

AN SPDE APPROACH TO PERTURBATION THEORY OF Φ_2^4 : ASYMPTOTICITY AND SHORT DISTANCE BEHAVIOR

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In this paper we study the perturbation theory of Φ_2^4 model on the whole plane via stochastic quantization. We use integration by parts formula (i.e., Dyson–Schwinger equations) to generate the perturbative expansion for the k -point correlation functions, and prove bounds on the remainder of the truncated expansion using PDE estimates; this in particular proves that the expansion is asymptotic. Furthermore, we derive short distance behaviors of the 2-point function and the connected 4-point function, also via suitable Dyson–Schwinger equations combined with PDE arguments.

1. Introduction. Physicists’ approach to quantum field theory often relies on perturbation theory, which is a formal series expansion in terms of a certain coupling constant; see, for instance the standard book [40], Chapter 7. The first goal of this paper is to discuss this perturbative approach and provide some estimates on the expansion using SPDE methods. We will focus on the $\lambda\Phi^4$ model in two space dimensions, formally given by

$$(1.1) \quad d\nu \propto \exp\left(-\frac{\lambda}{4} \int_{\mathbb{R}^2} :\Phi(x)^4: dx\right) d\mu,$$

where $d\mu$ is the Gaussian free field with mean zero and variance $\frac{1}{2}(m - \Delta)^{-1}$ for $m > 0$. In physics, interesting quantities such as correlations are calculated by formally Taylor expanding the exponential density in the coupling constant λ , and then applying the Wick theorem for Gaussian measures (which yields Feynman diagrams). Jaffe [28] proved that this formal expansion is a divergent series for any fixed $\lambda > 0$.

A toy model to illustrate this divergent perturbation theory is a Φ^4 model in “zero dimension”: consider the following function (“partition function”):

$$\mathcal{Z}(\lambda) = \int_{\mathbb{R}} e^{-x^2} e^{-\lambda x^4} dx.$$

One has $\mathcal{Z}(0) = \sqrt{\pi}$. For $\lambda > 0$, one can formally calculate it by the expansion $e^{-\lambda x^4} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} x^{4n}$, since there is exact formula for Gaussian moments, that is,

$$(1.2) \quad \mathcal{Z}(\lambda) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{(-\lambda)^n}{n!} x^{4n} e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{(4n-1)!!\sqrt{\pi}}{2^{2n}}.$$

This series is divergent for any fixed $\lambda > 0$. However, despite its divergence, this perturbative approach is widely used for other quantum field theories, such as quantum electrodynamics (QED), gauge theory, or the so-called “standard model” in general that comprises these models; and these perturbative calculations yield numerical values which agree with the experiments at extremely high accuracy; see, for example, [5], equation (4), for calculation of

the electron's magnetic moment in QED which agrees with the experiment to eleven decimal places.

This means that practically it is still often useful to calculate physical quantities by truncating this series. For instance in the above toy model, $\mathcal{Z}(0.01) \approx 1.7597$, and the first two terms on the RHS of (1.2) are approximately $1.7725 - 0.0133 = 1.7592$ which is close to the exact value.¹ We also refer to another interesting example along this line with illuminating discussion in [18], Section 6.1.²

We remark that the model (1.1) we consider here is a type of bosonic models, whereas the perturbation series for some of the fermionic models actually converge, see [19] (and more recently [1], Sections B,D, in the setting of stochastic quantization).

The above discussion indicates that *asymptoticity* is a more relevant notion than convergence in perturbation theory. Asymptoticity means that truncating the series after a *fixed* finite number of terms provides a good approximation to $\mathcal{Z}(\lambda)$ as λ tends towards zero. For the Φ_2^4 model Dimock [14] proved that the perturbative expansion of correlation functions (Schwinger functions) is asymptotic. His proof crucially relies on the constructive field theory results and methods by Glimm, Jaffe, Spencer [21]. The asymptoticity of Φ_3^4 was also proved in [17], Theorem 3, see also [30], Section IV.2. We remark that a key input to the proof in [14] is a uniform exponential cluster property proven in [21], from which Dimock managed to deduce that correlation functions are smooth in $\lambda \in [0, \lambda_0]$ where $\lambda_0 > 0$ is sufficiently small, and asymptoticity essentially follows from the Taylor remainder theorem. The smallness of λ is mainly due to requirement for the cluster property to hold. This is also the case for $d = 3$. In fact [17] proved cluster property in 3d in detail, and only mentioned that smoothness in λ and asymptoticity follow analogously as in 2d [14]. [30], Section IV.2, provided more details but the basic mechanism is also similar to [14]. The relation between smoothness of correlations in coupling constants and cluster properties was observed in statistical mechanics, see [29]. Using the results of [14], Eckmann–Epstein–Fröhlich [16] proved that in weakly coupled $P(\Phi)_2$ theories, perturbation theory in the coupling constant is asymptotic to the S-matrix elements; and the result was extended to Φ_3^4 in [15] where it was shown that the time-ordered functions and S-matrix elements admit the standard perturbation series as asymptotic expansions.

Another approach to asymptotic perturbation theory for the (one and two-component) Φ^4 model in 2d and 3d was developed by Bovier and Felder in [6], using an infinite family of skeleton inequalities. The leading orders of these skeleton inequalities were proved by Brydges–Fröhlich–Sokal in [8, 10] and applied to construct $\Phi_{2,3}^4$ in [9].

In this paper we give a new proof of asymptoticity for Φ_2^4 based on SPDE methods. More precisely, we consider the stochastic quantization of the Φ_2^4 model (i.e., the dynamical Φ_2^4 model) on $\mathbb{R}^+ \times \mathbb{R}^2$:

$$(1.3) \quad (\partial_t - \Delta + m)\Phi = -\frac{1}{2}\lambda:\Phi^3: + \xi,$$

with ξ space–time white noise on $\mathbb{R}^+ \times \mathbb{R}^2$ and $:\Phi^3:$ is the Wick product defined in Section 2. The solution theory of equation (1.3) is now well developed: see [4] where martingale solutions are constructed and [12] where strong solutions are addressed, as well as the more

¹Of course since the series is divergent, for the fixed $\lambda = 0.01$ the partial sum of (1.2) will not be infinitely close to the exact value of $\mathcal{Z}(0.01)$; indeed the $n = 70$ th term on the RHS is approximately equal to 0.9 and a truncation up to such a high order certainly loses accuracy.

²The author explains that convergence and effectiveness of approximation are completely different. For example, $\sum_0^\infty \frac{(-100)^n}{n!}$ converges to $e^{-100} \approx 10^{-44}$, but unfortunately its first few partial sums are 1, -99 , 4901 , $-161,765\frac{2}{3}$.

recent approach to global well-posedness in [34]. It is obtained in [35] that the marginal distribution of the stationary solution to (1.3) is given by Φ_2^4 field. Hence, we could use (1.3) to study the Φ_2^4 field.

Another ingredient in our approach is the *integration by parts (IBP)* formula w.r.t. the Φ_2^4 field, also called Dyson–Schwinger equations. In our SPDE approach it seems not easy to establish cluster properties or prove smoothness in λ over the entire plane, so we do not follow the methodology in [14]. Instead, we iteratively apply IBP to generate the perturbation theory (which is similar to [9] and [6]), which after a certain order yields a remainder represented as integrals of the field Φ and Gaussian covariances. Our strategy is then to introduce graphic representation for such a remainder, and then apply PDE a priori estimates to control the graphs. To this end we will carry out some manipulations on the graphs, reduce graphs to trees and design inductive arguments.

As the Φ_2^4 field does not have full support on function space, we consider the following approximation: we start with Gibbs measures $(\nu_{M,\varepsilon})_{M,\varepsilon}$ on $\Lambda_{M,\varepsilon} = (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^2$ given by

$$\mathrm{d}\nu_{M,\varepsilon} \propto \exp\left\{-\varepsilon^2 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{\lambda}{4}\Phi^4 + \left(-\frac{3}{2}\lambda a_{M,\varepsilon} + m\right)\Phi^2 + |\nabla_\varepsilon \Phi|^2\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\Phi(x),$$

where ∇_ε denotes the discrete gradient and $a_{M,\varepsilon}$ are renormalization constants. We will use the stochastic quantization equation for $\nu_{M,\varepsilon}$ to prove that $\{\nu_{M,\varepsilon}\}_{M,\varepsilon}$ form a tight set in Section 2 (see also [23] for the proof of 3d case). We still use (M, ε) to denote the subsequence such that $\nu_{M,\varepsilon} \circ (\mathcal{E}^\varepsilon)^{-1}$ converge to ν weakly with $\mathcal{E}^\varepsilon$ being the extension operator given in (A.4). Here ν could be every accumulation point and all the theorems hold for every ν .

The k -point correlations for $\nu_{M,\varepsilon}$ —for which we will expand using integration by parts formula—are given by

$$S_{\lambda,M,\varepsilon}^k(x_1, \dots, x_k) = \int \left[\prod_{i=1}^k \Phi(x_i) \right] \nu_{M,\varepsilon}(\mathrm{d}\Phi).$$

Letting $M \rightarrow \infty, \varepsilon \rightarrow 0$ we obtain the k -point function for ν

(1.4) $\langle S_\lambda^{v,k}, \varphi \rangle \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \int \int_{\mathbb{R}^{2k}} \left(\prod_{i=1}^k \Phi(x_i) \right) \varphi(x_1, \dots, x_k) \prod_{i=1}^k \mathrm{d}x_i \, \mathrm{d}\nu_{M,\varepsilon}(\Phi),$

for $\varphi \in \mathcal{S}(\mathbb{R}^{2k})$ with compact support Fourier transform, where the existence of the limit will be discussed in Theorem 2.5. We also give the definition of $S_\lambda^{v,k}$ using the extension operator in Section 3.1, which coincides with the above definition (1.4). Our first main result gives the asymptotic expansion of $S_\lambda^{v,k}$.

THEOREM 1.1. *It holds that for $\varphi \in \mathcal{S}(\mathbb{R}^{2k})$*

(1.5)
$$\langle S_\lambda^{v,k}, \varphi \rangle = \sum_{n=0}^N \frac{\lambda^n}{n!} \langle F_n^k, \varphi \rangle + \lambda^{N+1} \langle R_{N+1}^{v,k}, \varphi \rangle,$$

with F_n^k only depending on the Green function of Gaussian free field and

$$|\langle R_{N+1}^{v,k}, \varphi \rangle| \lesssim 1 + \lambda^{4k+12(N+1)}$$

with the proportional constant independent of λ .³

³Here the proportional constant may depend on N . We remark that the terms $\lambda^{4k+12(N+1)}, \lambda^{27}$ and λ^{54} in these theorems provide controls for large λ , but these exponents may not be optimal (see Remark 2.4).

The complete version of Theorem 1.1 is given in Theorem 3.9 and Proposition 3.10. Note that [14] only amounts to $\lambda \langle R_{N+1}^k, \varphi \rangle \rightarrow 0$ as $\lambda \downarrow 0$, and the result in [6] also assumes small λ regime, whereas we give a more precise bound here which holds for arbitrary $\lambda > 0$. Moreover, we obtain (see Theorem 3.9 and Proposition 3.10) that (1.5) holds in $\mathbf{C}^{-\gamma}(\rho^\ell)$ for $\gamma > 0$, some $\ell > 0$ and polynomial weight ρ and the same bound holds for $\|R_{N+1}^k\|_{\mathbf{C}^{-\gamma}(\rho^\ell)}$. Here $\mathbf{C}^{-\gamma}(\rho^\ell)$ is the weighted Besov–Hölder space (see the Appendix for definition).

Note that the expansion (1.5) is precisely the Taylor expansion in the coupling constant λ in the aforementioned perturbation theory. Namely, by taking the n th derivative on the RHS of (1.5) and evaluating at $\lambda = 0$, and invoking the above bound on R_{N+1}^k , we have

$$\left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} \langle S_\lambda^{v,k}, \varphi \rangle = \langle F_n^k, \varphi \rangle \quad \forall n \leq N.$$

In physics these derivatives on the LHS result in expectations of Gaussian products and are typically computed by Wick theorems and Feynman diagrams.

REMARK 1.2. By [28] each F_n^k has a lower bound depending on n that is not summable, which implies the expansion is a divergent series.

As the second goal of the paper, our strategy of combining the integration by parts formula and SPDE uniform estimates also allow us to find the leading short distance behavior for the 2-point correlation and the connected 4-point correlation. Let $C_{M,\varepsilon}$ (resp. C_ε) be the Green function of $2(m - \Delta_\varepsilon)$ on $\Lambda_{M,\varepsilon}$ (resp. on $\Lambda_\varepsilon = \varepsilon\mathbb{Z}^2$). We also use C to denote the Green function of $2(m - \Delta)$ on $\mathbb{R}^2$.

THEOREM 1.3. Let $\rho(x) = (1 + |hx|^2)^{-\delta/2}$ be a polynomial weight with $h > 0$ small enough and $\delta > 1/2$. It holds that for $\gamma > 0$ and some $\ell > 0$

$$(1.6) \quad \|S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon}\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon,\rho^\ell)} \lesssim \lambda^2 + \lambda^{27},$$

with the proportional constant independent of M, ε . Furthermore taking tight limit, it holds that

$$\|S_\lambda^{v,2} - C\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^2,\rho^\ell)} \lesssim \lambda^2 + \lambda^{27}.$$

Here $\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)$ is the discrete Besov–Hölder space and we refer to the Appendix for its definition.

REMARK 1.4. By Theorem 1.3 we can write $S_\lambda^{v,2}$ as a function on $\mathbb{R}^2$ and the leading short distance behavior of $S_\lambda^{v,2}$ is identical to that of C up to a $\mathbf{C}^{2-\gamma}$ part (although each of them are singular at origin).

We define the following connected 4-point function:

$$\begin{aligned} U_{M,\varepsilon}^4(x_1, x_2, x_3, x_4) &\stackrel{\text{def}}{=} S_{\lambda,M,\varepsilon}^4(x_1, x_2, x_3, x_4) - S_{\lambda,M,\varepsilon}^2(x_1, x_2)S_{\lambda,M,\varepsilon}^2(x_3, x_4) \\ &\quad - S_{\lambda,M,\varepsilon}^2(x_1, x_3)S_{\lambda,M,\varepsilon}^2(x_2, x_4) - S_{\lambda,M,\varepsilon}^2(x_1, x_4)S_{\lambda,M,\varepsilon}^2(x_2, x_3). \end{aligned}$$

THEOREM 1.5. Let $\rho(x) = (1 + |hx|^2)^{-\delta/2}$ be a polynomial weight with $h > 0$ small enough and $\delta > 0$. It holds that for $\gamma > 0$ and some $\ell > 0$

$$\|U_{M,\varepsilon}^4\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^4,\rho^\ell)} \lesssim \lambda + \lambda^{54},$$

and

$$\left\| U_{M,\varepsilon}^4 + 6\lambda \int_{\Lambda_{M,\varepsilon}} \prod_{i=1}^4 C_{M,\varepsilon}(x_i - z) \, \mathrm{d}z \right\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)} \lesssim \lambda^2 + \lambda^{54},$$

with the proportional constant independent of M, ε . Moreover, as in (1.4) the tight limit of $U_{M,\varepsilon}^4$ exists which we denote by $U_\lambda^{\nu,4}$ and

$$\|U_\lambda^{\nu,4}\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^8, \rho^\ell)} \lesssim \lambda + \lambda^{54},$$

and

$$\left\| U_\lambda^{\nu,4} + 6\lambda \int \prod_{i=1}^4 C(x_i - z) \, \mathrm{d}z \right\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^8, \rho^\ell)} \lesssim \lambda^2 + \lambda^{54}.$$

The proof of the above two results are given in Section 5. We will see in the proof of Theorem 1.5 that the expansion of $U_\lambda^{\nu,4}$ cancels some irregular part, and as a result, the regularity of $U_\lambda^{\nu,4}$ is $\mathbf{C}^{2-\gamma}$ which is better than the regularity of $S_\lambda^{\nu,4}$ which is $\mathbf{C}^{-\gamma}$, for $\gamma > 0$. Here $\mathbf{C}^{-\gamma}$ is the optimal regularity of $S_\lambda^{\nu,4}$ if considering in Besov–Hölder space since the Green’s function C , which is not continuous, appears in the expansion of $S_\lambda^{\nu,4}$.

Similar results were also obtained in [9], Theorem 1.1, but in our results above we obtain bounds under stronger regularity norms; moreover [9] requires $\lambda > 0$ to be sufficiently small whereas we do not.

The short distance behavior of Φ_3^4 on the torus has also been studied by Brydges–Dimock–Hurd, see [7], Theorem 9, where the $L^p, p < \frac{3}{2}$ -norm of truncated correlations are controlled. Here we consider the case of Φ_2^4 on the full plane and we are able to estimate the $\mathbf{C}^{2-\gamma}$ -Hölder norm of $U_\lambda^{\nu,4}$ and $S_\lambda^{\nu,2} - C$. We refer to the remark at the end of [7] for a discussion on regularity and how singular the truncated correlation functions can be at coinciding points.

We remark that as far as we understand, our technique in proving the above theorems appears to be more robust than skeleton inequalities used in earlier literature (e.g., [6] and [9]), in the sense that at least in 2d, it is valid for all values of the coupling constant λ and can be very easily adapted to N component models for any $N > 1$ (PDE estimates for such models were discussed in [36], Section 2 and Section 6.2).

Finally we mention that it would be interesting to see whether our methodology could be used to deduce similar results in 3d. For 3d stochastic quantization, namely the dynamical Φ_3^4 model, we refer to the construction of local solutions [11, 24, 25] and global solutions [2, 22, 23, 33]. These works, followed by [3, 27, 32, 36, 37], including the present paper, show that the usage of SPDE methods is not only successful in constructing Euclidean QFTs but also proving their axioms and properties. As remarked in [23] deducing properties about correlations from the dynamics would be a very interesting and challenging problem, and our results in the present paper could be viewed as a step toward this direction. It would be also interesting to study connected correlations of higher orders. We leave these generalizations to further work.

2. Uniform estimates via SPDEs. In this section we study Φ_2^4 field via stochastic quantization. We will obtain tightness of the discrete Φ_2^4 field, which gives a construction of Φ_2^4 field in the continuous setting. We also give some uniform estimates via SPDEs, especially how these estimates depend on the parameter λ .

We consider the Φ_2^4 model in both discrete and continuous settings. In particular, we set $\Lambda_\varepsilon := \varepsilon \mathbb{Z}^2$ for $\varepsilon = 2^{-N}, N \in \mathbb{N} \cup \{0\}$ and the periodic lattice $\Lambda_{M,\varepsilon} := \varepsilon \mathbb{Z}^2 \cap \mathbb{T}_M^2 =$

$\varepsilon\mathbb{Z}^2 \cap [-\frac{M}{2}, \frac{M}{2}]^2$. We use $\mathcal{S}(\mathbb{R}^d)$ to denote the class of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the dual space of $\mathcal{S}(\mathbb{R}^d)$ called the Schwartz generalized function space. Given a Banach space E with a norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to E , equipped with the supremum norm $\|f\|_{C_T E} = \sup_{t \in [0, T]} \|f(t)\|_E$ and we write $L_T^p E = L^p([0, T]; E)$ for the space of L^p functions from $[0, T]$ to E , equipped with the L^p norm $\|f\|_{L_T^p E} = (\int_0^T \|f(t)\|_E^p dt)^{1/p}$. We also use $o(\varepsilon)$ to denote $a(\varepsilon) \in \mathbb{R}$ satisfying $\lim_{\varepsilon \rightarrow 0} \frac{a(\varepsilon)}{\varepsilon} = 0$. In this paper we use the polynomial weight on $\mathbb{R}^d$ given by

$$\rho(x) = (1 + |hx|^2)^{-\delta/2}, \quad \delta, h \geq 0.$$

It is easy to see that

$$(2.1) \quad |\nabla \rho(x)| \lesssim |h| \rho(x).$$

We use $B_{p,q}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)$, $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ to denote the weighted Besov spaces on Λ_ε^k , $k \in \mathbb{N}$, and we recall their definition in the [Appendix](#). We also write $B_{p,q}^{\alpha,\varepsilon}(\rho) = B_{p,q}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)$ if there is no confusion. Note that if $\varepsilon = 0$, $B_{p,q}^{\alpha,\varepsilon}(\rho)$ is the classical weighted Besov space $B_{p,q}^\alpha(\rho)$ on $\mathbb{R}^{2k}$. Set $H^{\alpha,\varepsilon}(\rho) = B_{2,2}^{\alpha,\varepsilon}(\rho)$, $\mathbf{C}^{\alpha,\varepsilon}(\rho) = B_{\infty,\infty}^{\alpha,\varepsilon}(\rho)$ and $H^\alpha(\rho) = B_{2,2}^\alpha(\rho)$, $\mathbf{C}^\alpha(\rho) = B_{\infty,\infty}^\alpha(\rho)$. We also use $\mathbf{C}_s^{\alpha,\varepsilon}$ to denote the Besov–Hölder space w.r.t. each component on Λ_ε^k for $k \in \mathbb{N}$ with definition given in the [Appendix](#). The duality on Λ_ε is given by

$$\langle f, g \rangle_\varepsilon \stackrel{\text{def}}{=} \varepsilon^2 \sum_{x \in \Lambda_\varepsilon} f(x) g(x),$$

with $\langle \cdot, \cdot \rangle$ as the usual duality between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. We also write $L^{p,\varepsilon}$ for the L^p space on Λ_ε endowed with the norm

$$\|f\|_{L^{p,\varepsilon}} = \left(\varepsilon^2 \sum_{x \in \Lambda_\varepsilon} |f(x)|^p \right)^{1/p},$$

with the usual modification for $p = \infty$. We refer to the [Appendix](#) for discrete Besov embedding, duality, interpolation, which will be used below.

We start with Gibbs measures $(\nu_{M,\varepsilon})_{M,\varepsilon}$ on $\Lambda_{M,\varepsilon}$ given by

$$(2.2) \quad d\nu_{M,\varepsilon} \propto \exp \left\{ -\varepsilon^2 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{\lambda}{4} \Phi^4 + \left(-\frac{3}{2} \lambda a_{M,\varepsilon} + m \right) \Phi^2 + |\nabla_\varepsilon \Phi|^2 \right] \right\} \prod_{x \in \Lambda_{M,\varepsilon}} d\Phi(x),$$

where

$$\nabla_\varepsilon f(x) = \left(\frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon} \right)_{i=1,2}$$

denotes the discrete gradient and $a_{M,\varepsilon}$ are renormalization constants defined below. Here $(e_i)_{i=1,2}$ is the canonical basis in $\mathbb{R}^2$. We write

$$\Delta_\varepsilon f(x) = \varepsilon^{-2} (f(x + \varepsilon e_i) + f(x - \varepsilon e_i) - 2f(x)), \quad x \in \Lambda_\varepsilon$$

as the discrete Laplacian on Λ_ε and $\mathcal{L}_\varepsilon := \partial_t + m - \Delta_\varepsilon$. We also use $\Phi_{M,\varepsilon}$ to denote the stationary solution to the discrete stochastic quantization equation on $\Lambda_{M,\varepsilon}$

$$(2.3) \quad \mathcal{L}_\varepsilon \Phi_{M,\varepsilon} + \frac{\lambda}{2} \Phi_{M,\varepsilon}^3 - \frac{3}{2} \lambda a_{M,\varepsilon} \Phi_{M,\varepsilon} = \xi_{M,\varepsilon},$$

where $\xi_{M,\varepsilon}$ is a discrete approximation of a space–time white noise ξ on $\mathbb{R}^+ \times \mathbb{R}^2$ constructed as follows: Let ξ_M denote the periodization of ξ on $\mathbb{R}^+ \times \mathbb{T}_M^2$ and define the spatial discretization by

$$\xi_{M,\varepsilon}(t,x) := \varepsilon^{-2} \langle \xi_M(t,\cdot), 1_{|\cdot-x|_\infty \leq \varepsilon/2} \rangle, \quad (t,x) \in \mathbb{R}^+ \times \Lambda_{M,\varepsilon},$$

where $|x|_\infty = |x_1| \vee |x_2|$ for $x = (x_1, x_2)$. The law of $\Phi_{M,\varepsilon}$ at every time $t \geq 0$ is given by $\nu_{M,\varepsilon}$ defined in (2.2), which is the unique invariant measure to the finite-dimensional gradient system (2.3) (see, e.g., [13]). We extend all the functions and distributions to Λ_ε by periodic extension.

