Geometry & Topology

Volume 27 (2023)

Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles

Davesh Maulik
Junliang Shen

msp

Geometry & Topology 27:4 (2023) 1539–1586
DOI: 10.2140/gt.2023.27.1539
Published: 15 June 2023

Cohomological –independence for moduli of one-dimensional sheaves and moduli of Higgs bundles

DAVESH MAULIK
JUNLIANG SHEN

We prove that the intersection cohomology (together with the perverse and the Hodge filtrations) for the moduli space of one-dimensional semistable sheaves supported in an ample curve class on a toric del Pezzo surface is independent of the Euler characteristic of the sheaves. We also prove an analogous result for the moduli space of semistable Higgs bundles with respect to an effective divisor D of degree $\deg(D) > 2g-2$. Our results confirm the cohomological –independence conjecture by Bousseau for $\mathbb{P}^2$, and verify Toda's conjecture for Gopakumar–Vafa invariants for certain local curves and local surfaces.

For the proof, we combine a generalized version of Ngô's support theorem, a dimension estimate for the stacky Hilbert–Chow morphism, and a splitting theorem for the morphism from the moduli stack to the good GIT quotient.

14H60, 14J60, 14N35; 32S60

0 Introduction

We work over the complex numbers $\mathbb{C}$.

0.1 Cohomological χ–independence

Let C be a nonsingular irreducible projective curve of genus $g \geq 2$. The moduli space $N_{n,\chi}$ of (slope-)semistable vector bundles $\mathcal{E}$ with

$$\mathrm{rank}(\mathcal{E}) = n \quad \text{and} \quad \chi(\mathcal{E}) = \chi$$

is an irreducible projective variety, whose topology has been studied intensively for decades. When we fix the rank n, tensor product and duality induce natural isomorphisms between the moduli spaces indexed by different Euler characteristics (or degrees):

$$N_{n,\chi} \cong N_{n,\chi+n}, \qquad N_{n,\chi} \cong N_{n,-\chi+(2-2g)n}. \tag{1}$$

Under the assumption $\gcd(n,\chi) = 1$, so that the moduli spaces $N_{n,\chi}$ are nonsingular, Harder and Narasimhan proved in [20, Theorem 3.3.2] that the Poincaré polynomials of $N_{n,\chi}$ are distinct unless the moduli spaces are related via (1).

In this paper, we are interested in moduli spaces where the cohomological information does not depend on the Euler characteristic χ. More precisely, we consider the following two types of moduli spaces $M^L_{\beta,\chi}$ and $M^D_{n,\chi}$:

(A) $M^L_{\beta,\chi}$ is the moduli space of 1–dimensional semistable sheaves $\mathcal{F}$ with

$$[\mathrm{supp}(\mathcal{F})] = \beta \quad \text{and} \quad \chi(\mathcal{F}) = \chi$$

on a nonsingular toric del Pezzo surface S. Here the semistability is with respect to a polarization L on S, $\mathrm{supp}(-)$ denotes the Fitting support, and β is an ample curve class.

(B) $M^D_{n,\chi}$ is the moduli space of semistable Higgs bundles $(\mathcal{E},\theta)$ with respect to an effective divisor D of degree $\deg(D) > 2g-2$ on C with

$$\mathrm{rank}(\mathcal{E}) = n \quad \text{and} \quad \chi(\mathcal{E}) = \chi.$$

We refer to Section 2 for more details on these moduli spaces. When χ is chosen so that there are no strictly semistable objects, the moduli spaces $M^L_{\beta,\chi}$ and $M^D_{n,\chi}$ are nonsingular, and we consider their singular cohomology. However, for arbitrary values of χ, these moduli spaces can be singular, due to the presence of strictly semistable objects. In this case, it is more natural for us to study their intersection cohomology. Our main result states that, unlike the case of curves, the intersection cohomology of these spaces is independent of the choice of χ:

Theorem 0.1 For any $\chi, \chi' \in \mathbb{Z}$, there are isomorphisms of graded vector spaces

$$\mathrm{IH}^*(M_{\beta,\chi}) \cong \mathrm{IH}^*(M_{\beta,\chi'}), \qquad \mathrm{IH}^*(M_{n,d}) \cong \mathrm{IH}^*(M_{n,d'}),$$

where $\mathrm{IH}^*(-)$ denotes the intersection cohomology. Moreover, these isomorphisms respect perverse and Hodge filtrations carried by these vector spaces.

This phenomenon is surprising, since there is no direct geometric relationship other than those parallel to (1) between these moduli spaces with different Euler characteristics, and the result applies to both smooth and singular moduli spaces. For example in the case (B), the moduli space is nonsingular if and only if $\gcd(n, d) = 1$. Nevertheless, the result on intersection cohomology holds uniformly. Regarding the second part of the theorem and compatibility with filtrations, see Theorem 0.4 for further refinements.

Theorem 0.1 proves the cohomological χ–independence conjecture (see Bousseau [3, Conjecture 0.4.3]) of the moduli space of 1–dimensional semistable sheaves on $\mathbb{P}^2$, which further proves [2, Conjecture 0.4.2] on the BPS numbers of the log K3 surface $(\mathbb{P}^2, E)$; see Bousseau [2, Theorem 0.4.5]. Its refinement (Theorem 0.4) proves Toda's conjecture [48, Conjecture 1.2] on the Gopakumar–Vafa invariants in the cases of certain local curves and local toric del Pezzo surfaces with ample curve classes; see Theorem 0.6. In case (A), when $S = \mathbb{P}^2$, it was proven by Bousseau [2, Theorem 0.5.2] that the dependence of the (intersection) Betti numbers on χ only relies on $\gcd(\deg(\beta), \chi)$, using connections with Gromov–Witten theory for the log K3 surface $(\mathbb{P}^2, E)$ and scattering diagrams. In case (B), when $\gcd(n, d) = 1$, the equality of Poincaré polynomials was proved by a direct calculation in work of Mozgovoy and Schiffmann [39] and Mellit [36], as well as in Groechenig, Wyss and Ziegler [19] by p–adic integration. We discuss connections between our theorems and enumerative geometry in Section 0.3 in more detail.

Remark 0.2 By Demazure [15], a nonsingular del Pezzo surface belongs to one of the following types:

(a) $\mathbb{P}^2$.

(b) $\mathbb{P}^1 \times \mathbb{P}^1$.

(c) The blow-up of $\mathbb{P}^2$ at n very general points with $1 \le n \le 8$.

Hence Theorem 0.1 recovers the case when a del Pezzo surface belongs to (a), (b), or (c) with $n \le 3$. We note that the Fano condition is essential (see Section 0.4), but the toric condition is due to a technical result (Proposition 2.6), which we expect to hold for all del Pezzo surfaces. In other words, if the inequality (40) of Proposition 2.6 is proven for any del Pezzo surface S, then Theorem 0.1 (as well as Theorem 0.4 below) also holds for any del Pezzo surface.

Remark 0.3 Although we will not require it further, our proof of Theorem 0.1 actually provides a natural isomorphism between these spaces, well-defined up to a scalar, which is compatible with the perverse and Hodge filtrations.

0.2 A support theorem

Comparing to $N_{n,\chi}$, a key feature of a moduli space M of type (A) or (B) is that it admits a morphism $h: M \to B$ that behaves like a completely integrable system. Here h is the Hilbert–Chow morphism

$$h: M^L_{\beta,\chi} \to B = \mathbb{P}H^0(S, \mathcal{O}_S(\beta)), \qquad \mathcal{F} \mapsto \mathrm{supp}(\mathcal{F}), \tag{2}$$

in the case (A), and the Hitchin fibration

$$h: M_{n,\chi} \to B = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(iD)), \qquad (\mathcal{E},\theta) \mapsto \mathrm{char}(\theta), \tag{3}$$

in the case (B). In either case, there is a maximal Zariski open subset $U \subset B$ parametriz-ing nonsingular curves in the linear system $|\beta|$ or nonsingular spectral curves over C. We denote by $\pi: \mathcal{C} \to U$ the smooth map given by the universal curve over U.

Theorem 0.4 Let M be a moduli space of (A) or (B), and let $h: M \to B$ be the morphism given by (2) or (3), respectively. Let $\pi: \mathcal{C} \to U \subset B$ be the universal curve of genus g. Then there is an isomorphism

$$Rh_* IC_M \simeq \bigoplus_{i=0}^{2g} IC\Big(\bigwedge\nolimits^{i} R^1\pi_*\mathbb{Q}_{\mathcal{C}}\Big)[-i+g] \tag{4}$$

in the bounded derived category $D^b\mathrm{MHM}(B)$ of mixed Hodge modules on B.

Since the righthand side of (4) clearly does not depend on L or χ, Theorem 0.4 implies Theorem 0.1 immediately by taking global cohomology. The sheaf-theoretic nature of (4) further yields refinements of Theorem 0.1 involving perverse and Hodge filtrations.

Although Theorem 0.4 concerns mixed Hodge modules, it suffices to work with perverse sheaves for the proof. In fact, it is not difficult to check (4) over U:

$$Rh_*\mathbb{Q}_{h^{-1}(U)} \simeq \bigoplus_{i=0}^{2d} \wedge^i R^1 h_*\mathbb{Q}[-i], \tag{5}$$

an isomorphism which only concerns the variation of Hodge structures of abelian varieties; see Proposition 2.2. In view of the decomposition theorem of Saito [42] for Hodge modules, to prove (4) from (5), we only need to verify that every semisimple component of Rh_*IC_M has full support B. This can be checked completely via the decomposition theorem due to Beĭlinson, Bernstein and Deligne [1] of Rh_*IC_M in terms of (shifts of) semisimple perverse sheaves. In particular, Theorem 0.4 can be viewed as a support theorem for the moduli spaces (A) and (B).

Ngô [40] introduced a support theorem, which determines the supports of the direct image complex $Rf_*\mathbb{Q}$ for certain morphisms $f: M \to B$ called weak abelian fibrations. It played a crucial role in his proof of the fundamental lemma of the Langlands program. After that, support theorems become powerful tools in various branches of mathematics; see for example Maulik and Shen [31], Maulik and Yun [33], Migliorini and Shende [37], Yun [49], Yun and Zhang [50], de Cataldo, Hausel and Migliorini [5] and de Cataldo, Rapagnetta and Saccà [7].

In our proof of Theorem 0.4, we systematically develop techniques for applying Ngô's support theorem to a more general setup. More precisely, we do not assume that the total space M is nonsingular, and we work with more general objects $K \in D_c^b(M)$ than the trivial local system $\mathbb{Q}$ on M. Theorem 1.1 reduces a support inequality of Ngô type to a relative dimension bound (see the condition (c)) for the complex Rf_*K. Then we introduce techniques to check this bound when M is a moduli space of type (A) or (B), and K is the intersection cohomology complex IC_M.

0.3 Enumerative geometry

The cohomological χ–independence phenomenon is expected to be part of a much more general phenomenon in the context of enumerative geometry of curves on Calabi–Yau 3–folds, specifically the proposal for Gopakumar-Vafa invariants developed in Maulik and Toda [32] and Toda [48].

Let X be a Calabi–Yau 3–fold with $\beta \in H_2(X;\mathbb{Z})$ a curve class, and let $\omega \in \mathrm{Pic}(X)_{\mathbb{C}}$ be an element in the complexified ample cone of X. Following Davison and Meinhardt [14], and conditional on the conjectural existence of a certain orientation, Toda introduced in [48] the BPS sheaf

$$\phi_{\mathrm{BPS}} \in \mathrm{Perv}(M_{\beta,\chi}),$$

which is a perverse sheaf on the moduli space $M_{\beta,\chi}$ of ω–semistable sheaves on X. Consider the Hilbert–Chow map

$$h\colon M_{\beta,\chi} \to \mathrm{Chow}_\beta(X), \quad F \mapsto \mathrm{supp}(F).$$

For any $\gamma \in \mathrm{Chow}_\beta(X)$, the Gopakumar–Vafa (GV) invariant (see [48, Definition 1.1]) is defined by the identity

$$\Omega_\gamma(\chi,\omega) := \sum_{i\in\mathbb{Z}} \chi\big({}^pH^i(Rh_*\phi_{\mathrm{BPS}})|_\gamma\big)\, y^i \in \mathbb{Z}[y, y^{-1}]. \tag{6}$$

If ω is chosen so there are no strictly semistables, this definition specializes to the definition of Gopakumar–Vafa invariants in Maulik and Toda [32]. For any choice of χ and ω, these invariants are conjectured to encode the same information as the Gromov–Witten invariants of X in the curve class β and arbitrary genus. Since the latter invariants are independent of χ and ω, in order for this conjecture to be well-posed, the Gopakumar–Vafa invariants should be independent of this extra data as well. More precisely, in [48, Conjecture 1.2] Toda made the following conjecture concerning the structure of GV invariants, extending [32, Conjecture 3.3].

Conjecture 0.5 (Toda) *The invariant (6) is independent of χ and ω.*

The invariants (6) specialize to a certain case of the Joyce–Song generalized Donaldson–Thomas (DT) invariants [24], and Conjecture 0.5 is expected to refine the Joyce–Song conjecture [24, Conjecture 6.20] on the generalized DT invariants, which in turn implies the strong rationality conjecture for Pandharipande–Thomas invariants, in Pandharipande and Thomas [41] and Toda [47].

Although currently the existence of the BPS sheaf is conjectural for most cases, it is known to exist for local curve and surface geometries; Meinhardt [34, Theorem 1.1] proved that when X is a local curve $\mathrm{Tot}_C(\mathcal{O}_C(D) \oplus K_C(-D))$ with $\deg(D) > 2g-2$ or a local del Pezzo surface $\mathrm{Tot}(K_S)$, the BPS sheaf coincides with the intersection cohomology complex of the moduli space.

Theorem 0.6 *Conjecture 0.5 holds when X is a local curve $\mathrm{Tot}_C(\mathcal{O}_C(D) \oplus K_C(-D))$ with D effective of $\deg(D) > 2g-2$ and $\beta = n[C]$, or a local toric del Pezzo surface $\mathrm{Tot}(K_S)$ and β is an ample curve class on S.*

In fact, Toda showed in [48, Theorem 7.3] that (6) is independent of the stability parameter under certain conditions which hold for local curves and local surfaces. Hence we may assume that is given by a rational polarization, and the χ–independence of (6) follows from the χ–independence of the complex $Rf_* IC_M$ for a moduli space of type (A) or (B) with the Hilbert–Chow map (2) or (3), respectively. The latter is given by Theorem 0.4.

