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An alternative formalism for modeling spin

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Abstract We present an alternative formalism for modeling spin. The ontological elements of this formalism are base-2 sequences of length n. The machinery necessary to model physics is then developed by considering correlations between base-2 sequences. Upon choosing a reference base-2 sequence, a relational system of numbers can be defined, which we interpret as quantum numbers. Based on the properties of these relational quantum numbers, the selection rules governing interacting spin systems are derived from first principles. A tool for calculating the associated probabilities, which are the squared Clebsch–Gordan coefficients in quantum mechanics, is also presented. The resulting model offers a vivid information theoretic picture of spin and interacting spin systems. Importantly, this model is developed without making any assumptions about the nature of space-time, which presents an interesting opportunity to study emergent space-time models.

1 Introduction

Historically, discovering new ways of obtaining established results has been an effective means of making progress in physics. The epitome of this is Hamilton's reformulation of Newtonian mechanics. Superficially, a reproduction of a known result, especially one obtained nearly a century ago, seems inconsequential. Though occasionally, as was the case for Hamilton's insight, the manner in which the old result is reproduced can open new avenues of thought and exploration. It is in this vein that we present here an alternative formalism for modeling spin, which emerges upon the consideration of two point correlations between base-2 sequences. The information theoretic roots of this alternative formalism paint a completely new picture of the conceptually elusive, but physically ubiquitous quantity known as spin.

The information we obtain about physical systems requires measurement, which inevitably involves one or more quantum mechanical interactions [1–3]. While one cannot say with certainty if interactions in nature are discrete or continuous at the fundamental level, the observable outcome of any interaction is always discrete. For this reason, the results of any conceivable physical experiment can be reduced to counting. This fact stands in stark contrast with the uncountable sets universally employed by modern theories, which are based on continuous functions satisfying differential equations. This tension between the countable nature of empirical data and the uncountable sets that form the foundations of modern theories is not simply a matter of improving precision or collecting more data [4]. Rather, it exists because of a fundamental difference between our experience of the physical world and the theories we use to model those experiences. This simple observation leads us to the following quote from Niels Bohr:

> "It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature." [5]

If we cannot prove that nature is continuous, then perhaps we should explore theories which do not require it be so. The quantum revolution of the twentieth century was a direct consequence of the observed discreteness of interactions [6]. However, quantum mechanics (QM) was built with the classical Hamiltonian in mind [7]. This approach resulted in a strong dependence of the theory on uncountable sets. While quantum gravity is generally considered to be the final piece of the quantum revolution [8–12], there remain significant questions regarding the nature of the quantum state in QM [13–17]. This less appreciated use of uncountable sets in physics was a primary motivation for the development of the alternative formalism presented here, which has the ability to reproduce predictions from QM under a continuum limit, while also revealing important geometric properties and selection rules in the finite regime.

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For nearly a century, there has been a perpetual debate regarding the reality of the quantum state in QM [18–20]. That is, does the quantum state represent something truly physical, or is it epistemic? Much of this debate occurs within the context of the standard Dirac formalism for QM, which involves Hilbert spaces, the Schrodinger equation, the Born rule, etc. [21,22]. Applying the various no-go theorems that have resulted from this debate to an alternative formalism is not generally useful, especially when that alternative formalism does not assume a preexisting space-time, as will be the case here. However, even within epistemic interpretations of the quantum state, there is still some notion of an ontic state, where the quantum state is simply an ensemble of these ontic states. This conceptual picture of the quantum state is precisely the one that develops within the formalism to be introduced here, where ontic states are modeled by sequences of finite group elements, beginning with the group Z_2. The information stored in the ordering of these finite group elements is then hidden, or coarse-grained away, leading to non-determinism in the resulting model.

While non-determinism is certainly a central feature of QM, one should not lose sight of the profound role determinism plays in nature. As one might imagine, incorporating the correct non-deterministic and deterministic features into a single cohesive model for spin is no small task. Yet, the formalism to be introduced here manages this feat quite naturally. For example, quantities like total spin, which is an emergent and relational property of two point correlations between base-2 sequences, can be conserved by considering permutations of the underlying sequences. The selection rules obeyed by interacting spin systems can be recovered by considering three point correlations between base-2 sequences, along with simple arithmetic arguments. Of course, these selections rules include deterministic equations associated with the conservation of angular momentum within interacting spin systems. Thus, important laws of nature arise naturally within this formalism, rather than being asserted through axioms or principles.

The probability coefficients obeyed by interacting spin systems, which are the squared Clebsch–Gordan coefficients in QM, represent an important test case for the development of this formalism and the subsequent model. As previously mentioned, we make no assumptions about the nature of space-time. Rather, our intention is to use calculations, such as the probability coefficients for interacting spin systems, to guide our development of space-time. The result of this calculation is a simple closed form expression, coupled with a vivid conceptual picture which involves two observers, one associated with each of the constituents involved in a spin interaction experiment. These observers, which we call Alice and Bob, each construct their own epistemic ensemble, which encodes the knowledge each has about the physical systems involved in the experiment. The probability coefficients are then found by counting paths between their ensembles, such that certain quantum numbers are conserved.

In recent decades, several serious research efforts have been made towards producing an alternative to QM [23–30]. Through unique combinations of motivations, development strategies, and results, each of these efforts have contributed significantly to a shifting paradigm, at least within the small community of active researchers in this field. For those familiar with these efforts, the existence of a theory beyond QM is not some faint notion, but a plausible and attainable reality. Given the foundational role that QM plays in science and technology, as well as the considerable challenges facing these fields today, the pace of scientific discourse regarding this matter must increase. What differentiates the formalism presented here from these previous efforts is its unique combination of simplicity and modeling power. With a small number of mathematical tools, it has the ability to produce the selection rules and probability coefficients associated with a real experiment, while refraining from making any assumptions about the nature of space-time. In other words, the formalism and subsequent model introduced here not only offers an interesting information theoretic picture of the quantum state as well as interactions, but it also has clear predictive power and the potential to inform important next steps in the development of an emergent space-time.

This paper is broken into six sections, including the introduction. In Sect. 2, the foundations of the alternative formalism will be introduced, which involves base-2 sequences and correlations between them. In Sect. 3, our definition of quantum numbers will be introduced, along with the notation necessary to label sequences, or sets of sequences, using these quantum numbers. The properties of these quantum numbers are then explored in Sect. 4, which leads to the derivation of the selection rules for interacting spin systems. The probabilities associated with interacting spin systems are then calculated in Sect. 5. Finally, the implications of this work, as well as some ideas regarding future work, are discussed in Sect. 6.

2 Sequences

The building block of this formalism is the base-2 sequence. A base-2 sequence is a list comprised of two distinct symbols, where the symbols may be repeated and order matters. The symbols used here are 0 and 1, which are the members of the finite group Z_2 [31]:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

A base-2 sequence can be of any length, which is denoted as n. For a given length n, there will be 2^n unique sequences. The set of all such sequences is denoted as $S^1(n)$. In physics, it is well known that most of the information contained in a composite system does not lie in its subsystems, but actually in the correlations between its subsystems [32]. For this reason, we are motivated to introduce the set $S^2(n)$, which is the set of all two point correlations between base-2 sequences. An element of the set $S^2(4)$ is given here:

$$\begin{pmatrix} 1\,0 \\ 0\,1 \\ 1\,1 \\ 1\,0 \end{pmatrix} \in S^2(n=4). \tag{2}$$

Individual elements of these sets are denoted as $s^2 \in S^2(n)$, where n has been suppressed. Using this notation, an element of $S^2(n)$ can be constructed using two elements of $S^1(n)$ like so, where the $\otimes$ symbol is used to denote the correlation operator:

$$s^1 \otimes s'^1 = s^2. \tag{3}$$

A more explicit representation of the operation shown in Eq. (3) is given here, where a particular example of s^1 and s'^1 has been chosen:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\,0 \\ 0\,0 \\ 0\,1 \\ 1\,1 \\ 0\,0 \\ 1\,0 \end{pmatrix}. \tag{4}$$