The aim of this section is to show that $\{\nu_{M,\varepsilon}\}_{M,\varepsilon}$ (properly extended to $\mathcal{S}'(\mathbb{R}^2)$) is tight in $\mathbf{C}^{-\gamma}(\rho)$ for $\gamma > 0$ and some polynomial weight ρ . Every accumulation point ν is an invariant measure of the following SPDE on $\mathbb{R}^+ \times \mathbb{R}^2$:

$$(2.4) \qquad \qquad \qquad \mathcal{L}\Phi + \frac{\lambda}{2}:\Phi^3: = \xi,$$

with $\mathcal{L} = \partial_t - \Delta + m$. Here $:\Phi^3:$ is the Wick power defined by

$$:\Phi^3: = \lim_{\substack{\varepsilon \rightarrow 0 \\ M \rightarrow \infty}} \mathcal{E}^\varepsilon(\Phi_{M,\varepsilon}^3 - 3a_{M,\varepsilon}\Phi_{M,\varepsilon}),$$

where $\mathcal{E}^\varepsilon$ is an extension operator from $B_{p,q}^{\alpha,\varepsilon}(\rho)$ to $B_{p,q}^\alpha(\rho)$ for some $\alpha < 0$ defined in (A.4) in the Appendix.

Here we understand (2.4) decomposed as the following two equations: $\Phi = Z + Y$

$$(2.5) \qquad \mathcal{L}Z = \xi, \qquad \mathcal{L}Y = -\frac{\lambda}{2}(Y^3 + 3Y^2Z + 3Y:Z^2: + :Z^3:),$$

where

$$\begin{aligned} :Z^2: &\stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}^\varepsilon:Z_{M,\varepsilon}^2: \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}^\varepsilon(Z_{M,\varepsilon}^2 - a_{M,\varepsilon}), \\ :Z^3: &\stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}^\varepsilon:Z_{M,\varepsilon}^3: \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}^\varepsilon(Z_{M,\varepsilon}^3 - 3a_{M,\varepsilon}Z_{M,\varepsilon}). \end{aligned}$$

Here $Z_{M,\varepsilon}$ are stationary solutions to

$$(2.6) \qquad \qquad \qquad \mathcal{L}_\varepsilon Z_{M,\varepsilon} = \xi_{M,\varepsilon},$$

$a_{M,\varepsilon} = \mathbf{E}Z_{M,\varepsilon}^2(0)$ and the limits are in $C([0, T]; \mathbf{C}^{-\gamma}(\rho^\kappa))$ **P**-a.s. for any $\gamma, \kappa > 0$ and $T > 0$. By standard renormalization calculation we have that for every $p \geq 1, \kappa, \gamma > 0$ and $T > 0$

$$(2.7) \qquad \mathbf{E}\|Z_{M,\varepsilon}\|_{C_T\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p + \mathbf{E}\|:Z_{M,\varepsilon}^2:\|_{C_T\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p + \mathbf{E}\|:Z_{M,\varepsilon}^3:\|_{C_T\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \lesssim 1,$$

with the proportional constant independent of ε, M , and

$$(2.8) \qquad \begin{aligned} &\mathbf{E}\|\mathcal{E}^\varepsilon Z_{M,\varepsilon} - Z\|_{C_T\mathbf{C}^{-\gamma}(\rho^\kappa)}^p + \mathbf{E}\|\mathcal{E}^\varepsilon:Z_{M,\varepsilon}^2: - :Z^2:\|_{C_T\mathbf{C}^{-\gamma}(\rho^\kappa)}^p \\ &\quad + \mathbf{E}\|\mathcal{E}^\varepsilon:Z_{M,\varepsilon}^3: - :Z^3:\|_{C_T\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \rightarrow 0, \end{aligned}$$

as $M \rightarrow \infty, \varepsilon \rightarrow 0$ (cf. [23, 34, 39]). Global well-posedness of equations (2.5) and hence equation (2.4) have been obtained in [34]. In the following we take $(\Phi_{M,\varepsilon}, Z_{M,\varepsilon})$ as joint stationary processes satisfying equations (2.3) and (2.6), respectively, which could be constructed by [36], Lemma 5.7. Hence, $Y_{M,\varepsilon} \stackrel{\text{def}}{=} \Phi_{M,\varepsilon} - Z_{M,\varepsilon}$ satisfy the following equation on $\Lambda_{M,\varepsilon}$:

$$(2.9) \qquad \mathcal{L}_\varepsilon Y_{M,\varepsilon} = -\frac{\lambda}{2}(Y_{M,\varepsilon}^3 + 3Y_{M,\varepsilon}^2Z_{M,\varepsilon} + 3Y_{M,\varepsilon}:Z_{M,\varepsilon}^2: + :Z_{M,\varepsilon}^3:).$$

In the following we derive uniform estimates in both parameters M, ε for $Y_{M,\varepsilon}$. To this end, we recall that all the distributions above are extended periodically to the full lattice Λ_ε .

For general $p \geq 2$ we have the following L^p uniform bounds.

LEMMA 2.1. *For the polynomial weight $\rho(x) = (1 + |hx|^2)^{-\delta/2}$ and $h > 0$ small enough, it holds that for every $p \geq 1$, $\delta > \frac{1}{2}$ and $0 < \sigma < 1$*

$$(2.10) \quad \mathbf{E} \|Y_{M,\varepsilon} \rho^2\|_{L^{2,\varepsilon}}^{2p} + \mathbf{E} \|Y_{M,\varepsilon} \rho^2\|_{H^{1-\sigma,\varepsilon}}^2 \|Y_{M,\varepsilon} \rho^2\|_{L^{2,\varepsilon}}^{2(p-1)} \lesssim \lambda^p + \lambda^{3p}.$$

Furthermore, for every $p_0 \geq 2$ there is $h > 0$ small enough such that for the weight $\rho(x) = (1 + |hx|^2)^{-\delta/2}$ and every $2 \leq p \leq p_0$ with p even, $p\delta > 2$

$$(2.11) \quad \mathbf{E} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}} + \mathbf{E} \|\nabla_\varepsilon Y_\varepsilon\|^2 |Y_\varepsilon|^{p-2} \rho^p\|_{L^{1,\varepsilon}} \lesssim \lambda + \lambda^{2+p/2}.$$

Here all the proportional constants are independent of M, ε .

PROOF. We omit the subscript M for notational simplicity, and all the proportional constants in the proof are independent of ε and M . We first note that for the canonical basis $\{e_i\}_{i=1}^2$ in $\mathbb{R}^2$ and even p

$$(2.12) \quad \begin{aligned} & \nabla_\varepsilon Y_\varepsilon(x) \cdot \nabla_\varepsilon Y_\varepsilon^{p-1}(x) \\ &= \sum_{i=1}^2 (\nabla_\varepsilon^i Y_\varepsilon(x))^2 \left(\sum_{\ell=2}^p Y_\varepsilon^{p-\ell}(x + \varepsilon e_i) Y_\varepsilon^{\ell-2}(x) \right) \\ &\geq \frac{1}{2} \left(|\nabla_\varepsilon Y_\varepsilon(x)|^2 |Y_\varepsilon(x)|^{p-2} + \sum_{i=1}^2 (\nabla_\varepsilon^i Y_\varepsilon(x))^2 Y_\varepsilon^{p-2}(x + \varepsilon e_i) \right), \end{aligned}$$

where $\nabla_\varepsilon^i Y_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} (Y_\varepsilon(x + \varepsilon e_i) - Y_\varepsilon(x))$ and we used $Y_\varepsilon^{p-\ell}(x + \varepsilon e_i) Y_\varepsilon^{\ell-2}(x) \geq 0$ for ℓ even and

$$Y_\varepsilon^{p-\ell}(x + \varepsilon e_i) Y_\varepsilon^{\ell-2}(x) \leq \frac{1}{2} Y_\varepsilon^{p-\ell+1}(x + \varepsilon e_i) Y_\varepsilon^{\ell-3}(x) + \frac{1}{2} Y_\varepsilon^{p-\ell-1}(x + \varepsilon e_i) Y_\varepsilon^{\ell-1}(x)$$

for ℓ odd in the last step. Taking the $L^{2,\varepsilon}$ inner product with $Y_\varepsilon^{p-1} \rho^p$ on both sides of (2.9) and using (2.12) we obtain

$$(2.13) \quad \begin{aligned} & \frac{1}{p} \frac{d}{dt} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}} + \frac{1}{2} (D_1 + D_2) + m \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}}^2 + \frac{\lambda}{2} \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}} \\ &\leq \langle [\nabla_\varepsilon, \rho^p] Y_\varepsilon^{p-1}, \nabla_\varepsilon Y_\varepsilon \rangle_\varepsilon - \frac{3\lambda}{2} \langle \rho^p Y_\varepsilon^{p+1}, Z_\varepsilon \rangle_\varepsilon \\ &\quad - \frac{3\lambda}{2} \langle \rho^p Y_\varepsilon^p, :Z_\varepsilon^2: \rangle_\varepsilon - \frac{\lambda}{2} \langle \rho^p Y_\varepsilon^{p-1}, :Z_\varepsilon^3: \rangle_\varepsilon \\ &\stackrel{\text{def}}{=} \sum_{k=1}^4 I_k, \end{aligned}$$

where

$$D_1 \stackrel{\text{def}}{=} \|\nabla_\varepsilon Y_\varepsilon\|^2 |Y_\varepsilon|^{p-2} \rho^p\|_{L^{1,\varepsilon}}, \quad D_2 \stackrel{\text{def}}{=} \left\| \sum_{i=1}^2 (\nabla_\varepsilon^i Y_\varepsilon)^2 \cdot Y_\varepsilon^{p-2}(\cdot + \varepsilon e_i) \rho^p \right\|_{L^{1,\varepsilon}}.$$

We also set $B \stackrel{\text{def}}{=} \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}}$. For I_1 we have

$$I_1 \leq C_\rho \|\rho^p Y_\varepsilon^p\|_{L^{1,\varepsilon}}^{1/2} D_1^{1/2} \leq C_{\rho,\gamma} \|\rho^p Y_\varepsilon^p\|_{L^{1,\varepsilon}} + \gamma D_1,$$

where the constants $C_\rho, C_{\rho,\gamma}$ depend on $\|\rho^{-p}[\nabla_\varepsilon, \rho^p]\|_{L^\infty, \varepsilon}$, which for given p_0 by (2.1) could be chosen sufficiently small for $p \leq p_0$ by choosing h small enough. By Lemma A.4 we have for $0 < s < \frac{1}{3}, \kappa > 0$

$$I_2 \lesssim \lambda \|Y_\varepsilon^{p+1}\|_{B_{1,1}^{s,\varepsilon}(\rho^{p-\kappa})} \|Z_\varepsilon\|_{C^{-s,\varepsilon}(\rho^\kappa)},$$

and we use Lemma A.3 to have

$$(2.14) \quad \|Y_\varepsilon^{p+1}\|_{B_{1,1}^{s,\varepsilon}(\rho^{p-\kappa})} \lesssim \|Y_\varepsilon^{p+1}\|_{B_{1,1}^{-s,\varepsilon}(\rho^{p-2\kappa})}^{1/2} \|Y_\varepsilon^{p+1}\|_{B_{1,1}^{3s,\varepsilon}(\rho^p)}^{1/2}.$$

For the first factor on the RHS of (2.14) we use Hölder's inequality to have

$$\|Y_\varepsilon^{p+1} \rho^{p-2\kappa}\|_{B_{1,1}^{-s,\varepsilon}} \lesssim \|Y_\varepsilon^{p+1} \rho^{p-2\kappa}\|_{L^{1,\varepsilon}} \lesssim \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}}^{\frac{p+1}{p+2}} \|\rho^\theta\|_{L^{1,\varepsilon}}^{\frac{1}{p+2}} \lesssim B^{\frac{p+1}{p+2}},$$

where $\theta = (p-2\kappa)(p+2) - p(p+1) = (1-2\kappa)p - 4\kappa > 2/\delta$ for $\kappa > 0$ small enough. For the second factor on the RHS of (2.14) we use (iii) in Lemma A.5 to obtain

$$\begin{aligned} \|Y_\varepsilon^{p+1}\|_{B_{1,1}^{3s,\varepsilon}(\rho^p)} &\lesssim \|Y_\varepsilon^{p+1}\|_{L^{1,\varepsilon}(\rho^p)} + \|\nabla_\varepsilon Y_\varepsilon^{p+1}\|_{L^{1,\varepsilon}(\rho^p)} \\ &\lesssim \|Y_\varepsilon^{p+1}\|_{L^{1,\varepsilon}(\rho^p)} + (D_1^{1/2} + D_2^{1/2}) B^{1/2}, \end{aligned}$$

where we use (2.1) to have

$$\begin{aligned} \|\nabla_\varepsilon Y_\varepsilon^{p+1}\|_{L^{1,\varepsilon}(\rho^p)} &\lesssim \sum_{i=1}^2 \|\nabla_\varepsilon^i Y_\varepsilon\| (Y_\varepsilon^p(\cdot + \varepsilon e_i) + Y_\varepsilon^p) \rho^p \|_{L^{1,\varepsilon}} \\ (2.15) \quad &\lesssim D_2^{1/2} \left(\sum_{i=1}^2 \|Y_\varepsilon\|^{p+2}(\cdot + \varepsilon e_i) \rho^p \|_{L^{1,\varepsilon}} \right)^{1/2} + D_1^{1/2} B^{1/2} \\ &\lesssim (D_1^{1/2} + D_2^{1/2}) B^{1/2}, \end{aligned}$$

in the last step. Substituting the above calculations into (2.14) and using Young's inequality we obtain for $\gamma > 0$

$$\begin{aligned} I_2 &\lesssim \lambda (B^{\frac{p+1}{p+2}} + (D_1^{\frac{1}{4}} + D_2^{\frac{1}{4}}) B^{\frac{1}{4} + \frac{p+1}{2(p+2)}}) \|Z_\varepsilon\|_{C^{-s,\varepsilon}(\rho^\kappa)} \\ &\leq \gamma \lambda B + \lambda C_\gamma \|Z_\varepsilon\|_{C^{-s,\varepsilon}(\rho^\kappa)}^{p+2} + \gamma D_1 + \gamma D_2 + \lambda^{2+p/2} C_\gamma \|Z_\varepsilon\|_{C^{-s,\varepsilon}(\rho^\kappa)}^{2(p+2)}. \end{aligned}$$

Similarly, for I_3 we use Lemma A.4 to have

$$I_3 \lesssim \lambda \|Y_\varepsilon^p\|_{B_{1,1}^{s,\varepsilon}(\rho^{p-\kappa})} \|Z_\varepsilon^2\|_{C^{-s,\varepsilon}(\rho^\kappa)},$$

and we apply Lemma A.3 to obtain for $0 < s < \frac{1}{3}$

$$(2.16) \quad \|Y_\varepsilon^p\|_{B_{1,1}^{s,\varepsilon}(\rho^{p-\kappa})} \lesssim \|Y_\varepsilon^p\|_{B_{1,1}^{-s,\varepsilon}(\rho^{p-2\kappa})}^{1/2} \|Y_\varepsilon^p\|_{B_{1,1}^{3s,\varepsilon}(\rho^p)}^{1/2}.$$

For the first factor on the RHS of (2.16) we use Hölder's inequality to have

$$\|Y_\varepsilon^p \rho^{p-2\kappa}\|_{B_{1,1}^{-s,\varepsilon}} \lesssim \|Y_\varepsilon^p \rho^{p-2\kappa}\|_{L^{1,\varepsilon}} \lesssim \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}}^{\frac{p}{p+2}} \|\rho^{\theta_1}\|_{L^{1,\varepsilon}}^{\frac{2}{p+2}} \lesssim B^{\frac{p}{p+2}},$$

where $\theta_1 = p(1-\kappa) - 2\kappa > 2/\delta$ for $\kappa > 0$ small enough. For the second factor on the RHS of (2.16) we use (iii) in Lemma A.5 to obtain

$$\begin{aligned} \|Y_\varepsilon^p\|_{B_{1,1}^{3s,\varepsilon}(\rho^p)} &\lesssim \|Y_\varepsilon^p\|_{L^{1,\varepsilon}(\rho^p)} + \|\nabla_\varepsilon Y_\varepsilon^p\|_{L^{1,\varepsilon}(\rho^p)} \\ &\lesssim \|Y_\varepsilon^p\|_{L^{1,\varepsilon}(\rho^p)} + (D_1^{1/2} + D_2^{1/2}) \|Y_\varepsilon^p\|_{L^{1,\varepsilon}(\rho^p)}^{1/2} \\ &\lesssim B^{\frac{p}{p+2}} + (D_1^{1/2} + D_2^{1/2}) B^{\frac{p}{2(p+2)}}, \end{aligned}$$

where we use a similar calculation as (2.15) in the second step. Combining the above calculations and Young's inequality we obtain

$$\begin{aligned} I_3 &\lesssim \lambda(B^{\frac{p}{p+2}} + (D_1^{\frac{1}{4}} + D_2^{\frac{1}{4}})B^{\frac{3p}{4(p+2)}}) \|Z_\varepsilon^2\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)} \\ &\leq \gamma \lambda B + \lambda C_\gamma \|Z_\varepsilon^2\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}^{\frac{p+2}{2}} + \gamma D_1 + \gamma D_2 + C_\gamma \lambda^{\frac{p}{6} + \frac{4}{3}} \|Z_\varepsilon^2\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}^{\frac{2(p+2)}{3}}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_4 &\lesssim \lambda(B^{\frac{p-1}{p+2}} + (D_1^{\frac{1}{4}} + D_2^{\frac{1}{4}})B^{\frac{3p-4}{4(p+2)}}) \|Z_\varepsilon^3\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)} \\ &\leq \gamma \lambda B + \lambda C_\gamma \|Z_\varepsilon^3\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}^{\frac{p+2}{3}} + \gamma D_1 + \gamma D_2 + C_\gamma \lambda^{\frac{p+12}{10}} \|Z_\varepsilon^3\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}^{\frac{2(p+2)}{5}}. \end{aligned}$$

Combining the above estimates and choosing γ small enough to absorb γD_1 , γD_2 and $\gamma \lambda B$ to the LHS of (2.13) we obtain

$$\begin{aligned} (2.17) \quad &\frac{1}{p} \frac{d}{dt} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}} + \frac{1}{4} D_1 + \frac{1}{4} D_2 + \frac{m}{2} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}}^2 + \frac{\lambda}{4} \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}} \\ &\leq (\lambda + \lambda^{2+p/2}) C(\mathbb{Z}_\varepsilon), \end{aligned}$$

where $C(\mathbb{Z}_\varepsilon)$ denotes a polynomial in the terms of

$$\|Z_{M,\varepsilon}\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}, \quad \|Z_{M,\varepsilon}^2\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}, \quad \|Z_{M,\varepsilon}^3\|_{\mathbf{C}^{-s,\varepsilon}(\rho^\kappa)}$$

and $\mathbf{EC}(\mathbb{Z}_\varepsilon) \lesssim 1$ with the proportional constant independent of ε , M . Recall that $(\Phi_\varepsilon, Z_\varepsilon)$ are stationary process, so are Y_ε . Integrating over t and taking expectation we obtain

$$\begin{aligned} (2.18) \quad &\frac{1}{p} \mathbf{E} \|Y_\varepsilon^p(t) \rho^p\|_{L^{1,\varepsilon}} + \frac{1}{4} \int_0^t \mathbf{E} (D_1 + D_2) ds \\ &+ \frac{m}{2} \int_0^t \mathbf{E} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}}^2 ds + \frac{\lambda}{4} \int_0^t \mathbf{E} \|Y_\varepsilon^{p+2} \rho^p\|_{L^{1,\varepsilon}} ds \\ &\leq C_{m,\rho} (\lambda + \lambda^{2+p/2}) + \frac{1}{p} \mathbf{E} \|Y_\varepsilon^p(0) \rho^p\|_{L^{1,\varepsilon}}, \end{aligned}$$

with the constant $C_{m,\rho} \geq 0$. Using stationarity of Y_ε and dividing t on both sides of (2.18) we obtain

$$\begin{aligned} &\frac{1}{4} \mathbf{E} \| |\nabla_\varepsilon Y_\varepsilon(0)|^2 |Y_\varepsilon(0)|^{p-2} \rho^p \|_{L^{1,\varepsilon}} + \frac{m}{2} \mathbf{E} \|Y_\varepsilon^p(0) \rho^p\|_{L^{1,\varepsilon}}^2 \\ &= \frac{1}{4t} \int_0^t \mathbf{E} \| |\nabla_\varepsilon Y_\varepsilon|^2 |Y_\varepsilon|^{p-2} \rho^p \|_{L^{1,\varepsilon}} ds + \frac{m}{2t} \int_0^t \mathbf{E} \|Y_\varepsilon^p \rho^p\|_{L^{1,\varepsilon}}^2 ds \\ &\leq C_{m,\rho} (\lambda + \lambda^{2+p/2}) + \frac{1}{pt} \mathbf{E} \|Y_\varepsilon^p(0) \rho^p\|_{L^{1,\varepsilon}}. \end{aligned}$$

The last term can be absorbed by the LHS for t large enough. Then (2.11) follows. Moreover, by Lemma A.5 we have for $\sigma \in (0, 1)$ and any polynomial weight ρ

$$(2.19) \quad \|Y_\varepsilon\|_{B_{2,2}^{1-\sigma,\varepsilon}(\rho)} \lesssim \|Y_\varepsilon\|_{B_{2,2}^{-\sigma,\varepsilon}(\rho)} + \|\nabla_\varepsilon Y_\varepsilon\|_{B_{2,2}^{-\sigma,\varepsilon}(\rho)} \lesssim \|Y_\varepsilon\|_{L^{2,\varepsilon}(\rho)} + \|\nabla_\varepsilon Y_\varepsilon\|_{L^{2,\varepsilon}(\rho)}.$$

Choosing $p = 2$, replacing ρ by ρ^2 with $\delta > \frac{1}{2}$ in (2.17) and using (2.19) we obtain for some $c_0 > 0$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|Y_\varepsilon(t) \rho^2\|_{L^{2,\varepsilon}}^2 + c_0 \|Y_\varepsilon \rho^2\|_{H^{1-\sigma,\varepsilon}}^2 + \frac{m}{4} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^2 + \frac{\lambda}{4} \|Y_\varepsilon \rho\|_{L^{4,\varepsilon}}^4 \\ &\leq (\lambda + \lambda^3) C(\mathbb{Z}_\varepsilon). \end{aligned}$$

Furthermore, we consider $\frac{d}{dt}\|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2p}$ and have for every $p > 1$

$$\begin{aligned} & \frac{1}{2p} \frac{d}{dt} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2p} + c_0 \|Y_\varepsilon \rho^2\|_{H^{1-\sigma,\varepsilon}}^2 \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2(p-1)} \\ & \quad + \frac{m}{4} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2p} + \frac{\lambda}{4} \|Y_\varepsilon \rho\|_{L^{4,\varepsilon}}^4 \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2(p-1)} \\ & \lesssim (\lambda + \lambda^3) C(\mathbb{Z}_\varepsilon) \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2(p-1)} \lesssim (\lambda^p + \lambda^{3p}) C(\mathbb{Z}_\varepsilon) + \frac{m}{8} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2p}, \end{aligned}$$

which by taking stationary initial condition and similar argument as above gives (2.10). $\square$

By Lemma 2.1 we could give uniform control for the nonlinear terms on the right-hand side of (2.9). Our aim is to do asymptotic expansion of k -point function up to λ^{N+1} for $k, N \in \mathbb{N}$. In the following we fix a polynomial weight and several parameters:

$$(2.20) \quad \rho(x) = (1 + |hx|^2)^{-\delta/2}, \quad \delta > 1/2, \quad p_0 > (9k(N+1)) \vee (96), \quad p_0 \text{ even},$$

with $h > 0$ small enough such that the bounds (2.10) hold for every $p \geq 1$ and (2.11) holds for $1 \leq p \leq p_0$ and even p . In the subsequent sections we also consider functions on Λ_ε^k and we also abuse the notation ρ to denote the weight in $\mathbb{R}^{2k}$. In this case we say ρ in (2.20) means that $h > 0$ satisfies the same condition as in (2.20) and it is easy to find that ρ could be controlled by the corresponding weight in $\mathbb{R}^2$.