Cohomological χ–independence for Higgs bundles has also been studied systematically with connections to Kac polynomials and quivers. See Schiffmann [44] for more details. For contractible curves on Calabi–Yau threefolds, cohomological χ–independence has been studied by Davison [12], who has proposed a representation-theoretic approach in that case via the cohomological Hall algebra. For other places where GV invariants arise geometrically, see Shen and Yin [45] and Chuang, Diaconescu and Pan [9] for connections with hyper-Kähler geometries (de Cataldo, Hausel and Migliorini [5]) and the $P = W$ conjecture (de Cataldo, Maulik and Shen [6]), respectively.

0.4 K3 surfaces and O’Grady 10

As illustrated in the following example of K3 surfaces, the “Fano” condition for the surface S in (A) and the condition $\deg(D) > 2g - 2$ for Higgs bundles in (B) are essential for the χ–independence to hold for intersection cohomology groups.

Let

$$(S, L) \quad \text{with } L = \mathcal{O}_S(\beta), \ \beta^2 = 2,$$

be a general polarized K3 surface of degree 2. The linear system $|\beta|$ is 2–dimensional whose general member is a genus 2 nonsingular curve. The linear system $|2\beta|$ is 5–dimensional. We consider the moduli space of semistable sheaves on S supported in the curve class 2β.

If $\chi = 1$, the moduli space $M^L_{2\beta,1}$ is nonsingular and deformation equivalent to the Hilbert scheme of 5 points on a K3 surface. When $\chi = 0$, the moduli space $M^L_{2\beta,0}$ is singular which admits a symplectic resolution. The resolved variety provides O’Grady’s 10–dimensional “sporadic” example of compact hyper-Kähler manifolds. As a key step in their analysis of the topology of the O’Grady 10 variety, de Cataldo, Rapagnetta and Saccà [7] study the fibrations (2):

$$\begin{tikzcd} M^L_{2\beta,0} \arrow[dr, dash, "h_0"'] & & M^L_{2\beta,1} \arrow[dl, dash, "h_1"] \\ & {|2\beta|} & \end{tikzcd}$$

where $h_i(F) = \mathrm{supp}(F)$. Combining Corollary 3.6.5 and Proposition 4.7.2 of [7], we observe that

$$\mathcal{R}h_1 IC = \mathcal{R}h_0 IC \oplus S[3]; \tag{7}$$

where S is a semisimple object supported on the divisor $\mathrm{Sym}^2(|\beta|) \subset |2\beta|$. Furthermore, by [7, Proposition 4.6.1], the object S (which is denoted by $S_{\dagger}$ in [7]) has nontrivial global cohomology. Hence we see from (7) that the χ–independence fails for the $K3$ surface S both sheaf theoretically (Theorem 0.4) and cohomologically (Theorem 0.1).

A similar phenomenon as above is expected to hold for the Higgs bundles with $D = K_C$. Failure of the χ–independence for the "Calabi–Yau" case is due to the fact that the BPS sheaf is different from the intersection cohomology complex on the moduli space.

Plan of the paper

In Section 1, we formulate and prove a generalized version of Ngô's support theorem, which applies to singular varieties and more general complexes. In order to apply this support theorem to intersection cohomology complexes, we need to prove a bound for IC-complexes (which holds automatically in the smooth case). This is accomplished in Sections 2 and 3, where we combine techniques from algebraic stacks, nilpotent Higgs bundles, moduli of framed objects, and unbounded complexes. Then in Section 4, we follow a strategy of Chaudouard and Laumon to show that the support inequalities are sufficient to deduce our theorems for moduli of 1–dimensional sheaves and Higgs bundles.

Acknowledgements We are grateful to Bhargav Bhatt, Pierrick Bousseau and Johan de Jong for helpful discussions, and Pinka and Peter Pinkerton for further assistance. We would like to thank Pierrick Bousseau and Tudor Padurariu for their careful reading of an early draft of the paper and pointing out several typos. We also thank the anonymous referee for careful reading and numerous useful suggestions. Shen was supported by the NSF grant DMS-2134315.

1 A support theorem for self-dual complexes

1.1 Overview

The purpose of this section is to formulate and prove a generalized version of Ngô's support theorem for self-dual complexes. Throughout Section 1, until Section 1.7, we assume that the base field k is a finite field with $\bar{k}$ its algebraic closure. We assume that l is a prime number coprime to the characteristic of k when we work with l–adic sheaves. For notational convenience, we omit Tate twists when it does not cause confusion.

Let B be a scheme over k. Let $g\colon P \to B$ be a smooth B–group scheme with geometrically connected fibers, and let $f\colon M \to B$ be a proper morphism with M quasi-projective. Assume that the group scheme P acts on M via

$$a\colon P \times_B M \to M. \tag{8}$$

We say that the triple (M, P, B) is a weak abelian fibration of relative dimension d if

(i) every fiber of the map g is pure of dimension d, and M has pure dimension

$$\dim M = d + \dim B; \tag{9}$$

(ii) the action (8) of P on M has affine stabilizers, and

(iii) the Tate module $T_{\bar{\mathbb{Q}}_l}(P)$ associated with the group scheme P is polarizable.

The notion of weak abelian fibration was introduced by Ngô [40] modeled on Hitchin's integrable systems [21; 22]. We refer to Section 1.3 for a brief review of Tate modules and their polarizations.

For a closed point $s \in B$, we denote by $\delta(s)$ the dimension of the affine part of the algebraic group P_s. This defines an upper-semicontinuous function

$$\delta\colon B \to \mathbb{N}, \quad s \mapsto \delta(s).$$

For a closed subvariety $Z \subset B$, we define $\delta(Z)$ to be the minimal value of the function δ on Z.

The following is our main theorem.

Theorem 1.1 *Let (M, P, B) be a weak abelian fibration of relative dimension d. Let $K \in D_c^b(M, \bar{\mathbb{Q}}_l)$ be a P–equivariant bounded complex satisfying the following properties:*

(a) **Decomposition theorem** *The direct image complex admits a (noncanonical) decomposition*

$$Rf_*K \simeq \bigoplus_i {}^pH^i(Rf_*K)[-i]. \tag{10}$$

Moreover, after a base change to $B_{\bar{k}} = B \times_k \bar{k}$, the perverse sheaves ${}^{p}\mathcal{H}^i(Rf_*\mathcal{K})$ are semisimple of the form

$${}^{p}\mathcal{H}^i(Rf_*\mathcal{K}) = \bigoplus_{\alpha} \mathrm{IC}_{Z_{\alpha,i}}(\mathcal{L}_{\alpha,i}),$$

where $Z_{\alpha,i}$ is a closed irreducible subvariety of $B_{\bar{k}}$ and each $\mathcal{L}_{\alpha,i}$ is a pure simple local system of weight i on an open dense subset of $Z_{\alpha,i}$. We call these $Z_{\alpha,i}$ the supports of the decomposition (10).

(b) **Duality** We have an isomorphism

$$\mathbb{D}(\mathcal{K}) \simeq \mathcal{K}[2\dim M]$$

with $\mathbb{D}(-)$ the dualizing functor on M.

(c) **Relative dimension bound** For the standard truncation functor $\tau_{>\bullet}(-)$, we have

$$\tau_{>2d}(Rf_*\mathcal{K}) = 0.$$

Then for any support Z of the decomposition (10), we have the inequality

$$\operatorname{codim} Z \leq \delta_Z. \tag{11}$$

In [40], Ngô worked with the trivial local system $\overline{\mathbb{Q}}_\ell$ on M, where he assumed that the conditions (a) and (b) hold. Furthermore, he assumed that every fiber of f is pure of dimension d, where the condition (c) follows automatically by the base change. Therefore, Theorem 1.1 is a generalization of Ngô's support theorem [40, Theorem 7.2.1 and Proposition 7.2.2]. We note that (c) is a crucial condition for the support theorem to hold for general $\mathcal{K}$ as in Theorem 1.1. We first illustrate this in the following special case of Theorem 1.1 — the Goresky–MacPherson inequality.

1.2 The Goresky–MacPherson inequality

If the group scheme P is affine and its action on M is trivial, Theorem 1.1 then specializes to the following theorem, which is known as the Goresky–MacPherson inequality when M is nonsingular and $\mathcal{K} = \overline{\mathbb{Q}}_\ell$.

Theorem 1.2 *Let $f : M \to B$ be a proper map with $\dim M = \dim B + d$. Assume $\mathcal{K} \in D^b(M, \overline{\mathbb{Q}}_\ell)$ satisfies (a), (b) and (c) of Theorem 1.1. Then any support Z of (10) satisfies the inequality*

$$\operatorname{codim} Z \leq d.$$

We first provide a proof of Theorem 1.2 since it contains the main ingredients in the proof of Theorem 1.1, and in particular demonstrates the role played by the conditions (a), (b) and (c).

Proof Let Z be a support. We write

$$\mathrm{occ}(Z) := \{ i \in \mathbb{Z} : {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathcal{K}) \text{ contains a simple factor with support } Z \},$$
$$\mathrm{amp}(Z) := \max(\mathrm{occ}(Z)) - \min(\mathrm{occ}(Z)).$$

By (b), the set $\mathrm{occ}(Z)$ is symmetric with respect to the integer $\dim M$. This allows us to pick $m \in \mathrm{occ}(Z)$ with $m \geq \dim M$. In particular, we have ${}^{\mathfrak{p}}\mathcal{H}^m(Rf_*\mathcal{K}) \neq 0$. Hence by (a) there exists an open subset $U \subset Z$ and a local system $\mathcal{L}$ on U such that the shifted perverse sheaf

$$(\mathcal{L}[\dim Z])[-m] = \mathcal{L}[\dim Z - m]$$

is a direct-sum component of the complex $(Rf_*\mathcal{K})|_U$. We obtain that

$$\mathcal{H}^{m-\dim Z}(Rf_*\mathcal{K}) \neq 0 \in D^b_c(B, \overline{\mathbb{Q}}_\ell). \tag{12}$$

By (12) and the condition (c), we conclude that

$$\dim M - \dim Z \leq m - \dim Z \leq 2d,$$

where the first inequality follows from the choice of m. This completes the proof of Theorem 1.2 thanks to (9). □

As observed by Ngô [40, Proposition 7.3.2], for a weak abelian fibration (M, P, B) and an object $\mathcal{K}$ as in Theorem 1.1, if we have

$$\mathrm{amp}(Z) \geq 2(d - \delta_Z), \tag{13}$$

then the integer m in the proof of Theorem 1.2 can be chosen so that

$$m \geq \dim M + (d - \delta_Z).$$

An identical argument as above implies (11).

In conclusion, the following proposition implies Theorem 1.1.

Proposition 1.3 Under the assumption of Theorem 1.1, the inequality (13) holds for any support Z of $Rf_*\mathcal{K}$.

The rest of Section 1 is devoted to proving Proposition 1.3. We will further reduce Proposition 1.3 to a fiberwise "freeness" statement as stated in Proposition 1.5. Essentially, the arguments of [40, Section 7] can be modified to prove this more generalized version of "freeness" under our assumptions. We point out the necessary modifications (see Propositions 1.4 and 1.6) and sketch all major steps in the proof, which follows [40, Section 7], for the reader's convenience.

1.3 Actions of the group scheme

As part of the data of a weak abelian fibration (M, P, B), the group scheme $g: P \to B$ is smooth over B with d–dimensional geometrically connected fibers, which defines a complex

$$h_P := Rg_! \overline{\mathbb{Q}}_\ell[2d]$$

on the base B. The stalk of each cohomology sheaf $\mathcal{H}^{-i}(h_P)$ over a closed point $s \in B$ computes the i^{th} homology of the group P_s,

$$\mathcal{H}^{-i}(h_P)_s = H_c^{2d-i}(P_s, \overline{\mathbb{Q}}_\ell) = H_i(P_s, \overline{\mathbb{Q}}_\ell).$$

The Tate module associated with $g: P \to B$ is defined to be

$$T_{\overline{\mathbb{Q}}_\ell}(P) := \mathcal{H}^{-1}(h_P). \tag{14}$$

Note that the complex $h_{(-)}$ and the sheaf $T_{\overline{\mathbb{Q}}_\ell}(-)$ are defined for any smooth group scheme over B. In our setting, as shown in [40, Section 7.4.3], the group structure $P \times_B P \to P$ induces a convolution product

$$\star: h_P \otimes h_P \to h_P.$$

Furthermore, it is also shown there that equation (14) extends to a natural isomorphism

$$h_P = \bigoplus \wedge^i T_{\overline{\mathbb{Q}}_\ell}(P)[i],$$

compatible with the multiplication action on each side.

We consider the Chevalley decomposition of the nonsingular commutative group P_s

$$1 \to R_s \to P_s \to A_s \to 1 \tag{15}$$

over any geometric point $s \in B$ where R_s is affine and A_s is an abelian variety. This induces the short exact sequence of Tate modules

$$0 \to T_{\overline{\mathbb{Q}}_\ell}(R_s) \to T_{\overline{\mathbb{Q}}_\ell}(P_s) \to T_{\overline{\mathbb{Q}}_\ell}(A_s) \to 0. \tag{16}$$

Following [40, Section 7.1.4], we say that the Tate module (14) is polarizable if étale locally there exists a bilinear form

$$T_{\overline{\mathbb{Q}}_\ell}(P) \times T_{\overline{\mathbb{Q}}_\ell}(P) \to \overline{\mathbb{Q}}_\ell$$

which induces a nondegenerate pairing on $T_{\overline{\mathbb{Q}}_\ell}(A_s)$ for any $s \in B$ via the quotient map of (16).

The following proposition generalizes the cap product action constructed in Section 7.4.2 of [40].

