ORIGINAL ARTICLE

An Optimal Volume Growth Estimate for Noncollapsed Steady Gradient Ricci Solitons

Richard H. Bamler[1] · Pak-Yeung Chan[2] · Zilu Ma[3] · Yongjia Zhang[4]

Abstract
In this paper, we prove a volume growth estimate for steady gradient Ricci solitons with bounded Nash entropy. We show that such a steady gradient Ricci soliton has volume growth rate no smaller than $r^{\frac{n+1}{2}}$. This result not only improves the estimate in (Chan et al., arXiv:2107.01419, 2021, Theorem 1.3), but also is optimal since the Bryant soliton and Appleton's solitons (Appleton, arXiv:1708.00161, 2017) have exactly this growth rate.

Keywords Ricci flow · Ricci soliton · Singularity model

Mathematics Subject Classification 53E20

✉ Zilu Ma
zilu.ma@rutgers.edu

Richard H. Bamler
rbamler@berkeley.edu

Pak-Yeung Chan
pachan@ucsd.edu

Yongjia Zhang
sunzhang91@sjtu.edu.cn

[1] Department of Mathematics, University of California Berkeley, Berkeley, CA 94720, USA

[2] Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA

[3] Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

[4] School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China

1 Introduction

The Ricci flow has been a powerful tool in settling various longstanding problems in geometry and topology, among which the most well-known ones are the geometrization and the Poincaré conjectures. The success of the Hamilton–Perelman program [24, 33–35] in dimension 3 suggests that the analysis of singularity formation plays a central role in the study of the Ricci flow. In the Hamilton–Perelman program, a *singularity model* is understood to be an ancient solution arising as the smooth limit of a scaled sequence of a Ricci flow forming a finite-time singularity (see below for more details). Among all singularity models, the most important ones are the shrinking and steady *gradient Ricci solitons*. Perelman's canonical neighborhood theorem shows that a 3-dimensional Ricci flow becomes locally close to a singularity model wherever the curvature is large. However, due to the lack of the Hamilton–Ivey pinching estimate, this canonical neighborhood theorem is generally not true in higher dimensions.

Recently, the first-named author [2–4] established a new theory about weak limits of Ricci flows on closed manifolds. This theory sheds more light on the formation of singularities in dimension 4 and higher. Indeed, the three last-named authors have already employed these methods in the study of ancient solutions and singularities of the Ricci flow; see, for instance, [14–16, 29]. Very recently, the first-named author [5] proved that the fundamental group of a noncollapsed ancient Ricci flow is finite. In this paper, we shall study the volume growth of steady gradient Ricci solitons using these techniques.

Let us recall the definition of gradient Ricci soliton. A triple (M^n, g, f) is a called a gradient Ricci soliton if

$$\text{Ric} + \nabla^2 f = \frac{\kappa}{2} g, \tag{1.1}$$

for some constant $\kappa \in \mathbb{R}$. The soliton is called *shrinking* if $\kappa > 0$, *steady* if $\kappa = 0$, and *expanding* if $\kappa < 0$. Any soliton canonically induces a Ricci flow, called the *canonical form*. Precisely, if we define Φ_t and g_t by

$$\frac{\partial}{\partial t} \Phi_t = \frac{1}{1 - \kappa t} \nabla f \circ \Phi_t,$$
$$\Phi_0 = \text{id},$$
$$g_t = (1 - \kappa t)\Phi_t^* g, \tag{1.2}$$

then g_t moves by the Ricci flow. In the shrinking ($\kappa > 0$) and steady ($\kappa = 0$) case, the canonical form not only is a self-similar ancient solution moving by diffeomorphism, but also often arises as a singularity model. For instance, the blow-up limit at every Type I singularity is (the canonical form of) a shrinking gradient Ricci soliton (cf. [22]), and a degenerate neck-pinch (cf. [23]) is modeled on a Bryant soliton. We remark that by the recent work of Choi–Haslhofer [17], if we consider the more general *singular Ricci flow* (cf. [6, 26]) instead of Ricci flow, then there could possibly be non-solitonic blow-up limits.

The study of steady gradient Ricci solitons is important not only for the understanding of the formation of Type-II singularities in particular, but also for the understanding

of ancient Ricci flows in general. For instance, a steady soliton may arise as a sequential limit from a shrinking soliton with exactly quadratic curvature growth (cf. [18]); the only positively curved ancient, noncompact and noncollapsed Ricci flow in dimension 3 is the Bryant soliton (cf. [7]).