COROLLARY 2.2. *For the polynomial weight and p_0 in (2.20), it holds that for $1 \leq p \leq p_0/3$, $0 < \beta < \frac{1}{4p}$, $\kappa > 0$*

$$(2.21) \quad \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^p \lesssim \lambda^p + \lambda^{4p},$$

where the proportional constant is independent of ε, M .

PROOF. We consider the case for p even and the general case follows by Hölder's inequality. We also omit the subscript M for notational simplicity and all the proportional constants in the proof are independent of ε and M . We first prove that for any $0 < s < \frac{1}{2p}$ that

$$\begin{aligned} (2.22) \quad & \mathbf{E} \|Y_{M,\varepsilon}^3\|_{L^{p,\varepsilon}(\rho^3)}^p + \mathbf{E} \|Y_{M,\varepsilon}^2 Z_{M,\varepsilon}\|_{B_{1+s,\infty}^{-s,\varepsilon}(\rho^{4+\kappa})}^p + \mathbf{E} \|Y_{M,\varepsilon} Z_{M,\varepsilon}^2\|_{H^{-s,\varepsilon}(\rho^{2+\kappa})}^p \\ & \lesssim \lambda + \lambda^{3p}. \end{aligned}$$

By Lemma 2.1 we have for $3p \leq p_0$

$$\mathbf{E} \|Y_\varepsilon^3\|_{L^{p,\varepsilon}(\rho^3)}^p \lesssim \mathbf{E} \|Y_\varepsilon\|_{L^{3p,\varepsilon}(\rho)}^{3p} \lesssim \lambda + \lambda^{2+\frac{3p}{2}}.$$

For the second term in (2.22) we use Lemma A.6 to have for $s > \gamma > 0$

$$\begin{aligned} & \|Y_\varepsilon^2 Z_\varepsilon\|_{B_{1+s,\infty}^{-s,\varepsilon}(\rho^{4+\kappa})} \lesssim \|Z_\varepsilon\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)} \|Y_\varepsilon^2\|_{B_{1+s,\infty}^s(\rho^4)} \lesssim \|Z_\varepsilon\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)} \|Y_\varepsilon^2\|_{B_{1,1}^{3s,\varepsilon}(\rho^4)} \\ & \lesssim \|Z_\varepsilon \rho^\kappa\|_{\mathbf{C}^{-\gamma,\varepsilon}} \|Y_\varepsilon \rho^2\|_{H^{4s,\varepsilon}} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}} \\ & \lesssim \|Z_\varepsilon\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)} \|Y_\varepsilon \rho^2\|_{H^{1-\sigma,\varepsilon}}^{2\theta_1} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{2-2\theta_1}, \end{aligned}$$

where we choose $0 < \sigma < 1$ small enough such that $\theta_1 = \frac{2s}{1-\sigma} < 1$ and we use Lemma A.5 in the second inequality and Lemma A.3 in the last inequality. Thus by (2.10) in Lemma 2.1 and (2.7) we obtain for σ small satisfying $2sp < 1 - \sigma$,

$$\mathbf{E} \|Y_\varepsilon^2 Z_\varepsilon\|_{B_{1+s,\infty}^{-s,\varepsilon}(\rho^{4+\kappa})}^p \lesssim \mathbf{E} \|Z_\varepsilon\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \|Y_\varepsilon \rho^2\|_{H^{1-\sigma,\varepsilon}}^{2p\theta_1} \|Y_\varepsilon \rho^2\|_{L^{2,\varepsilon}}^{p(2-2\theta_1)} \lesssim \lambda^p + \lambda^{3p},$$

where we use Hölder's inequality in the last step.

For the third term in (2.22) we use Lemma A.6, Lemma A.3, (2.10), (2.7) and Hölder's inequality to have for $sp < 2(1 - \sigma)$

$$\begin{aligned} \mathbf{E} \|Y_\varepsilon : Z_\varepsilon^2 : \|_{H^{-s, \varepsilon}(\rho^{2+\kappa})}^p &\lesssim \mathbf{E} \|Y_\varepsilon\|_{H^{s, \varepsilon}(\rho^2)}^p \|Z_\varepsilon^2 : \|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^\kappa)}^p \\ &\lesssim \mathbf{E} \|Y_\varepsilon\|_{H^{1-\sigma, \varepsilon}(\rho^2)}^{p\theta_1/2} \|Y_\varepsilon\|_{L^{2, \varepsilon}(\rho^2)}^{p(1-\theta_1/2)} \|Z_\varepsilon^2 : \|_{\mathbf{C}^{-\gamma}(\rho^\kappa)}^p \lesssim \lambda^{\frac{p}{2}} + \lambda^{\frac{3p}{2}}. \end{aligned}$$

Now (2.22) follows.

In the following we prove (2.21) by Schauder estimate Lemma A.7. Set

$$F_\varepsilon \stackrel{\text{def}}{=} Y_\varepsilon^3 + 3Y_\varepsilon^2 Z_\varepsilon + 3Y_\varepsilon : Z_\varepsilon^2 : + :Z_\varepsilon^3 :.$$

We choose $\alpha, s > 0$ such that

$$(2.23) \quad \frac{2}{p} < \alpha < \frac{1}{4p} + \frac{2}{p} < \frac{s(1-s)}{1+s} + \frac{2}{p} < 2,$$

where the last inequality holds by $sp < \frac{1}{2}$. By Schauder's estimate Lemma A.7

$$\|Y_\varepsilon\|_{L_T^p B_{p,p}^{\alpha, \varepsilon}(\rho)} \lesssim \|Y_\varepsilon(0)\|_{B_{p,p}^{\alpha-\frac{2}{p}, \varepsilon}(\rho)} + \lambda \|F_\varepsilon\|_{L_T^p B_{p,p}^{\alpha-2, \varepsilon}(\rho)},$$

with the proportional constant independent of T . By Lemma A.5 and the choice of α in (2.23) we have the following embedding:

$$\begin{aligned} B_{1+s, \infty}^{-s, \varepsilon}(\rho^{4+\kappa}) &\subset B_{p,p}^{\alpha-2, \varepsilon}(\rho^{4+\kappa}), & H^{-s, \varepsilon}(\rho^{2+\kappa}) &\subset B_{p,p}^{\alpha-2, \varepsilon}(\rho^{4+\kappa}), \\ L^{p, \varepsilon}(\rho^3) &\subset B_{p,p}^{\alpha-2, \varepsilon}(\rho^{4+\kappa}). \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbf{E} \|Y_\varepsilon(0)\|_{B_{p,p}^{\alpha, \varepsilon}(\rho^{4+\kappa})}^p \\ &= \frac{1}{T} \mathbf{E} \|Y_\varepsilon\|_{L_T^p B_{p,p}^{\alpha, \varepsilon}(\rho^{4+\kappa})}^p \lesssim \frac{1}{T} \mathbf{E} \|Y_\varepsilon(0)\|_{B_{p,p}^{\alpha-\frac{2}{p}, \varepsilon}(\rho^{4+\kappa})}^p + \frac{\lambda^p}{T} \mathbf{E} \|F_\varepsilon\|_{L_T^p B_{p,p}^{\alpha-2, \varepsilon}(\rho^{4+\kappa})}^p \\ &\lesssim \frac{1}{T} \mathbf{E} \|Y_\varepsilon(0)\|_{B_{p,p}^{\alpha-\frac{2}{p}, \varepsilon}(\rho^{4+\kappa})}^p + \lambda^p \mathbf{E} \|Y_{M, \varepsilon}^3\|_{L^{p, \varepsilon}(\rho^3)}^p + \lambda^p \mathbf{E} \|Y_{M, \varepsilon}^2 Z_{M, \varepsilon}\|_{B_{1+s, \infty}^{-s, \varepsilon}(\rho^{4+\kappa})}^p \\ &\quad + \lambda^p \mathbf{E} \|Y_{M, \varepsilon} : Z_{M, \varepsilon}^2 : \|_{H^{-s, \varepsilon}(\rho^{2+\kappa})}^p + \lambda^p \mathbf{E} \| : Z_{M, \varepsilon}^3 : \|_{\mathbf{C}^{-s, \varepsilon}(\rho^\kappa)}^p \\ &\lesssim \frac{1}{T} \mathbf{E} \|Y_\varepsilon(0)\|_{B_{p,p}^{\alpha-\frac{2}{p}, \varepsilon}(\rho^{4+\kappa})}^p + \lambda^p + \lambda^{4p}, \end{aligned}$$

where we use (2.22) in the last step. Choosing T large enough and using Besov embedding Lemma A.5, we obtain (2.21). $\square$

Define

$$\begin{aligned} \Phi_{M, \varepsilon} &= Y_{M, \varepsilon} + Z_{M, \varepsilon}, \\ (2.24) \quad : \Phi_{M, \varepsilon}^2 : &\stackrel{\text{def}}{=} \Phi_{M, \varepsilon}^2 - a_{M, \varepsilon} = Y_{M, \varepsilon}^2 + 2Y_{M, \varepsilon} Z_{M, \varepsilon} + : Z_{M, \varepsilon}^2 :, \\ : \Phi_{M, \varepsilon}^3 : &\stackrel{\text{def}}{=} \Phi_{M, \varepsilon}^3 - 3a_{M, \varepsilon} \Phi_{M, \varepsilon} = Y_{M, \varepsilon}^3 + 3Z_{M, \varepsilon} Y_{M, \varepsilon}^2 + 3 : Z_{M, \varepsilon}^2 : Y_{M, \varepsilon} + : Z_{M, \varepsilon}^3 :. \end{aligned}$$

Then we can use the right-hand sides to control $\mathbf{E} \| : \Phi_{M, \varepsilon}^i : \|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^9)}^p$ for $i = 1, 2, 3$. More precisely we have the following result which will be useful in the subsequent sections.

COROLLARY 2.3. *For the polynomial weight and p_0 as in (2.20), it holds for $0 < \gamma < \frac{1}{2}$, $\kappa > 0$, $1 \leq p < p_0/9$ that*

$$\mathbf{E} \|\Phi_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{4+\kappa})}^p + \mathbf{E} \|\Phi_{M,\varepsilon}^2\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^9)}^p + \mathbf{E} \|\Phi_{M,\varepsilon}^3\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{13})}^p \lesssim 1 + \lambda^{12p},$$

with the proportional constant independent of M, ε .

PROOF. It suffices to consider the case that p even since the general case follows by Hölder's inequality. In the following we consider each term separately and we choose $\kappa > 0$ small enough. By (2.21) we have for $\gamma > 0$, $0 < \beta < \frac{1}{4p}$, $3p < p_0$:

$$(2.25) \quad \mathbf{E} \|\Phi_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{4+\kappa})}^p \lesssim \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^p + \mathbf{E} \|Z_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \lesssim 1 + \lambda^{4p}.$$

We use Lemma A.6, (2.21) and (2.7) to have for $0 < \beta < \frac{1}{8p}$, $6p < p_0$

$$\begin{aligned} & \mathbf{E} \|\Phi_{M,\varepsilon}^2\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^9)}^p \\ & \lesssim \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^{2p} + \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^p \|Z_{M,\varepsilon}\|_{\mathbf{C}^{-\beta/2,\varepsilon}(\rho^\kappa)}^p + \mathbf{E} \|Z_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \\ & \lesssim 1 + \lambda^{8p}, \end{aligned}$$

where we use Hölder's inequality in the last step. Similarly for the last term we use Lemma A.6, Hölder's inequality and (2.21) to have for $\gamma > 0$ and $0 < \beta < \frac{1}{12p}$, $9p < p_0$

$$\begin{aligned} \mathbf{E} \|\Phi_{M,\varepsilon}^3\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{13})}^p & \lesssim \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^{3p} + \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^{2p} \|Z_{M,\varepsilon}\|_{\mathbf{C}^{-\beta/2,\varepsilon}(\rho^\kappa)}^p \\ & \quad + \mathbf{E} \|Y_{M,\varepsilon}\|_{\mathbf{C}^{\beta,\varepsilon}(\rho^{4+\kappa})}^p \|Z_{M,\varepsilon}^2\|_{\mathbf{C}^{-\beta/2,\varepsilon}(\rho^\kappa)}^p + \mathbf{E} \|Z_{M,\varepsilon}^3\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \\ & \lesssim 1 + \lambda^{12p}. \end{aligned}$$

The corollary then follows from the above bounds. $\square$

REMARK 2.4. The powers for the weight and λ in Corollary 2.3 are not optimal. If we use $L^{p,\varepsilon}$ -norm in (2.11) to bound $\mathbf{C}^{-\gamma,\varepsilon}$ we have lower powers with the cost that γ depends on p_0 .

We consider the tight limit of

$$(\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Z_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Y_{M,\varepsilon}(t))_{M,\varepsilon}$$

and we denote by (Φ, Z, Y) the canonical representative of the random variables under consideration (i.e., the canonical process on the canonical probability space with the limiting measure).

THEOREM 2.5. *For the polynomial weight and p_0 as in (2.20), it holds that the family*

$$(\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Z_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Y_{M,\varepsilon}(t))_{M,\varepsilon}$$

is tight in $(\mathbf{C}^{-\gamma}(\rho^{4+2\kappa}), \mathbf{C}^{-\gamma}(\rho^{2\kappa}), \mathbf{C}^\gamma(\rho^{4+\kappa}))$ for $0 < \gamma, \kappa < \frac{1}{8}$. Moreover, the first marginal ν of every tight limit μ is an invariant measure of (2.4) and satisfies for every $p \geq 1$, $\sigma, \gamma > 0$ with $3p < p_0$

$$\mathbf{E}^\nu \|\Phi\|_{\mathbf{C}^{-\gamma}(\rho^{4+\kappa})}^p \lesssim 1 + \lambda^{4p}, \quad \mathbf{E}^\mu \|Y\|_{H^{1-\sigma}(\rho^2)}^2 \lesssim \lambda^2, \quad \mathbf{E}^\mu \|Y\|_{L^p(\rho)}^p \lesssim \lambda + \lambda^{2+\frac{p}{2}}.$$

PROOF. By Lemma A.8 and (2.10) we know that for $p \geq 1$, $Y_{M,\varepsilon} = \Phi_{M,\varepsilon} - Z_{M,\varepsilon}$ satisfies

$$\mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon}\|_{L^2(\rho^2)}^{2p} + \mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon}\|_{H^{1-\sigma}(\rho^2)}^2 \|\mathcal{E}^\varepsilon Y_{M,\varepsilon}\|_{L^2(\rho^2)}^{2(p-1)} \lesssim \lambda^p + \lambda^{3p}.$$

By Corollary 2.2 and (2.25) we have for $\gamma > 0$, $p \geq 2$, $0 < \beta < \frac{1}{4p}$, $3p < p_0$

$$\mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon}\|_{\mathbf{C}^\beta(\rho^{4+\kappa})}^p + \mathbf{E} \|\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma}(\rho^{4+\kappa})}^p \lesssim 1 + \lambda^{4p}.$$

Then the tightness of $(\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Z_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Y_{M,\varepsilon}(t))_{M,\varepsilon}$ in $(\mathbf{C}^{-\gamma-\kappa}(\rho^{4+2\kappa}), \mathbf{C}^{-\gamma}(\rho^\kappa), \mathbf{C}^{\beta-\kappa}(\rho^{4+2\kappa}))$ follows from compact embedding Lemma A.5 and the moment bound follows from lower semicontinuity of the norm and (2.25), (2.11) and (2.10) with $p = 1$. We may extract a converging subsequence, which is still denoted as $(\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Z_{M,\varepsilon}(t), \mathcal{E}^\varepsilon Y_{M,\varepsilon}(t))_{M,\varepsilon}$, modify the stochastic basis, and find random variables $(\Phi, Z, Y) \in (\mathbf{C}^{-\gamma}(\rho^{4+\kappa}), \mathbf{C}^{-\gamma}(\rho^\kappa), \mathbf{C}^\beta(\rho^{4+\kappa}))$ such that for $p \geq 2$, $0 < \beta < \frac{1}{4p}$, $3p < p_0$

$$\mathbf{E} \|\mathcal{E}^\varepsilon \Phi_{M,\varepsilon} - \Phi\|_{\mathbf{C}^{-\gamma-\kappa}(\rho^{4+2\kappa})}^p \rightarrow 0, \quad \mathbf{E} \|\mathcal{E}^\varepsilon Z_{M,\varepsilon} - Z\|_{\mathbf{C}^{-\gamma}(\rho^\kappa)}^p \rightarrow 0, \quad M \rightarrow \infty, \varepsilon \rightarrow 0,$$

and

$$\mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon} - Y\|_{\mathbf{C}^{\beta-\kappa}(\rho^{4+2\kappa})}^p \rightarrow 0, \quad M \rightarrow \infty, \varepsilon \rightarrow 0.$$

Hence, by Lemma A.9, Lemma A.6 and (2.21), (2.7), (2.8) we have for $\gamma > 0$, $\frac{1}{8p} > \beta > 0$, $3p < p_0$

$$\begin{aligned} & \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon} Z_{M,\varepsilon}) - YZ\|_{\mathbf{C}^{-\gamma}(\rho^{4+3\kappa})}^p \\ & \lesssim \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon} Z_{M,\varepsilon}) - \mathcal{E}^\varepsilon Y_{M,\varepsilon} \mathcal{E}^\varepsilon Z_{M,\varepsilon}\|_{\mathbf{C}^{-\gamma}(\rho^{4+3\kappa})}^p + \mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon} \mathcal{E}^\varepsilon Z_{M,\varepsilon} - YZ\|_{\mathbf{C}^{-\gamma}(\rho^{4+3\kappa})}^p \\ & \lesssim o(\varepsilon) + \mathbf{E} \|\mathcal{E}^\varepsilon Y_{M,\varepsilon} - Y\|_{\mathbf{C}^\beta(\rho^{4+2\kappa})}^p \|\mathcal{E}^\varepsilon Z_{M,\varepsilon}\|_{\mathbf{C}^{-\beta/2}(\rho^\kappa)}^p \\ & \quad + \mathbf{E} \|Y\|_{\mathbf{C}^\beta(\rho^{4+2\kappa})}^p \|\mathcal{E}^\varepsilon Z_{M,\varepsilon} - Z\|_{\mathbf{C}^{-\beta/2}(\rho^\kappa)}^p \\ & \rightarrow 0, \quad \text{as } M \rightarrow \infty, \varepsilon \rightarrow 0. \end{aligned}$$

Similarly we have that as $M \rightarrow \infty$, $\varepsilon \rightarrow 0$, for $9p < p_0$, $\gamma > 0$

$$(2.26) \quad \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon}^2) - Y^2\|_{\mathbf{C}^{-\gamma}(\rho^{8+4\kappa})}^p + \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon}^3) - Y^3\|_{\mathbf{C}^{-\gamma}(\rho^{12+6\kappa})}^p \rightarrow 0,$$

$$(2.27) \quad \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon}^2 Z_{M,\varepsilon}) - Y^2 Z\|_{\mathbf{C}^{-\gamma}(\rho^{8+5\kappa})}^p + \mathbf{E} \|\mathcal{E}^\varepsilon (Y_{M,\varepsilon} Z_{M,\varepsilon}^2) - Y Z^2\|_{\mathbf{C}^{-\gamma}(\rho^{4+3\kappa})}^p \rightarrow 0,$$

which implies that

$$(2.28) \quad \begin{aligned} & \mathbf{E} \|\mathcal{E}^\varepsilon : \Phi_{M,\varepsilon}^2 : - : \Phi^2 : \|_{\mathbf{C}^{-\gamma}(\rho^{8+4\kappa})}^p \\ & + \mathbf{E} \|\mathcal{E}^\varepsilon : \Phi_{M,\varepsilon}^3 : - : \Phi^3 : \|_{\mathbf{C}^{-\gamma}(\rho^{12+6\kappa})}^p \rightarrow 0, \quad M \rightarrow \infty, \varepsilon \rightarrow 0, \end{aligned}$$

where

$$\begin{aligned} : \Phi^2 : & \stackrel{\text{def}}{=} Y^2 + 2YZ + :Z^2:, \\ : \Phi^3 : & \stackrel{\text{def}}{=} Y^3 + 3ZY^2 + 3:Z^2:Y + :Z^3:. \end{aligned}$$

Since $Y_{M,\varepsilon}$ satisfies equation (2.9), by Corollary 2.3 it is easy to obtain for $0 \leq s \leq t \leq T$

$$\mathbf{E} \|Y_{M,\varepsilon}(t) - Y_{M,\varepsilon}(s)\|_{\mathbf{C}^{-2-\gamma,\varepsilon}(\rho^{13})}^p \lesssim (1 + \lambda^{12p})(t-s)^p,$$

which combined with (2.7), (2.8) implies that the tightness of the stochastic process $(\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}, \mathcal{E}^\varepsilon Y_{M,\varepsilon}, \mathcal{E}^\varepsilon Z_{M,\varepsilon})$ in $C_T \mathbf{C}^{-2-2\gamma}(\rho^{13+\kappa}) \times C_T \mathbf{C}^{-2-2\gamma}(\rho^{13+\kappa}) \times C_T \mathbf{C}^{-\gamma}(\rho^\kappa)$. Since $\Phi_{M,\varepsilon}$ satisfies (2.3), we know $\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}$ satisfies the following equation:

$$\mathcal{E}^\varepsilon \Phi_{M,\varepsilon}(t) = \mathcal{E}^\varepsilon Y_{M,\varepsilon}(0) + \int_0^t (\Delta_\varepsilon \mathcal{E}^\varepsilon Y_{M,\varepsilon} + \mathcal{E}^\varepsilon : \Phi_{M,\varepsilon}^3 :) \, ds + \mathcal{E}^\varepsilon Z_{M,\varepsilon}(t).$$

For smooth test functions φ it is each to check $\Delta_\varepsilon \varphi \rightarrow \Delta \varphi$ for $\varepsilon \rightarrow 0$. By the convergence in (2.28), the nonlinear term $\mathcal{E}^\varepsilon : \Phi_{M,\varepsilon}^3 :$ also converges in $L^p(\Omega, L_T^p \mathbf{C}^{-\gamma}(\rho^{12+6\kappa}))$, which implies that the limiting process Φ satisfies equation (2.4). As $\Phi_{M,\varepsilon}$ is stationary solution to (2.3), ν is an invariant measure to (2.4). $\square$

REMARK 2.6. By a similar argument as in [23] we could prove that every ν is translation invariant and reflection positive and every ν satisfies an integration by parts formula, which has the same form as in [20], Chapter 12. As Φ is not a usual function, we will not use the integration by parts formula for ν directly but use the associated one for $\nu_{M,\varepsilon}$ instead in the subsequent sections.

3. Integration by parts and graphs. The purpose of this section is to study the perturbative expansion for the k -point correlation $S_{\lambda,M,\varepsilon}^k$ and apply the estimates from the previous section to obtain a bound on the remainder of this expansion. To this end we actually consider an equivalent expansion using integration by parts (“IBP” for short in the sequel), that is, Dyson–Schwinger equations. We also use the same notation as in Section 2.

