



Research paper



The relationship between mechanism geometry and the centers of stiffness and compliance

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ABSTRACT

A significant amount of research has been directed toward developing a more intuitive appreciation of spatial elastic behavior. Results of these analyses have often been described in terms of the elastic behavior (stiffness or compliance) *centers*. This paper investigates the properties of centers of stiffness and compliance and provides a fresh view of elastic center locations, specifically, the locus of centers associated with a given mechanism's topology and geometry. We show that the location of the center of stiffness (compliance) for a set of elastic components connected in parallel (in serial) can be described in terms similar to the location of the center of mass for a set of mass particles. This provides a physical interpretation of the centers associated with a compliant behavior, and a useful guide in the design of mechanisms that realize desirable compliant behaviors.

1. Introduction

For stable physical interaction, some form of passive compliance is needed in robotic manipulation. A general model of passive compliant behavior is a rigid body suspended by passive elastic components. For small displacements, the force–deflection relationship is characterized by a 6×6 matrix $\mathbf{K}$ which maps the body motion to the force/torque applied to the body [1]:

$$\mathbf{w} = \mathbf{K}\mathbf{t}, \quad (1)$$

where $\mathbf{t}$ is the body twist motion (a 6-vector), and $\mathbf{w}$ is the wrench (force/torque; a 6-vector) applied to the body. If twist $\mathbf{t}$ and wrench $\mathbf{w}$ are expressed in Plücker's axis and ray coordinates respectively, the stiffness matrix $\mathbf{K}$ is symmetric positive semi-definite (PSD) [2]. If the body suspension is fully elastic, the behavior can be equivalently represented by the compliance matrix $\mathbf{C}$, the inverse of $\mathbf{K}$.

In compliance analysis, some specific points, defined as “centers”, are of general interest. The remote center of compliance (RCC) device was shown to be useful in robotic assembly [3,4]. The RCC is an example of elastic mechanisms for which the center location is determined by the intersection of elastic component axes. This type of center corresponds to a very small set of compliant behaviors.

Loncaric [2] defined the *center of stiffness* in a more general way: the center of stiffness is defined to be the origin of the coordinate frame at which the stiffness matrix has symmetric off-diagonal blocks. Similarly, the *center of compliance* is defined as the location of the coordinate frame at which the compliance matrix has symmetric off-diagonal blocks. It was proved that, every fully elastic behavior has a center of stiffness and a center of compliance. Lipkin and Patterson [5] defined the *center of elasticity* to be the center

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of the three wrench-compliant axes associated with the elastic behavior. The location of each of these three centers is, in the generic case, unique.

Many researchers have investigated the properties of these three types of centers. In [6–8], behavior centers were analyzed using screw theory and the mathematical relationships between these types of centers were investigated. In [9], the uniqueness of the stiffness center was addressed. In [10], it was shown that any stiffness/compliance can be realized with a parallel/serial mechanism having concurrent spring/joint axes intersecting at the associated center.

In recent work, screw theory was used in the analysis and synthesis of compliant mechanisms using flexures as joints [11–13]. In [14–16], compliance synthesis of parallel mechanisms having a specified topology was addressed. In our recent work [17–20], geometric approaches to the realization of general planar compliant behavior were developed for planar mechanisms, each having a different number (3–6) of elastic components. Although the relationship between mechanism geometry and possible compliance center locations was addressed in each, the results presented in [16–19] were only for planar cases. Very little work has considered the relationships between mechanism geometry and the locations of the elastic behavior centers for spatial cases.

The work presented here is motivated by the desire for a better physical understanding of *spatial* compliant behavior, especially in terms of behavior centers. This paper addresses the relationship between mechanism geometry and the centers of stiffness and compliance defined by Loncaric [2]. A physical interpretation of the mathematically defined center locations is helpful in the design of compliant mechanisms for specific applications. The main contributions of the paper are:

- The identification of the relationship between a compliant mechanism description (configuration and component elastic properties) and the center location of the compliant behavior realized by the mechanism;
- The identification of the analogy between the center of compliance/stiffness and the center of mass, which provides a more intuitive physical interpretation of the centers associated with a compliant behavior;
- The development of a means to determine the locus of stiffness (compliance) centers associated with a configuration of a parallel mechanism (serial mechanism), which bounds a mechanism's ability to realize a compliant behavior and provides guidance in the geometrical design of a compliant mechanism.

The paper is outlined as follows. Section 2 reviews the existing definitions of the center of stiffness and the center of compliance. Screw representations of compliant components in parallel and serial mechanisms, and the realization of a compliant behavior using these two types of mechanisms are also reviewed. In Section 3, properties of the stiffness centers and compliance centers are investigated in terms of parallel and serial mechanisms that can be used to realize the behavior. In Section 4, the relationship between the center of stiffness (compliance) and its realization in a fully parallel mechanism (serial mechanism) is presented. In Section 5, a discussion and brief conclusion are provided.

2. Background

In this section, existing concepts of centers of stiffness and compliance are summarized. Screw representations of parallel and serial mechanism configurations and the realization of elastic behaviors are also reviewed.

2.1. Center of stiffness and center of compliance

Consider a general spatial stiffness matrix $\mathbf{K}$ that maps twist $\mathbf{t}$ (described in Plücker's axis coordinates) to wrench $\mathbf{w}$ (described in Plücker's ray coordinates). The partitioned form of $\mathbf{K}$ is:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix}, \quad (2)$$

where $\mathbf{K} \in \mathbb{R}^{6 \times 6}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D} \in \mathbb{R}^{3 \times 3}$.