Unlike shrinking solitons, though, the geometric characterizations of steady solitons are less complete. This, to some extent, is reflected by the newer examples constructed by Appleton [1] and Lai [27]. Some previous constructions of steady solitons using ODE method also include [8–10, 20, 21, 25, 36, 38]. Furthermore, shrinking solitons are automatically strongly noncollapsed (cf. [13, 28]), but this is obviously not true for steady solitons. In fact, the cigar soliton of Hamilton—the first steady soliton ever found—and the 3-dimensional flying wings of Lai [27], conjectured by Hamilton, are collapsed. Up to this point, the volume growth estimates of steady solitons are also less sharp than that of shrinking solitons. Munteanu and Wang [32] showed that the volume of the geodesic ball of a noncompact gradient shrinker grows at least linearly in the radius, i.e., $|B(p, r)| \geq Cr$, where $C = C(n)e^{c(n)\mu}$ depends on the dimension and the shrinker entropy μ, and $c(n) > 1$. Using their Sobolev inequality, Li and Wang [28, Proposition 6] provided a better constant $C = c(n)e^{\mu}$. This estimate is optimal since it is satisfied by cylinders. However, the same technique does not yield an equally nice volume growth estimate for steady solitons. Indeed, the three last named authors [15] proved that a Sobolev inequality on a steady soliton implies that the volume growth rate is at least $r^{\frac{n}{2}}$, but this is not optimal since the Bryant soliton has volume growth rate $r^{\frac{n+1}{2}}$.

In this paper, we prove an optimal volume growth estimate for steady gradient Ricci solitons with bounded Nash entropy. First of all, we recall some known results on the volume growth rate for steady solitons. Besides the $r^{\frac{n}{2}}$ volume growth rate lower bound mentioned above (cf. [15]), Munteanu–Šešum [31] showed that a steady soliton has at least linear volume growth, Cui [19] proved a volume growth lower bound for steady Kähler Ricci solitons with positive Ricci curvature. The *optimal* volume growth lower bound proved in this paper says that a steady gradient Ricci soliton with *bounded Nash entropy* has volume growth rate no smaller than $r^{\frac{n+1}{2}}$. Since the Bryant soliton (cf. [11]) as well as Appleton's solitons ([1], they are asymptotic to quotients of the Bryant soliton) have exactly this volume growth rate, our result is optimal indeed. As a consequence, a steady gradient Ricci soliton with volume growth strictly slower than $r^{\frac{n+1}{2}}$ cannot arise as a singularity model (see below).

Throughout the paper, we shall assume that (M^n, g, f) is a complete steady gradient Ricci soliton normalized in the way that

$$\text{Ric} = \nabla^2 f, \quad R + |\nabla f|^2 = 1. \tag{1.3}$$

Here, for the notational simplicity, we have reversed the sign of f in (1.1). Then the 1-parameter family of diffeomorphisms Φ_t defined in (1.2) is now the group of diffeomorphisms generated by $-\nabla f$ with $\Phi_0 = \text{id}$. We shall still use $g_t = \Phi_t^* g$ to denote the canonical form of the steady soliton.

Let us fix a point $o \in M$, and we shall impose one more condition on the steady soliton in question, namely, a uniformly bounded Nash entropy:

$$\mathcal{N}_{o,0}(\tau) \geq -Y \quad \text{for all } \tau > 0, \tag{1.4}$$

where $Y \in (0, \infty)$ is a constant and $\mathcal{N}$ should be regarded as the Nash entropy of the canonical form. We refer the readers to [2] for the definitions. We shall denote by $|\Omega|_g$ the volume of a measurable subset $\Omega \subset M$ relative to the metric g and by $B_r(x)$ or $B(x, r)$ the geodesic ball centered at x with radius r. With these preparation, our main theorem is stated as follows.

Theorem 1.1 *Suppose that (M^n, g, f) is a complete steady gradient Ricci soliton normalized as in* (1.3) *and the canonical form $(M^n, g_t)_{t \in \mathbb{R}}$ satisfies* (1.4). *Additionally, assume that **either one** of the following conditions is true*:

(1) $(M^n, g_t)_{t \in \mathbb{R}}$ *arises as a singularity model*; *or*
(2) (M^n, g) *has bounded curvature.*

Then

$$c(n, \mu_\infty) r^{\frac{n+1}{2}} \leq |B_r(o)| \leq C(n, \mu_\infty) r^n \quad \text{for all } r > \bar{r}(n, \mu_\infty),$$

where $\mu_\infty := \inf_{\tau > 0} \mathcal{N}_{o,0}(\tau) = \lim_{\tau \to \infty} \mathcal{N}_{o,0}(\tau) > -\infty$ and $c(n, \mu_\infty)$ and $C(n, \mu_\infty)$ are positive constants of the form

$$c(n, \mu_\infty) = \frac{c(n)}{\sqrt{1 - \mu_\infty}} e^{\mu_\infty}, \quad C(n, \mu_\infty) = C(n) e^{\mu_\infty}.$$

Furthermore, the upper bound is also true for all $r > 0$ (instead of $r \geq \bar{r}(n, \mu_\infty)$).

