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Stability of Peaked Solitary Waves for a Class of Cubic Quasilinear Shallow-Water Equations

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This paper is concerned with two classes of cubic quasilinear equations, which can be derived as asymptotic models from shallow-water approximation to the 2D incompressible Euler equations. One class of the models has homogeneous cubic nonlinearity and includes the integrable modified Camassa–Holm (mCH) equation and Novikov equation, and the other class encompasses both quadratic and cubic nonlinearities. It is demonstrated here that both these models possess localized peaked solutions. By constructing a Lyapunov function, these peaked waves are shown to be dynamically stable under small perturbations in the natural energy space H^1, without restriction on the sign of the momentum density. In particular, for the homogeneous cubic nonlinear model, we are able to further incorporate a higher-order conservation law to conclude orbital stability in $H^1 \cap W^{1,4}$. Our analysis is based on a strong use of the conservation laws, the introduction of certain auxiliary functions, and a refined continuity argument.

1 Introduction

Solitary waves are solutions to a time-dependent problem that carry finite energy, remain spatially localized, and evolve by translating at a fixed velocity without altering their shapes. They arise from precise balance between dispersion and nonlinear

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effects and play a substantial role in the study of more general solutions in the limit of large time. Their importance is also manifested in the well-known "soliton resolution conjecture" (see, e.g., [23]), asserting that in general, solutions resolve into a superposition of weakly interacting solitary waves and decaying dispersive waves. Hence, it is natural to investigate the stability of solitary waves in order to understand this asymptotic decomposition.

1.1 Cubic nonlinear shallow-water model equations

We consider two families of quasilinear shallow-water equations recently derived in [4] as asymptotic shallow-water model equations for the 2D full water wave dynamics. These two families are derived in a scaling regime corresponding to waves with relatively large amplitude, which is in contrast with the classical Korteweg–de Vries (KdV) weakly nonlinear scaling.

To introduce the equations of the present study, let us briefly recall the modeling process of [4]. The common procedure in the water wave modeling involves relating two independent nondimensional parameters

$$\varepsilon = \frac{a}{h_0} \ll 1, \qquad \mu = \frac{h_0^2}{\lambda^2} \ll 1,$$

where a, h_0, and λ are the typical amplitude of the wave, the depth of the water, and the wavelength, respectively. The balance between the nonlinearity parameter ε and dispersion parameter μ responsible for generation of interesting nonlinear phenomena is usually quantified to be a power law scaling between ε and μ in the asymptotic regime $\varepsilon, \mu \ll 1$. For instance, the KdV weakly nonlinear scaling corresponds to $\varepsilon = O(\mu)$, and the so-called Camassa–Holm (CH) scaling regime for shallow–water waves of *moderate amplitude* amounts to asking $\varepsilon = O(\mu^{1/2})$.

The next level of nonlinearity-enhancing scaling proposed in [4] aims at incorporating higher-order nonlinearity to capture more pronounced nonlinear behavior, for example, the *curvature blow-up*, that is, the 2nd derivative of solution becomes unbounded in finite time while the solution and its gradient remain bounded.

Setting $\varepsilon = O(\mu^{2/5})$ and expanding the equation for the scaled surface elevation η, it follows that

$$\begin{aligned} 2(\eta_x + \eta_t) + \frac{1}{3}\mu\eta_{xxx} + 3\varepsilon\eta\eta_x - \frac{3}{4}\varepsilon^2\eta^2\eta_x + \frac{3}{8}\varepsilon^3\eta^3\eta_x + \varepsilon\mu\left(\frac{23}{12}\eta_x\eta_{xx} + \frac{5}{6}\eta\eta_{xxx}\right) \\ + \frac{115}{192}\varepsilon^4\eta^4\eta_x + \varepsilon^2\mu\left(\frac{23}{16}\eta\eta_x\eta_{xx} + \frac{29}{8}\eta^2\eta_{xxx} + \frac{3}{4}\eta_x^3\right) = 0 + O(\varepsilon^5, \mu^2). \end{aligned} \tag{1.1}$$

We then adapt the idea of [1] to expand η in terms of another function u, which is related to the horizontal velocity of the fluid, together with its derivatives, using the so-called Kodama transformation [14]. In particular, the expansion takes the following form:

$$\eta \sim u + \varepsilon A + \mu B + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \varepsilon^3 K + \varepsilon^2 \mu G + \varepsilon\mu^2 H, \tag{1.2}$$

where

$$A := \lambda_1 u^2, \quad B := \lambda_2 u_{xx}, \quad E := \lambda_3 u^3, \quad K = \lambda_0 u^4, \quad C := \lambda_4 u_x^2 + \lambda_5 u u_{xx},$$
$$D := \lambda_6 u_{xxxx}, \quad G := \lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx}, \quad H := \lambda_9 u_x u_{xxx} + \lambda_{10} u u_{xxxx} + \lambda_{11} u_{xx}^2.$$

This Kodama transformation produces sufficient degrees of freedom to allow one to derive the desired family of cubic nonlinear asymptotic model equations. For example (see [4] for more details), setting

$$\lambda_1 = \frac{k_1}{2} + \frac{189}{20}, \quad \lambda_2 = \frac{k_1}{6} + \frac{179}{60}, \quad \lambda_3 = \frac{23}{5} + \frac{k_1}{4},$$
$$\lambda_0 = \frac{3}{19}k_1^3 + \frac{13083}{1520}k_1^2 + \frac{1189081}{7600}k_1 + \frac{108125767}{114000},$$
$$\lambda_4 = -\frac{1}{6}k_1^2 - \frac{671}{120}k_1 - \frac{56327}{1200}, \quad \lambda_5 = -\frac{1}{6}k_1^2 - \frac{67}{15}k_1 - \frac{30437}{1200},$$

where λ_7, λ_8 are completely free, and $\lambda_6, \lambda_9, \lambda_{10}, \lambda_{11}$ are uniquely determined from λ_7 and λ_8, and $k_1 \approx -15.1765$ is the unique real root of

$$2000k_1^3 + 106200k_1^2 + 1871550k_1 + 10934031 = 0,$$

one can derive the following model equation consisting of quadratic terms being characteristic for the Camassa–Holm (CH) equation together with cubic nonlinear terms known from the Novikov equation and the modified Camassa–Holm (mCH) equation:

$$m_t + u_x - \frac{\mu}{4}u_{xxx} + \frac{\varepsilon}{2}(2u_x m + u m_x) + \frac{k_1\varepsilon^2}{4}\left(\left(u^2 - \frac{5}{12}\mu u_x^2\right)m\right)_x$$
$$+ \frac{69\varepsilon^2}{20}(u^2 m_x + 3u u_x m) = 0 + O(\varepsilon^5, \mu^2),$$

where $m = u - \frac{5}{12}\mu u_{xx}$ is called the momentum density. Via a further scaling

$$u \to 2\varepsilon^{-1}u, \quad t \to \left(\frac{5}{12}\mu\right)^{-\frac{1}{2}} t, \quad x \to \left(\frac{5}{12}\mu\right)^{-\frac{1}{2}} x,$$

and a formal scaling limit consideration $t \to \delta^{-2}t$ and $u \to \delta^{-1}u$ then sending $\delta \to 0$, the quadratic CH terms can be removed, yielding the so-called mCH–Novikov equation

$$m_t + k_1\left((u^2 - u_x^2)m\right)_x + \frac{69}{5}(u^2 m_x + 3u u_x m) = 0,$$

where, abusing notation, the momentum density becomes $m = u - u_{xx}$.

On the other hand, considering

$$\eta = u + \frac{97}{20}\varepsilon u^2 + \frac{29}{20}\mu u_{xx} + \varepsilon\mu\left(\frac{1261}{600}uu_{xx} - \frac{10373}{1200}u_x^2\right) + \frac{23}{10}\varepsilon^2 u^3 + \frac{13067089}{114000}\varepsilon^3 u^4$$

in (1.2) and performing a similar scaling and formal limit procedure, another cubic nonlinear equation can be derived as

$$m_t + u_x - \frac{3}{5}u_{xxx} + (2u_x m + u m_x) + \frac{46}{5}\left(\left(u^2 - \frac{1}{4}(u^2)_{xx}\right)u\right)_x = 0.$$

One of the purposes of this work is to investigate some qualitative properties of the above two equations, while not restricting ourselves with the explicit coefficients of the cubic and quadratic terms. In particular, we will study the following two equations:

$$m_t + k_1\left((u^2 - u_x^2)m\right)_x + k_2(u^2 m_x + 3uu_x m) = 0, \tag{1.3}$$

and

$$m_t + k_1\left(2u_x m + u m_x\right) + k_2\left(\left(u^2 - \frac{1}{4}(u^2)_{xx}\right)u\right)_x = 0, \tag{1.4}$$

where $m = u - u_{xx}$ and k_1 and k_2 are two arbitrary constants. Further motivation to consider (1.4) is explained in Section 3.

1.2 Peaked solitary waves

Mathematically, equation (1.3) can be viewed as a combination of the mCH equation [8, 21, 22] (corresponding to $k_1 = 1$, $k_2 = 0$) and the Novikov equation [20] (corresponding to $k_1 = 0$, $k_2 = 1$). Equation (1.4) generalizes the well-known CH equation [2, 9] when $k_1 = 1$ and $k_2 = 0$.

