \Delta^-, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}, \end{cases} \tag{7.19}$$

or

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e.,, } w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+3) = 1, \quad \alpha \in \Delta^+, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}. \end{cases} \tag{7.20}$$

It follows from Lemma A.6 that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_\alpha w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_{\alpha+\beta} s_\alpha w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} \underbrace{s_\beta s_{\alpha+\beta} s_\alpha w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(\alpha) = \operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(\beta) \quad \text{and} \quad \operatorname{sgn}(\alpha+\beta) = \operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = \operatorname{sgn}(w_s^{-1}\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19), which we can verify in exactly the same way as in Subcase 4.1.

Subcase 4.6 (to be paired with Subcase 6.6 below)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+1) = 1, \quad \beta \in \Delta^-, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}, \end{cases} \tag{7.21}$$

or

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+3) = 1, \quad \beta \in \Delta^+, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}. \end{cases} \tag{7.22}$$

It follows from Lemma A.7 that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} \underbrace{s_\alpha w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_{\alpha+\beta} s_\alpha w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} \underbrace{s_\beta s_{\alpha+\beta} s_\alpha w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(\alpha) = -\operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(\beta) \quad \text{and} \quad \operatorname{sgn}(\alpha+\beta) = \operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = \operatorname{sgn}(w_s^{-1}\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19), which we can verify in exactly the same way as in Subcase 4.1.

Subcase 4.7 (to be paired with Subcase 6.7 below)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\} \text{ (i.e., } -w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+1) = 1, \quad \alpha, \beta \in \Delta^+, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}, \end{cases} \tag{7.23}$$

or

$$\begin{cases} S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+3\} \text{ (i.e., } w_s^{-1}\alpha \text{ is a simple root),} \\ \mathbf{b}(s+3) = 1, \quad \alpha, \beta \in \Delta^-, \quad \text{and} \quad w_{s+1} \xrightarrow[\mathsf{B}]{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_{s+1} = w_{s+2}. \end{cases} \tag{7.24}$$

It follows from Lemma A.3 that

$$\underbrace{w_s}_{=v_s} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_\beta w_s}_{=:v_{s+1}} \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} \underbrace{s_{\alpha+\beta} s_\beta w_s}_{=:v_{s+2}} \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} \underbrace{s_\alpha s_{\alpha+\beta} s_\beta w_s}_{=w_{s+3}=v_{s+3}} \quad \text{in QBG}(W),$$

and that

$$\operatorname{sgn}(w_s^{-1}\alpha) = -\operatorname{sgn}(w_s^{-1}\beta) = -\operatorname{sgn}(w_s^{-1}(\alpha+\beta)) = -\operatorname{sgn}(\alpha) = -\operatorname{sgn}(\beta) = -\operatorname{sgn}(\alpha+\beta).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18), which we can verify in exactly the same way as in Cases 1 and 2.

Subcase 4.8. Assume that $(\mathbf{w}, \mathbf{b})$ does not satisfy any of equations (7.10)–(7.24); notice that $S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset$ or $\mathbf{b}(t) = 0$ for $t \in S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$. Then, we define v_{s+1} and v_{s+2} by the following directed path in QBG(W):

$$(w_s =) \quad v_s = v_{s+1} \xrightarrow{|w_{s+1}^{-1}(\alpha+\beta)|} s_{\alpha+\beta} v_{s+1} = v_{s+2} = v_{s+3} \quad (= w_{s+3}).$$

We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$. If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} \neq \emptyset$, then we set $\mathbf{c}(t) = 0$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18), which we can verify in exactly the same way as in Cases 1 and 2.

Case 5. Assume that $\#\{s+1 \le t \le s+3 \mid w_{t-1} \neq w_t\} = 2$; then the sequence

$$(w_s, w_{s+1}, w_{s+2}, w_{s+3}\,;\, w_s^{-1}\gamma_{s+1}, w_{s+1}^{-1}\gamma_{s+2}, w_{s+2}^{-1}\gamma_{s+3}) \tag{7.25}$$

is identical to one of the following:

(a) $(w_s,\ s_\alpha w_s,\ s_{\alpha+\beta} s_\alpha w_s,\ s_{\alpha+\beta} s_\alpha w_s\,;\, w_s^{-1}\alpha,\ w_s^{-1}\beta,\ w_s^{-1}\alpha)$; note that $w_{s+3} = s_\alpha s_\beta w_s$.
(b) $(w_s,\ w_s,\ s_{\alpha+\beta} w_s,\ s_\beta s_{\alpha+\beta} w_s\,;\, w_s^{-1}\alpha,\ w_s^{-1}(\alpha+\beta),\ -w_s^{-1}\alpha)$; note that $w_{s+3} = s_\alpha s_\beta w_s$.
(c) $(w_s,\ s_\alpha w_s,\ s_\alpha w_s,\ s_\beta s_\alpha w_s\,;\, w_s^{-1}\alpha,\ w_s^{-1}\beta,\ w_s^{-1}(\alpha+\beta))$; note that $w_{s+3} = s_\beta s_\alpha w_s$.

In Case 5, we make frequent uses of a fundamental fact about the existence and uniqueness of a label-increasing or label-decreasing directed path in the quantum Bruhat graph with respect to a fixed reflection order; see, for example, [LNS^3, Theorem 7.3].

Subcase 5.1. Assume that $\operatorname{sgn}(w_s^{-1}\alpha) = \operatorname{sgn}(w_s^{-1}\beta)$. We fix a reflection order $\lhd$ on Δ^+ such that $|w_s^{-1}\alpha| \lhd |w_s^{-1}(\alpha+\beta)| \lhd |w_s^{-1}\beta|$.

(5.1a). If the sequence in (7.25) is of the form (a), then we have the label-increasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s = s_\alpha s_\beta w_s, \tag{7.26}$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+3\}$. It follows that there exists a unique label-decreasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\alpha s_\beta w_s = w_{s+3} \tag{7.27}$$

or

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s = w_{s+3}. \tag{7.28}$$

If (7.27) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_\beta w_s, s_\alpha s_\beta w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}) \\ \iff\ &S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+2\}$, then we define $\mathbf{c}(s+2) := \mathbf{b}(s+3)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18). Indeed, we can easily show the equalities in (3.18), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show that $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that X in (7.8) is equal to Y in (7.9). We set $\epsilon := \mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta)$. Recall that $\delta_{\mathsf{P}} = 1$ (resp., $= 0$) if a statement P is true (resp., false). We have

$$\begin{aligned} X_1 &= \langle \mathrm{wt}_{s+1}(\mathbf{w}),\, \mathrm{qwt}^\vee_{s+1}(\mathbf{w}, \mathbf{b})\rangle + \langle \mathrm{wt}_{s+2}(\mathbf{w}),\, \mathrm{qwt}^\vee_{s+2}(\mathbf{w}, \mathbf{b})\rangle \\ &= \langle -l'_{s+1} w_s^{-1}\gamma_{s+1},\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b}) + \delta_{s+1 \in T^-(\mathbf{w})} |w_{s+1}^{-1}\gamma_{s+1}|^\vee \rangle \\ &\quad + \langle -l'_{s+2} w_{s+1}^{-1}\gamma_{s+2},\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b}) + \delta_{s+1 \in T^-(\mathbf{w})} |w_{s+1}^{-1}\gamma_{s+1}|^\vee + \delta_{s+2 \in T^-(\mathbf{w})} |w_{s+2}^{-1}\gamma_{s+2}|^\vee \rangle \\ &= \langle -l'_{s+1} w_s^{-1}\alpha,\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b}) + \delta_{s+1 \in T^-(\mathbf{w})} |w_s^{-1}\alpha|^\vee \rangle \\ &\quad + \langle -l'_{s+2} w_s^{-1}\beta,\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b}) + \delta_{s+1 \in T^-(\mathbf{w})} |w_s^{-1}\alpha|^\vee + \delta_{s+2 \in T^-(\mathbf{w})} |w_s^{-1}\beta|^\vee \rangle \\ &= \langle -l'_{s+1} w_s^{-1}\alpha - l'_{s+2} w_s^{-1}\beta,\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b})\rangle \\ &\quad - 2\delta_{s+1 \in T^-(\mathbf{w})}\epsilon l'_{s+1} + \delta_{s+1 \in T^-(\mathbf{w})}\epsilon l'_{s+2} - 2\delta_{s+2 \in T^-(\mathbf{w})}\epsilon l'_{s+2}, \end{aligned}$$