A *singularity model* is an ancient solution $(M^n, g_t)_{t \in (-\infty, 0]}$ arising as a blow-up limit of a compact Ricci flow $(\overline{M}^n, \overline{g}_t)_{t \in [0, T)}$ around its singular time. A singularity model in the sense of Hamilton [24] is a smooth Cheeger–Gromov–Hamilton limit, whereas a singularity model in the sense of [4] is an $\mathbb{F}$-limit (cf. [3]). In fact, by [4, Theorem 2.5], a *smooth* singularity model in the sense of [4] is also a singularity model in the sense of Hamilton (but the reverse is not true). In the assumption of Theorem 1.1 (1), the singularity model can be either in the sense of Hamilton or in the sense of [4].

Remarks

1. The bounded Nash entropy assumption (1.4) implies strong noncollapsing, and it is obvious that (1.4) holds on every singularity model. Yet it is an interesting question to ask whether (1.4) is equivalent to the (either strong or weak) noncollapsing condition on every steady soliton.
2. μ_∞ in the statement of Theorem 1.1 is the shrinker entropy of any tangent flow at infinity of the ancient solution $(M^n, g_t)_{t \in (-\infty]}$ given by [4, Theorem 2.40]. Moreover, the value of μ_∞ is independent of the choice of the point o. This can be seen from [29, Proposition 4.6].

3. The volume growth upper bound is a direct consequence of [2, Theorem 8.1] and we will leave the detailed proof to the reader; in this paper we shall only prove the volume growth lower bound.
4. The volume growth upper bound is also sharp since it is satisfied by the steady Gaussian soliton. This conclusion is in the spirit of a similar result in the shrinking case (cf. [12, 30]). Previous works on the volume growth upper bound for steady Ricci solitons include [31, 37].
5. It is proved in [18] that a 4-dimensional steady gradient Ricci soliton which arises as a singularity model must have bounded curvature. As a consequence, if $n = 4$, then case (2) in the statement of Theorem 1.1 is redundant.
6. The reverse of the theorem is false, i.e., if the volume growth rate of a steady soliton is at least $r^{(n+1)/2}$, it may still be collapsed. For example, the cross product of Hamilton's cigar soliton and $\mathbb{R}^k$ ($k \geq 1$) has volume growth rate r^{k+1} but the total dimension is $k + 2$ and it is collapsed.

2 Proofs

Roughly speaking, we prove the main theorem by packing balls centered at ℓ-centers, namely, the points at which Perelman's [33] reduced distance function almost attains its minimum (see below for the definition). Since, as it is shown by the three last-named authors [14, Proposition 5.6], ℓ-centers are always close to H_n-centers (cf. [2, Definition 3.10]), a ball centered at an ℓ-center must have a volume lower bound estimate as given by [2, Theorem 6.2]. This is the argument which proves the optimal volume growth lower bound.

Since the canonical form of the steady soliton (M^n, g, f) moves only by diffeomorphism, we may work with Perelman's $\mathcal{L}$-geometry [33, §7] on the background of the static manifold (M^n, g).

2.1 Perelman's $\mathcal{L}$-Geometry on Steady Solitons

As mentioned before, we will use g_t to represent the canonical form of the steady soliton (M^n, g, f) satisfying the conditions of Theorem 1.1. Recall that Perelman defined the $\mathcal{L}$-length in [33, §7]. For any $\tau > 0$, and any piecewise smooth curve $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$,

$$\mathcal{L}(\Gamma) := \int_0^\tau \sqrt{s}(R_{g-s} + |\dot{\Gamma}|^2_{g-s})(\Gamma(s)) \, ds.$$

To reinterpret the $\mathcal{L}$-geometry on the static background (M, g), let

$$\gamma(s) = \Phi_{-s}(\Gamma(s)) \quad \text{for } s \in [0, \tau].$$

Then

$$\dot{\gamma} = \nabla f|_\Gamma + \Phi_{-s*}(\dot{\Gamma}),$$

and

$$\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{s}\big(R_g + |\dot{\gamma} - \nabla f|_g^2\big)(\gamma(s))\, ds,$$

and this expression only uses the static metric g. If we perform a change of variables: $u = \sqrt{s}$, and write $\tilde{\gamma}(u) = \gamma(u^2)$, then

$$\mathcal{L}(\Gamma) = \int_0^{\sqrt{\tau}} \left(\frac{1}{2}\left|\dot{\tilde{\gamma}} - 2u\nabla f\right|^2 + 2u^2 R(\tilde{\gamma}(u))\right) du.$$