Proposition 1.4 Let (M, P, B) be a weak abelian fibration of relative dimension d, and let $\mathcal{K} \in D_c^b(M, \overline{\mathbb{Q}}_\ell)$ be a P–equivariant object. Then the P–action (8) on M induces an action of Λ_P on $Rf_*\mathcal{K}$,

$$c : \Lambda_P \otimes Rf_*\mathcal{K} \to Rf_*\mathcal{K}. \tag{17}$$

Furthermore, the compositions $c \circ (c \otimes \mathrm{id})$ and $c \circ (\mathrm{id} \otimes c)$ define the same morphism

$$\Lambda_P \otimes \Lambda_P \otimes Rf_*\mathcal{K} \to Rf_*\mathcal{K}.$$

Proof The trace map

$$Ra_!\overline{\mathbb{Q}}_\ell[2d] \to \overline{\mathbb{Q}}_\ell$$

on M associated with (8) induces a morphism

$$Ra_!\overline{\mathbb{Q}}_\ell[2d] \otimes \mathcal{K} \to \mathcal{K}. \tag{18}$$

We consider the Cartesian diagram

$$\begin{tikzcd} P \times_B M \arrow[r, "p_M"] \arrow[d, "p_P"'] & M \arrow[d, "f"] \\ P \arrow[r, "g"] & B \end{tikzcd} \tag{19}$$

with p_M and p_P the projections. The lefthand side of (18) is equal to

$$Ra_!(\overline{\mathbb{Q}}_\ell[2d] \otimes a^*\mathcal{K}) = Ra_!(\overline{\mathbb{Q}}_\ell[2d] \otimes p_M^*\mathcal{K}).$$

Here we used the projection formula and the isomorphism $a^*\mathcal{K} \simeq p_M^*\mathcal{K}$ given by the P–equivariance of $\mathcal{K}$. Hence we obtain the morphism

$$Ra_!(\overline{\mathbb{Q}}_\ell[2d] \otimes p_M^*\mathcal{K}) \to \mathcal{K}.$$

Applying the functor Rf_* to the morphism above and noticing that $Rf_* = Rf_!$, we have

$$Rf_! Ra_!(\overline{\mathbb{Q}}_\ell[2d] \otimes p_M^* \mathcal{K}) \to Rf_* \mathcal{K};$$

where the lefthand side can be rewritten as

$$\begin{aligned} Rf_! Ra_!(\overline{\mathbb{Q}}_\ell[2d] \otimes p_M^* \mathcal{K}) &= Rf_! Rp_{M!}(\overline{\mathbb{Q}}_\ell[2d] \otimes p_M^* \mathcal{K}) = Rf_!(Rp_{M!}\overline{\mathbb{Q}}_\ell[2d] \otimes \mathcal{K}) \\ &= Rf_!(Rp_{M!}(p_P^*\overline{\mathbb{Q}}_\ell[2d]) \otimes \mathcal{K}) \\ &= Rf_!(f^* Rg_! \overline{\mathbb{Q}}_\ell[2d] \otimes \mathcal{K}) \\ &= Rg_! \overline{\mathbb{Q}}_\ell[2d] \otimes Rf_! \mathcal{K}; \end{aligned}$$

where the first equality follows from

$$f \circ a = g \times_B f : P \times_B M \to B;$$

the second equality is given by the projection formula, the third equality follows from $p_P^* \overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell$, the fourth equality is the base change

$$Rp_{M!} p_P^* = f^* Rg_!$$

with respect to the diagram (19), and the last equality is again given by the projection formula.

To show the second claim of the proposition, we apply the same construction from above to the commutative diagram

$$\begin{array}{ccc} P \times_B P \times_B M & \xrightarrow{\mathrm{id}_P \times a} & P \times_B M \\ \Big\downarrow{\scriptstyle m \times \mathrm{id}_M} & & \Big\downarrow{\scriptstyle a} \\ P \times_B M & \xrightarrow{a} & M \end{array} \tag{20}$$

Again using the trace map, each path defines a morphism

$$\mathfrak{f}_P \otimes \mathfrak{f}_P \otimes Rf_* \mathcal{K} \to Rf_* \mathcal{K}:$$

The path via the lower-left corner gives the morphism $c \circ (c \otimes \mathrm{id})$ and the path via the upper-right corner gives the morphism $c \circ (\mathrm{id} \otimes c)$. The equivariant structure on $\mathcal{K}$ implies a cocycle condition on the isomorphism after pullback to $P \times_B P \times_B M$; this cocycle condition implies that these two morphisms agree.

This completes the proof of Proposition 1.4. □

1.4 Actions on each support and freeness

We denote by I the set of the supports $Z_\lambda \subset B \times_k \bar{k}$ of Rf_*K. In view of the condition (a) of Theorem 1.1, we have a canonical decomposition of the perverse sheaf ${}^pH^i(Rf_*K)$ in terms of the supports

$$ {}^pH^i(Rf_*K) = \bigoplus_{\lambda \in I} K^i_\lambda, $$

where K^i_λ has support Z_λ indexed by $\lambda \in I$. We collect all the direct summands of Rf_*K with support Z_λ,

$$ K_\lambda := \bigoplus_i K^i_\lambda. \tag{21} $$

In the following four steps, we prove that Proposition 1.3 can be reduced to a freeness property concerning stalks of (21).

Step 1 For any support Z_λ of Rf_*K, we may find an open dense subset $V_\lambda \subset Z_\lambda$ such that

(i) the restriction of K^i_λ to V_λ is of the form $L^i_\lambda[\dim V_\lambda]$ with L^i_λ a pure local system of weight i,

(ii) the restriction P_λ of the group scheme P to the support Z_λ admits a Chevalley decomposition

$$ 1 \to R_\lambda \to P_\lambda \to A_\lambda \to 1, \tag{22} $$

whose induced short exact sequence of Tate modules

$$ 0 \to T_{\bar{\mathbb{Q}}_\ell}(R_\lambda) \to T_{\bar{\mathbb{Q}}_\ell}(P_\lambda) \to T_{\bar{\mathbb{Q}}_\ell}(A_\lambda) \to 0 $$

satisfies that $T_{\bar{\mathbb{Q}}_\ell}(R_\lambda)$ is a pure local system of weight -2, and

(iii) for any other support $Z_{\lambda'}$, we have $Z_{\lambda'} \cap V_\lambda = \emptyset$ unless $Z_\lambda \subset Z_{\lambda'}$.

Since (i) and (iii) are standard and (ii) only concerns the group scheme P, this follows identically from [40, Section 7.4.8].

Step 2 In [40, Section 7.4.6], we replace $Rf_!\bar{\mathbb{Q}}_\ell$ by $Rf_!K$, and we replace the cap product action

$$ \Lambda_P \otimes Rf_!\mathbb{Q}_\ell^X \to Rf_!\mathbb{Q}_\ell^X $$

of [40, Section 7.4.2] by the action (17) constructed in Proposition 1.4:

$$ \Lambda_P \otimes Rf_!K \to Rf_!K. $$

As a consequence, for each $\lambda \in I$ and i we obtain an morphism

$$T_{\overline{\mathbb{Q}}_\ell}(P_\lambda) \otimes K^i_\lambda \to K^{i-1}_\lambda. \tag{23}$$

The last statement follows from an identical argument as in Sections 7.4.6 and 7.4.7 of [40]. More precisely, perverse truncation functors yield

$$T_{\overline{\mathbb{Q}}_\ell}(P_\lambda) \otimes {}^pH^i(Rf_!K) \to {}^pH^{i-1}(Rf_!K);$$

which can be further written as

$$\bigoplus_{\lambda \in I} T_{\overline{\mathbb{Q}}_\ell}(P_\lambda) \otimes K^i_\lambda \to \bigoplus_{\lambda \in I} K^{i-1}_\lambda$$

in terms of the supports Z_λ. This gives the canonical morphism (23).

Step 3 Now we combine Steps 1 and 2. Consider the restriction of K_λ to V_λ of Step 1,

$$L_\lambda := \bigoplus_i L_\lambda[-i].$$

Using the last part of Proposition 1.4, the morphisms (23) extend to an action of the local system of graded algebras $\wedge^\bullet_{P_\lambda} = \bigoplus \bigwedge^i T_{\overline{\mathbb{Q}}_\ell}(P_\lambda)[-i]$ on L_λ.

As explained in [40, Section 7.4.9], the first paragraph of page 121, an argument using weights shows that (23) passes through an action of the abelian variety part $T_{\overline{\mathbb{Q}}_\ell}(A_\lambda)$ of the Tate module $T_{\overline{\mathbb{Q}}_\ell}(P_\lambda)$. As a result, we have a graded module structure on L_λ of the graded algebra $\wedge^\bullet_{A_\lambda}$ associated with the abelian scheme A_λ in (22),

$$\wedge^\bullet_{A_\lambda} \otimes L_\lambda \to L_\lambda. \tag{24}$$

Note that we use the assumption that k is a finite field here.

Step 4 As commented in the paragraph after [40, Proposition 7.4.10], Proposition 1.3 can be deduced from the following proposition.

Proposition 1.5 We follow the same notation as in Steps 1–3 above. Let $u_\lambda \in V_\lambda$ be any geometric point. Then the stalk $L_{\lambda;u_\lambda}$ of L_λ is a free graded module of the graded algebra $\wedge^\bullet_{A_\lambda;u_\lambda}$ under the action (24).

We complete the proof of Proposition 1.5 in the next two sections.

1.5 Freeness

In this section we prove the following proposition, generalizing [40, Proposition 7.5.1]. Then in Section 1.6 we eventually reduce Proposition 1.5 to Proposition 1.6.

Proposition 1.6 Assume X is projective over $\bar{k}$ and admits an action of an abelian variety A over $\bar{k}$ with finite stabilizers. Let $E \in D_c^b(X, \bar{\mathbb{Q}}_\ell)$ be an A–equivariant object. Then the graded cohomology group

$$\bigoplus_i H^i(X, E)[-i] \tag{25}$$

is naturally a free graded module of the graded algebra $\Lambda_A = \bigoplus_i H^i(A, \bar{\mathbb{Q}}_\ell)[-i]$.

Proof Since the A–action preserves the connected components of X, we may assume that X is connected. We consider the quotient map

$$q\colon X \to Y := X/A$$

with X/A an Artin stack with finite inertia. Thanks to the projectivity of A, the morphism q is smooth and proper. For the A–equivariant object E, there exists an object E' on Y such that

$$q^*E' = E;$$

and the projection formula yields

$$Rq_*E = Rq_*\bar{\mathbb{Q}}_\ell \otimes E'. \tag{26}$$

In particular, the complex Rq_*E admits a natural Λ_A–action through the first factor of the righthand side of (26). This shows that (25) is a natural graded Λ_A–module.

Now since q is smooth and proper, we have a decomposition[1]

$$Rq_*\bar{\mathbb{Q}}_\ell \simeq \bigoplus_i R^iq_*\bar{\mathbb{Q}}_\ell[-i]. \tag{27}$$

Moreover, we consider the Cartesian diagram, with all the arrows smooth maps,

$$\begin{tikzcd} A \times X \arrow[r, "q'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & Y \end{tikzcd}$$

[1]As explained in the proof of [40, Proposition 7.5.1], the decomposition here is induced by the cup-product with an relative ample class. We also refer to [46] as a general reference for the decomposition theorem for Artin stacks with affine stabilizers.

By the base change, we obtain the canonical A–equivariant isomorphism of local systems

(28) $$q^* R^i q_* \mathbb{Q}_\ell \xrightarrow{\simeq} R^i q'_*(q'^* \mathbb{Q}_\ell)|_X.$$

We have that $q'^*\mathbb{Q}_\ell$ is a trivial local system of rank 1 and an A–equivariant structure on it is trivial by the connectedness of A; cf [51, Lemma A.1.2]. In particular, $q^* R^i q_* \mathbb{Q}_\ell$ is a trivial local system equipped with the trivial A–equivariant structure by (28). Consequently, each $R^i q_* \mathbb{Q}_\ell$ is canonically isomorphic to the trivial local system taking values in $H^i(A, \mathbb{Q}_\ell)$. The filtration $(Rq_* \mathbb{Q}_\ell) \otimes E'$ of $Rq_* E$ induces the spectral sequence

$$H^j(Y, R^i q_* \mathbb{Q}_\ell \otimes E') = H^j(Y, E') \otimes H^i(A, \mathbb{Q}_\ell) \Rightarrow H^{i+j}(X, E);$$

which degenerates thanks to (26) and the decomposition (27). Hence we obtain a filtration stable under the Λ_A–action, whose graded pieces are the free graded Λ_A modules

$$H^j(Y, E') \otimes \bigoplus_i H^i(A, \mathbb{Q}_\ell).$$

This proves the freeness of the entire module $H^*(X, E) = \bigoplus_i H^i(X, E)[-i]$. □

1.6 Proof of Proposition 1.5

We deduce Proposition 1.5 from Proposition 1.6 by a descending induction on the dimension of the support Z_γ. This is parallel to [40, Section 7.7].

We complete the induction in the following three steps.

Step A The induction base follows from Proposition 1.6, which we explain as follows. We assume $Z_{\gamma_0} = B$ and V_{γ_0} is an open dense subset of Z_{γ_0} as in Step 1 of Section 1.4. All the other Z_γ with $\gamma \neq \gamma_0$ do not intersect with V_{γ_0}, and

$${}^pH^i(Rf_*K)|_{V_{\gamma_0}} = L^i_{\gamma_0}[\dim B].$$

Therefore, for any geometric point u_{γ_0} of V_{γ_0} with

$$i_{u_{\gamma_0}} : M_{u_{\gamma_0}} \hookrightarrow M$$

the corresponding fiber, we have the identification

(29) $$\bigoplus_i L^i_{\gamma_0, u_{\gamma_0}}[-i + \dim B] = \bigoplus_i H^i(M_{u_{\gamma_0}}, i^*_{u_{\gamma_0}} K)[-i]$$

by the base change. Parallel to Step 3 of Section 1.4, (29) admits a natural $\pi_0(A_{\lambda_0,u_{\lambda_0}})$–action induced by the action of $\pi_0(P_{\lambda_0,u_{\lambda_0}})$.

As explained in the last paragraph of [40, Section 7.7.1], we may assume that the geometric point u_{λ_0} is defined over a finite field. So there exists a quasi-lifting $A_{\lambda_0,u_{\lambda_0}} \to P_{\lambda_0,u_{\lambda_0}}$ (see [40, Proposition 7.5.3]) such that the $\pi_0(A_{\lambda_0,u_{\lambda_0}})$–action on (29) is induced by the $A_{\lambda_0,u_{\lambda_0}}$–action on $M_{u_{\lambda_0}}$. By the axiom (ii) of weak abelian fibrations, the $A_{\lambda_0,u_{\lambda_0}}$–action on $M_{u_{\lambda_0}}$ passing through $P_{\lambda_0,u_{\lambda_0}}$ has finite stabilizers. Hence Proposition 1.6 implies that (29) is free over $\pi_0(A_{\lambda_0,u_{\lambda_0}})$. This completes the proof of the induction base.