Like their ancestors—the CH, mCH and Novikov equations, the two equations (1.3) and (1.4) both exhibit nonlinear dispersion, which enables them to support a remarkable class of non-smooth soliton-like solutions, namely, the peaked solitary waves of the form

$$\varphi_c(x - ct) := ae^{-|x-ct|};$$

see Theorem 2.1 and Theorem 3.1. These peaked solitary waves are also a common characteristics shared by many dual integrable nonlinear systems, like the CH, mCH, and Novikov equations. Indeed, (1.3) can be put into the form

$$\frac{\partial m}{\partial t} = J_1 \frac{\delta E}{\delta m}, \tag{1.5}$$

with the Hamiltonian operator

$$J_1 = -k_1 \partial_x m \partial_x^{-1} m \partial_x - 2k_2 (3m\partial_x + 2m_x)(4\partial_x - \partial_x^3)^{-1}(3m\partial_x + m_x),$$

and the corresponding Hamiltonian functional

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\, \mathrm{d}x.$$

In addition, another conserved quantity of (1.3) is $F_1(u) = \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4\right) \mathrm{d}x$ (see Lemma 2.1). For (1.4), it is found that the Hamiltonian functional $E(u)$ is again conserved quantity. It is also observed that (1.4) can be rewritten as

$$\frac{\partial m}{\partial t} = J_2 \frac{\delta F_2}{\delta m}, \quad \text{with} \quad J_2 = -\frac{1}{4}\partial_x (1 - \partial_x^2), \tag{1.6}$$

and the corresponding Hamiltonian functional is given by

$$F_2(u) = 2k_1 I_1(u) + k_2 I_2(u), \tag{1.7}$$

where

$$I_1(u) = \int_{\mathbb{R}} (u^3 + u u_x^2)\, \mathrm{d}x, \qquad I_2(u) = \int_{\mathbb{R}} (u^4 + u^2 u_x^2)\, \mathrm{d}x. \tag{1.8}$$

At this point, we have not fully exploited to see if the model equations (1.3) and (1.4) can be put into a bi-Hamiltonian form, or if they admit a Lax pair, and hence integrable. This is certainly an interesting direction to go but it is beyond the scope of this paper.

1.3 Orbital stability in H^1 space

The primary goal of the present paper is to investigate the dynamical stability of the peaked solutions for these two model equations (1.3) and (1.4). A common strategy for studying the stability of solitary waves of such systems is to exploit the Hamiltonian structure. However, many solitary waves are not local minimizers of the energy but are instead indefinite energy saddles. Fortunately, in a number of cases, the Hamiltonian system is *canonical* and the solitary waves can be thought of as local extrema of the energy subject to the constraint of a fixed momentum, another conserved quantity generated by translation symmetry. Such a fact was exploited in two seminal papers of Grillakis–Shatah–Strauss [10, 11] to develop a powerful tool to determine stability or instability. Among other hypotheses, one crucial assumption needed for application of the machinery of [10, 11] is that the spectrum of the linearized Hamiltonian at the solitary wave consists of finitely many negative eigenvalues, zero, and a subset of the positive real axis separated uniformly away from the origin.

1.3.1 *Lyapunov method*

Note that the peaked solitary wave φ_c is a global weak solution. Its non-smoothness property leads to a degeneracy in the linearized Hamiltonian, making the spectral analysis and hence the approach in [10] difficult to apply. Such a difficulty has been long observed for many quasilinear dispersive equations admitting peaked solutions. Constantin–Strauss [7] introduced a new idea in the spirit of the Lyapunov method to establish the H^1-orbital stability of the CH peakons. Their idea relies crucially on two special conserved quantities $E(u)$ and $F(u)$, one (say, E) being the H^1-energy, and the key observation is that the H^1 distance of the perturbed solutions to the peaked wave is controlled by the difference between the corresponding energies, with an error given by the pointwise difference between the peaks of the solution u and the peaked wave φ_c, that is

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - z)\|_{H^1}^2 + 4\left(u(z) - M_{\varphi_c}\right)$$

for any $z \in \mathbb{R}$, where M_{φ_c} denotes the peak of φ_c. Then a Lyapunov function can be constructed via the introduction of some suitable auxiliary functions. Through this Lyapunov function, one obtains an inequality relating the maximum of the perturbed solution with the conserved quantities

$$\left|M_u - M_{\varphi_c}\right| \lesssim |E(u) - E(\varphi_c)| + |F(u) - F(\varphi_c)|,$$

where M_u is the peaks of u. Finally since the difference terms on the right-hand side can be made small according to the initial perturbation, $\left|M_u - M_{\varphi_c}\right|$ will be small, proving stability.

The idea of [7] has been successfully applied to many other peakon equations, like the ones with quadratic nonlinearity including the Degasperis–Procesi (DP) equation [15] and the μ-CH equation [5]; and the cubic nonlinear models, for example, the Novikov equation [17] and the generalized mCH equation [16].

1.3.2 *Sign of m*

The nonlinear terms in (1.3) are all cubic, leading one to reexamine the approach of [17]. Among the assumptions of [17] on the initial perturbation, a crucial one is the sign condition, namely $m_0(x) = (1-\partial_x^2)u_0(x) \geq 0$. Such a sign condition will be preserved for later time, and thus the solution satisfies that $u(t,x) \geq 0$. This allows one to find the required auxiliary function

$$h(x) := u^2(x) \mp \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x),$$

and use h to bound the functional $F(u)$ by $E(u)$ and the peak value M_u as

$$F(u) - \frac{4}{3}M_u^4 \leq \frac{4}{3}\max_{x\in\mathbb{R}}(u^2)\left(E(u) - 2M_u^2\right) = \frac{4}{3}M_u^2\left(E(u) - 2M_u^2\right), \tag{1.9}$$

cf. [17, Lemma 9]. One of our goals here is to remove this sign constraint on the initial data. Indeed, the structure of equation (1.3) suggests us to work with the same auxiliary function h. But we see that (1.9) still holds if the positivity assumption on u is replaced by

$$M_u + \min_{x\in\mathbb{R}} u \geq 0. \tag{1.10}$$

On the other hand, since $\min_{x\in\mathbb{R}} \varphi_c = 0$ we know that $M_{\varphi_c} + \min_{x\in\mathbb{R}} \varphi_c > 0$, and hence (1.10) holds with a strict inequality at initial time if the initial data u_0 is sufficiently close to φ_c. Continuity then guarantees that this property will propagate for some time, which in turn ensures stability over that time period. But the H^1-orbital stability then implies that (1.10) holds over this time period as well. This way the same argument repeats, and so one achieves stability over the entire time of existence, cf. Theorem 2.2.

We would like to point out that our new method in handling equation (1.3) can be used to treat the Novikov equation ($k_1 = 0$) and the mCH equation ($k_2 = 0$). As a result, the assumption on the initial momentum density $m_0 \geq 0$ used in [17] can be removed, at the price that the global solution in [17, Theorem 1] being replaced by a local one.

1.3.3 *A new auxiliary function*

Equation (1.4) carries a similar structure as the generalized mCH equation [16] in the sense that the nonlinearities contain both quadratic and cubic terms. It was discovered in [16] that the interaction between the quadratic and the cubic terms can be quite subtle and hence requires a rather delicate analysis. In particular, an auxiliary function of the form $h(x) := 2k_1 u(x) + k_2\left(u^2(x) \mp \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x)\right)$ is used and the positivity of the initial momentum density m_0 is still needed to ensure an estimate like (1.9). However, for our equation (1.4), the sign-preservation property of m fails to be true and thus assuming $m_0 \geq 0$ is never enough to infer the positivity of the solution for later time. What turns out to make the argument work is that we may consider a different auxiliary function

$$h(x) := 2k_1 u(x) + k_2 u^2(x).$$

When the two parameters k_1 and k_2 are "cooperative", namely, $k_1, k_2 > 0$, we can easily bound h as $h \leq 2k_1 M_u + k_2 M_u^2$, provided that (1.10) holds. Once such a bound for h is available, a similar estimate of the form (1.9) can be proved; see Lemma 3.5. As a result, the orbital stability of the peakons can be established with the help of a similar continuity argument as before, cf. Theorem 3.2 (1).

On the other hand, when k_1 and k_2 are "uncooperative" in the sense that $k_1 > 0$ but $k_2 < 0$, it becomes less clear whether h can be bounded in terms of M_u. However, by restricting the wave speed, one can prove that for small perturbations h will be increasing; see Lemma 3.4 part (2). This immediately implies that $h \leq 2k_1 M_u + k_2 M_u^2$, and the orbital stability follows, cf. Theorem 3.2 (2).

1.4 Orbital stability in a finer energy space $H^1 \cap W^{1,4}$

The H^1-topology used in the above stability results naturally arises from the conservation of the H^1-energy $E(u)$ for both equations (1.3) and (1.4). On the other hand, though not having been explicitly analyzed, traditional characteristics method seems to indicate that strong solutions of (1.3) and (1.4) can exhibit finite-time wave-breaking (i.e., derivative blow-up) for well-chosen initial data.(The wave-breaking for (1.3) has recently been studied in [4].) Such a feature in turn suggests a strong instability property of the peakons under the Lipschitz metric for perturbations as the strong solutions. Therefore, it would be interesting to investigate the stability issue under a certain topology that is between H^1 and $W^{1,\infty}$. We would like to also point out that the strong $W^{1,\infty}$ instability for H^1-stable peakons under weak-solution perturbations has been confirmed for the CH equation [18, 19] and the Novikov equation [6] recently.

A natural way to seek an intermediate topology is to examine the higher-order conservation laws. It turns out that the conserved quantity $F_1(u)$ of equation (1.3) together with the H^1 conservation gives $W^{1,4}$ control of solutions, cf. (2.36). In fact, one can prove that for a perturbation u of the peaked wave φ_c,

$$|F_1(u-\varphi_c)| \lesssim |F_1(u) - F_1(\varphi_c)| + f(\|u-\varphi_c\|_{H^1})$$

for some polynomial function f, cf. (2.38). From this, one can deduce $W^{1,4}$ stability of the peakons even for initial perturbation that is only H^1 close to the peaked solitary waves; see Theorem 2.3.

Applying the same idea to (1.4), on the other hand, would not generate a finer topology than H^1, since the conservation law F_2 does not provide a stronger norm than E.

1.5 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we first state the local well-posedness result of the initial-value problem associated with (1.3) and then establish the existence of peaked solutions and prove their orbital stability in $H^1 \cap W^{1,4}$. The similar discussion for equation (1.4) is performed in Section 3 to yield the H^1-orbital stability for the corresponding peaked waves. In the Appendix, we provide some technical details of existence of the peaked waves.