Loncaric [2] defined the *center of stiffness* as the location at which the 3×3 off-diagonal block of the stiffness matrix, $\mathbf{B}$, is symmetric. For the generic case, the position of the stiffness center relative to the frame at which $\mathbf{K}$ is described is calculated using

$$\mathbf{r}_k = -[\mathbf{A} - \text{trace}(\mathbf{A})\mathbf{I}]^{-1}\mathbf{b}, \quad (3)$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{b}$ is the 3-vector associated with the anti-symmetric part of $\mathbf{B}$ [2]. Note that the stiffness center location only depends on matrices $\mathbf{A}$ and $\mathbf{B}$.

Eq. (3) presented in [2] was obtained using a Lie group approach. The derivation process was to eliminate the skew-symmetric part in the off-diagonal block by coordinate transformation. Although Eq. (3) identifies a coordinate frame location at which $\mathbf{K}$ has a specific form, the geometric significance of this equation is not evident due to the matrix inverse operation.

Similarly, the center of compliance is defined as the location at which the 3×3 off-diagonal block of the compliance matrix is symmetric. If a compliance matrix is partitioned in the form:

$$\mathbf{C} = \begin{bmatrix} \mathbf{E} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{H} \end{bmatrix}, \quad (4)$$

the position of the compliance center is

$$\mathbf{r}_c = -[\mathbf{H} - \text{trace}(\mathbf{H})\mathbf{I}]^{-1}\mathbf{g}, \quad (5)$$

where $\mathbf{g}$ is the 3-vector associated with the anti-symmetric part of $\mathbf{G}$ [2]. The compliance center location only depends on matrices $\mathbf{H}$ and $\mathbf{G}$.

It is known [2] that, for the generic case, the two centers are unique and non-coincident.

2.2. Compliant behavior and compliant mechanisms

Consider a parallel mechanism having n springs. The geometry of each spring can be described by a unit screw defined as the *spring wrench* [21]. In Plücker's ray coordinates, a general *screw spring* wrench has the form

$$\mathbf{w}_s = \begin{bmatrix} \mathbf{n} \\ h\mathbf{n} + \mathbf{r} \times \mathbf{n} \end{bmatrix}, \quad (6)$$

where $\mathbf{n}$ is a unit 3-vector indicating the direction of the spring axis, h is the pitch of the screw spring, and $\mathbf{r}$ is a position vector from the coordinate frame to an arbitrary point on the spring axis. When $h = 0$, $\mathbf{w}_s$ represents a conventional (simple) line spring.

For a torsional spring, the spring wrench is

$$\mathbf{w}_\tau = \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \end{bmatrix}, \quad (7)$$

where the unit 3-vector $\mathbf{n}$ indicates the direction of the rotation axis. Note that, since $\mathbf{w}_\tau$ is a free vector, it can be located anywhere in space.

If a 6×6 stiffness matrix $\mathbf{K}$ is realized with a passive n -spring parallel system, then,

$$\mathbf{K} = k_1 \mathbf{w}_1 \mathbf{w}_1^T + k_2 \mathbf{w}_2 \mathbf{w}_2^T + \cdots + k_n \mathbf{w}_n \mathbf{w}_n^T, \quad (8)$$

where $k_i \geq 0$ is the spring stiffness associated with $\mathbf{w}_i$ [21].

Dual to a parallel mechanism, the geometry of a joint in a serial mechanism can be described by a unit twist defined as the *joint twist* [22]. In Plücker's axis coordinates, a general *screw joint* twist has the form:

$$\mathbf{t}_s = \begin{bmatrix} h\mathbf{n} + \mathbf{r} \times \mathbf{n} \\ \mathbf{n} \end{bmatrix}, \quad (9)$$

where the unit 3-vector $\mathbf{n}$ indicates the direction of the twist axis, h is the pitch of the screw joint, and $\mathbf{r}$ is the position vector from the coordinate frame to an arbitrary point on the twist axis. When $h = 0$, the joint is a conventional (simple) revolute joint. When $h = \infty$, the joint is a conventional (simple) prismatic joint. For a prismatic joint, the joint twist is:

$$\mathbf{t}_p = \begin{bmatrix} \mathbf{n} \\ \mathbf{0} \end{bmatrix},$$

where the unit 3-vector $\mathbf{n}$ indicates the direction of the prismatic axis. Note that, since $\mathbf{t}_p$ is a free vector, it can be located anywhere in space.

If a 6×6 compliance matrix $\mathbf{C}$ is realized with a passive n -joint serial system, then,

$$\mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T + \cdots + c_n \mathbf{t}_n \mathbf{t}_n^T, \quad (10)$$

where $c_i \geq 0$ is the joint compliance associated with $\mathbf{t}_i$ [22].

3. Center locations relative to the axes of compliant components

In this section, the relationship between the locations of the behavior centers and the screw axes of the compliant components is presented.

3.1. Compliant component axes relative to the behavior centers

Consider an elastic behavior described by stiffness matrix $\mathbf{K}$. When the behavior $\mathbf{K}$ is described with a frame located at the center of stiffness C_k , the stiffness matrix has symmetric off-diagonal blocks, $\mathbf{B} = \mathbf{B}^T$ and

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}. \quad (11)$$

Proposition 1. *If a stiffness matrix $\mathbf{K}$ is realized with a parallel mechanism whose geometry is described by a set of spring wrenches $\mathbf{w}_i$ ($n = 1, 2, \dots, n$), then,*

$$k_1 \mathbf{r}_{k1} + k_2 \mathbf{r}_{k2} + \cdots + k_n \mathbf{r}_{kn} = \mathbf{0}, \quad (12)$$

where each $k_i \geq 0$ is the value of stiffness associated with $\mathbf{w}_i$ and each $\mathbf{r}_{ki}$ is the perpendicular vector from the center of stiffness C_k to the spring axis of $\mathbf{w}_i$.