Step B Since Proposition 1.5 is a local statement, we may work with a strictly Henselian base. Assume that B_λ is the strict Henselization of a geometric point u_λ defined over a finite field lying in $V_\lambda \subset Z_\lambda$. By the choice of V_λ in Step 1 of Section 1.4, the stalk $K^i_{\lambda_0,u_\lambda}$ is nonzero only if Z_λ is strictly contained in Z_{λ_0}. In this case, the induction assumption implies that, for any $m \in \mathbb{Z}$, the graded $\overline{\mathbb{Q}}_\ell$–vector space

$$\bigoplus_i H^m(K^i_{\lambda_0,u_\lambda})[-i]$$

is equipped with a natural free $\pi_0(A_{\lambda,u_\lambda})$–action induced by (17). This is explained in [40, Proposition 7.7.4], which essentially relies on the polarizability of P, ie the axiom (iii) of weak abelian fibrations. (Since this part only concerns the group scheme P, the proof of [40, Proposition 7.7.4] applies identically here.)

Step C We complete the induction argument.

The condition (a) of Theorem 1.1 — ie the decomposition theorem for Rf_*K — implies the degeneracy of the spectral sequence

$$H^j({}^pH^i(Rf_*K)_{u_\lambda}) \Rightarrow H^{i+j}(M_{u_\lambda}, K_{u_\lambda}), \tag{30}$$

where $u_\lambda : M_{u_\lambda} \hookrightarrow M$ is the geometric fiber over u_λ. This induces a $\pi_0(A_{\lambda,u_\lambda})$–stable filtration $F^\bullet H$ on the total cohomology

$$H := \bigoplus_i H^i(M_{u_\lambda}, K_{u_\lambda})[-i],$$

whose m^{th} graded piece is

$$F^mH/F^{m+1}H = \bigoplus_i H^m({}^pH^i(Rf_*K)_{u_\lambda})[-i-m]. \tag{31}$$

In addition, we have the following:

(a) By picking a quasi-lifting $A_{\gamma;u_\gamma} \to P_{\gamma;u_\gamma}$ as in the last paragraph of Step A, it follows from Proposition 1.6 and the P–equivariance of $\mathcal{K}$ that H is a free graded $\mathbb{Q}[A_{\gamma;u_\gamma}]$–module.

(b) Since
$$\bigoplus_i {}^pH^i(Rf_*\mathcal{K}|_{B_\gamma})[-i] = \bigoplus_{\gamma'} \bigoplus_i \mathcal{K}^i_{\gamma';u_\gamma}[-i];$$
the graded piece (31), as a graded $\mathbb{Q}[A_{\gamma;u_\gamma}]$–module, is a direct sum of the graded $\mathbb{Q}[A_{\gamma;u_\gamma}]$–modules
$$\bigoplus_{i} H^m(\mathcal{K}^i_{\gamma';u_\gamma})[-i]$$
over all γ' with $Z_\gamma \subsetneq Z_{\gamma'}$.

(c) By Step B, the induction assumption implies that each
$$\bigoplus_i H^m(\mathcal{K}^i_{\gamma';u_\gamma})[-i]$$
is a free graded $\mathbb{Q}[A_{\gamma;u_\gamma}]$–module when $Z_\gamma \subsetneq Z_{\gamma'}$.

(d) The graded $\mathbb{Q}[A_{\gamma;u_\gamma}]$–module vanishes:
$$\bigoplus_i H^m(\mathcal{K}^i_{\gamma;u_\gamma})[-i] = \bigoplus_i H^{m+\dim V_\gamma}(\mathcal{L}^i_{\gamma;u_\gamma})[-i] = 0$$
if $m \neq -\dim V_\gamma$ for degree reasons, since $\mathcal{L}^i_{\gamma;u_\gamma}$ is a skyscraper sheaf supported at u_γ.

Recall the filtration $F^\bullet H$ associated with the spectral sequence (30), whose graded pieces are given by (31). We arrive at exactly the situation of the last paragraph of [40, page 131]: the spectral sequence (30) induces a 3–layer filtration of $\mathbb{Q}[A_{\gamma;u_\gamma}]$–modules
$$0 \subset F^{n+1}H \subset F^nH \subset H; \qquad n = -\dim V_\gamma = -\dim Z_\gamma;$$
where

- H is free by (a), and
- $F^{n+1}H$ and H/F^nH are free by (c) and (d). In fact, (d) ensures that
$$\bigoplus_i H^m(\mathcal{K}^i_{\gamma;u_\gamma})[-i]$$
vanishes when $m \neq n$, and therefore $F^mH/F^{m+1}H$ is free by (c).

This implies the freeness for
$$F^nH/F^{n+1}H=\bigoplus_i L_{\alpha;u_\alpha}[-i-n]\oplus\bigoplus_{\alpha'\succ\alpha}\bigoplus_i H^n(K_{i,\alpha';u_\alpha})[-i-n];$$
which completes the induction; see [40, pages 131–132]. □

Remark 1.7 We fixed some minor typos in [40, Section 7.7.2]: the correct formula for the m^{th} graded piece [40, page 131, line 9] of the Leray spectral sequence is
$$H^m\Big(\bigoplus_n {}^pH^n(Rf_*\mathbb{Q}_\ell)|_{s_0}[-n]\Big);$$
which is not equal to
$$H^m\Big(\bigoplus {}^pH^n(Rf_*\mathbb{Q}_\ell[-n])|_{s_0}\Big)[m-n]$$
as stated in [40]. As consequences, the following statements in the last paragraph of [40, page 131] are incorrect:

- For $\alpha'\succ\alpha$, we have that $H^m\big(\bigoplus_{n\in\mathbb{Z}} K^n_{\alpha';u_\alpha}[-n]\big)$ is a free $\Lambda_{A_{u_\alpha}}$–module.
- For $\alpha=\alpha'$, we have that $H^m(K_{\alpha;u_\alpha})=0$ unless $m=-\dim Z_\alpha$.

Their corrected versions are given in (b), (c) and (d) of Step C above.

1.7 Spread out for $\mathbb{C}$

As a corollary of Theorem 1.1, the following theorem concerns the intersection cohomology complex of a weak abelian fibration (M,P,B) over the complex numbers $\mathbb{C}$.

Theorem 1.8 Suppose that (M,P,B) is a weak abelian fibration over $\mathbb{C}$ of relative dimension d, ie the triple satisfies (i)–(iii) of Section 1.1. Assume that

$$\tau_{>2d}(Rf_*\mathrm{IC}_M[-\dim M])=0. \tag{32}$$

Then any support Z of the decomposition for $Rf_*\mathrm{IC}_M$ satisfies the inequality
$$\operatorname{codim} Z\geq\delta_Z.$$

Proof By a standard spreading out argument (see for example [1, Section 6]), we may reduce Theorem 1.8 to the same statement over finite fields. More precisely, we spread out the weak abelian fibration (M,P,B) over $\operatorname{Spec} R$ where R is a DVR of characteristic 0, such that the geometric fiber over a general prime $p\in\operatorname{Spec} R$ is a weak abelian fibration in characteristic p as in the beginning of Section 1.1.

Moreover, the condition (32) holds over a general prime in $\mathrm{Spec}\, R$. Therefore, if a support Z of the decomposition theorem associated with (M, P, B) violates the inequality $\mathrm{codim}\, Z \geq i_Z$, then by spreading out, this inequality is violated by a support over a general prime $\mathfrak{p} \in \mathrm{Spec}\, R$ as well, which contradicts our assumption that Theorem 1.8 holds over finite fields..

In our setting, note that the complex

$$K = IC_M[\dim M]$$

is P–equivariant, which satisfies (a)–(c) of Theorem 1.1 by the decomposition theorem, Verdier duality, and the condition (32), respectively. Hence we conclude Theorem 1.8 from Theorem 1.1. □

In order to apply the support theorem to the intersection cohomology complex for a weak abelian fibration with singular ambient space M, the crucial point is to verify the "relative dimension bound" (32). We discuss systematically in the next two sections how to obtain such a bound for the moduli of 1–dimensional sheaves and the moduli of semistable Higgs bundles.

2 Moduli of 1–dimensional sheaves and Higgs bundles

2.1 Overview

Throughout the rest of the paper, we work over the complex numbers $\mathbb{C}$. We show in this section that the morphisms (2) and (3) admit the structures of weak abelian fibrations.

A crucial technical result is Proposition 2.6, concerning a dimension bound for certain moduli of pure 1–dimensional sheaves. As a consequence, we verify in Theorem 2.3 the irreducibility of the moduli spaces $M^L_{\beta,\chi}$ of (A), which may be of independent interest.

The dimension bound given by Proposition 2.6 will be used again in Section 3, which plays an important role in the proof of our main theorems.

2.2 Curves in del Pezzo surfaces

Let S be a del Pezzo surface, ie a nonsingular projective surface with $-K_S$ ample.

Lemma 2.1 Let E be an effective divisor on S. Then

$$\dim H^1(S, \mathcal{O}_S(E)) = \dim H^2(S, \mathcal{O}_S(E)) = 0.$$

In particular, we have $\dim H^0(S, \mathcal{O}_S(E)) = \frac{1}{2}E\cdot(E - K_S) + 1$.

Proof By Serre duality, we obtain

$$H^2(S, \mathcal{O}_S(E)) = H^0(S, \mathcal{O}_S(K_S - E))^\vee = 0.$$

Now we prove the vanishing of

$$H^1(S, \mathcal{O}_S(E))^\vee = H^1(S, \mathcal{O}_S(K_S - E)). \tag{33}$$

We consider the short exact sequence

$$0 \to \mathcal{O}_S(K_S - E) \to \mathcal{O}_S(K_S) \to \mathcal{O}_E(K_S) \to 0,$$

which induces the long exact sequence

$$\cdots \to H^0(S, \mathcal{O}_E(K_S)) \to H^1(S, \mathcal{O}_S(K_S - E)) \to H^1(S, \mathcal{O}_S(K_S)) \to \cdots. \tag{34}$$

The vanishing $H^0(S, \mathcal{O}_E(K_S)) = 0$ follows from $\deg_E(K_S) < 0$, and Serre duality yields the vanishing $H^1(S, \mathcal{O}_S(K_S)) = 0$. Hence (34) implies the vanishing of (33).

The last statement follows from the Riemann–Roch formula. □

Let β be an ample and effective class on S. Then Lemma 2.1 implies that the base $B = \mathbb{P}H^0(S, \mathcal{O}_S(\beta))$ of (2) is of dimension

$$\dim B = \tfrac{1}{2}\beta(\beta - K_S).$$

We define $\pi\colon \mathcal{C}_B \to B$ to be the universal curve for the linear system $|\beta|$. Since β is ample, it is basepoint free on the del Pezzo surface S. Hence the Bertini theorem implies that a general member of $|\beta|$ is a nonsingular and integral curve of genus

$$g_\beta = \tfrac{1}{2}\beta(\beta + K_S) + 1.$$

In particular, there exists a Zariski open dense subset $U \subset S$ such that the restriction of $\mathcal{C}_B$ to U is smooth,

$$\pi\colon \mathcal{C} \to U \subset B.$$

We consider the relative degree-0 Picard variety

$$P := \mathrm{Pic}^0(\mathcal{C}_B/B)$$

parametrizing line bundles on the fibers of $\pi\colon \mathcal{C}_B \to B$ whose restrictions to each irreducible component are of degree 0. The projection morphism

$$\pi_P\colon P \to B$$

has fibers of pure dimension g_β. The restriction of P to U gives a smooth abelian scheme

$$\pi_P\colon P_U := \mathrm{Pic}^0(\mathcal{C}_U/U) \to U.$$

We also consider the relative degree e Picard variety over U

$$p_e : P^e_U := \mathrm{Pic}^e(C_U/U) \to U$$

for any integer e. We recall the following well-known fact [5, Lemma 1.3.5] concerning the variation of Hodge structures for Picard varieties of smooth curves.

Proposition 2.2 For any $e \in \mathbb{Z}$, we have an isomorphism of variations of Hodge structures on U:

$$R^i p_{e*}\mathbb{Q}_{P^e_U} \simeq \bigwedge\nolimits^i R^1\pi_*\mathbb{Q}_C.$$

2.3 Moduli spaces of 1–dimensional sheaves

Now assume that S is a toric del Pezzo surface with a polarization L. The moduli space $M^L_{\beta,\chi}$ parametrizes S–equivalence classes of pure 1–dimensional (Gieseker-)semistable sheaves F on S with

$$\mathrm{supp}(F) = \beta, \quad \chi(F) = \chi.$$

Here the semistability is with respect to the slope function

$$\mu(E) = \frac{\chi(E)}{c_1(E)\cdot L}.$$

We recall the Hilbert–Chow morphism

$$h : M^L_{\beta,\chi} \to B, \quad F \mapsto \mathrm{supp}(F),$$

defined by taking the Fitting support [29]. The open subvariety $h^{-1}(U) \subset M^L_{\beta,\chi}$ parametrizes line bundles supported on the nonsingular curves in $|\beta|$. Hence every fiber of h over a closed point $b \in U$ is an abelian variety of dimension g_β, and we have

$$h^{-1}(U) = \mathrm{Pic}^e(C_U/U), \quad e = \chi - 1 + g_\beta. \tag{35}$$

The moduli space $M^L_{\beta,\chi}$ can be viewed as a compactification of the relative Picard variety (35).

The following theorem is of independent interest, and we postpone its proof to Section 2.6.

Theorem 2.3 The moduli space $M^L_{\beta,\chi}$ is irreducible of dimension

$$\dim M^L_{\beta,\chi} = \beta^2 + 1 = \dim B + g_\beta.$$

The group scheme $\pi_P \colon P \to B$ acts naturally and fiberwise on the moduli space $M_{\beta,\chi}$ via tensor product,

$$\mathcal{L}\cdot\mathcal{F} = \mathcal{L}\otimes\mathcal{F} \quad \text{for } \mathcal{L}\in P_b = \pi_P^{-1}(b),\ \mathcal{F}\in h^{-1}(b),$$

with $b \in B$ a closed point.

Proposition 2.4 The triple $(M^L_{\beta,\chi}, P, B)$ with h and π_P above form a weak abelian fibration of relative dimension g_β.