For any $x \in M$ and $\tau > 0$, we define

$$L(\Phi_\tau(x), \tau) := \inf_\Gamma \mathcal{L}(\Gamma),$$

where the infimum is taken over all $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$ and $\Gamma(\tau) = \Phi_\tau(x)$. On the static metric background, we may define an equivalent function:

$$\Lambda(x, \tau) := L(\Phi_\tau(x), \tau) = \inf \int_0^\tau \sqrt{s}\big(R_g + |\dot{\gamma} - \nabla f|_g^2\big)(\gamma(s))\, ds, \qquad (2.1)$$

where the infimum is taken over all $\gamma : [0, \tau] \to M$ with $\gamma(0) = o$ and $\gamma(\tau) = x$, and a curve at which the above infimum is attained shall be called a Λ-**geodesic**. Accordingly, define

$$\lambda(x, \tau) := \ell(\Phi_\tau(x), \tau) := \frac{1}{2\sqrt{\tau}}\Lambda(x, \tau).$$

Arguing as Perelman in [33, Section 7.1], we have that, for any $\tau > 0$, there is a point $p_\tau \in M$ such that $\lambda(p_\tau, \tau) = \ell(\Phi_\tau(p_\tau), \tau) \le n/2$. Any such point p_τ is called an ℓ-**center** at time $-\tau$. Note that in our current case we are considering the ℓ-center on a static metric background, hence it differs from the ℓ-center defined in [14] by a diffeomorphism.

2.2 Locations of ℓ-Centers

Lemma 2.1 $\lambda(o, \tau) \ge \tau/12$, *for any* $\tau > 0$.

Proof Let $\gamma : [0, \tau] \to M$ be a loop at o and let $\tilde{\gamma} : [0, \sqrt{\tau}] \to M$ be the reparametrization: $\tilde{\gamma}(u) = \gamma(u^2)$. Then

$$\int_0^\tau \sqrt{s}(R + |\dot{\gamma} - \nabla f|^2) = \int_0^{\sqrt{\tau}} \left(\frac{1}{2}\left|\dot{\tilde{\gamma}} - 2u\nabla f\right|^2 + 2u^2 R(\tilde{\gamma}(u))\right) du$$

$$= \int_0^{\sqrt{\tau}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}|^2 - 2u(f \circ \tilde{\gamma} - f(o))' + 2u^2\right) du$$

$$= \frac{2}{3}\tau^{3/2} + \int_0^{\sqrt{\tau}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}|^2 + 2(f \circ \tilde{\gamma}(u) - f(o)) \right) du,$$

$$(2.2)$$

where in the second equality we have applied (1.3). Let $F(u) = f \circ \tilde{\gamma}(u) - f(o)$ and define

$$L := \sup_{u \in [0, \sqrt{\tau}]} \operatorname{dist}(o, \tilde{\gamma}(u)) =: \operatorname{dist}(o, \tilde{\gamma}(u_1)),$$

for some $u_1 \in [0, \sqrt{\tau}]$. Then we have

$$\frac{1}{2}\int_0^{\sqrt{\tau}} |\dot{\tilde{\gamma}}|^2 \geq \frac{1}{2}\int_0^{u_1} |\dot{\tilde{\gamma}}|^2 + \frac{1}{2}\int_{u_1}^{\sqrt{\tau}} |\dot{\tilde{\gamma}}|^2$$

$$\geq \frac{L^2}{2}\left(\frac{1}{u_1} + \frac{1}{\sqrt{\tau} - u_1} \right) \geq \frac{2L^2}{\sqrt{\tau}},$$

where we have applied the Cauchy–Schwarz inequality (e.g., $L^2 \leq (\int_0^{u_1} |\dot{\tilde{\gamma}}|)^2 \leq \int_0^{u_1} |\dot{\tilde{\gamma}}|^2 \cdot \int_0^{u_1} 1^2$). Since $|\nabla f| \leq 1$ by (1.3), we have

$$|F(u)| \leq \operatorname{dist}(\tilde{\gamma}(u), o) \leq L, \quad \forall u \in [0, \sqrt{\tau}],$$

and thus

$$\int_0^{\sqrt{\tau}} 2(f \circ \tilde{\gamma}(u) - f(o)) du \geq -2L\sqrt{\tau}.$$