$$\begin{aligned} X_2 &= \delta_{s+1 \in T^-(\mathbf{w})} \,\mathrm{sgn}(\alpha) l'_{s+1} + \delta_{s+2 \in T^-(\mathbf{w})} \,\mathrm{sgn}(\alpha+\beta) l'_{s+2}, \\ X_3 &= \delta_{s+3 \in S(\mathbf{w})} \mathbf{b}(s+3) l'_{s+3}; \end{aligned}$$

recall that $X = X_1 - X_2 - X_3$. Also, we have

$$\begin{aligned} Y_1 &= \langle \mathrm{wt}_{s+1}(\mathbf{v}),\, \mathrm{qwt}^\vee_{s+1}(\mathbf{v}, \mathbf{c})\rangle + \langle \mathrm{wt}_{s+3}(\mathbf{v}),\, \mathrm{qwt}^\vee_{s+3}(\mathbf{v}, \mathbf{c})\rangle \\ &= \langle -k'_{s+1} v_s^{-1}\beta_{s+1},\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c}) + \delta_{s+1 \in T^-(\mathbf{v})} |v_{s+1}^{-1}\beta_{s+1}|^\vee \rangle \\ &\quad + \big\langle -k'_{s+3} v_{s+2}^{-1}\beta_{s+3},\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c}) + \\ &\quad\quad \delta_{s+1 \in T^-(\mathbf{v})} |v_{s+1}^{-1}\beta_{s+1}|^\vee + \delta_{s+3 \in T^-(\mathbf{v})} |v_{s+3}^{-1}\beta_{s+3}|^\vee - \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) v_{s+2}^{-1}\beta_{s+2}^\vee \big\rangle \\ &= \langle -k'_{s+1} w_s^{-1}\beta,\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c}) + \delta_{s+1 \in T^-(\mathbf{v})} |w_s^{-1}\beta|^\vee \rangle \\ &\quad + \big\langle -k'_{s+3} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c}) + \\ &\quad\quad \delta_{s+1 \in T^-(\mathbf{v})} |w_s^{-1}\beta|^\vee + \delta_{s+3 \in T^-(\mathbf{v})} |w_s^{-1}(\alpha+\beta)|^\vee - \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) w_s^{-1}\alpha^\vee \big\rangle \\ &= \langle -k'_{s+1} w_s^{-1}\beta - k'_{s+3} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c})\rangle \\ &\quad - 2\delta_{s+1 \in T^-(\mathbf{v})}\epsilon k'_{s+1} - \delta_{s+1 \in T^-(\mathbf{v})}\epsilon k'_{s+3} - 2\delta_{s+3 \in T^-(\mathbf{v})}\epsilon k'_{s+3} + \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) k'_{s+3}, \end{aligned}$$

$$\begin{aligned} Y_2 &= \delta_{s+1 \in T^-(\mathbf{v})} \,\mathrm{sgn}(\beta) k'_{s+1} + \delta_{s+3 \in T^-(\mathbf{v})} \,\mathrm{sgn}(\alpha) k'_{s+3}, \\ Y_3 &= \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) k'_{s+2}; \end{aligned}$$

recall that $Y = Y_1 - Y_2 - Y_3$. Here we note that $\mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b}) = \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c})$. By (7.4), we see that

$$\begin{gathered} \langle -l'_{s+1} w_s^{-1}\alpha - l'_{s+2} w_s^{-1}\beta,\, \mathrm{qwt}^\vee_s(\mathbf{w}, \mathbf{b})\rangle = \langle -k'_{s+1} w_s^{-1}\beta - k'_{s+3} w_s^{-1}(\alpha+\beta),\, \mathrm{qwt}^\vee_s(\mathbf{v}, \mathbf{c})\rangle, \\ -\delta_{s+3 \in S(\mathbf{w})} \mathbf{b}(s+3) l'_{s+3} = \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) k'_{s+3} - \delta_{s+2 \in S(\mathbf{v})} \mathbf{c}(s+2) k'_{s+2}. \end{gathered}$$

Hence, in order to show that $X = Y$, we need to show that

$$\begin{aligned} &\delta_{s+1 \in T^-(\mathbf{w})}(-2\epsilon l'_{s+1} + \epsilon l'_{s+2} - \mathrm{sgn}(\alpha) l'_{s+1}) + \delta_{s+2 \in T^-(\mathbf{w})}(-2\epsilon l'_{s+2} - \mathrm{sgn}(\alpha+\beta) l'_{s+2}) \\ &\quad = \delta_{s+1 \in T^-(\mathbf{v})}(-2\epsilon k'_{s+1} - \epsilon k'_{s+3} - \mathrm{sgn}(\beta) k'_{s+1}) + \delta_{s+3 \in T^-(\mathbf{v})}(-2\epsilon k'_{s+3} - \mathrm{sgn}(\alpha) k'_{s+3}). \end{aligned} \tag{7.29}$$

Since both the label-increasing directed path (7.26) and the label-decreasing directed path (7.27) are shortest directed paths from w_s to $s_\alpha s_\beta w_s = s_\beta s_{\alpha+\beta} w_s$ in $\mathrm{QBG}(W)$, the sum of the labels of quantum edges in (7.26) is identical to that in (7.27); see, for example, [LNS[3], Proposition 8.1]. From this fact, together with the assumption that $\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta)$, we deduce that

$$\begin{aligned} T^-(\mathbf{w}) \cap \{s+1, s+2, s+3\} &= \emptyset,\ \{s+2\},\ \text{or } \{s+1, s+2\}, \\ T^-(\mathbf{v}) \cap \{s+1, s+2, s+3\} &= \emptyset,\ \{s+1\},\ \text{or } \{s+3\}, \end{aligned}$$

and that

$$T^-(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\},\ \{s+1, s+2\})$$

$$\iff T^{-}(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\},\ \{s+3\}).$$

We show (7.29) in the case that $\epsilon = 1$ and $T^{-}(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+2\}$; the proofs in the other cases are similar or simpler. In this case, note that $\delta_{s+1 \in T^{-}(\mathbf{w})} = 0$, $\delta_{s+2 \in T^{-}(\mathbf{w})} = 1$, $\delta_{s+1 \in T^{-}(\mathbf{v})} = 1$, and $\delta_{s+3 \in T^{-}(\mathbf{v})} = 0$. Also, since $(s_\alpha w_s)^{-1}(\alpha + \beta) = w_s^{-1}\beta \in \Delta^{+}$ and the edge $s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s$ is a quantum edge, we deduce from Lemma 2.1 that $\alpha + \beta \in \Delta^{-}$, and hence $\mathrm{sgn}(\alpha+\beta) = -1$. Similarly, since $w_s^{-1}\beta \in \Delta^{+}$ and the edge $w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s$ is a quantum edge, we deduce from Lemma 2.1 that $\beta \in \Delta^{-}$, and hence $\mathrm{sgn}(\beta) = -1$. Thus, equation (7.29) (which we need to show) follows from these equalities and (7.4).