2 The mCH–Novikov Equation

This section is focused on the existence and stability of peaked solitary waves for the mCH–Novikov equation (1.3).

2.1 Local well-posedness and conservation laws

A necessary ingredient in our stability analysis is the local well-posedness theory for the initial-value problem

$$\begin{cases} m_t + k_1\left((u^2-u_x^2)m\right)_x + k_2(u^2 m_x + 3uu_x m) = 0, \quad t>0,\ x\in\mathbb{R}, \\ m = u - u_{xx}, \\ u(0,x) = u_0(x), \qquad x\in\mathbb{R}. \end{cases} \tag{2.1}$$

Recall that the inverse operator $(1-\partial_x^2)^{-1}$ can be obtained by convolution with the corresponding Green's function such that

$$u = (1-\partial_x^2)^{-1} m = p * m, \quad \text{where} \quad p(x) = \frac{1}{2}e^{-|x|}, \tag{2.2}$$

and $*$ denotes the convolution product.

Applying now the operator $(1-\partial_x^2)^{-1}$ to equation in (2.1), it follows that

$$u_t + k_1\left(u^2 - \frac{1}{3}u_x^2\right)u_x + k_1 p_x * \left(\frac{2}{3}u^3 + uu_x^2\right) + \frac{k_1}{3}p * u_x^3$$
$$+ k_2 u^2 u_x + k_2 p_x * \left(u^3 + \frac{3}{2}uu_x^2\right) + \frac{k_2}{2}p * u_x^3 = 0. \tag{2.3}$$

We will start by considering solutions to the above problem (2.1) in Sobolev spaces with sufficiently high regularity. The precise definition of such solutions is given below.

Definition 2.1 (Strong solutions). If $u \in C\left([0,T), H^s(\mathbb{R})\right) \cap C^1\left([0,T), H^{s-1}(\mathbb{R})\right)$ with $s > \frac{5}{2}$ and some $T > 0$ satisfies (2.1), then u is called a strong solution on $[0,T)$. If u is a strong solution on $[0,T)$ for every $T > 0$, then it is called a global strong solution.

Remark 2.1. The regularity requirement that $u \in H^{5/2+}$ comes from applying transport theory to (2.3). We see from (2.3) that equation (1.3) can be reformulated in a transport type with the transport velocity $k_1\left(u^2 - \frac{1}{3}u_x^2\right) + k_2u^2$. Standard transport theory requires a control on $\left\|k_1\left(u^2 - \frac{1}{3}u_x^2\right) + k_2u^2\right\|_{W^{1,\infty}}$, which, by Sobolev embedding, amounts to asking $u \in H^{5/2+}$.

The argument for establishing the local well-posedness of strong solutions to (2.1) is now fairly standard. For example, one can follow the same approach as in [12]. Hence, we will only state the result without proof.

Proposition 2.1. If $s > \frac{5}{2}$ and $u_0 \in H^s(\mathbb{R})$, then there exists a time $T > 0$ such that the initial-value problem (2.1) has a unique strong solution $u \in C\left([0,T), H^s(\mathbb{R})\right) \cap C^1\left([0,T), H^{s-1}(\mathbb{R})\right)$. Further, the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $H^s(\mathbb{R})$ into $u \in C\left([0,T), H^s(\mathbb{R})\right) \cap C^1\left([0,T), H^{s-1}(\mathbb{R})\right)$.

Regarding stability, as explained in the Introduction, certain conserved quantities of the equation play a crucial role. For this reason, we give the following result. Its proof can be seen in Appendix A.

Lemma 2.1. For the strong solutions u obtained in Proposition 2.1, the following functionals

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\,\mathrm{d}x, \quad F_1(u) = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)\mathrm{d}x \tag{2.4}$$

are conserved, that is $\frac{d}{dt}E(u) = \frac{d}{dt}F_1(u) = 0$ for all $t \in [0,T)$.

2.2 Existence of peaked solitary waves

As is mentioned in the Introduction, the peaked solitary waves have low regularity and hence cannot be regarded as strong solutions to (2.1).

The following theorem proves the existence of the peaked solutions to equation (2.3). Details of the proof can be found in Appendix A.

Theorem 2.1. The function

$$u(t,x) = \varphi_c(x-ct) := ae^{-|x-ct|}, \tag{2.5}$$

is a peaked solution to (2.3) provided that

(1) $2k_1 + 3k_2 \neq 0$, $\frac{3c}{2k_1+3k_2} > 0$ and $a = \pm\sqrt{\frac{3c}{2k_1+3k_2}} \neq 0$; or
(2) $2k_1 + 3k_2 = c = 0$ and $a \neq 0$.

Remark 2.2. Note that case (1) in Theorem 2.1 generates a pair of peaked ($a > 0$) and anti-peaked ($a < 0$) solutions both moving at speed $c \neq 0$, whereas case (2) corresponds to a stationary peaked solution.

Remark 2.3. If one imposes the ansatz that the solution u of equation (2.3) is a linear superposition of N peakons

$$u(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}, \tag{2.6}$$

then a direct computation shows that the position functions $q_i(t)$ and the amplitude functions $p_i(t)$ satisfy the following dynamical system:

$$\dot{p}_i = k_2 p_i \sum_{j,l=1}^{N} p_j p_l \,\mathrm{sign}(q_l - q_i) e^{-|q_l-q_i|-|q_j-q_i|},$$

$$\dot{q}_i = \frac{2k_1}{3} p_i^2 + 2k_1 \sum_{j=1, j\neq i}^{N} p_i p_j e^{-|q_j-q_i|} + 4k_1 \sum_{\substack{1\le j<i,\\ i<l\le N}} p_j p_l e^{-|q_j-q_l|} + k_2 \sum_{j,l=1}^{N} p_j p_l e^{-|q_l-q_i|-|q_j-q_i|}.$$

2.3 H^1-orbital stability

The main goal in this subsection is to prove the orbital stability for the single peaked solutions obtained in Theorem 2.1 in the natural H^1 energy space suggested by the conservation law E as in (2.4). We will only discuss the case when $a > 0$. The case for anti-peakons ($a < 0$) can be treated by exploiting the invariance of equation in (2.1) under the transformation $u \to -u$.

Recall Theorem 2.1. It is obvious that $\varphi_c(x) \in H^1(\mathbb{R})$ has the peak at $x = 0$ and a simple computation reveals

$$\max_{x\in\mathbb{R}} \varphi_c(x) = \varphi_c(0) = a := \begin{cases} \sqrt{\frac{3c}{2k_1+3k_2}}, & 2k_1 + 3k_2 \neq 0 \text{ and } \frac{3c}{2k_1+3k_2} > 0, \\ \in \mathbb{R}^+, & 2k_1 + 3k_2 = c = 0, \end{cases} \tag{2.7}$$

$$E(\varphi_c) = 2a^2, \qquad F_1(\varphi_c) = \frac{4}{3}a^4. \tag{2.8}$$

Define the following functionals:

$$\widetilde{M}(t) = \max_{x\in\mathbb{R}}\{u(t,x)\}, \qquad \widetilde{m}(t) = \min_{x\in\mathbb{R}}\{u(t,x)\}, \tag{2.9}$$

for every $t \in [0, T^*)$, where $T^* > 0$ is the maximal existence time of solutions u to initial-value problem (2.1).

The H^1-orbital stability of the peaked waves is given as follows.

Theorem 2.2 (H^1-orbital stability). Let $\varphi_c(x - ct) = ae^{-|x-ct|}$ be the peaked solutions given in Theorem 2.1. Assume that the initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$. Then φ_c is H^1-orbitally stable in the following sense: $\exists\, 0 < \delta_0 \ll 1$ such that if

$$\|u_0 - \varphi_c\|_{H^1} < a\delta, \qquad 0 < \delta < \delta_0, \tag{2.10}$$

then the corresponding solution $u(t,x)$ to (2.1) satisfies

$$\sup_{t\in[0,T^*)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1} < 2\left(3a + C(u_0)^{1/4}\right)\delta^{1/4}, \tag{2.11}$$

where $\xi(t)$ is the point at which the solution $u(t,x)$ achieves its maximum and the constant

$$C(u_0) := \frac{2\sqrt{2}a}{3}\|u_{0x}\|^2_{L^\infty}\|u_{0x}\|_{L^2}. \tag{2.12}$$

Remark 2.4. It is easy to check that the mCH–Novikov equation in (2.1) has the sign-persistence property: if the initial data $m_0 = (1 - \partial_x^2)u_0 \geq 0$ (*or* ≤ 0), then the corresponding solution satisfies that $m(t,x) \geq 0$ (*or* ≤ 0). Therefore, if one assumes in addition that $m_0 \geq 0$ (*or* ≤ 0), then the sign property on m implies that $|u_x| \leq |u|$. Hence, in (2.12), we have from (2.17) that

$$\|u_{0x}\|_{L^\infty}^2 \|u_{0x}\|_{L^2} \leq 2E(u_0)^{3/2} < 2\left(E(\varphi_c) + 4a^2\delta\right)^{3/2} < 6a^3.$$

Therefore, $C(u_0)$ in (2.12) can be replaced by

$$C(u_0) = 4\sqrt{2}a^4.$$

Remark 2.5. Our stability result is established in the H^1-metric, which is below the regularity index H^s for strong solutions as given in Proposition 2.1. The issue of extending our result to replace the H^1-metric by the H^s-metric is much more delicate. On the other hand, one may consider an H^1-stability result for suitable weak solutions. One of the main ingredients in the proof of Theorem 2.1 is the use of the two conservation laws. Hence, once a weak solution theory is established so that E and F_1 are conserved, it seems plausible that the same stability property holds for those weak solutions.

The proof of Theorem 2.2 is approached via a series of lemmas.