Let $C_\varepsilon(x) := \frac{1}{2}(m - \Delta_\varepsilon)^{-1}(x)$ be the Green function of discrete Laplacian Δ_ε on Λ_ε and $C_{M,\varepsilon}$ is the periodic Green function. Choosing $a_{M,\varepsilon} = C_{M,\varepsilon}(0)$ as the Wick constant, we recall

$$:\Phi_{M,\varepsilon}^3: = \Phi_{M,\varepsilon}^3 - 3a_{M,\varepsilon} \Phi_\varepsilon, \qquad :\Phi_{M,\varepsilon}^2: = \Phi_{M,\varepsilon}^2 - a_{M,\varepsilon}.$$

Recall the following IBP formula with respect to $\nu_{M,\varepsilon}$ (see, e.g., [23], Section 6, or [36], Appendix C) for $F(\Phi_{M,\varepsilon}) = f(\Phi_{M,\varepsilon}(z_1), \dots, \Phi_{M,\varepsilon}(z_n))$, $n \in \mathbb{N}$, $z_i \in \Lambda_{M,\varepsilon}$ with smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ having polynomial growth first order derivative

$$\begin{aligned} \mathbf{E} \left(\frac{\delta F(\Phi_{M,\varepsilon})}{\delta \Phi_{M,\varepsilon}(z)} \right) &= \int \left(\frac{\delta F(\Phi)}{\delta \Phi(z)} \right) \nu_{M,\varepsilon}(\mathrm{d}\Phi) = \int F(\Phi) \left(\frac{\delta V_{M,\varepsilon}(\Phi)}{\delta \Phi(z)} \right) \nu_{M,\varepsilon}(\mathrm{d}\Phi) \\ (3.1) \qquad &= 2\mathbf{E}(F(\Phi_{M,\varepsilon})(m - \Delta_\varepsilon)\Phi_{M,\varepsilon}(z)) + \lambda \mathbf{E}(F(\Phi_{M,\varepsilon}) : \Phi_{M,\varepsilon}(z)^3 :), \end{aligned}$$

where $V_{M,\varepsilon}(\Phi) = \sum_{\Lambda_{M,\varepsilon}} [\frac{\lambda}{4} \Phi^4 + (-\frac{3}{2} \lambda a_{M,\varepsilon} + m) \Phi^2 + |\nabla_\varepsilon \Phi|^2]$ and

$$\frac{\delta F(\Phi_{M,\varepsilon})}{\delta \Phi_{M,\varepsilon}(z)} = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \left(F \left(\Phi_{M,\varepsilon} + \eta \frac{e_z}{\varepsilon^2} \right) - F(\Phi_{M,\varepsilon}) \right),$$

for $e_z : \Lambda_{M,\varepsilon} \rightarrow [0, 1]$, $e_z(z) = 1$, $e_z(y) = 0$ for $y \neq z$. We write (3.1) in terms of Green’s function $C_{M,\varepsilon}$:

$$\begin{aligned} \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x - z) \mathbf{E} \left(\frac{\delta F(\Phi_{M,\varepsilon})}{\delta \Phi_{M,\varepsilon}(z)} \right) \mathrm{d}z \\ (3.2) \qquad &= \mathbf{E}(\Phi_{M,\varepsilon}(x) F(\Phi_{M,\varepsilon})) \\ &\quad + \lambda \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x - z) \mathbf{E}(F(\Phi_{M,\varepsilon}) : \Phi_{M,\varepsilon}(z)^3 :) \mathrm{d}z \end{aligned}$$

for any $x \in \Lambda_{M,\varepsilon}$ and we use $\int_{\Lambda_{M,\varepsilon}} f(z) \, \mathrm{d}z$ to denote $\varepsilon^2 \sum_{z \in \Lambda_{M,\varepsilon}} f(z)$.

3.1. *k*-Point correlation. Consider the *k*-point correlation given by

$$S_{\lambda, M, \varepsilon}^k(x_1, \dots, x_k) = \mathbf{E} \left[\prod_{i=1}^k \Phi_{M, \varepsilon}(x_i) \right] \in \mathbf{C}^{-\gamma, \varepsilon}(\rho^\ell),$$

for $\gamma > 0$, some $\ell > 0$ and the polynomial weight ρ .

In the following, let Φ denote the stationary solution to (2.4) with marginal distribution given by ν obtained in Theorem 2.5. We still use (M, ε) to denote the subsequence such that $\nu_{M, \varepsilon} \circ (\mathcal{E}^\varepsilon)^{-1}$ converge to ν weakly with $\mathcal{E}^\varepsilon$ being the extension operator given in (A.4). Define

$$\begin{aligned} (3.3) \quad \langle S_{\lambda}^{\nu, k}, \varphi \rangle &\stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathbf{E} \int_{\mathbb{R}^{2k}} \left(\prod_{i=1}^k \mathcal{E}^\varepsilon \Phi_{M, \varepsilon}(x_i) \right) \varphi(x_1, \dots, x_k) \prod_{i=1}^k dx_i \\ &= \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \langle \mathcal{E}_k^\varepsilon S_{\lambda, M, \varepsilon}^k, \varphi \rangle, \end{aligned}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^{2k})$, where we use the Fubini Theorem in the second equality. Recall $\mathcal{E}_k^\varepsilon$ is introduced in (A.5) in the Appendix to extend functions on Λ_ε^k and by (A.6) this coincides with the one given in (1.4). By Theorem 2.5 and Lemmas A.1, A.8 we know

$$(3.4) \quad \lim_{M \rightarrow \infty, \varepsilon \rightarrow 0} \mathcal{E}_k^\varepsilon S_{\lambda, M, \varepsilon}^k = S_{\lambda}^{\nu, k} \quad \text{in } \mathbf{C}^{-\gamma}(\rho^\ell),$$

for $\gamma > 0$, some $n > 0$ and the polynomial weight ρ .

In the following we expand $S_{\lambda, M, \varepsilon}^k$ by applying (3.2) with suitable choices of test functions F . We introduce the following shorthand notation:

$$(3.5) \quad \mathcal{I}_\varepsilon f(x) \stackrel{\text{def}}{=} \int_{\Lambda_{M, \varepsilon}} C_{M, \varepsilon}(x - y) f(y) dy.$$

Before the proof, let us start by an example on how the iteration of this formula gives the desired expansion for $S_{\lambda, M, \varepsilon}(x_0 - y_0) = S_{\lambda, M, \varepsilon}^k(x_0 - y_0)$ with $k = 2$, $x_0, y_0 \in \Lambda_{M, \varepsilon}$. We first describe the initial steps of this iteration:

(1) Taking $F(\Phi_{M, \varepsilon}) = \Phi_{M, \varepsilon}(y_0)$ in (3.2) with $x = x_0$, we have

$$(3.6) \quad C_{M, \varepsilon}(x_0 - y_0) = S_{\lambda, M, \varepsilon}(x_0 - y_0) + \lambda \mathbf{E}(\Phi_{M, \varepsilon}(y_0) \mathcal{I}_\varepsilon(:\Phi_{M, \varepsilon}^3:)(x_0)).$$

This gives us the leading order expansion of the form $S_{\lambda, M, \varepsilon} = C_{M, \varepsilon} + O(\lambda)$ where $O(\lambda)$ refers to the last term and we will prove in Proposition 3.10 below that the $\mathbf{C}^{-\gamma, \varepsilon}(\rho^\ell)$ -norm of this term is bounded by $\lambda(1 + \lambda^{16})$ for $\gamma > 0$, some $\ell > 0$ and the polynomial weight ρ in (2.20).

(2) We can further expand the last term of (3.6): taking $F(\Phi_{M, \varepsilon}) = \lambda \mathcal{I}_\varepsilon(:\Phi_{M, \varepsilon}^3:)(x_0)$ in (3.2)

$$\begin{aligned} (3.7) \quad & 3\lambda \int_{\Lambda_{M, \varepsilon}} C_{M, \varepsilon}(y_0 - z) C_{M, \varepsilon}(x_0 - z) \mathbf{E}(:\Phi_{M, \varepsilon}(z)^2:) dz \\ &= \lambda \mathbf{E}(\Phi_{M, \varepsilon}(y_0) \mathcal{I}_\varepsilon(:\Phi_{M, \varepsilon}^3:)(x_0)) \\ & \quad + \lambda^2 \mathbf{E}(\mathcal{I}_\varepsilon(:\Phi_{M, \varepsilon}^3:)(x_0) \mathcal{I}_\varepsilon(:\Phi_{M, \varepsilon}^3:)(y_0)). \end{aligned}$$

Substituting this into (3.6) we see that in order to get the expansion up to the next order, we need to apply IBP again to the LHS in (3.7). Choose $F(\Phi_{M, \varepsilon}) = \Phi_{M, \varepsilon}(x)$. The LHS of (3.2)

only gives a Wick constant which can be absorbed into the RHS, so we get

$$(3.8) \quad 0 = \mathbf{E}(\cdot \Phi_{M,\varepsilon}^2(x) \cdot) + \lambda \mathbf{E}(\Phi_{M,\varepsilon}(x) \mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(x)) \cdot),$$

which implies that

$$\begin{aligned} & 3\lambda \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(y_0 - z) C_{M,\varepsilon}(x_0 - z) \mathbf{E}(\cdot \Phi_{M,\varepsilon}(z)^2 \cdot) \, dz \\ &= -3\lambda^2 \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(y_0 - z) C_{M,\varepsilon}(x_0 - z) \mathbf{E}(\Phi_{M,\varepsilon}(z) \mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(z)) \cdot) \, dz. \end{aligned}$$

Therefore we have

$$(3.9) \quad \begin{aligned} S_{\lambda,M,\varepsilon}(x_0 - y_0) &= C_{M,\varepsilon}(x_0 - y_0) + \lambda^2 \mathbf{E}(\mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(x_0)) \mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(y_0)) \cdot) \\ &\quad + 3\lambda^2 \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(y_0 - z) C_{M,\varepsilon}(x_0 - z) \mathbf{E}(\Phi_{M,\varepsilon}(z) \mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(z)) \cdot) \, dz. \end{aligned}$$

This gives us the next order expansion $S_{\lambda,M,\varepsilon} = C_{M,\varepsilon} + O(\lambda^2)$, that is, the order λ term is zero (see Proposition 3.10 below for the control of the term containing $\Phi_{M,\varepsilon}$).

REMARK 3.1. We will iteratively apply IBP as above to obtain higher order expansions. Remark that alternatively, if we replace all the $\Phi_{M,\varepsilon}$ in (3.9) by the Gaussian field $Z_{M,\varepsilon}$, using

$$\mathbf{E}(Z_{M,\varepsilon}(z) \mathcal{I}_\varepsilon(\cdot Z_{M,\varepsilon}^3(\cdot)(z)) \cdot) = 0,$$

and pretending that the error of this replacement is order $O(\lambda^3)$, then we can write $S_{\lambda,M,\varepsilon}(x_0 - y_0)$ as

$$(3.10) \quad \begin{aligned} & C_{M,\varepsilon}(x_0 - y_0) + \lambda^2 \mathbf{E}(\mathcal{I}_\varepsilon(\cdot Z_{M,\varepsilon}^3(\cdot)(x_0)) \mathcal{I}_\varepsilon(\cdot Z_{M,\varepsilon}^3(\cdot)(y_0)) \cdot) + O(\lambda^3) \\ &= C_{M,\varepsilon}(x_0 - y_0) \\ &\quad + 6\lambda^2 \int_{\Lambda_{M,\varepsilon} \times \Lambda_{M,\varepsilon}} C_{M,\varepsilon}(y_0 - w) C_{M,\varepsilon}(x_0 - z) C_{M,\varepsilon}(z - w)^3 \, dz \, dw + O(\lambda^3). \end{aligned}$$

This is the expansion at one more order.

REMARK 3.2. The procedure illustrated above for 2-point correlation also applies to k -point correlations for general k . For example, taking $F(\Phi_{M,\varepsilon}) = \prod_{i=2}^k \Phi_{M,\varepsilon}(x_i)$ we have

$$(3.11) \quad \begin{aligned} & \sum_{i=2}^k C_{M,\varepsilon}(x_1 - x_i) S_{\lambda,M,\varepsilon}^{k-2}(x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \\ &= S_{\lambda,M,\varepsilon}^k(x_1, \dots, x_k) + \lambda \mathbf{E} \left(\prod_{i=2}^k \Phi_{M,\varepsilon}(x_i) \mathcal{I}_\varepsilon(\cdot \Phi_{M,\varepsilon}^3(\cdot)(x_1)) \cdot \right) \, dz. \end{aligned}$$

Graphic notation. To iterate the above procedure in a more systematic way, it will be convenient to introduce some graphic notation. We denote $C_{M,\varepsilon}$ by a line, and $\Phi_{M,\varepsilon}$ by a

tiny wavy line. Then one can write the computation (3.6)–(3.10) above graphically as

$$\begin{aligned}
 S_{\lambda, M, \varepsilon}^2 &= \text{Diagram 1} \\
 &= \text{Diagram 2} - \lambda \text{Diagram 3} \\
 &= \text{Diagram 4} - 3\lambda \text{Diagram 5} + \lambda^2 \text{Diagram 6} \\
 &= \text{Diagram 7} + \lambda^2 \text{Diagram 8} + 3\lambda^2 \text{Diagram 9} \\
 &= \text{Diagram 10} + 6\lambda^2 \text{Diagram 11} + O(\lambda^3)
 \end{aligned}
 \tag{3.12}$$

REMARK 3.3. A very helpful intuition is that at each step of IBP, we simply pick up a point which has a wavy line (which corresponds to x in (3.2)), and then connect it to the other existing points with a wavy line (LHS of (3.2)), and also connect it to a new point with 3 new wavy lines (namely create a new factor $\mathcal{I}_\varepsilon(\cdot\Phi_{M,\varepsilon}^3\cdot)$ corresponding to the last term of (3.2)).

For instance, (3.11) amounts to picking up the point x_1 , or graphically:

$$\begin{aligned}
 S_{\lambda, M, \varepsilon}^4 &= \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} - \lambda \text{Diagram 5}
 \end{aligned}
 \tag{3.13}$$

We also find that for $\mathbf{E}[\Phi_{M,\varepsilon}^2(x)\cdot]$, (3.8) holds and for $\mathbf{E}[\Phi_{M,\varepsilon}^3(x)\cdot]$ we use (3.2) with $F(\Phi_{M,\varepsilon}) = \Phi_{M,\varepsilon}^2(x)$ to have

$$0 = \mathbf{E}[\Phi_{M,\varepsilon}^3(x)\cdot] + \lambda \mathbf{E}[\Phi_{M,\varepsilon}^2(x)\mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3\cdot)(x)].$$

Then there is no need to connect x to itself.

In general, given a graph G , we write $G = (V_G, E_G)$ or simply $G = (V, E)$ where V is the set of vertices and E is the set of edges. We denote by $|V|$, $|E|$ the cardinalities of these sets, namely the number of vertices and edges. Here for any two distinct vertices $u, v \in V$, we allow multiple edges between u and v (namely we allow “multigraphs” in the language of graph theory). However, we will assume throughout the paper that our graphs do not have self-loops, that is, there is not any edge of the form $\{u, u\}$ for $u \in V$.

DEFINITION 3.4. For each $\ell \geq 0$, we define $\mathcal{H}_\ell^k$ to be the set of all the graphs $G = (V, E)$ such that $|V| = \ell + k$ which has k “special points” $\{u_m^*, m = 1, \dots, k\}$ in V with $\deg(u_m^*) \in \{0, 1\}$, and such that $\deg(v) \in \{1, 2, 3, 4\}$ for every $v \in V \setminus \{u_m^*, m = 1, \dots, k\}$. We then define $\mathcal{H} := \bigcup_{\ell \geq 0} \mathcal{H}_\ell^k$.

We then define $\mathcal{G}_\ell^k$ to be the set of all the graphs $G = (V, E) \in \mathcal{H}_\ell^k$ such that $\deg(v) = 4$ for every $v \in V \setminus \{u_m^*, m = 1, \dots, k\}$ and $\deg(u_m^*) = 1$ for $m = 1, \dots, k$. We then write $\mathcal{G} := \bigcup_{\ell \geq 0} \mathcal{G}_\ell^k$. Clearly, for $G \in \mathcal{H}_\ell^k$ if we write

$$n_\Phi(G) := 4\ell + k - \sum_{v \in V} \deg(v),$$

then $\mathcal{G} = \{G \in \mathcal{H} : n_\Phi(G) = 0\} \subset \mathcal{H}$.

For any such graph we will write $V_G^\partial = \{u_m^*, m = 1, \dots, k\}$ and $V_G^0 = V_G \setminus V_G^\partial$.

Remark that the graphs in (3.12) are all elements of $\mathcal{H}$, where those without any tiny wavy line are elements of $\mathcal{G}$.

We define a mapping from $\mathcal{H}_\ell^k$ to the set of all functions in $\{x_{u_m^*}\}_{m=1}^k$, which maps $G = (V, E) \in \mathcal{H}_\ell^k$ to

(3.14)

$$\begin{aligned} \mathbf{I}_G(x_1, \dots, x_k) &= \int \left(\prod_{\{u,v\} \in E_G} C_{M,\varepsilon}(x_u, x_v) \right) \\ &\quad \times \mathbf{E} \left(\prod_{u_m^* \in V_G^\partial} \Phi_{M,\varepsilon}(x_m)^{1-\deg(u_m^*)} \prod_{z \in V_G^0} : \Phi_{M,\varepsilon}(x_z)^{4-\deg(z)} : \right) \prod_{z \in V_G^0} dx_z, \end{aligned}$$

where $x_m = x_{u_m^*}$. In particular when $G \in \mathcal{G}_\ell^k$, $\mathbf{I}_G$ only depends on $C_{M,\varepsilon}$, and not on $\Phi_{M,\varepsilon}$ (which is then the usual convention for “Feynman diagrams” in physics).

We will sometimes say that G is the graph associated with the function $\mathbf{I}_G$. More generally when a function F is a linear combination of functions of $\mathbf{I}_G$ ($G \in \mathcal{H}'$) for a finite collection of graphs $\mathcal{H}' \subset \mathcal{H}$ we say that the graphs in $\mathcal{H}'$ are the graphs associated with the function F . Whenever it is clear from the context we will sometimes use the same notation (such as x) to denote both a vertex of a graph and the point in $\mathbb{R}^2$ that is parametrized by x .

LEMMA 3.5. *For any $k \geq 1$ and $N \geq -1$, we have the following representation for the k -point correlation:*

(3.15)

$$S_{\lambda,M,\varepsilon}^k = \sum_{n=0}^N \frac{\lambda^n}{n!} F_{n,M,\varepsilon}^k + \lambda^{N+1} R_{N+1,M,\varepsilon}^k,$$

where the graphs associated with $F_{n,M,\varepsilon}^k$ belong to $\mathcal{G}_n^k$, and the graphs associated with $R_{N+1,M,\varepsilon}^k$ belong to $\mathcal{H}_{N+1}^k$. The functions $F_{n,M,\varepsilon}^k$ and $R_{N+1,M,\varepsilon}^k$ are independent of λ .

We remark that in the lemma, when $N = -1$, in which case the “empty” sum in (3.15) is understood as 0 by standard convention, (3.15) trivially holds which states that $S_{\lambda,M,\varepsilon}^k = R_{0,M,\varepsilon}^k$ where $R_{0,M,\varepsilon}^k$ can be indeed associated with a graph in $\mathcal{H}_0^k$, that is, the graph with only k vertices and no edge. We also note that as an example the third line of (3.12) shows that (3.15) holds for $k = 2$ and $N = 1$, where $F_{0,M,\varepsilon} = C_{M,\varepsilon}$ and $F_{1,M,\varepsilon} = 0$.

PROOF OF LEMMA 3.5. In the proof we omit M, ε for notational simplicity. By the $\Phi \rightarrow -\Phi$ symmetry, $S_{\lambda,M,\varepsilon}^k = 0$ if k is odd, so nothing needs to be proven. For $k \in 2\mathbb{N}$, we will prove by induction in N that the lemma holds with R_{N+1}^k having the following form:

(3.16)

$$R_{N+1}^k = \sum_{\substack{G \in \mathcal{H}_{N+1}^k \\ n_\Phi(G) \in [0,m] \cap 2\mathbb{Z}}} r_G \mathbf{I}_G$$

for some $m \in 2\mathbb{Z}$ which may depend on N , and some coefficients $r_G \in \mathbb{R}$.

As remarked above, the lemma holds with $N = -1$, with $R_0^k = S_\lambda^k = \mathbf{I}_G$, where $G = (V, E) \in \mathcal{H}_0^k$ with V consisting of the k special points only and E being empty.

Assume that for a fixed integer $N \geq 0$ we have already shown that

(3.17)

$$S_\lambda^k = \sum_{n=0}^{N-1} \frac{\lambda^n}{n!} F_n^k + \lambda^N R_N^k, \quad R_N^k = \sum_{\substack{G \in \mathcal{H}_N^k \\ n_\Phi(G) \in [0,m] \cap 2\mathbb{Z}}} r_G \mathbf{I}_G$$

we then prove that the same holds with N replaced by $N + 1$, with updated values of m and r_G .

To this end we use IBP to decrease the value of $n_\Phi(G)$ for the graphs $G \in \mathcal{H}_N^k$ above, so that we are only left with graphs in $\mathcal{G}_N^k$ (i.e., $n_\Phi = 0$), at the cost of producing other graphs in $\mathcal{H}_{N+1}^k$ which however are multiplied by the parameter λ and which will be defined as R_{N+1}^k .

More precisely we claim that for each $G \in \mathcal{H}_N^k$ such that $n_\Phi(G) = m$, the term $\mathbf{I}_G$ in (3.17) can be written as

$$(3.18) \quad \mathbf{I}_G = \sum_{G' \in \mathcal{H}_N^k, n_\Phi(G')=m-2} a_{G'} \mathbf{I}_{G'} + \lambda \sum_{G'' \in \mathcal{H}_{N+1}^k} b_{G''} \mathbf{I}_{G''}$$

for some $a_{G'}, b_{G''} \in \mathbb{R}$.

Assuming (3.18), we plug all these $\mathbf{I}_G$ with $n_\Phi(G) = m$ into (3.17), which allows us to write

$$(3.19) \quad R_N^k = \sum_{\substack{G \in \mathcal{H}_N^k \\ n_\Phi(G) \in [0, m-2] \cap 2\mathbb{Z}}} \bar{r}_G \mathbf{I}_G + \lambda \sum_{G \in \mathcal{H}_{N+1}^k} \bar{r}_G \mathbf{I}_G$$

for some coefficients $\bar{r}_G \in \mathbb{R}$. We then iterate this argument and plug these into (3.19) to obtain (3.19) with an even lower value of m (and new coefficients $\bar{r}_G \in \mathbb{R}$). After $m/2$ iterations, we get

$$(3.20) \quad R_N^k = \sum_{G \in \mathcal{G}_N^k} r'_G \mathbf{I}_G + \lambda \sum_{G \in \mathcal{H}_{N+1}^k} r'_G \mathbf{I}_G$$

for some r'_G . We then see that the lemma holds for N with $F_1, \dots, F_{N-1}$ remaining the same,

$$F_N = N! \sum_{G \in \mathcal{G}_N^k} r'_G \mathbf{I}_G \quad \text{and} \quad R_{N+1}^k = \sum_{G \in \mathcal{H}_{N+1}^k} r'_G \mathbf{I}_G.$$

Since all the graphs in the last sum are finite graphs, R_{N+1}^k has the form (3.16) for some m .

It remains to prove (3.18), for which it will be helpful to recall the intuition explained in Remark 3.3.

