MATHEMATICS OF COMPUTATION
Volume 92, Number 344, November 2023, Pages 2439–2473
https://doi.org/10.1090/mcom/3857
Article electronically published on June 9, 2023

ROBUST BPX PRECONDITIONER FOR FRACTIONAL LAPLACIANS ON BOUNDED LIPSCHITZ DOMAINS

JUAN PABLO BORTHAGARAY, RICARDO H. NOCHETTO, SHUONAN WU, AND JINCHAO XU

Abstract. We propose and analyze a robust Bramble-Pasciak-Xu (BPX) preconditioner for the integral fractional Laplacian of order $s \in (0,1)$ on bounded Lipschitz domains. Compared with the standard BPX preconditioner, an additional scaling factor $1-\widetilde{\gamma}^s$, for some fixed $\widetilde{\gamma} \in (0,1)$, is incorporated to the coarse levels. For either quasi-uniform grids or graded bisection grids, we show that the condition numbers of the resulting systems remain uniformly bounded with respect to both the number of levels and the fractional power.

1. Introduction

Given $s \in (0,1)$, the fractional Laplacian of order s in $\mathbb{R}^d$ is the pseudodifferential operator with symbol $|\xi|^{2s}$. That is, denoting the Fourier transform by $\mathcal{F}$, for every function $v : \mathbb{R}^d \to \mathbb{R}$ in the Schwartz class $\mathcal{S}$ it holds that

$$\mathcal{F}\left((-\Delta)^s v\right)(\xi) = |\xi|^{2s}\mathcal{F}(v)(\xi).$$

Upon inverting the Fourier transform, one obtains the following equivalent expression:

$$(1.1)\qquad (-\Delta)^s v(x) = C(d,s)\ \text{p.v.}\int_{\mathbb{R}^d} \frac{v(x)-v(y)}{|x-y|^{d+2s}}dy, \qquad C(d,s) = \frac{2^{2s}s\Gamma(s+\frac{d}{2})}{\pi^{d/2}\Gamma(1-s)}.$$

The constant $C(d,s) \simeq s(1-s)$ compensates the singular behavior of the integrals for $s \to 0$ (as $|y| \to \infty$) and for $s \to 1$ (as $y \to x$), and yields [26, Proposition 4.4]

$$(1.2)\qquad \lim_{s\to 0}(-\Delta)^s v(x) = v(x), \quad \lim_{s\to 1}(-\Delta)^s v(x) = -\Delta v(x), \quad \forall v \in C_0^\infty(\mathbb{R}^d).$$

From a probabilistic point of view, the fractional Laplacian is related to a simple random walk with arbitrarily long jumps [51], and is the infinitesimal generator of a $2s$-stable Lévy-process [8]. Thus, the fractional Laplacian has been widely utilized to model jump processes arising in social and physical environments, such as finance [24], predator search patterns [48], or ground-water solute transport [7].

There exist several nonequivalent definitions of a fractional Laplace operator $(-\Delta)^s$ on a bounded domain $\Omega \subset \mathbb{R}^d$ (see [10, 11]). Our emphasis in this paper

Received by the editor June 2, 2021, and, in revised form, July 15, 2022, December 28, 2022, and March 7, 2023.

2020 *Mathematics Subject Classification.* Primary 65F08, 65N30, 65M50, 35R11.

The first author was supported in part by NSF grant DMS-1411808 and Fondo Vaz Ferreira grant 2019-068. The second author was supported in part by NSF grant DMS-1411808. The third author was supported in part by the National Natural Science Foundation of China grant No. 12222101 and No. 11901016. The fourth author was supported in part by NSF grant DMS-1819157.

©2023 American Mathematical Society

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

is on the homogeneous Dirichlet problem for the *integral* (or restricted) fractional Laplacian: given $f\colon \Omega \to \mathbb{R}$, one seeks $u\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.3}$$

where the pointwise definition of $(-\Delta)^s u(x)$ is given by (1.1) for $x \in \Omega$. Consequently, the integral fractional Laplacian on Ω maintains the probabilistic interpretation and corresponds to a *killed* Lévy process [8, 22]. It is noteworthy that, as the underlying stochastic process admits jumps of arbitrary length, for the integral fractional Laplacian the standard Dirichlet conditions need to be replaced by suitable *volume constraints* on the complement of the domain Ω, e.g. $u = 0$ in Ω^c. In contrast, the spectral Laplacian for $s \in (0, 1)$ and the censored (or regional) Laplacian for $s \in (\frac{1}{2}, 1)$ admit Dirichlet boundary conditions on $\partial\Omega$. Despite their strikingly different boundary behavior, we show in Section 9 that the three operators are spectrally equivalent in bounded Lipschitz domains, an interesting property already known for the integral and spectral operators [22].

Weak solutions to (1.3) are the minima of the functional $v \mapsto \frac{1}{2}|v|^2_{H^s(\mathbb{R}^d)} - \int_\Omega fv$ on the zero-extension space $\widetilde{H}^s(\Omega)$ (see Section 2.2). In accordance with (1.2) restricted to any $v \in C_0^\infty(\Omega)$, it holds that if $v \in \widetilde{H}^\sigma(\Omega)$ for some $\sigma > 0$, then [39]

$$\lim_{s \to 0^+} |v|_{H^s(\mathbb{R}^d)} = \|v\|_{L^2(\Omega)}, \tag{1.4}$$

while if $v \in L^2(\mathbb{R}^d)$ is such that $\operatorname{supp} v \subset \overline{\Omega}$ and $\lim_{s \to 1^-} |v|_{H^s(\mathbb{R}^d)}$ exists and is finite, then $v \in H^1_0(\Omega)$ and [17]

$$\lim_{s \to 1^-} |v|_{H^s(\mathbb{R}^d)} = |v|_{H^1(\Omega)}. \tag{1.5}$$

We emphasize that the presence of a scaling factor $C(d, s) \simeq s(1 - s)$ in the fractional-order seminorm $|v|_{H^s(\mathbb{R}^d)}$ is fundamental for (1.4) and (1.5) to hold. Consider a discretization of (1.3) using standard linear Lagrangian finite element space on a mesh $\mathcal{T}$ (denoted by $\mathbb{V}(\mathcal{T})$, see details in Section 2.2) whose elements have maximum and minimum size $h_{\max}$ and $h_{\min}$ respectively, and denote by $\mathbf{A}$ the corresponding stiffness matrix. Then, as shown in [4], the condition number of $\mathbf{A}$ obeys the relation

$$\operatorname{cond}(\mathbf{A}) \lesssim \left(\dim \mathbb{V}(\mathcal{T})\right)^{2s/d} \left(\frac{h_{\max}}{h_{\min}}\right)^{d-2s} \tag{1.6}$$

for $0 < s < 1$ with $2s < d$, and one can remove the factor involving $\frac{h_{\max}}{h_{\min}}$ by preconditioning $\mathbf{A}$ by a diagonal scaling. On non-quasi-uniform grids, the hidden constant in the critical case $2s = d$ is worse by a logarithmic factor.

We point out that solutions to (1.3) generically exhibit the boundary behavior $u \simeq d(\cdot, \partial\Omega)^s$, and thus graded meshes towards $\partial\Omega$ are required to recover optimal convergence rates. The relation (1.6) shows that even in the limit $s \to 0$, in which the fractional Laplacian approaches the identity (cf. (1.2)), the use of graded grids may give rise to ill-conditioned matrices; this could be cured though by diagonal scaling (see Section 8.2).

In recent years, efficient finite element discretizations of (1.3) have been examined in several papers. Adaptive algorithms have been considered in [2, 28, 32], and a posteriori error analysis has been addressed in [30, 44]. Standard finite element discretizations of the fractional Laplacian give rise to full stiffness matrices; matrix

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

compression techniques have been proposed and studied in [3, 37, 61]. For the efficient resolution of the discrete problems, operator preconditioners have been considered in [33].

In this work, we propose a multilevel BPX preconditioner (cf. [6, 18, 31, 49, 53]) $\mathbf{B}$ for the solution of (1.3) that yields $\mathrm{cond}(\mathbf{BA}) \lesssim 1$. In general, our result follows from the general theory for multigrid preconditioners (cf. [34, 54, 55, 58]). An important consequence of (1.4) and (1.5) is that, on any given grid, the stiffness matrices associated with integral fractional Laplacians of order s approach either the standard mass matrix (as $s \to 0$) or the stiffness matrix corresponding to the Laplacian (as $s \to 1$), the latter because the canonical basis functions of $\mathbb{V}(\mathcal{T})$ are Lipschitz and $W_0^{1,\infty}(\Omega) \subset \widetilde{H}^s(\Omega)$. This is consistent with (1.6): for example, on quasi-uniform grids of size h, such a formula yields $\mathrm{cond}(\mathbf{A}) \simeq h^{-2s}$.

Based on the above observations, one of our main goals is to obtain a preconditioner that is *uniform with respect to s as well as with respect to the number of levels $\bar{J}$.* For such a purpose, we need to weigh the contributions of the coarser levels differently to the finest level. On a family of quasi-uniform grids $\{\overline{\mathcal{T}}_k\}_{k=0}^{\bar{J}}$ with size $\bar{h}_k$, we shall consider a preconditioner in the operator form (cf. (4.2))

$$\overline{B} = \overline{I}_{\bar{J}} \bar{h}_{\bar{J}}^{2s} \overline{Q}_{\bar{J}} + (1 - \widetilde{\gamma}^s) \sum_{k=0}^{\bar{J}-1} \overline{I}_k \bar{h}_k^{2s} \overline{Q}_k, \tag{1.7}$$

with an arbitrary parameter $\widetilde{\gamma} \in (0, 1)$. Above, $\overline{Q}_k$ and $\overline{I}_k$ are suitable L^2-projection and inclusion operators, respectively. Clearly, if $s \in (0, 1)$ is fixed, then the factor $1 - \widetilde{\gamma}^s$ is equivalent to a constant. However, such a factor tends to 0 as $s \to 0$, and this correction is fundamental for the resulting condition number to be uniformly bounded with respect to s.

We now present a simple numerical example to illustrate this point. Let $\Omega = (-1, 1)^2$, $f = 1$, $s = 10^{-1}, 10^{-2}$, and choose either $\widetilde{\gamma} = 0$ (i.e., no correction) or $\widetilde{\gamma} = \frac{1}{2}$ in the preconditioner above to compute finite element solutions to (1.3) on a sequence of nested grids. The left panel in Table 1 shows the condition numbers of the preconditioned linear systems. It is apparent that setting $\widetilde{\gamma} = \frac{1}{2}$ gives rise to a more robust behavior with respect to both s and the number of levels $\bar{J}$.