Lemma 2.2. For any $u \in H^1(\mathbb{R})$ and $z \in \mathbb{R}$, we have

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - z)\|_{H^1}^2 + 4a\left(u(z) - a\right). \tag{2.13}$$

Proof. Using integration by parts and (2.8), it follows that

$$\begin{aligned}
&\|u - \varphi_c(\cdot - z)\|_{H^1}^2 \\
&\quad = E(u) + E(\varphi_c(\cdot - z)) - 2a\int_{\mathbb{R}} u(x)\varphi(x-z)\,\mathrm{d}x \\
&\quad - 2a\int_{-\infty}^{z} u_x(x)\varphi(x-z)\,\mathrm{d}x + 2a\int_{z}^{\infty} u_x(x)\varphi(x-z)\,\mathrm{d}x \\
&\quad = E(u) + E(\varphi_c) - 4au(z) = E(u) - E(\varphi_c) - 4a\left(u(z) - a\right).
\end{aligned}$$

Consequently, we have established the lemma. ■

The following lemma is essential to derive the orbital stability of φ_c.

Lemma 2.3. Assume that $u \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, and $\widetilde{M}$, $\widetilde{m}$ are defined in (2.9).

(1) If $\widetilde{M} + \widetilde{m} \geq 0$, then $F_1(u) \leq \frac{4}{3}\widetilde{M}^2 E(u) - \frac{4}{3}\widetilde{M}^4$.

(2) If $\widetilde{M} + \widetilde{m} \leq 0$, then $F_1(u) \leq \frac{4}{3}\widetilde{m}^2 E(u) - \frac{4}{3}\widetilde{m}^4$.

Proof. (1) Since $\widetilde{M} + \widetilde{m} \geq 0$, then there exists $\xi \in \mathbb{R}$ such that $\widetilde{M} = u(\xi)$. Let us define

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi, \\ u(x) + u_x(x), & x > \xi, \end{cases} \tag{2.14}$$

and a direct computation gives rise to

$$\begin{aligned} \int_{\mathbb{R}} g^2(x)\,\mathrm{d}x &= \int_{\mathbb{R}} (u^2 + u_x^2)\,\mathrm{d}x - 2\int_{-\infty}^{\xi} u u_x\,\mathrm{d}x + 2\int_{\xi}^{\infty} u u_x\,\mathrm{d}x \\ &= E(u) - 2u^2(\xi) = E(u) - 2\widetilde{M}^2. \end{aligned} \tag{2.15}$$

On the other hand, we define $h(x)$ by

$$h(x) = \begin{cases} u^2(x) - \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x), & x < \xi, \\ u^2(x) + \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x), & x > \xi. \end{cases}$$

Direct computation yields that

$$\int_{\mathbb{R}} h(x)g^2(x)\,\mathrm{d}x = F_1(u) - \frac{4}{3}\widetilde{M}^4.$$

Since $\widetilde{M} + \widetilde{m} \geq 0$ implies that $u^2 \leq \widetilde{M}^2$, together with $h - \frac{4}{3}u^2 = -\frac{1}{3}u^2 \pm \frac{2}{3}uu_x - \frac{1}{3}u_x^2 = -\frac{1}{3}(u \pm u_x)^2 \leq 0$, it follows that

$$h(x) \leq \frac{4}{3}u^2 \leq \frac{4}{3}\widetilde{M}^2. \tag{2.16}$$

Combining (2.15) and (2.16), we deduce that

$$F_1(u) - \frac{4}{3}\widetilde{M}^4 = \int_{\mathbb{R}} h(x)g^2(x)\,\mathrm{d}x \leq \frac{4}{3}E(u)\widetilde{M}^2 - \frac{8}{3}\widetilde{M}^4,$$

thereby concluding part (1) of the lemma.

Part (2) of the lemma can be proved in a similar way. ■

Remark 2.6. Note that the functions g and h are zero when u is replaced by $-\varphi_c$. Indeed, this requirement is crucial to construct a Lyapunov function for the stability.

Lemma 2.4. Let $u \in H^s(\mathbb{R})$, $s > \frac{5}{2}$ and assume $\|u - \varphi_c\|_{H^1} < a\delta$, with $0 < \delta \ll 1$. Then

$$|E(u) - E(\varphi_c)| \leq 4a^2\delta, \tag{2.17}$$

$$|F_1(u) - F_1(\varphi_c)| \leq \left(C(u) + 17a^4\right)\delta, \tag{2.18}$$

where $C(u) := \frac{2\sqrt{2}a}{3}\|u_x\|^2_{L^\infty}\|u_x\|_{L^2}$.

Proof. Using the relation (2.15), for any $u \in H^1(\mathbb{R})$, it is inferred that

$$\sup_{x\in\mathbb{R}} |u(x)| \leq \frac{\sqrt{2}}{2}E(u)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}\|u\|_{H^1},$$

with equality holding if and only if u is a multiple of some translate of $e^{-|x|}$.

From the assumption on $\|u - \varphi_c\|_{H^1}$, it follows that

$$\begin{aligned}
\left|E(u) - E(\varphi_c)\right| &= \left|(\|u\|_{H^1} + \|\varphi_c\|_{H^1})(\|u\|_{H^1} - \|\varphi_c\|_{H^1})\right| \\
&\leq (\|u - \varphi_c\|_{H^1} + 2\|\varphi_c\|_{H^1})\|u - \varphi_c\|_{H^1} \\
&\leq \left(a\delta + 2\sqrt{2}a\right)a\delta < 4a^2\delta,
\end{aligned}$$

and

$$\begin{aligned}
\left|F_1(u) - F_1(\varphi_c)\right| &= \left|\int_{\mathbb{R}}\left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)\,\mathrm{d}x - \int_{\mathbb{R}}\left(\varphi_c^4 + 2\varphi_c^2\varphi_{cx}^2 - \frac{1}{3}\varphi_{cx}^4\right)\,\mathrm{d}x\right| \\
&\leq \int_{\mathbb{R}}\left|u^4 + 2u^2u_x^2 - \varphi_c^4 - 2\varphi_c^2\varphi_{cx}^2\right|\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}}\left|u_x^4 - \varphi_{cx}^4\right|\mathrm{d}x \\
&\leq \int_{\mathbb{R}}\left|(u^2 - \varphi_c^2)(u^2 + 2u_x^2)\right|\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}}\left|u_x^4 - \varphi_{cx}^4\right|\mathrm{d}x \\
&\quad + \int_{\mathbb{R}}\varphi_c^2\left|(u^2 + 2u_x^2 - \varphi_c^2 - 2\varphi_{cx}^2\right|\mathrm{d}x \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We compute I_1 as follows:

$$\begin{aligned} I_1 &\le 2\int_{\mathbb{R}} |u+\varphi_c|\cdot|u-\varphi_c|\cdot(u^2+u_x^2)\,\mathrm{d}x \\ &\le 2\left(\|u\|_{L^\infty}+\|\varphi_c\|_{L^\infty}\right)\cdot\|u-\varphi_c\|_{L^\infty}\int_{\mathbb{R}}(u^2+u_x^2)\,\mathrm{d}x \\ &\le \left(\sqrt{2}\|u\|_{H^1}+2a\right)\|u-\varphi_c\|_{H^1}\cdot E(u) \\ &\le \|u-\varphi_c\|_{H^1}\left(E(\varphi_c)+4a^2\delta\right)\cdot\left(\sqrt{2}\|u-\varphi_c\|_{H^1}+4a\right) \\ &\le 2a^4\delta(4+\sqrt{2}\delta)(2\delta+1). \end{aligned} \tag{2.19}$$

In a similar manner,

$$\begin{aligned} I_3 &\le a^2\int_{\mathbb{R}}\left|(u-\varphi_c)^2+2(u_x-\varphi_{cx})^2+2\varphi_c(u-\varphi_c)+4\varphi_{cx}(u_x-\varphi_{cx})\right|\mathrm{d}x \\ &\le 2a^2\left(\|u-\varphi_c\|_{H^1}^2+2\|\varphi_c\|_{H^1}\|u-\varphi_c\|_{H^1}\right)\le 2a^4\delta(\delta+2\sqrt{2}). \end{aligned} \tag{2.20}$$

For the term I_2, by the Hölder and Young inequalities, it follows that

$$\begin{aligned} I_2 &= \frac{1}{3}\int_{\mathbb{R}}\left|(u_x^2+\varphi_{cx}^2)(u_x+\varphi_{cx})(u_x-\varphi_{cx})\right|\mathrm{d}x \\ &\le \frac{1}{3}\left(\int_{\mathbb{R}}(u_x^2+\varphi_{cx}^2)^2(u_x+\varphi_{cx})^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(u_x-\varphi_{cx})^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\le \frac{2\sqrt{2}}{3}\left(\int_{\mathbb{R}}(u_x^6+\varphi_{cx}^6)\,\mathrm{d}x\right)^{\frac{1}{2}}\|u-\varphi_c\|_{H^1}. \end{aligned} \tag{2.21}$$

Since $u\in H^s(\mathbb{R})\subset W^{1,\infty}(\mathbb{R})$, $s>\frac{5}{2}$, we have

$$\|u_x\|_{L^6}\le\|u_x\|_{L^\infty}^{\frac{2}{3}}\|u_x\|_{L^2}^{\frac{1}{3}}.$$

We also know that $\|\varphi_{cx}\|_{L^6}^6=\frac{1}{3}a^6$. Hence, plugging the above into (2.21), there appears the relation

$$I_2\le\left(C(u)+\frac{2\sqrt{6}}{9}a^4\right)\delta,\quad\text{where}\;\; C(u)=\frac{2\sqrt{2}a}{3}\|u_x\|_{L^\infty}^2\|u_x\|_{L^2}. \tag{2.22}$$

In view of (2.19), (2.20), and (2.22), we conclude that

$$\left|F_1(u) - F_1(\varphi_c)\right| \le \left(C(u) + 17a^4\right)\delta.$$

This completes the proof of the lemma. ■

Lemma 2.5. Assume that $u(x) \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, which satisfies (2.17) and (2.18) with $0 < \delta \ll 1$. Then we have the following:

(1) If $\widetilde{M}(t) + \widetilde{m}(t) \ge 0$, then

$$\left|\widetilde{M} - a\right| < \sqrt{\left(21a^2 + \frac{3}{4a^2}C(u)\right)\delta}. \tag{2.23}$$

(2) If $\widetilde{M}(t) + \widetilde{m}(t) < 0$, then

$$\left|\widetilde{m} + a\right| < \sqrt{\left(21a^2 + \frac{3}{4a^2}C(u)\right)\delta}. \tag{2.24}$$

Proof. (1) If $\widetilde{M}(t) + \widetilde{m}(t) \ge 0$, it then follows from Lemma 2.3 (1) that

$$\widetilde{M}^4 + \frac{3}{4}F_1(u) - E(u)\widetilde{M}^2 \le 0. \tag{2.25}$$

Hence, we define the function $f_u(y)$ by

$$f_u(y) := y^4 + \frac{3}{4}F_1(u) - E(u)y^2, \quad y \in \mathbb{R}. \tag{2.26}$$

Recalling (2.8), a direct calculation reveals that

$$f_{\varphi_c}(y) = y^4 - 2a^2y^2 + a^4 = (y+a)^2(y-a)^2. \tag{2.27}$$

From (2.26), there appears the relation

$$f_{\varphi_c}(\widetilde{M}) = f_u(\widetilde{M}) + \widetilde{M}^2\left(E(u) - E(\varphi_c)\right) - \frac{3}{4}\left(F_1(u) - F_1(\varphi_c)\right),$$

which, together with (2.25) and (2.27), yields

$$\left(\widetilde{M} + a\right)^2\left(\widetilde{M} - a\right)^2 \le \widetilde{M}^2\left(E(u) - E(\varphi_c)\right) - \frac{3}{4}\left(F_1(u) - F_1(\varphi_c)\right). \tag{2.28}$$

On the other hand, using the relation

$$E(u) - 2\widetilde{M}^2 = \int_{\mathbb{R}} g^2(x)\,\mathrm{d}x \geq 0,$$

and the assumption (2.17), we discover that

$$0 < \widetilde{M}^2 \leq \frac{E(u)}{2} \leq a^2(2\delta + 1) < 2a^2. \tag{2.29}$$

Hence, in view of (2.28) and (2.29), we conclude that

$$a\left|\widetilde{M} - a\right| < \sqrt{\left(21a^4 + \frac{3}{4}C(u)\right)\delta},$$

which implies (2.23).

Part (2) of the lemma can be proved in a similar way and the detail is omitted, thereby concluding the proof of Lemma 2.5. ■

We are now in the position to give a proof of the H^1-stability result.

Proof of Theorem 2.2. Applying Lemma 2.1, we see that

$$E(u(t,\cdot)) = E(u_0) \ \text{ and } \ F_1(u(t,\cdot)) = F_1(u_0), \ \ t \in [0, T^*).$$

Therefore from assumption (2.10), it is easy to see that the conclusion of Lemma 2.4 holds. Assumption (2.10) implies that

$$\|u_0 - \varphi_c\|_{L^\infty} < a\delta \ll a.$$

By (2.7), it follows that

$$\widetilde{M}(0) = \max_{x\in\mathbb{R}} u_0(x) \geq u_0(0) > \varphi_c(0) - a\delta = a(1-\delta) > 0.$$

If $\widetilde{m}(0) = \min_{x\in\mathbb{R}} u_0(x) \geq 0$, the obviously

$$\widetilde{M}(0) + \widetilde{m}(0) > a(1-\delta) > 0.$$

If $\widetilde{m}(0) < 0$, then there exists some $\eta \in \mathbb{R}$ such that $u_0(\eta) = \widetilde{m}(0)$. This way we know that

$$\widetilde{m}(0) = \min_{x\in\mathbb{R}} u_0(x) = u_0(\eta) > \varphi_c(\eta) - a\delta > -a\delta.$$

So we still have

$$\widetilde{M}(0) + \widetilde{m}(0) > a(1-\delta) - a\delta > 0.$$

Therefore in any case we know that

$$\widetilde{M}(0) + \widetilde{m}(0) > 0.$$

Furthermore, by continuity, there exits a $T_0 > 0$ such that

$$\widetilde{M}(t) + \widetilde{m}(t) > 0, \tag{2.30}$$

for all $t \in [0, T_0]$. In this way, Lemma 2.3 (1) and Lemma 2.5 (1) hold true for $t \in [0, T_0]$.

Reading off (2.28) and using (2.29), we have

$$\begin{aligned} a\,|u(t,\xi(t)) - a| &< \sqrt{2a^2\left(E(u) - E(\varphi_c)\right) - \frac{3}{4}\left(F_1(u) - F_1(\varphi_c)\right)} \\ &= \sqrt{2a^2\left(E(u_0) - E(\varphi_c)\right) - \frac{3}{4}\left(F_1(u_0) - F_1(\varphi_c)\right)} \\ &< \sqrt{\left(21a^4 + \frac{3}{4}C(u_0)\right)\delta}, \end{aligned} \tag{2.31}$$

for any $t \in [0, T_0]$. Now replacing z by ξ in Lemma 2.2, there obtains the equality

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1}^2 = E(u_0) - E(\varphi_c) - 4a\left(u(t,\xi(t)) - a\right).$$

This, together with the estimates (2.17) and (2.31) leads to that for $t \in [0, T_0]$,

$$\begin{aligned} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1} &\le \sqrt{|E(u_0) - E(\varphi_c)| + 4a\,|u(t,\xi(t)) - a|} \\ &< 2\left(3a + C(u_0)^{1/4}\right)\delta^{1/4}. \end{aligned} \tag{2.32}$$

An important consequence of (2.32) is that we now claim that (2.30) holds for all $t \in [0, T^*)$. If not, then there exists some $T \in (0, T^*)$ such that (2.30) holds for all $t \in [0, T)$, but

$$\widetilde{M}(T) + \widetilde{m}(T) = 0. \tag{2.33}$$

This implies that (2.32) holds for $t \in [0, T)$. So when δ is sufficiently small so that

$$2\left(3a + C(u_0)^{1/4}\right)\delta^{1/4} < \frac{a}{4},$$

we know that over $t \in [0, T)$,

$$\widetilde{M}(t) + \widetilde{m}(t) > \frac{a}{2}.$$

Thus, a continuity argument indicates that $\widetilde{M}(T) + \widetilde{m}(T) \ge \frac{a}{2}$ with $a > 0$, which contradicts (2.33). Therefore, (2.32) holds for all $t \in [0, T^*)$, and hence we obtain (2.11). ■

2.4 $W^{1,4}$-orbital stability

Note that the conservation of F_1 together with E as in (2.4) provides a control of $\|u_x\|_{L^4}$; see, for example, [3]. In fact, we show that the H^1-stability obtained in the previous subsection can be further improved to the stability in the space $H^1 \cap W^{1,4}$. Again we only consider the case $a > 0$ here.

Theorem 2.3 ($W^{1,4}$-orbital stability). Let the assumptions of Theorem 2.2 hold. Then φ_c is $W^{1,4}$-orbitally stable in the following sense: $\exists\, 0 < \delta_0 \ll 1$ such that if

$$\|u_0 - \varphi_c\|_{H^1} < a\delta, \qquad 0 < \delta < \delta_0,$$

then the corresponding solution $u(t,x)$ to (2.1) satisfies

$$\sup_{t\in[0,T^*)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{W^{1,4}} < C_1(u_0)\delta^{1/16} + C_2(u_0)\delta^{1/4}, \tag{2.34}$$

where $\xi(t)$ is the point at which the solution $u(t,x)$ achieves its maximum and the constants C_1 and C_2 depend on a, $\|u_{0x}\|_{L^\infty}$ and $\|u_{0x}\|_{L^2}$.

Proof. Let us denote

$$v(t,\cdot) := u(t,\cdot + \xi(t)) - \varphi_c,$$

where $\xi(t)$ is the point at which $u(t,x)$ attains its maximum. In view of Theorem 2.2, we know that

$$\|v\|_{H^1} < K\delta^{1/4}. \tag{2.35}$$

Following [3,(2.7)-(2.8)], it is found that

$$\|v_x\|_{L^4}^4 \le 3\left(\|v\|_{H^1}^4 - F_1(v)\right), \quad \|v_x\|_{L^3}^3 \le \sqrt{3}\|v\|_{H^1}\sqrt{\|v\|_{H^1}^4 - F_1(v)}. \tag{2.36}$$

Plugging $u = v + \varphi_c$ into $F_1(u)$ and using the fact that $\|\varphi_c\|_{L^\infty} = \|\varphi_{cx}\|_{L^\infty} = \|\varphi_c\|_{L^2} = \|\varphi_{cx}\|_{L^2} = a$ yields after a direct computation that

$$\begin{aligned}
|F_1(v)| \le{}& \left|F_1(u) - F_1(\varphi_c)\right| \\
&+ 2\left|\int_{\mathbb{R}} \left(2v^2 v_x\varphi_{cx} + v^2\varphi_{cx}^2 + 2vv_x^2\varphi_c + 4vv_x\varphi_c\varphi_{cx} + 2v\varphi_c\varphi_{cx}^2 + v_x^2\varphi_c^2 + 2v_x\varphi_c^2\varphi_{cx}\right) \mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}} \left(4v^3\varphi_c + 6v^2\varphi_c^2 + 4v\varphi_c^3\right) \mathrm{d}x\right| + \frac{1}{3}\left|\int_{\mathbb{R}} \left(4v_x^3\varphi_{cx} + 6v_x^2\varphi_{cx}^2 + 4v_x\varphi_{cx}^3\right) \mathrm{d}x\right| \\
\le{}& \left|F_1(u) - F_1(\varphi_c)\right| + \frac{4}{3}\left|\int_{\mathbb{R}} v_x^3\varphi_{cx}\, \mathrm{d}x\right| + 14a^3\|v\|_{H^1} + 20a^2\|v\|_{H^1}^2 + 12a\|v\|_{H^1}^3.
\end{aligned}$$