Simply put, base-2 sequences are the bricks of this formalism, while the correlation operator is the mortar. This correlation operation can also be thought of as an increase in basis. While an element of $S^1(n)$ is a sequence written in base-2, elements of $S^2(n)$ can be thought of as sequences written in base-4, where the new basis elements, or symbols, are the members of the group $Z_2 \otimes Z_2$. While one can always use the base-2 representation, it will be conceptually beneficial to introduce alternative symbols for the basis elements of $S^2(n)$; $00 = A$, $11 = B$, $10 = C$, and $01 = D$. With this notation in hand, Eq. (4) can be rewritten as follows:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C \\ A \\ D \\ B \\ A \\ C \end{pmatrix}. \tag{5}$$

More generally, the approach taken in this formalism is to construct random base-2 $n \times d$ matrices by gluing together d base-2 sequences of length n using the $\otimes$ operator. It may be useful to imagine each base-2 sequence as a point in some
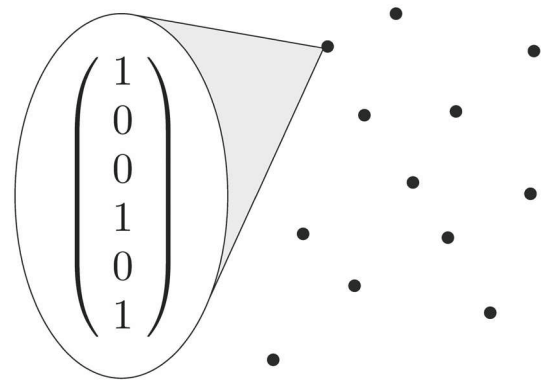

Fig. 1 A random selection of base-2 sequences are visualized as points in some abstract space, where the position of each point has no physical significance

abstract space. Though the details of that space, as well as the distribution of the points within it, have no physical significance just yet (Fig. 1). The distribution of points within this abstract space is related to the issue of ordering sets. Given a set of base-2 sequences, which one should come first? Binary languages in computer science offer perfectly reasonable answers to this question. However, those approaches to ordering base-2 sequences rely on the information stored in the ordering of the base-2 basis elements, which we plan to hide, or coarse-grain away. In the following section, an ordering scheme will be introduced that can survive such a step.

3 Quantum numbers

Within this model, the information stored in the configuration of the basis elements comprising a sequence is hidden, or coarse-grained away. This means that the successful ordering scheme will only be a partial ordering of base-2 sequences, rather than a total ordering. This is an essential feature of this formalism, which leads directly to non-determinism in the subsequent model.

The ordering scheme employed here requires the introduction of a relational system of numbers. This number system is a function of the reference sequence, which is a particular base-2 sequence chosen from the set $S^1(n)$, and is denoted as s_0^1. Using the correlation operation, this reference sequence is then used to construct elements of the set $S^2(n)$, which are base-4 sequences. The number of times a particular basis element appears in a sequence is called a count. For each relationship between the reference sequence and another element of $S^1(n)$, there are four associated counts. These four counts are denoted as $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, where the tilde notation has been introduced to distinguish each count from its associated base-4 basis element.

From these four counts, a relational set of measures can be defined for each base-4 sequence, which we interpret as quantum numbers. It will be shown that the quantum numbers $j = \frac{\tilde{C}+\tilde{D}}{2}$ and $m = \frac{\tilde{C}-\tilde{D}}{2}$ share important properties with total spin and the z-component of spin, respectively [21]. Moreover, the quantum number j, which is closely related to the Hamming distance in computer science, is a metric. This means that for any choice of three base-2 sequences, one can be placed at each of the vertices of a triangle, where j is the length of the edge connecting two vertices. This feature endows this formalism with important geometric properties.

A complete set of quantum numbers allows one to determine the number of times each basis element appears within a particular sequence. To make j and m complete, the quantum numbers $g = \frac{\tilde{A}+\tilde{B}}{2}$ and $l = \frac{\tilde{A}-\tilde{B}}{2}$, which do not yet have established physical analogues, must be included. Thus, the complete set of quantum numbers for a particular base-4 sequence is as follows:

$$j = \frac{\tilde{C} + \tilde{D}}{2} \tag{6}$$

$$m = \frac{\tilde{C} - \tilde{D}}{2} \tag{7}$$

$$g = \frac{\tilde{A} + \tilde{B}}{2} \tag{8}$$

$$l = \frac{\tilde{A} - \tilde{B}}{2} \tag{9}$$

$$-j \le m \le j \tag{10}$$

$$-g \le l \le g. \tag{11}$$

The quantum numbers defined in Eqs. (6)–(9) will serve as ordering parameters. Notationally, these ordering parameters can be used to distinguish one set of sequences from another. In the case of base-2 sequences, the subset of $S^1(n)$ containing all base-2 sequences with the quantum numbers j, m, g, and l, as determined by the chosen reference sequence s_0^1, is denoted as follows: $S^1(j, m, g, l) \subset S^1(n)$. Note that $n = 2j + 2g = \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D}$, making explicit mention of n unnecessary if both j and g are given. An element of the subset $S^1(j, m, g, l)$ can then be denoted by including subscripts like so: $s_{j,m,g,l}^1 \in S^1(j, m, g, l)$. With this notation in hand, the correlation operation can be defined as follows:

$$s_0^1 \otimes s_{j,m,g,l}^1 = s_{j,m,g,l}^2. \tag{12}$$

Equation (12) raises an important issue, which is that the quantum numbers j, m, g, and l can be used to label base-2 sequences like $s_{j,m,g,l}^1$, as well as base-4 sequences like $s_{j,m,g,l}^2$. When used to label base-2 sequences, these quantum numbers are functions of the chosen reference sequence s_0^1, resulting in a relational ordering scheme. This just means that the quantum numbers j, m, g, and l associated with a particular base-2 sequence may vary depending on the reference
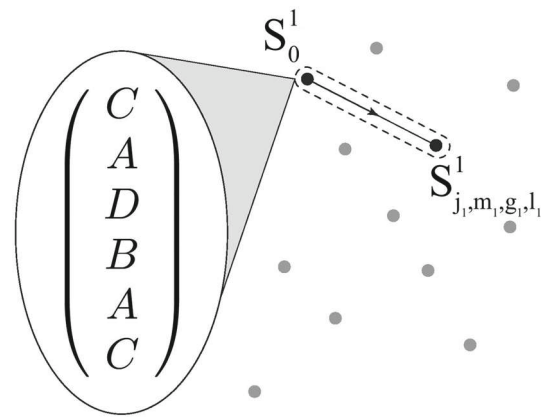

Fig. 2 A correlation of two base-2 sequences, which is a base-4 sequence and an element of $S^2(n)$, can be visualized as a directed edge connecting two vertices. By choosing a reference sequence, we can assign quantum numbers to the remaining base-2 sequences and order the points introduced in Fig. 1 accordingly. For the case shown here, the quantum numbers are $j_{01} = \frac{3}{2}$, $m_{01} = +\frac{1}{2}$, $g_{01} = \frac{3}{2}$, and $l_{01} = +\frac{1}{2}$

sequence. On the other hand, the subset of base-4 sequences associated with j, m, g, and l will include all possible two point correlations between base-2 sequences that result in those quantum numbers.