In summary, we have

$$\int_0^{\tau} \sqrt{s}(R + |\dot{\gamma} - \nabla f|^2) \geq \frac{2}{3}\tau^{3/2} + \frac{2L^2}{\sqrt{\tau}} - 2L\sqrt{\tau}$$

$$= \frac{2}{3}\tau^{3/2} + \frac{2}{\sqrt{\tau}}(L^2 - L\tau)$$

$$= \frac{1}{6}\tau^{3/2} + \frac{2}{\sqrt{\tau}}\left(L - \frac{\tau}{2} \right)^2 \geq \frac{1}{6}\tau^{3/2},$$

and the conclusions follow by taking the infimum on the left hand side.   □

The following lemma is straightforward and is similar to the standard triangle inequality; cf. [14, §4, Claim 3].

Lemma 2.2 *For any $x, y \in M, \tau > 0$ and any $\delta \in (0, 1)$,*

$$\lambda(x, (1 + \delta)^2 \tau) \leq \lambda(y, \tau) + \frac{\operatorname{dist}^2(x, y)}{\delta\tau} + 5\delta\tau.$$

Proof Let $\gamma_1 : [0, \tau] \to M$ be a minimizing Λ-geodesic from o to y, namely, a curve at which the infimum in (2.1) is attained. Let $\tilde{\gamma}_2 : [\sqrt{\tau}, (1+\delta)\sqrt{\tau}] \to M$ be a minimizing g-geodesic from y to x with constant speed. Define $\gamma_2 : [\tau, (1 + \delta)^2\tau] \to M$ by $\gamma_2(s) = \tilde{\gamma}_2(\sqrt{s})$.

$$
\begin{aligned}
\Lambda(x, (1 + \delta)^2\tau) &\leq \int_0^\tau \sqrt{s}(R + |\dot{\gamma}_1 - \nabla f|^2)(\gamma_1(s))\,ds \\
&\quad + \int_\tau^{(1+\delta)^2\tau} \sqrt{s}(R + |\dot{\gamma}_2 - \nabla f|^2)(\gamma_2(s))\,ds \\
&\leq \Lambda(y, \tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}_2|^2 + 2u|\dot{\tilde{\gamma}}_2||\nabla f| + 2u^2(R + |\nabla f|^2)\right)du \\
&\leq \Lambda(y, \tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}} \left(|\dot{\tilde{\gamma}}_2|^2 + 4u^2\right)du \\
&\leq \Lambda(y, \tau) + \frac{\mathrm{dist}^2(x, y)}{\delta\sqrt{\tau}} + 4\frac{(1 + \delta)^3 - 1}{3}\tau^{3/2} \\
&\leq \Lambda(y, \tau) + \frac{\mathrm{dist}^2(x, y)}{\delta\sqrt{\tau}} + 10\delta\tau^{3/2}.
\end{aligned}
$$

The conclusion follows by dividing $2(1 + \delta)\sqrt{\tau}$ on both sides. $\square$

Lemma 2.3 *There is a universal constant $\alpha \in (0, 1)$, such that for any $\tau \geq \bar{\tau}(n)$ and any ℓ-center p_τ, we have*

$$\mathrm{dist}(p_\tau, o) \geq \alpha\tau.$$

Proof By Lemmas 2.1 and 2.2, for any $\delta \in (0, 1)$, if $\tau \geq \bar{\tau}(n, \delta)$, then we have

$$
\begin{aligned}
\frac{(1 + \delta)^2\tau}{12} &\leq \lambda(o, (1 + \delta)^2\tau) \leq \lambda(p_\tau, \tau) + \frac{\mathrm{dist}^2(p_\tau, o)}{\delta\tau} + 5\delta\tau \\
&\leq \frac{\mathrm{dist}^2(p_\tau, o)}{\delta\tau} + 10\delta\tau,
\end{aligned}
$$

where we have used the fact that $\lambda(p_\tau, \tau) \leq \frac{n}{2}$. We may take, e.g., $\delta = 10^{-3}$ to obtain the inequality. $\square$

Lemma 2.4 *For any $\tau \geq \bar{\tau}(n)$, there is $x_\tau \in M$ such that $\mathrm{dist}(x_\tau, o) = \tau$ and $\lambda(x_\tau, \tau_0) \leq C$ for some $\tau_0 \in [c\tau, \tau/\alpha]$, where $c > 0$ and $C < \infty$ are dimensional constants and α is given by Lemma 2.3.*