Proof We need to check (i)–(iii) of Section 1.1. Recall from Section 2.2 that the group scheme $\pi_P \colon P \to B$ is smooth and its fibers are of pure dimension g_β. Hence the condition (i) follows from Theorem 2.3. The affineness of the stabilizers (condition (ii)) is proven in [7, Lemma 3.5.4], and the polarizability of the Tate module (condition (iii)) associated with the group scheme P is given by [4, Theorem 3.3.1] as explained in [7, Lemma 3.5.5]. □

2.4 Moduli stacks

For the polarized surface (S, L), the moduli of semistable sheaves can be constructed as a GIT-quotient of the corresponding Quot-scheme (denoted by Quot),

$$M^L_{\beta,\chi} = \mathrm{Quot}^{ss}/GL_m,$$

where the semistable part of the Quot scheme Quot^{ss} and m rely on the Hilbert polynomial $\dim H^0(S, \mathcal{F}\otimes L^{\otimes n})$ of a semistable sheaf $\mathcal{F}$ with $\mathrm{supp}(\mathcal{F}) = \beta$ and $\chi(\mathcal{F}) = \chi$. We also consider the moduli stack of semistable sheaves

$$\mathcal{M}^L_{\beta,\chi} = [\mathrm{Quot}^{ss}/GL_m]$$

such that the natural projection

$$q \colon \mathcal{M}^L_{\beta,\chi} \to M^L_{\beta,\chi}$$

induces a good moduli space of the Artin stack $\mathcal{M}^L_{\beta,\chi}$.

Lemma 2.5 The stack $\mathcal{M}^L_{\beta,\chi}$ is nonsingular of dimension

$$\dim \mathcal{M}^L_{\beta,\chi} = \beta^2.$$

Proof The obstruction space for a semistable sheaf $\mathcal{F} \in \mathcal{M}^L_{\beta,\chi}$ is

(36) $$\mathrm{Ext}^2_S(\mathcal{F},\mathcal{F}) = \mathrm{Hom}_S(\mathcal{F},\mathcal{F}\otimes\omega_S)^\vee, \quad \omega_S = \mathcal{O}_S(K_S).$$

We prove in the following that (36) vanishes.

By the semicontinuity of $\mathrm{Hom}_S(F, F \otimes \omega_S)$, it suffices to show the vanishing

$$\mathrm{Hom}_S(F, F \otimes \omega_S) = 0$$

when F is a polystable sheaf on S. Hence we only need to prove the vanishing

$$\mathrm{Hom}_S(F_1, F_2 \otimes \omega_S) = 0 \tag{37}$$

for two stable sheaves F_1 and F_2 on S with the same slope

$$\mu(F_1) = \mu(F_2).$$

Since $-K_S$ is effective for the del Pezzo surface S, we have a short exact sequence

$$0 \to F_2 \otimes \omega_S \to F_2 \to F_2|_E \to 0,$$

where E is a curve in the linear system $|-K_S|$. The induced long exact sequence gives

$$0 \to \mathrm{Hom}_S(F_1, F_2 \otimes \omega_S) \to \mathrm{Hom}_S(F_1, F_2) \to \mathrm{Hom}_S(F_1, F_2|_E). \tag{38}$$

When $F_1 \ncong F_2$, by the stability we have $\mathrm{Hom}_S(F_1, F_2) = 0$. When $F_1 = F_2$, the second map of (38) is injective:

$$\mathrm{Hom}_S(F_1, F_1) = \mathbb{C} \cdot \mathrm{id} \hookrightarrow \mathrm{Hom}_S(F_1, F_1|_E).$$

In particular, (37) vanishes in either case. This implies the vanishing of the obstruction (36) and proves that $M^L_{\beta,\chi}$ is nonsingular. Consequently, we have

$$\dim M^L_{\beta,\chi} = \dim \mathrm{Ext}^1_S(F, F) - \dim \mathrm{Hom}_S(F, F) = -\chi(F, F) = \beta^2. \qquad \square$$

Combining with the Hilbert–Chow morphism $h\colon M^L_{\beta,\chi} \to B$, we obtain a morphism

$$h_M\colon M^L_{\beta,\chi} \to B. \tag{39}$$

Proposition 2.6 Let S be a toric del Pezzo surface. For any closed point $b \in B$, we have the following dimension bound for the fiber of (39):

$$\dim h_M^{-1}(b) \le \tfrac{1}{2}\beta \cdot (\beta + K_S). \tag{40}$$

When b represents an integral nonsingular curve, then $h_M^{-1}(b)$ is exactly a connected component of its Picard stack whose dimension $g_\beta - 1$ matches the righthand side of (40).

We prove Proposition 2.6 in Section 2.5. Then in Section 2.6 we use Proposition 2.6 to complete the proof of Theorem 2.3.

2.5 Proof of Proposition 2.6

We reduce Proposition 2.6 to a dimension bound for the nilpotent cone of Higgs bundles.

Let C be a nonsingular curve of genus g, and let D be a degree d effective divisor on C with

$$d > 2g-2. \tag{41}$$

We denote by $M^{\mathrm{nil}}_{n,\chi}$ the moduli stack of nilpotent Higgs bundles

$$(\mathcal{E},\theta) \quad \text{with } \theta: \mathcal{E} \to \mathcal{E}\otimes \mathcal{O}_C(D),\ \mathrm{rank}(\mathcal{E}) = n,\ \chi(\mathcal{E}) = \chi,$$

where we do not impose any (semi)stability conditions.

The stack $M^{\mathrm{nil}}_{n,\chi}$ is essentially the central fiber of the (stacky) Hitchin fibration. Alternatively, by the spectral correspondence, $M^{\mathrm{nil}}_{n,\chi}$ parametrizes pure 1–dimensional sheaves $\mathcal{F}$ with

$$\mathrm{supp}(\mathcal{F}) = nC, \quad \chi(\mathcal{F}) = \chi,$$

on the total space $\mathrm{Tot}(\mathcal{O}_C(D))$ of the line bundle $\mathcal{O}_C(D)$. Here the spectral correspondence is induced by the pushforward along the standard projection $\mathrm{Tot}(\mathcal{O}_C(D)) \to C$.

Proposition 2.7 (cf [8]) We have

$$\dim M^{\mathrm{nil}}_{n,\chi} \leq n(g-1) + \tfrac{1}{2}n(n-1)d. \tag{42}$$

Proof The dimension formula for the stack of the nilpotent cone and the comparison to the righthand side of (42) are given in lines 2 and 6 of [8, page 725, Section 10]. Although it is assumed in the beginning of [8] that the curve C has genus $g \geq 2$, the dimension calculation of [8, Section 10] does not require this constraint as long as (41) holds.[2] □

Now we prove Proposition 2.6.

We consider the maximal open torus $T \subset S$ whose action on S induces T–actions on both the moduli stack $M^L_{\beta,\chi}$ and the base B. By a semicontinuity argument (cf [18, Proof of Corollary 1]), it suffices to show (40) for all T–fixed points $b \in B$. Since we are only concerned with dimension counts, we prove the following stronger statement for toric divisors without imposing (semi)stability conditions.

[2] An alternative proof of this dimension bound can be obtained using the method of [43, Proposition 3.1]. We note that the last equation of [8, Section 10] shows that (42) still holds if $d = 2g-2$.

Claim For an effective divisor
$$E = \sum_i n_i E_i, \quad n_i > 0,$$
with each E_i a nonsingular irreducible toric divisor, we have

$$\dim \mathcal{M}_{E,\chi} \le \tfrac{1}{2} E\cdot(E + K_S). \tag{43}$$

Here $\mathcal{M}_{E,\chi}$ stands for the moduli stack of pure 1–dimensional sheaves supported on $E \subset S$.

We prove (43) by induction on the number of the irreducible components $\{E_i\}$.

For the induction base, we consider $E = nE'$ with E' irreducible. Then $E' \simeq \mathbb{P}^1$ and the normal bundle $\mathcal{O}_{E'}(d)$ of E' in S satisfies

$$d = E'^2 = -2 + E'\cdot(-K_S) > -2. \tag{44}$$

Since the formal neighborhood of an irreducible toric divisor only depends on the degree of the normal bundle, the thickened curve $E = nE' \subset S$ is isomorphic to the n^{th} thickening
$$nE' \subset \mathrm{Tot}(\mathcal{O}_{E'}(d))$$
of the 0–section in the total space of $\mathcal{O}_{E'}(d)$. Hence by Proposition 2.7, where the condition (41) is guaranteed by (44), we have
$$\dim \mathcal{M}_{nE',\chi} \le -n + \tfrac{1}{2}n(n-1)d = nE'\cdot(nE' + K_S)/2.$$
Here we used $E'^2 = d$ and $E'\cdot K_S = -d-2$ in the last identity. This proves the induction base.

To complete the induction, we assume that $E = E' + E''$. Here
$$E' = \sum_i n_i E'_i \quad \text{and} \quad E'' = \sum_i m_i E''_i,$$
with E'_i, E''_j irreducible toric divisors satisfying $E'_i \neq E''_j$ for any i, j.

Lemma 2.8 Let F be a pure 1–dimensional sheaf supported on E. Then there exists a canonical short exact sequence
$$0 \to F' \to F \to F'' \to 0,$$
where F' and F'' are pure 1–dimensional sheaves supported on E' and E'', respectively.

Proof We take

$$F'' := (F|_{E''})/(\text{maximal 0–dimensional subsheaf of } F|_{E''}) \in \mathrm{Coh}(E''), \quad F' := \mathrm{Ker}(F \to F|_{E''} \to F'') \in \mathrm{Coh}(E').$$

Since F' is a subsheaf of F, it is pure, supported on E'. □

Two sheaves F' and F'' (as in Lemma 2.8) with different supports satisfy

$$\mathrm{Hom}_S(F'', F') = \mathrm{Hom}_{E'}(i^*F'', F') = 0,$$

where $i : E' \hookrightarrow S$ is the embedding and i^*F'' is 0–dimensional on E'. Serre duality further implies

$$\mathrm{Ext}^2_S(F'', F') = \mathrm{Hom}_S(F', F'' \otimes \omega_S)^\vee = 0.$$

Hence by Lemma 2.8, after decomposing the stack $\mathfrak{M}_{E,\chi}$ into strata, we obtain a morphism to

$$\bigsqcup_{\chi'+\chi''=\chi} \mathfrak{M}_{E',\chi'} \times \mathfrak{M}_{E'',\chi''},$$

whose closed fiber over (F', F'') has dimension upper bound

$$\dim \mathrm{Ext}^1(F'', F') = -\chi(F'', F') = E' \cdot E''.$$

Combining with the induction assumption on the dimensions of $\mathfrak{M}_{E',\chi'}$ and $\mathfrak{M}_{E'',\chi''}$, we conclude that

$$\dim \mathfrak{M}_{E,\chi} \le \tfrac{1}{2}E' \cdot (E' + K_S) + \tfrac{1}{2}E'' \cdot (E'' + K_S) + E' \cdot E'' = \tfrac{1}{2}E \cdot (E + K_S).$$

This completes the induction. □

2.6 Proof of Theorem 2.3

We first prove the irreducibility of $M^L_{\beta,\chi}$. Equivalently, we prove the irreducibility of the stack $\mathfrak{M}^L_{\beta,\chi}$.

Recall the open subset $U \subset B$ formed by nonsingular curves in the linear system $|\beta|$. The open substack $h_{\mathfrak{M}}^{-1}(U)$ parametrizes line bundles on these curves with Euler characteristic χ. In particular, $h_{\mathfrak{M}}^{-1}(U)$ is Zariski open and dense in an irreducible component of the relative Picard stack associated with the universal curve $\mathcal{C} \to U$. Assume $\mathfrak{M}^L_{\beta,\chi}$ has another irreducible component $\mathfrak{M}'$ which does not contain $h_{\mathfrak{M}}^{-1}(U)$. By Lemma 2.5 it has dimension

$$\dim \mathfrak{M}' = \beta^2;$$

and it maps to the complement $B \setminus U$ under the morphism (39). This implies that a general fiber of

$$h_M|_{M'} : M' \to B \setminus U$$

has dimension at least

$$\dim M' - (\dim B - 1) = \beta^2 - \tfrac{1}{2}\beta\cdot(\beta - K_S) + 1 > \tfrac{1}{2}\beta\cdot(\beta + K_S),$$

which contradicts Proposition 2.6. This completes the proof of the irreducibility of $M^L_{\beta,\chi}$. □

2.7 Higgs bundles

Most of the statements for $M_{\beta,\chi}$ discussed above hold identically for the moduli spaces $M_{n,\chi}$ of Higgs bundles in the case of (B). This is due to the fact that $M_{n,\chi}$ can be viewed as the moduli space of 1–dimensional semistable sheaves $\mathcal{F}$ on $\mathrm{Tot}(\mathcal{O}_C(D))$ with

$$[\mathrm{supp}(\mathcal{F})] = n[C], \quad \chi(\mathcal{F}) = \chi,$$

via the spectral correspondence. We summarize these results in the following, for the reader's convenience.

Recall the universal spectral curve

$$\pi : \mathcal{C}_B \to B$$

with $\pi_P : P = \mathrm{Pic}^0(\mathcal{C}_B/B) \to B$ the relative degree 0 Picard variety. Similar to the case of $M^L_{\beta,\chi}$, the group scheme P acts on $M_{n,\chi}$ via tensor product

$$\mathcal{L}\cdot\mathcal{F} = \mathcal{L}\otimes\mathcal{F}, \qquad \text{with } \mathcal{L}\in\pi_P^{-1}(b) \text{ and } \mathcal{F}\in h^{-1}(b) \text{ for all } b\in B.$$

Here we view a Higgs bundle as a pure 1–dimensional coherent sheaf supported on the spectral curve

$$\pi^{-1}(b) \subset \mathrm{Tot}(\mathcal{O}_C(D)).$$

The moduli stack of semistable Higgs bundles admits a morphism

$$q : \mathcal{M}_{n,\chi} \to M_{n,\chi},$$

which induces

$$h_{\mathcal{M}} = h\circ q : \mathcal{M}_{n,\chi} \to B.$$

Proposition 2.9 Assume $\deg(D) = d > 2g-2$. The following statements hold:

(a) The fiber of $h_{\mathcal{M}}$ over a closed point $b\in B$ satisfies

$$\dim h_{\mathcal{M}}^{-1}(b) \leq n(g-1) + \tfrac{1}{2}n(n-1)d.$$

(b) The stack $\mathcal{M}^{\chi}_{n,L}$ is irreducible and nonsingular of dimension

$$\dim \mathcal{M}^{\chi}_{n,L} = n^2 d.$$

(c) The moduli space $M^{\chi}_{n,L}$ is irreducible of dimension

$$\dim M^{\chi}_{n,L} = n^2 d + 1.$$

(d) The triple $(M^{\chi}_{n,L}, P, B)$ form a weak abelian fibration of relative dimension

$$g_n = n(g-1) + \tfrac{1}{2}n(n-1)d + 1.$$

Proof These statements are parallel to Proposition 2.7, Lemma 2.5, Theorem 2.3, and Proposition 2.4. Statement (a) follows from a semicontinuity argument and Proposition 2.7. Statement (b) follows from Serre duality for semistable Higgs bundles [39, Corollary 2.6]. Statement (c) follows from (a) and (b), as explained in Section 2.6. Statement (d) is deduced by an identical proof as for Proposition 2.4. □

2.8 Assumptions on the curve class β

In Section 2, the ampleness assumption of the curve class β is used for the following properties:

(I) The linear system $|\beta|$ is basepoint free.