Another aspect to take into account in (1.3) is the low regularity of solutions [14, 36, 46], which calls for graded grids in numerical computation [1, 13]. However, graded grids give rise to worse-conditioned matrices, as described by (1.6). This work also addresses preconditioning on graded bisection grids that can be employed to obtain the refinement as needed. Our algorithm on graded bisection grids builds on the subspace decomposition introduced in [21], which leads to optimal multilevel methods for classical ($s = 1$) problems. Our theory on graded bisection grids, however, differs from the existing ones [21, 31, 52] to account for the uniformity with respect to s. As illustrated by the right panel in Table 1, including a correction factor on the coarser scales leads to a more robust preconditioner. This confirms the practical value of the modification in addition to its theoretical value.

We now briefly discuss the main difficulty of our analysis for graded bisection grids. We rely on the theory of subspace correction [54], but the presence of the scaling factor $1 - \widetilde{\gamma}^s$ on coarse meshes complicates the stable decomposition. Since the subspaces generated by bisection are local and non-nested, the decomposition in [21, 31, 52] directly applies to the difference of some local operators (called slicing operators in [21]), but the technique used in [21] yields a stability constant

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

TABLE 1. Condition numbers with BPX preconditioner without ($\widetilde{\gamma} = 0$) and with ($\widetilde{\gamma} = \frac{1}{2}$) a correction factor. We display results on a family of uniformly refined grids (top panel), and on a sequence of suitably graded bisection grids (bottom panel).

Uniform grids				
DOFs	$s = 10^{-1}$		$s = 10^{-2}$	
	$\tilde{\gamma} = 0$	$\tilde{\gamma} = \frac{1}{2}$	$\tilde{\gamma} = 0$	$\tilde{\gamma} = \frac{1}{2}$
225	8.66	2.92	12.11	3.70
961	10.81	3.00	15.97	3.80
3969	12.66	3.03	19.66	3.83
16129	14.28	3.03	23.27	3.84
Graded bisection grids				
DOFs	$s = 10^{-1}$		$s = 10^{-2}$	
	$\tilde{\gamma} = 0$	$\tilde{\gamma} = \frac{1}{2}$	$\tilde{\gamma} = 0$	$\tilde{\gamma} = \frac{1}{2}$
161	8.06	3.85	11.48	5.06
853	10.80	4.21	15.86	5.14
2265	13.43	4.55	20.72	5.38
9397	15.53	4.74	24.71	5.55

depending on $(1 - \widetilde{\gamma}^s)^{-1}$ that blows up as $s \to 0$. Instead, we develop in Section 3.2 a new tool called *s-uniform decomposition* on nested spaces. Invoking this tool, we construct in Section 7 a stable decomposition on a sequence of auxiliary nested subspaces for bisection grids that leads to the desired L^2-stable decomposition to the local subspaces. The resulting BPX counterpart of (1.7) is robust with respect to the number J of levels and the fractional order s, and applies as well to the spectral and censored Laplacians in view of their spectral equivalence to the integral Laplacian alluded to earlier.

This paper is organized as follows. Section 2 collects preliminary material about the interpolation spaces and the finite element discretization of (1.3). Next, in Section 3 we discuss general aspects of the method of subspace corrections and introduce an *s-uniform decomposition* that plays a central role in our analysis. As an application, we introduce a BPX preconditioner for quasi-uniform grids in Section 4, and prove that it leads to condition numbers uniformly bounded with respect to the number of refinements $\bar{J}$ and the fractional power s. Afterwards, we delve into the preconditioning of systems arising from graded bisection grids. For that purpose, Section 5 offers a review of the bisection method with novel twists and proposes a BPX preconditioner on graded bisection grids. Sections 6 and 7 provide the technical analysis of the s-uniform decomposition for graded bisection grids. Section 8 presents some numerical experiments that illustrate the uniform performance of the BPX preconditioners with respect to s and the number of levels. Finally, Section 9 discusses BPX preconditioners for the spectral and censored fractional Laplacians.

2. PRELIMINARIES

In this section, we set the notation used in the rest of the paper regarding Sobolev spaces and recall some preliminary results about their interpolation. We refer to [13] for the basic definitions we use here. We are particularly concerned

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

with the zero-extension Sobolev space $\widetilde{H}^\sigma(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^\sigma(\mathbb{R}^d)}}$, which is the set of functions in $H^\sigma(\mathbb{R}^d)$ whose support is contained in Ω. Given $u, v \in \widetilde{H}^\sigma(\Omega)$, we define the (scaled) inner product $(\cdot,\cdot)_\sigma \colon \widetilde{H}^\sigma(\Omega) \times \widetilde{H}^\sigma(\Omega) \to \mathbb{R}$ to be

$$(2.1)\quad (u,v)_\sigma := (u,v)_{H^\sigma(\mathbb{R}^d)} = \frac{C(d,\sigma)}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2\sigma}}\, dx\, dy,$$

where $C(d,\sigma)$ is the constant from (1.1). We point out that the integration in (2.1) takes place in $\left(\Omega \times \mathbb{R}^d\right) \cup \left(\mathbb{R}^d \times \Omega\right)$ because functions in $\widetilde{H}^\sigma(\Omega)$ vanish in Ω^c. We further write the scaled Gagliardo seminorm $|u|_\sigma = (u,u)_\sigma^{1/2} = |u|_{H^\sigma(\mathbb{R}^d)}$, and let $\|u\|_0 := \|u\|_{L^2(\Omega)}$.

For convenience, we write $X \lesssim Y$ (resp. $X \gtrsim Y$) to indicate $X \le CY$ (resp. $CX \ge Y$), where C denotes, if not specified, a generic positive constant that may stand for different values at its different occurrences but is independent of the fractional power s or the number of levels; this implies the independence of the mesh-size for quasi-uniform grids or the dimension of the FEM-space for graded bisection grids. The notation $X \simeq Y$ means both $X \lesssim Y$ and $X \gtrsim Y$ hold.

2.1. **Interpolation and fractional Sobolev spaces.** An important feature of the fractional Sobolev scale is that it can be equivalently defined by *interpolation* of integer-order spaces. This, along with the observation that the norm equivalence constants are uniform with respect to s, is fundamental for our work. In view of the applications below, we now recall the abstract setting for two separable Hilbert spaces $X^1 \subset X^0$ with X^1 continuously embedded and dense in X^0. Following [38, Section 2.1], the inner product in X^1 can be represented by a self-adjoint and coercive operator $S : D(S) \to X^0$ with domain $D(S) \subset X^1$ dense in X^0, i.e. $(v,w)_{X^1} = (Sv, w)_{X^0}$ for all $v \in D(S), w \in X^1$. Invoking the spectral decomposition of self-adjoint operators [59], we let $\Lambda : X^1 \to X^0$ be the square root of S, which in turn is self-adjoint, coercive, and satisfies

$$(2.2)\qquad (v,w)_{X^1} = (\Lambda v, \Lambda w)_{X^0} \quad \forall v, w \in X^1.$$

Suppose further that the spectrum $\{\lambda_k\}_{k=1}^\infty$ of Λ is discrete and the corresponding eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ form a complete orthonormal basis for X^0; hence $\Lambda v = \sum_{k=1}^\infty \lambda_k v_k \varphi_k$ for all $v = \sum_{k=1}^\infty v_k \varphi_k \in X^1$. Then, we can define a fractional power $s \in (0,1)$ of Λ as follows:

$$(2.3)\qquad \Lambda^s v := \sum_{k=1}^\infty \lambda_k^s v_k \varphi_k \quad \text{if} \quad \|\Lambda^s v\|_{X_0}^2 := \sum_{k=1}^\infty \lambda_k^{2s} v_k^2 < \infty.$$

On the other hand, we can construct intermediate spaces by the K-method. Given $s \in (0,1)$, we consider the interpolation space $(X^0, X^1)_{s,2}$ with norm

$$(2.4)\qquad \|v\|_{(X^0,X^1)_{s,2}} := \left(\frac{2\sin(\pi s)}{\pi} \int_0^\infty t^{-1-2s} K_2(v,t)^2 dt \right)^{\frac12},$$

where $K_2(t,v) := \inf_{v=v^0+v^1} \left(\|v^0\|_{X^0}^2 + t^2 \|v^1\|_{X^1}^2 \right)^{\frac12}$. The following result [38, Theorem 15.1] gives an intrinsic spectral equivalence between the interpolation by K-method and spectral theory.

Theorem 2.1 (Intrinsic spectral equivalence)**.** *Let $X^1 \subset X^0$ be two Hilbert spaces with X^1 continuously embedded and dense in X^0. Let the self-adjoint and coercive*

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

operator $\Lambda : X^1 \to X^0$ satisfy (2.2) *and have a discrete, complete and orthonormal set of eigenpairs $(\lambda_k, \varphi_k)_{k=1}^\infty$ in X^0. Given $s \in (0,1)$, for any $v \in X^0$ with $\|\Lambda^s v\|_{X^0} < \infty$ we have*

$$\|\Lambda^s v\|_{X_0} = \|v\|_{(X^0,X^1)_{s,2}}.$$

We now apply Theorem 2.1 (intrinsic spectral equivalence) to L^2-based Sobolev spaces. Let $X^0 = \widetilde{L}^2(\Omega)$ and $X^1 = \widetilde{H}^1(\Omega)$ denote the spaces of functions in $L^2(\Omega)$ and $H_0^1(\Omega)$ extended by zero to Ω^c, respectively, and let the inner product in X^1 be given by $(v,w)_{X^1} = \int_{\mathbb{R}^d} \nabla v \cdot \nabla w = \int_\Omega \nabla v \cdot \nabla w$. The corresponding operator S equals the Laplacian $-\Delta$ with zero Dirichlet condition on the bounded Lipschitz domain Ω; S thus admits a set of eigenpairs $\{\widehat{\lambda}_k, \widehat{\varphi}_k\}_{k=1}^\infty$ where the eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ are extended by zero to Ω^c and form a complete orthonormal set in $\widetilde{L}^2(\Omega)$ [29, Section 6.5.1 and Appendix D.5]. Therefore, the corresponding eigenpairs $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ of $\Lambda = (-\Delta)^{\frac{1}{2}}$ satisfy $\lambda_k = \widehat{\lambda}_k^{1/2}$ and $\varphi_k = \widehat{\varphi}_k$, whence

$$\|\Lambda^s v\|_0^2 = \|(-\Delta)^{\frac{s}{2}} v\|_0^2 = \sum_{k=1}^\infty \lambda_k^{2s} v_k^2 = \sum_{k=1}^\infty \widehat{\lambda}_k^s v_k^2 \tag{2.5}$$

is the norm square of the interpolation space $\widetilde{H}^s(\Omega) = \big(\widetilde{L}^2(\Omega), \widetilde{H}^1(\Omega)\big)_{s,2}$. Since this norm is uniformly equivalent to the scaled Gagliardo norm

$$|v|_s \simeq \|\Lambda^s v\|_0 \quad \forall v \in \widetilde{H}^s(\Omega) \tag{2.6}$$

(cf. [40, Theorem B.8, Theorem B.9] and [20]), (2.5) yields the following equivalence.