If (7.28) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, w_s, s_{\alpha+\beta} w_s, s_\beta s_{\alpha+\beta} w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, and that

$$\begin{aligned} & S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}) \\ & \iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+1\}$, then we define $\mathbf{c}(s+1) := \mathbf{b}(s+3)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19). Indeed, we can easily show the equalities in (3.19), except for $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$. In order to show that $\deg(\mathbf{v}, \mathbf{c}) = \deg(\mathbf{w}, \mathbf{b})$, it suffices to show that $\deg'(\mathbf{v}, \mathbf{c}) = \deg'(\mathbf{w}, \mathbf{b})$. For this equality, we claim that X in (7.8) is equal to Z in (7.12). As above, we have

$$\begin{aligned} X_1 &= \langle -l'_{s+1} w_s^{-1}\alpha - l'_{s+2} w_s^{-1}\beta,\ \mathrm{qwt}_s^{\vee}(\mathbf{w}, \mathbf{b}) \rangle \\ &\quad - 2\delta_{s+1 \in T^{-}(\mathbf{w})} \epsilon l'_{s+1} + \delta_{s+1 \in T^{-}(\mathbf{w})} \epsilon l'_{s+2} - 2\delta_{s+2 \in T^{-}(\mathbf{w})} \epsilon l'_{s+2}, \\ X_2 &= \delta_{s+1 \in T^{-}(\mathbf{w})} \mathrm{sgn}(\alpha) l'_{s+1} + \delta_{s+2 \in T^{-}(\mathbf{w})} \mathrm{sgn}(\alpha+\beta) l'_{s+2}, \\ X_3 &= \delta_{s+3 \in S(\mathbf{w})} \mathbf{b}(s+3) l'_{s+3}; \end{aligned}$$

recall that $X = X_1 - X_2 - X_3$. By computations similar to the above, we have

$$\begin{aligned} Z_1 &= \langle -l'_{s+2} w_s^{-1}(\alpha+\beta) + l'_{s+3} w_s^{-1}\alpha,\ \mathrm{qwt}_s^{\vee}(\mathbf{v}, \mathbf{c}) \rangle \\ &\quad - 2\delta_{s+2 \in T^{-}(\mathbf{w})} \epsilon l'_{s+2} + \delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+2} \\ &\quad + \delta_{s+2 \in T^{-}(\mathbf{v})} \epsilon l'_{s+3} + 2\delta_{s+3 \in T^{-}(\mathbf{v})} \epsilon l'_{s+3} - 2\delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+3}, \\ Z_2 &= \delta_{s+2 \in T^{-}(\mathbf{v})} \mathrm{sgn}(\alpha+\beta) l'_{s+2} + \delta_{s+3 \in T^{-}(\mathbf{v})} \mathrm{sgn}(\beta) l'_{s+3}, \\ Z_3 &= \delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+1}; \end{aligned}$$

recall that $Z = Z_1 - Z_2 - Z_3$. Note that

$$\langle -l'_{s+1} w_s^{-1}\alpha - l'_{s+2} w_s^{-1}\beta,\ \mathrm{qwt}_s^{\vee}(\mathbf{w}, \mathbf{b}) \rangle = \langle -l'_{s+2} w_s^{-1}(\alpha+\beta) + l'_{s+3} w_s^{-1}\alpha,\ \mathrm{qwt}_s^{\vee}(\mathbf{v}, \mathbf{c}) \rangle,$$

$$\begin{aligned} & - \delta_{s+3 \in S(\mathbf{w})} \mathbf{b}(s+3) l'_{s+3} \\ & = \delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+2} - 2\delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+3} - \delta_{s+1 \in S(\mathbf{v})} \mathbf{c}(s+1) l'_{s+1}. \end{aligned}$$

Hence, in order to show that $X = Z$, we need to show that

$$\begin{aligned} & \delta_{s+1 \in T^{-}(\mathbf{w})} (-2\epsilon l'_{s+1} + \epsilon l'_{s+2} - \mathrm{sgn}(\alpha) l'_{s+1}) + \delta_{s+2 \in T^{-}(\mathbf{w})} (-2\epsilon l'_{s+2} - \mathrm{sgn}(\alpha+\beta) l'_{s+2}) \\ & = \delta_{s+2 \in T^{-}(\mathbf{w})} (-2\epsilon l'_{s+2} + \epsilon l'_{s+3} - \mathrm{sgn}(\alpha+\beta) l'_{s+2}) + \delta_{s+3 \in T^{-}(\mathbf{v})} (2\epsilon l'_{s+3} - \mathrm{sgn}(\beta) l'_{s+3}). \end{aligned} \tag{7.30}$$

Since both the label-increasing directed path (7.26) and the label-decreasing directed path (7.28) are shortest directed paths from w_s to $s_\alpha s_\beta w_s = s_\beta s_{\alpha+\beta} w_s$ in $\mathrm{QBG}(W)$, the sum of the labels of quantum edges in (7.26) is identical to that in (7.28); see, for example, [LNS^3, Proposition 8.1]. From this fact, together with the assumption that $\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta)$, we deduce that

$$\begin{aligned} & T^{-}(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset,\ \{s+1\},\ \text{or } \{s+1, s+2\}, \\ & T^{-}(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset,\ \{s+3\},\ \text{or } \{s+2\}, \end{aligned}$$

and

$$T^{-}(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\},\ \{s+1, s+2\})$$

$$\iff T^-(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\},\ \{s+2\}).$$

We show (7.30) in the case that $\epsilon = 1$ and $T^-(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \{s+1\}$; the proofs in the other cases are similar or simpler. In this case, note that $\delta_{s+1 \in T^-(\mathbf{w})} = 1$, $\delta_{s+2 \in T^-(\mathbf{w})} = 0$, $\delta_{s+2 \in T^-(\mathbf{v})} = 0$, and $\delta_{s+3 \in T^-(\mathbf{v})} = 1$. Also, since $w_s^{-1}\alpha \in \Delta^+$ and the edge $w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s$ is a quantum edge, we deduce from Lemma 2.1 that $\alpha \in \Delta^-$, and hence $\mathrm{sgn}(\alpha) = -1$. Similarly, since $(s_{\alpha+\beta} w_s)^{-1}(\beta) = -w_s^{-1}\alpha \in \Delta^-$ and the edge $s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s$ is a quantum edge, we deduce from Lemma 2.1 that $\beta \in \Delta^+$, and hence $\mathrm{sgn}(\beta) = 1$. Thus, equation (7.30) (which we need to show) follows from these equalities and (7.4).

(5.1b). If the sequence in (7.25) is of the form (b), then we have the label-decreasing directed path

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+1\}$. It follows that there exists a unique label-increasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s = w_{s+3}. \tag{7.31}$$

If we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\alpha w_s, s_{\alpha+\beta} s_\alpha w_s, s_{\alpha+\beta} s_\alpha w_s)$, then we see that $\mathbf{v} = (v_0, v_1, \ldots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, and that

$$\begin{aligned} & S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}) \\ \iff & S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+3\}$, then we define $\mathbf{c}(s+3) := \mathbf{b}(s+1)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

(5.1c). If the sequence in (7.25) is of the form (c), then we have the label-increasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\beta s_\alpha w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+2\}$. It follows that there exists a unique label-decreasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}\alpha|} s_{\alpha+\beta} s_\beta w_s = w_{s+3}. \tag{7.32}$$

If we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_{\alpha+\beta} s_\beta w_s, s_{\alpha+\beta} s_\beta w_s)$, then we see that $\mathbf{v} = (v_0, v_1, \ldots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} & S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}) \\ \iff & S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+3\}$, then we define $\mathbf{c}(s+3) := \mathbf{b}(s+2)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 5.2. Assume that $\mathrm{sgn}(w_s^{-1}\alpha) = -\mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta))$. We fix a reflection order $\lhd$ on Δ^+ such that $|w_s^{-1}(\alpha+\beta)| \lhd |w_s^{-1}\alpha| \lhd |w_s^{-1}\beta|$.

(5.2a). If the sequence in (7.25) is of the form (a), then we have the label-increasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s = s_\alpha s_\beta w_s,$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+3\}$. It follows that there exists a unique label-decreasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\alpha s_\beta w_s = w_{s+3}. \tag{7.33}$$

Define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_\beta w_s, s_\alpha s_\beta w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+2\}$, then we define $\mathbf{c}(s+2) := \mathbf{b}(s+3)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

(5.2b). If the sequence in (7.25) is of the form (b), then we have the label-increasing directed path

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+1\}$. It follows that there exists a unique label-decreasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\alpha s_\beta w_s = w_{s+3}. \tag{7.34}$$

Define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_\beta w_s, s_\alpha s_\beta w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+2\}$, then we define $\mathbf{c}(s+2) := \mathbf{b}(s+1)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

(5.2c). If the sequence in (7.25) is of the form (c), then we have the label-decreasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\beta s_\alpha w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+2\}$. It follows that there exists a unique label-increasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\beta|} s_\alpha s_{\alpha+\beta} w_s = w_{s+3}. \tag{7.35}$$

Define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, w_s, s_{\alpha+\beta} w_s, s_\alpha s_{\alpha+\beta} w_s)$. It is easily seen that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+1\}$, then we define $\mathbf{c}(s+1) := \mathbf{b}(s+2)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 5.3. Assume that $\mathrm{sgn}(w_s^{-1}\alpha) = -\mathrm{sgn}(w_s^{-1}\beta) = -\mathrm{sgn}(w_s^{-1}(\alpha+\beta))$. We fix a reflection order $\lhd$ on Δ^+ such that $|w_s^{-1}(\alpha+\beta)| \lhd |w_s^{-1}\beta| \lhd |w_s^{-1}\alpha|$.