Fixing G , we can assume that $n_\Phi(G) \geq 2$ (otherwise if $n_\Phi(G) = 0$ there is nothing to prove). We then know that there is a vertex $x \in V_G^0$ with $\deg(x) < 4$ or $x \in V_G^\partial$ with $\deg(x) = 0$. Fix any such vertex x . Applying (3.2) with x therein being this vertex x , we have

$$(3.21) \quad \mathbf{I}_G = \left(\sum_{\substack{z \in V_G^\partial \\ \deg z=0, z \neq x}} (1 - \deg(z)) \mathbf{I}_{G_{xz}^+} \right) + \left(\sum_{\substack{z \in V_G^0 \\ \deg z < 4, z \neq x}} (4 - \deg(z)) \mathbf{I}_{G_{xz}^-} \right) - \lambda \mathbf{I}_{G_{xz}^+},$$

where the graph G_{xz}^+ is defined by “adding a new vertex z ”:

$$V_{G_{xz}^+} = V_G \cup \{z\}, \quad E_{G_{xz}^+} = E_G \cup \{\{x, z\}\}, \quad (z \notin V_G)$$

and the graph G_{xz}^- is defined by “connecting x and some $z \in V_G$ by a new edge”

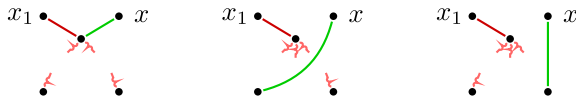
$$V_{G_{xz}^-} = V_G, \quad E_{G_{xz}^-} = E_G \cup \{\{x, z\}\}, \quad (z \in V_G).$$

This is precisely in the form (3.18), since $n_\Phi(G_{xz}^-) = n_\Phi(G) - 2$. $\square$

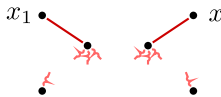
We remark that from the above proof, it is clear that all the graphs associated with $F_{n,M,\varepsilon}^k$ and $R_{N+1,M,\varepsilon}^k$ are obtained by the procedure described below (3.21), starting from the “initial graph” with $|V| = k$ and $|E| = 0$. For convenience of estimating the graphs later, we color the edges of these graphs as follows. In the above proof, when we use (3.21) each time,

- we use the red color for the new line $\{x, z\}$ appearing in G_{xz}^+ ;
- we use the green color for the new line $\{x, z\}$ appearing in G_{xz}^- .

REMARK 3.6. Just as an example for (3.21) in the above proof: if G is the last graph in (3.13), then x can be any point except for the upper-left one. Taking for instance x to be the upper-right point, then G_{xz}^- will be the following graphs:



and G_{xz}^+ is then



Note that we also color the edge connected to x_1 by red because of the way it arises in (3.13).

The reason for coloring the edges is that we will reduce graphs in $\mathcal{H}_N^k$ to trees by “cutting out” the green lines.

LEMMA 3.7. For each graph G associated with $F_{n,M,\varepsilon}^k$ or $R_{N+1,M,\varepsilon}^k$ in Lemma 3.5, denote by $\widehat{G}$ the subgraph formed by all its red lines.

Then, $\widehat{G}$ has exactly k connected components; each connected component is a rooted tree, and has exactly one vertex in V_G^∂ which is regarded as the root of the tree.⁴

PROOF. This is obvious by construction. Indeed, for the initial case where $|V_G| = k$ and $|E_G| = 0$, we have that $\widehat{G}$ is just k trivial trees. Assuming that we already know that a graph G satisfies the property in the lemma, then, when a new vertex z is added which creates a new red edge $\{x, z\}$ as in (3.21), it simply adds one more edge to one of the k red trees. (See Figure 1 for an illustration.) $\square$

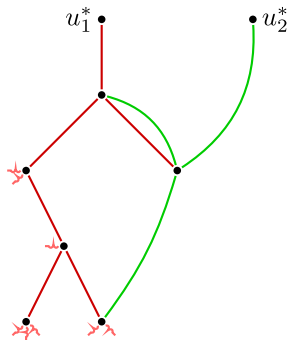


FIG. 1. An illustration of trees formed by red lines in the case of 2-point correlation. Here $u_1^*, u_2^* \in V_G^\partial$ and all other vertices are in V_G^0 . One of the two trees, that is, the tree which only contains one single vertex v , is trivial.

⁴Recall that in graph theory, a tree is a graph without any cycle, and a rooted tree is a tree in which one vertex is designated to be the root. In particular a graph consisting of only a single vertex and no edge is trivially a rooted tree.

LEMMA 3.8. *For the polynomial weight ρ in (2.20), it holds that for $\gamma > 0$ and some $\ell > 0$ and $n \leq N + 1$*

$$\lim_{M \rightarrow \infty, \varepsilon \rightarrow 0} \mathcal{E}_k^\varepsilon F_{n,M,\varepsilon}^k = F_n^k \quad \text{in } \mathbf{C}^{-\gamma}(\rho^\ell),$$

and

$$\lim_{M \rightarrow \infty, \varepsilon \rightarrow 0} \mathcal{E}_k^\varepsilon R_{n,M,\varepsilon}^k = R_n^{v,k} \quad \text{in } \mathbf{C}^{-\gamma}(\rho^\ell),$$

with $\mathcal{E}_k^\varepsilon$ being the extension operator defined in (A.5). Here F_n^k can be written as integrals of the Green function C of $\frac{1}{2}(m - \Delta)$ and $R_n^{v,k}$ only depends on C , Φ . The associated graph of F_n^k is the same as $F_{n,M,\varepsilon}^k$ with $C_{M,\varepsilon}$ in (3.14) replaced by the Green function C and the sum on $\Lambda_{M,\varepsilon}$ replaced by the integral on $\mathbb{R}^2$.

The proof of Lemma 3.8 is postponed to Section 4.2. In the proof we could also give the explicit formula of $R_n^{v,k}$. Formally the associated graph of $R_n^{v,k}$ is the same as $R_{n,M,\varepsilon}$ with $C_{M,\varepsilon}$ and $\Phi_{M,\varepsilon}$ in (3.14) replaced by the Green function C of $\frac{1}{2}(m - \Delta)$ and Φ , respectively and the sum on $\Lambda_{M,\varepsilon}$ replaced by the integral on $\mathbb{R}^2$.

The following gives the main result of this section.

THEOREM 3.9. *It holds that*

$$(3.22) \quad S_\lambda^{v,k} = \sum_{n=0}^N \frac{\lambda^n}{n!} F_n^k + \lambda^{N+1} R_{N+1}^{v,k},$$

with F_n^k , $R_{N+1}^{v,k}$ given in Lemma 3.8. This equality holds in $\mathbf{C}^{-\gamma}(\rho^\ell)$ for $\gamma > 0$, some $\ell > 0$ and polynomial weight ρ in (2.20).

PROOF. By the definition of S_λ^v in (3.3), (3.4), (3.15) we have

$$\begin{aligned} S_\lambda^{v,k} &= \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}_k^\varepsilon S_{\lambda,M,\varepsilon}^k \\ &= \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \sum_{n=0}^N \frac{\lambda^n}{n!} \mathcal{E}_k^\varepsilon F_{n,M,\varepsilon}^k + \lambda^{N+1} \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \mathcal{E}_k^\varepsilon R_{N+1,M,\varepsilon}^k \\ &= \sum_{n=0}^N \frac{\lambda^n}{n!} F_n^k + \lambda^{N+1} R_{N+1}^{v,k}, \end{aligned}$$

where we use Lemmas 3.8 to obtain the last equality. $\square$

Now to prove (1.5), we only need to show the following.

PROPOSITION 3.10. *For the polynomial weight ρ in (2.20) and $n \leq N + 1$, it holds that for $\gamma > 0$ and some $\ell > 0$*

$$\|R_n^{v,k}\|_{\mathbf{C}^{-\gamma}(\rho^\ell)} \lesssim 1 + \lambda^{4k+12n}.$$

PROOF. The result follows from Lemma 4.2 below and Lemma 3.8. $\square$

4. Proof of results in Section 3.

4.1. *Uniform bound on the remainder R_N .* In this section we give uniform estimates for $\mathbf{I}_G$ defined in (3.14) with $G \in \mathcal{H}$. For later use we construct the following random variables such that $\{Z_{M,\varepsilon}^{(i)}\}_{i \in \mathbb{N}}$ are i.i.d. random variables and each $Z_{M,\varepsilon}^{(i)}$ is an independent copy of $Z_{M,\varepsilon}$. We introduce D_i as follows: let $D_0 \stackrel{\text{def}}{=} \{1\}$ and

$$(4.1) \quad \begin{aligned} D_1 &\stackrel{\text{def}}{=} \{\Phi_{M,\varepsilon}, Z_{M,\varepsilon}^{(i)}, i \in \mathbb{N}\}, \\ D_2 &\stackrel{\text{def}}{=} \{:\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)}:, :\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}:, :Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}:, i, j \in \mathbb{N}, i \neq j\}, \\ D_3 &\stackrel{\text{def}}{=} \{:\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}:, :\Phi_{M,\varepsilon}^3 Z_{M,\varepsilon}^{(i)}:, :\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}:, :Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)} Z_{M,\varepsilon}^{(k)}:, \\ &\quad i, j, k \in \mathbb{N}, i \neq j \neq k\}, \end{aligned}$$

where $:\Phi_{M,\varepsilon}^i:$ are defined in (2.24) and for $i, j, k \in \mathbb{N}, i \neq j \neq k$

$$\begin{aligned} :\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)}: &\stackrel{\text{def}}{=} \Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)}, & :Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}: &\stackrel{\text{def}}{=} Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}, \\ :\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}: &\stackrel{\text{def}}{=} :\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}:, & :\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}: &\stackrel{\text{def}}{=} \Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)}, \\ :Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)} Z_{M,\varepsilon}^{(k)}: &\stackrel{\text{def}}{=} Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)} Z_{M,\varepsilon}^{(k)}. \end{aligned}$$

By probabilistic calculation we have that for $\kappa, \gamma > 0$ and $p > 1$

$$(4.2) \quad \mathbf{E} \|Z_{M,\varepsilon}^{(i)}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p + \mathbf{E} \|:\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)}:\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p + \mathbf{E} \|:Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)} Z_{M,\varepsilon}^{(k)}:\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \lesssim 1,$$

where $i \neq j \neq k$ and the proportional constant is independent of ε, M and there exist random variables $Z^{(i)}, :Z^{(i)} Z^{(j)}:, :Z^{(i)} Z^{(j)} Z^{(k)}: \in L^p(\Omega, C_T \mathbf{C}^{-\gamma}(\rho^\kappa))$ such that

$$\begin{aligned} \mathbf{E} \|\mathcal{E}^\varepsilon Z_{M,\varepsilon}^{(i)} - Z^{(i)}\|_{C_T \mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p + \mathbf{E} \|\mathcal{E}^\varepsilon :\Phi_{M,\varepsilon} Z_{M,\varepsilon}^{(i)}: - :Z^{(i)} Z^{(j)}:\|_{C_T \mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \\ + \mathbf{E} \|\mathcal{E}^\varepsilon :Z_{M,\varepsilon}^{(i)} Z_{M,\varepsilon}^{(j)} Z_{M,\varepsilon}^{(k)}: - :Z^{(i)} Z^{(j)} Z^{(k)}:\|_{C_T \mathbf{C}^{-\gamma,\varepsilon}(\rho^\kappa)}^p \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, M \rightarrow \infty. \end{aligned}$$

(cf. [23, 34, 39]). By direct calculation we have

$$(4.3) \quad C_{M,\varepsilon}(x - y) = \mathbf{E}[Z_{M,\varepsilon}(x) Z_{M,\varepsilon}(y)].$$

COROLLARY 4.1. *For the polynomial weight ρ and p_0 in (2.20), it holds for any $f_i \in D_i$, $1 < p < p_0/9$, $\gamma > 0$ that*

$$\mathbf{E} \|f_1\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^5)}^p + \mathbf{E} \|f_2\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^9)}^p + \mathbf{E} \|f_3\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{13})}^p \lesssim 1 + \lambda^{12p},$$

with the proportional constant independent of M, ε .

PROOF. Similar as before we omit the superscript M in the proof for notational simplicity. By Corollary 2.3 the desired estimate holds for $f_i = :\Phi^i:$. For $f_1 = Z_\varepsilon^{(i)}$, $f_2 = Z_\varepsilon^{(i)} Z_\varepsilon^{(j)}$ and $f_3 = Z_\varepsilon^{(i)} Z_\varepsilon^{(j)} Z_\varepsilon^{(k)}$, $i \neq j \neq k$, the result follows from (4.2). For other cases we write

$$\begin{aligned} :\Phi_\varepsilon Z_\varepsilon^{(i)}: &= Y_\varepsilon Z_\varepsilon^{(i)} + :Z_\varepsilon Z_\varepsilon^{(i)}:, \\ :\Phi_\varepsilon Z_\varepsilon^{(i)} Z_\varepsilon^{(j)}: &= Y_\varepsilon :Z_\varepsilon^{(i)} Z_\varepsilon^{(j)}: + :Z_\varepsilon Z_\varepsilon^{(i)} Z_\varepsilon^{(j)}:, \\ :\Phi_\varepsilon^2 Z_\varepsilon^{(i)}: &= Y_\varepsilon^2 Z_\varepsilon^{(i)} + 2Y_\varepsilon :Z_\varepsilon Z_\varepsilon^{(i)}: + :Z_\varepsilon^2 Z_\varepsilon^{(i)}:, \end{aligned}$$

where $:Z_\varepsilon^2 Z_\varepsilon^{(i)}: := :Z_\varepsilon^2: Z_\varepsilon^{(i)}$ and by probabilistic calculation for $\gamma, \kappa > 0$, $p > 1$

$$\mathbf{E} \| :Z_\varepsilon^2 Z_\varepsilon^{(i)}: \|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^\kappa)}^p \lesssim 1.$$

By (4.2), the same calculation in Corollary 2.3 and Lemma A.6, the desired estimates also hold for the terms $:\Phi_\varepsilon Z_\varepsilon^{(i)}:$, $:\Phi_\varepsilon Z_\varepsilon^{(i)} Z_\varepsilon^{(j)}:$ and $:\Phi_\varepsilon^2 Z_\varepsilon^{(i)}:$, which implies the result. $\square$

To state the following lemma for $G \in \mathcal{H}$ with $\mathbf{I}_G$ in (3.14), we extend it as periodic functions on Λ_ε .

LEMMA 4.2. *For each $G \in \mathcal{H}_n^k$, $n \leq N + 1$, that appears in Lemma 3.5, letting $\mathbf{I}_G$ be the function introduced in (3.14) and ρ be polynomial weight as in (2.20) it holds that for $\gamma > 0$ and some $\ell > 0$*

$$(4.4) \quad \|\mathcal{E}_k^\varepsilon \mathbf{I}_G\|_{\mathbf{C}^{-\gamma}(\rho^\ell)} \lesssim 1 + \lambda^{4k+12n},$$

where the proportional constant is independent of ε , M and $\mathcal{E}_k^\varepsilon$ is the extension operator introduced in (A.5) in the Appendix.

PROOF. In the proof we use ρ^ℓ to denote weight and ℓ may change from line to line. We also abuse the notation of ρ for the weight in different spatial dimensions. By Lemma 3.7 we have $\widehat{G}$ which is a union of k rooted trees, formed by the red edges in G .

Step 1. Reduction to trees.

It will turn out that trees are easier to estimate than general graphs. To reduce our analysis for $\mathbf{I}_G$ to the case of trees, we proceed as follows.

Recall the definition (3.14) for $\mathbf{I}_G$. For each of the green lines $\{u, v\} \in E_G$, we use (4.3) to replace each $C_{M, \varepsilon}(x_u, x_v)$ in (3.14) by $\mathbf{E}[Z_{M, \varepsilon}^{(i)}(x_u) Z_{M, \varepsilon}^{(i)}(x_v)]$ with $\{Z_{M, \varepsilon}^{(i)}\}_i$ given at the beginning of this section. Each $C_{M, \varepsilon}(x_u, x_v)$ corresponds to different $Z_{M, \varepsilon}^{(i)}$. This can be done for all the green lines since our graph is finite. The resulting function is the same as $\mathbf{I}_G$.

Note that for periodic functions f on Λ_ε , $\mathcal{I}_\varepsilon(f)$ introduced in (3.5) could be written as

$$\mathcal{I}_\varepsilon(f) = \int_{\Lambda_\varepsilon} C_\varepsilon(x - y) f(y) dy.$$

Since $\widehat{G}$ is a disjoint union of k trees (by Lemma 3.7), that is, $\widehat{G} = \bigsqcup_{i=1}^k T_i$, where each T_i is a tree, we know that $\mathbf{I}_G$ is the expectation of a product of k functions, and each of these k functions have the following form:

$$(4.5) \quad \begin{aligned} F_T(x_{u^*}) &= \int \prod_{\{u, v\} \in E_T} C_\varepsilon(x_u, x_v) \\ &\times \left(f_1(x_{u^*})^{1-\deg_T(u^*)} \prod_{v \in V_T \setminus \{u^*\}} f_v(x_v) g_v^T(x_v) \right) \prod_{v \in V_T \setminus \{u^*\}} dx_v, \end{aligned}$$

where $f_1 \in D_1$, $f_v \in \bigcup_{i=0}^3 D_i$, $g_v^T = 1$ and $\deg_T(v)$ denotes the degree of v in the graph T . Here and below we just write T for T_i to simplify the notation, and we have introduced the function g_v^T for the purpose of induction later.

For instance, the graph in Figure 1 then reduces to the one shown in Figure 2 where each tiny green wavy line denotes a factor of $Z_{M, \varepsilon}^{(i)}$.

We first consider the degenerate case separately where $\deg(u^*) = 0$, namely F_T is simply $f_1(x_{u^*})$ with $f_1 \in D_1$. We use (2.25) and (4.2) and easily find for $\gamma > 0$ and $3p < p_0$ with p_0 in (2.20)

$$(4.6) \quad \mathbf{E} \| Z_{M, \varepsilon} \rho \|_{\mathbf{C}^{-\gamma, \varepsilon}}^p + \mathbf{E} \| \Phi_\varepsilon \rho^5 \|_{\mathbf{C}^{-\gamma, \varepsilon}}^p \lesssim 1 + \lambda^{4p}.$$

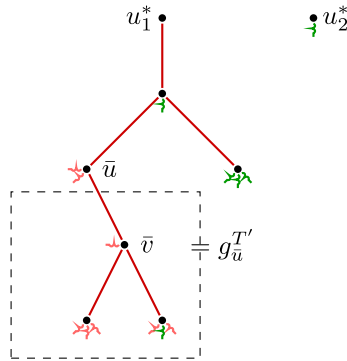


FIG. 2. An illustration for reducing graphs to trees, and inductive integrations.

In the following we only need to consider the case that $\deg(u^*) = 1$ that is, nontrivial trees. We claim that for $\gamma > 0$ and ℓ large enough,

$$(4.7) \qquad \|F_T\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)} \lesssim \prod_{v \in V_T} \|f_v\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{13})},$$

with $f_v \in \bigcup_{i=0}^3 D_i$ defined in (4.1). By the reduction we have for $\gamma > 0$

$$(4.8) \qquad \begin{aligned} \|\mathbf{I}_G\|_{\mathbf{C}_s^{-\gamma,\varepsilon}(\Lambda_\varepsilon^k, \rho^\ell)} &\lesssim \mathbf{E} \prod_{i=1}^k \|F_{T_i}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\ell)} \\ &\lesssim \prod_{\deg(u_i^*)=1} (\mathbf{E} \|F_{T_i}\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)}^k)^{\frac{1}{k}} \prod_{\deg(u_i^*)=0} (\mathbf{E} \|F_{T_i}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^5)}^k)^{\frac{1}{k}}, \end{aligned}$$

where $\mathbf{C}_s^{-\gamma,\varepsilon}(\Lambda_\varepsilon^k, \rho^\ell)$ is the Hölder space for each variable on Λ_ε^k introduced in (A.2) in the Appendix. By the claim (4.7) and Corollary 4.1, we deduce that for $\deg(u_i^*) = 1$, $\gamma > 0$

$$(4.9) \qquad (\mathbf{E} \|F_{T_i}\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)}^k)^{1/k} \lesssim 1 + \lambda^{12n_i},$$

where n_i is given by the number of vertices in V_0 for the tree T_i and $\sum_{i=1}^k n_i \leq n$ and we use $9kn_i \leq 9k(N+1) < p_0$ in (2.20). For the case that $\deg(u_i^*) = 0$, by (4.6),

$$(4.10) \qquad (\mathbf{E} \|F_{T_i}\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^5)}^k)^{1/k} \lesssim 1 + \lambda^4.$$

As a result, (4.4) follows from Lemma A.8, Lemma A.1 and arbitrary γ . It remains to prove (4.7).

Step 2. Estimate of each tree.

Fixing a rooted tree T as above, we will integrate the variables in F_T in (4.5) from the leaves⁵ of the tree T and estimate the effect of the integrations. More precisely, we claim that for every subtree $\bar{T}$ of T which contains the root u^* , (4.5) still holds with T and g^T on the RHS replaced by $\bar{T}$ and $g^{\bar{T}}$, where the functions g (which depend on $\bar{T}$) are such that $\|g\|_{\mathbf{C}^{s,\varepsilon}(\rho^\ell)}$ is bounded by $\prod_{j \in I_g} \|f_j\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^{13})}$, $f_j \in \bigcup_{i=0}^3 D_i$ for some index set $I_g \subset \mathbb{N}$ and $s > \gamma > 0$, $n > 0$.

We prove by induction downwards on the value of $|V_{\bar{T}}|$. We already have that (4.5) holds for $\bar{T} = T$ where $g = 1$ trivially satisfies the claimed bound.

⁵Recall that in graph theory a leaf of a rooted tree is a vertex of degree 1 and is not the root. Any nontrivial rooted tree must have at least one leaf.

Assume that (4.5) holds with $F_T(x_{u^*})$ on the LHS, and the RHS written as the integral in terms of a subtree $\bar{T}$. Let $\bar{v}$ be a leaf of $\bar{T}$. Then there is a unique $\bar{u} \in V_{\bar{T}}$ such that $\bar{e} \stackrel{\text{def}}{=} \{\bar{u}, \bar{v}\} \in E_{\bar{T}}$. We then have

$$(4.11) \quad \begin{aligned} F_T(x_{u^*}) &= \int \prod_{\{u,v\} \in E_{\bar{T}} \setminus \{\bar{e}\}} C_\varepsilon(x_u, x_v) \prod_{v \in V_{\bar{T}} \setminus \{u^*, \bar{v}\}} (f_v(x_v) g_v^{\bar{T}}(x_v)) \\ &\quad \times \left(\int C_\varepsilon(x_{\bar{u}}, x_{\bar{v}}) (f_{\bar{v}}(x_{\bar{v}}) g_{\bar{v}}^{\bar{T}}(x_{\bar{v}})) dx_{\bar{v}} \right) \prod_{v \in V_{\bar{T}} \setminus \{u^*, \bar{v}\}} dx_v, \end{aligned}$$

and we now estimate the integration over $\bar{v}$ in the parenthesis, which will imply that (4.5) holds for the subtree $T' = (V_{\bar{T}} \setminus \{\bar{v}\}, E_{\bar{T}} \setminus \{\bar{e}\})$ with $g_{\bar{u}}^{T'}(x_{\bar{u}})$ redefined as

$$(4.12) \quad g_{\bar{u}}^{T'}(x_{\bar{u}}) = \left(\int C_\varepsilon(x_{\bar{u}}, x_{\bar{v}}) (f_{\bar{v}}(x_{\bar{v}}) g_{\bar{v}}^{\bar{T}}(x_{\bar{v}})) dx_{\bar{v}} \right) g_{\bar{u}}^{\bar{T}}(x_{\bar{u}})$$

and f_v, g_v for $v \neq \bar{u}$ left unchanged. (See Figure 2 for an illustration.)