Proposition 2.1 (Norm equivalence)**.** *The following equivalence is uniform in $s \in (0,1)$,*

$$|v|_s^2 \simeq \sum_{k=1}^\infty \widehat{\lambda}_k^s v_k^2 \quad \forall v \in \widetilde{H}^s(\Omega). \tag{2.7}$$

2.2. **Variational formulation and finite element discretization.** Since a Poincaré inequality is valid in $\widetilde{H}^s(\Omega)$ (cf. [1, Prop. 2.4], for example), the map $u \mapsto (u,u)_s$ is an inner product on $\widetilde{H}^s(\Omega)$. Given $f \in H^{-s}(\Omega)$, the dual of $\widetilde{H}^s(\Omega)$, the weak formulation of the homogeneous Dirichlet problem (1.3) reads: find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := (u,v)_s = \langle f, v\rangle_{s,\Omega} \quad \forall v \in \widetilde{H}^s(\Omega), \tag{2.8}$$

where $\langle \cdot,\cdot\rangle_{s,\Omega}$ stands for the duality pairing between $H^{-s}(\Omega)$ and $\widetilde{H}^s(\Omega)$. Existence and uniqueness of solutions of (2.8) is a consequence of the Riesz representation theorem.

Given a conforming and shape-regular triangulation $\mathcal{T}$ of Ω, we consider discrete spaces consisting of continuous piecewise linear functions that vanish on $\partial\Omega$,

$$\mathbb{V}(\mathcal{T}) = \{v_h \in C(\overline{\Omega})\colon v_h|_T \in P_1(T)\ \forall T \in \mathcal{T},\ v_h|_{\partial\Omega} = 0\}. \tag{2.9}$$

It is clear that $\mathbb{V}(\mathcal{T}) \subset \widetilde{H}^s(\Omega)$, independently of the value of s. Therefore, we can pose a conforming discretization of (2.8): we seek $u_h \in \mathbb{V}(\mathcal{T})$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle_{s,\Omega} \quad \forall v_h \in \mathbb{V}(\mathcal{T}). \tag{2.10}$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Remark 1 (Necessity of graded grids). Convergence rates in the energy norm are derived by using suitable interpolation estimates [1, 23] and regularity results [11, Theorem 3.7]. As shown in [1], conforming finite element approximations over quasi-uniform grids converge at most with order $\frac{1}{2}$ in the energy norm, due to the boundary behavior (9.3) [10, 36, 46]. To mitigate such a low convergence rate one can incorporate a mesh *grading towards* $\partial\Omega$. This idea was exploited in [1] (see also [11, 16]), where the regularity of the solution is characterized in weighted Sobolev spaces, with the weight being a power of $d(\cdot, \partial\Omega)$. In fact, let us define the *patch* of a closed element $\tau \in \mathcal{T}$

$$S_\tau := \bigcup\{\tau' \in \mathcal{T} \colon \tau' \cap \tau \neq \varnothing\}.$$

Given a grading parameter $\mu \geq 1$ and a mesh size parameter h, we assume that, for all $\tau \in \mathcal{T}$, the element size $h_\tau = |\tau|^{1/d}$ satisfies

$$h_\tau \simeq \begin{cases} h^\mu & \text{if } S_\tau \cap \partial\Omega \neq \varnothing, \\ h\, d(\tau, \partial\Omega)^{(\mu-1)/\mu} & \text{otherwise.} \end{cases} \tag{2.11}$$

If the right-hand side f is sufficiently smooth, it turns out that the optimal choice for μ is $\frac{d}{d-1}$. We refer to [1, 11, 16] for details.

We conclude this section with a *fractional local inverse estimate* valid on arbitrary, shape-regular grids $\mathcal{T}$. The proof hinges on localization of fractional norms to element patches. Because the Gagliardo seminorm $|\cdot|_\sigma$ involves integration on Ω^c, we introduce the *extended patch* of a closed element $\tau \in \mathcal{T}$,

$$\widetilde{S}_\tau := \begin{cases} S_\tau & \text{if } \tau \cap \partial\Omega = \varnothing, \\ B_\tau & \text{otherwise,} \end{cases} \tag{2.12}$$

where $B_\tau = B(x_\tau, Ch_\tau)$ is the ball of center x_τ and radius Ch_τ, with x_τ being the barycenter of τ, and $C = C(\sigma)$ a shape regularity dependent constant such that $S_\tau \subset B_\tau$. We assume $\widetilde{S}_\tau$ is a Lipschitz set, so that the space $H^\beta(\widetilde{S}_\tau)$ defined by interpolation between $H^1(\widetilde{S}_\tau)$ and $L^2(\widetilde{S}_\tau)$ is well-defined and has a seminorm equivalent to the (scaled) Gagliardo seminorm

$$v \mapsto |v|_{H^\beta(\widetilde{S}_\tau)} := \left(\frac{C(d,\beta)}{2} \iint_{\widetilde{S}_\tau \times \widetilde{S}_\tau} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2\beta}}\, \mathrm{d}y\, \mathrm{d}x \right)^{1/2}.$$

Lemma 2.1 (Local inverse inequality). *Let $\sigma \in [0,1]$ and $\beta \in [0,\sigma]$, and assume $\widetilde{S}_\tau$ is Lipschitz for every $\tau \in \mathcal{T}$. Then,*

$$|v|_\sigma \lesssim \left(\sum_{\tau \in \mathcal{T}} h_\tau^{2(\beta-\sigma)} |v|^2_{H^\beta(\widetilde{S}_\tau)} \right)^{\frac{1}{2}} \quad \forall v \in \mathbb{V}(\mathcal{T}), \tag{2.13}$$

where the hidden constant depends on the spatial dimension, shape-regularity constant and Lipschitz constant of $\widetilde{S}_\tau$, but is uniformly bounded with respect to σ and β.

Proof. We decompose the scaled seminorm $|v|_\sigma$ locally according to [15, Lemma 4.1] for $\sigma < 1$,

$$|v|^2_\sigma \leq \frac{C(d,\sigma)}{2} \sum_{\tau \in \mathcal{T}} \left(\iint_{\tau \times \widetilde{S}_\tau} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\sigma}}\, \mathrm{d}y\, \mathrm{d}x + \frac{C}{\sigma h_\tau^{2\sigma}} \|v\|^2_{L^2(\tau)} \right) \quad \forall v \in \mathbb{V}(\mathcal{T}),$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where $C(d,\sigma)$ is taken as in (2.1) and the constant C depends only on d and the shape-regularity constant of $\mathcal{T}$. We next exploit the local quasi-uniformity of $\mathcal{T}$ and operator interpolation theory (cf. [50, Chapter 34 & 36]) applied to the estimates

$$|v|_{H^1(\widetilde{S}_\tau)} \lesssim h_\tau^{-1}\|v\|_{L^2(\widetilde{S}_\tau)}, \qquad \|v\|_{L^2(\widetilde{S}_\tau)} \le \|v\|_{L^2(\widetilde{S}_\tau)},$$

to deduce the local inverse estimate with hidden constant insensitive to σ

$$|v|_{H^\sigma(\widetilde{S}_\tau)} \lesssim h_\tau^{-\sigma}\|v\|_{L^2(\widetilde{S}_\tau)},$$

because $\widetilde{S}_\tau$ is Lipschitz. Applying again operator interpolation theory to this estimate and $|v|_{H^\sigma(\widetilde{S}_\tau)} \le |v|_{H^\sigma(\widetilde{S}_\tau)}$ gives

$$|v|_{H^\sigma(\widetilde{S}_\tau)} \lesssim h_\tau^{\beta-\sigma}|v|_{H^\beta(\widetilde{S}_\tau)}, \quad \beta\in[0,\sigma]$$

and leads to the desired estimate (2.13) for $\sigma<1$. The case $\sigma=1$ is simpler because the seminorm $|v|_1$ is local. In fact, it hinges on the local inverse estimate $|v|_{H^1(\widetilde{S}_\tau)} \lesssim h_\tau^{\beta-1}|v|_{H^\beta(\widetilde{S}_\tau)}$, which in turn results from operator interpolation applied to the estimates $|v|_{H^1(\widetilde{S}_\tau)} \lesssim h_\tau^{-1}\|v\|_{L^2(\widetilde{S}_\tau)}$ and $|v|_{H^1(\widetilde{S}_\tau)} \le |v|_{H^1(\widetilde{S}_\tau)}$. □

3. s-Uniform additive multilevel preconditioning

Let $(\cdot,\cdot)$ be the L^2-inner product in Ω and $V:=\mathbb{V}(\mathcal{T})$ denote the discrete space. Let $A: V\to V$ be the symmetric positive definite (SPD) operator defined by $(Au,v):=a(u,v)$ for any $u,v\in V$, and let $\widetilde{f}\in V$ be given by $(\widetilde{f},v)=\langle f,v\rangle_{s,\Omega}$ for any $v\in V$. With this notation at hand, the discretization (2.10) leads to the following linear equation in V

$$Au=\widetilde{f}. \tag{3.1}$$

In this section, we give some general and basic results that will be used to construct and analyze the s-uniform additive multilevel preconditioners for (3.1).

3.1. Space decomposition.

We now invoke the *method of subspace corrections* [27, 54, 55, 58, 60]. We first decompose the space V as the sum $V=\sum_{j=0}^J V_j$ of subspaces $V_j\subset V$. For $j=0,1,\ldots,J$, we consider the following operators:

- $Q_j\colon V\to V_j$ is the L^2-projection operator defined by $(Q_jv,v_j)=(v,v_j)$ for all $v\in V, v_j\in V_j$;
- $I_j\colon V_j\to V$ is the natural inclusion operator given by $I_jv_j=v_j$ for all $v_j\in V_j$;
- $R_j\colon V_j\to V_j$ is an approximate inverse of the restriction of A to V_j (often known as smoother); we set $\|v_j\|^2_{R_j^{-1}}:=(R_j^{-1}v_j,v_j)$ for all $v_j\in V_j$ provided that R_j is SPD on V_j.

A straightforward calculation shows that $Q_j=I_j^t$ because $(Q_jv,v_j)=(v,I_jv_j)=(I_j^tv,v_j)$ for all $v\in V, v_j\in V_j$. Let the fictitious space be $\underset{\sim}{V}=V_0\times V_1\times\ldots\times V_J$. Then, the Parallel Subspace Correction (PSC) preconditioner $B\colon V\to V$ is defined by

$$B:=\sum_{j=0}^J I_jR_jQ_j=\sum_{j=0}^J I_jR_jI_j^t. \tag{3.2}$$

Lemmas 3.1 and 3.2 follow from the general theory of preconditioning techniques based on fictitious or auxiliary spaces [34, 41, 54–56, 58].