(5.3a). If the sequence in (7.25) is of the form (a), then we have the label-decreasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s = s_\alpha s_\beta w_s,$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+3\}$. It follows that there exists a unique label-increasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s = w_{s+3}. \tag{7.36}$$

Define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, w_s, s_{\alpha+\beta} w_s, s_\beta s_{\alpha+\beta} w_s)$. It is easily seen that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, and that

$$S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\})$$

$$\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}).$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+1\}$, then we define $\mathbf{c}(s+1) := \mathbf{b}(s+3)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

(5.3b). If the sequence in (7.25) is of the form (b), then we have the label-increasing directed path

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+1\}$. It follows that there exists a unique label-decreasing directed path from w_s to w_{s+3}, which is of the form:

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s = s_\alpha s_\beta w_s \tag{7.37}$$

or

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\alpha s_\beta w_s = w_{s+3}. \tag{7.38}$$

If (7.37) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\alpha w_s, s_{\alpha+\beta} s_\alpha w_s, s_{\alpha+\beta} s_\alpha w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+3\}$, then we define $\mathbf{c}(s+3) := \mathbf{b}(s+1)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

If (7.38) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_\beta w_s, s_\alpha s_\beta w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+2\}$, then we define $\mathbf{c}(s+2) := \mathbf{b}(s+1)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

(5.3c). If the sequence in (7.25) is of the form (c), then we have the label-decreasing directed path

$$w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_\beta s_\alpha w_s = w_{s+3},$$

where $S(\mathbf{w}) \cap \{s+1, s+2, s+3\}$ is either $\emptyset$ or $\{s+2\}$. It follows that there exists a unique label-increasing directed path from w_s to w_{s+3}, which is either of the form:

$$w_s \xrightarrow{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s \xrightarrow{|w_s^{-1}\beta|} s_\alpha s_{\alpha+\beta} w_s = w_{s+3} \tag{7.39}$$

or

$$w_s \xrightarrow{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow{|w_s^{-1}\alpha|} s_{\alpha+\beta} s_\beta w_s = w_{s+3}. \tag{7.40}$$

If (7.39) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, w_s, s_{\alpha+\beta} w_s, s_\alpha s_{\alpha+\beta} w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+1\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+1\}$, then we define $\mathbf{c}(s+1) := \mathbf{b}(s+2)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

If (7.40) holds, then we define $(v_s, v_{s+1}, v_{s+2}, v_{s+3}) := (w_s, s_\beta w_s, s_{\alpha+\beta} s_\beta w_s, s_{\alpha+\beta} s_\beta w_s)$. We see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$, and that

$$\begin{aligned} &S(\mathbf{w}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+2\}) \\ &\iff S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset \text{ (resp., } \{s+3\}). \end{aligned}$$

If $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \{s+3\}$, then we define $\mathbf{c}(s+3) := \mathbf{b}(s+2)$. In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Case 6. Assume that $\#\{s+1 \le t \le s+3 \mid w_{t-1} \ne w_t\} = 3$, i.e., $w_s \ne w_{s+1} \ne w_{s+2} \ne w_{s+3}$.

Subcase 6.1 (to be paired with Subcase 4.1)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} s_\alpha w_s.$$

It follows from Lemma A.1 that

$$\begin{cases} w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\alpha| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta} w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\alpha \text{ is a simple root}, \\ \{s+3\} & \text{if } w_s^{-1}\alpha \text{ is a simple root}. \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

Subcase 6.2 (to be paired with Subcase 4.2)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta} w_s) < \ell(w_s).$$

It follows from Lemma A.2 that

$$\begin{cases} w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\alpha| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\alpha) = -\mathrm{sgn}(w_s^{-1}\beta) = -\mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta} w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\alpha \text{ is a simple root}, \\ \{s+3\} & \text{if } w_s^{-1}\alpha \text{ is a simple root}. \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

Subcase 6.3 (to be paired with Subcase 4.3)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_{\alpha+\beta} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta} s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta} w_s) < \ell(w_s).$$

It follows from Lemma A.4 that

$$\begin{cases} w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta} w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\beta| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\alpha) = -\mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta} w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\beta \text{ is a simple root}, \\ \{s+3\} & \text{if } w_s^{-1}\beta \text{ is a simple root}. \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 6.4 (to be paired with Subcase 4.4)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta}w_s) < \ell(w_s).$$

It follows from Lemma A.5 that

$$\begin{cases} w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\beta| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\alpha) = -\,\mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta}w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\beta \text{ is a simple root,} \\ \{s+3\} & \text{if } w_s^{-1}\beta \text{ is a simple root.} \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 6.5 (to be paired with Subcase 4.5)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta}w_s) > \ell(w_s).$$

It follows from Lemma A.6 that

$$\begin{cases} w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\alpha| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta}w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\beta \text{ is a simple root,} \\ \{s+3\} & \text{if } w_s^{-1}\beta \text{ is a simple root.} \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

Subcase 6.6 (to be paired with Subcase 4.6)**.** Assume that $(\mathbf{w}, \mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta}w_s) > \ell(w_s).$$

It follows from Lemma A.7 that

$$\begin{cases} w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\alpha| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta}w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\beta \text{ is a simple root,} \\ \{s+3\} & \text{if } w_s^{-1}\beta \text{ is a simple root.} \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ satisfies (3.19).

Subcase 6.7 (to be paired with Subcase 4.7)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s \quad \text{and} \quad \ell(s_{\alpha+\beta}w_s) > \ell(w_s).$$

It follows from Lemma A.3 that

$$\begin{cases} w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}(\alpha+\beta)|} s_{\alpha+\beta}w_s = w_{s+3} \quad \text{in } \mathrm{QBG}(W), \\ |w_s^{-1}\beta| \text{ is a simple root, and} \\ \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)). \end{cases}$$

If we set $v_{s+1} := w_s$ and $v_{s+2} := s_{\alpha+\beta}w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$. Also, we have

$$S(\mathbf{v}) = \begin{cases} \{s+1\} & \text{if } -w_s^{-1}\beta \text{ is a simple root,} \\ \{s+3\} & \text{if } w_s^{-1}\beta \text{ is a simple root.} \end{cases}$$

We set $\mathbf{c}(t) := 1$ for $t \in S(\mathbf{v}) \cap \{s+1, s+2, s+3\}$. Then, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 6.8 (to be paired with Subcase 6.9 below)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s.$$

It follows from Lemma A.9 that

$$w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_\alpha s_{\alpha+\beta}s_\beta w_s,$$

and that

$$\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)) = \mathrm{sgn}(\alpha) = -\mathrm{sgn}(\beta) = -\mathrm{sgn}(\alpha+\beta).$$

If we set $v_{s+1} := s_\beta w_s$ and $v_{s+2} := s_{\alpha+\beta}s_\alpha w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 6.9 (to be paired with Subcase 6.8)**.** Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s.$$

It follows from Lemma A.9 that

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\alpha|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{Q}]{|w_s^{-1}\beta|} s_\alpha s_{\alpha+\beta}s_\beta w_s,$$

and that

$$\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)) = -\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = -\mathrm{sgn}(\alpha+\beta).$$

If we set $v_{s+1} := s_\beta w_s$ and $v_{s+2} := s_{\alpha+\beta}s_\alpha w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w},\mathbf{b}) := (\mathbf{v},\mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Subcase 6.10. Assume that $(\mathbf{w},\mathbf{b})$ satisfies

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_\beta s_{\alpha+\beta}s_\alpha w_s.$$

It follows from Lemma A.10 that

$$w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_\beta w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\alpha|} s_{\alpha+\beta}s_\alpha w_s \xrightarrow[\mathsf{B}]{|w_s^{-1}\beta|} s_\alpha s_{\alpha+\beta}s_\beta w_s,$$

and that

$$\mathrm{sgn}(w_s^{-1}\alpha) = \mathrm{sgn}(w_s^{-1}\beta) = \mathrm{sgn}(w_s^{-1}(\alpha+\beta)) = \mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = \mathrm{sgn}(\alpha+\beta).$$

If we set $v_{s+1} := s_\beta w_s$ and $v_{s+2} := s_{\alpha+\beta} s_\alpha w_s$, then we see that $\mathbf{v} = (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$; notice that $S(\mathbf{v}) \cap \{s+1, s+2, s+3\} = \emptyset$, and hence $\mathbf{c}$ is defined only by (7.6). In this case, $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfies (3.18).