The integration over $\bar{v}$ in the parenthesis is in one of the following forms:

$$(4.13) \quad \mathcal{I}_\varepsilon(g)(x_{\bar{u}}), \quad \mathcal{I}_\varepsilon(f_3)(x_{\bar{u}}), \quad \mathcal{I}_\varepsilon(f_2 g)(x_{\bar{u}}), \quad \mathcal{I}_\varepsilon(f_1 g)(x_{\bar{u}}),$$

with $f_i \in D_i$. By (A.7) and Lemma A.6, we have that for $1 > s > \gamma > 0$,

$$(4.14) \quad \begin{aligned} \|\mathcal{I}_\varepsilon(g)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^\ell)} &\lesssim \|g\|_{\mathbf{C}^{s, \varepsilon}(\rho^\ell)}, \\ \|\mathcal{I}_\varepsilon(f_3)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^{13})} &\lesssim \|f_3\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^{13})}, \\ \|\mathcal{I}_\varepsilon(g f_2)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^{\ell+13})} &\lesssim \|g\|_{\mathbf{C}^{s, \varepsilon}(\rho^n)} \|f_2\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^{13})}, \\ \|\mathcal{I}_\varepsilon(g f_1)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^{\ell+13})} &\lesssim \|g\|_{\mathbf{C}^{s, \varepsilon}(\rho^n)} \|f_1\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^{13})}. \end{aligned}$$

By the inductive assumption on g , we have that the left-hand sides are bounded by

$$\prod_{j \in I_g} \|f_j\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^\ell)} \|f_{\bar{v}}\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^{13})},$$

with $f_{\bar{v}} \in \bigcup_{i=0}^3 D_i$. This implies that (4.5) holds for the smaller subtree T' , namely the number of vertices decreases by one.

By inductively decreasing the value of $|V_{\bar{T}}|$, we know that (4.5) holds for the subtree $\bar{T}$ which only has 2 vertices including u^* . Now F_T is simply *equal to* one of the cases in (4.13), so we can just bound F_T by (4.14). On the other hand, integration over one vertices counts for at most one $\|f_i\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^{13})}$, which gives (4.7). $\square$

4.2. Convergence of $F_{n, M, \varepsilon}^k$ and $R_{n, M, \varepsilon}^k$. In this section we give the proof of Lemma 3.8, that is, the discrete integral converges to the corresponding continuous one. By the uniform bound in Section 4.1 we already know that each $\|\mathcal{E}_k^\varepsilon F_{n, M, \varepsilon}^k\|_{\mathbf{C}^{-\gamma}(\rho^\ell)}$ and $\|\mathcal{E}_k^\varepsilon R_{n, M, \varepsilon}^k\|_{\mathbf{C}^{-\gamma}(\rho^\ell)}$ are uniformly bounded, which implies there exists a convergent subsequence. Now, we want to give the explicit formula of the limit.

PROOF OF LEMMA 3.8. Since we consider the limits of $\mathcal{E}_k^\varepsilon F_{n, M, \varepsilon}^k$ and $\mathcal{E}_k^\varepsilon R_{n, M, \varepsilon}^k$, which only depend on the law of $\Phi_{M, \varepsilon}$ and $Z_{M, \varepsilon}$, we can assume the setting as in the proof of Theorem 2.5, that is, we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ and we have random variables $(\Phi_{M, \varepsilon}, Z_{M, \varepsilon}, Y_{M, \varepsilon})$ and (Φ, Z, Y) such that the convergence in (2.26)–(2.28) holds. We also have all the random variables

$$\{Z_{M, \varepsilon}^{(i)}, \cdot, Z_{M, \varepsilon}^{(i)} Z_{M, \varepsilon}^{(j)}, \cdot, \cdot, Z_{M, \varepsilon}^{(i)} Z_{M, \varepsilon}^{(j)} Z_{M, \varepsilon}^{(k)}, \cdot, \cdot, i \neq j \neq k\}$$

and their continuous limit $\{Z^{(i)}, :Z^{(i)}Z^{(j)}:, :Z^{(i)}Z^{(j)}Z^{(k)}:, i \neq j \neq k\}$ on the same stochastic basis, which can be done by the Skorohod theorem. As a result, every element in $\bigcup_{i=1}^3 D_i$ and their continuous limit can be realized on this stochastic basis. We also view $\Phi_{M,\varepsilon}, Y_{M,\varepsilon}$ and $Z_{M,\varepsilon}^{(i)}$ as periodic functions on Λ_ε .

Combining the proof in Theorem 2.5 and Corollary 4.1 we know that for every element $f_i \in D_i, \mathcal{E}^\varepsilon f_i$ converges to the corresponding continuous one in $L^p(\Omega; \mathbf{C}^{-\gamma}(\rho^{13}))$ for $\gamma > 0, p > 1$ with $9p < p_0$ for p_0 in (2.20). For example, we consider $:\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}:$ and by similar calculation as (2.27) we obtain

$$\mathbf{E}\|\mathcal{E}^\varepsilon(Y_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}) - Y^2 Z^{(i)}\|_{\mathbf{C}^{-\gamma}(\rho^{13})}^p \lesssim o(\varepsilon),$$

for $\gamma > 0$, where the proportional constant is independent of M . For other terms in $:\Phi_{M,\varepsilon}^2 Z_{M,\varepsilon}^{(i)}:$ we have similar convergence. In the following n may change for different convergences.

For each graph we consider each $\mathbf{I}_G$ and as in the proof of Lemma 4.2 we know

$$\mathbf{I}_G(x_1, \dots, x_k) = \mathbf{E}\left(\prod_{i=1}^k F_{T_i}(x_i)\right),$$

with F_{T_i} defined in (4.5) and T_i being trees and subgraphs for G . By the uniform bounds obtained in (4.10) and (4.9), we first consider the convergence of $\mathcal{E}^\varepsilon F_{T_i}$ in $L^k(\Omega, \mathbf{C}^{-\gamma}(\rho^\ell))$ for each tree T_i and $\gamma > 0$. For the degenerate case where $\deg(u^*) = 0$, that is, F_{T_i} is just $f_1(x_i)$ with $f_1 \in D_1$, we have proved the convergence.

In the following we consider the nontrivial case, that is, $\deg(u^*) = 1$. The idea is to use Lemma A.10, Lemma A.9 and uniform estimate in Corollary 4.1 to exchange the extension operator $\mathcal{E}^\varepsilon$ to every leaf in the tree up to an $o(\varepsilon)$ constant. We consider the subtree $\bar{T}$ of T_i and we want to prove for each subtree $\bar{T}$ as in Step 2 in the proof of Lemma 4.2 and $0 < \gamma < s < 1$

(4.15)

$$\begin{aligned} &\mathbf{E}\left\|\mathcal{E}^\varepsilon F_{T_i}(x_{u^*})\right. \\ &\quad \left.- \int \prod_{\{u,v\} \in E_{\bar{T}}} C(x_u - x_v) \prod_{v \in V_{\bar{T}} \setminus \{u^*\}} (\mathcal{E}^\varepsilon f_v)(x_v) (\mathcal{E}^\varepsilon g_v^{\bar{T}})(x_v) \prod_{v \in V_{\bar{T}} \setminus \{u^*\}} \mathrm{d}x_v\right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \\ &\quad \lesssim o(\varepsilon). \end{aligned}$$

We do induction on $|V_{\bar{T}}|$ and we start from the subtree $\bar{T}$ containing the root u^* with $|V_{\bar{T}}| = 2$ and by Step 2 in the proof of Lemma 4.2, we have

$$F_{T_i}(x_{u^*}) = \int C_\varepsilon(x_{u^*}, x_v) f_v(x_v) g_v^{\bar{T}}(x_v) \mathrm{d}x_v,$$

with $f_v \in \bigcup_{i=0}^3 D_i$ and $g_v^{\bar{T}}$ as in the proof of Lemma 4.2. By Lemma A.10 and uniform bounds of $\mathcal{E}^\varepsilon F_{T_i}$ we know that for $s > \gamma > 0$ and some $n > 0$

(4.16)

$$\mathbf{E}\left\|\mathcal{E}^\varepsilon F_{T_i}(x_{u^*}) - \int C(x_{u^*} - x_v) \mathcal{E}^\varepsilon(f_v(x_v) g_v^{\bar{T}}(x_v)) \mathrm{d}x_v\right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \lesssim o(\varepsilon).$$

Using Lemma A.9 we find

$$\mathbf{E}\left\|\mathcal{E}^\varepsilon F_{T_i}(x_{u^*}) - \int C(x_{u^*} - x_v) (\mathcal{E}^\varepsilon f_v)(x_v) (\mathcal{E}^\varepsilon g_v^{\bar{T}})(x_v) \mathrm{d}x_v\right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \lesssim o(\varepsilon).$$

Hence (4.15) holds with $\bar{T}$ satisfying $|V_{\bar{T}}| = 2$. Suppose that (4.15) holds with $\bar{T}$ satisfying $|V_{\bar{T}}| \leq n$. Now we consider the subgraph $\bar{T}$ with $|V_{\bar{T}}| = n + 1$. By the Step 2 in the proof of Lemma 4.2 we could find a $\bar{v}$ as a leaf of $\bar{T}$ and a unique $\bar{u} \in V_{\bar{T}}$ such that (4.11) holds. We set $g_{\bar{u}}^{T'}(x_{\bar{u}})$ as in (4.12) and use induction for $T' = \bar{T} \setminus \{\bar{v}\}$ to have

$$\begin{aligned} & \mathbf{E} \left\| \mathcal{E}^\varepsilon F_{T_i}(x_{u^*}) \right. \\ & \quad \left. - \int \prod_{\{u,v\} \in E_{\bar{T}} \setminus \{\bar{v}\}} C(x_u - x_v) \prod_{v \in V_{\bar{T}} \setminus \{u^*, \bar{v}\}} (\mathcal{E}^\varepsilon f_v)(x_v) (\mathcal{E}^\varepsilon g_{\bar{v}}^{T'})(x_{\bar{v}}) \prod_{v \in V_{\bar{T}} \setminus \{u^*, \bar{v}\}} dx_v \right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \\ & \lesssim o(\varepsilon). \end{aligned}$$

For $\mathcal{E}^\varepsilon g_{\bar{u}}^{T'}(x_{\bar{u}})$ we use Lemma A.10 and Lemma A.9 to have

$$\mathbf{E} \left\| \mathcal{E}^\varepsilon g_{\bar{u}}^{T'}(x_{\bar{u}}) - \mathcal{E}^\varepsilon g_{\bar{u}}^{\bar{T}}(x_{\bar{u}}) \int C(x_{\bar{u}} - x_{\bar{v}}) (\mathcal{E}^\varepsilon f_{\bar{v}})(x_{\bar{v}}) (\mathcal{E}^\varepsilon g_{\bar{v}}^{\bar{T}})(x_{\bar{v}}) dx_{\bar{v}} \right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \lesssim o(\varepsilon).$$

Hence (4.15) holds for $\bar{T}$ satisfying $|V_{\bar{T}}| = n + 1$ and for $\bar{T} = T_i$, which combined with the convergence of $\mathcal{E}^\varepsilon f_v$ implies that

$$\begin{aligned} (4.17) \quad & \mathbf{E} \left\| \mathcal{E}^\varepsilon F_{T_i}(x_{u^*}) - \int \prod_{\{u,v\} \in E_{T_i}} C(x_u - x_v) \prod_{v \in V_{T_i} \setminus \{u^*\}} \bar{f}_v(x_v) \prod_{v \in V_{T_i} \setminus \{u^*\}} dx_v \right\|_{\mathbf{C}^{2-s}(\rho^\ell)}^k \\ & \lesssim o(\varepsilon), \end{aligned}$$

with $\bar{f}_v = \lim_{\varepsilon \rightarrow 0} \mathcal{E}^\varepsilon f_v$. Here for the second term we abuse the notation $\bar{f}_v(x_v)$ which means a distribution in $\mathbf{C}^{-\gamma}(\rho^\ell)$ and the integral of $C(x_u - x_v)$ w.r.t. x_v means the operator $\frac{1}{2}(m - \Delta)^{-1}$ acts on the corresponding distributions $\bar{f}_v$. Now we obtain the convergence of each F_{T_i} and give the explicit formula for the limit F_n^k and $R_n^{v,k}$.

We use G to denote the graph associated with F_n^k . To write F_n^k as the integrals of C , that is,

$$\int \left(\prod_{\{u,v\} \in E_G} C(x_u - x_v) \right) \prod_{v \in V_G^0} dx_v,$$

which we denote by $\tilde{F}_n^k$, we only need to replace all the correlation of $Z^{(i)}$ in F_n^k by C . However, since $Z^{(i)}$ is not a usual function, we need more argument to explain this replacement and it suffices to prove $\langle F_n^k, \varphi \rangle = \langle \tilde{F}_n^k, \varphi \rangle$ with $0 \leq \varphi \in \mathcal{S}(\mathbb{R}^{2k})$. As $Z^{(i)}$ is not a function, we first replace each $Z^{(i)}$ in F_n^k by $Z^{(i)} * \varphi_\varepsilon$ with $\{\varphi_\varepsilon\}_\varepsilon$ being standard mollifiers and we denote the resulting function by $F_n^{k,\varepsilon}$. This means we replace each $\bar{f}_v$ in the second term of (4.17) by f_v^ε , which converges to $\bar{f}_v$ in $\mathbf{C}^{-\gamma}(\rho^\ell)$. To compare F_n^k with $F_n^{k,\varepsilon}$ we are in a similar situation as the convergence of $F_{n,M,\varepsilon}^k$ to F_n^k with the sum on Λ_ε replaced by the integral on $\mathbb{R}^2$. Then the same argument as reduction to trees and induction on subtrees implies that

$$|\langle F_n^{k,\varepsilon} - F_n^k, \varphi \rangle| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

For $F_n^{k,\varepsilon}$ we could replace each $\mathbf{E}[(Z^{(i)} * \varphi_\varepsilon)(x)(Z^{(i)} * \varphi_\varepsilon)(y)]$ by $\tilde{C}_\varepsilon(x - y) \stackrel{\text{def}}{=} (\varphi_\varepsilon * C * \varphi_\varepsilon)(x - y)$ by Fubini's theorem. Now we check that $F_n^{k,\varepsilon}$ converge to $\tilde{F}_n^k$. For each graph G associated with F_n^k we abuse the notation m to denote the Lebesgue measure $\prod_{v \in V_G} dx_v$. We then choose a finite reference measure μ which is absolutely continuous w.r.t. m and has the C associated with the red lines and φ as the density. Then it suffices to check that all the green lines (means $\tilde{C}_\varepsilon$) are uniform integrable under this finite measure μ since $\tilde{C}_\varepsilon$ converges to C

in measure. To obtain uniform integrability we prove that the integral of the square of all the green lines w.r.t. μ is uniformly bounded, which follows by the same argument as what we proved in Lemma 4.2 since adding extra $Z^{(i)}$ to f_v at each point will not influence our proof. Thus

$$|\langle F_n^{k,\varepsilon} - \tilde{F}_n^k, \varphi \rangle| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

which implies $F_n^k = \tilde{F}_n^k$. Now the result follows. $\square$

We remark that one could also use [26], Theorem A.3, to prove the convergence of $F_n^{k,\varepsilon}$ to $\tilde{F}_n^k$, though it would require some adaptation since the setting is slightly different, for example, C would need to be decomposed as a compact supported function plus a smooth remainder.

5. Short distance behavior of correlations. In this section we use integration by parts formula and similar estimates to derive the short distance behavior of the 2-point correlation function and 4-point correlation function.

To compare $S(x_0 - y_0) - C(x_0 - y_0)$ we use integration by parts for the last term of (3.9). Choosing $F(\Phi_{M,\varepsilon}) = \lambda \mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(x)$ in (3.2) we obtain

$$\begin{aligned} & 3\lambda \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x - z)^2 \mathbf{E}(\Phi_{M,\varepsilon}(z)^2) \, dz \\ &= \lambda \mathbf{E}(\Phi_{M,\varepsilon}(x) \mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(x)) + \lambda^2 \mathbf{E}((\mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(x))^2). \end{aligned}$$

Substitute this into (3.9) we obtain

$$\begin{aligned} & S_{\lambda,M,\varepsilon}^2(x_0 - y_0) \\ &= C_{M,\varepsilon}(x_0 - y_0) + \lambda^2 \mathbf{E}(\mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(x_0) \mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(y_0)) \\ (5.1) \quad & + 9\lambda^2 \int_{\Lambda_\varepsilon^2} C_{M,\varepsilon}(y_0 - z) C_{M,\varepsilon}(x_0 - z) C_{M,\varepsilon}(z - z_1)^2 \mathbf{E}(\Phi_{M,\varepsilon}(z_1)^2) \, dz_1 \, dz \\ & - 3\lambda^3 \int_{\Lambda_\varepsilon} C_{M,\varepsilon}(y_0 - z) C_{M,\varepsilon}(x_0 - z) \mathbf{E}((\mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(z))^2) \, dz. \end{aligned}$$

In the following we give the proof of Theorem 1.3. In the proof we view all the functions on $\Lambda_{M,\varepsilon}$ as periodic functions on Λ_ε and we will use Lemma A.2 to bound each term.

PROOF OF THEOREM 1.3. In the proof we choose $p_0 > 18$ for the weight ρ in (2.20) and use ρ^ℓ to denote the weight and ℓ may change from line to line. We also abuse the notation of ρ for different spatial dimensions. We write the last three terms in (5.1) as $\sum_{i=1}^3 I_i$, which are graphically

$$(5.2) \quad \lambda^2 \begin{array}{c} x_0 \bullet \\ \text{blue arc} \\ \text{red arc} \end{array} + 9\lambda^2 \begin{array}{c} x_0 \bullet \quad y_0 \bullet \\ \text{blue arcs} \\ \text{red arc} \end{array} - 3\lambda^3 \begin{array}{c} x_0 \bullet \quad y_0 \bullet \\ \text{blue arcs} \\ \text{red arcs} \end{array}$$

It suffices to calculate the $\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)$ -norm for each graph and $\gamma > 0$. By Corollary 2.3 and (A.7) we know for $\gamma > 0$ and some $\ell > 0$

$$\|I_1\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^2,\rho^\ell)} \lesssim \lambda^2 \mathbf{E} \|\Phi_{M,\varepsilon}^3\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\ell)}^2 \lesssim \lambda^2 + \lambda^{26}.$$

Similarly, by translation invariance,

$$J(z) \stackrel{\text{def}}{=} \mathbf{E}(\mathcal{I}_\varepsilon(\Phi_{M,\varepsilon}^3)(z)^2)$$

is a constant independent of z , and we can write I_3 as $-3\lambda^3(C_{M,\varepsilon} * C_{M,\varepsilon})(x_0 - y_0)J$. By Corollary 2.3 and (A.7) we know for $\gamma > 0$ and some $\ell > 0$

$$\|J\|_{L^{\infty,\varepsilon}(\rho^n)} \lesssim 1 + \lambda^{24}.$$

Moreover, by (A.8), $C_{M,\varepsilon} * C_{M,\varepsilon}(x_0 - y_0) \in \mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^2)$, which implies

$$\|I_3\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^2, \rho^n)} \lesssim \lambda^3 + \lambda^{27}.$$

For I_2 we set

$$J_1 \stackrel{\text{def}}{=} \int C_{M,\varepsilon}(z - z_2)^2 \mathbf{E}(\cdot; \Phi_{M,\varepsilon}(z_2)^2) dz_2,$$

which is also a constant by translation invariance. Then we could write I_2 as $9\lambda^2(C_{M,\varepsilon} * C_{M,\varepsilon})(x_0 - y_0)J_1$ and by (A.8) and Corollary 2.3 we have for $\gamma > 0$

$$|J_1| \lesssim \|J_1\|_{L^{\infty,\varepsilon}(\rho^\ell)} \lesssim \mathbf{E}\|\Phi_{M,\varepsilon}^2\|_{\mathbf{C}^{-\gamma,\varepsilon}(\rho^\ell)} \lesssim 1 + \lambda^{12}.$$

Using $(C_{M,\varepsilon} * C_{M,\varepsilon})(x_0 - y_0) \in \mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^2)$ we deduce

$$\|I_2\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^2, \rho^\ell)} \lesssim \lambda^2 + \lambda^{14}.$$

Now the first result follows from Lemma A.1 and the fact that we can view $S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon}$ as a function of one variable by translation invariance.

For the second result, note that by Lemma A.8 $\|\mathcal{E}^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon})\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^4, \rho^\ell)}$ is uniformly bounded w.r.t. ε, M with $\mathcal{E}^\varepsilon$ defined for functions on Λ_ε^2 and there exists a subsequence converges, which is still denoted by ε, M . We would like to prove the limit of $\mathcal{E}^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon})$ given by $S_\lambda^{\nu,2} - C$. By Lemma 3.8 we know the limit of $\mathcal{E}_2^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon})$ given by $S_\lambda^{\nu,2} - C$. Moreover, by (A.6) we know for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ with compact support Fourier transform

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \langle \mathcal{E}^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon}), \varphi \rangle \\ &= \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \langle S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon}, \varphi \rangle_\varepsilon \\ &= \lim_{\varepsilon \rightarrow 0, M \rightarrow \infty} \langle \mathcal{E}_2^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon}), \varphi \rangle = \langle S_\lambda^{\nu,2} - C, \varphi \rangle, \end{aligned}$$

which by lower semicontinuity of the norm implies

$$\|S_\lambda^{\nu,2} - C\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^4, \rho^\ell)} \lesssim \lambda^2 + \lambda^{27}.$$

Since

$$\mathcal{E}_2^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon})(x, y) = \mathcal{E}_2^\varepsilon(S_{\lambda,M,\varepsilon}^2 - C_{M,\varepsilon})(x - y, 0),$$

the limit $S_\lambda^{\nu,2} - C$ also satisfies the same property and can be viewed as a function on $\mathbb{R}^2$. Hence,

$$\|S_\lambda^{\nu,2} - C\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^2, \rho^\ell)} \lesssim \lambda^2 + \lambda^{27}. \quad \square$$

In the following we consider the connected 4-point function and we give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. In the proof we omit the subscript M, ε and $\Lambda_{M,\varepsilon}$ for notational simplicity. We also write $S(x, y) = S_{\lambda,M,\varepsilon}^2(x, y)$. We choose $p_0 > 36$ for the weight

ρ in (2.20) and will use ρ^ℓ to denote the weight and ℓ may change from line to line. We introduce the following notation:

$$\mathcal{I}f(x) \stackrel{\text{def}}{=} \int C(x-y)f(y)dy, \quad \mathcal{J}f(x, x_1) \stackrel{\text{def}}{=} \int C(x-z)C(x_1-z)f(z)dz,$$

and

$$\mathcal{J}_1f(x, x_1, x_2) \stackrel{\text{def}}{=} \int C(x-z)C(x_1-z)C(x_2-z)f(z)dz.$$

The idea of the proof is to use the integration by parts formula for $S_\lambda^4(x_1, x_2, x_3, x_4)$ and we will find some $C(x_i - x_j)$, $i, j \in \{1, 2, 3, 4\}$ appear. However, C is not in the Hölder space $\mathbf{C}^{2-\gamma, \varepsilon}(\rho^\ell)$, $\gamma > 0$. By (3.6) and Theorem 1.3 we find

$$(5.3) \quad S(x_i - x_j) = C(x_i - x_j) - R(x_i - x_j),$$

with $R \in \mathbf{C}^{2-\gamma, \varepsilon}(\Lambda_\varepsilon^2, \rho^\ell)$. Here by (3.6)

$$(5.4) \quad R(x_i - x_j) = \lambda \mathbf{E}(\Phi(x_j)\mathcal{I}(\Phi^3:)(x_i)).$$

Hence, we use $C(x_i - x_j)$ from $S(x_i - x_j)$ to cancel with the corresponding $C(x_i - x_j)$ in the decomposition of S_λ^4 .