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Lemma 3.1 (Identity for PSC)**.** *If R_j is SPD on V_j for $j = 0, 1, \ldots, J$, then B defined in* (3.2) *is also SPD under the inner product $(\cdot, \cdot)$. Furthermore,*

$$(B^{-1}v, v) = \inf_{\sum_{j=0}^{J} v_j = v} \sum_{j=0}^{J} (R_j^{-1} v_j, v_j) \quad \forall v \in V. \tag{3.3}$$

Lemma 3.2 (Estimate on $\mathrm{cond}(BA)$)**.** *If the operator B in* (3.2) *satisfies*

(A1) *Stable decomposition: for every $v \in V$, there exists $(v_j)_{j=0}^{J} \in \underset{\sim}{V}$ such that $\sum_{j=0}^{J} v_j = v$ and*

$$\sum_{j=0}^{J} \|v_j\|_{R_j^{-1}}^2 \leq c_0 \|v\|_A^2, \tag{3.4}$$

where $\|v\|_A^2 = (Av, v)$, then $\lambda_{\min}(BA) \geq c_0^{-1}$;

(A2) *Boundedness: For every $(v_j)_{j=0}^{J} \in \underset{\sim}{V}$ there holds*

$$\left\| \sum_{j=0}^{J} v_j \right\|_A^2 \leq c_1 \sum_{j=0}^{J} \|v_j\|_{R_j^{-1}}^2, \tag{3.5}$$

then $\lambda_{\max}(BA) \leq c_1$. Consequently, if B satisfies (A1) *and* (A2)*, then* $\mathrm{cond}(BA) \leq c_0 c_1$.

3.2. **Instrumental tools for s-uniform preconditioner.** We assume that the spaces $\{V_j\}_{j=0}^{J}$ are nested, i.e.

$$V_{j-1} \subset V_j \quad \forall 1 \leq j \leq J.$$

With the convention that $Q_{-1} = 0$, we consider the L^2-slicing operators

$$\widetilde{Q}_j \colon V \to V_j : \quad \widetilde{Q}_j := Q_j - Q_{j-1} \quad (j = 0, 1, \ldots, J).$$

Clearly, the L^2-orthogonality implies that $Q_k Q_j = Q_{k \wedge j}$, where $k \wedge j := \min\{k, j\}$. Hence,

$$\widetilde{Q}_j Q_k = Q_k \widetilde{Q}_j = \begin{cases} \widetilde{Q}_j & j \leq k, \\ 0 & j > k, \end{cases} \qquad \widetilde{Q}_k \widetilde{Q}_j = \delta_{kj} \widetilde{Q}_k. \tag{3.6}$$

Lemma 3.3 plays a key role in the analysis of an s-uniform preconditioner, which is obtained by using the identity of PSC (3.3) and reordering the BPX preconditioner [18, 53].

Lemma 3.3 (s-Uniform decomposition)**.** *Given $\gamma \in (0, 1)$, $s \in (0, 1]$, it holds that, for every $v \in V$,*

$$\sum_{j=0}^{J} \gamma^{-2sj} \|(Q_j - Q_{j-1})v\|_0^2 = \inf_{\substack{v_j \in V_j \\ \sum_{j=0}^{J} v_j = v}} \left[\gamma^{-2sJ} \|v_J\|_0^2 + \sum_{j=0}^{J-1} \frac{\gamma^{-2sj}}{1 - \gamma^{2s}} \|v_j\|_0^2 \right].$$

Proof. This proof is an application of Lemma 3.1 (Identity for PSC). Taking

$$B = \sum_{j=0}^{J} \gamma^{2sj} (Q_j - Q_{j-1}),$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

by the L^2-orthogonality (3.6), we easily see that $B^{-1} = \sum_{j=0}^{J} \gamma^{-2sj}(Q_j - Q_{j-1})$ and

$$(B^{-1}v, v) = \sum_{j=0}^{J} \gamma^{-2sj} \|(Q_j - Q_{j-1})v\|_0^2.$$

On the other hand, to identify R_j we reorder the sum in the definition of B

$$B = \sum_{j=0}^{J} \gamma^{2sj}(Q_j - Q_{j-1}) = \gamma^{2sJ} Q_J + \sum_{j=0}^{J-1} (1-\gamma^{2s})\gamma^{2sj} Q_j = \sum_{j=0}^{J} I_j R_j Q_j,$$

where

$$R_j v_j := \begin{cases} (1-\gamma^{2s})\gamma^{2sj} v_j & j = 0, \dots, J-1, \\ \gamma^{2sj} v_j & j = J, \end{cases}$$

for all $v_j \in V_j$. Finally, the identity (3.3) of PSC gives the desired result. $\square$

Lemma 3.4 is useful to obtain stable decompositions in fractional-order norms. The proof is a direct application of space interpolation theory and is therefore omitted here.

Lemma 3.4 (s-Uniform interpolation)**.** *Assume that the spaces $\{V_j\}_{j=0}^{J}$ are nested, and*

$$\sum_{j=0}^{J} \gamma^{-2j} \|(Q_j - Q_{j-1})v\|_0^2 \lesssim |v|_1^2 \qquad \forall v \in V. \tag{3.7}$$

Then, the following inequality holds for $s \in [0,1]$, with the hidden constant independent of s and J,

$$\sum_{j=0}^{J} \gamma^{-2sj} \|(Q_j - Q_{j-1})v\|_0^2 \lesssim |v|_s^2 \qquad \forall v \in V.$$

4. s-Uniform BPX preconditioner for quasi-uniform grids

We propose and study a BPX preconditioner [6, 18, 31, 49, 53] for the solution of the systems arising from the finite element discretizations (2.10) on quasi-uniform grids. We emphasize that, in contrast to [6, 18, 31], the proposed preconditioner is *uniform* with respect to both the number of levels and the order s. To this end, we introduce a new factor for coarse spaces which differs from the original BPX preconditioners.

Consider a family of uniformly refined grids $\{\overline{\mathcal{T}}_k\}_{k=0}^{\bar{J}}$ on Ω, where $\overline{\mathcal{T}}_0 = \mathcal{T}_0$ is a quasi-uniform initial triangulation. On each of these grids we define the space $\overline{V}_k := \mathbb{V}(\overline{\mathcal{T}}_k)$ according to (2.9). Let $\overline{V} = \overline{V}_{\bar{J}}$ and $\overline{A}$ be the SPD operator on $\overline{V}$ associated with $a(\cdot,\cdot)$: $(\overline{A}v, w) = a(v, w)$ for all $v, w \in \overline{V}$. Let the grid size be $\bar{h}_k \simeq \gamma^k$, where $\gamma \in (0,1)$ is a fixed constant. For instance, we have $\gamma = \frac{1}{2}$ for uniform refinement, in which each simplex is refined into 2^d children, and $\gamma = (\frac{1}{2})^{1/d}$ for uniform bisection, in which each simplex is refined into 2 children.

Let $\overline{Q}_k : \overline{V} \to \overline{V}_k$ and $\overline{I}_k : \overline{V}_k \to \overline{V}$ be the L^2-projection and inclusion operators defined in Section 3.1, and let $\overline{Q}_{-1} := 0$. Let $\widetilde{\gamma} \in (0,1)$ be a fixed constant; it can

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

be taken equal to γ but this is not needed. For every $v_k \in \overline{V}_k, k = 0, \dots, \bar{J}$, we define $\overline{R}_k : \overline{V}_k \to \overline{V}_k$ to be

$$\overline{R}_k v_k := \begin{cases} (1-\widetilde{\gamma}^s)\bar{h}_k^{2s} v_k & k = 0, \dots, \bar{J}-1, \\ \bar{h}_k^{2s} v_k & k = \bar{J}. \end{cases} \tag{4.1}$$

We now introduce the BPX preconditioner and study its properties in the sequel

$$\overline{B} := \sum_{k=0}^{\bar{J}} \overline{I}_k \overline{R}_k \overline{I}_k^t = \overline{I}_{\bar{J}} \bar{h}_{\bar{J}}^{2s} \overline{Q}_{\bar{J}} + (1-\widetilde{\gamma}^s) \sum_{k=0}^{\bar{J}-1} \overline{I}_k \bar{h}_k^{2s} \overline{Q}_k. \tag{4.2}$$

Our next goal is to prove Theorem 4.1, namely that $\overline{B}$ satisfies the two necessary conditions (3.4) and (3.5) of Lemma 3.2 (estimate of $\operatorname{cond}(BA)$) uniformly in $\bar{J}$ and s over quasi-uniform grids. We observe that the scaling $(1-\widetilde{\gamma}^s)^{-1} > 1$ makes it easier to prove the boundedness (3.5) but complicates the stable decomposition (3.4).

Theorem 4.1 (Uniform preconditioning on quasi-uniform grids)**.** *Let Ω be a bounded Lipschitz domain and $s \in (0,1)$. Consider discretizations to* (1.3) *using piecewise linear Lagrangian finite elements on quasi-uniform grids. Then, the preconditioner* (4.2) *satisfies* $\operatorname{cond}(\overline{BA}) \lesssim 1$, *where the hidden constant is uniform with respect to both $\bar{J}$ and s.*

We start with a norm equivalence for discrete functions. We rely on operator interpolation and the decomposition for $s = 1$ [12, 25, 45, 54], which was proposed earlier in [18, 53] with a removable logarithmic factor. A similar result, for the interpolation norm of $(L^2(\Omega), H_0^1(\Omega))_{s,2}$, was given in [56, Theorem 10.5].

Theorem 4.2 (Norm equivalence)**.** *Let Ω be a bounded Lipschitz domain and $s \in [0,1]$. If $\overline{V} = \overline{V}_{\bar{J}}$ and $\overline{Q}_k : \overline{V} \to \overline{V}_k$ denotes the L^2-projection operators onto discrete spaces $\overline{V}_k$, and $\overline{Q}_{-1} := 0$, then for any $v \in \overline{V}$ the decomposition $v = \sum_{k=0}^{\bar{J}} (\overline{Q}_k - \overline{Q}_{k-1}) v$ satisfies*

$$|v|_s^2 \simeq \sum_{k=0}^{\bar{J}} \bar{h}_k^{-2s} \|(\overline{Q}_k - \overline{Q}_{k-1}) v\|_0^2. \tag{4.3}$$

The equivalence constant hidden in (4.3) *is independent of s and $\bar{J}$.*

Proof. We show (4.3) in the entire space $\widetilde{H}^s(\Omega)$, namely $\overline{J} = \infty$. We consider the self-adjoint operator $\Lambda = \sum_{k=0}^{\infty} \bar{h}_k^{-1} (\overline{Q}_k - \overline{Q}_{k-1}) : \widetilde{H}_0^1(\Omega) \to \widetilde{L}^2(\Omega)$, which induces a norm in $\widetilde{H}_0^1(\Omega)$ equivalent to the standard H^1-norm according to [12, 45, 54]

$$\|\Lambda v\|_0^2 = \sum_{k=0}^{\infty} \bar{h}_k^{-2} \|(\overline{Q}_k - \overline{Q}_{k-1}) v\|_0^2 \simeq \|v\|_1^2 \quad \forall v \in \widetilde{H}_0^1(\Omega).$$

Therefore, Theorem 2.1 (intrinsic spectral equivalence) implies that $\|\Lambda^s v\|_0$ defines a norm on the interpolation space $\widetilde{H}^s(\Omega)$ that is uniformly equivalent to the interpolation norm, whence $|v|_s \simeq \|\Lambda^s v\|_0$ by virtue of (2.6). It remains to characterize $\|\Lambda^s v\|_0$.