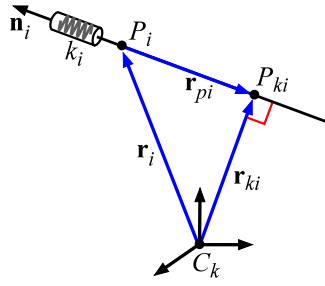


Fig. 1. Coordinate frame at the center of stiffness. Geometric relation: $\mathbf{r}_{ki}$ is the perpendicular vector from the stiffness center C_k to the spring wrench axis $\mathbf{w}_i$ and $\mathbf{r}_{ki} = \mathbf{r}_i + \mathbf{r}_{pi}$.

Proof. Suppose a set of springs with stiffness k_i and spring wrench $\mathbf{w}_i$ realizes a stiffness $\mathbf{K}$. Then,

$$\mathbf{K} = k_1 \mathbf{w}_1 \mathbf{w}_1^T + k_2 \mathbf{w}_2 \mathbf{w}_2^T + \cdots + k_n \mathbf{w}_n \mathbf{w}_n^T. \quad (13)$$

In the frame at the stiffness center C_k , let

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ h_i \mathbf{n}_i + \mathbf{r}_i \times \mathbf{n}_i \end{bmatrix}, \quad (14)$$

where $\mathbf{n}_i$ is a unit 3-vector (indicating the direction of the spring), h_i is the pitch of the screw, and $\mathbf{r}_i$ is the vector from the coordinate origin C_k to an arbitrary point P_i on the spring axis. Let P_{ki} be the point on the axis of spring wrench $\mathbf{w}_i$ associated with the perpendicular position vector $\mathbf{r}_{ki}$ from the center of stiffness, and let $\mathbf{r}_{pi}$ be the vector from P_i to P_{ki} as shown in Fig. 1. Then,

$$\mathbf{r}_{pi} = -(\mathbf{r}_i \cdot \mathbf{n}_i) \mathbf{n}_i, \quad (15)$$

and

$$\mathbf{r}_{ki} = \mathbf{r}_i + \mathbf{r}_{pi}. \quad (16)$$

Consider the rank-1 stiffness matrix $\mathbf{K}_i$ associated with spring wrench $\mathbf{w}_i$ in the partitioned form:

$$\mathbf{K}_i = k_i \mathbf{w}_i \mathbf{w}_i^T = \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{B}_i^T & \mathbf{D}_i \end{bmatrix}, \quad (17)$$

where:

$$\begin{aligned} \mathbf{A}_i &= k_i \mathbf{n}_i \mathbf{n}_i^T, \\ \mathbf{B}_i &= k_i h_i \mathbf{n}_i \mathbf{n}_i^T + k_i \mathbf{n}_i (\mathbf{r}_i \times \mathbf{n}_i)^T, \\ \mathbf{D}_i &= k_i (h_i \mathbf{n}_i + \mathbf{r}_i \times \mathbf{n}_i) (h_i \mathbf{n}_i + \mathbf{r}_i \times \mathbf{n}_i)^T. \end{aligned}$$

The stiffness matrix $\mathbf{K}$ is the sum of $\mathbf{K}_i$:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \sum_{i=1}^n \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{B}_i^T & \mathbf{D}_i \end{bmatrix}. \quad (18)$$

The 3×3 off-diagonal block $\mathbf{B}$ in $\mathbf{K}$ is:

$$\mathbf{B} = \sum_{i=1}^n (k_i h_i \mathbf{n}_i \mathbf{n}_i^T + k_i \mathbf{n}_i (\mathbf{r}_i \times \mathbf{n}_i)^T). \quad (19)$$

Denote:

$$\mathbf{B}_1 = \sum_{i=1}^n k_i h_i \mathbf{n}_i \mathbf{n}_i^T, \quad \mathbf{B}_2 = \sum_{i=1}^n k_i \mathbf{n}_i (\mathbf{r}_i \times \mathbf{n}_i)^T,$$

then,

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2, \quad (20)$$

and since $\mathbf{B}$ and $\mathbf{B}_1$ are symmetric, $\mathbf{B}_2$ must be symmetric. Thus, at the center of stiffness,

$$\mathbf{B}_2 - \mathbf{B}_2^T = \mathbf{0}, \quad (21)$$

which yields

$$\sum_{i=1}^n k_i [\mathbf{n}_i (\mathbf{r}_i \times \mathbf{n}_i)^T - (\mathbf{r}_i \times \mathbf{n}_i) \mathbf{n}_i^T] = \mathbf{0}. \quad (22)$$

Using the identity, $\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = [\mathbf{a} \times \mathbf{b}]_{\times}$, where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary 3-vectors and $[\mathbf{a}]_{\times}$ is the 3×3 skew symmetric matrix associated with the cross product operation of $\mathbf{a}$, Eq. (22) can be expressed as:

$$\sum_{i=1}^n k_i (\mathbf{r}_i \times \mathbf{n}_i) \times \mathbf{n}_i = \mathbf{0}. \quad (23)$$

Using the triple cross-product identity and noting that each $\mathbf{n}_i$ is a unit vector, Eq. (23) yields

$$\begin{aligned} \sum_{i=1}^n k_i [(\mathbf{r}_i \cdot \mathbf{n}_i) \mathbf{n}_i - (\mathbf{n}_i \cdot \mathbf{n}_i) \mathbf{r}_i] &= \mathbf{0}, \\ \sum_{i=1}^n k_i [(\mathbf{r}_i \cdot \mathbf{n}_i) \mathbf{n}_i - \mathbf{r}_i] &= \mathbf{0}. \end{aligned} \quad (24)$$

By Eqs. (15) and (16),

$$\mathbf{r}_{ki} = \mathbf{r}_i - (\mathbf{r}_i \cdot \mathbf{n}_i) \mathbf{n}_i, \quad i = 1, 2, \dots, n.$$

Thus,

$$\sum_{i=1}^n k_i \mathbf{r}_{ki} = \mathbf{0},$$

which proves Eq. (12). $\square$

By duality, the results for stiffness and parallel mechanisms hold for compliance and serial mechanisms.