Proof Let $\gamma : [0, \tau/\alpha] \to M$ be a minimizing Λ-geodesic from o to $p := p_{\tau/\alpha}$. By Lemma 2.3, $\mathrm{dist}(p, o) \geq \tau$. So we can define

$$\tau_0 := \sup\{s \in [0, \tau/\alpha] : \mathrm{dist}(\gamma(s), o) \leq \tau\}, \quad x_\tau := \gamma(\tau_0).$$

We first show that $\tau_0 \geq c\tau$ for some universal constant $c > 0$. Define $\tilde{\gamma} : [0, \sqrt{\tau/\alpha}] \to M$ by $\tilde{\gamma}(u) = \gamma(u^2)$. Note that, arguing in the same way as (2.2), we have

$$\frac{1}{2}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \leq \Lambda(p, \tau/\alpha) + \int_0^{\sqrt{\tau_0}} 2u\langle \dot{\tilde{\gamma}}, \nabla f \rangle$$

$$\leq n\sqrt{\tau/\alpha} + \frac{1}{4}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 + 4\int_0^{\sqrt{\tau_0}} u^2,$$

$$\frac{1}{4}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \leq n\sqrt{\tau/\alpha} + \frac{4}{3}\tau_0^{3/2}.$$

It follows that

$$\frac{1}{4}\tau^2 = \frac{1}{4}\text{dist}(o, x_\tau)^2 \leq \frac{1}{4}\left(\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|\right)^2 \leq \frac{1}{4}\sqrt{\tau_0}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2$$

$$\leq n\sqrt{\tau_0\tau/\alpha} + \frac{4}{3}\tau_0^2 \leq \frac{1}{8}\tau^2 + \frac{4}{3}\tau_0^2,$$

if $\tau \geq \bar{\tau}(n)$. Hence $\tau_0 \geq c\tau$ for some dimensional constant $c > 0$. Then

$$\lambda(x_\tau, \tau_0) \leq \frac{\sqrt{\tau/\alpha}}{\sqrt{\tau_0}}\lambda(p, \tau/\alpha) \leq \frac{n}{2\sqrt{c\alpha}}.$$

$\square$

Lemma 2.5 *Suppose that (M^n, g, f) satisfies the assumptions in Theorem 1.1. Then for any $\tau \geq \bar{\tau}(n)$, there is $x_\tau \in M$ such that $\text{dist}(x_\tau, o) = \tau$ and*

$$|B(x_\tau, \sqrt{A\tau})| \geq ce^{\mu_\infty}\tau^{n/2},$$

where $A = C_n(1 - \mu_\infty), c = c(n) > 0$.

Proof Let x_τ, τ_0 be given by Lemma 2.4. Recall that $c\tau \leq \tau_0 \leq \tau/\alpha$ and $\lambda(x_\tau, \tau_0) \leq C$, for some dimensional constants c, C and α is given by Lemma 2.3.

It suffices to show that

$$\left|B_{g-\tau_0}\left(y_\tau, \sqrt{\alpha A\tau_0}\right)\right|_{g-\tau_0} \geq c_n e^{\mu_\infty}\tau_0^{n/2}, \tag{2.3}$$

where $y_\tau = \Phi_{\tau_0}(x_\tau)$. Because once we can show (2.3), we have

$$\left|B_g\left(x_\tau, \sqrt{A\tau}\right)\right|_g \geq \left|B_g\left(x_\tau, \sqrt{\alpha A\tau_0}\right)\right|_g = \left|B_{g-\tau_0}\left(y_\tau, \sqrt{\alpha A\tau_0}\right)\right|_{g-\tau_0}$$

$$\geq c_n e^{\mu_\infty}\tau_0^{n/2} \geq c_n e^{\mu_\infty}\tau^{n/2}, \tag{2.4}$$

where we used the fact that $\tau/\alpha \geq \tau_0 \geq c\tau$ for some dimensional constant $c > 0$. We leave the details of the proof of (2.4) to the reader.

Now we prove (2.3). Let $(z, -\tau_0)$ be an H_n-center of $(o, 0)$. By [33, 9.5] and [2, Theorem 7.2] (or [14, Theorem 3.2]), we have

$$(4\pi\tau_0)^{-n/2}e^{-C} \le (4\pi\tau_0)^{-n/2}e^{-\ell(y_\tau, \tau_0)} \le K(o, 0 \mid y_\tau, -\tau_0)$$

$$\le C_n e^{-\mu_\infty} \tau_0^{-n/2} \exp\left(-\frac{\mathrm{dist}^2_{-\tau_0}(y_\tau, z)}{9\tau_0}\right),$$

where K is the fundamental solution to the conjugate heat equation, and we also used Lemma 2.4 and the fact that $\ell(y_\tau, \tau_0) = \lambda(x_\tau, \tau_0) \le C$. Hence