We notice that $\widetilde{V}_k := (\overline{Q}_k - \overline{Q}_{k-1}) \widetilde{H}_0^1(\Omega)$ is an eigenspace of Λ with eigenvalue $\lambda_k = \bar{h}_k^{-1}$ and $\widetilde{H}_0^1(\Omega) = \oplus_{k=0}^{\infty} \widetilde{V}_k$ is an L^2-orthogonal decomposition. Consequently, (2.3) yields $\|\Lambda^s v\|_0^2 = \sum_{k=0}^{\infty} \bar{h}_k^{-2s} \|(\overline{Q}_k - \overline{Q}_{k-1}) v\|_0^2$ and thus (4.3), as asserted. $\square$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Corollary 4.1 (Stable decomposition). *Let $\tilde{\gamma} \in (0,1)$ be a fixed constant. For every $v \in \overline{V} = \overline{V}_{\bar{J}}$ and $s \in (0,1]$, there exists a decomposition $(v_0, \ldots, v_{\bar{J}}) \in \overline{V}_0 \times \ldots \times \overline{V}_{\bar{J}}$, such that $\sum_{k=0}^{\bar{J}} v_k = v$ and*

$$\bar{h}_{\bar{J}}^{-2s} \|v_{\bar{J}}\|_0^2 + \frac{1}{1-\widetilde{\gamma}^s} \sum_{k=0}^{\bar{J}-1} \bar{h}_k^{-2s} \|v_k\|_0^2 \simeq |v|_s^2.$$

Proof. This is a direct consequence of Lemma 3.3 (s-uniform decomposition) and Theorem 4.2 (norm equivalence) because $\bar{h}_k \simeq \gamma^k$ and $1-\gamma^s \simeq 1-\tilde{\gamma}^s$ uniformly in $s \in [0,1]$. □

We now prove the boundedness estimate in Lemma 3.2 (estimate on cond(BA)) with a constant independent of both $\bar{J}$ and s.

Proposition 4.1 (Boundedness). *Let $\tilde{\gamma} \in (0,1)$ be a fixed constant, $s \in (0,1]$. The preconditioner $\overline{B}$ in* (4.2) *satisfies* (3.5)*, namely*

$$\Big| \sum_{k=0}^{\bar{J}} v_k \Big|_s^2 \leq c_1 \bigg(\bar{h}_{\bar{J}}^{-2s} \|v_{\bar{J}}\|_0^2 + \frac{1}{1-\widetilde{\gamma}^s} \sum_{k=0}^{\bar{J}-1} \bar{h}_k^{-2s} \|v_k\|_0^2 \bigg),$$

where $\widetilde{\gamma} \in (0,1)$ can be taken arbitrarily and the constant c_1 is independent of $\bar{J}$ and s.

Proof. Let $v := \sum_{k=0}^{\bar{J}} v_k$. Then, we use Theorem 4.2 (norm equivalence), the fact that $\bar{h}_k \simeq \gamma^k$ and Lemma 3.3 (s-uniform decomposition) to write

$$\begin{aligned}
\Big| \sum_{k=0}^{\bar{J}} v_k \Big|_s^2 = |v|_s^2 &\simeq \sum_{k=0}^{\bar{J}} \bar{h}_k^{-2s} \|(\overline{Q}_k - \overline{Q}_{k-1})v\|_0^2 \\
&\simeq \sum_{k=0}^{\bar{J}} \gamma^{-2sk} \|(\overline{Q}_k - \overline{Q}_{k-1})v\|_0^2 \\
&= \inf_{\substack{w_k \in \overline{V}_k \\ \sum_{k=0}^{\bar{J}} w_k = v}} \left[\gamma^{-2s\bar{J}} \|w_{\bar{J}}\|_0^2 + \sum_{k=0}^{\bar{J}-1} \frac{\gamma^{-2sk}}{1-\gamma^{2s}} \|w_k\|_0^2 \right].
\end{aligned}$$

Therefore, upon setting $w_k = v_k$ for $k = 0, \ldots \bar{J}$ above, we deduce that

$$\begin{aligned}
\Big| \sum_{k=0}^{\bar{J}} v_k \Big|_s^2 &\lesssim \gamma^{-2s\bar{J}} \|v_{\bar{J}}\|_0^2 + \sum_{k=0}^{\bar{J}-1} \frac{\gamma^{-2sk}}{1-\gamma^{2s}} \|v_k\|_0^2 \\
&\leq c_1 \bigg(\bar{h}_{\bar{J}}^{-2s} \|v_{\bar{J}}\|_0^2 + \frac{1}{1-\widetilde{\gamma}^s} \sum_{k=0}^{\bar{J}-1} \bar{h}_k^{-2s} \|v_k\|_0^2 \bigg).
\end{aligned}$$

The proof is thus complete. □

Remark 2 (Dependence on s). The standard BPX preconditioner reads [6, 18, 31]

$$\overline{B}_{\text{std}} = \sum_{k=0}^{\bar{J}} \overline{I}_k \bar{h}_k^{2s} \overline{Q}_k : \overline{V} \to \overline{V}. \tag{4.4}$$

To explore sensitivity with respect to s, we consider the limiting case $s = 0$ and express A as the identity matrix and $\overline{R}_k : \overline{V}_k \to \overline{V}_k$ as the identity operator for all

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$k = 0, \dots \bar{J}$. In view of Lemma 3.2, by decomposing any $v \in \overline{V}$ as $v = \sum_{k=0}^{\bar{J}} \widetilde{Q}_k v$ it follows that $\lambda_{\min}(\overline{B}_{\rm std}\overline{A}) = 1$. Additionally, for any $v \in \overline{V}_0$ at the coarsest level we have $\overline{B}_{\rm std}\overline{A}v = (\bar{J}+1)v$ and thus $\lambda_{\max}(\overline{B}_{\rm std}\overline{A}) \geq \bar{J}+1$. This implies that the condition number $\mathrm{cond}(\overline{B}_{\rm std}\overline{A})$ blows up as $s \to 0$ and $\bar{J} \to \infty$; this is observed in the experimental results reported in Table 1.

5. s-Uniform BPX preconditioner for graded bisection grids

In this section, we briefly review the bisection method with emphasis on graded grids following [21], and present new notions. We also refer to [42, 43, 57] for additional details.

We then design a BPX preconditioner for the integral fractional Laplacian (1.1) on graded bisection grids that is uniform with respect to both the number of levels J and fractional order s. The boundedness can be proved by combining the BPX preconditioner on quasi-uniform grids with the theory for graded bisection grids from [21, 31, 42, 43, 52].

5.1. **Bisection rules and compatible bisections.** For each closed simplex $\tau \in \mathcal{T}$ and a refinement edge e, the pair (τ, e) is called *labeled simplex*, and $(\mathcal{T}, \mathcal{L}) := \{(\tau, e) : \tau \in \mathcal{T}\}$ is called a *labeled triangulation*. For a labeled triangulation $(\mathcal{T}, \mathcal{L})$, and $\tau \in \mathcal{T}$, a *bisection* $b_\tau : \{(\tau, e)\} \mapsto \{(\tau_1, e_1), (\tau_2, e_2)\}$ is a map that encodes the refinement procedure. The formal addition is defined as follows:

$$\mathcal{T} + b_\tau := (\mathcal{T}, \mathcal{L}) \setminus \{(\tau, e)\} \cup \{(\tau_1, e_1), (\tau_2, e_2)\}.$$

For an ordered sequence of bisections $\mathcal{B} = (b_{\tau_1}, b_{\tau_2}, \dots, b_{\tau_N})$, we set

$$\mathcal{T} + \mathcal{B} := ((\mathcal{T} + b_{\tau_1}) + b_{\tau_2}) + \cdots + b_{\tau_N}.$$

Given an initial grid $\mathcal{T}_0$, the set of conforming grids obtained from $\mathcal{T}_0$ using the bisection method is defined as

$$\mathbb{T}(\mathcal{T}_0) := \{\mathcal{T} = \mathcal{T}_0 + \mathcal{B} \colon \mathcal{B} \text{ is a bisection sequence and } \mathcal{T} \text{ is conforming}\}.$$

The bisection method considered in this paper is assumed to satisfy the following two properties, which are valid for a variety of bisection grids [21].

- (A1) Shape regularity: $\mathbb{T}(\mathcal{T}_0)$ is shape regular.
- (A2) Conformity of uniform refinement: $\overline{\mathcal{T}}_k := \overline{\mathcal{T}}_{k-1} + \{b_\tau : \tau \in \overline{\mathcal{T}}_{k-1}\} \in \mathbb{T}(\mathcal{T}_0)\ \forall k \geq 1$.

We denote by $\mathcal{N}(\mathcal{T})$ the set of vertices of the mesh $\mathcal{T}$, and define the *first ring* of either a vertex $p \in \mathcal{N}(\mathcal{T})$ or an edge $e \in \mathcal{E}(\mathcal{T})$ as

$$\mathcal{R}_p = \{\tau \in \mathcal{T} \mid p \in \tau\}, \quad \mathcal{R}_e = \{\tau \in \mathcal{T} \mid e \subset \tau\},$$

and the *local patch* of either p or e as $\omega_p = \cup_{\tau \in \mathcal{R}_p} \tau$, and $\omega_e = \cup_{\tau \in \mathcal{R}_e} \tau$. An edge e is called *compatible* if e is the refinement edge of τ for all $\tau \in \mathcal{R}_e$. Let p be the midpoint of a compatible edge e and $\mathcal{R}_p$ be the ring of p in $\mathcal{T} + \{b_\tau : \tau \in \mathcal{R}_e\}$. Given a compatible edge e, a *compatible bisection* is a mapping $b_e : \mathcal{R}_e \to \mathcal{R}_p$. The addition is thus defined by

$$\mathcal{T} + b_e := \mathcal{T} + \{b_\tau : \tau \in \mathcal{R}_e\} = \mathcal{T} \setminus \mathcal{R}_e \cup \mathcal{R}_p,$$

which preserves the conformity of triangulations. Figure 5.1 depicts the two possible configurations of a compatible bisection b_{e_j} in 2D.