Thus we have defined $\mathsf{YB}(\mathbf{w}, \mathbf{b})$ for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$. Also, we define $\mathsf{YB}(\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma) \sqcup \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ for $(\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ by interchanging α and β in Cases 1–6, and then define $\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Xi)$ as in (7.7) (with Γ replaced by Ξ). We can verify that these subsets and the map $(\mathbf{w}, \mathbf{b}) \mapsto \mathsf{YB}(\mathbf{w}, \mathbf{b})$ satisfy conditions (1)–(3). This completes the proof of Theorem 3.12 in the case that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$.

7.2. **In type $A_1 \times A_1$.** We give a proof of Theorem 3.12 in the case that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$. Assume that $\Gamma \in \mathsf{AP}(\lambda)$ is of the form

$$\Gamma : A_\circ = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda,$$

with $\gamma_{s+1} = \alpha$, $\gamma_{s+2} = \beta$, i.e.,

$$\cdots \xrightarrow{\gamma_s} A_s \xrightarrow{\alpha} A_{s+1} \xrightarrow{\beta} A_{s+2} \xrightarrow{\gamma_{s+3}} A_{s+3} \cdots \qquad \text{(in } \Gamma).$$

Then, $\Xi = (\beta_1, \dots, \beta_m)$, where $\beta_k := \gamma_k$ for $1 \le k \le m$ with $k \ne s+1, s+2$, and $\beta_{s+1} = \beta$, $\beta_{s+2} = \alpha$; note that Ξ is an alcove path from $A_\circ$ to A_λ of the form:

$$\Xi : A_\circ = A_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} A_s \xrightarrow{\beta} B_{s+1} \xrightarrow{\alpha} A_{s+2} \xrightarrow{\gamma_{s+3}} \cdots \xrightarrow{\gamma_m} A_m = A_\lambda$$

for some alcove B_{s+1}.

Now, for $\mathbf{w} = (w_0, w_1, \dots, w_m) \in \mathbf{QW}_{\lambda,w}(\Gamma)$, we set

$$v_k := w_k \qquad \text{for } 0 \le k \le m \text{ with } k \ne s+1,$$

and define v_{s+1} as follows:

(i) if $w_s = w_{s+1} = w_{s+2}$, then we define v_{s+1} by $v_s = v_{s+1} = v_{s+2}$;

(ii) if $w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s = w_{s+1} = w_{s+2}$, then we define v_{s+1} by $v_s = v_{s+1} \xrightarrow{|v_{s+1}^{-1}\alpha|} s_\alpha v_{s+1} = v_{s+2}$;

(iii) if $w_s = w_{s+1} \xrightarrow{|w_{s+1}^{-1}\beta|} s_\beta w_{s+1} = w_{s+2}$, then we define v_{s+1} by $v_s \xrightarrow{|v_s^{-1}\beta|} s_\beta v_s = v_{s+1} = v_{s+2}$;

(iv) if $w_s \xrightarrow{|w_s^{-1}\alpha|} s_\alpha w_s = w_{s+1} \xrightarrow{|w_{s+1}^{-1}\beta|} s_\beta w_{s+1} = w_{s+2}$, then we define v_{s+1} by $v_s \xrightarrow{|v_s^{-1}\beta|} s_\beta v_s = v_{s+1} \xrightarrow{|v_{s+1}^{-1}\alpha|} s_\alpha v_{s+1} = v_{s+2}$.

Then it follows that $\mathbf{v} := (v_0, v_1, \dots, v_m) \in \mathbf{QW}_{\lambda,w}(\Xi)$. Also, for $\mathbf{b} : S(\mathbf{w}) \to \{0, 1\}$, we define $\mathbf{c} : S(\mathbf{v}) \to \{0, 1\}$ as follows. Noting that $S(\mathbf{w}) \setminus \{s+1, s+2\} = S(\mathbf{v}) \setminus \{s+1, s+2\}$, we set

$$\mathbf{c}|_{S(\mathbf{v}) \setminus \{s+1,s+2\}} := \mathbf{b}|_{S(\mathbf{w}) \setminus \{s+1,s+2\}}. \tag{7.41}$$

Notice that $s+1 \in S(\mathbf{w}) \cap \{s+1, s+2\}$ if and only if $s+2 \in S(\mathbf{v}) \cap \{s+1, s+2\}$; in this case, we set $\mathbf{c}(s+2) := \mathbf{b}(s+1)$. Similarly, notice that $s+2 \in S(\mathbf{w}) \cap \{s+1, s+2\}$ if and only if $s+1 \in S(\mathbf{v}) \cap \{s+1, s+2\}$; in this case, we set $\mathbf{c}(s+1) := \mathbf{b}(s+2)$. Then we see that $\mathsf{YB}(\mathbf{w}, \mathbf{b}) := (\mathbf{v}, \mathbf{c}) \in \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$.

As in the case that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$, we can verify that the map YB is a bijection from $\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Gamma) := \widetilde{\mathbf{QW}}_{\lambda,w}(\Gamma)$ to $\widetilde{\mathbf{QW}}^{(0)}_{\lambda,w}(\Xi) := \widetilde{\mathbf{QW}}_{\lambda,w}(\Xi)$ satisfying (3.18). This completes the proof of Theorem 3.12 in the case that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$.

Thus we have established Theorem 3.12.

Appendix A. Technical lemmas.

In this section, we assume that $\mathfrak{g}$ is simply-laced, and that $\alpha, \beta \in \Delta$ are such that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$.

Lemma A.1. *Let $w \in W$. Then,*

$$w \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{Q}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$;
(3) *we have the quantum edge* $w \xrightarrow[\mathsf{Q}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(w\alpha) = \mathrm{sgn}(w\beta) = \mathrm{sgn}(w(\alpha+\beta)) = -\mathrm{sgn}(\alpha) = -\mathrm{sgn}(\beta) = -\mathrm{sgn}(\alpha+\beta).$$

Proof. We may assume that $\alpha \in \Delta^+$, since

$$\begin{cases} \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1 \iff \langle -\alpha, -\beta^\vee \rangle = \langle -\beta, -\alpha^\vee \rangle = -1; \\ w \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{Q}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha \\ \qquad \iff w \xrightarrow[\mathsf{Q}]{|-\alpha|} ws_{-\alpha} \xrightarrow[\mathsf{Q}]{|-\beta|} ws_{-\alpha}s_{-\beta} \xrightarrow[\mathsf{Q}]{|-\alpha|} ws_{-\alpha}s_{-\beta}s_{-\alpha}; \\ \text{condition (1) (resp., (2), (3)) holds for } \alpha \text{ and } \beta \\ \qquad \iff \text{condition (1) (resp., (2), (3)) holds for } -\alpha \text{ and } -\beta. \end{cases} \tag{A.1}$$

We set $v := ws_\alpha s_\beta s_\alpha = ws_{\alpha+\beta}$.

We first prove the "only if" part. We have

$$\begin{aligned} \ell(v) &= \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1 - 2\langle \rho, |\beta|^\vee \rangle + 1 - 2\langle \rho, \alpha^\vee \rangle + 1 \\ &= \ell(w) - 2\langle \rho, 2\alpha^\vee + |\beta|^\vee \rangle + 3. \end{aligned} \tag{A.2}$$

Also, we have

$$\ell(v) = \ell(ws_{\alpha+\beta}) \geq \ell(w) - \ell(s_{\alpha+\beta}) = \ell(w) - 2\langle \rho, |\alpha+\beta|^\vee \rangle + 1. \tag{A.3}$$

Combining (A.2) and (A.3), we obtain

$$\langle \rho, |\alpha+\beta|^\vee \rangle - \langle \rho, 2\alpha^\vee + |\beta|^\vee \rangle + 1 \geq 0. \tag{A.4}$$

Suppose, for a contradiction, that β is negative. If $\alpha+\beta$ is positive, then we see by (A.4) that $\langle \rho, -\alpha^\vee + 2\beta^\vee \rangle + 1 \geq 0$, which contradicts the inequalities $\langle \rho, \alpha^\vee \rangle \geq 1$ and $\langle \rho, \beta^\vee \rangle \leq -1$. If $\alpha+\beta$ is negative, then we see by (A.4) that $\langle \rho, -3\alpha^\vee \rangle + 1 \geq 0$, which also contradicts the inequality $\langle \rho, \alpha^\vee \rangle \geq 1$. Hence β is positive, which shows (2). We see by (A.4) that $-\langle \rho, \alpha^\vee \rangle + 1 \geq 0$, which shows (1). In addition, since equality holds in the inequality $-\langle \rho, \alpha^\vee \rangle + 1 \geq 0$, we deduce that the inequality in (A.3) is, in fact, equality, which implies (3). This proves the "only if" part.