(II) A general curve in $|\beta|$ is integral and nonsingular.

We may replace the ampleness assumption for β by the conditions (I) and (II) above.

Proposition 2.10 *Theorem 2.3 and Proposition 2.4 hold for any curve class β which contains an integral curve in the linear system $|\beta|$.*

Proof Assume $C_0 \in |\beta|$ is integral. By the adjunction formula, either $C_0 \cong \mathbb{P}^1$ is an exceptional divisor or $C_0^2 \geq 0$. In the first case, the moduli space is a reduced point. In the second case, we obtain that the divisor C is integral and nef. Therefore (I) and (II) follow from [16, Corollary 4.7] and the Bertini theorem. □

3 Intersection cohomology complexes

We prove in this section a support inequality for the moduli spaces $M_{\beta,\chi}$ and $M^{\chi}_{n,L}$.

Theorem 3.1 *Let $h\colon M \to B$ be the morphism (2) or (3). We define the δ–function on B from the associated group scheme P as in Section 1. Then any support Z of $Rh_* IC_M$ satisfies*

$$\operatorname{codim} Z \geq \delta_Z.$$

By Theorem 1.8 and Propositions 2.4 and 2.9(d), it suffices to prove the following proposition concerning the intersection cohomology complex.

Proposition 3.2 We have

$$(45)\qquad {}^{\mathfrak{p}}\mathcal{H}^{>2R}(Rh_* IC_M[-\dim M]) = 0, \qquad R := \dim M - \dim B.$$

3.1 Sketch of the proof of Proposition 3.2

Although Proposition 3.2 only concerns bounded complexes on schemes, our proof relies on unbounded complexes on Artin stacks. From now on, we work with the derived category $D_c(-;\mathbb{Q}_\ell)$ of constructible sheaves with $\mathbb{Q}_\ell$–coefficients for Artin stacks as in [26; 27]. We denote by $D^b_c(-;\mathbb{Q}_\ell)$, $D^-(-;\mathbb{Q}_\ell)$ and $D^+(-;\mathbb{Q}_\ell)$ the subcategories of complexes which are bounded, bounded from above, and bounded from below, respectively. In this section, we assume that all Artin stacks are of finite type. We use the six operations for Artin stacks following [26; 27; 28]. Furthermore, by [28], we also have the perverse t–structure in the unbounded setting.

Recall the morphism from the moduli stack to the moduli space of 1–dimensional semistable sheaves/Higgs bundles

$$q : \mathcal{M} \to M.$$

The composition of q and $h : M \to B$ induces a morphism

$$h_{\mathcal{M}} = h \circ q : \mathcal{M} \to B.$$

We consider the (unbounded) complexes

$$Rh_{\mathcal{M}!}\mathbb{Q}_\ell \in D^-(B;\mathbb{Q}_\ell), \qquad Rq_*\mathbb{Q}_\ell \in D^+(M;\mathbb{Q}_\ell).$$

We first prove Proposition 3.2 assuming the following two propositions which concern the stack $\mathcal{M}$.

Proposition 3.3 We have

$$(46)\qquad {}^{\mathfrak{p}}\mathcal{H}^{>2R-2}(Rh_{\mathcal{M}!}\mathbb{Q}_\ell) = 0.$$

Proposition 3.4 There exists a splitting

$$(47)\qquad Rq_*\mathbb{Q}_\ell \simeq IC_M[-\dim M] \oplus E \in D^+(M;\mathbb{Q}_\ell).$$

Proof of Proposition 3.2 Applying the dualizing functor to the isomorphism (47), we obtain

$$\mathbb{D}(Rq_*\mathbb{Q}_\ell) \simeq IC_M[\dim M] \oplus E', \qquad E' \in D^-(M;\mathbb{Q}_\ell).$$

Since M is nonsingular, the lefthand side is isomorphic to

$$Rq_!\mathbb{D}(\mathbb{Q}_\ell) = Rq_!\mathbb{Q}_\ell[2\dim M] = Rq_!\mathbb{Q}_\ell[2\dim M - 2].$$

Combining the two equations above, we conclude that

$$Rq_!\mathbb{Q}_\ell \simeq IC_M[-\dim M + 2] \oplus \mathcal{K}, \qquad \mathcal{K} \in D(M;\mathbb{Q}_\ell).$$

Hence, thanks to properness of $h: M \to B$, we have $Rh_! = Rh_*$, and the lefthand side of (45) (shifted by degree 2) is a direct sum component of the lefthand side of (46). In particular, Proposition 3.2 follows from Proposition 3.3. □

In the rest of Section 3, we prove Propositions 3.3 and 3.4.

3.2 Proof of Proposition 3.3

Proposition 3.3 is a consequence of the following well-known vanishing and the dimension bounds (Proposition 2.6 for (A) and Proposition 2.9(a) for (B)) obtained in Section 2.

Lemma 3.5 Let Y be an irreducible Artin stack of dimension r. Then for $n > 2r$ we have the following vanishing for compactly supported cohomology:

$$H_c^n(Y;\mathbb{Q}_\ell) = 0. \tag{48}$$

Proof In the special case when Y is nonsingular, (48) follows from the Verdier duality

$$H_c^n(Y;\mathbb{Q}_\ell)^\vee = H^{2r-n}(Y;\mathbb{Q}_\ell) = 0, \qquad 2r - n < 0.$$

In general, since we are only concerned with the constructible sheaf $\mathbb{Q}_\ell$, we may assume that Y is reduced. Then by stratifying Y into locally closed nonsingular substacks and the excision sequences ([28, Example 2.1(iv)]), we reduce (48) for general Y to the nonsingular ones. □

Let $b \in B$ be a closed point. We denote by M_b the substack

$$M_b := h_M^{-1}(b) \subset M.$$

Propositions 2.6 and 2.9(a) yield

$$\dim M_b \leq R - 1, \qquad \text{where } R = \dim M - \dim B.$$

Combining with Lemma 3.5, we conclude that the complex

$$(Rh_{\mathcal{M}!}\mathbb{Q}_l)_b = H_c(\mathcal{M}_b, \mathbb{Q}_l)$$

is concentrated in degrees $\le 2(R-1)$ for any closed point $b \in B$. In particular,

$$(\tau_{>2R-2}(Rh_{\mathcal{M}!}\mathbb{Q}_l))_b = \tau_{>2R-2}((Rh_{\mathcal{M}!}\mathbb{Q}_l)_b) = 0 \quad \text{for all } b \in B.$$

This completes the proof of Proposition 3.3. □

3.3 Moduli of framed objects

The main difficulty for proving Proposition 3.4 is the nonproperness of the morphism $q: \mathcal{M} \to M$. In order to apply the decomposition theorem [1] to q, we use the moduli of framed objects [34] to "approximate" the stack $\mathcal{M}$. See [10; 11; 13; 14; 35; 38; 30] for applications of such techniques in the study of quivers representations and Donaldson–Thomas theory.

Let M and $\mathcal{M}$ be the moduli space and the moduli stack of (A) or (B) in Section 0.1. Since in either case M can be realized as a moduli space of semistable sheaves on an algebraic surface, we obtain (see Section 2.4) that M can be realized as a GIT-quotient of a Quot-scheme

$$M = \mathrm{Quot}^{ss}/\!/GL_m,$$

where the semistable locus $\mathrm{Quot}^{ss} \subset \mathrm{Quot}$ is with respect to a GL_m–linearized polarization $\mathcal{L}_m$ on Quot. The morphism q is induced by the morphism from the stack to the corresponding good GIT-quotient:

$$\mathcal{M} = [\mathrm{Quot}^{ss}/GL_m] \xrightarrow{q} \mathrm{Quot}^{ss}/\!/GL_m = M. \tag{49}$$

Proposition 3.6 *For any $N > 0$, there exist a nonsingular scheme M_f and a nonsingular Artin stack $\mathcal{X}_f$ with a commutative diagram*

$$\begin{tikzcd} M_f \arrow[rr, hook, "j"] \arrow[dr, "p_M"'] & & \mathcal{X}_f \arrow[dl, "p_X"] \\ & M & \end{tikzcd} \tag{50}$$

satisfying the following properties:

(a) p_X *is an affine space bundle,*

(b) $j: M_f \hookrightarrow \mathcal{X}_f$ *is an open immersion,*

(c) the composition $M_f \xrightarrow{p_M} M \xrightarrow{q} \mathbf{M}$ is projective,

and (d) for the complement $Z_f := \mathfrak{X}_f \setminus M_f$, we have

$$\mathrm{codim}_{\mathfrak{X}_f}(Z_f) > N.$$

Proof We complete the proof of Proposition 3.2 in the following three steps.

Step 1 (quiver moduli) For a fixed integer $f > m$, let Q_f be a quiver of two vertices P_1 and P_2 such that the dimension vector is $(1, m)$ and there are f arrows from P_1 to P_2. Following King [25], the representation space

$$A := \mathrm{Hom}(\mathbb{C}, \mathbb{C}^m)^f \cong \mathbb{C}^{mf}$$

of the quiver Q_f admits a natural action of the group

$$G_m := GL_1 \times GL_m.$$

Moreover, for any $\ell > 0$, the character

$$\chi : G_m \to \mathbb{C}^*, \qquad (g_1, g_m) \mapsto \det(g_1)^{-m} \cdot \det(g_m), \qquad \text{where } g_i \in GL_i,$$

yields a stability condition on A. Here the stability is given by GIT associated with the trivial line bundle $\mathcal{O}_A$, equipped with the G_m–linearization induced by χ. We denote by $A^{ss} \subset A$ the semistable locus with respect to χ.

Claim We have

$$\mathrm{codim}_A(A \setminus A^{ss}) \to \infty \quad \text{when } f \to \infty.$$

Proof If we view A as the parameter space of $m \times f$ matrices, the GIT-unstable loci are contained in the determinantal variety $D_{m-1} \subset A$ formed by matrices of rank $< m$. Hence we have

$$\dim(A \setminus A^{ss}) \le \dim D_{m-1} = (m-1)(f+1),$$

which implies that

$$\mathrm{codim}_A(A \setminus A^{ss}) \ge mf - (m-1)(f+1) = f + 1 - m \to \infty$$

when $f \to \infty$. □

Step 2 (moduli of framed objects) The moduli space of framed objects [34] combines the quotients (49) and the quiver Q_f, which provides the scheme M_f and the stack $\mathfrak{X}_f$ for Proposition 3.6. We recall the construction as follows.

Consider the natural $\mathbb{G}_m$–action on the product

$$\mathrm{Quot} \times \mathbb{A},$$

where the action on the first factor passes through the obvious GL_m–action and the action on the second factor is given in Step 1. Since the diagonal torus

$$GL_1 \hookrightarrow GL_1 \times GL_m = \mathbb{G}_m$$

acts trivially on both Quot and $\mathbb{A}$, the $\mathbb{G}_m$–action on $\mathrm{Quot} \times \mathbb{A}$ further passes through a $P\mathbb{G}_m (:= \mathbb{G}_m/GL_1)$–action.

We choose $a > 0$ such that $a > m$, and we consider the $\mathbb{G}_d$–linearization

$$\mathcal{L}^{a,\chi} := \mathcal{L}_m^{\otimes a} \boxtimes \mathcal{O}_{\mathbb{A}} \tag{51}$$

on $\mathrm{Quot} \times \mathbb{A}$. Let

$$(\mathrm{Quot} \times \mathbb{A})^{ss} \subset \mathrm{Quot} \times \mathbb{A}$$

be the GIT-semistable locus associated with (51). By a calculation using the Hilbert–Mumford criterion, it was proven in [34] (under a more general setup) that we have the open immersions

$$\mathrm{Quot}^{ss} \times \mathbb{A}^{ss} \subset (\mathrm{Quot} \times \mathbb{A})^{ss} \subset \mathrm{Quot}^{ss} \times \mathbb{A}. \tag{52}$$

Here the first inclusion is given in [34, Remark 3.40] and the second inclusion is given in [34, Proposition 3.39].

We define

$$M_f := (\mathrm{Quot} \times \mathbb{A})^{ss}/\mathbb{G}_m = (\mathrm{Quot} \times \mathbb{A})^{s}/P\mathbb{G}_m, \qquad \mathcal{X}_f := (\mathrm{Quot}^{ss} \times \mathbb{A})/P\mathbb{G}_m. \tag{53}$$

Note that by [34, Proposition 3.39], the scheme M_f can be interpreted as the moduli of framed objects, which is nonsingular by [34, Corollary 3.41].

Step 3 (properties) We show that M_f and $\mathcal{X}_f$ defined in (53) fit into the commutative diagram (50) and satisfy the properties (a)–(c). Moreover, they also satisfy (d) when $f \to \infty$.

The open immersion

$$j : M_f \hookrightarrow \mathcal{X}_f$$

is induced by the second inclusion of (52). The quotient stack $\mathcal{X}_f$ admits a natural map to $\mathcal{M}$ via the natural projection

$$p_{\mathcal{X}} : \mathcal{X}_f = (\mathrm{Quot}^{ss} \times \mathbb{A})/P\mathbb{G}_m \to \mathrm{Quot}^{ss}/P\mathbb{G}_m = \mathcal{M};$$

which is an A–bundle. By setting $p_M = p_X \circ j$, we obtain the commutative diagram (50) and (a) and (b) immediately. The property (c) follows from [34, Theorem 3.42].

We note that the morphism from M_f to M has a natural geometric interpretation. In fact, we have

$$M = (\mathrm{Quot}^{ss} A)/\mathbb{G}_m$$

by [34, last paragraph before Section 3.7]. Therefore the contraction

$$q \circ p_M : M_f = (\mathrm{Quot}\, A)^{ss}/\mathbb{G}_m \to (\mathrm{Quot}^{ss} A)/\mathbb{G}_m = M$$

can be viewed as a variation of GIT from the $\mathbb{G}_m$–linearization (49) to the $\mathbb{G}_m$–linearization

$$L^{a;0} := L_m^{\otimes a} \boxtimes \mathcal{O}_A$$

with the trivial $\mathbb{G}_m$–action on the second factor.