On a more technical note, the position of the reference sequence within the correlation shown in Eq. (12) is important due to the asymmetry of the $C = 10$ and $D = 01$ basis elements under the commutation operation. Under this operation, the counts $\tilde{C}$ and $\tilde{D}$ are exchanged, implying the quantum number m must change sign according to Eq. (7). Notationally, subscripts can be added to each quantum number to convey the orientation of the correlation like so: $s_0^1 \otimes s_{j_1,m_1,g_1,l_1}^1 = s_{j_{01},m_{01},g_{01},l_{01}}^2$. Again, the only quantum number that changes sign under the exchange of these indices is m: $m_{01} = -m_{10}$. The picture associated with the operation in Eq. (12), which can be visualized as a directed edge connecting two vertices, is given in Fig. 2.

The physical interpretation of the operation shown in Eq. (12) is a single measurement. We read the expression $s_0^1 \otimes s_{j_1,m_1,g_1,l_1}^1 = s_{j_{01},m_{01},g_{01},l_{01}}^2$ as follows: the sequence to the left of the $\otimes$ symbol "looks" at the sequence to the right and "sees" the quantum numbers j, m, g, and l. Importantly, what the reference sequence "sees" is not actually the other base-2 sequence, but rather the coarse-grained relationship between the sequences. From this picture, an interesting question arises. Given two base-2 sequences with the quantum numbers j, m, g, l and j', m', g', l', as determined by a common reference sequence, which quantum numbers describe their relationship? As will be shown in the following section, the answer to this question contains the selection rules for interacting spin systems.

4 Selection rules

In this section, a single reference sequence is used to determine the quantum numbers j, m, g, and l for two different base-2 sequences. Independently, these operations take the following form, where the choice of indices will be discussed shortly:

$$s^1_{j_1,m_1,g_1,l_1} \otimes s^1_0 = s^2_{j_{10},m_{10},g_{10},l_{10}} \tag{13}$$

$$s^1_0 \otimes s^1_{j_2,m_2,g_2,l_2} = s^2_{j_{02},m_{02},g_{02},l_{02}}. \tag{14}$$

Using the quantum numbers j_{10}, m_{10}, g_{10}, l_{10}, j_{02}, m_{02}, g_{02}, and l_{02}, we can infer some properties of the following relationship:

$$s^1_{j_1,m_1,g_1,l_1} \otimes s^1_{j_2,m_2,g_2,l_2} = s^2_{j_{12},m_{12},g_{12},l_{12}}. \tag{15}$$

Recall that the ordering of the indices on each quantum number only impacts the sign of m. The choice of index orderings in Eqs. (13)–(15) has been made for pedagogical reasons, but any other ordering is equally valid (there are eight unique choices). By simple arguments (see Appendix A), we can prove the following relationships between the quantum numbers j_{10}, m_{10}, g_{10}, l_{10}, j_{02}, m_{02}, g_{02}, and l_{02} and j_{12}, m_{12}, g_{12}, and l_{12}, where it is assumed that $n \geq 2j_{10} + 2j_{02}$:

$$n = 2(j_{10} + g_{10}) = 2(j_{02} + g_{02}) \tag{16}$$

$$m_{12} = m_{10} + m_{02} = l_{02} - l_{10} \tag{17}$$

$$l_{12} = l_{10} + m_{02} = l_{02} - m_{10} \tag{18}$$

$$|j_{10} - j_{02}| \leq j_{12} \leq j_{10} + j_{02} \tag{19}$$

$$\frac{n}{2} - j_{10} - j_{02} \leq g_{12} \leq \frac{n}{2} - |j_{10} - j_{02}|. \tag{20}$$

Equations (10), (17), and (19) contain the selection rules governing interacting spin systems in QM [33]. Because there are three base-2 sequences involved, the true object of interest in this section is a three point correlation between base-2 sequences, where the set of all such correlations is denoted as $S^3(n)$. As with $S^2(n)$, which can be interpreted as the set of all base-4 sequences, $S^3(n)$ can be interpreted as the set of all base-8 sequences, where the basis elements are members of the group $Z_2 \otimes Z_2 \otimes Z_2$. Rather than introducing new symbols for each of these eight basis elements, as done for base-4 sequences, the base-2 representation will be used: 000, 111, 101, 010, 100, 011, 001, and 110. A visualization of a three point correlation among base-2 sequences is offered in Fig. 3, which takes the form of a directed graph. Based on the choice of index orderings made in Eqs. (13)–(15), the base-4 basis element associated with each of the two point correlations of interest can be identified as follows, where $X \in \{0, 1\}$:

$$s^2_{j_{10},m_{10},g_{10},l_{10}} \to \underline{X}X\underline{X}$$

$$s^2_{j_{02},m_{02},g_{02},l_{02}} \to X\underline{X}\underline{X}$$

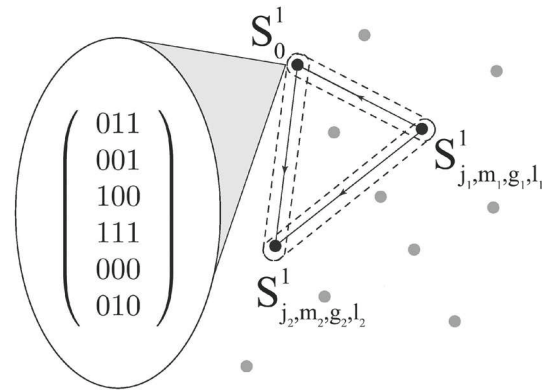$$s^2_{j_{12},m_{12},g_{12},l_{12}} \to \underline{X}X\underline{X}.$$

Fig. 3 A correlation of three base-two sequences, which is a base-8 sequence and an element of $S^3(n)$, can be visualized as a directed graph with three vertices and three directed edges. Among the rules governing the quantum numbers associated with this graph are those for interacting spin systems in QM. For the case shown here, the quantum numbers are $j_{10} = \frac{3}{2}$, $m_{10} = -\frac{1}{2}$, $g_{10} = \frac{3}{2}$, $l_{10} = +\frac{1}{2}$, $j_{02} = 1$, $m_{02} = 0$, $g_{02} = 2$, $l_{02} = 0$, and $j_{12} = \frac{3}{2}$, $m_{12} = -\frac{1}{2}$, $g_{12} = \frac{3}{2}$, $l_{12} = +\frac{1}{2}$

Table 1 Base-8 quantum numbers

$k = \widetilde{010}$	$l_{12} = \frac{\widetilde{000}+\widetilde{010}-\widetilde{111}-\widetilde{101}}{2}$
$j_{10} = \frac{\widetilde{100}+\widetilde{101}+\widetilde{011}+\widetilde{010}}{2}$	$j_{02} = \frac{\widetilde{110}+\widetilde{101}+\widetilde{001}+\widetilde{010}}{2}$
$m_{10} = \frac{\widetilde{100}+\widetilde{101}-\widetilde{011}-\widetilde{010}}{2}$	$m_{02} = \frac{\widetilde{110}+\widetilde{010}-\widetilde{001}-\widetilde{101}}{2}$
$j_{12} = \frac{\widetilde{100}+\widetilde{110}+\widetilde{011}+\widetilde{001}}{2}$	$m_{12} = \frac{\widetilde{100}+\widetilde{110}-\widetilde{011}-\widetilde{001}}{2}$
$n = \widetilde{100} + \widetilde{011} + \widetilde{110} + \widetilde{001} + \widetilde{101} + \widetilde{010} + \widetilde{000} + \widetilde{111}$	

Table 2 Base-8 counts

$\widetilde{010} = k$	$\widetilde{101} = j_{10} + j_{02} - j_{12} - k$
$\widetilde{100} = m_{10} - j_{02} + j_{12} + k$	$\widetilde{011} = j_{10} - m_{10} - k$
$\widetilde{110} = j_{02} + m_{02} - k$	$\widetilde{001} = j_{12} + k - m_{02} - j_{10}$
$\widetilde{111} = \frac{n}{2} - l_{12} - j_{10} - j_{02} + k$	$\widetilde{000} = \frac{n}{2} - j_{12} + l_{12} - k$

Because of these relations, base-8 counts can be associated with base-4 counts like so: $\tilde{C}_{10} = \widetilde{100} + \widetilde{101}$, $\tilde{D}_{12} = \widetilde{001} + \widetilde{011}$, etc.. This enables us to define a complete set of base-8 quantum numbers that include base-4 quantum numbers like j_{10}, m_{10}, j_{02}, and m_{02}. In fact, defining a complete set of base-8 quantum numbers only requires the introduction of one new quantum number, which can also be interpreted as a count:

$$k = \widetilde{010}. \tag{21}$$

The complete set of base-8 quantum numbers to be used to label base-8 sequences are n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12}, l_{12}, and k, which are defined in Table 1, along with m_{12} for completeness. Table 2 provides the map from quantum numbers back to base-8 counts. The definitions offered in these tables will vary depending on how one orders the indices in Eqs. (13)–(15), though the results obtained herein hold for any choice.