Note that we have

$$\left|\frac{4}{3}\int_{\mathbb{R}} v_x^3\varphi_{cx}\,dy\right| \le \frac{4a}{3}\|v_x\|_{L^3}^3 \le \frac{4a}{\sqrt{3}}\|v\|_{H^1}\sqrt{\|v\|_{H^1}^4 - F_1(v)}.$$

Thus, from (2.35) for $\delta \ll 1$ sufficiently small it follows that

$$|F_1(v)| \le \left|F_1(u) - F_1(\varphi_c)\right| + \frac{4a}{\sqrt{3}}\|v\|_{H^1}\sqrt{\|v\|_{H^1}^4 - F_1(v)} + 15a\|v\|_{H^1}. \tag{2.37}$$

On the other hand, from Lemma 2.4, we have

$$|F_1(u) - F_1(\varphi_c)| = |F_1(u_0) - F_1(\varphi_c)| \le \left(C(u_0) + 17a^4\right)\delta.$$

Plugging the above into (2.37) yields that

$$|F_1(v)| \le \frac{4a}{\sqrt{3}}\|v\|_{H^1}\sqrt{\|v\|_{H^1}^4 - F_1(v)} + L \le \frac{8a^2}{3}\|v\|_{H^1}^2 + \frac{1}{2}\|v\|_{H^1}^4 + \frac{1}{2}|F_1(v)| + L$$

where $L := 15aK\delta^{1/4} + \left(C(u_0) + 17a^4\right)\delta < 15aK\delta^{1/4} + K^4\delta$. Hence, we have

$$|F_1(v)| \le \frac{16a^2}{3}\|v\|_{H^1}^2 + \|v\|_{H^1}^4 + 2L. \tag{2.38}$$

Therefore from (2.36), it is inferred that

$$\|v_x\|_{L^4}^4 \le 16a^2\|v\|_{H^1}^2 + 6\|v\|_{H^1}^4 + 6L \le \tilde{C}_1(u_0)\delta^{1/4} + \tilde{C}_2(u_0)\delta$$

where $\tilde{C}_1(u_0) := 90aK + 12a^2K$ and $\tilde{C}_2(u_0) := 24a^2K + 12K^4$. Moreover, it is noted that

$$\|v\|_{L^4} \le \|v\|_{H^1} \le K\delta^{1/4}.$$

Combining the above leads to (2.34). This completes the proof of Theorem 2.3. ■

3 The Extended Cubic CH Equation

We now turn our attention to the stability analysis for the extended cubic CH equation. It is known that the classical CH equation without the linear terms u_x and u_{xxx} possesses peaked localized solitons. Analogously we will focus on the equation (1.4), which neglects the linear terms u_x and u_{xxx}, although the linear term u_{xxx} could be removed by the Galilean transformation $u(t,x) \to u(t, x-\kappa t)$ with a suitable parameter κ.

3.1 Existence of peaked solitary waves

Consider the initial-value problem

$$\begin{cases} m_t + k_1\left(2u_x m + u m_x\right) + k_2\left((u^2 - \frac{1}{4}(u^2)_{xx})u\right)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, & \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Applying the operator $(1-\partial_x^2)^{-1}$ to above equation in (3.1) yields the following nonlocal equation:

$$u_t + k_1 u u_x + k_1 p_x * \left(u^2 + \frac{1}{2}u_x^2\right) + \frac{k_2}{2}u^2 u_x + \frac{k_2}{2}p_x * \left(u u_x^2 + \frac{5}{3}u^3\right) = 0. \tag{3.2}$$

The following local well-posedness results of strong solutions can be obtained by applying a Galerkin-type approximation method, which is established by Hilmonas and Holliman [13]. The proof is thus omitted.

Proposition 3.1. If $s > \frac{3}{2}$ and $u_0 \in H^s(\mathbb{R})$, then there exists $T > 0$ and a unique strong solution $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ of the initial-value problem (3.1). Further, the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $H^s(\mathbb{R})$ into $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$.

Similarly as in the previous section, we record the important conservation laws.

Lemma 3.1. For a strong solution u obtained in Proposition 3.1, the following functionals

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\,dx, \quad F_2(u) = 2k_1 \int_{\mathbb{R}} (u^3 + u u_x^2)\,dx + k_2 \int_{\mathbb{R}} (u^4 + u^2 u_x^2)\,dx \tag{3.3}$$

are conserved, that is, $\frac{d}{dt}E(u) = \frac{d}{dt}F_2(u) = 0$, for all $t \in [0,T)$.

Proof. The conservation of $E(u)$ can be proved by multiplying equation in (3.1) by u and integrating over $\mathbb{R}$ and then integrating by parts. The conservation of $F_2(u)$ is an easy consequence of the Hamiltonian structure of equation in (3.1), cf. (1.6). In fact, we have

$$\frac{dF_2(u)}{dt} = \left\langle \frac{\delta F_2}{\delta u},\, u_t \right\rangle = \left\langle (1-\partial_x^2)\frac{\delta F_2}{\delta m},\, u_t \right\rangle = \left\langle \frac{\delta F_2}{\delta m},\, m_t \right\rangle = \left\langle \frac{\delta F_2}{\delta u},\, J_2 \frac{\delta F_2}{\delta u} \right\rangle = 0,$$

and this completes the proof of the lemma. ■

The existence of the single peaked solutions to equation (3.2) is given below. Details of the proof can be found in Appendix A.

Theorem 3.1. Assume $k_2 \neq 0$. The equation (3.2) admits the single peakon of the following forms:

(1) If $c \neq 0$ and $k_1^2 + 2k_2 c \geq 0$, then the single peaked solutions have the form

$$u(t,x) = \varphi_c(x-ct) := ae^{-|x-ct|}, \quad \text{with } a = \frac{-k_1 \pm \sqrt{k_1^2 + 2k_2 c}}{k_2} =: a_\pm \neq 0. \tag{3.4}$$

(2) If $c = 0$ and $k_1 \neq 0$, then the single peaked solutions take the form

$$u(t,x) = a\varphi(x) := ae^{-|x|}, \quad \text{with } a = -\frac{2k_1}{k_2} \neq 0. \tag{3.5}$$

3.2 H^1-orbital stability of $\varphi_c(x-ct)$

The focus of this subsection is the stability analysis with peakons of the form (3.4). For simplicity, we will only consider the case when $a = a_+ > 0$, since the other case can be easily handled by using invariance of the cubic CH equation in (3.1) under the transformation $u \to -u$ and $k_1 \to -k_1$.

It is easy to check that

$$\max_{x\in\mathbb{R}}\{\varphi_c(x)\} = \varphi_c(0) = a_+, \quad E(\varphi_c) = \|\varphi_c\|_{H^1}^2 = 2a_+^2, \quad F_2(\varphi_c) = \frac{8}{3}k_1 a_+^3 + k_2 a_+^4.$$

The main result of this subsection is the following:

Theorem 3.2 (H^1-orbital stability). Consider $\varphi_c = a_+ e^{-|x-ct|}$ the peaked solutions defined in (3.4). Then φ_c is orbitally stable in the following sense. Assume that the initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. There exists some $0 < \delta_0 \ll 1$ such that if

$$\|u_0 - \varphi_c\|_{H^1} < a_+\delta, \qquad 0 < \delta < \delta_0, \tag{3.6}$$

then

(1) when $k_1 > 0$, $k_2 > 0$, the corresponding solution $u(t)$ of (3.1) satisfies

$$\sup_{t\in[0,T^*)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1} < 2a_+ \left(\frac{86k_1 + 75k_2 a_+}{8k_1 + 3k_2 a_+}\right)^{1/4} \delta^{1/4}. \tag{3.7}$$

(2) when $k_1 > 0$, $k_2 < 0$, and $0 < c < -\frac{4k_1^2}{9k_2}$, the corresponding solution $u(t)$ of (3.1) satisfies

$$\sup_{t\in[0,T^*)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1} < 2a_+ \left(\frac{41k_1 + 36|k_2|a_+}{2k_1}\right)^{1/4} \delta^{1/4},$$

where $T^* > 0$ is the maximal existence time of the solution $u(t,x)$ and $\xi(t) \in \mathbb{R}$ is the point at which the solution $u(t,x)$ achieves its maximum.

The proof of this theorem is achieved via a series of lemmas.

Lemma 3.2. For any $u \in H^1(\mathbb{R})$ and $z \in \mathbb{R}$, we have

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - z)\|_{H^1}^2 + 4a_+ \left(u(z) - a_+\right).$$

Proof. The proof follows exactly along the same line as for the proof of Lemma 2.2. ■

Lemma 3.3. Let $u \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Assume $\|u - \varphi_c\|_{H^1} < a_+\delta$, with $0 < \delta \ll 1$. Then

$$|E(u) - E(\varphi_c)| \leq C_2\delta, \qquad |F_2(u) - F_2(\varphi_c)| \leq C_3\delta, \tag{3.8}$$

where $C_2 := 4a_+^2$, $C_3 := \left(10|k_1| + 8|k_2|a_+\right) a_+^3$.