To be more precise, for $S_\lambda^4 = \mathbf{E}[\prod_{i=1}^4 \Phi(x_i)]$ we use integration by parts formula (3.11) with $k = 4$ to have

$$\begin{aligned} S_\lambda^4(x_1, x_2, x_3, x_4) &= C(x_1 - x_2)S(x_3 - x_4) + C(x_1 - x_3)S(x_2 - x_4) + C(x_1 - x_4)S(x_2 - x_3) \\ &\quad - \lambda \mathbf{E}(\Phi(x_2)\Phi(x_3)\Phi(x_4)\mathcal{I}(\Phi^3:)(x_1)). \end{aligned}$$

By (5.3) this implies that U^4 can be written as

$$(5.5) \quad R(x_1 - x_2)C(x_3 - x_4) + R(x_1 - x_3)C(x_2 - x_4) + R(x_1 - x_4)C(x_2 - x_3)$$

$$(5.6) \quad - \lambda \mathbf{E}(\Phi(x_2)\Phi(x_3)\Phi(x_4)\mathcal{I}(\Phi^3:)(x_1)) - \bar{U}$$

and by Lemma A.1

$$\begin{aligned} \bar{U}(x_1, x_2, x_3, x_4) &= R(x_1 - x_2)R(x_3 - x_4) + R(x_1 - x_3)R(x_2 - x_4) + R(x_1 - x_4)R(x_2 - x_3) \\ &\in \mathbf{C}^{2-\gamma, \varepsilon}(\Lambda_\varepsilon^4, \rho^\ell). \end{aligned}$$

By Theorem 1.3 we have

$$\|\bar{U}\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)} \lesssim \lambda^4 + \lambda^{54}.$$

Since by (A.7) $\mathcal{I}$ has smoothing effect, we use IBP for the first term in (5.6) with the isolated point x_4 to have two terms, which by (5.4) cancel with the first two terms in (5.5). As a result, we obtain that U^4 can be written as

$$(5.7) \quad R(x_1 - x_4)C(x_2 - x_3) + \lambda^2 \mathbf{E}(\Phi(x_2)\Phi(x_3)\mathcal{I}(\Phi^3:)(x_1)\mathcal{I}(\Phi^3:)(x_4))$$

$$(5.8) \quad - 3\lambda \mathbf{E}(\Phi(x_2)\Phi(x_3)\mathcal{J}(\Phi^2:)(x_1, x_4)) - \bar{U}.$$

We further apply IBP for each term in (5.7)–(5.8). For $R(x_1 - x_4)C(x_2 - x_3)$ in (5.7) we use integration by parts (3.7) for R with x_4 to have

$$\begin{aligned} (5.9) \quad R(x_1 - x_4)C(x_2 - x_3) &= 3C(x_2 - x_3)\lambda \mathbf{E}(\mathcal{J}(\Phi^2:)(x_1, x_4)) \\ &\quad - \lambda^2 C(x_2 - x_3)\mathbf{E}(\mathcal{I}(\Phi^3:)(x_1)\mathcal{I}(\Phi^3:)(x_4)), \end{aligned}$$

where the second line cancels with one term obtained from the second term in (5.7) by IBP. In fact using IBP for x_2 the second term in (5.7) could be written as

$$(5.10) \quad \lambda^2 C(x_2 - x_3) \mathbf{E}(\mathcal{I}(\Phi^3)(x_1) \mathcal{I}(\Phi^3)(x_4)) \\ + 3\lambda^2 \mathbf{E}(\Phi(x_3) \mathcal{J}(\Phi^2)(x_2, x_4) \mathcal{I}(\Phi^3)(x_1))$$

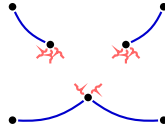
$$(5.11) \quad + 3\lambda^2 \mathbf{E}(\Phi(x_3) \mathcal{J}(\Phi^2)(x_1, x_2) \mathcal{I}(\Phi^3)(x_4))$$

$$(5.12) \quad - \lambda^3 \mathbf{E}(\Phi(x_3) \mathcal{I}(\Phi^3)(x_2) \mathcal{I}(\Phi^3)(x_1) \mathcal{I}(\Phi^3)(x_4)).$$

As mentioned above the first term cancels with the second term for $R(x_1 - x_4)C(x_2 - x_3)$. By using IBP for the isolated point x_3 again, the term in (5.12) could be written as

$$(5.13) \quad - 3\lambda^3 \mathbf{E}(\mathcal{J}(\Phi^2)(x_2, x_3) \mathcal{I}(\Phi^3)(x_1) \mathcal{I}(\Phi^3)(x_4)) \\ - 3\lambda^3 \mathbf{E}(\mathcal{I}(\Phi^3)(x_2) \mathcal{J}(\Phi^2)(x_1, x_3) \mathcal{I}(\Phi^3)(x_4)) \\ - 3\lambda^3 \mathbf{E}(\mathcal{I}(\Phi^3)(x_2) \mathcal{I}(\Phi^3)(x_1) \mathcal{J}(\Phi^2)(x_3, x_4)),$$

which share the following same graph:



and one term like $\lambda^4 \mathbf{E}(\prod_{i=1}^4 \mathcal{I}(\Phi^3)(x_i))$. By (A.7) and Corollary 2.3 we know for $\gamma > 0$ and $9p < p_0$

$$(5.14) \quad \mathbf{E} \|\mathcal{I}(\Phi^3)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^\ell)}^p \lesssim \mathbf{E} \|\Phi^3\|_{\mathbf{C}^{-\gamma, \varepsilon}(\rho^\ell)}^p \lesssim 1 + \lambda^{12p},$$

and

$$(5.15) \quad \mathbf{E} \|\mathcal{J}(\Phi^2)\|_{\mathbf{C}^{2-\gamma, \varepsilon}(\rho^\ell)}^p \lesssim \mathbf{E} \|\Phi^2\|_{\mathbf{C}^{-\gamma/2, \varepsilon}(\rho^\ell)}^p \lesssim 1 + \lambda^{12p},$$

which by Lemma A.1 implies that the $\mathbf{C}^{2-\gamma, \varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)$ -norm for the term in (5.12) is uniformly bounded by $\lambda^3 + \lambda^{52}$ up to a multiplicative constant. Here we abuse the notation of ρ for different spatial dimensions.

Now we consider the first term in (5.8) and we apply IBP for the isolated point x_2 to write it as

$$(5.16) \quad - 3\lambda C(x_2 - x_3) \mathbf{E}(\mathcal{J}(\Phi^2)(x_1, x_4)) - 6\lambda \mathbf{E}(\Phi(x_3) \mathcal{J}_1(\Phi)(x_1, x_2, x_4)) \\ + 3\lambda^2 \mathbf{E}(\Phi(x_3) \mathcal{I}(\Phi^3)(x_2) \mathcal{J}(\Phi^2)(x_1, x_4)).$$

We find that the first term cancels with the first term on the RHS of (5.9). We apply IBP for the second term for the isolated point x_3 and obtain

$$(5.17) \quad - 6\lambda \int C(x_1 - z) C(x_4 - z) C(x_2 - z) C(x_3 - z) dz$$

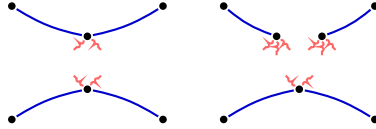
$$(5.18) \quad + 6\lambda^2 \mathbf{E}(\mathcal{I}(\Phi^3)(x_3) \mathcal{J}_1(\Phi)(x_1, x_2, x_4)).$$

We apply IBP for the terms in (5.10), (5.11) and (5.16) for x_3 , which have the same graph, and we obtain three terms (see (5.20) below) similar to (5.18) and the following terms:

$$9\lambda^2 \mathbf{E}(\mathcal{J}(\Phi^2)(x_2, x_4) \mathcal{J}(\Phi^2)(x_1, x_3)) - 3\lambda^3 \mathbf{E}(\mathcal{J}(\Phi^2)(x_2, x_4) \mathcal{I}(\Phi^3)(x_1) \mathcal{I}(\Phi^3)(x_3)) \\ + 9\lambda^2 \mathbf{E}(\mathcal{J}(\Phi^2)(x_1, x_2) \mathcal{J}(\Phi^2)(x_3, x_4))$$

$$\begin{aligned} & -3\lambda^3 \mathbf{E}(\mathcal{J}(\cdot\!:\!\Phi^2\!:\!)(x_1, x_2)\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_4)\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_3)) \\ & +9\lambda^2 \mathbf{E}(\mathcal{J}(\cdot\!:\!\Phi^2\!:\!)(x_1, x_4)\mathcal{J}(\cdot\!:\!\Phi^2\!:\!)(x_2, x_3)) \\ & -3\lambda^3 \mathbf{E}(\mathcal{J}(\cdot\!:\!\Phi^2\!:\!)(x_1, x_4)\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_2)\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_3)), \end{aligned}$$

where they have the following two graphs:



By (5.14) and (5.15) $\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)$ -norm of all the above terms are uniformly bounded by $\lambda^2 + \lambda^{39}$ up to a multiplicative constant.

Now the only term in U^4 not in $\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)$ is given by the following terms in (5.17) and (5.18) and similar terms as (5.18):

$$(5.19) \quad -6\lambda \int C(x_1 - z)C(x_4 - z)C(x_2 - z)C(x_3 - z) \, \mathrm{d}z$$

$$\begin{aligned} (5.20) \quad & +6\lambda^2 \mathbf{E}(\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_3)\mathcal{J}_1(\Phi)(x_1, x_2, x_4)) + 6\lambda^2 \mathbf{E}(\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_1)\mathcal{J}_1(\Phi)(x_2, x_3, x_4)) \\ & +6\lambda^2 \mathbf{E}(\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_2)\mathcal{J}_1(\Phi)(x_1, x_3, x_4)) + 6\lambda^2 \mathbf{E}(\mathcal{I}(\cdot\!:\!\Phi^3\!:\!)(x_4)\mathcal{J}_1(\Phi)(x_1, x_2, x_3)). \end{aligned}$$

The term in (5.19) belongs to $\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)$ by (A.9) and Lemma A.1 and could be bounded by λ up to a multiplicative constant. The last four terms have the same graph. We only consider the term in (5.20) and the other terms could be estimated similarly. By (A.9) and Corollary 2.3 we know for $\gamma > 0$ and $3p < p_0$

$$\mathbf{E}\|\mathcal{J}_1(\Phi)\|_{\mathbf{C}^{2-\gamma,\varepsilon}(\rho^\ell)}^p \lesssim \mathbf{E}\|\Phi\|_{\mathbf{C}^{-\gamma/2,\varepsilon}(\rho^\ell)}^p \lesssim 1 + \lambda^{4p},$$

which combined with (5.14) and Lemma A.1 implies that the $\mathbf{C}^{2-\gamma,\varepsilon}(\Lambda_\varepsilon^4, \rho^\ell)$ -norm of the terms in (5.20) are uniformly bounded. Now the first result follows.

Using Lemma A.8 we obtain $\|\mathcal{E}^\varepsilon U_{M,\varepsilon}^4\|_{\mathbf{C}^{2-\gamma}(\mathbb{R}^8, \rho^\ell)}$ is uniformly bounded w.r.t. ε, M with $\mathcal{E}^\varepsilon$ defined for functions on Λ_ε^4 . We define the limit of a subsequence as $U_\lambda^{\nu,4}$, which is also the limit of $\mathcal{E}_4^\varepsilon U_{M,\varepsilon}^4$ by similar argument as in the proof of Theorem 1.3. On the other hand, by (A.9) and Lemmas A.1, A.8,

$$\mathcal{E}^\varepsilon \mathcal{J}_2(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} \mathcal{E}^\varepsilon \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x_1 - z)C_{M,\varepsilon}(x_4 - z)C_{M,\varepsilon}(x_2 - z)C_{M,\varepsilon}(x_3 - z) \, \mathrm{d}z$$

is uniform bounded in $\mathbf{C}^{2-\gamma}(\rho^\ell)$, which implies that there exists a subsequence, still denoted by ε, M , converges. Then by (A.6) for $\varphi(x_1, x_2, x_3, x_4) = \prod_{i=1}^4 \varphi_i(x_i)$ and $\varphi_i \in \mathcal{S}(\mathbb{R}^2)$ with Fourier transform compact support, it holds

$$\lim_{M \rightarrow \infty, \varepsilon \rightarrow 0} \langle \mathcal{E}^\varepsilon J_2, \varphi \rangle = \lim_{M \rightarrow \infty, \varepsilon \rightarrow 0} \langle \mathcal{E}_4^\varepsilon J_2, \varphi \rangle.$$

Then by a similar argument as in the proof in Section 4.2 for $\mathcal{E}_4^\varepsilon J_2$ we replace $C_{M,\varepsilon}(x_i - z)$ by $\mathbf{E}(Z_{M,\varepsilon}^{(i)}(x_i)Z_{M,\varepsilon}^{(i)}(z))$ for $i = 2, 3, 4$ and apply Lemmas A.9 and A.10 to obtain that $\mathcal{E}_4^\varepsilon \mathcal{J}_2$ converges to $\int C(x_1 - z)C(x_4 - z)C(x_2 - z)C(x_3 - z) \, \mathrm{d}z$ when testing positive φ . The lower-semicontinuity of the norm gives the result. $\square$

APPENDIX: DISCRETE BESOV SPACES

In this section we introduce Besov spaces on the lattice $\Lambda_\varepsilon = \varepsilon \mathbb{Z}^d$ where $\varepsilon = 2^{-N}$, $N \in \mathbb{N} \cup \{0\}$, from [23, 31] and Besov spaces on $\mathbb{R}^d$ from [38]. For $f \in \ell^1(\Lambda_\varepsilon)$ and $g \in L^1(\varepsilon^{-1}\mathbb{T}^d)$ we define the Fourier and the inverse Fourier transform as

$$\mathcal{F}f(\xi) = \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) e^{-2\pi i \xi \cdot x}, \quad \mathcal{F}^{-1}g(x) = \int_{\varepsilon^{-1}\mathbb{T}^d} g(\xi) e^{2\pi i \xi \cdot x} d\xi,$$

for $\xi \in \varepsilon^{-1}\mathbb{T}^d$, $x \in \Lambda_\varepsilon$. We use $\varepsilon = 0$ to denote the continuous setting with $\mathcal{F}_{\mathbb{R}^d}$ and $\mathcal{F}_{\mathbb{R}^d}^{-1}$ being the usual Fourier transform and its inverse on $\mathbb{R}^d$. Let $(\varphi_j)_{j \geq -1}$ be a dyadic partition of unity on $\mathbb{R}^d$. We define the dyadic partition of unity for $x \in \varepsilon^{-1}\mathbb{T}^d$:

$$(A.1) \quad \varphi_j^\varepsilon(x) = \begin{cases} \varphi_j(x), & j < j_\varepsilon, \\ 1 - \sum_{j < j_\varepsilon} \varphi_j(x), & j = j_\varepsilon. \end{cases}$$

Here $j_\varepsilon := \inf\{j : \text{supp } \varphi_j \cap \partial(\varepsilon^{-1}\mathbb{T}^d) \neq \emptyset\}$.

Now we define the Littlewood–Paley blocks for distributions on Λ_ε by

$$\Delta_j^\varepsilon f := \mathcal{F}^{-1}(\varphi_j^\varepsilon \mathcal{F}f),$$

which leads to the definition of weighted Besov spaces. Let ρ denote a polynomial weight of the form $\rho(x) = (1 + |x|^2)^{-\delta/2}$ for some $\delta \geq 0$. For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\varepsilon \in [0, 1]$ we define the weighted Besov spaces on Λ_ε given by the norm

$$\|f\|_{B_{p,q}^{\alpha,\varepsilon}(\Lambda_\varepsilon,\rho)} = \left(\sum_{-1 \leq j \leq j_\varepsilon} 2^{\alpha j q} \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)}^q \right)^{1/q} < \infty.$$

If $\varepsilon = 0$, $B_{p,q}^{\alpha,\varepsilon}(\rho)$ is the classical Besov space $B_{p,q}^\alpha(\rho)$ on $\mathbb{R}^d$. Similarly, we extend the definition of Besov space to functions on $\Lambda_\varepsilon^k = \varepsilon \mathbb{Z}^{dk}$, $k \in \mathbb{N}$, which is denoted by $B_{p,q}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)$. Whenever there is no confusion we also write $B_{p,q}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)$ as $B_{p,q}^{\alpha,\varepsilon}(\rho)$ for simplicity. We also set $\mathbf{C}^{\alpha,\varepsilon}(\rho) \stackrel{\text{def}}{=} B_{\infty,\infty}^{\alpha,\varepsilon}(\rho)$ and $H^{\alpha,\varepsilon}(\rho) \stackrel{\text{def}}{=} B_{2,2}^{\alpha,\varepsilon}(\rho)$. We also define the Besov–Hölder space w.r.t. each component in Λ_ε^k for $k \in \mathbb{N}$: for $f : \Lambda_\varepsilon^k \rightarrow \mathbb{R}$, $\alpha \in \mathbb{R}$,

$$(A.2) \quad \|f\|_{\mathbf{C}_s^{\alpha,\varepsilon}(\Lambda_\varepsilon^k,\rho)} \stackrel{\text{def}}{=} \sup_{-1 \leq j_i \leq j_\varepsilon} 2^{\alpha \sum_{i=1}^k j_i} \left\| \left(\prod_{i=1}^k \Delta_{j_i, x_i}^\varepsilon \right) f \right\|_{L^{\infty,\varepsilon}(\rho)} < \infty.$$

Here we write $x = (x_1, \dots, x_k) \in \Lambda_\varepsilon^k$ and $\Delta_{j_i, x_i}^\varepsilon$ means the Littlewood–Paley blocks for the i th component $x_i \in \Lambda_\varepsilon$. If $\varepsilon = 0$, $\mathbf{C}_s^\alpha(\mathbb{R}^{kd}, \rho)$ is the Besov space w.r.t. each component in $\mathbb{R}^d$. The duality on Λ_ε is given by

$$\langle f, g \rangle_\varepsilon \stackrel{\text{def}}{=} \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) g(x).$$

We also set

$$(f *_\varepsilon g)(x) \stackrel{\text{def}}{=} \varepsilon^d \sum_{y \in \Lambda_\varepsilon} f(x - y) g(y).$$

For the polynomial weight ρ , by [23], Lemma A.1, it holds for $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ that

$$(A.3) \quad \|f\|_{B_{p,q}^{\alpha,\varepsilon}(\rho)} \asymp \|f\rho\|_{B_{p,q}^{\alpha,\varepsilon}},$$

where the implicit constant is independent of ε .

For a function f on $\varepsilon^{-1}\mathbb{T}^d$ we use f_{ext} to denote its periodic extension to $\mathbb{R}^d$.

Extension operator:

We follow [31] to introduce the following extension operator. Recall φ_j^ε given in (A.1). We choose a symmetric function $\psi \in C_c(\mathbb{R}^d)$ satisfying the following property:

- $\sum_{k \in \mathbb{Z}^d} \psi(\cdot - k) = 1$,
- $\psi = 1$ on $\text{supp } \varphi_j$ for $j < j_1$,
- $(\text{supp } \psi \cap \text{supp } (\varphi_{j_1}^1)_{\text{ext}}) \setminus \mathbb{T}^d \neq \emptyset \Rightarrow j = j_1$.

Here $(\varphi_j^1)_{\text{ext}}$ means periodic extension to $\mathbb{R}^d$ of function φ_j^1 on $\mathbb{T}^d$. More precisely, in [31] such ψ is called a smearing function. It is easy to see that $\psi(\varepsilon \cdot)$ satisfies the same property on Λ_ε . Set $w^\varepsilon = \mathcal{F}_{\mathbb{R}^d}^{-1} \psi(\varepsilon \cdot)$ and define

$$(A.4) \quad \mathcal{E}^\varepsilon f(x) \stackrel{\text{def}}{=} \varepsilon^d \sum_{y \in \Lambda_\varepsilon} w^\varepsilon(x - y) f(y), \quad f \in B_{p,q}^{\alpha,\varepsilon}(\rho).$$

We also introduce the following extension operators for functions $f : \Lambda_\varepsilon^k \rightarrow \mathbb{R}$:

$$(A.5) \quad \mathcal{E}_k^\varepsilon f(x_1, \dots, x_k) \stackrel{\text{def}}{=} \varepsilon^{dk} \sum_{\substack{y_i \in \Lambda_\varepsilon, \\ i=1, \dots, k}} \left(\prod_{i=1}^k w^\varepsilon(x_i - y_i) \right) f(y_1, \dots, y_k).$$

It is easy to see that

$$(A.6) \quad \langle \mathcal{E}^\varepsilon f, g \rangle = \langle f, w^\varepsilon * g \rangle_\varepsilon, \quad f, g \in L^{2,\varepsilon}(\Lambda_\varepsilon).$$

We have the following elementary result for the Besov–Hölder space.

LEMMA A.1. For $\varepsilon \in [0, 1)$, if $\alpha < 0$, then $\mathbf{C}_s^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho) \subset \mathbf{C}^{k\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)$; if $\alpha > 0$, then

$$\|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)} \asymp \sum_{i=1}^k \|f\|_{\mathbf{C}_{x_i}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)}.$$

Here

$$\|f\|_{\mathbf{C}_{x_i}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)} := \sup_{x_j \in \Lambda_\varepsilon, j \neq i} \|f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_k)\|_{\mathbf{C}^{\alpha,\varepsilon}(\Lambda_\varepsilon, \rho(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_k))}.$$

PROOF. Let Δ_j^ε and $\Delta_{j_i, x_i}^\varepsilon$ the Littlewood–Paley blocks in Λ_ε^k and Λ_ε , respectively. By Littlewood–Paley decomposition we have

$$\begin{aligned} 2^{\alpha k j} \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}(\rho)} &\lesssim \sum_{\substack{-1 \leq j_i \leq j_\varepsilon, \\ i=1, \dots, k}} 2^{\alpha k j} \left\| \Delta_j^\varepsilon \left(\prod_{i=1}^k \Delta_{j_i, x_i}^\varepsilon \right) f \right\|_{L^{\infty,\varepsilon}(\rho)} \\ &\lesssim 2^{\alpha k j} \sum_{j_i \lesssim j} \left\| \left(\prod_{i=1}^k \Delta_{j_i, x_i}^\varepsilon \right) f \right\|_{L^{\infty,\varepsilon}(\rho)} \\ &\lesssim 2^{\alpha k j} \sum_{j_i \lesssim j} 2^{-\alpha \sum_{i=1}^k j_i} \|f\|_{\mathbf{C}_s^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)} \lesssim \|f\|_{\mathbf{C}_s^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)}, \end{aligned}$$

where in the second inequality we use $\varphi_j(\xi_1, \dots, \xi_k) \prod_{i=1}^k \varphi_{j_i}(\xi_i) \neq 0$ only if $j_i \lesssim j$. Here and in the following we abuse the notation φ_j in different dimensions. Thus the first result

follows from the definition. In the following we prove the second one. By definition we have

$$\begin{aligned}
 & \|f\|_{\mathbf{C}_{x_i}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k,\rho)} \\
 &= \sup_{x_j \in \Lambda_\varepsilon, j \neq i} \sup_l 2^{l\alpha} \|\Delta_{l,x_i}^\varepsilon f(x_1, \dots, x_{i-1}, \cdot, x_i, \dots, x_k)\|_{L^\infty(\Lambda_\varepsilon, \rho(x_1, \dots, x_{i-1}, \cdot, x_i, \dots, x_k))} \\
 &= \sup_l 2^{l\alpha} \|\Delta_{l,x_i}^\varepsilon f\|_{L^\infty(\Lambda_\varepsilon^k, \rho)} \lesssim \sup_l 2^{l\alpha} \sum_{l \lesssim j} \|\Delta_{l,x_i}^\varepsilon \Delta_j^\varepsilon f\|_{L^\infty(\Lambda_\varepsilon^k, \rho)} \\
 &\lesssim \sup_l 2^{l\alpha} \sum_{l \lesssim j} \|\Delta_j^\varepsilon f\|_{L^\infty(\Lambda_\varepsilon^k, \rho)} \lesssim \sup_l 2^{l\alpha} \sum_{l \lesssim j} 2^{-j\alpha} \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)} \lesssim \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)},
 \end{aligned}$$

where in the first inequality we used $\varphi_j(\xi_1, \dots, \xi_k) \varphi_l(\xi_i) \neq 0$ only if $l \lesssim j$ and in the last inequality we use $\alpha > 0$. For the converse part we have

$$\begin{aligned}
 \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)} &= \sup_j 2^{j\alpha} \|\Delta_j^\varepsilon f\|_{L^\infty(\Lambda_\varepsilon^k, \rho)} \lesssim \sup_j 2^{j\alpha} \sum_{\substack{-1 \leq j_i \leq j_\varepsilon, \\ i=1, \dots, k}} \left\| \Delta_j^\varepsilon \left(\prod_{i=1}^k \Delta_{j_i, x_i}^\varepsilon \right) f \right\|_{L^\infty, \varepsilon(\Lambda_\varepsilon^k, \rho)} \\
 &\lesssim \sum_i \sum_{j_i \sim j} 2^{\alpha j} \left\| \Delta_j^\varepsilon \left(\prod_{i=1}^k \Delta_{j_i, x_i}^\varepsilon \right) f \right\|_{L^\infty, \varepsilon(\Lambda_\varepsilon^k, \rho)} \\
 &\lesssim \sum_i \sum_{j_i \sim j} 2^{\alpha j} \|\Delta_{j_i, x_i}^\varepsilon f\|_{L^\infty, \varepsilon(\Lambda_\varepsilon^k, \rho)} \lesssim \sum_{i=1}^k \|f\|_{\mathbf{C}_{x_i}^{\alpha,\varepsilon}(\Lambda_\varepsilon^k, \rho)},
 \end{aligned}$$

where we used $\varphi_j(\xi_1, \dots, \xi_k) \prod_{i=1}^k \varphi_{j_i}(\xi_i) \neq 0$ only if there exists one $j_i \sim j$ in the third inequality. Thus the second result follows. $\square$

By similar argument as in the proof of [23], Lemma A.16, and [31], Proposition 3.6, we obtain the following estimate. We recall that we view all the functions on $\Lambda_{M,\varepsilon}$ as periodic functions on Λ_ε .