In all cases but one, the quantum numbers defined in Table 1 can be found by collecting the three complete sets of base-4 quantum numbers $(j_{10}, m_{10}, g_{10}, l_{10})$, $(j_{02}, m_{02}, g_{02}, l_{02})$ and $(j_{12}, m_{12}, g_{12}, l_{12})$, with the only exception being k. As discussed in Sect. 3, base-4 quantum numbers arise from the operation depicted in Eq. (12), which is interpreted as a single measurement. The fact that the base-8 quantum number k cannot be determined by collecting a group of individual measurements suggests that it is non-local within this model, while the other seven quantum numbers n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12}, and l_{12} are local. As will be seen in the following section, the non-local quantum number k will play an important role in the phenomenon of interference.

5 Probabilities

The physical scenario of interest in this section is one in which a system with spin quantum numbers (j_{12}, m_{12}) is comprised of, or decays into two systems with spin quantum numbers (j_{10}, m_{10}) and (j_{02}, m_{02}). The question of interest is this: Given the priors j_{10}, j_{02}, j_{12}, and m_{12}, what is the probability of observing a particular combination of m_{10} and m_{02}?

Answering this question within the model developed here will require the construction of two sets of base-8 sequences, one associated with the experiment used to collect the quantum number m_{10} and one for m_{02}. These two sets can be interpreted as epistemic ensembles representing the knowledge of observers named Alice and Bob, where Alice is responsible for collecting m_{10} and Bob m_{02}. The probabilities of interest can then be calculated by counting the number of unique ways to pair base-8 sequences from Alice's ensemble with those in Bob's, while accounting for a form of interference. In particular, we will be interested in those pairs which share the same combination of local quantum numbers n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12}, and l_{12}, where interference is driven by the difference between Alice's and Bob's value of the non-local quantum number k.

A single pair of sequences from separate ensembles is interpreted as a path within this model. That is, the probabilities being calculated in this section are related to counting local quantum number conserving paths between Alice's and Bob's ensembles. Path interference is then driven by a measure of disagreement between Alice and Bob regarding the value of the non-local quantum number k. Specifically, paths for which $k^B - k^A$ is odd interfere destructively with those in which it is even, where the superscript indicates which ensemble each k is associated with. For each of these paths, there is an associated map which connects Alice's and Bob's sequences under the addition modulo two operation (see Appendix B). The maps of interest in this calculation,

which conserve local quantum numbers, generate permutations of the underlying base-2 sequences.

Now that the general framework of this calculation has been established, all that remains is to construct Alice's and Bob's ensembles, which will require the introduction of two combinatorial tools [34]. One which will simply count the number of base-8 sequences associated with a particular combination of quantum numbers, and one that will account for the fact that Alice and Bob are actually performing a measurement on part of the total system.

The number of sequences associated with a particular combination of quantum numbers can be found by counting permutations. For base-8 sequences, this can be accomplished by using the following combinatorial tool, where Table 2 can be used to convert from quantum numbers to counts:

$$\Phi(n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12}, l_{12}, k) = \frac{n!}{\widetilde{010}!\,\widetilde{101}!\,\widetilde{100}!\,\widetilde{011}!\,\widetilde{001}!\,\widetilde{011}!\,\widetilde{000}!\,\widetilde{111}!}. \quad (22)$$

In cases where not all quantum numbers are known, Eq. (22) can be summed over for all possible combinations of the unknown quantum numbers. For the calculation of interest in this section, the priors j_{10}, j_{02}, and j_{12}, along with a particular combination of m_{10} and m_{02} constitute five of the eight quantum numbers necessary to qualify as complete. In addition to these, we will also require that all sequences share a common length n, where the only restriction will be that $n \geq 2j_{10} + 2j_{02}$. The two remaining quantum numbers l_{12} and k must then be summed over, where the bounds of these sums can be found in Appendix C.

To account for Alice's and Bob's measurement of m_{10} and m_{02}, respectively, one additional combinatorial tool must be introduced. The purpose of this tool is to modify the information encoded into the base-8 sequences being counted by Eq. (22). This modification pertains to the base-4 basis elements associated with the quantum numbers m_{10} and m_{02}, which are (C_{10}, D_{10}) and (C_{02}, D_{02}), respectively. This combinatorial tool takes the following form, which has the effect of holding these base-4 basis elements fixed when counting base-8 permutations:

$$F^A(n, j_{10}, m_{10}) = \frac{\tilde{C}_{10}!\,\tilde{D}_{10}!\,(n - \tilde{C}_{10} - \tilde{D}_{10})!}{n!} \quad (23)$$

$$F^B(n, j_{02}, m_{02}) = \frac{\tilde{C}_{02}!\,\tilde{D}_{02}!\,(n - \tilde{C}_{02} - \tilde{D}_{02})!}{n!}. \quad (24)$$

For clarity, these expressions can also be written in terms of base-8 counts like so:

$$F^A(n, j_{10}, m_{10}) = \frac{(\widetilde{101} + \widetilde{100})!\,(\widetilde{010} + \widetilde{011})!\,(\widetilde{000} + \widetilde{111} + \widetilde{110} + \widetilde{001})!}{n!} \quad (25)$$
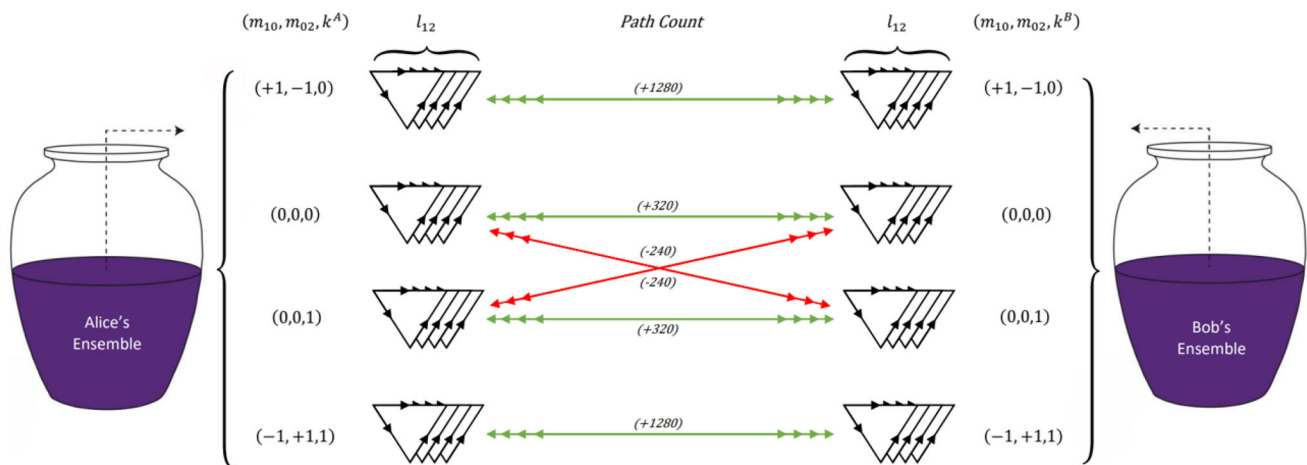
Fig. 4 An example of the path counting procedure between Alice's and Bob's ensemble for the priors $n = 6$, $j_{10} = 1$, $j_{02} = 1$, $j_{12} = 1$, and $m_{12} = 0$. Each triangle represents a subset of $S^3(n)$ (the set of all base-8 sequences) with a unique combination of eight quantum num-