Proof. The 1st part of (3.8) is just (2.17). As for the 2nd estimate, it is noted that

$$|F_2(u) - F_2(\varphi_c)| \leq 2|k_1||I_1(u) - I_1(\varphi_c)| + |k_2||I_2(u) - I_2(\varphi_c)|,$$

where the functionals I_1 and I_2 are defined in (1.8). In view of [7, Lemma 3], it follows that

$$|I_1(u) - I_1(\varphi_c)| < a_+^3\delta \left(3\sqrt{2} + 3\delta + \frac{\delta^2}{\sqrt{2}}\right) < 5a_+^3\delta.$$

Next, similar to (2.19) and (2.20), a calculation reveals that

$$\begin{aligned} |I_2(u) - I_2(\varphi_c)| &\leq \int_{\mathbb{R}} \left[|(u^2 - \varphi_c^2)(u^2 + u_x^2)| + \varphi_c^2(u^2 + u_x^2 - \varphi_c^2 - \varphi_{cx}^2)\right] \mathrm{d}x \\ &< a_+^4\delta(4 + \sqrt{2}\delta)(2\delta + 1) + a_+^4\delta(\delta + 2\sqrt{2}) < 8a_+^4\delta. \end{aligned}$$

Putting the above together, we complete the proof of the lemma. ■

Lemma 3.4. Assume that $u \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and $\|u - \varphi_c\|_{H^1} < a_+\delta$, with $0 < \delta \ll 1$. Furthermore, assume that one of the following two conditions holds:

(1) $k_1 > 0$, $k_2 > 0$, and $\widetilde{M} + \widetilde{m} \geq 0$.
(2) $k_1 > 0$, $k_2 < 0$ and $0 < c \leq -\frac{k_1^2}{2k_2}$.

Then we have

$$2k_1 u + k_2 u^2 \leq 2k_1 \widetilde{M} + k_2 \widetilde{M}^2, \tag{3.9}$$

where recall that the constants $\widetilde{M} = \max_{x\in\mathbb{R}} u(x) \geq 0$ and $\widetilde{m} = \min_{x\in\mathbb{R}} u(x) \in \mathbb{R}$.

Proof. (1) If $k_1 > 0$, $k_2 > 0$ and $\widetilde{M} + \widetilde{m} \geq 0$, then it is easy to see that

$$2k_1 u + k_2 u^2 \leq 2k_1 \widetilde{M} + k_2 \max_{x\in\mathbb{R}}\{u^2(x)\} \leq 2k_1 \widetilde{M} + k_2 \widetilde{M}^2.$$

(2) In this case, we have

$$0 < \frac{\sqrt{2}}{2}\|\varphi_c\|_{H^1} = a_+ = \frac{-k_1 + \sqrt{k_1^2 + 2k_2 c}}{k_2} < \frac{k_1}{|k_2|}. \tag{3.10}$$

Then choosing δ small enough, from (3.8), (3.10), and the Sobolev embedding, it transpires that

$$\|u\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\|u\|_{H^1} < \frac{\sqrt{2}}{2}\|\varphi_c\|_{H^1} + \sqrt{2}a_+\delta^{\frac{1}{2}} < \frac{k_1}{|k_2|}. \tag{3.11}$$

On the other hand, define the functional $f(u) := 2k_1 u + k_2 u^2$. In view of (3.10) and (3.11), a direct computation yields that

$$\frac{df}{du} = 2k_1 + 2k_2 u \geq 2k_1 - 2|k_2||u| > 0 \quad \text{for } \|u\|_{L^\infty} < \frac{k_1}{|k_2|}.$$

Hence, it follows that $f(u) \leq f(\widetilde{M})$. We thus finish the proof of the lemma. ■

Lemma 3.5. Under the conditions of Lemma 3.4, for $u \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, we have

$$F_2(u) \leq (2k_1 \widetilde{M} + k_2 \widetilde{M}^2)E(u) - \frac{4}{3}k_1 \widetilde{M}^3 - k_2 \widetilde{M}^4. \tag{3.12}$$

Proof. Taking $\xi \in \mathbb{R}$ such that $\widetilde{M} = u(\xi)$ and defining $g(x)$ as in (2.14), it is noted that

$$\int_{\mathbb{R}} g^2(x)\,\mathrm{d}x = E(u) - 2u^2(\xi) = E(u) - 2\widetilde{M}^2. \tag{3.13}$$

In addition, the auxiliary function $h(x)$ is defined by

$$h(x) := 2k_1 u(x) + k_2 u^2(x), \quad x \in \mathbb{R}.$$

It holds that

$$2k_1 \int_{\mathbb{R}} u(x)g^2(x)\,\mathrm{d}x = 2k_1 I_1(u) - \frac{8k_1}{3}\widetilde{M}^3, \quad k_2 \int_{\mathbb{R}} u^2(x)g^2(x)\,\mathrm{d}x = k_2 I_2(u) - k_2\widetilde{M}^4.$$

Thus, we have

$$\int_{\mathbb{R}} h(x)g^2(x)\,\mathrm{d}x = F_2(u) - \frac{8k_1}{3}\widetilde{M}^3 - k_2\widetilde{M}^4. \tag{3.14}$$

On account of Lemma 3.4, it then follows from (3.13) and (3.14) that

$$\begin{aligned} F_2(u) - \frac{8k_1}{3}\widetilde{M}^3 - k_2\widetilde{M}^4 &= \int_{\mathbb{R}} h(x)g^2(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left(2k_1 u + k_2 u^2\right) g^2(x)\,\mathrm{d}x \\ &\le \left(2k_1\widetilde{M} + k_2\widetilde{M}^2\right) E(u) - 4k_1\widetilde{M}^3 - 2k_2\widetilde{M}^4, \end{aligned}$$

which in turn implies that

$$F_2(u) \le \left(2k_1\widetilde{M} + k_2\widetilde{M}^2\right) E(u) - \frac{4}{3}k_1\widetilde{M}^3 - k_2\widetilde{M}^4.$$

Hence, we reach the conclusion of Lemma 3.5. ■

Lemma 3.6. Let $k_1 > 0$, $k_2 > 0$ and $\widetilde{M} + \widetilde{m} \ge 0$. Assume that $u \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and satisfies (3.8). Then

$$\left|\widetilde{M} - a_+\right| < \sqrt{\frac{78k_1 + 72k_2 a_+}{8k_1 + 3k_2 a_+}} \cdot a_+ \delta^{\frac{1}{2}}.$$

Proof. From (3.12), it follows that

$$k_1\left(\frac{4}{3}\widetilde{M}^3 - 2E(u)\widetilde{M} + 2I_1(u)\right) + k_2\left(\widetilde{M}^4 - E(u)\widetilde{M}^2 + I_2(u)\right) \le 0. \tag{3.15}$$

This motivates us to define the Lyapunov function $p_u(z)$ by

$$p_u(z) := k_1\left(\frac{4}{3}z^3 - 2E(u)z + 2I_1(u)\right) + k_2\left(z^4 - E(u)z^2 + I_2(u)\right). \tag{3.16}$$

Recall that $E(\varphi_c) = 2a_+^2$, $I_1(\varphi_c) = \frac{4}{3}a_+^3$ and $I_2(\varphi_c) = a_+^4$. We have

$$p_{\varphi_c}(z) = k_1\left(\frac{4}{3}z^3 - 2E(\varphi_c)z + \frac{8}{3}a_+^3\right) + k_2\left(z^4 - E(\varphi_c)z^2 + a_+^4\right)$$
$$= (z-a_+)^2\left(\frac{4}{3}k_1(z+2a_+) + k_2(z+a_+)^2\right).$$

We can also write

$$p_{\varphi_c}(\widetilde{M}) = p_u(\widetilde{M}) + 2k_1\widetilde{M}\left(E(u) - E(\varphi_c)\right) + k_2\widetilde{M}^2\left(E(u) - E(\varphi_c)\right) - \left(F_2(u) - F_2(\varphi_c)\right),$$

which together (3.15) yields

$$\begin{aligned}&(\widetilde{M} - a_+)^2\left(\frac{4}{3}k_1(\widetilde{M}+2a_+) + k_2(\widetilde{M}+a_+)^2\right)\\ &\quad\le \left(2k_1\widetilde{M} + k_2\widetilde{M}^2\right)\left|E(u) - E(\varphi_c)\right| + \left|F_2(u) - F_2(\varphi_c)\right|.\end{aligned} \tag{3.17}$$

Using the conditions $k_1 > 0$, $k_2 > 0$, it is determined that

$$\frac{4}{3}k_1(\widetilde{M}+2a_+) + k_2(\widetilde{M}+a_+)^2 \ge \frac{8}{3}k_1a_+ + k_2a_+^2. \tag{3.18}$$

On the other hand, (3.8) implies that

$$0 < \widetilde{M} \le \frac{\sqrt{2}}{2}E(u)^{\frac{1}{2}} \le \frac{\sqrt{2}}{2}\left(E(\varphi_c) + 4a_+^2\delta\right)^{\frac{1}{2}} = a_+(2\delta+1)^{\frac{1}{2}}. \tag{3.19}$$

Hence, in view of (3.17), (3.18), and (3.19), we conclude that

$$\begin{aligned}\left|\widetilde{M} - a_+\right| &\le \sqrt{\frac{3\left(2k_1\widetilde{M} + k_2\widetilde{M}^2\right)\left|E(u) - E(\varphi_c)\right| + 3\left|F_2(u) - F_2(\varphi_c)\right|}{8k_1a_+ + 3k_2a_+^2}}\\ &\le \sqrt{\frac{(6k_1\widetilde{M} + 3k_2\widetilde{M}^2)C_2\delta + 3C_3\delta}{8k_1a_+ + 3k_2a_+^2}} < \sqrt{\frac{78k_1 + 72k_2a_+}{8k_1 + 3k_2a_+}}\cdot a_+\delta^{\frac{1}{2}},\end{aligned}$$

which completes the proof of Lemma 3.6. ■

Lemma 3.7. Let $k_1 > 0$, $k_2 < 0$ and $0 < c \leq -\frac{4k_1^2}{9k_2}$. Assume that $u \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and satisfies (3.8). Then we have

$$|\widetilde{M} - a_+| < \sqrt{\frac{78k_1 + 72|k_2|a_+}{4k_1}} \cdot a_+\delta^{\frac{1}{2}}.$$

Proof. Using the similar arguments as Lemma 3.6, we replace (3.17) by

$$\begin{aligned}&(\widetilde{M} - a_+)^2\left(\frac{4}{3}k_1(\widetilde{M} + 2a_+) + k_2(\widetilde{M} + a_+)^2\right)\\ &\quad\leq \left(2k_1\widetilde{M} + |k_2|\widetilde{M}^2\right)\left|E(u) - E(\varphi_c)\right| + \left|F_2(u) - F_2(\varphi_c)\right|.\end{aligned} \tag{3.20}$$