$$\mathrm{dist}^2_{-\tau_0}(y_\tau, z) \le 9(-\mu_\infty + C_n)\tau_0.$$

We choose A so that

$$\alpha A = 18(-\mu_\infty + C_n) + 10H_n.$$

Note that, by [14, Proposition 3.3], [2, Theorem 6.2] also holds for Ricci flows with bounded curvature on compact intervals. All the results in [2] applies to singularity models (in the sense of Hamilton or in the sense of [4]), because of the smooth convergence. By [2, Theorem 6.2],

$$\left|\left(B_{g_{-\tau_0}}\left(y_\tau, \sqrt{\alpha A\tau_0}\right)\right)\right|_{-\tau_0} \ge \left|\left(B_{g_{-\tau_0}}\left(z, \sqrt{\alpha A\tau_0/2}\right)\right)\right|_{-\tau_0} \ge c(n)e^{\mu_\infty}\tau_0^{n/2}.$$

So we finished the proof of (2.3).   □

2.3 Proof of the Main Theorem

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\bar{\tau}(n) < \infty$ be given by Lemma 2.5. For each $r > 10A + \bar{\tau}(n)$, we construct a decreasing sequence $r = \tau_1 > \tau_2 > \cdots > \tau_N > 0$, such that $\tau_N < r/10$ and for $1 \le j \le N - 1$,

$$\tau_j - \tau_{j+1} = \sqrt{A\tau_j} + \sqrt{A\tau_{j+1}}.$$

As long as $\tau_j \ge r/10$, the above equation is solvable for positive τ_{j+1} since the discriminant $A + 4(\tau_j - \sqrt{A\tau_j}) = 4(\sqrt{\tau_j} - \sqrt{A}/2)^2 \ge 0$. Since $\tau_j \ge r/10$ and $r > 10A$, there is a unique positive solution for τ_{j+1}. Moreover, $\tau_j - \tau_{j+1} \ge \sqrt{A\tau_j} \ge \sqrt{Ar/10}$, hence we can find a finite positive integer N such that $0 < \tau_N < r/10$. For each j, by Lemma 2.5, there is $x_j \in M$ such that $\mathrm{dist}(x_j, o) = \tau_j$, and

$$\left|B\left(x_j, \sqrt{A\tau_j}\right)\right| \ge c(n)e^{\mu_\infty}\tau_j^{n/2}.$$

By the construction of $\{\tau_j\}$, the balls $\left\{ B\left(x_j, \sqrt{A\tau_j}\right)\right\}_{j=1}^N$ are pairwise disjoint. It follows that

$$
\begin{aligned}
|B_{2r}(o)| &\geq \sum_{j=1}^{N} \left| B\left(x_j, \sqrt{A\tau_j}\right)\right| \geq \sum_{j=1}^{N} c(n) e^{\mu_\infty} \tau_j^{n/2} \\
&\geq \frac{c(n)}{\sqrt{A}} e^{\mu_\infty} \sum_{j=1}^{N-1} \tau_j^{\frac{n-1}{2}} \left(\tau_j - \tau_{j+1}\right) \\
&\geq \frac{c(n)}{\sqrt{A}} e^{\mu_\infty} \sum_{j=1}^{N-1} \int_{\tau_{j+1}}^{\tau_j} \tau^{\frac{n-1}{2}}\, d\tau \\
&= \frac{c(n)}{\sqrt{A}} e^{\mu_\infty} \int_{\tau_N}^{r} \tau^{\frac{n-1}{2}}\, d\tau \\
&\geq \frac{c(n)}{\sqrt{A}} e^{\mu_\infty} r^{\frac{n+1}{2}}.
\end{aligned}
$$