bers. The probability of obtaining a particular combination of m_{10} and m_{02} can be found by dividing the number of paths associated with that combination by the total sum of paths for all combinations

$$
\begin{aligned}
&F^B(n, j_{02}, m_{02}) \\
&= \frac{(\widetilde{010} + \widetilde{110})!(\widetilde{101} + \widetilde{001})!(\widetilde{000} + \widetilde{111} + \widetilde{100} + \widetilde{011})!}{n!}.
\end{aligned}
$$
(26)

For a particular combination of m_{10} and m_{02}, the number of local quantum number conserving paths between Alice's and Bob's ensembles, while accounting for interference, is given by the following expression, where we have suppressed all arguments not being summed over:

$$
\begin{aligned}
&\Upsilon(n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12}) \\
&= \sum_{k^A, k^B} \sum_{l_{12}} (-1)^{(k^B - k^A)} \Phi(l_{12}, k^B) F^B \Phi(l_{12}, k^A) F^A.
\end{aligned}
$$
(27)

The closed form expression for calculating the probability of observing a particular combination of m_{10} and m_{02} is as follows, where the normalization is simply Eq. (27) summed over the allowed combinations of m_{10} and m_{02}, given the prior m_{12}:

$$
\begin{aligned}
&P(m_{10}, m_{02}|n, j_{10}, j_{02}, j_{12}, m_{12}) \\
&= \frac{\Upsilon(n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12})}{\sum_{m_1, m_2} \Upsilon(n, j_{10}, j_{02}, m_{10}, m_{02}, j_{12})}.
\end{aligned}
$$
(28)

A depiction of the calculation associated with Eq. (28) is offered in Fig. 4, in which a sample calculation is performed. The priors associated with this sample calculation are $n = 6$, $j_{10} = 1$, $j_{02} = 1$, $j_{12} = 1$, and $m_{12} = 0$. Given these priors, along with Eqs. (10) and (17), the three allowed combinations of m_{10} and m_{02} are $(+1, -1)$, $(0, 0)$, and $(-1, +1)$. By summing over the paths depicted in Fig. 4, the probability of obtaining a particular combination of m_{10} and m_{02} is as follows:

$$
P(+1, -1|6, 1, 1, 1, 0) = \frac{1280}{2720} = 0.470588
$$

$$
P(0, 0|6, 1, 1, 1, 0) = \frac{160}{2720} = 0.058824
$$

$$
P(-1, +1|6, 1, 1, 1, 0) = \frac{1280}{2720} = 0.470588.
$$

The difference between these predictions and those of QM, which are 0.5, 0.0, and 0.5 for $(+1, -1)$, $(0, 0)$, and $(-1, +1)$, respectively, are plotted as a function of n in Fig. 5. The deviation between the predictions of this model and that of QM can be made arbitrarily small by increasing n. In the limit that n goes to infinity, the number of sequences in Alice's and Bob's ensembles becomes uncountable. While this model cannot be falsified by studying deviations from QM, proving that n is finite is certainly possible.

Within Dirac's formalism for QM, the primary method of calculating these probabilities, which are the squared Clebsch–Gordan coefficients, is a recursive algorithm employing ladder operators. There is also a more technical derivation associated with tensor decomposition, which requires a background in representation theory. Regardless of the method of derivation, there is a closed form, or non-recursive method of calculating the square roots of these probabilities. This expression, which is Eq. (60) in Appendix D, is equivalent to Eq. (28) in the limit of large n (Fig. 5). Beyond issues of aesthetics, Eq. (60) also lacks any clear explanatory power within QM. For example, its not even obvious that it is a probability, whereas Eq. (28) clearly takes the form of a frequency. Finally, the method of calculating probabilities by counting paths between two epistemic ensembles appears to be a far more general framework than this particular calculation. One is free to encode a wide variety of physical
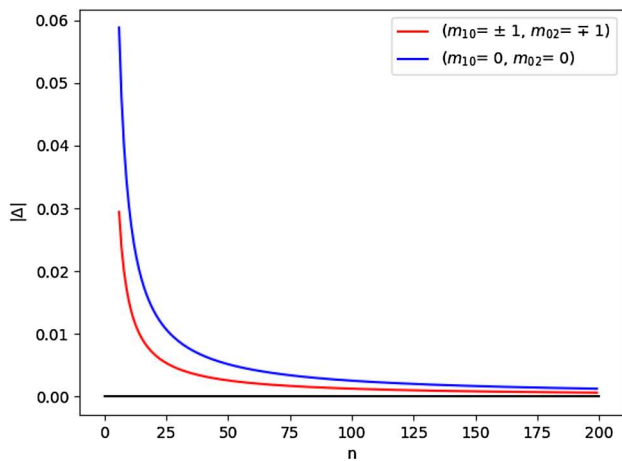
Fig. 5 The magnitude of the difference between the prediction yielded by Eq. (28) and that of QM, plotted as a function of the sequence length n, where $|\Delta| = |P(m_{10}, m_{02}|j_{10}, j_{02}, j_{12}, m_{12}) - \langle j_1, j_2, m_1, m_2|j_1, j_2, J, M\rangle^2|$

Fig. 6 A correlation of four base-two sequences, which is a base-16 sequence and an element of $S^4(n)$, can typically be visualized as a tetrahedron. These can be interpreted as correlations between two measurements involving separate reference sequences, one associated with Alice (A) and one with Bob (B)

scenarios into this scheme, which is of significant interest for future work.

6 Discussion

Why should spin be the focus of an alternative formalism for modeling quantum mechanical systems? We can certainly make a case that spin is among the most fundamental features of physical systems. Spin is even used as a building block for space-time itself [35,36]. However, the truth is that a model for spin was not the original objective of this research effort. Instead, it began as a deductive approach to discretizing the quantum state in QM, in which the starting point was the set of all base-2 sequences of length n. By considering two and three point correlations between the elements of the set $S^1(n)$, or the set of all base-2 sequences of length n, a relational set of quantum numbers emerged. The selection rules and probabilities for interacting spin systems then developed naturally by asking simple questions of the resulting formalism. Though a model for spin was not the original objective of this research effort, the manner in which it emerged is striking.

The results presented in this paper represent a small fraction of the modeling potential of this formalism. For example, one can consider higher order correlations between base-2 sequences. In the case of four point correlations between base-2 sequences, the associated geometric elements will typically be tetrahedra (Fig. 6). Though, unlike three point correlations, there is no guarantee that four randomly selected base-2 sequences will form a valid simplex. This leads to non-trivial behavior of geometric elements beyond two spatial
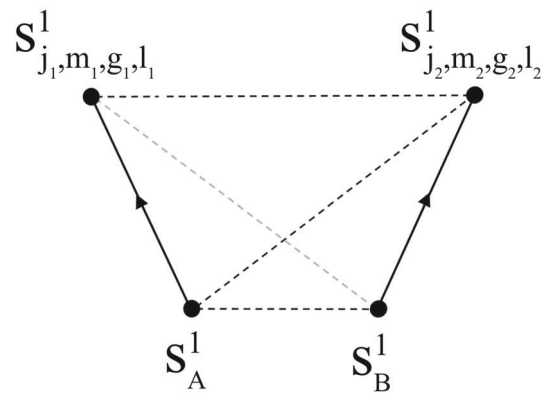
dimensions, which may shed some light on the importance of three spatial dimensions in physics. Each of these geometric elements will have quantum numbers beyond those associated with the lengths of its edges. In the case of four point correlations, there will be ten such quantum numbers.