It remains to prove (d). By (52) and the claim in Step 1, we have

$$\mathrm{codim}_{X_f}(Z_f) \geq \mathrm{codim}_A(A \setminus A^{ss}) \to \infty \qquad \text{when } f \to \infty. \qquad \square$$

3.4 Proof of Proposition 3.4

To construct the desired splitting, we follow the approach of [34]. Namely we use the construction in the previous section to approximate M with M_f and apply the decomposition theorem to the proper morphism $q \circ p_M : M_f \to M$.

Fix $N > 0$ and choose f as in Proposition 3.6. Let $i : Z_f \hookrightarrow M_f$ denote the closed immersion which has codimension larger than N. Consider the excision triangle on X_f,

$$i_! i^! \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to Rj_* j^* \mathbb{Q}_\ell \to i_! i^! \mathbb{Q}_\ell[1]. \tag{54}$$

Since X_f is nonsingular, we have

$$i^! \mathbb{Q}_\ell = \omega_{Z_f}[-2\dim X_f].$$

Also, from Section V.2 of [23], we have that the complex ω_{Z_f} is concentrated in degrees $[-2\dim Z_f, \infty)$. By combining these with the codimension bound for Z_f, we have that the complex $i^! \mathbb{Q}_\ell$ is supported in degrees $[2N, \infty)$.

If we push forward the excision triangle (54) to M along

$$q \circ p_X : X_f \to M,$$

we obtain the triangle

$$R(q \circ p_X)_* i_! i^! \mathbb{Q}_\ell \to R(q \circ p_X)_* \mathbb{Q}_\ell \to R(q \circ p_X)_* Rj_* j^* \mathbb{Q}_\ell \to R(q \circ p_X)_* i_! i^! \mathbb{Q}_\ell[1]. \tag{55}$$

Since the derived pushforward functor preserves the subcategory $D^{\geq 0}$ and its shifts, the leftmost term of (55) is supported in degrees $[2N, \infty)$ as well. Furthermore, by [28, Lemma 3.3], after changing N by a bounded amount, we have the same support result for perverse cohomology sheaves. In other words, we have the vanishing

$${}^p\tau_{<2N} R(q \circ p_X)_* i_! i^! \mathbb{Q}_\ell = 0$$

and, by (55), the quasi-isomorphism

$${}^p\tau_{<2N} R(q \circ p_X)_* \mathbb{Q}_\ell \to {}^p\tau_{<2N} R(q \circ p_X)_* Rj_* j^* \mathbb{Q}_\ell. \tag{56}$$

Finally, since $p_M : X_f \to M$ is an affine space bundle, so that $Rp_{X*} \mathbb{Q}_\ell = \mathbb{Q}_\ell$, we can rewrite (56) as

$${}^p\tau_{<2N} Rq_* \mathbb{Q}_\ell \to {}^p\tau_{<2N} R(q \circ p_M)_* \mathbb{Q}_\ell. \tag{57}$$

By the decomposition theorem for the proper, surjective morphism $q \circ p_M : M_f \to M$, the righthand side of (57) can be noncanonically written as a direct sum of its (shifted) perverse cohomology sheaves:

$${}^p\tau_{<2N} R(q \circ p_M)_* \mathbb{Q}_\ell \simeq \bigoplus_{k=\dim M}^{2N-1} P_k[-k]. \tag{58}$$

The lowest perverse cohomology sheaf $P_{\dim M}$ occurs in degree $\dim M$, because of surjectivity of the morphism. After restricting to an open subset $V \subset M$ over which $q \circ p_M$ is smooth, it is given by the shifted local system whose fiber over $x \in V$ is $H^0(M_{x,f}, \mathbb{Q}_\ell)$, with $M_{x,f}$ the closed fiber of M_f over x. In particular, $P_{\dim M}$ contains IC_M as a direct summand.

If we combine the splitting (58) with (57), we see that the composition

$$IC_M[-\dim M] \to P_{\dim M}[-\dim M] \to {}^p\tau_{<2N} Rq_* \mathbb{Q}_\ell$$

admits a splitting

$$u_N : {}^p\tau_{<2N} Rq_* \mathbb{Q}_\ell \to IC_M[-\dim M].$$

As Ext-groups between perverse sheaves vanish outside of bounded degree, for N sufficiently large, we have the stabilization

$$\mathrm{Hom}({}^p\tau_{\leq 2N} R\pi_*\mathbb{Q}_\ell, \mathrm{IC}_M[\dim M]) = \mathrm{Hom}({}^p\tau_{\leq 2N+1} R\pi_*\mathbb{Q}_\ell, \mathrm{IC}_M[\dim M]).$$

In other words, for the canonical morphism

$$v_N : {}^p\tau_{\leq 2N} R\pi_*\mathbb{Q}_\ell \to {}^p\tau_{\leq 2(N+1)} R\pi_*\mathbb{Q}_\ell,$$

the splittings u_N are compatible in the sense that $u_N = u_{N+1} \circ v_N$.

By [26, Lemma 4.3.2], we have that the unbounded complex $R\pi_*\mathbb{Q}_\ell$ is the homotopy colimit of its truncations, ie

$$R\pi_*\mathbb{Q}_\ell = \operatorname*{hocolim}_{N\to\infty} {}^p\tau_{\leq 2N} R\pi_*\mathbb{Q}_\ell.$$

So as a result, the splittings u_N yield a splitting

$$u : R\pi_*\mathbb{Q}_\ell \to \mathrm{IC}_M[\dim M],$$

and consequently, a direct summand decomposition (47) as desired. □

4 Proof of the main theorem

4.1 Overview

We complete the proof of Theorem 0.4. For the approach, we combine the support inequality of Theorem 3.1 and techniques of [8; 4].

4.2 δ–inequalities for integral curves

As in Section 2.2, we consider the relative degree 0 Picard variety

$$\mathrm{Pic}^0(C_B/B) \to B$$

associated with a family of curves $\pi_B : C_B \to B$. We obtain the δ–invariants computing the dimensions of the affine parts of the group schemes

$$\delta(b) := \dim(\mathrm{Pic}^0(C_b)^{\mathrm{aff}}), \quad b \in B,$$

where C_b is the fiber of π_B over b. For a closed subvariety $Z \subset B$, we define δ_Z to be $\delta(b)$ for a general point in Z.

In Section 4.2, we first focus on the case of a flat family of integral curves

$$\pi_B\colon C_B \to B$$

satisfying that the compactified Jacobian

$$\mathrm{Pic}^0(C_B/B) = \overline{J}_{C_B}$$

(parametrizing degree 0 torsion-free sheaves on the curves C_b) is nonsingular.

The following lemma is the "Severi inequality", which is parallel to [4, (41)] for Higgs bundles. The proof of [4, (41)] works identically here since the only structure used for the spectral curves $C_B \to B$ in [4] is the smoothness of $\overline{J}_{C_B}$. We give a proof for the reader's convenience.

Lemma 4.1 For any subvariety $Z \subset B$, we have

$$\operatorname{codim} Z \geq \delta_Z.$$

Proof Let $C_{\mathrm{univ}} \to B_{\mathrm{univ}}$ be a semiuniversal family of curves such that the family $C_B \to B$ is induced by a map

$$\phi\colon B \to B_{\mathrm{univ}}.$$

Let $B_{\mathrm{univ}}^{\delta} \subset B_{\mathrm{univ}}$ be the locus given by $\{b \in B : \delta(b) = \delta\}$. Since $\overline{J}_{C_B}$ is nonsingular, by the paragraph following [17, Theorem 2], the image $\phi(B) \subset B_{\mathrm{univ}}$ meets $B_{\mathrm{univ}}^{\delta}$ transversally. Hence for any irreducible subvariety $Z \subset B$ whose general points lie in $B_{\mathrm{univ}}^{\delta}$, we have

$$\dim Z \leq \dim(\phi(B) \cap B_{\mathrm{univ}}^{\delta}) = \dim \phi(B) + \dim B_{\mathrm{univ}}^{\delta} - \dim B_{\mathrm{univ}} \leq \dim B - \delta,$$ where

the equality follows from the transversality. We conclude that

$$\operatorname{codim} Z = \dim B - \dim Z \geq \delta = \delta_Z. \qquad \square$$

Our major application of Lemma 4.1 is for curves in a linear system on a del Pezzo surface.

Corollary 4.2 Let β be a curve class on a del Pezzo surface, let

$$\pi_B\colon C_B \to B = \mathbb{P}H^0(S, \mathcal{O}_S(\beta)) \tag{59}$$

be the universal curve in the linear system, and let $\pi_B^{\circ}\colon C^{\circ} \to B^{\circ}$ be the restriction of (59) to the subset $B^{\circ} \subset B$ of integral curves. Then for any irreducible subvariety $Z \subset B$ whose generic point lies in B°, we have

$$\operatorname{codim} Z \geq \delta_Z.$$

Proof Since the compactified Jacobian $\overline{J}\mathcal{C}_{\mathcal{C}^\circ}$ associated with $\pi_B^\circ : \mathcal{C}^\circ \to B^\circ$ is an open subvariety of the moduli of stable pure 1–dimensional sheaves supported in the class β, we deduce the smoothness of $\overline{J}\mathcal{C}_{\mathcal{C}^\circ}$ from the smoothness of the moduli stack (cf Lemma 2.5). Hence Corollary 4.2 follows by applying Lemma 4.1 to the family $\pi_B : \mathcal{C}^\circ \to B^\circ$. ☐

When the integral locus $B^\circ \subset B$ is nonempty, by Proposition 2.10 the moduli space $M^L_{\beta,\chi}$ is irreducible for any polarization L and $\chi \in \mathbb{Z}$ satisfying that

$$\dim M^L_{\beta,\chi} = \dim \overline{J}\mathcal{C}_{B^\circ} = \beta^2 + 1.$$

We define the following invariant associated with the curve class β:

(60) $$\epsilon_\beta := \dim \mathrm{Pic}^0(\mathcal{C}^\circ/B^\circ) - 2\dim B = \dim M^L_{\beta,\chi} - 2\dim B = 1 + \beta\cdot K_S,$$

where the last equality follows from Lemma 2.1.

4.3 δ–Inequalities for linear systems

In Section 4.3, we assume that β is an ample curve class on a del Pezzo surface S. We introduce a stratification of

$$B = \mathbb{P}H^0(S, \mathcal{O}_S(\beta))$$

analogous to the stratification introduced in [8, Section 9] and [4, Section 5.2] for Higgs bundles.

We consider the s–tuples

(61) $$\underline{\beta} = ((m_1,\beta_1),(m_2,\beta_2),\dots,(m_s,\beta_s)),$$

where $s \geq 1$, $m_i \geq 1$, and β_i are (not necessarily distinct) curve classes on S such that (i) $\sum_{i=1}^{s} m_i\beta_i = \beta$, and

(ii) there exists an integral curve in $|\beta_i|$ for each $1 \leq i \leq s$.

The objects (61) are called types of the curves in the linear system $B = |\beta|$. Two such objects

$$\underline{\beta} = ((m_1,\beta_1),(m_2,\beta_2),\dots,(m_s,\beta_s)),$$
$$\underline{\beta}' = ((m'_1,\beta'_1),(m'_2,\beta'_2),\dots,(m'_{s'},\beta'_{s'})),$$

are said to give the same type if $s = s'$ and there exists a bijection

$$\sigma : \{1,2,\dots,s\} \to \{1,2,\dots,s\}$$

such that $\beta_i = \beta'_{\sigma(i)}$ and $m_i = m'_{\sigma(i)}$. We have a stratification according to the types of the curves in $|\beta|$,

$$B = \bigsqcup_{\underline{\beta}} B_{\underline{\beta}},$$

where each $B_{\underline{\beta}}$ is a locally closed subset of B formed by curves in $|\beta|$ of type $\underline{\beta}$:

$$B_{\underline{\beta}} = \Big\{ E = \sum_i m_i E_i \in |\beta| : E_i \in |\beta_i|;\ E_i \text{ are distinct integral curves} \Big\}.$$

Proposition 4.3 *Let $Z \subset B$ be an irreducible subvariety whose general points have type*

$$\underline{\beta} = ((m_1,\beta_1),(m_2,\beta_2),\dots,(m_s,\beta_s)).$$

Then we have

$$\delta_{\underline{\beta}} + \operatorname{codim} Z \geq \sum_{i=1}^{s} \delta_{\beta_i} + l_Z.$$

Proof We apply a similar argument as in [4, Corollary 5.4.4] for Higgs bundles.

For a curve class β_i, we denote by $|\beta_i|'$ the open subvariety of $|\beta_i| = PH^0(S, \mathcal{O}_S(\beta))$ consisting of integral curves. We define

$$C'_{\beta_i} \to |\beta_i|', \quad \mathrm{Pic}_{\beta_i} \to |\beta_i|', \tag{62}$$

to be the universal curve and the corresponding relative degree 0 Picard variety over $|\beta_i|'$. For a type $\underline{\beta}$ as in (61), we have a finite morphism

$$\phi_{\underline{\beta}} : \widetilde{B}_{\underline{\beta}} := \prod_{i=1}^{s} |\beta_i| \to B, \quad (E_i)_{i=1}^{s} \mapsto \sum_{i=1}^{s} m_i E_i,$$

whose image is

$$\mathrm{Im}(\phi_{\underline{\beta}}) = \overline{B_{\underline{\beta}}} \subset B.$$

The morphism $\phi_{\underline{\beta}}$ sends the open subvariety $\prod_{i=1}^{s} |\beta_i|' \subset \widetilde{B}_{\underline{\beta}}$ to $B_{\underline{\beta}} \subset B$.

Now we assume that $\eta \in B$ is the generic point of Z. By the first two paragraphs of [4, Proof of Corollary 5.4.4], there exists a point

$$\eta' = (\eta_1,\eta_2,\dots,\eta_s) \in \phi^{-1}(\eta) \subset \widetilde{B}_{\underline{\beta}}, \quad \eta_i \in |\beta_i|',$$

satisfying that

(i) $\dim \overline{\{\eta\}} = \dim \overline{\{\eta'\}} = \dim Z$,

(ii) $\dim(\mathrm{Pic}_{\eta})^{ab} = \sum_{i=1}^{s} \dim(\mathrm{Pic}_{\beta_i,\eta_i})^{ab}$.