Next we prove the "if" part. Recall that $\alpha \in \Delta^+$, and hence $\beta \in \Delta^+$ by (2). It follows from (3) that

$$\ell(v) = \ell(ws_{\alpha+\beta}) = \ell(w) - 2\langle \rho, \alpha^\vee + \beta^\vee \rangle + 1. \tag{A.5}$$

Also, we see that

$$\begin{aligned} \ell(v) &= \ell(ws_\alpha s_\beta s_\alpha) \geq \ell(ws_\alpha s_\beta) - 2\langle \rho, \alpha^\vee \rangle + 1 \\ &\geq \ell(ws_\alpha) - 2\langle \rho, \beta^\vee \rangle + 1 - 2\langle \rho, \alpha^\vee \rangle + 1 \\ &\geq \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1 - 2\langle \rho, \beta^\vee \rangle + 1 - 2\langle \rho, \alpha^\vee \rangle + 1 \\ &= \ell(w) - 2\langle \rho, 2\alpha^\vee + \beta^\vee \rangle + 3. \end{aligned} \tag{A.6}$$

Combining (A.5) and (A.6), we obtain

$$\langle \rho,\, \alpha^\vee + \beta^\vee \rangle - \langle \rho,\, 2\alpha^\vee + \beta^\vee \rangle + 1 \le 0; \tag{A.7}$$

the left-hand side of (A.7) is equal to $-\langle \rho,\, \alpha^\vee \rangle + 1$, which is equal to 0 by (1). Hence the inequality in (A.7) is equality. Therefore, we see that all the inequalities in (A.6) are, in fact, equalities. This proves the "if" part.

Since $w \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha$ and $\alpha \in \Delta^+$, it follows from Lemma 2.1 that $w\alpha \in \Delta^-$. Similarly, since $ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha$ and $\alpha \in \Delta^+$, it follows from Lemma 2.1 that $w\beta = ws_\alpha s_\beta(\alpha) \in \Delta^-$. Hence we obtain $\mathrm{sgn}(w\alpha) = \mathrm{sgn}(w\beta) = -\,\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta)$. This completes the proof of the lemma. □

By arguments similar to those for Lemma A.1, we can prove the following lemmas.

Lemma A.2. *Let $w \in W$. Then,*

$$w \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{Q}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha s_\beta s_\alpha \quad \text{and} \quad \ell(ws_{\alpha+\beta}) < \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = \mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(\beta) = \mathrm{sgn}(w\beta)$;
(3) *we have the quantum edge* $w \xrightarrow[\mathsf{Q}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = -\,\mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(w\alpha) = \mathrm{sgn}(w\beta) = \mathrm{sgn}(w(\alpha+\beta)).$$

Lemma A.3. *Let $w \in W$. Then,*

$$w \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\beta s_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta s_\alpha s_\beta \quad \text{and} \quad \ell(ws_{\alpha+\beta}) > \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(w\beta)$;
(3) *we have the Bruhat edge* $w \xrightarrow[\mathsf{B}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = -\,\mathrm{sgn}(\alpha+\beta) = -\,\mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(w\beta) = -\,\mathrm{sgn}(w(\alpha+\beta)).$$

Lemma A.4. *Let $w \in W$. Then,*

$$w \xrightarrow[\mathsf{Q}]{|\beta|} ws_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\beta s_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta s_\alpha s_\beta \quad \text{and} \quad \ell(ws_{\alpha+\beta}) < \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = -\,\mathrm{sgn}(w\alpha)$;
(3) *we have the quantum edge* $w \xrightarrow[\mathsf{Q}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = -\,\mathrm{sgn}(\alpha+\beta) = -\,\mathrm{sgn}(w\alpha) = \mathrm{sgn}(w\beta) = \mathrm{sgn}(w(\alpha+\beta)).$$

Lemma A.5. *Let $w \in W$. Then,*

$$w \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\beta s_\alpha \xrightarrow[\mathsf{Q}]{|\beta|} ws_\beta s_\alpha s_\beta \quad \text{and} \quad \ell(ws_{\alpha+\beta}) < \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = \mathrm{sgn}(w\alpha)$;

(3) *we have the quantum edge* $w \xrightarrow[\mathsf{Q}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(\beta) = -\,\mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(w\beta) = \mathrm{sgn}(w(\alpha+\beta)).$$

Lemma A.6. *Let* $w \in W$. *Then,*

$$w \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha \quad \text{and} \quad \ell(ws_{\alpha+\beta}) > \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = \mathrm{sgn}(w\alpha)$;
(3) *we have the Bruhat edge* $w \xrightarrow[\mathsf{B}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(w\alpha) = \mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(w\beta) \quad \text{and} \quad \mathrm{sgn}(w(\alpha+\beta)) = \mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(\beta).$$

Lemma A.7. *Let* $w \in W$. *Then,*

$$w \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha s_\beta s_\alpha \quad \text{and} \quad \ell(ws_{\alpha+\beta}) > \ell(w)$$

if and only if the following conditions (1)–(3) *hold*:

(1) $|\alpha|$ *is a simple root*;
(2) $\mathrm{sgn}(\alpha) = \mathrm{sgn}(w\beta)$;
(3) *we have the Bruhat edge* $w \xrightarrow[\mathsf{B}]{|\alpha+\beta|} ws_{\alpha+\beta}$ *in* $\mathrm{QBG}(W)$.

In this case,

$$\mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(\alpha) = -\,\mathrm{sgn}(w\beta) \quad \text{and} \quad \mathrm{sgn}(w(\alpha+\beta)) = \mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(\beta).$$

Lemma A.8. *For any* $w \in W$, *we do not have the following directed path in* $\mathrm{QBG}(W)$:

$$w \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha. \tag{A.8}$$

Lemma A.9. *Let* $w \in W$. *Then,*

$$w \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{Q}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\alpha s_\beta s_\alpha$$

if and only if

$$w \xrightarrow[\mathsf{Q}]{|\beta|} ws_\beta \xrightarrow[\mathsf{Q}]{|\alpha|} ws_\beta s_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta s_\alpha s_\beta.$$

In this case,

$$\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = \mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(w\alpha) = -\,\mathrm{sgn}(w\beta) = -\,\mathrm{sgn}(w(\alpha+\beta)).$$

Lemma A.10. *Let* $w \in W$. *Then,*

$$w \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\alpha s_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\alpha s_\beta s_\alpha$$

if and only if

$$w \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta \xrightarrow[\mathsf{B}]{|\alpha|} ws_\beta s_\alpha \xrightarrow[\mathsf{B}]{|\beta|} ws_\beta s_\alpha s_\beta.$$

In this case,

$$\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = \mathrm{sgn}(\alpha+\beta) = \mathrm{sgn}(w\alpha) = \mathrm{sgn}(w\beta) = \mathrm{sgn}(w(\alpha+\beta)).$$

Appendix B. An example.