Four point correlations between base-2 sequences may also be thought of as two point correlations between base-4 sequences, which we interpret as measurements (Fig. 6). In other words, we may interpret four point correlations as relationships between two spin measurements, which are necessarily separated in space-time. It is this scenario which corresponds to Stern–Gerlach experiments involving sequences of detectors. Of particular interest are those cases in which two Stern–Gerlach detectors are rotated with respect to one another. A model for this physical scenario will enable us to address the issues of complementarity, as well as the violation of Bell's inequalities [37]. This, along with the accompanying geometric picture, will also inform the development of a model for space-time.

A motivating observation of the work presented here is the tension between the countability of empirical data and the uncountable sets employed by the theories tasked with modeling that data. What makes the approach taken here unique is that one need not choose between these two views of nature. As the length of sequences are taken to infinity, the number of unique sequences becomes uncountable, leading to continuous probability distributions. This implies that expectation values of any observable can then vary continuously, even if that observable is itself discrete. This feature offers the opportunity to develop discrete physics models in the finite n regime, while also studying the continuum limit of those models. This "continuization" approach can be contrasted with traditional methods of quantization, which

involve the discretization of continuous mathematical structures.

Though the results presented here are promising, there remain many important issues raised in the quantum foundations literature that have not been adequately addressed. These issues include the measurement problem, contextuality, and the role of complex numbers, among many others. Additionally, the precise relationship between various features of the formalism introduced here and QM must still be established. Addressing these issues will, in virtually every case, require a specific model for space-time. Though, the issues concerning quantum foundations will not be the only ones that must be addressed to further justify this research effort. Ultimately, this formalism is only viable if it has the capacity to support both space-time and matter degrees of freedom. For this reason, we must adopt a long term, collaborative approach to model development. The work presented here is intended to establish a general framework upon which such a model can be built.

The formalism and subsequent model we have introduced are rooted in information theory and have displayed clear predictive power. While these results recast important physics in a new and intriguing light, they are far from the end of the story. There remain important unanswered questions, as well as new questions which we have not yet thought to ask. Given the mathematical simplicity and vivid conceptual picture, we are optimistic that researchers from a broad range of backgrounds will find this effort both enticing and promising.

Appendix A: Derivation of the selection rules for interacting spin systems

A.1 Proof of $n = 2(j_{10} + g_{10}) = 2(j_{02} + g_{02})$

Proving the relation given in Eq. (16) requires us to add Eqs. (6) and (8), yielding the following:

$$j + g = \frac{\tilde{C} + \tilde{D}}{2} + \frac{\tilde{A} + \tilde{B}}{2}. \tag{29}$$

The length of a particular sequence is given by the total number of basis elements contained within that sequence. In the case of a base-4 sequence, that is given by $\tilde{A} + \tilde{B} + \tilde{C} + \tilde{D} = n$. Substituting this result into Eq. (29) yields:

$$j + g = \frac{n}{2}. \tag{30}$$

An obvious consequence of two base-4 sequences sharing a common base-2 reference sequence, as is the requirement in Sect. 4, is that both base-4 sequences must be the same length. This fact, together with Eq. (30) yields the result in Eq. (16):

$$n = 2(j_{10} + g_{10}) = 2(j_{02} + g_{02}). \tag{31}$$

A.2 Proof of the selection rules for m and l

The simplest path towards proving Eqs. (17) and (18) requires the introduction of the base-2 counts $\tilde{0}_0$, $\tilde{1}_0$, $\tilde{0}_1$, $\tilde{1}_1$, $\tilde{0}_2$, and $\tilde{1}_2$, where the subscripts indicate which base-2 sequence each count is associated with. Using the definition of the base-4 basis elements A, B, C, and D offered in Sect. 3, the base-4 counts can be expressed in terms of these base-2 counts like so:

$$\tilde{0}_0 = \tilde{A}_{10} + \tilde{C}_{10} = \tilde{A}_{02} + \tilde{D}_{02} \tag{32}$$

$$\tilde{1}_0 = \tilde{B}_{10} + \tilde{D}_{10} = \tilde{B}_{02} + \tilde{C}_{02} \tag{33}$$

$$\tilde{0}_1 = \tilde{A}_{10} + \tilde{D}_{10} = \tilde{A}_{12} + \tilde{D}_{12} \tag{34}$$

$$\tilde{1}_1 = \tilde{B}_{10} + \tilde{C}_{10} = \tilde{B}_{12} + \tilde{C}_{12} \tag{35}$$

$$\tilde{0}_2 = \tilde{A}_{02} + \tilde{C}_{02} = \tilde{A}_{12} + \tilde{C}_{12} \tag{36}$$

$$\tilde{1}_2 = \tilde{B}_{02} + \tilde{D}_{02} = \tilde{B}_{12} + \tilde{D}_{12}. \tag{37}$$

Using Eqs. (7), (32) and (34), the base-4 quantum numbers of interest can be expressed as follows:

$$m_{10} = \frac{\tilde{C}_{10} - \tilde{D}_{10}}{2} =$$
$$\frac{\tilde{0}_0 - \tilde{A}_{10} - \tilde{0}_1 + \tilde{A}_{10}}{2} = \frac{\tilde{0}_0 - \tilde{0}_1}{2}. \tag{38}$$

Alternatively, m_{10} can be defined as:

$$m_{10} = \frac{\tilde{C}_{10} - \tilde{D}_{10}}{2} =$$

$$\frac{\tilde{1}_1 - \tilde{B}_{10} - \tilde{1}_0 + \tilde{B}_{10}}{2} = \frac{\tilde{1}_1 - \tilde{1}_0}{2}. \tag{39}$$

By an identical procedure, the quantum number l_{10} can also be defined in terms of base-2 counts. Generalizing the indices, the following relations between base-4 quantum numbers and base-2 counts can be defined:

$$m_{\mu\nu} = \frac{\tilde{0}_\nu - \tilde{0}_\mu}{2} = \frac{\tilde{1}_\mu - \tilde{1}_\nu}{2} \tag{40}$$

$$l_{\mu\nu} = \frac{\tilde{0}_\nu - \tilde{1}_\mu}{2} = \frac{\tilde{0}_\mu - \tilde{1}_\nu}{2}. \tag{41}$$

Using Eqs. (40) and (41), with the appropriate choice of indices, Eq. (17) becomes:

$$m_{12} = m_{10} + m_{02}$$
$$\rightarrow \frac{\tilde{0}_2 - \tilde{0}_1}{2} = \frac{\tilde{0}_0 - \tilde{0}_1}{2} + \frac{\tilde{0}_2 - \tilde{0}_0}{2} \tag{42}$$
$$m_{12} = l_{02} - l_{10}$$
$$\rightarrow \frac{\tilde{0}_2 - \tilde{0}_1}{2} = \frac{\tilde{0}_2 - \tilde{1}_0}{2} - \frac{\tilde{0}_1 - \tilde{1}_0}{2}. \tag{43}$$

Equations (42) and (43) both evaluate to true statements, implying the relations given in Eq. (17) are proven. Using Eqs. (40) and (41), with the appropriate choice of indices, Eq. (18) becomes:

$$l_{12} = l_{10} + m_{02}$$
$$\rightarrow \frac{\tilde{0}_1 - \tilde{1}_2}{2} = \frac{\tilde{0}_1 - \tilde{1}_0}{2} + \frac{\tilde{1}_0 - \tilde{1}_2}{2} \tag{44}$$
$$l_{12} = l_{02} - m_{10}$$
$$\rightarrow \frac{\tilde{0}_1 - \tilde{1}_2}{2} = \frac{\tilde{0}_0 - \tilde{1}_2}{2} - \frac{\tilde{0}_0 - \tilde{0}_1}{2}. \tag{45}$$

Again, Eqs. (44) and (45) both evaluate to true statements, implying the relations given in Eq. (18) are proven.