We now introduce the concepts of *generation* and *level*. The generation $g(\tau)$ of any element $\tau \in \mathcal{T}_0$ is set to be 0, and the generation of any subsequent element τ

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

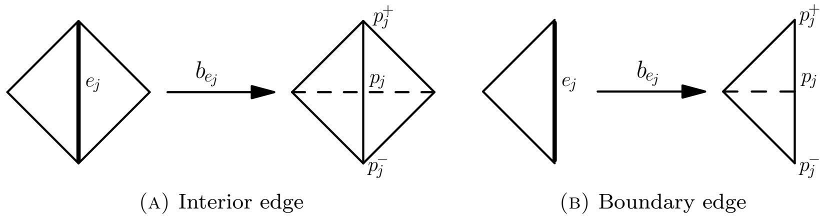


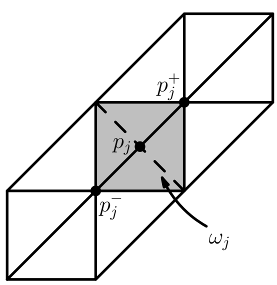
FIGURE 5.1. Two possible configurations of a compatible bisection b_{e_j} in 2D. The edge with boldface is the compatible refinement edge, and the dash-line represents the bisection.

is 1 plus the generation of its father. For any vertex p, the generation $g(p)$ of p is defined as the minimal integer k such that $p \in \mathcal{N}(\overline{\mathcal{T}}_k)$. Therefore, $g(\tau)$ and $g(p)$ are the minimal number of compatible bisections required to create τ and p from $\mathcal{T}_0$. Once p belongs to a bisection mesh, it will belong to all successive refinements; hence $g(p)$ is a static quantity insensitive to the level of resolution around p. To account for this issue, we define the *level* $\ell(p)$ of a vertex p to be the maximal generation of elements in the first ring $\mathcal{R}_p$; this is then a dynamic quantity that characterizes the level of resolution around p.

We then have the decomposition of bisection grids in terms of compatible bisections; see [21, Theorem 3.1].

Theorem 5.1 (Decomposition of bisection grids)**.** *Let $\mathcal{T}_0$ be a conforming mesh with initial labeling that enforces the bisection method to satisfy assumption* (A2)*, i.e. for all $k \geq 0$ all uniform refinements $\overline{\mathcal{T}}_k$ of $\mathcal{T}_0$ are conforming. Then for every $\mathcal{T} \in \mathbb{T}(\mathcal{T}_0)$, there exists a compatible bisection sequence $\mathcal{B} = (b_1, b_2, \dots, b_J)$ with $J = \#\mathcal{N}(\mathcal{T}) - \#\mathcal{N}(\mathcal{T}_0)$ such that*

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{B}. \tag{5.1}$$



- e_j: the refinement edge;
- p_j: the midpoint of e_j;
- p_j^-, p_j^+: two end points of e_j;
- ω_j: the patch of p_j (or ω_{p_j});
- $\widetilde{\omega}_j = \omega_{p_j} \cup \omega_{p_j^-} \cup \omega_{p_j^+}$;
- h_j: the local mesh size of ω_j;
- $\mathcal{T}_j = \mathcal{T}_0 + (b_1, \dots, b_j)$;
- $\mathcal{R}_j$: the first ring of p_j in $\mathcal{T}_j$.

FIGURE 5.2. Plot of local patch ω_j associated to a bisection node p_j, enlarged local patch $\widetilde{\omega}_j$ and definition of related quantities

For a compatible bisection b_j with refinement edge e_j, we introduce the *bisection triplet*

$$T_j := \{p_j, p_j^+, p_j^-\}, \tag{5.2}$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where p_j^- and p_j^+ are the end points of e_j and p_j is its middle point; see Figure 5.2. A vertex can be a middle point of a bisection solely once, when it is created, but instead it can be an end point of a refinement edge repeatedly; in fact this is the mechanism for the level to increment by 1. In addition, since $p_j^{\pm}$ already exist when p_j is created, it follows that

$$g_j := g(p_j) \geq g(p_j^{\pm}).$$

The notion of generation of the bisection is well-defined due to Lemma 5.1, see [21, Lemma 3.3].

Lemma 5.1 (Compatibility and generation)**.** *If $b_j \in \mathcal{B}$ is a compatible bisection, then all elements in $\mathcal{R}_j := \mathcal{R}_{p_j}$ have the same generation g_j.*

In light of the Lemma 5.1, we say that g_j is the generation of the compatible bisection $b_j : \mathcal{R}_{e_j} \to \mathcal{R}_{p_j}$. Because by assumption $h(\tau) \simeq 1$ for $\tau \in \mathcal{T}_0$, we have the following important relation between generation and mesh size:

$$h_j \simeq \gamma^{g_j}, \quad \text{with } \gamma = \left(\frac{1}{2}\right)^{1/d} \in (0,1).$$

Moreover, there exists a constant k_* depending on the shape regularity of $\mathbb{T}(\mathcal{T}_0)$ such that for every vertex $p \in \mathcal{N}(\mathcal{T}_j)$

$$\max_{\tau\in\mathcal{R}_p} g(\tau) - \min_{\tau\in\mathcal{R}_p} g(\tau) \leq k_*, \qquad \#\mathcal{R}_p \leq k_*. \tag{5.3}$$

Combining this geometric property with Lemma 5.1 (compatibility and generation), we deduce that

$$g_j - k_* \leq g(\tau) \leq g_j + k_* \quad \forall \tau \in \widetilde{\mathcal{R}}_j := \mathcal{R}_{p_j} \cup \mathcal{R}_{p_j^-} \cup \mathcal{R}_{p_j^+}. \tag{5.4}$$

Another ingredient for our analysis is the relation between the generation of compatible bisections and their local or enlarged patches [21, Lemmas 3.4 and 3.5].

Lemma 5.2 (Generation and patches)**.** *Let $\mathcal{T}_J = \mathcal{T}_0 + \mathcal{B} \in \mathbb{T}(\mathcal{T}_0)$ with compatible bisection sequence $\mathcal{B} = (b_1, \ldots, b_J)$. Then the following properties are valid:*

- *Nonoverlapping patches: For any $j \neq k$ and $g_j = g_k$, we have*

$$\mathring{\omega}_j \cap \mathring{\omega}_k = \varnothing.$$

- *Quasi-monotonicity: For any $j > i$ and $\mathring{\widetilde{\omega}}_j \cap \mathring{\widetilde{\omega}}_i \neq \varnothing$, we have*

$$g_j \geq g_i - 2k_*,$$

where k_* is the integer defined in (5.3).

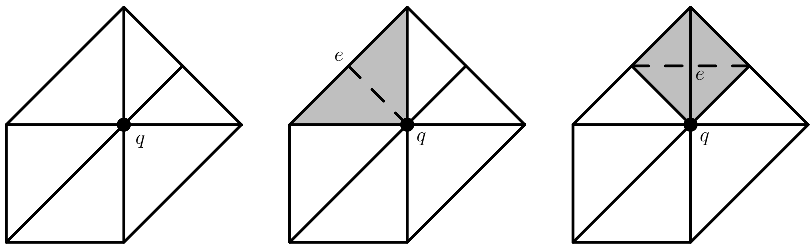
We now investigate the evolution of the level $\ell(p)$ of a generic vertex p of $\mathcal{T}$.

Lemma 5.3 (Levels of a vertex)**.** *If $q \in T_j \cap T_k$, where T_j is a bisection triplet and T_k is the next one to contain q after T_j, and $\ell_j(q)$ and $\ell_k(q)$ are the corresponding levels, then*

$$\ell_k(q) - \ell_j(q) \leq k_*,$$

where k_* is the integer given in (5.3).

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use



(A) $\mathcal{R}_q$ (B) Case 1: $q \notin e$ (C) Case 2: $q \in e$

FIGURE 5.3. Two cases of bisection in $\mathcal{R}_q$: The bisection edge e is on the boundary of the patch and q does not belong to the bisection triplet (B); the node q is an endpoint of e and belongs to the bisection triplet (C). The former can happen a fixed number k_* of times before the second takes place, where k_* depends on the shape regularity of $\mathbb{T}(\mathcal{T}_0)$.

Proof. Every time a bisection changes the ring $\mathcal{R}_q$, the level of q may increase at most by 1. If the refinement edge e of the bisection is on the boundary of the patch ω_q, then q does not belong to the bisection triplet; see Figure 5.3(middle). The number of such edges is smaller than a fixed integer k_* that only depends on the shape regularity of $\mathbb{T}(\mathcal{T}_0)$. Therefore, after at most k_* bisections the vertex q is an endpoint of a bisection triplet T_k; see Figure 5.3(right). This implies $\ell_k(q) \leq \ell_j(q) + k_*$ as asserted. ☐

We conclude this section with the following sequence of auxiliary meshes

$$\widehat{\mathcal{T}}_j := \widehat{\mathcal{T}}_{j-1} + \{b_i \in \mathcal{B} : g_i = j\} \quad j \geq 1, \qquad \widehat{\mathcal{T}}_0 := \mathcal{T}_0, \tag{5.5}$$

where $\mathcal{B}$ is the set of compatible bisections (5.1). Note that each bisection b_i in (5.1) does not require additional refinement beyond the refinement patch ω_i when incorporated in the order of the subscript i according to (5.1). This is not obvious in (5.5) because the bisections are now ordered by generation. The mesh $\widehat{\mathcal{T}}_j$ contains all elements τ of generation $g(\tau) \leq j$ leading to the finest graded mesh $\mathcal{T} = \mathcal{T}_J$. The sequence $\{\widehat{\mathcal{T}}_j\}_{j=1}^{\bar{J}}$ is never constructed but is useful for theoretical purposes in Section 7.

Lemma 5.4 (Conformity of $\widehat{\mathcal{T}}_j$). *The meshes $\widehat{\mathcal{T}}_j$ are conforming for all $j \geq 0$.*

Proof. We argue by induction. The starting mesh $\widehat{\mathcal{T}}_0$ is conforming by construction. Suppose that $\widehat{\mathcal{T}}_{j-1}$ is conforming. We observe that the bisections b_i with $g_i = j$ are disjoint according to Lemma 5.2 (generation and patches). Suppose that adding b_i does lead to further refinement beyond the refinement patch ω_i. If this were the case, then recursive bisection refinement would end up adding compatible bisections of generation strictly less than j that belong to the refinement chains emanating from ω_i [42, 43]. But such bisections are all included in $\widehat{\mathcal{T}}_{j-1}$ by virtue of (5.5). This shows that all bisections b_i with $g_i = j$ are compatible with $\widehat{\mathcal{T}}_{j-1}$ and yield local refinements that keep mesh conformity. ☐

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

5.2. **Space decomposition and BPX preconditioner.** Let $\mathcal{T}_j = \mathcal{T}_0 + \{b_1, \cdots, b_j\} \in \mathbb{T}(\mathcal{T}_0)$ be a conforming bisection grid obtained from $\mathcal{T}_0$ after $j \le J$ compatible bisections $\{b_i\}_{i=1}^j$ and let $\mathcal{N}_j = \mathring{\mathcal{N}}(\mathcal{T}_j)$ denote the set of interior vertices of $\mathcal{T}_j$. Let $\mathbb{V}(\mathcal{T}_j)$ be the finite element space of C^0 piecewise linear functions over $\mathcal{T}_j$ that vanish on $\partial\Omega$ and its nodal basis functions be $\phi_{j,p}$, namely $\mathbb{V}(\mathcal{T}_j) = \mathrm{span}\{\phi_{j,p} : p \in \mathcal{N}_j\}$. We define the *local* spaces

$$V_j = \mathrm{span}\{\phi_{j,q} : q \in T_j \cap \mathcal{N}_j\}, \quad j = 1, \dots, J \tag{5.6}$$

associated with each bisection triplet T_j. We observe that $\dim V_j \le 3$ and $\mathrm{supp}\phi \subset \widetilde{\omega}_j$ for $\phi \in V_j$ and $1 \le j \le J$; see Figure 5.2. We indicate by $V := \mathbb{V}(\mathcal{T}_J)$ the finite element space over the finest graded grid $\mathcal{T}_J$, with interior nodes $\mathcal{P} = \mathcal{N}_J$ and nodal basis functions ϕ_p,

$$V = \mathrm{span}\{\phi_p : p \in \mathcal{P}\}, \quad V_p = \mathrm{span}\{\phi_p\}; \tag{5.7}$$

hence $\dim V_p = 1$. Let h_p be the local grid size around p, which can be defined by $(|\omega_q|/\#\mathcal{R}_q)^{1/d}$ due to the shape regularity. Adding the spaces V_p and V_j yields the *space decomposition* of V

$$V = \sum_{p \in \mathcal{P}} V_p + \sum_{j=0}^{J} V_j. \tag{5.8}$$

We stress that the spaces V_j appear in the order of creation and not of generation, as is typical of adaptive procedures. Remarkably, the functions $\phi_{j,q}$ with $q = p_j^{\pm}$ depend on the order of creation of V_j (see Figure 5.2). Consequently, reordering of V_j by generation, which is convenient for analysis, must be performed with caution; see Sections 6 and 7.