In this appendix, we assume that $\mathfrak{g}$ is of type A_2, i.e., $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Applying Theorem 3.4 to the case that $\lambda = \varpi_1 + \varpi_2$, $\Gamma = (\theta, \alpha_2, \theta, \alpha_1) \in \mathsf{AP}_{\mathrm{red}}(\varpi_1 + \varpi_2)$, and $w = w_\circ$, we obtain in $\mathbb{K} \subset K_{H\times\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$,

$$\begin{aligned}
&\mathbf{e}^{\varpi_1+\varpi_2} \cdot [\mathcal{O}_{\mathbf{Q}_G(w_\circ)}] = [\mathcal{O}_{\mathbf{Q}_G(w_\circ)}(-\varpi_1 - \varpi_2)] \\
&+ q^2 \Big([\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}(\varpi_1 + \varpi_2)] - [\mathcal{O}_{\mathbf{Q}_G(s_2 t_{\theta^\vee})}(\varpi_1 + \varpi_2)] \\
&\qquad - [\mathcal{O}_{\mathbf{Q}_G(s_1 t_{\theta^\vee})}(\varpi_1 + \varpi_2)] + [\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\theta^\vee})}(\varpi_1 + \varpi_2)] \\
&\qquad + [\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\theta^\vee})}(\varpi_1 + \varpi_2)] - [\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\theta^\vee})}(\varpi_1 + \varpi_2)] \Big) \\
&+ q[\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\alpha_2^\vee})}(\varpi_1 - 2\varpi_2)] - q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\alpha_2^\vee})}(\varpi_1 - 2\varpi_2)] \\
&+ q[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\alpha_1^\vee})}(-2\varpi_1 + \varpi_2)] - q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\alpha_1^\vee})}(-2\varpi_1 + \varpi_2)] \\
&+ q[\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}] - q[\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\theta^\vee})}] - q[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\theta^\vee})}] + q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\theta^\vee})}].
\end{aligned} \tag{B.1}$$

Indeed, observe that

$$l_1 = l_2 = 1,\ l_3 = 2,\ l_4 = 1, \qquad l'_1 = 1,\ l'_2 = l'_3 = l'_4 = 0;$$

for the definitions of l_t and l'_t, see Section 2.2. Also, we see that

$$\begin{aligned}
&\mathbf{QW}_{\varpi_1+\varpi_2, w_\circ} = \\
&\big\{ \mathbf{w}_1 = (w_\circ, w_\circ, w_\circ, w_\circ, w_\circ),\ \mathbf{w}_2 = (w_\circ, e, e, e, e),\ \mathbf{w}_3 = (w_\circ, e, s_2, s_2, s_2), \\
&\mathbf{w}_4 = (w_\circ, e, s_2, s_2 s_1, s_2 s_1),\ \mathbf{w}_5 = (w_\circ, e, s_2, s_2 s_1, w_\circ),\ \mathbf{w}_6 = (w_\circ, e, s_2, s_2, s_1 s_2), \\
&\mathbf{w}_7 = (w_\circ, e, e, e, s_1),\ \mathbf{w}_8 = (w_\circ, w_\circ, s_1 s_2, s_1 s_2, s_1 s_2),\ \mathbf{w}_9 = (w_\circ, w_\circ, s_1 s_2, s_1, s_1), \\
&\mathbf{w}_{10} = (w_\circ, w_\circ, s_1 s_2, s_1, e),\ \mathbf{w}_{11} = (w_\circ, w_\circ, w_\circ, e, e),\ \mathbf{w}_{12} = (w_\circ, w_\circ, w_\circ, e, s_1), \\
&\mathbf{w}_{13} = (w_\circ, w_\circ, w_\circ, w_\circ, s_2 s_1) \big\},
\end{aligned}$$

and that

$$S(\mathbf{w}_j) = \begin{cases} \{2, 4\} & \text{if } j = 1, \\ \{3\} & \text{if } j = 8, \\ \{4\} & \text{if } j = 9, \\ \{2\} & \text{if } j = 11, 12, 13, \\ \emptyset & \text{otherwise;} \end{cases}$$

observe that $\#\widetilde{\mathbf{QW}}_{\varpi_1+\varpi_2, w_\circ} = 21$, which is greater than 15, the number of terms on the right-hand side of (B.1). Here, recall the definition of $\mathbf{G}(\mathbf{w}, \mathbf{b})$ for $(\mathbf{w}, \mathbf{b}) \in \widetilde{\mathbf{QW}}_{\lambda, w}$ from Theorem 3.4. Let us first compute $\mathbf{G}(\mathbf{w}, \mathbf{b})$ for $\mathbf{w} = \mathbf{w}_{10}, \mathbf{w}_{11}$. Note that $S(\mathbf{w}_{10}) = \emptyset$ and $S(\mathbf{w}_{11}) = \{2\}$. We have

$$\begin{gathered}
(-1)^{(\mathbf{w}_{10}, \emptyset)} = 1, \qquad (-1)^{(\mathbf{w}_{11}, 2\mapsto 0)} = 1, \qquad (-1)^{(\mathbf{w}_{11}, 2\mapsto 1)} = -1, \\
\mathrm{qwt}(\mathbf{w}_{10}, \emptyset) = 2\alpha_1 + \alpha_2, \quad \mathrm{qwt}(\mathbf{w}_{11}, 2 \mapsto 0) = \theta, \quad \mathrm{qwt}(\mathbf{w}_{11}, 2 \mapsto 1) = 2\alpha_1 + \alpha_2, \\
\mathrm{wt}(\mathbf{w}) = w_\circ^{-1}(\varpi_1 + \varpi_2) \quad \text{for } \mathbf{w} = \mathbf{w}_{10}, \mathbf{w}_{11}, \\
\deg(\mathbf{w}_{10}, \emptyset) = 3, \qquad \deg(\mathbf{w}_{11}, 2 \mapsto 0) = 1, \qquad \deg(\mathbf{w}_{11}, 2 \mapsto 1) = 3.
\end{gathered}$$

Therefore,

$$\begin{aligned}
&\mathbf{G}(\mathbf{w}_{10}, \emptyset) = q^3 [\mathcal{O}_{\mathbf{Q}_G(t_{2\alpha_1^\vee + \alpha_2^\vee})}(-\varpi_1 + 2\varpi_2)], \\
&\mathbf{G}(\mathbf{w}_{11}, 2 \mapsto 0) = q[\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}], \qquad \mathbf{G}(\mathbf{w}_{11}, 2 \mapsto 1) = -q^3 [\mathcal{O}_{\mathbf{Q}_G(t_{2\alpha_1^\vee + \alpha_2^\vee})}(-\varpi_1 + 2\varpi_2)];
\end{aligned}$$

notice that $\mathbf{G}(\mathbf{w}_{10}, \emptyset) + \mathbf{G}(\mathbf{w}_{11}, 2 \mapsto 1) = 0$.

Next, let us compute $\mathbf{G}(\mathbf{w}, \mathbf{b})$ for $\mathbf{w} = \mathbf{w}_9, \mathbf{w}_{12}$. Note that $S(\mathbf{w}_9) = \{4\}$ and $S(\mathbf{w}_{12}) = \{2\}$. We have

$$\begin{gathered}
(-1)^{(\mathbf{w}_9, 4\mapsto 0)} = 1, \qquad (-1)^{(\mathbf{w}_9, 4\mapsto 1)} = -1, \\
(-1)^{(\mathbf{w}_{12}, 2\mapsto 0)} = -1, \qquad (-1)^{(\mathbf{w}_{12}, 2\mapsto 1)} = 1,
\end{gathered}$$

$$\begin{aligned}
\mathrm{qwt}(\mathbf{w}_9, 4 \mapsto 0) &= \theta, \qquad \mathrm{qwt}(\mathbf{w}_9, 4 \mapsto 1) = 2\alpha_1 + \alpha_2, \\
\mathrm{qwt}(\mathbf{w}_{12}, 2 \mapsto 0) &= \theta, \qquad \mathrm{qwt}(\mathbf{w}_{12}, 2 \mapsto 1) = 2\alpha_1 + \alpha_2, \\
\mathrm{wt}(\mathbf{w}) &= w_\circ^{-1}(\varpi_1 + \varpi_2) \quad \text{for } \mathbf{w} = \mathbf{w}_9,\ \mathbf{w}_{12}, \\
\deg(\mathbf{w}_9, 4 \mapsto 0) &= 1, \qquad \deg(\mathbf{w}_9, 4 \mapsto 1) = 3, \\
\deg(\mathbf{w}_{12}, 2 \mapsto 0) &= 1, \qquad \deg(\mathbf{w}_{12}, 2 \mapsto 1) = 3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{G}(\mathbf{w}_9, 4 \mapsto 0) = q[\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}], \qquad \mathbf{G}(\mathbf{w}_9, 4 \mapsto 1) = -q^3[\mathcal{O}_{\mathbf{Q}_G(t_{2\alpha_1^\vee + \alpha_2^\vee})}(-\varpi_1 + 2\varpi_2)], \\
&\mathbf{G}(\mathbf{w}_{12}, 2 \mapsto 0) = -q[\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}], \qquad \mathbf{G}(\mathbf{w}_{12}, 2 \mapsto 1) = q^3[\mathcal{O}_{\mathbf{Q}_G(t_{2\alpha_1^\vee + \alpha_2^\vee})}(-\varpi_1 + 2\varpi_2)];
\end{aligned}$$

notice that $\mathbf{G}(\mathbf{w}_9, 4 \mapsto 0) + \mathbf{G}(\mathbf{w}_{12}, 2 \mapsto 0) = 0$ and $\mathbf{G}(\mathbf{w}_9, 4 \mapsto 1) + \mathbf{G}(\mathbf{w}_{12}, 2 \mapsto 1) = 0$.