A.3 Proof of the selection rules for j and g

As defined in Table 1, the quantum number j_{12} can be expressed in terms of base-8 counts like so:

$$j_{12} = \frac{\widetilde{100} + \widetilde{110} + \widetilde{011} + \widetilde{001}}{2}. \tag{46}$$

As an explicit example, an element of $S^3(n = 4)$ is offered, where brackets around the base-2 basis elements in $s^1_{j_1,m_1,g_1,l_1}$ and $s^1_{j_2,m_2,g_2,l_2}$ that contribute to the quantum numbers j_{10} and j_{02} have been introduced:

$$\begin{pmatrix} 1 & 1 & [0] \\ 1 & 1 & 1 \\ [1] & 0 & 0 \\ [0] & 1 & 1 \end{pmatrix}. \tag{47}$$

In this element of $S^3(n = 4)$, the bracketed base-2 elements in $s^1_{j_1,m_1,g_1,l_1}$ and $s^1_{j_2,m_2,g_2,l_2}$ do not overlap with one another. This implies that the quantum number j_{12} between

$s^1_{j_1,m_1,g_1,l_1}$ and $s^1_{j_2,m_2,g_2,l_2}$ is simply $j_{12} = j_{10} + j_{02} = \frac{(2+1)}{2} = \frac{3}{2}$. On the other hand, we could have the following situation:

$$\begin{pmatrix} [0] & 1 & [0] \\ 1 & 1 & 1 \\ [1] & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{48}$$

The difference here is that one of the bracketed base-2 basis elements from $s^1_{j_1,m_1,g_1,l_1}$ now overlaps one from $s^1_{j_2,m_2,g_2,l_2}$. This implies that the quantum number j_{12} between $s^1_{j_1,m_1,g_1,l_1}$ and $s^1_{j_2,m_2,g_2,l_2}$ is now $j_{12} = j_{10} + j_{02} - 1 = \frac{(2+1)}{2} - 1 = \frac{1}{2}$. In other words, given the quantum numbers j_{10} and j_{02}, we can have either $j_{12} = \frac{3}{2}$ or $j_{12} = \frac{1}{2}$. In general, the allowed range of the quantum number j_{12} is as follows, which is Eq. (19):

$$|j_{10} - j_{02}| \leq j_{12} \leq j_{10} + j_{02}. \tag{49}$$

In the case that $n < 2(j_{10} + j_{02})$, an overlap is guaranteed. Because $s^2_{j_{10},m_{10},g_{10},l_{10}}$ and $s^2_{j_{02},m_{02},g_{02},l_{02}}$ share a common reference sequence, the base-4 basis elements that can overlap in the resulting base-8 sequence are (A_{10}, A_{02}), (B_{10}, B_{02}), (C_{10}, A_{02}), (D_{10}, B_{02}), (A_{10}, D_{02}), (B_{10}, C_{02}), (C_{10}, D_{02}), and (D_{10}, C_{02}). The (C_{10}, D_{02}) and (D_{10}, C_{02}) cases correspond to the base-8 basis elements 101 and 010 respectively, which are precisely the overlap scenarios of interest when considering j_{12}. Therefore, the maximum number of overlaps that may occur are limited by the sum $min\left[\tilde{D}_{10}, \tilde{C}_{02}\right] + min\left[\tilde{C}_{10}, \tilde{D}_{02}\right]$. Each overlap leads to a reduction in j_{12} by one, leading to the following expression:

$$j_{12,min} = j_{10} + j_{02}$$
$$-min[\tilde{D}_{10}, \tilde{C}_{02}] - min[\tilde{C}_{10}, \tilde{D}_{02}]. \tag{50}$$

(C_{10}, A_{02}), (D_{10}, B_{02}), (A_{10}, D_{02}), and (B_{10}, C_{02}) correspond to the base-8 basis elements 100, 011, 001, and 110, respectively. This implies that these overlap scenarios all contribute to j_{12}. However, if $n < 2(j_{10} + j_{02})$, then it is guaranteed that either $\tilde{D}_{10} > \tilde{B}_{02}$ or $\tilde{C}_{10} > \tilde{A}_{02}$, or equivalently $\tilde{B}_{10} < \tilde{C}_{02}$ or $\tilde{A}_{10} < \tilde{D}_{02}$. This implies that (C_{10}, D_{02}) and or (D_{10}, C_{02}) overlap scenarios must occur. This allows us to define the following expression:

$$j_{12,max} = j_{10} + j_{02}$$
$$-max\left[0, \tilde{D}_{10} - \tilde{B}_{02}\right] - max\left[0, \tilde{C}_{10} - \tilde{A}_{02}\right]. \tag{51}$$

Using the relation between j, g, and n offered in Eq. (30), the results derived for j_{12} can be used to derive the corresponding results for g_{12}.

Appendix B: An example of maps

Within this formalism, a map connects two sequences of equal basis and length via element-wise addition modulo two, which is denoted by the $\oplus$ symbol. That is, given the proper map, any initial sequence can be mapped to any final sequence like so, where the basis of these sequences is 2^d:

$$s^d_{initial} \oplus s^d_{map} = s^d_{final}. \tag{52}$$

As a more concrete example of the operation shown in Eq. (52), a particular choice of the initial and final sequence is made, where $s^2_{initial} = s^2_{\frac{1}{2},+\frac{1}{2},\frac{5}{2},+\frac{1}{2}}$ and $s^2_{final} = s^2_{\frac{1}{2},-\frac{1}{2},\frac{5}{2},-\frac{1}{2}}$:

$$\begin{pmatrix} A \\ A \\ C \\ B \\ B \\ A \end{pmatrix} \oplus \begin{pmatrix} B \\ A \\ C \\ A \\ A \\ D \end{pmatrix} = \begin{pmatrix} B \\ A \\ A \\ B \\ B \\ D \end{pmatrix}. \tag{53}$$

Expressing these base-4 sequences using the base-2 representation, we have:

$$\begin{pmatrix} 00 \\ 00 \\ 10 \\ 11 \\ 11 \\ 00 \end{pmatrix} \oplus \begin{pmatrix} 11 \\ 00 \\ 10 \\ 00 \\ 00 \\ 01 \end{pmatrix} = \begin{pmatrix} 11 \\ 00 \\ 00 \\ 11 \\ 11 \\ 01 \end{pmatrix}. \tag{54}$$

The example shown here has the effect of conserving the quantum numbers j and g, but not the quantum numbers m and l. Maps which conserve all quantum numbers are permutations.