Proposition 2. *If a compliance matrix $\mathbf{C}$ is realized with a serial mechanism whose configuration is described by a set of joint twists $\mathbf{t}_i$ ($i = 1, 2, \dots, n$), then,*

$$c_1 \mathbf{r}_{c1} + c_2 \mathbf{r}_{c2} + \dots + c_n \mathbf{r}_{cn} = \mathbf{0}, \quad (25)$$

where each $c_i \geq 0$ is the joint compliance associated with $\mathbf{t}_i$ and each $\mathbf{r}_{ci}$ is the perpendicular vector from the center of compliance C_c to the joint axis of $\mathbf{t}_i$. $\square$

Proposition 1 reveals the relationship between the location of the stiffness center and a set of springs connected in parallel that realize the behavior. The stiffness weighted average distance from the stiffness center to the spring axes is zero.

This result indicates the analogy between the stiffness center of a set of springs connected in parallel and the mass center of a set of particle masses. For a set of particles ($m_1, m_2, \dots, m_n$), if $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)$ are the corresponding position vectors from the center of mass G to the locations of particles, then,

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \dots + m_n \mathbf{r}_n = \mathbf{0}, \quad (26)$$

which indicates that the mass weighted average distance from the mass center to the particles is zero. Thus, Eq. (12) for the stiffness center is analogous to Eq. (26) for the mass center.

Dual to the stiffness and parallel mechanism relationship, an analogy between the compliance center associated with a set of compliant joints and the mass center associated a set of mass particles can be drawn in a similar way.

There is one significant difference between the relation between the elastic behavior centers and elastic components and the relation between mass centers and particle masses. Note that in an arbitrary coordinate frame, the location of the mass center can be expressed as the mass weighted average of particle position:

$$\mathbf{r}_G = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \dots + m_n \mathbf{r}_n}{m_1 + m_2 + \dots + m_n}, \quad (27)$$

where $\mathbf{r}_i$ is the position vector of mass m_i . The location of stiffness center $\mathbf{r}_k$ (or compliance center $\mathbf{r}_c$), however, cannot be interpreted in terms of the vectors $\mathbf{r}_{ki}$ (or $\mathbf{r}_{ci}$) in the same form of Eq. (27). This is because, unlike the particle mass case (in which each $\mathbf{r}_i$ is the position of a point), each $\mathbf{r}_{ki}$ ($\mathbf{r}_{ci}$) is the perpendicular vector from the frame to a line (screw axis). If the origin of the frame is changed, the point on the spring axis associated with the perpendicular vector is also changed. However, with some modification, the location of stiffness center $\mathbf{r}_k$ (and compliance center $\mathbf{r}_c$) can be expressed in an arbitrary frame in a form similar to Eq. (27) as shown below.

3.2. Centers in an arbitrary frame

The results described above apply to the perpendicular position vectors of the spring/joint axes relative to the center of stiffness/compliance. The following results apply to position vectors of spring/joint axes and the position vector of stiffness/compliance center in an arbitrary frame.

Suppose P_{ki} is the point on the spring wrench $\mathbf{w}_i$ associated with the perpendicular distance vector $\mathbf{r}_{ki}$ from the stiffness center C_k as illustrated in Fig. 2. For each spring wrench $\mathbf{w}_i$, point P_{ki} is unique. In an arbitrary frame, we have:

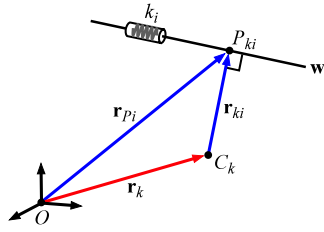


Fig. 2. Stiffness center in an arbitrary coordinate frame. Geometric relation: $\mathbf{r}_{Pi} = \mathbf{r}_k + \mathbf{r}_{ki}$.

Proposition 3. In an arbitrary frame, if $\mathbf{r}_k$ is the position vector identifying the location of stiffness center C_k , and $\mathbf{r}_{Pi}$ is the position vector from the frame origin O to point P_{ki} , then

$$\mathbf{r}_k = \frac{k_1 \mathbf{r}_{P1} + k_2 \mathbf{r}_{P2} + \cdots + k_n \mathbf{r}_{Pn}}{k_1 + k_2 + \cdots + k_n}. \quad (28)$$

Proof. From the geometry illustrated in Fig. 2,

$$\mathbf{r}_{Pi} = \mathbf{r}_k + \mathbf{r}_{ki}. \quad (29)$$

Thus,

$$\sum_{i=1}^n k_i \mathbf{r}_{Pi} = \sum_{i=1}^n k_i (\mathbf{r}_k + \mathbf{r}_{ki}) = \left(\sum_{i=1}^n k_i \right) \mathbf{r}_k + \sum_{i=1}^n k_i \mathbf{r}_{ki}. \quad (30)$$

By Proposition 1, the second sum in Eq. (30) is zero. Hence,

$$\left(\sum_{i=1}^n k_i \right) \mathbf{r}_k = \sum_{i=1}^n k_i \mathbf{r}_{Pi}, \quad (31)$$

which proves Eq. (28). $\square$

Similarly, suppose that a compliance matrix $\mathbf{C}$ is realized with an n -joint serial mechanism at a configuration described by joint twists $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n)$ and that P_{ci} is the point on the axis of twist $\mathbf{t}_i$ associated with the perpendicular distance vector from the compliance center to the axis. In an arbitrary frame, we have:

Proposition 4. In an arbitrary frame, if $\mathbf{r}_c$ is the position vector identifying the location of the compliance center C_c , and $\mathbf{r}_{Pi}$ is position vector from the frame origin O to point P_{ci} , then

$$\mathbf{r}_c = \frac{c_1 \mathbf{r}_{P1} + c_2 \mathbf{r}_{P2} + \cdots + c_n \mathbf{r}_{Pn}}{c_1 + c_2 + \cdots + c_n}. \quad (32)$$

Note that, unlike Eq. (27) for the mass center of a particle mass system, Eq. (28) (or Eq. (32)) cannot be used directly to find the stiffness (or compliance) center location because the screw axis perpendicular position vectors are not known. Propositions 3 and 4 are merely generalizations of Propositions 1 and 2 that take the form of Eq. (27). These results, however, can be used to find behavior center locations directly from a description of the mechanism geometry and component stiffnesses (as shown below).