From the conditions $k_1 > 0$, $k_2 < 0$, and $0 < c < \frac{-4k_1^2}{9k_2}$, a direct calculation yields that $0 < 2a_+ < \frac{-4k_1}{3k_2}$. Choosing δ sufficiently small, we have from (3.11) that

$$0 < \widetilde{M} + a_+ \leq 2a_+ + \sqrt{2}a_+\delta^{\frac{1}{2}} < -\frac{4k_1}{3k_2},$$

from which we deduce that

$$\frac{4}{3}k_1(\widetilde{M} + 2a_+) + k_2(\widetilde{M} + a_+)^2 > \frac{4k_1}{3}a_+. \tag{3.21}$$

Hence, in view of (3.19), (3.20), and (3.21), we therefore conclude that

$$|\widetilde{M} - a| \leq \sqrt{\frac{(6k_1\widetilde{M} + 3|k_2|\widetilde{M}^2)C_2\delta + 3C_3\delta}{4k_1a_+}} < \sqrt{\frac{78k_1 + 72|k_2|a_+}{4k_1}} \cdot a_+\delta^{\frac{1}{2}}.$$

This completes the proof of Lemma 3.7. ■

Proof of Theorem 3.2. Assume $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $u \in C([0, T^*), H^s(\mathbb{R})) \cap C^1([0, T^*), H^{s-1}(\mathbb{R}))$ be the corresponding solution of initial-value problem (3.1) on the line with $T^* > 0$ being the maximal existence time of the solution. From Lemma 3.1, it is noted that

$$E(u(t,\cdot)) = E(u_0) \ \text{ and } \ F_2(u(t,\cdot)) = F_2(u_0), \quad t \in [0, T^*). \tag{3.22}$$

(1) Applying (3.22) together with (3.6) implies that Lemma 3.3 holds. Furthermore, a similar argument as in the proof of Theorem 2.2 suggests the existence of $T_0 > 0$

such that

$$\widetilde{M}(t)+\widetilde{m}(t)>0, \quad \text{for} \quad t\in[0,T_0],$$

which allows us to apply Lemma 3.6 to obtain

$$\left|u(t,\xi(t))-a_+\right|<\sqrt{\frac{78k_1+72k_2a_+}{8k_1+3k_2a_+}}\cdot a_+\delta^{\frac{1}{2}}, \tag{3.23}$$

for any $t\in[0,T_0]$, where $u(t,\xi(t))=\widetilde{M}(t)$. Moreover, utilizing Lemma 3.2, we have

$$\begin{aligned}\|u(t,\cdot)-\varphi_c(\cdot-\xi(t))\|_{H^1}&\leq\sqrt{|E(u_0)-E(\varphi_c)|+4a_+|u(t,\xi(t))-a_+|}\\&<2a_+\left(1+\frac{78k_1+72k_2a_+}{8k_1+3k_2a_+}\right)^{1/4}\delta^{1/4},\end{aligned} \tag{3.24}$$

for $t\in[0,T_0]$. Again a similar continuity argument as is performed in the proof of Theorem 2.2 implies that T_0 can be pushed all the way until T^*, which means that (3.24) holds for all $t\in[0,T^*)$. Thus, we complete the proof of part (1) of Theorem 3.2.

(2) Similarly, one can apply Lemma 3.3 here. Moreover, since in this case

$$0<c<-\frac{4k_1^2}{9k_2} \quad\Longrightarrow\quad 0<c<-\frac{k_1^2}{2k_2},$$

Lemma 3.7 can be applied. Then the rest of the proof can be done in a similar approach, and hence we omit it here. ■

4 Appendix A

For the readers' convenience, we provide the details about the proofs of Lemma 2.1, Theorem 2.1, and Theorem 3.1 in this appendix.

Proof of Lemma 2.1. The conservation of $E(u)$ can be obtained by multiplying equation in (2.1) by u and integrating over $\mathbb{R}$ and then using integration by parts. On the other hand, taking the inner products between equation in (2.1) and $4(1-\partial_x^2)^{-1}\left((u^2-u_x^2)m\right)$, then we have

$$\begin{aligned}0=&4\langle m_t,(1-\partial_x^2)^{-1}\left((u^2-u_x^2)m\right)\rangle+4k_1\langle\left((u^2-u_x^2)m\right)_x,(1-\partial_x^2)^{-1}\left((u^2-u_x^2)m\right)\rangle\\&+4k_2\langle u^2m_x+3uu_xm,(1-\partial_x^2)^{-1}\left((u^2-u_x^2)m\right)\rangle=:P_1+P_2+P_3.\end{aligned} \tag{4.1}$$

For the terms P_1 and P_2, a direct calculation gives that

$$P_1 = 4\int_{\mathbb{R}} u_t(u^2 - u_x^2)m\,\mathrm{d}x = \frac{d}{dt}\int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)\mathrm{d}x, \tag{4.2}$$

and

$$\begin{aligned} P_2 &= 4k_1 \int_{\mathbb{R}} \left((u^2 - u_x^2)m\right)_x (1-\partial_x^2)^{-1}\left((u^2 - u_x^2)m\right)\mathrm{d}x \\ &= 2k_1 \int_{\mathbb{R}} \left(\left((1-\partial_x^2)^{-\frac{1}{2}}\left((u^2-u_x^2)m\right)\right)^2\right)_x \mathrm{d}x = 0. \end{aligned} \tag{4.3}$$

Applying integration by parts, we discover that

$$\begin{aligned} P_3 &= 4k_2\langle u^2 m_x + 3uu_x m, (1-\partial_x^2)^{-1}\left((u^2-u_x^2)m\right)\rangle \\ &= 4k_2\langle u^2u_x, (u^2-u_x^2)m\rangle + 2k_2\langle (1-\partial_x^2)^{-1}u_x^3, (u^2-u_x^2)m\rangle \\ &\quad +4k_2\langle (1-\partial_x^2)^{-1}\partial_x\left(u^3 + \tfrac{3}{2}uu_x^2\right), (u^2-u_x^2)m\rangle = 0, \end{aligned} \tag{4.4}$$

where we have used the operator formula $(1-\partial_x^2)^{-1}\partial_x^2 = -1 + (1-\partial_x^2)^{-1}$. Plugging (4.2)-(4.4) into (4.1), we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)\mathrm{d}x = 0. \tag{4.5}$$

This completes the proof of Lemma 2.1. ■

Proof of Theorem 2.1. Recall the definition (2.5). For simplicity, we drop the subscript in φ_c. We have that

$$\varphi'(x) = -\mathrm{sgn}(x)\varphi(x), \qquad p(x) = \frac{1}{2a}\varphi(x).$$

Plugging in the ansatz (2.5) into (2.3) and computing the convolution terms, we have

$$\begin{aligned} &p' * \left[\left(\frac{2k_1}{3} + k_2\right)\varphi^3 + \left(k_1 + \frac{3k_2}{2}\right)\varphi(\varphi')^2\right] + \left(\frac{k_1}{3} + \frac{k_2}{2}\right)p * (\varphi')^3 \\ &= \frac{1}{2a}\left(\frac{k_1}{3} + \frac{k_2}{2}\right)\left(5\varphi' * \varphi^3 + \varphi * (\varphi^2\varphi')\right) = \frac{4(2k_1+3k_2)}{9a}\varphi' * \varphi^3. \end{aligned}$$

A direct computation yields that

$$\varphi' * \varphi^3 = -(\mathrm{sgn}(x)\varphi) * \varphi^3 = \frac{3a}{4}\mathrm{sgn}(x)\varphi(\varphi^2 - a^2).$$

For the local terms in (2.3), we have

$$-c\varphi' + k_1\left(\varphi^2\varphi' - \frac{1}{3}(\varphi')^3\right) + k_2\varphi^2\varphi' = \operatorname{sgn}(x)\varphi\left(c - \frac{2k_1+3k_2}{3}\varphi^2\right).$$

Putting together, we find the equation for φ to be

$$\operatorname{sgn}(x)\varphi\left(c - \frac{2k_1+3k_2}{3}a^2\right) = 0, \tag{4.6}$$

which leads to the two cases (1) $a = \pm\sqrt{\frac{3c}{2k_1+3k_2}}$ and (2) $a \neq 0$ with $2k_1+3k_2 = c = 0$. This completes the proof of Theorem 2.1. ■

Proof of Theorem 3.1. Similar to the approach in Theorem 2.1, we plug the function φ into (3.2). A direct calculation then reveals

$$-c\varphi' + k_1\varphi\varphi' + \frac{k_2}{2}\varphi^2\varphi' = \operatorname{sgn}(x)\varphi\left(c - k_1\varphi - \frac{k_2}{2}\varphi^2\right)$$

and

$$p' * \left(k_1\varphi^2 + \frac{k_1}{2}(\varphi')^2 + \frac{k_2}{2}\varphi(\varphi')^2 + \frac{5}{6}\varphi^3\right) = \operatorname{sgn}(x)k_1\left(\varphi^2 - a\varphi\right) + \operatorname{sgn}(x)\frac{k_2}{2}\left(\varphi^3 - a^2\varphi\right),$$

where use has been made of the equalities $\varphi' * \varphi^2 = \frac{2a}{3}\operatorname{sgn}(x)\cdot\left(\varphi^2 - a\varphi\right)$. In view of the above two identities, we deduce that the equation for φ is

$$\operatorname{sgn}(x)\left(c - k_1 a - \frac{k_2}{2}a^2\right)\varphi = 0.$$

Solving the above, we have that (1) $c \neq 0$ and $a = \frac{-k_1\pm\sqrt{k_1^2+2k_2c}}{k_2}$, or (2) $c = 0$ and $a = -\frac{2k_1}{k_2}$. ■

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