Appendix C: Derivation of summation limits

C.1 Derivation of k_{min} and k_{max}

The two overlap scenarios discussed in Appendix A.3 that lead to cases in which $j_{12} < j_{10} + j_{02}$ are (C_{10}, D_{02}) and (D_{10}, C_{02}), which correspond to the base-8 basis elements 101 and 010, respectively. In cases where the quantum numbers j_{10}, j_{02}, and j_{12} are all known, there still may be a range

of possible values for the counts $\widetilde{010}$ and $\widetilde{101}$, where the count $\widetilde{010}$ is associated with the quantum number k. It is convenient to introduce the quantum number $X = k + \widetilde{101}$, where $j_{12} = j_{10} + j_{02} - X$. This relation implies that for fixed j_{10}, j_{02}, and j_{12}, the quantum number X is also fixed. Ignoring X for the time being, we have $\widetilde{101}_{max} = min\left[\tilde{C}_{10}, \tilde{D}_{02}\right]$. For a given X, k_{min} must be equivalent to $X - \widetilde{101}_{max}$. This allows us to define k_{min}:

$$k_{min} = max\left[0, X - min\left[\tilde{C}_{10}, \tilde{D}_{02}\right]\right]. \tag{55}$$

Again ignoring X, we have $k_{max} = min\left[\tilde{C}_{02}, \tilde{D}_{10}\right]$, which implies the following:

$$k_{max} = min\left[X, min\left[\tilde{C}_{02}, \tilde{D}_{10}\right]\right]. \tag{56}$$

Thus, given the quantum numbers $j_{10}, m_{10}, j_{02}, m_{02}$, and j_{12}, we can define bounds on the allowed values of k.

C.2 Derivation of $l_{12,min}$ and $l_{12,max}$

From Table 1, the definition of l_{12} in terms of base-8 counts is as follows:

$$l_{12} = \frac{\widetilde{000} + \widetilde{010} - \widetilde{111} - \widetilde{101}}{2}. \tag{57}$$

Given the priors n, j_{10}, j_{02}, j_{12}, and a particular combination of k^A and k^B, along with their definition in terms of base-8 counts given in Table 1, the bounds on l_{12} are as follows:

$$l_{12,min} = -\frac{n}{2} + j_{12} + max(k^A, k^B) \tag{58}$$

$$l_{12,max} = \frac{n}{2} - j_{12}$$
$$- max(j_{10} + j_{02} - j_{12} - k^A, j_{10} + j_{02} - j_{12} - k^B). \tag{59}$$

Appendix D: The standard closed form Clebsch–Gordan coefficients

The closed form expression for the Clebsch–Gordan coefficients within QM takes the following form, where z may take on any value for which no factorials have negative arguments [38]:

$$\langle j_1 j_2 J M | j_1 j_2 m_1 m_2 \rangle$$
$$= \sqrt{\frac{(2J+1)(j_1 + j_2 - J)!(J + j_1 - j_2)!(J + j_2 - j_1)!}{(j_1 + j_2 + J + 1)!}}$$
$$\cdot \sum_z (-1)^z \frac{\sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(J + M)!(J - M)!}}{z!(j_1 + j_2 - J - z)!(j_1 - m_1 - z)!(j_2 + m_2 - z)!(J - j_2 + m_1 + z)!(J - j_1 - m_2 + z)!}. \tag{60}$$

References

1. C.E. Shannon, *The Mathematical Theory of Communication* (University of Illinois Press, Urbana, 1949–1962)
2. J. Stone, *Information Theory: A Tutorial Introduction* (Sebtel Press, Upper Saddle River, 2015)
3. J. Pierce, *An Introduction to Information Theory: Symbols, Signals & Noise*, 2nd rev. edn. (Dover Publications, New York, 1980)
4. J.W. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton University Press, Princeton, 1979)
5. A. Petersen, The philosophy of Niels Bohr. Bull. At. Sci. **19**, 8–14 (1963)
6. A. Einstein, *The Collected Papers of Albert Einstein* (Princeton University Press, Princeton, 1987)
7. J. von Neumann, N.A. Wheeler, R.T. Beyer, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 2018)
8. P.E. Gibbs, The small scale structure of space-time: a bibliographical review (1995). arXiv:hep-th/9506171
9. S. Surya, The causal set approach to quantum gravity. Living Rev. Relativ. **22** (2019). arXiv:1903.11544
10. C. Rovelli, Loop quantum gravity. Living Rev. Relativ. **1**, 1–75 (1998). arXiv:gr-qc/9710008
11. S. Mukhi, String theory: a perspective over the last 25 years. Class. Quantum Gravity **28**, 153001 (2011). arXiv:1110.2569
12. R. Loll, Quantum gravity from causal dynamical triangulations: a review. Class. Quantum Gravity **37**, 013002 (2019). arXiv:1905.08669
13. L. Hardy, Are quantum states real? Int. J. Mod. Phys. B **27**, 1345012 (2012). arXiv:1205.1439
14. S. Hossenfelder, T.N. Palmer, Rethinking superdeterminism. Front. Phys. **8**, 473–476 (2020). arXiv:1912.06462
15. L. Ballentine, The statistical interpretation of quantum mechanics. Rev. Mod. Phys. **42**, 358–381 (1970)
16. C.A. Fuchs, QBism, the perimeter of quantum Bayesianism (2010). arXiv:1003.5209
17. L. Smolin, A real ensemble interpretation of quantum mechanics. Found. Phys. **42**, 1239–1261 (2012). arXiv:1104.2822
18. N. Harrigan, R.W. Spekkens, Einstein, incompleteness, and the epistemic view of quantum states. Found. Phys. **40**, 125 (2010). arXiv:0706.2661
19. M.S. Leifer, Is the quantum state real? An extended review of psi-ontology theorems. Quanta **3**, 67 (2014). arXiv:1409.1570
20. M.F. Pusey, J. Barrett, T. Rudolph, On the reality of the quantum state. Nat. Phys. **8**, 475–478 (2012). arXiv:1111.3328
21. D. Griffiths, *Introduction to Quantum Mechanics*, 2nd edn. (Pearson Prentice Hall, Upper Saddle River, 2005)
22. J. Sakurai, *Modern Quantum Mechanics*, rev edn. (Addison-Wesley Pub. Co., Reading, 1994–1995)
23. S. Adler, *Quantum Theory as an Emergent Phenomenon (the Statistical Mechanics of Matrix Models as the Precursor of Quantum Field Theory)* (Cambridge University Press, Cambridge, 2004)
24. G. 't Hooft, The cellular automaton interpretation of quantum mechanics (2014). arXiv:1405.1548
25. R.W. Spekkens, In defense of the epistemic view of quantum states: a toy theory. Phys. Rev. A **75**, 032110 (2007). arXiv:quant-ph/0401052
26. G. Chiribella, G.M. D'Ariano, P. Perinotti, Informational derivation of quantum theory. Phys. Rev. A **84**, 012311 (2010). arXiv:1011.6451
27. C. Rovelli, Relational quantum mechanics. Int. J. Theor. Phys. **35**, 1637–1678 (1996). arXiv:quant-ph/9609002
28. L. Hardy, Quantum theory from five reasonable axioms (2001). arXiv:quant-ph/0101012
29. T.N. Palmer, Discretisation of the Bloch sphere, fractal invariant sets and Bell's theorem. Proc. R. Soc. A Math. Phys. Eng. Sci. **476** (2020)
30. L.N. Chang, D. Minic, T. Takeuchi, Spekkens' toy model, finite field quantum mechanics, and the role of linearity. J. Phys. Conf. Ser. **1275**, 012036 (2019). arXiv:1903.06337
31. A. Zee, *Group Theory in a Nutshell for Physicists* (Princeton University Press, Princeton, 2016)
32. D. Page, Average entropy of subsystem. Phys. Rev. Lett. **71**, 1291–1294 (1993). arXiv:gr-qc/9305007
33. A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1985)
34. T. Faticoni, *Combinatorics: An Introduction*, 1st edn. (Wiley, Somerset, 2014)
35. R. Penrose, Angular momentum: an approach to combinatorial space-time, in *Quantum Theory and Beyond* (1971), pp. 151–180
36. C. Rovelli, L. Smolin, Spin networks and quantum gravity. Phys. Rev. D **52**, 5743 (1995)
37. D. Minic, S. Powers, D. Stojkovic, T. Takeuchi, Bell's inequalities. Coming soon (2022)
38. G. Racah, Theory of complex spectra II. Phys. Rev. **62**, 438–462 (1942)