LEMMA A.2. For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $d = 2$

$$(A.7) \quad \|(m - \Delta_\varepsilon)^{-1} f\|_{B_{p,q}^{\alpha+2,\varepsilon}(\rho)} \lesssim \|f\|_{B_{p,q}^{\alpha,\varepsilon}(\rho)}.$$

Moreover, set

$$\mathcal{J}f(x_1, x_2) = \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x_1 - z) C_{M,\varepsilon}(x_2 - z) f(z) \, dz,$$

and

$$\mathcal{J}_1 f(x_1, x_2, x_3) = \int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x_1 - z) C_{M,\varepsilon}(x_2 - z) C_{M,\varepsilon}(x_3 - z) f(z) \, dz.$$

If $\alpha < 0$, $\gamma > 0$, then for $\rho(x_1, x_2) = \rho_1(x_1)$

$$(A.8) \quad \|\mathcal{J}f\|_{\mathbf{C}^{\alpha+2-\gamma,\varepsilon}(\rho)} \lesssim \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\rho_1)},$$

and for $\rho(x_1, x_2, x_3) = \rho_1(x_1)$

$$(A.9) \quad \|\mathcal{J}_1 f\|_{\mathbf{C}^{\alpha+2-\gamma,\varepsilon}(\rho)} \lesssim \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\rho_1)}.$$

Here all the proportional constants are independent of ε and M .

PROOF. We only give the proof of (A.8) and (A.7), (A.9) follow similarly. Since $C_{M,\varepsilon} = \sum_{l \in M\mathbb{Z}^2} C_\varepsilon(\cdot + l)$, it is easy to see that

$$\int_{\Lambda_{M,\varepsilon}} C_{M,\varepsilon}(x_1 - z) C_{M,\varepsilon}(x_2 - z) f(z) dz = \int_{\Lambda_\varepsilon} C_\varepsilon(x_1 - z) C_{M,\varepsilon}(x_2 - z) f(z) dz,$$

with f being the periodic extension from $\Lambda_{M,\varepsilon}$ to Λ_ε . By the definition of the Besov space it is sufficient to prove for $-1 \leq j \leq j_\varepsilon$

$$(A.10) \quad \|\Delta_j^\varepsilon \mathcal{J}f\|_{L^\infty(\rho)} \lesssim 2^{(-2+\gamma-\alpha)j} \|f\|_{\mathbf{C}^{\alpha,\varepsilon}(\rho_1)}.$$

We have for $\xi, \eta \in \varepsilon^{-1}\mathbb{T}^2$

$$\mathcal{F}(\mathcal{J}f)(\xi, \eta) = \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta) \mathcal{F}f(\xi + \eta),$$

which combined with $\sum_{-1 \leq l \leq j_\varepsilon} \varphi_l = 1$ implies that

$$\begin{aligned} \Delta_j^\varepsilon \mathcal{J}f &= \mathcal{F}^{-1}(\varphi_j^\varepsilon(\xi, \eta) \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta) \mathcal{F}f(\xi + \eta)) \\ &= \sum_{-1 \leq l \leq j_\varepsilon} \mathcal{F}^{-1}(\varphi_j^\varepsilon(\xi, \eta) \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta) \varphi_l^\varepsilon(\xi + \eta) \mathcal{F}f(\xi + \eta)) \\ &= \sum_{l \lesssim j} \mathcal{F}^{-1}(\varphi_j^\varepsilon(\xi, \eta) \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta) \varphi_l^\varepsilon(\xi + \eta) \mathcal{F}f(\xi + \eta)) \\ &= \sum_{l \lesssim j} \mathcal{F}^{-1}(\varphi_j^\varepsilon(\xi, \eta) \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta)) *_\varepsilon \mathcal{F}^{-1}(\varphi_l^\varepsilon(\xi + \eta) \mathcal{F}f(\xi + \eta)) \\ &\stackrel{\text{def}}{=} \sum_{l \lesssim j} I_1 *_\varepsilon I_2. \end{aligned}$$

Here we abuse the notation φ_j^ε for different spatial dimensions and in the third equality we used on the support of φ_j^ε , $|\xi + \eta| \lesssim 2^j$. Moreover, by a similar argument as [31], Proposition 3.6, we can use the smear function ψ introduced in the definition of extension operator $\mathcal{E}^\varepsilon$ and rewrite I_1 as the Fourier transform on $\mathbb{R}^4$. More precisely,

$$(A.11) \quad I_1 = \mathcal{F}_{\mathbb{R}^4}^{-1}(\psi(\varepsilon\xi, \varepsilon\eta)(\varphi_j^\varepsilon)_{\text{ext}}(\xi, \eta) \mathcal{F}C_\varepsilon(\xi) \mathcal{F}C_{M,\varepsilon}(\eta)).$$

Here $(\varphi_j^\varepsilon)_{\text{ext}}$ means periodic extension of φ_j^ε from $\varepsilon^{-1}\mathbb{T}^2$ to $\mathbb{R}^2$ and we extend $\mathcal{F}C_\varepsilon, \mathcal{F}C_{M,\varepsilon}$ to $\mathbb{R}^2$ by the following formula:

$$\mathcal{F}C_\varepsilon(\xi_1, \xi_2) = \frac{1}{2[m + 4(\sin^2(\varepsilon\pi\xi_1) + \sin^2(\varepsilon\pi\xi_2))/\varepsilon^2]} =: \frac{1}{2(m + l^\varepsilon(\xi))},$$

and by the Poisson summation formula

$$(A.12) \quad \mathcal{F}C_{M,\varepsilon}(\xi) = \frac{1}{M^2} \mathcal{F}C_\varepsilon(\xi) \sum_{n \in \frac{1}{M}\mathbb{Z}^2} \delta_0(\xi - n),$$

where δ_0 is the Dirac measure at 0. It is easy to see that

$$I_2(x_1, x_2) = \Delta_l^\varepsilon f(x_1) \varepsilon^{-2} \mathbf{1}_{x_1=x_2}, \quad x_1, x_2 \in \Lambda_\varepsilon.$$

Thus we use $\rho(x_1) \lesssim \rho(y_1)(1 + |x_1 - y_1|^\beta)$ for some $\beta > 0$ to obtain

$$\|\Delta_j^\varepsilon \mathcal{J}f\|_{L^{\infty,\varepsilon}(\rho)} \lesssim \sup_{y_1, y_2 \in \Lambda_\varepsilon} \sum_{l \lesssim j} \varepsilon^2 \sum_{x_1 \in \Lambda_\varepsilon} |I_1(y_1 - x_1, y_2 - x_1)(1 + |y_1 - x_1|^\beta)| \|\Delta_l^\varepsilon f\|_{L^{\infty,\varepsilon}(\rho_1)}.$$

On the other hand since $\alpha < 0$

$$\sum_{l \lesssim j} \|\Delta_l^\varepsilon f\|_{L^{\infty, \varepsilon}(\rho_1)} \lesssim \sum_{l \lesssim j} 2^{-l\alpha} \|f\|_{\mathbf{C}^{\alpha, \varepsilon}(\rho_1)} \lesssim 2^{-j\alpha} \|f\|_{\mathbf{C}^{\alpha, \varepsilon}(\rho_1)},$$

and

$$I_1(x) = 2^{2j} \bar{V}_j(2^j x_1, x_2),$$

with

$$\bar{V}_j = \mathcal{F}_{\mathbb{R}^4}^{-1}(\psi(\varepsilon 2^j \xi, \varepsilon \eta)(\varphi_j^\varepsilon)_{\text{ext}}(2^j \xi, \eta) \mathcal{F} C_\varepsilon(2^j \xi) \mathcal{F} C_{\varepsilon, M}(\eta)).$$

Now it suffices to prove for $N \in \mathbb{N}$

$$(A.13) \quad (1 + |x_1|^2)^N \bar{V}_j(x) \lesssim 2^{(-2+\gamma)j}.$$

Note that by (A.12)

$$\bar{V}_j(x_1, x_2) = \frac{1}{M^2} \sum_{n \in \frac{1}{M} \mathbb{Z}^2} \int_{\mathbb{R}^2} \psi(\varepsilon 2^j \xi, \varepsilon n)(\varphi_j^\varepsilon)_{\text{ext}}(2^j \xi, n) \mathcal{F} C_\varepsilon(2^j \xi) \mathcal{F} C_\varepsilon(n) e^{2\pi i(\xi x_1 + n x_2)} d\xi_1.$$

By [31], Lemma 3.5, and using $l_\varepsilon(2^j \xi) = 2^{2j} l_{\varepsilon 2^j}(\xi)$ and $\varepsilon 2^j \lesssim 1$

$$|\partial_\xi^N(\psi(\varepsilon 2^j \xi, \varepsilon n)(\varphi_j^\varepsilon)_{\text{ext}}(2^j \xi, n) \mathcal{F} C_\varepsilon(2^j \xi))| \lesssim 2^{-2j} 1_{|\xi| \lesssim 1},$$

which combined with

$$\begin{aligned} & |(1 + |x_1|^2)^N \bar{V}_j(x)| \\ & \lesssim \frac{1}{M^2} \sum_{n \in \frac{1}{M} \mathbb{Z}^2} \left| \int e^{2\pi i x_1 \cdot \xi} (1 - \Delta_\xi)^N (\psi(\varepsilon 2^j \xi, \varepsilon n) \varphi_j^\varepsilon(2^j \xi, n) \mathcal{F}(C_\varepsilon(2^j \xi))) d\xi \right| \mathcal{F} C_\varepsilon(n), \end{aligned}$$

implies that

$$|(1 + |x_1|^2)^N \bar{V}_j(x)| \lesssim 2^{-2j} \left(\frac{1}{M^2} \sum_{n \in \frac{1}{M} \mathbb{Z}^2, |n| \lesssim 2^j} \frac{1}{m + |n|^2} \right) \lesssim 2^{(-2+\gamma)j}.$$

Thus (A.13) and (A.10) holds, which implies (A.8). $\square$

We recall the following results for discrete Besov embedding, duality, interpolation and all the results hold for $\varepsilon = 0$, that is, the continuous setting. The following interpolation inequality are used frequently, which is an easy consequence of Hölder's inequality and the corresponding definition (see [23], Lemma A.3, for the proof).

LEMMA A.3. *Let ρ be a polynomial weight and $\theta \in [0, 1]$. Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ and $\delta, \delta_1, \delta_2 \in \mathbb{R}$ satisfy*

$$\delta = \theta \delta_1 + (1 - \theta) \delta_2, \quad \alpha = \theta \alpha_1 + (1 - \theta) \alpha_2,$$

and $p, q, p_1, q_1, p_2, q_2 \in [1, \infty]$ satisfy

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1 - \theta}{q_2}.$$

Then it holds that

$$(A.14) \quad \|f\|_{B_{p,q}^{\alpha,\varepsilon}(\rho^\delta)} \leq \|f\|_{B_{p_1,q_1}^{\alpha_1,\varepsilon}(\rho^{\delta_1})}^\theta \|f\|_{B_{p_2,q_2}^{\alpha_2,\varepsilon}(\rho^{\delta_2})}^{1-\theta}.$$

LEMMA A.4. *Let ρ be a polynomial weight. Let $\alpha \in \mathbb{R}$ and $p_1, q_1, p_2, q_2 \in [1, \infty]$ satisfy*

$$1 = \frac{1}{p_1} + \frac{1}{p_2}, \quad 1 = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then it holds that

$$\langle f, g \rangle_\varepsilon \lesssim \|f\|_{B_{p_1, q_1}^{\alpha, \varepsilon}(\rho)} \|g\|_{B_{p_2, q_2}^{-\alpha, \varepsilon}(\rho^{-1})}.$$

PROOF. See [23], Lemma A.2. $\square$

We recall the following Besov embedding theorems (cf. [31], Lemma 2.22).

LEMMA A.5.

(i) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$, $\rho_2 \lesssim \rho_1$. Then $B_{p_1, q_1}^{\alpha_1, \varepsilon}(\rho_1)$ is continuously embedded in $B_{p_2, q_2}^{\alpha_2, \varepsilon}(\rho_2)$ for $\alpha_2 - \frac{d}{p_2} \leq \alpha_1 - \frac{d}{p_1}$. Furthermore, if $\alpha_2 - \frac{d}{p_2} < \alpha_1 - \frac{d}{p_1}$ and $\lim_{|x| \rightarrow \infty} \rho_2(x)/\rho_1(x) = 0$, the embedding is compact.*

(ii) *Let $1 \leq p < \infty$, $\varepsilon > 0$. Then $B_{2,1}^{0, \varepsilon}(\rho) \subset L^{2, \varepsilon}(\rho) \subset B_{2,2}^{0, \varepsilon}(\rho)$ and $B_{p,1}^{0, \varepsilon}(\rho) \subset L^{p, \varepsilon}(\rho) \subset B_{p, \infty}^{0, \varepsilon}(\rho)$.*

(iii) *Let $\gamma \in (0, 1)$, $p \in [1, \infty]$. Then*

$$\|f\|_{B_{p,p}^{1-\gamma, \varepsilon}} \lesssim \|\nabla_\varepsilon f\|_{B_{p,p}^{-\gamma, \varepsilon}} + \|f\|_{B_{p,p}^{-\gamma, \varepsilon}}.$$

PROOF. See [31], Lemma 2.22, for the proof of (i). (ii) and (iii) follow from [23], Lemma A.4, Lemma A.5. $\square$

Recall the following result on the bounds for powers of functions (cf. [23], Lemma A.7, [31], Lemma 4.2).

LEMMA A.6. *Let $\alpha > 0$. Let ρ_1, ρ_2 be polynomial weights. Then for every $\beta > 0$ it holds that*

$$\|f^2\|_{B_{1,1}^{\alpha, \varepsilon}(\rho_1 \rho_2)} \lesssim \|f\|_{L^{2, \varepsilon}(\rho_1)} \|f\|_{H^{\alpha+2\beta, \varepsilon}(\rho_2)}.$$

For $p \in [1, \infty]$, $\gamma < 0 < \alpha$ with $\alpha + \gamma > 0$ and $\beta > 0$ it holds that

$$\|fg\|_{B_{p, \infty}^{\gamma, \varepsilon}(\rho_1 \rho_2)} \lesssim \|f\|_{B_{p, \infty}^{\alpha, \varepsilon}(\rho_1)} \|g\|_{C^{\gamma+\beta, \varepsilon}(\rho_2)}.$$

Now we prove the following Schauder estimate for discrete heat semigroup $P_t^\varepsilon = e^{t(\Delta_\varepsilon - m)}$.

LEMMA A.7. *Let ρ be a polynomial weight and $\alpha \in \mathbb{R}$, $p \in [1, \infty]$, $T > 0$. Then it holds that for $f \in L_T^p B_{p,p}^{\alpha-2, \varepsilon}(\rho)$, $g \in B_{p,p}^{\alpha-2/p, \varepsilon}(\rho)$*

$$\left\| \int_0^\cdot P_{\cdot-s}^\varepsilon f \, ds \right\|_{L_T^p B_{p,p}^{\alpha, \varepsilon}(\rho)} \lesssim \|f\|_{L_T^p B_{p,p}^{\alpha-2, \varepsilon}(\rho)},$$

and

$$\|P_\cdot^\varepsilon g\|_{L_T^p B_{p,p}^{\alpha, \varepsilon}(\rho)} \lesssim \|g\|_{B_{p,p}^{\alpha-2/p, \varepsilon}(\rho)},$$

where the proportional constants are independent of T and ε .

PROOF. By [23], Lemma A.16, we have that for $c > 0$

$$\|\Delta_j^\varepsilon P_t^\varepsilon f\|_{L^{p,\varepsilon}(\rho)} \lesssim e^{-ct(2^{2j}+m)} \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)},$$

with the proportional constant independent of ε and t ,⁶ which implies that

$$\begin{aligned} & \left\| \int_0^\cdot P_{\cdot-s}^\varepsilon f \right\|_{L_T^p B_{p,p}^{\alpha,\varepsilon}(\rho)}^p \\ &= \sum_j 2^{\alpha p j} \int_0^T \left\| \int_0^t \Delta_j^\varepsilon P_{t-s}^\varepsilon f \, ds \right\|_{L^{p,\varepsilon}(\rho)}^p dt \\ &\lesssim \sum_j 2^{\alpha j p} \int_0^T \left(\int_0^t \|\Delta_j^\varepsilon P_{t-s}^\varepsilon f\|_{L^{p,\varepsilon}(\rho)} \, ds \right)^p dt \\ &\lesssim \sum_j 2^{\alpha j p} \int_0^T \left(\int_0^t e^{-c(t-s)(2^{2j}+m)} \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)} \, ds \right)^p dt \\ &\lesssim \sum_j 2^{\alpha j p} \int_0^T \left(\int_0^t e^{-c(t-s)(2^{2j}+m)} \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)}^p \, ds \right) \left(\int_0^t e^{-c(t-s)(2^{2j}+m)} \, ds \right)^{p-1} dt \\ &\lesssim \sum_j 2^{\alpha j p} 2^{-2j(p-1)} \int_0^T \int_s^T e^{-c(t-s)(2^{2j}+m)} \, dt \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)}^p \, ds \\ &\lesssim \sum_j 2^{(\alpha-2)jp} \int_0^T \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}(\rho)}^p \, ds, \end{aligned}$$

where we change the order of the integral in the fourth inequality. The first result then follows from the definition of the Besov space. Similarly we have

$$\begin{aligned} \|P_{\cdot}^\varepsilon g\|_{L_T^p B_{p,p}^{\alpha,\varepsilon}(\rho)}^p &\leq \sum_j 2^{\alpha j p} \int_0^T \|\Delta_j^\varepsilon P_t^\varepsilon g\|_{L^{p,\varepsilon}(\rho)}^p \, dt \\ &\lesssim \sum_j 2^{\alpha j p} \int_0^T e^{-ctp(2^{2j}+m)} \|\Delta_j^\varepsilon g\|_{L^{p,\varepsilon}(\rho)}^p \, dt \\ &\lesssim \sum_j 2^{jp\alpha-2j} \|\Delta_j^\varepsilon g\|_{L^{p,\varepsilon}(\rho)}^p. \end{aligned}$$

Thus the result follows. $\square$

Now we recall the following property of $\mathcal{E}^\varepsilon$ in Besov spaces.

LEMMA A.8. For any $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ the family of extension operators

$$\mathcal{E}^\varepsilon : B_{p,q}^{\alpha,\varepsilon}(\rho) \rightarrow B_{p,q}^\alpha(\rho), \quad \mathcal{E}_k^\varepsilon : \mathbf{C}_s^{\alpha,\varepsilon}(\rho) \rightarrow \mathbf{C}_s^\alpha(\rho),$$

defined in (A.4) and (A.5) are uniformly bounded in ε .

PROOF. See [31], Lemma 2.24, for the proof of the result for $\mathcal{E}^\varepsilon$. The result for $\mathcal{E}_k^\varepsilon$ follows from a similar argument. $\square$

⁶[23], Lemma A.16, only proves the result for $p = 1$ and the same result also holds for general $p \in [1, \infty]$ by exactly the same argument

LEMMA A.9. For $p \in [1, \infty]$, $\gamma < 0 < \alpha$ with $\alpha + \gamma > 0$ and $\beta > 0$ it holds that

$$\|\mathcal{E}^\varepsilon(fg) - \mathcal{E}^\varepsilon f \mathcal{E}^\varepsilon g\|_{B_{p,\infty}^{\gamma,\varepsilon}(\rho_1\rho_2)} \lesssim o(\varepsilon) \|f\|_{B_{p,\infty}^{\alpha,\varepsilon}(\rho_1)} \|g\|_{C^{\gamma+\beta,\varepsilon}(\rho_2)}.$$

PROOF. See [31], Lemma 4.2. $\square$

We prove the following result by the same argument as in [31], Theorem 5.13.

LEMMA A.10. Assume that

$$(m - \Delta_\varepsilon)u_\varepsilon = f_\varepsilon$$

and ρ be some polynomial weight. It holds that for $\alpha \in \mathbb{R}$ and $\delta > 0$

$$\|\mathcal{E}^\varepsilon u_\varepsilon - (m - \Delta)^{-1} \mathcal{E}^\varepsilon f_\varepsilon\|_{C^{\alpha+2-\delta}(\rho)} \lesssim \varepsilon^\delta \|f_\varepsilon\|_{C^{\alpha,\varepsilon}(\rho)}.$$

PROOF. It is easy to see that

$$(A.15) \quad (m - \Delta_\varepsilon)\mathcal{E}^\varepsilon u_\varepsilon = \mathcal{E}^\varepsilon(m - \Delta_\varepsilon)u_\varepsilon = \mathcal{E}^\varepsilon f_\varepsilon.$$

Moreover, by [31], Lemma 3.4, and Lemma A.8, (A.7) we know for any $\delta > 0$

$$\begin{aligned} \|(m - \Delta_\varepsilon)\mathcal{E}^\varepsilon u_\varepsilon - (m - \Delta)\mathcal{E}^\varepsilon u_\varepsilon\|_{C^{\alpha-\delta}(\rho)} &\lesssim \varepsilon^\delta \|\mathcal{E}^\varepsilon u_\varepsilon\|_{C^{\alpha+2}(\rho)} \lesssim \varepsilon^\delta \|u_\varepsilon\|_{C^{\alpha+2,\varepsilon}(\rho)} \\ &\lesssim \varepsilon^\delta \|f_\varepsilon\|_{C^{\alpha,\varepsilon}(\rho)}, \end{aligned}$$

which implies the result by (A.15) and the Schauder estimate for $(m - \Delta)^{-1}$. $\square$

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