Here for any connected commutative group scheme P, we use the notation P^{ab} to denote its abelian variety part in the Chevalley decomposition (15). Since by definition (see (60)) we have

$$\dim (\mathrm{Pic}_{\beta,\chi})^{ab} = g_{\beta} + \dim B - \delta_Z; \qquad \dim (\mathrm{Pic}_{\beta_i,\chi_i})^{ab} = g_{\beta_i} + \dim |\beta_i| - \delta_{\overline{\{\beta_i\}}},$$

we obtain from (ii) that

$$\delta_Z + \sum_{i=1}^{s} g_{\beta_i} - g_{\beta} = \dim B - \sum_{i=1}^{s} \dim |\beta_i| + \sum_{i=1}^{s} \delta_{\overline{\{\beta_i\}}}. \tag{63}$$

Applying Corollary 4.2 to (62), we have

$$\delta_{\overline{\{\beta_i\}}} \geq \dim |\beta_i| - \dim \overline{\{\beta_i\}},$$

which implies that the righthand side of (63) is

$$\dim B - \sum_{i=1}^{s} \dim \overline{\{\beta_i\}} = \dim B - \dim Z = \operatorname{codim} Z. \qquad \square$$

4.4 Higgs bundles

The analog of Proposition 4.3 for the moduli of Higgs bundles is exactly the inequality (75) of [4, Corollary 5.4.4]. Although the paper [4] works with Higgs bundles with $\gcd(n,\chi) = 1$, Corollary 5.4.4 only concerns the group scheme — the degree 0 Picard variety associated with the universal spectral curve, which is not constrained by the coprime assumption.

We now rewrite the inequality (75) of [4, Corollary 5.4.4] in Proposition 4.4 parallel to the form of Proposition 4.3.

Consider the Hitchin fibration

$$h: M_{n,\chi} \to B$$

associated with C, n, and an effective divisor D with degree $\deg(D) > 2g-2$. Recall from [4, Section 5.2] that the Hitchin base

$$B = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(iD))$$

admits a stratification

$$B = \bigsqcup_{(n;m)} B_{n;m}. \tag{64}$$

Here a type of spectral curve is given by $(n; m)$ with

$$s \geq 1; \qquad n = (n_1, n_2, \ldots, n_s); \qquad m = (m_1, m_2, \ldots, m_s); \qquad \sum_{i=1}^{s} m_i n_i = n;$$

and $B_{n;m}$ are formed by spectral curves $E \subset \mathrm{Tot}(\mathcal{O}_C(D))$ of the form $E = \sum_i m_i E_i$, with E_i distinct integral spectral curves that are degree n_i covers of the zero section C. This actually coincides with the notion of (61) since the class of any spectral curve in the surface $\mathrm{Tot}(\mathcal{O}_C(D))$ is of the form $\gamma_i = n_i[C]$ with $[C]$ the curve class of the zero section $C \subset \mathrm{Tot}(\mathcal{O}_C(D))$. We refer to [4, Section 5] for more details about the stratification (64).

We define the invariant, similar to (60),

$$\delta_n = \dim M_{n,\chi} - 2\dim B = 1 + (2g - 2 - \deg(D))n;$$

where we use the dimension formulas of [4, (77)] in the last identity.

Proposition 4.4 *Let $Z \subset B$ be an irreducible subvariety whose general points have type $(n; m)$. Then we have*

$$\delta_n + \operatorname{codim} Z \geq \sum_{i=1}^{s} \delta_{n_i} + \iota_Z.$$

Here ι_Z is defined via the relative degree 0 Picard variety associated with the spectral curves.

4.5 Proof of Theorem 0.4

We complete the proof of Theorem 0.4 in this section. Let

$$h : M^L_{\gamma,\chi} \to B$$

be the morphism (2). By Lemma 2.5, the open subvariety of stable sheaves

$$M^{L,s}_{\gamma,\chi} \subset M^L_{\gamma,\chi}$$

is nonsingular. So we have

$$(IC_{M^L_{\gamma,\chi}})|_{M^{L,s}_{\gamma,\chi}} = \mathbb{Q}[\dim M^L_{\gamma,\chi}].$$

In particular, the restriction of the direct image complex $Rh_* IC_{M^L_{\gamma,\chi}}$ to the open subset $U \subset B$ of nonsingular curves in $|\gamma|$ satisfies

$$Rh_* IC_{M^L_{\gamma,\chi}}|_U \simeq \bigoplus_{i=0}^{2d} \wedge^i R^1 \mathbb{Q}[\dim M^L_{\gamma,\chi} - i]. \tag{65}$$

Here $W := C \to U \subset B$, and (65) is an isomorphism of variations of Hodge structures by Proposition 2.2. Hence, in order to prove Theorem 0.4, it suffices to show that the lefthand side of (4), as a bounded complex of perverse sheaves, has full support B.

Assume that the irreducible subvariety $Z \subset B$ is a support whose general point has type $\underline{\gamma}$. We combine the inequalities of Proposition 4.3 and Theorem 3.1 to obtain

$$\delta_\gamma + \operatorname{codim} Z \geq \sum_{i=1}^{s} \delta_{\gamma_i} + r_Z \geq \sum_{i=1}^{s} \delta_{\gamma_i} + \operatorname{codim} Z,$$

which implies $\delta_\gamma \geq \sum_{i=1}^{s} \delta_{\gamma_i}$. Therefore,

$$1 - s \geq \Big(\gamma - \sum_i \gamma_i\Big) \cdot (-K_S). \tag{66}$$

Since $-K_S$ is ample and $\gamma - \sum_i \gamma_i \geq 0$, the only possibility for (66) to hold is $s = 1$ and $m_i = 1$. Equivalently, we have $Z = B$. This completes the proof of Theorem 0.4 for $M_{\gamma,\chi}$.

The proof for $M_{n,\chi}$ is identical, where we apply Theorem 3.1 and Proposition 4.4. □

References

[1] A A Beĭlinson, J Bernstein, P Deligne, Faisceaux pervers, from "Analysis and topology on singular spaces, I", Astérisque 100, Soc. Math. France, Paris (1982) 5–171 MR Zbl

[2] P Bousseau, A proof of N Takahashi's conjecture for $(\mathbb{P}^2, E)$ and a refined sheaves/Gromov–Witten correspondence, preprint (2020) arXiv 1909.02992v2

[3] P Bousseau, Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$, J. Algebraic Geom. 31 (2022) 593–686 MR Zbl

[4] M A de Cataldo, A support theorem for the Hitchin fibration: the case of SL_n, Compos. Math. 153 (2017) 1316–1347 MR Zbl

[5] M A A de Cataldo, T Hausel, L Migliorini, Topology of Hitchin systems and Hodge theory of character varieties: the case A_1, Ann. of Math. 175 (2012) 1329–1407 MR Zbl

[6] M A de Cataldo, D Maulik, J Shen, Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture, J. Amer. Math. Soc. 35 (2022) 911–953 MR Zbl

[7] M A A de Cataldo, A Rapagnetta, G Saccà, The Hodge numbers of O'Grady 10 via Ngô strings, J. Math. Pures Appl. 156 (2021) 125–178 MR Zbl

[8] P-H Chaudouard, G Laumon, Un théorème du support pour la fibration de Hitchin, Ann. Inst. Fourier (Grenoble) 66 (2016) 711–727 MR Zbl

[9] W-Y Chuang, D-E Diaconescu, G Pan, BPS states and the P D W conjecture, from "Moduli spaces" (L Brambila-Paz, O García-Prada, P Newstead, R P Thomas, editors), London Math. Soc. Lecture Note Ser. 411, Cambridge Univ. Press (2014) 132–150 MR Zbl

[10] B Davison, Cohomological Hall algebras and character varieties, Internat. J. Math. 27 (2016) art. id. 1640003 MR Zbl

[11] B Davison, The integrality conjecture and the cohomology of preprojective stacks, preprint (2017) arXiv 1602.02110v3

[12] B Davison, Refined invariants of finite-dimensional Jacobi algebras, preprint (2019) arXiv 1903.00659

[13] B Davison, BPS Lie algebras and the less perverse filtration on the preprojective CoHA, preprint (2020) arXiv 2007.03289v2

[14] B Davison, S Meinhardt, Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras, Invent. Math. 221 (2020) 777–871 MR Zbl

[15] M Demazure, Surfaces de del Pezzo, I–V, from "Séminaire sur les singularités des surfaces" (M Demazure, H C Pinkham, B Teissier, editors), Lecture Notes in Math. 777, Springer (1980) MR Zbl

[16] S Di Rocco, k–very ample line bundles on del Pezzo surfaces, Math. Nachr. 179 (1996) 47–56 MR Zbl

[17] B Fantechi, L Göttsche, D van Straten, Euler number of the compactified Jacobian and multiplicity of rational curves, J. Algebraic Geom. 8 (1999) 115–133 MR Zbl

[18] V Ginzburg, The global nilpotent variety is Lagrangian, Duke Math. J. 109 (2001) 511–519 MR Zbl

[19] M Groechenig, D Wyss, P Ziegler, Mirror symmetry for moduli spaces of Higgs bundles via p–adic integration, Invent. Math. 221 (2020) 505–596 MR Zbl

[20] G Harder, M S Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Ann. 212 (1975) 215–248 MR Zbl

[21] N J Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1987) 59–126 MR Zbl

[22] N Hitchin, Stable bundles and integrable systems, Duke Math. J. 54 (1987) 91–114 MR Zbl

[23] B Iversen, Cohomology of sheaves, Springer (1986) MR Zbl

[24] D Joyce, Y Song, A theory of generalized Donaldson–Thomas invariants, Mem. Amer. Math. Soc. 1020, Amer. Math. Soc., Providence, RI (2012) MR Zbl

[25] A D King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. 45 (1994) 515–530 MR Zbl

[26] Y Laszlo, M Olsson, The six operations for sheaves on Artin stacks, I: Finite coefficients, Publ. Math. Inst. Hautes Études Sci. 107 (2008) 109–168 MR Zbl

[27] Y Laszlo, M Olsson, The six operations for sheaves on Artin stacks, II: Adic coefficients, Publ. Math. Inst. Hautes Études Sci. 107 (2008) 169–210 MR Zbl

[28] Y Laszlo, M Olsson, Perverse t–structure on Artin stacks, Math. Z. 261 (2009) 737–748 MR Zbl

[29] J Le Potier, Faisceaux semi-stables de dimension 1 sur le plan projectif, Rev. Roumaine Math. Pures Appl. 38 (1993) 635–678 MR Zbl

[30] J Manschot, S Mozgovoy, Intersection cohomology of moduli spaces of sheaves on surfaces, Selecta Math. 24 (2018) 3889–3926 MR Zbl

[31] D Maulik, J Shen, Endoscopic decompositions and the Hausel–Thaddeus conjecture, Forum Math. Pi 9 (2021) art. id. e8, MR Zbl

[32] D Maulik, Y Toda, Gopakumar–Vafa invariants via vanishing cycles, Invent. Math. 213 (2018) 1017–1097 MR Zbl

[33] D Maulik, Z Yun, Macdonald formula for curves with planar singularities, J. Reine Angew. Math. 694 (2014) 27–48 MR Zbl

[34] S Meinhardt, Donaldson–Thomas invariants vs intersection cohomology for categories of homological dimension one, preprint (2015) arXiv 1512.03343

[35] S Meinhardt, M Reineke, Donaldson–Thomas invariants versus intersection cohomology of quiver moduli, J. Reine Angew. Math. 754 (2019) 143–178 MR Zbl

[36] A Mellit, Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures), Invent. Math. 221 (2020) 301–327 MR Zbl

[37] L Migliorini, V Shende, A support theorem for Hilbert schemes of planar curves, J. Eur. Math. Soc. 15 (2013) 2353–2367 MR Zbl

[38] S Mozgovoy, M Reineke, Intersection cohomology of moduli spaces of vector bundles over curves, preprint (2015) arXiv 1512.04076

[39] S Mozgovoy, O Schiffmann, Counting Higgs bundles and type A quiver bundles, Compos. Math. 156 (2020) 744–769 MR Zbl

[40] B C Ngô, Le lemme fondamental pour les algèbres de Lie, Publ. Math. Inst. Hautes Études Sci. 111 (2010) 1–169 MR Zbl

[41] R Pandharipande, R P Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009) 407–447 MR Zbl

[42] M Saito, Mixed Hodge modules, Publ. Res. Inst. Math. Sci. 26 (1990) 221–333 MR Zbl

[43] O Schiffmann, Indecomposable vector bundles and stable Higgs bundles over smooth projective curves, Ann. of Math. 183 (2016) 297–362 MR Zbl

[44] O G Schiffmann, Kac polynomials and Lie algebras associated to quivers and curves, from "Proceedings of the International Congress of Mathematicians, II: Invited lectures" (B Sirakov, P N de Souza, M Viana, editors), World Sci., Hackensack, NJ (2018) 1393–1424 MR Zbl

[45] J Shen, Q Yin, Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds, Duke Math. J. 171 (2022) 209–241 MR Zbl

[46] S Sun, Generic base change, Artin's comparison theorem, and the decomposition theorem for complex Artin stacks, J. Algebraic Geom. 26 (2017) 513–555 MR Zbl

[47] Y Toda, Stability conditions and curve counting invariants on Calabi–Yau 3–folds, Kyoto J. Math. 52 (2012) 1–50 MR Zbl

[48] Y Toda, Gopakumar–Vafa invariants and wall-crossing, J. Differential Geom. 123 (2023) 141–193 MR

[49] Z Yun, Langlands duality and global Springer theory, Compos. Math. 148 (2012) 835–867 MR Zbl

[50] Z Yun, W Zhang, Shtukas and the Taylor expansion of L–functions, Ann. of Math. 186 (2017) 767–911 MR Zbl

[51] X Zhu, An introduction to affine Grassmannians and the geometric Satake equivalence, from "Geometry of moduli spaces and representation theory" (R Bezrukavnikov, A Braverman, Z Yun, editors), IAS/Park City Math. Ser. 24, Amer. Math. Soc., Providence, RI (2017) 59–154 MR Zbl

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, United States

Department of Mathematics, Yale University
New Haven, CT, United States

maulik@mit.edu, junliang.shen@yale.edu

Proposed: Dan Abramovich
Seconded: Gang Tian, Richard P Thomas

Received: 10 January 2021
Revised: 1 October 2021

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2023 is US $740/year for the electronic version, and $1030/year (C$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 27 Issue 4 (pages 1273–1655) 2023