Let Q_p (resp. Q_j) and I_p (resp. I_j) be the L^2-projection and inclusion operators to and from the discrete spaces V_p (resp. V_j), defined in Section 3.1. Inspired by the definition (4.1), we now define the subspace smoothers to be

$$\begin{aligned} R_j v_j &:= (1 - \widetilde{\gamma}^s) h_j^{2s} v_j \quad && \forall v_j \in V_j, \\ R_p v_p &:= h_p^{2s} v_p && \forall v_p \in V_p, \end{aligned}$$

where R_p plays the role of the finest scale whereas R_j represents the intermediate scales. That is, the intermediate spaces V_j are viewed as "coarse spaces" and are scaled by an additional factor $1 - \widetilde{\gamma}^s$. This in turn induces the following BPX preconditioner on graded bisection grids

$$B = \sum_{p \in \mathcal{P}} I_p R_p I_p^t + \sum_{j=0}^{J} I_j R_j I_j^t = \sum_{p \in \mathcal{P}} I_p h_p^{2s} Q_p + (1 - \widetilde{\gamma}^s) \sum_{j=0}^{J} I_j h_j^{2s} Q_j. \tag{5.9}$$

6. Boundedness: Proof of (3.5) for graded bisection grids

Let $\bar{J} = \max_{\tau \in \mathcal{T}_J} g_\tau$ denote the maximal generation of elements in $\mathcal{T}_J$. This quantity is useful next to reorder the spaces V_j by generation because $g_j \le \bar{J}$.

Proposition 6.1 (Boundedness)**.** *Assume the extended patch $\widetilde{S}_\tau$ defined in* (2.12) *is Lipschitz for every $\tau \in \mathcal{T}_j$ with a uniform Lipschitz constant. Let $v = \sum_{p \in \mathcal{P}} v_p +$*

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$\sum_{j=0}^{J} v_j$ be a decomposition of $v \in V$ according to (5.8). *Then, there exists a constant $c_1 > 0$ independent of J and s such that*

$$|v|_s^2 \le c_1 \left(\sum_{p\in\mathcal{P}} h_p^{-2s} \|v_p\|_0^2 + \frac{1}{1-\widetilde{\gamma}^s} \sum_{j=0}^{J} h_j^{-2s} \|v_j\|_0^2 \right), \tag{6.1}$$

whence the preconditioner B in (5.9) *satisfies $\lambda_{\max}(BA) \le c_1$.*

Proof. We resort to Lemma 2.1 (local inverse inequality) with $\sigma = s$ and $\beta = 0$, which is valid on the graded grid $\mathcal{T}_J$, to write

$$|v|_s^2 = \left| \sum_{p\in\mathcal{P}} v_p + \sum_{j=0}^{J} v_j \right|_s^2 \lesssim \left| \sum_{p\in\mathcal{P}} v_p \right|_s^2 + \left| \sum_{j=0}^{J} v_j \right|_s^2 \lesssim \sum_{p\in\mathcal{P}} h_p^{-2s} \|v_p\|_0^2 + \left| \sum_{j=0}^{J} v_j \right|_s^2. \tag{6.2}$$

In order to deal with the last term, we reorder the functions v_j by generation and observe that $\operatorname{supp} v_j \subset \widetilde{\omega}_j$. We thus define $w_k = \sum_{g_j=k} v_j$ and use (5.4) to infer that $w_k \in \overline{V}_{k+k_*} = \mathbb{V}(\overline{\mathcal{T}}_{k+k_*})$. Similar to the proof of Proposition 4.1, using Theorem 4.2 (norm equivalence), the fact that $\bar{h}_k \simeq \gamma^k$ and Lemma 3.3 (s-uniform decomposition), we have

$$\begin{aligned}
\left| \sum_{j=0}^{J} v_j \right|_s^2 &= \left| \sum_{k=0}^{\bar{J}} \sum_{g_j=k} v_j \right|_s^2 = \left| \sum_{k=0}^{\bar{J}} w_k \right|_s^2 \\
&\simeq \sum_{\ell=0}^{\bar{J}+k_*} \gamma^{-2s\ell} \left\| (\overline{Q}_\ell - \overline{Q}_{\ell-1}) \sum_{k=0}^{\bar{J}} w_k \right\|_0^2 \\
&= \inf_{\substack{z_\ell \in \overline{V}_\ell \\ \sum_{\ell=0}^{\bar{J}+k_*} z_\ell = \sum_{k=0}^{\bar{J}} w_k}} \left[\gamma^{-2s(\bar{J}+k_*)} \|z_{\bar{J}+k_*}\|_0^2 + \sum_{\ell=0}^{\bar{J}+k_*-1} \frac{\gamma^{-2s\ell}}{1-\gamma^{2s}} \|z_\ell\|_0^2 \right].
\end{aligned}$$

Choosing $z_\ell = 0$ for $\ell \le k_* - 1$ and $z_\ell = w_{\ell-k_*} \in \overline{V}_\ell$ for $\ell \ge k_*$ we get

$$\left| \sum_{j=0}^{J} v_j \right|_s^2 \lesssim \frac{\gamma^{-2sk_*}}{1-\gamma^s} \sum_{k=0}^{\bar{J}} \gamma^{-2sk} \|w_k\|_0^2.$$

In view of Lemma 5.2, we see that the enlarged patches $\widetilde{\omega}_j$ and $\widetilde{\omega}_i$ have finite overlap depending only on shape regularity of $\mathbb{T}(\mathcal{T}_0)$ provided $g_j = g_i$, whence

$$\|w_k\|_0^2 \lesssim \sum_{g_j=k} \|v_j\|_0^2.$$

This in conjunction with $\frac{1-\widetilde{\gamma}^s}{1-\gamma^s} \simeq 1$ and the fact that k_* is uniformly bounded yields

$$\left| \sum_{j=0}^{J} v_j \right|_s^2 \lesssim \frac{1}{1-\widetilde{\gamma}^s} \sum_{k=0}^{\bar{J}} \gamma^{-2sk} \sum_{g_j=k} \|v_j\|_0^2 \simeq \frac{1}{1-\widetilde{\gamma}^s} \sum_{j=0}^{J} h_j^{-2s} \|v_j\|_0^2. \tag{6.3}$$

Combining (6.2) and (6.3) leads to (6.1) as asserted. Finally, the estimate $\lambda_{\max}(BA) \le c_1$ follows directly from Lemma 3.2 (estimate on $\operatorname{cond}(BA)$). ∎

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

7. Stable decomposition: proof of (3.4) for graded bisection grids

We start with a review of the case of quasi-uniform grids in Corollary 4.1 (stable decomposition) and a roadmap of our approach. We point out that robustness with respect to both J and s, most notably the handling of the factor $(1-\widetilde{\gamma}^s)^{-1}$ on coarse levels, is due to the combination of Lemma 3.3 (s-uniform decomposition) and Theorem 4.2 (norm equivalence), which in turn relies on Lemma 3.4 (s-uniform interpolation). Since Lemma 3.4 fails on graded bisection grids, applying Lemma 3.3 to such grids faces two main difficulties:

(a) Theorem 4.2 does not hold even for $s = 1$;
(b) the spaces V_j in (5.6) and V_p in (5.7) are locally supported, while the s-uniform interpolation requires nested spaces (see Lemma 3.4). To overcome these difficulties, we create a family of nested spaces $\{W_k\}_{k=0}^{\bar{J}}$ with $W_{\bar{J}} = V$ upon grouping indices according to generation and level around k: if

$$\mathcal{J}_k := \{0 \le j \le J : \quad g_j \le k\}, \qquad \mathcal{P}_k := \{p \in \mathcal{P} : \quad \ell(p) \le k\}, \tag{7.1}$$

then we define W_k to be

$$W_k := \sum_{j\in\mathcal{J}_k} V_j + \sum_{p\in\mathcal{P}_k} V_p. \tag{7.2}$$

Our approach consists of three steps. The first step, developed in Section 7.1, is to derive a global decomposition based on W_k. Since the levels within W_k are only bounded above, to account for coarse levels we invoke a localization argument based on a slicing Scott-Zhang operator as in [21] (or [31, 52]), which gives the stability result (3.7) on $\{W_k\}_{k=0}^{\bar{J}}$ via Lemma 3.3 (s-uniform decomposition) for $s = 1$; we bridge the gap to $0 < s < 1$ via Lemma 3.4 (s-uniform interpolation). The space W_k is created for theoretical convenience, but never constructed in practice, because there is no obvious underlying graded bisection grid on which the functions of W_k are piecewise linear. This complicates the stable decomposition of W_k into local spaces and requires a characterization of W_k in terms of the space $\widehat{V}_k = \mathbb{V}(\widehat{\mathcal{T}}_k)$ of piecewise linear functions over $\widehat{\mathcal{T}}_k$. The second step in Section 7.2 consists of proving

$$\widehat{V}_k \subset W_k \subset \widehat{V}_{k+k_*},$$

where k_* is constant. Therefore, the space W_k of unordered bisections of generation and level $\le k$ is equivalent, up to level k_*, to the space $\widehat{V}_k$ of ordered bisections of generation $\le k$; note that the individual spaces V_j might not coincide though. In the last step, performed in Section 7.3, we construct a stable decomposition for graded bisection grids and associated BPX preconditioner $\widehat{B}$. We also show that $\widehat{B}$ is equivalent to B in (5.9).