3.3. Center location from mechanism screws

Using the results presented above, the location of the stiffness/compliance center can be determined from the mechanism geometry alone without calculating the stiffness matrix.

Consider an n -spring parallel mechanism described by spring wrenches $\mathbf{w}_i$ ($i = 1, 2, \dots, n$) with line-of-action $\mathbf{s}_i$ (spring axis) of each $\mathbf{w}_i$ expressed as:

$$\mathbf{s}_i = \begin{bmatrix} \mathbf{n}_i \\ \mathbf{r}_i \times \mathbf{n}_i \end{bmatrix}, \quad i = 1, 2, \dots, n, \quad (33)$$

where $\mathbf{n}_i$ is the unit direction vector of the spring axis $\mathbf{w}_i$, and $\mathbf{r}_i$ is the perpendicular vector from arbitrary coordinate frame origin O to the spring axis $\mathbf{s}_i$ at point T_i as illustrated in Fig. 3.

In order to determine the location of the stiffness center, consider an arbitrary point P in space having position vector

$$\mathbf{r} = [x, y, z]^T \quad (34)$$

in an arbitrary coordinate frame $Oxyz$. Let $\mathbf{r}_{Pi}$ (from P to P_i) be the perpendicular vector from P to spring axis $\mathbf{s}_i$, $\mathbf{r}_{Ti}$ be the vector from T_i to P , and $\mathbf{r}_{ni}$ be the vector from T_i to P_i along $\mathbf{n}_i$ as illustrated in Fig. 3.

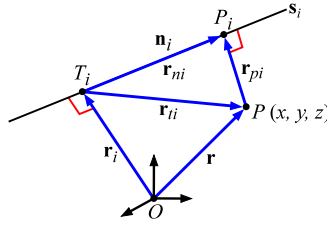


Fig. 3. The perpendicular vector from an arbitrary point P to spring axis s_i . Vector $\mathbf{r}_{pi}$ is a linear function of $\mathbf{r} = [x, y, z]^T$ determined by Eq. (35).

Since $\mathbf{n}_i$ is a unit vector and $\mathbf{r}_{pi} \perp \mathbf{n}_i$, vector $\mathbf{r}_{ni}$ can be expressed as:

$$\mathbf{r}_{ni} = (\mathbf{n}_i \cdot \mathbf{r}_{ti})\mathbf{n}_i = \mathbf{n}_i^T (\mathbf{r} - \mathbf{r}_i)\mathbf{n}_i = \mathbf{n}_i\mathbf{n}_i^T (\mathbf{r} - \mathbf{r}_i).$$

Using the vector relations:

$$\mathbf{r}_{pi} = \mathbf{r}_{ni} - \mathbf{r}_{ti}, \quad \mathbf{r}_{ti} = \mathbf{r} - \mathbf{r}_i,$$

vector $\mathbf{r}_{pi}$ can be expressed as:

$$\mathbf{r}_{pi} = (\mathbf{n}_i\mathbf{n}_i^T - \mathbf{I})(\mathbf{r} - \mathbf{r}_i), \quad i = 1, 2, \dots, n, \quad (35)$$

where $\mathbf{I}$ is the 3×3 identity matrix.

Since each $\mathbf{r}_{pi}$ in Eq. (35) is the perpendicular vector from point P to the spring axis s_i , if P is the stiffness center, by Proposition 1, the n vectors $\mathbf{r}_{p1}, \mathbf{r}_{p2}, \dots, \mathbf{r}_{pn}$ must satisfy

$$k_1\mathbf{r}_{p1} + k_2\mathbf{r}_{p2} + \dots + k_n\mathbf{r}_{pn} = \mathbf{0}. \quad (36)$$

For given spring axes ($s_1, s_2, \dots, s_n$) and spring rates ($k_1, k_2, \dots, k_n$), Eq. (36) is a linear equation for the position vector $\mathbf{r}$ of the stiffness center. If we denote:

$$\mathbf{Q}_i = \mathbf{n}_i\mathbf{n}_i^T - \mathbf{I}, \quad (37)$$

and

$$\mathbf{Q} = \sum_{i=1}^n k_i \mathbf{Q}_i, \quad \mathbf{v} = \sum_{i=1}^n k_i \mathbf{Q}_i \mathbf{r}_i,$$

then, Eq. (36) can be written as

$$\sum_{i=1}^n k_i \mathbf{Q}_i (\mathbf{r} - \mathbf{r}_i) = \mathbf{0} \implies \mathbf{Q}\mathbf{r} = \mathbf{v}.$$

In the generic case, the 3×3 matrix $\mathbf{Q}$ is full rank. Hence, the location of the stiffness center C_k is uniquely determined by:

$$\mathbf{r}_k = \mathbf{Q}^{-1}\mathbf{v}. \quad (38)$$

Using this result, the location of the stiffness center can be determined directly from the screws used in the behavior realization without first calculating the stiffness matrix then using Eq. (5).