$\square$

References

1. Appleton, A.: A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four. arXiv:1708.00161 (2017)
2. Bamler, R.H.: Entropy and heat kernel bounds on a Ricci flow background. arXiv:2008.07093 (2020)
3. Bamler, R.H.: Compactness theory of the space of super Ricci flows. arXiv:2008.09298 (2020)
4. Bamler, R.H.: Structure theory of non-collapsed limits of Ricci flows. arXiv:2009.03243 (2020)
5. Bamler, R.H.: On the fundamental group of non-collapsed ancient Ricci flows. arXiv:2110.02254 (2021)
6. Bamler, R.H., Kleiner, B.: Uniqueness and stability of Ricci flow through singularities. arXiv:1709.04122 (2017)
7. Brendle, S.: Ancient solutions to the Ricci flow in dimension 3. Acta Math. **225**, 1–102 (2020)
8. Bryant, R.: Ricci flow solitons in dimension three with SO(3)-symmetries. http://www.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf (2005)
9. Buzano, M., Dancer, A.S., Wang, M.: A family of steady Ricci solitons and Ricci-flat metrics. Commun. Anal. Geom. **23**(3), 611–638 (2015)
10. Cao, H.-D.: Existence of gradient Kähler–Ricci solitons. In: Elliptic and Parabolic Methods in Geometry (Minneapolis, MN, 1994), pp. 1–16. A K Peters, Wellesley (1996)
11. Cao, H.-D.: Recent progress on Ricci solitons. arXiv:0908.2006 (2009)
12. Cao, H.-D., Zhou, D.T.: On complete gradient shrinking Ricci solitons. J. Differ. Geom. **85**, 175–186 (2010)
13. Carrillo, J.A., Ni, L.: Sharp logarithmic Sobolev inequalities on gradient solitons and applications. Commun. Anal. Geom. **17**, 721–753 (2009)
14. Chan, P.-Y., Ma, Z.L., Zhang, Y.J.: Ancient Ricci flows with asymptotic solitons. arXiv:2106.06904 (2021)
15. Chan, P.-Y., Ma, Z.L., Zhang, Y.J.: A uniform Sobolev inequality for ancient Ricci flows with bounded Nash entropy. arXiv:2107.01419 (2021)

16. Chan, P.-Y., Ma, Z.L., Zhang, Y.J.: On Ricci flows with closed and smooth tangent flows. arXiv:2109.14763 (2021)
17. Choi, B., Haslhofer, R.: A note on blowup limits in 3d Ricci flow. arXiv:2109.13875 (2021)
18. Chow, B., Freedman, M., Shin, H., Zhang, Y.J.: Curvature growth of some 4-dimensional gradient Ricci soliton singularity models. Adv. Math. **372**, Art. No. 107303 (2020)
19. Cui, X.: On curvature, volume growth and uniqueness of steady Ricci solitons. Ph.D. thesis, Lehigh University, Bethlehem (2016)
20. Dancer, A.S., Wang, M.K.Y.: Some new examples of non-Kähler Ricci solitons. Math. Res. Lett. **16**(2), 349–363 (2009)
21. Dancer, A.S., Wang, M.K.Y.: On Ricci solitons of cohomogeneity one. Ann. Glob. Anal. Geom. **39**(3), 259–292 (2011)
22. Enders, J., Müller, R., Topping, P.M.: On type-I singularities in Ricci flow. Commun. Anal. Geom. **19**(5), 905–922 (2011)
23. Gu, H.-L., Zhu, X.-P.: The existence of Type II singularities for the Ricci flow on $\mathbb{S}^{n+1}$. Commun. Anal. Geom. **16**(3), 467–494 (2008)
24. Hamilton, R.: The formation of singularities in the Ricci flow. Surv. Differ. Geom. **2**, 7–136 (1995)
25. Ivey, T.: New examples of complete Ricci solitons. Proc. Am. Math. Soc. **122**(1), 241–245 (1994)
26. Kleiner, B., Lott, J.: Singular Ricci flows I. Acta Math. **219**(1), 65–134 (2017)
27. Lai, Y.: A family of 3d steady gradient solitons that are flying wings. arXiv:2010.07272 (2020)
28. Li, Y., Wang, B.: Heat kernel on Ricci shrinkers. Calc. Var. Partial Differ. Equ. **59**(6), Art. No. 194 (2020)
29. Ma, Z.L., Zhang, Y.J.: Perelman's entropy on ancient Ricci flows. J. Funct. Anal. **281**(9), Art. No. 109195 (2021)
30. Munteanu, O.: The volume growth of complete gradient shrinking Ricci solitons. arXiv:0904.0798 (2009)
31. Munteanu, O., Šešum, N.: On gradient Ricci solitons. J. Geom. Anal. **23**(2), 539–561 (2013)
32. Munteanu, O., Wang, J.P.: Analysis of weighted Laplacian and applications to Ricci solitons. Commun. Anal. Geom. **20**(1), 55–94 (2012)
33. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math.DG/0211159 (2002)
34. Perelman, G.: Ricci flow with surgery on three-manifolds. arXiv:math.DG/0303109 (2003)
35. Perelman, G.: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:math/0307245 (2003)
36. Stolarski, M.: Steady Ricci solitons on complex line bundles. arXiv:1511.04087v2 (2017)
37. Wei, G.F., Wu, P.: On volume growth of gradient steady Ricci solitons. Pac. J. Math. **265**(1), 233–241 (2013)
38. Wink, M.: Cohomogeneity one Ricci solitons from Hopf fibrations. arXiv:1706.09712v2 (2019)