By similar computations, we deduce that

$$\begin{aligned}
&\mathbf{G}(\mathbf{w}_1, 2 \mapsto 0, 4 \mapsto 0) = [\mathcal{O}_{\mathbf{Q}_G(w_\circ)}(-\varpi_1 - \varpi_2)], \\
&\mathbf{G}(\mathbf{w}_1, 2 \mapsto 1, 4 \mapsto 0) = -q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\alpha_1^\vee})}(-2\varpi_1 + \varpi_2)], \\
&\mathbf{G}(\mathbf{w}_1, 2 \mapsto 0, 4 \mapsto 1) = -q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\alpha_2^\vee})}(\varpi_1 - 2\varpi_2)], \\
&\mathbf{G}(\mathbf{w}_1, 2 \mapsto 1, 4 \mapsto 1) = q[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\theta^\vee})}],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{G}(\mathbf{w}_2, \emptyset) = q^2[\mathcal{O}_{\mathbf{Q}_G(t_{\theta^\vee})}(\varpi_1 + \varpi_2)], \qquad \mathbf{G}(\mathbf{w}_3, \emptyset) = -q^2[\mathcal{O}_{\mathbf{Q}_G(s_2 t_{\theta^\vee})}(\varpi_1 + \varpi_2)], \\
&\mathbf{G}(\mathbf{w}_4, \emptyset) = q^2[\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\theta^\vee})}(\varpi_1 + \varpi_2)], \qquad \mathbf{G}(\mathbf{w}_5, \emptyset) = -q^2[\mathcal{O}_{\mathbf{Q}_G(w_\circ t_{\theta^\vee})}(\varpi_1 + \varpi_2)], \\
&\mathbf{G}(\mathbf{w}_6, \emptyset) = q^2[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\theta^\vee})}(\varpi_1 + \varpi_2)], \qquad \mathbf{G}(\mathbf{w}_7, \emptyset) = q^2[\mathcal{O}_{\mathbf{Q}_G(s_1 t_{\theta^\vee})}(\varpi_1 + \varpi_2)],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{G}(\mathbf{w}_8, 3 \mapsto 0) = q[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\alpha_1^\vee})}(-2\varpi_1 + \varpi_2)], \qquad \mathbf{G}(\mathbf{w}_8, 3 \mapsto 1) = -q[\mathcal{O}_{\mathbf{Q}_G(s_1 s_2 t_{\theta^\vee})}], \\
&\mathbf{G}(\mathbf{w}_{13}, 2 \mapsto 0) = q[\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\alpha_2^\vee})}(\varpi_1 - 2\varpi_2)], \qquad \mathbf{G}(\mathbf{w}_{13}, 2 \mapsto 1) = -q[\mathcal{O}_{\mathbf{Q}_G(s_2 s_1 t_{\theta^\vee})}].
\end{aligned}$$

Thus we obtain (B.1), as desired.

References

[ACT] D. Anderson, L. Chen, and H.-H. Tseng, On the finiteness of quantum K-theory of a homogeneous space, *Int. Math. Res. Not.* **2020** (2020), https://doi.org/10.1093/imrn/rnaa108.

[BCMP] A. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin, A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties, *Algebr. Geom.* **5** (2018), 568–595.

[BFP] F. Brenti, S. Fomin, and A. Postnikov, Mixed Bruhat operators and Yang-Baxter equations for Weyl groups, *Int. Math. Res. Not.* **1999** (1999), 419–441.

[FK] I. Fischer and M. Konvalinka, A bijective proof of the ASM theorem, Part I: The operator formula, *Electron. J. Combin.* **27** (2020), Paper P3.35.

[Hi] H. Hiller, "Geometry of Coxeter groups", Research Notes in Mathematics Vol. 54, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[Hu] J. E. Humphreys, "Reflection Groups and Coxeter Groups", Cambridge Studies in Advanced Mathematics Vol. 29, Cambridge University Press, Cambridge, 1990.

[HL] C. Hohlweg and J.-P. Labbé, On inversion sets and the weak order in Coxeter groups, *European J. Combin.* **55** (2016), 1–19.

[Kat1] S. Kato, Loop structure on equivariant K-theory of semi-infinite flag manifolds, preprint, arXiv:1805.01718.

[Kat2] S. Kato, Frobenius splitting of Schubert varieties of semi-infinite flag manifolds, *Forum Math., Pi* **9** (2021), Paper No.e56.

[Kat3] S. Kato, On quantum K-groups of partial flag manifolds, preprint, arXiv:1906.09343.

[KoLN] T. Kouno, C. Lenart, and S. Naito, New structure on the quantum alcove model with applications to representation theory and Schubert calculus, preprint, arXiv:2105.02546.

[KaNS] S. Kato, S. Naito, and D. Sagaki, Equivariant K-theory of semi-infinite flag manifolds and the Pieri-Chevalley formula, *Duke Math. J.* **169** (2020), 2421–2500.

[KoNOS] T. Kouno, S. Naito, D. Orr, and D. Sagaki, Inverse K-Chevalley formulas for semi-infinite flag manifolds, I: minuscule weights in ADE type, *Forum Math., Sigma* **9** (2021), Paper No.e51.

[LS] T. Lam and M. Shimozono, Quantum cohomology of G/P and homology of affine Grassmannian, *Acta Math.* **204** (2010), 49–90.

[L] C. Lenart, On the combinatorics of crystal graphs. II. The crystal commutor, *Proc. Amer. Math. Soc.* **136** (2008), 825–837.

[LNS] C. Lenart, S. Naito, and D. Sagaki, A general Chevalley formula for semi-infinite flag manifolds and quantum K-theory, preprint, arXiv:2010.06143.

[LL] C. Lenart and A. Lubovsky, A uniform realization of the combinatorial R-matrix for column shape Kirillov-Reshetikhin crystals, *Adv. Math.* **334** (2018), 151–183.

[LNS^3] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, A uniform model for Kirillov-Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph, *Int. Math. Res. Not.* **2015** (2015), 1848–1901.

[LP1] C. Lenart and A. Postnikov, Affine Weyl groups in K-theory and representation theory, *Int. Math. Res. Not.* **2007** (2007), Art. ID rnm038, 65 pp.

[LP2] C. Lenart and A. Postnikov, A combinatorial model for crystals of Kac-Moody algebras, *Trans. Amer. Math. Soc.* **360** (2008), 4349–4381.

[M] I.G. Macdonald, "Affine Hecke Algebras and Orthogonal Polynomials", Cambridge Tracts in Mathematics Vol. 157, Cambridge University Press, Cambridge, 2003.

[NOS] S. Naito, D. Orr, and D. Sagaki, Chevalley formula for anti-dominant weights in the equivariant K-theory of semi-infinite flag manifolds, *Adv. Math.* **387** (2021), 107828.

[O] D. Orr, Equivariant K-theory of the semi-infinite flag manifold as a nil-DAHA module, preprint, arXiv:2001.03490.

[S] J. R. Stembridge, Combinatorial models for Weyl characters, *Adv. Math.* **168** (2002), 96–131.

(Cristian Lenart) Department of Mathematics and Statistics, State University of New York at Albany, Albany, NY 12222, U.S.A.
Email address: clenart@albany.edu

(Satoshi Naito) Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan.
Email address: naito@math.titech.ac.jp

(Daniel Orr) Department of Mathematics (MC 0123), 460 McBryde Hall, Virginia Tech, 225 Stanger St., Blacksburg, VA 24061, U.S.A.
Email address: dorr@vt.edu

(Daisuke Sagaki) Department of Mathematics, Faculty of Pure and Applied Sciences, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan.
Email address: sagaki@math.tsukuba.ac.jp