By duality, if the joint axis twist is given in the form of Eq. (9), and $\mathbf{r}_i$ is the perpendicular vector from the frame to the axis of $\mathbf{t}_i$, then the location of the stiffness center C_c is uniquely determined by:

$$\mathbf{r}_c = \mathbf{P}^{-1}\mathbf{u}, \quad (39)$$

where

$$\mathbf{P} = \sum_{i=1}^n c_i \mathbf{Q}_i, \quad \mathbf{u} = \sum_{i=1}^n c_i \mathbf{Q}_i \mathbf{r}_i,$$

and $\mathbf{Q}_i$ is the 3×3 matrix defined in Eq. (37).

4. Stiffness center locus from mechanism topology and geometry

In this section, the geometric description of the stiffness center locus for parallel mechanisms having different numbers of springs is presented. The locus for a given mechanism topology does not involve solving Eq. (38), only a geometric analysis of Proposition 1. Since torsional springs only contribute to the 3×3 diagonal block matrix $\mathbf{D}$ of $\mathbf{K}$ in Eq. (2), they have no influence on the location of the stiffness center. In the following, only springs having the line-of-action in the form of Eq. (33) are considered.

For a single spring, the stiffness center can be anywhere on the spring axis.

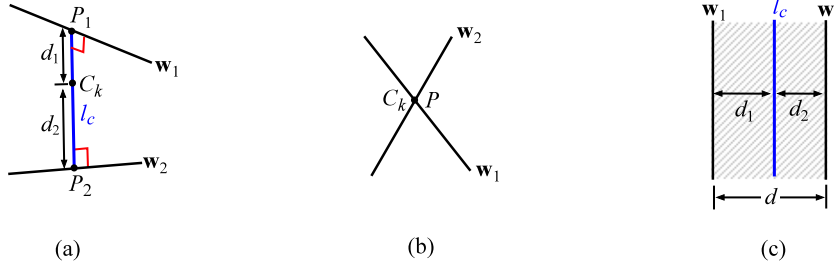


Fig. 4. The center of stiffness associated with a 2-spring system. (a) Generic case. The locus of stiffness centers is a line segment $\overline{P_1P_2}$ on the common perpendicular of the two spring axes. (b) The two spring axes intersect. The stiffness center is located at the intersection P . (c) The two spring axes are parallel. The stiffness center locus is the area between the two spring axes.

4.1. Systems with 2 springs

Consider a 2-spring system having spring wrenches $\mathbf{w}_1$ and $\mathbf{w}_2$. By Proposition 1,

$$k_1 \mathbf{r}_{k1} + k_2 \mathbf{r}_{k2} = \mathbf{0}, \quad (40)$$

where $\mathbf{r}_{k1}$ and $\mathbf{r}_{k2}$ are the perpendicular vectors from C_k to screw axes $\mathbf{s}_1$ and $\mathbf{s}_2$. Eq. (40) indicates that vectors $\mathbf{r}_{k1}$ and $\mathbf{r}_{k2}$ must be collinear. Consider the following 3 cases.

Case 1: Spring axes are not coplanar (generic case). For this case, the stiffness center C_k must be located on the common perpendicular l_c of the two axes of $\mathbf{w}_1$ and $\mathbf{w}_2$. Since both k_1 and k_2 are non-negative, C_k must be located on the line segment of l_c between P_1 and P_2 as shown in Fig. 4a. Thus, for a generic 2-spring system, the locus of stiffness centers is line segment $\overline{P_1P_2}$ on the common perpendicular of the two springs axes.

If the values of k_1 and k_2 are specified, the location of C_k on l_c is uniquely determined by Eq. (40). The location of the stiffness center can be identified on the line segment $\overline{P_1P_2}$ by the distance from P_1 or P_2 :

$$d_1 = \frac{k_2}{k_1 + k_2} d, \quad d_2 = \frac{k_1}{k_1 + k_2} d, \quad (41)$$

where d_i is the distance from P_i to C_k and d is the length of line segment $\overline{P_1P_2}$ as illustrated in Fig. 4a.

Now consider the two cases when the two spring wrenches are coplanar.

Case 2: Spring axes intersect at P . For this case, the center of stiffness must be located at intersection point P (Fig. 4b) regardless of the values of k_1 and k_2 .

Case 3: Spring axes are parallel. The locus of stiffness center C_k is the area between the two spring axes as illustrated in Fig. 4c. If the two spring rates are specified, the distances of C_k to the two spring axes (Fig. 4c) can be calculated using Eq. (41). The center can be anywhere on this line between and parallel to the two spring axes.

4.2. Systems with 3 springs

Consider a 3-spring system described by spring wrench $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$. In order to obtain the locus of stiffness centers, let $\mathbf{r} = [x, y, z]^T$ be the position vector of the stiffness center C_k from an arbitrarily specified frame. Then, using Eq. (35), the perpendicular vector from C_k to the spring axis of $\mathbf{w}_i$ is:

$$\mathbf{r}_{ki} = (\mathbf{n}_i \mathbf{n}_i^T - \mathbf{I})(\mathbf{r} - \mathbf{r}_i), \quad i = 1, 2, 3, \quad (42)$$

where $\mathbf{r}_i$ is the perpendicular vector from the coordinate frame to the spring axis of $\mathbf{w}_i$ as illustrated in Fig. 3. By Proposition 1,

$$k_1 \mathbf{r}_{k1} + k_2 \mathbf{r}_{k2} + k_3 \mathbf{r}_{k3} = \mathbf{0} \quad (43)$$

must be satisfied by some values of k_1 , k_2 and k_3 . Thus, the three vectors $\mathbf{r}_{k1}$, $\mathbf{r}_{k2}$ and $\mathbf{r}_{k3}$ must be co-planar, which means

$$(\mathbf{r}_{k1} \times \mathbf{r}_{k2}) \cdot \mathbf{r}_{k3} = 0. \quad (44)$$

Since each $\mathbf{r}_{ki}$ in Eq. (44) is a linear function in (x, y, z) , Eq. (44) defines a cubic surface in space. The stiffness center must be on this surface. It can be seen that the three spring axes $\mathbf{s}_i$ and the three common perpendiculars l_{ij} formed by any two lines $\mathbf{s}_i$ and $\mathbf{s}_j$ are also on this cubic surface.

It should be noted that not every point on the surface defined by Eq. (44) is the location of a stiffness center associated with the 3-spring system. Since each k_i in Eq. (43) is nonnegative, the locus of stiffness center locations is bounded. The boundary of this area is determined by the three spring axes $\mathbf{s}_i$ and the three common perpendiculars l_{ij} .

Suppose P_{ij} and P_{ji} are the two points respectively on $\mathbf{s}_i$ and $\mathbf{s}_j$ associated with the common perpendicular l_{ij} as illustrated in Fig. 5. The six line segments $\overline{P_{12}P_{21}}$, $\overline{P_{21}P_{23}}$, $\overline{P_{23}P_{32}}$, $\overline{P_{32}P_{31}}$, $\overline{P_{31}P_{13}}$, $\overline{P_{13}P_{12}}$ are on the surface and form a closed polygonal chain Γ . The locus of stiffness center locations is the area on the cubic surface enclosed by Γ (as shown in Fig. 5).

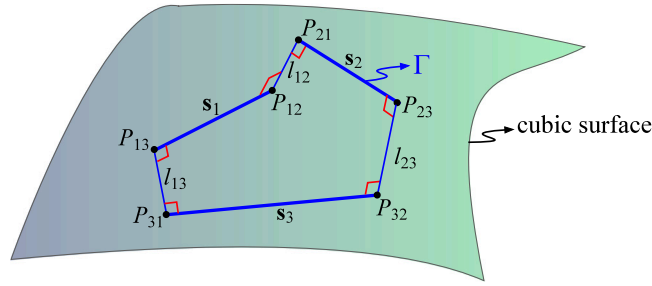


Fig. 5. The locus of stiffness centers is the area on the cubic surface enclosed by the closed polygonal chain with vertices P_{12} , P_{21} , P_{23} , P_{32} , P_{31} and P_{13} .

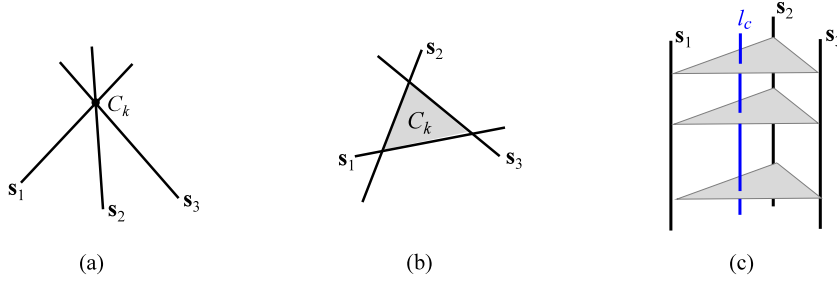


Fig. 6. Three special cases. (a) The three spring axes are concurrent. The stiffness center is located at the intersection point. (b) The three spring axes are co-planar. The stiffness center is located inside the triangle formed by the three spring axes. (c) The three spring axes are parallel. The stiffness center is located anywhere on a line l_c inside the triangular prism formed by the three spring axes.

In some special cases, the surface defined in Eq. (44) degenerates to a single point or other simpler space. Consider the following 3 cases.

Case 1: Spring axes are concurrent. For this case, the stiffness center C_k is located at the intersection point regardless of the values of k_1 , k_2 and k_3 as illustrated in Fig. 6a.

Case 2: Spring axes are co-planar. For this case, the locus of stiffness centers is a triangular area enclosed by the 3 spring axes as illustrated in Fig. 6b. For any given set of non-negative (k_1, k_2, k_3) , the center is located at a point within the triangle.

Case 3: Spring axes are parallel. For this case, the locus of stiffness centers is the space of a triangular prism formed by the three spring axes. For any given set of non-negative (k_1, k_2, k_3) , the center is located anywhere on a straight line l_c parallel to the three spring axes inside the triangular prism as illustrated in Fig. 6c.

4.3. Systems with 4 springs

Consider a system having 4 springs. For every combination of three spring axes (s_i, s_j, s_q) , a cubic surface can be obtained using Eq. (44):

$$(\mathbf{r}_{ki} \times \mathbf{r}_{kj}) \cdot \mathbf{r}_{kq} = 0. \quad (45)$$

An area Γ_{ijq} on the surface bounded by the six line segments is determined by the three spring axes and the three common perpendiculars (as described in Section 4.2). In the generic case, there are 4 distinct cubic surface areas (one for each 3-spring combination). Since every two areas share three of their boundary line segments, the union of the 4 areas:

$$\Gamma_{1234} = \cup \Gamma_{ijq} \quad (46)$$

is a closed surface in space. The space $\mathbb{V}$ enclosed by Γ_{1234} is the locus of stiffness center locations for the 4-spring system.

Because of the shared lines, $\mathbb{V}$ has 4 faces and 10 edges (as illustrated in Fig. 7) that are uniquely determined by the locations of the 4 spring axes. Each face Γ_{ijq} is a portion of the cubic surface defined in Eq. (44) by three spring axes, with 3 spring axes and the 3 common perpendiculars being its edges.

4.4. Systems with more than 4 springs

For an n -spring system with $n > 4$, every combination of four spring axes (s_i, s_j, s_q, s_r) forms a space $\mathbb{V}_{ijqr}$ enclosed by 4 faces as described in Section 4.3. An n -spring system has

$$m = C_n^4 = \frac{n!}{4!(n-4)!} \quad (47)$$

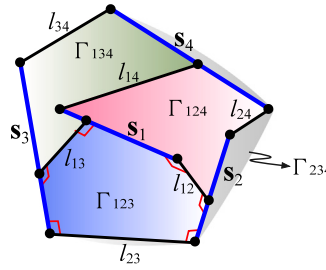


Fig. 7. Locus of stiffness center locations associated with a 4-spring system. Locus space $\mathbb{V}$ is enclosed by portions of 4 cubic surfaces determined by the 4 combinations of any 3 spring axes.

bounded spaces. The union of all m such spaces is the locus of the stiffness centers associated with the n -spring system:

$$\mathbb{V} = \cup \mathbb{V}_{ijqr}. \quad (48)$$

It can be proved that, in the generic case, $\mathbb{V}$ is a connected 3-dimensional bounded space.

4.5. Compliance center locus and n -joint serial mechanisms

In physics, a compliance is defined as the inverse of the stiffness. As such, a compliance matrix $\mathbf{C}$ is normally full-rank. In an n -joint serial compliant mechanism described by joint twists $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n)$, each joint having joint compliance c_i contributes a rank-1 PSD matrix

$$\mathbf{C}_i = c_i \mathbf{t}_i \mathbf{t}_i^T \quad (49)$$

defined as the *joint compliance* matrix [22]. A compliance matrix $\mathbf{C}$ associated with an n -joint serial mechanism can be defined using Eq. (10). As such, all results for the locus of center locations described in Sections 4.1–4.4 for bounding $\mathbf{C}_k$ in n -spring parallel mechanisms apply to bounding $\mathbf{C}_c$ in n -joint serial mechanisms.

5. Discussion and conclusion

In this section, a discussion of the results is provided and a conclusion is then presented.

5.1. Discussion

Propositions 1–4 identify the analogy between the stiffness/compliance center for a system of elastic components and the mass center for system of mass particles. Eq. (12) indicates that the center of stiffness is closer to the spring axis that has the highest spring rate. When the stiffness of a spring increases, the stiffness center moves toward the axis of that spring. Similarly, for a compliance associated with a serial mechanism, the compliance center is closer to the joint axis that has the highest joint compliance. When the compliance of a joint increases, the compliance center moves toward the joint axis of that joint. In most cases, the two centers are not at the same location in space.

In [7], the relationship between the locations of wrench-compliant axes and the stiffness center was investigated. It was shown [7] that, if $\mathbf{r}_{ki}$ is the perpendicular vector from the stiffness center to wrench-compliant axis $\mathbf{w}_{fi}$, then

$$k_{f1} \mathbf{r}_{k1} + k_{f2} \mathbf{r}_{k2} + k_{f3} \mathbf{r}_{k3} = \mathbf{0}, \quad (50)$$

where k_{fi} is the translational stiffness associated with $\mathbf{w}_{fi}$. As shown in [23], a stiffness $\mathbf{K}$ can be expressed as

$$\mathbf{K} = \sum_{i=1}^3 k_{fi} \mathbf{w}_{fi} + \sum_{i=1}^3 k_{\gamma i} \mathbf{w}_{\gamma i}, \quad (51)$$

where each $\mathbf{w}_{fi}$ is the eigenwrench (unit screw of the wrench-compliant axis), each $\mathbf{w}_{\gamma i}$ is a unit couple in the form of Eq. (7) (along the direction of an eigentwist), and each $k_{\gamma i}$ is the rotational stiffness associated with $\mathbf{w}_{\gamma i}$. Thus, $\mathbf{K}$ can be viewed as being represented (and therefore realized) by 3 screw springs along the 3 wrench-compliant axes and 3 torsional springs. It can be seen that Eq. (50) is a special case of **Proposition 1** that applies only to the eigenwrenches of the behavior.

The center of elasticity defined in [5] is the geometric center of the three eigenwrenches, which is independent of the three stiffnesses k_{fi} 's. Thus, the center of elasticity is not directly related to the set of compliant component locations of mechanisms that realize the behavior.

It is known [2,9] that, in the generic case, the location of the stiffness center in space is unique. As shown in [9], when non-uniqueness occurs, the stiffness center either can be located anywhere in space or is located on a straight line. In terms of the stiffness associated with an n -spring system, using the result of **Proposition 1**, it can be proved that the location of the stiffness

center is not unique *if and only if* the behavior is realized with either a set of torsional springs or a set of springs having parallel spring axes. For the first case, the locus of stiffness centers is the entire space. For the second case, the locus of stiffness centers is a straight line parallel to and surrounded by the spring axes.

For a given mechanism, the space of possible locations of the center of stiffness/compliance for all possible spring/joint stiffness values can be determined using the results of Section 4. The space is bounded by the locations of spring/joint axes and their common perpendiculars (lines in space). For a mechanism with specified geometry, if a desired compliant behavior has a center outside the space for that mechanism (as described in Section 4), the behavior cannot be realized by the mechanism regardless of the values of spring/joint stiffness. Thus, for a given mechanism, the space of compliant behaviors is very limited even if each spring/joint stiffness is infinitely adjustable.

It should be noted that the restrictions identified in Section 4 on the locus of behavior centers are only necessary conditions for the realization of the elastic behavior. A compliant behavior that has a center within the locus does not guarantee that it can be realized with the mechanism. To ensure the realization of a compliant behavior, additional conditions are needed [17,20,24].

5.2. Conclusion

In this paper, the relationships between the mechanism geometry and the centers of stiffness and compliance are investigated. The results provide a more intuitive physical interpretation of behavior centers and clearly show an analogy between the center of stiffness/compliance realized with a parallel/serial mechanism and the center of mass for a particle mass system. Using the theory, the location of the stiffness/compliance center can be determined from the mechanism geometry and the value of each spring/joint stiffness. The locus of stiffness/compliance center locations can also be determined from the mechanism geometry for all possible values of spring/joint stiffnesses. This ability is useful as a guide in the design and construction of mechanisms that realize desired compliant behaviors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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