7.1. Global L^2-orthogonal decomposition of W_k.

We recall that the Scott-Zhang quasi-interpolation operator $S_j : V \to \mathbb{V}(\mathcal{T}_j)$ can be defined at a node $p \in \mathcal{P}$ through the dual basis function on arbitrary elements $\tau \subset \mathcal{R}_p$ [21, 47]. We exploit this flexibility to define a suitable quasi-interpolation operator S_j as follows provided $S_{j-1} : V \to \mathbb{V}(\mathcal{T}_{j-1})$ is already known. Since $\mathcal{T}_j = \mathcal{T}_{j-1} + b_j$ and the compatible bisection b_j changes $\mathcal{T}_{j-1}$ locally in the bisection patch ω_{p_j} associated with the new vertex p_j, we set $S_j v(p) := S_{j-1} v(p)$ for all $p \in \mathcal{N}_j \setminus T_j$, where T_j is the bisection triplet (5.2). We next define $S_j v(p_j)$ using a simplex $\tau \in \mathcal{R}_j$ newly

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

created by the bisection b_j. If $p = p_j^{\pm} \in T_j$ and $\tau \in \mathcal{T}_{j-1}$ is the simplex used to define $S_{j-1}v(p)$, then we define $S_jv(p)$ according to the following rules:

(1) if $\tau \subset \omega_p(\mathcal{T}_j)$ we keep the nodal value of $S_{j-1}v$, i.e. $S_jv(p) = S_{j-1}v(p)$;
(2) otherwise we choose a new $\tau \subset \omega_p(\mathcal{T}_j) \cap \omega_p(\mathcal{T}_{j-1})$ to define $S_jv(p)$;

note that $\tau \in \mathcal{R}_j$ in case (2). Once $\tau \in \mathcal{T}_j$ has been chosen, the definition of $S_jv(p)$ for $p \in T_j$ is the same as in [19,47]. This construction guarantees the local stability bound [47]

$$h_p^{d/2}|S_jv(p)| \lesssim \|v\|_{0,\omega_p} \quad \forall p \in \mathcal{N}_j, \tag{7.3}$$

where the index 0 stands for the L^2 norm, and that the slicing operator $S_j - S_{j-1}$ is supported in the enlarged patch $\widetilde{\omega}_j$, namely

$$(S_j - S_{j-1})v \in V_j \quad \forall 1 \le j \le J. \tag{7.4}$$

We note that (7.3) and (7.4) are the only desired properties in Lemma 7.1. Other constructions of S_j (c.f. [31,52]) can also be applied.

Lemma 7.1 (Stable L^2-orthogonal decomposition). *Let $\widehat{Q}_k : V \to W_k$ be the L^2-orthogonal projection operator onto W_k and $\widehat{Q}_{-1} = 0$. For any $v \in V$, the global L^2-orthogonal decomposition $v = \sum_{k=0}^{\bar{J}}(\widehat{Q}_k - \widehat{Q}_{k-1})v$ satisfies*

$$\sum_{k=0}^{\bar{J}} \gamma^{-2sk}\|(\widehat{Q}_k - \widehat{Q}_{k-1})v\|_0^2 \lesssim |v|_s^2, \tag{7.5}$$

where the hidden constant is independent of $0 \le s \le 1$ and $\bar{J}$.

Proof. We rely on the auxiliary spaces $\overline{V}_k = \mathbb{V}(\overline{\mathcal{T}}_k)$ defined over uniformly refined meshes $\overline{\mathcal{T}}_k$ of $\mathcal{T}_0$ for $0 \le k \le \bar{J}$. Let $\overline{Q}_k : \overline{V}_{\bar{J}} \to \overline{V}_k$ denote the L^2-orthogonal projection operator onto $\overline{V}_k$ and consider the global L^2-orthogonal decomposition $v = \sum_{k=0} \bar{v}_k$ of any $v \in V \subset \overline{V}_{\bar{J}}$, where $\bar{v}_k := (\overline{Q}_k - \overline{Q}_{k-1})v$. This decomposition is stable in H^1 [12,45,54]

$$\sum_{k=0}^{\bar{J}} \gamma^{-2k}\|\bar{v}_k\|_0^2 \lesssim |v|_1^2.$$

If g_j is the generation of bisection b_j and $g_j > k$, then $\bar{v}_k$ is piecewise linear in ω_{e_j} (the patch of the refinement edge e_j), whence $(S_j - S_{j-1})\bar{v}_k = 0$ and the slicing operator detects frequencies $k \ge g_j$. Consider now the decomposition $v = \sum_{k=0}^{\bar{J}} v_k$ of $v \in V$ where

$$v_k := \sum_{g_j=k}(S_j - S_{j-1})v = \sum_{g_j=k}(S_j - S_{j-1})\sum_{\ell=k}^{\bar{J}}\bar{v}_\ell \in W_k. \tag{7.6}$$

In view of Lemma 5.2 (generation and patches) and the shape regularity of $\mathbb{T}(\mathcal{T}_0)$, enlarged patches $\widetilde{\omega}_j$ with the same generation $g_j = k$ have a finite overlapping property. This, in conjunction with (7.3) and (7.4) as well as the L^2-orthogonality of $\{\bar{v}_\ell\}_{\ell=k}^{\bar{J}}$, yields

$$\|v_k\|_0^2 \lesssim \sum_{g_j=k}\Big\|(S_j - S_{j-1})\sum_{\ell=k}^{\bar{J}}\bar{v}_\ell\Big\|_{0,\widetilde{\omega}_j}^2 \lesssim \sum_{g_j=k}\Big\|\sum_{\ell=k}^{\bar{J}}\bar{v}_\ell\Big\|_{0,\widetilde{\omega}_j}^2 \lesssim \Big\|\sum_{\ell=k}^{\bar{J}}\bar{v}_\ell\Big\|_0^2 = \sum_{\ell=k}^{\bar{J}}\|\bar{v}_\ell\|_0^2.$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

We use Lemma 3.3 (s-uniform decomposition) with $s=1$, together with (7.6), to obtain

$$\sum_{k=0}^{\bar J}\gamma^{-2k}\|(\widehat Q_k-\widehat Q_{k-1})v\|_0^2=\inf_{\substack{w_k\in W_k\\ \sum_{k=0}^{\bar J}w_k=v}}\Big[\gamma^{-2\bar J}\|w_{\bar J}\|_0^2+\sum_{k=0}^{\bar J-1}\frac{\gamma^{-2k}}{1-\gamma^2}\|w_k\|_0^2\Big]$$
$$\le\gamma^{-2\bar J}\|v_{\bar J}\|_0^2+\sum_{k=0}^{\bar J-1}\frac{\gamma^{-2k}}{1-\gamma^2}\|v_k\|_0^2.$$

Employing the preceding estimate of $\|v_k\|_0^2$ and reordering the sum implies

$$\sum_{k=0}^{\bar J}\gamma^{-2k}\|(\widehat Q_k-\widehat Q_{k-1})v\|_0^2\lesssim\gamma^{-2\bar J}\|\bar v_{\bar J}\|_0^2+\sum_{k=0}^{\bar J-1}\frac{\gamma^{-2k}}{1-\gamma^2}\sum_{\ell=k}^{\bar J}\|\bar v_\ell\|_0^2$$
$$\le\gamma^{-2\bar J}\|\bar v_{\bar J}\|_0^2+\sum_{\ell=0}^{\bar J}\sum_{k=0}^{\ell}\frac{\gamma^{-2k}}{1-\gamma^2}\|\bar v_\ell\|_0^2$$
$$=\gamma^{-2\bar J}\|\bar v_{\bar J}\|_0^2+\sum_{\ell=0}^{\bar J}\frac{\gamma^{-2\ell}-\gamma^2}{(1-\gamma^2)^2}\|\bar v_\ell\|_0^2$$
$$\lesssim\sum_{\ell=0}^{\bar J}\gamma^{-2\ell}\|\bar v_\ell\|_0^2\lesssim|v|_1^2.$$

Hence, we have shown that (7.5) holds for $s=1$. The desired estimate for arbitrary $0\le s\le1$ follows by Lemma 3.4 (s-uniform interpolation). □

As a consequence of Lemma 3.3 (s-uniform decomposition) and Lemma 7.1 (stable L^2-orthogonal decomposition), we deduce the following property.

Corollary 7.1 (s-Uniform decomposition on W_k)**.** *For every $v\in V$, there exists a decomposition $v=\sum_{k=0}^{\bar J}w_k$ with $w_k\in W_k$ for all $k=0,1,\dots,\bar J$ and*

$$\gamma^{-2s\bar J}\|w_{\bar J}\|_0^2+\sum_{k=0}^{\bar J-1}\frac{\gamma^{-2sk}}{1-\gamma^{2s}}\|w_k\|_0^2\lesssim|v|_s^2.$$

7.2. **Characterization of W_k.** We now study the geometric structure of the spaces W_k, defined in (7.2), which is useful in the construction of a stable decomposition of V. Recalling the definition of $\widehat{\mathcal T}_k$ in (5.5), our first goal is to compare W_k with the space

$$\widehat V_k:=\mathbb V(\widehat{\mathcal T}_k)$$

of C^0 piecewise linear functions over $\widehat{\mathcal T}_k$ that have vanishing trace. We will show below

$$\widehat V_k\subset W_k;\tag{7.7}$$

see Lemmas 7.3 and 7.4. We start with the set of interior vertices of W_k,

$$\mathcal V_k:=\mathcal B_k\cup\mathcal P_k,\qquad\mathcal B_k:=\bigcup\Big\{T_j:j\in\mathcal J_k\Big\},\quad\mathcal P_k=\Big\{p\in\mathcal P:\ell(p)\le k\Big\}.$$

Lemma 7.2 (Geometric structure of W_k)**.** *Functions in W_k are C^0 piecewise linear on the auxiliary mesh $\widehat{\mathcal T}_{k+k_*}$, where k_* is given in* (5.3)*. Equivalently, $W_k\subset\widehat V_{k+k_*}$.*

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. We examine separately each vertex $q \in \mathcal{V}_k$. If $q \in \mathcal{P}_k$, then $\ell(q) \leq k$ and all elements $\tau \in \mathcal{R}(q)$ have generation $g(\tau) \leq k$ by definition of level; hence $\tau \in \widehat{\mathcal{T}}_k$ for all $\tau \in \mathcal{R}(q)$. If $q \in \mathcal{B}_k \setminus \mathcal{P}_k$ instead, then the patch of q shares elements with that of the bisection node p_j

$$\min_{\tau \in \mathcal{R}_j(q)} g(\tau) \leq g(p_j) = g_j \leq k,$$

where $\mathcal{R}_j(q)$ is the ring of elements containing q in the mesh $\mathcal{T}_j$. Property (5.3) yields

$$\max_{\tau \in \mathcal{R}_j(q)} g(\tau) \leq \min_{\tau \in \mathcal{R}_j(q)} g(\tau) + k_* \leq k + k_*.$$

It turns out that all elements $\tau \in \widetilde{\mathcal{R}}_j$, the enlarged ring around p_j, have generation $g(\tau) \leq k + k_*$, whence $\tau \in \widehat{\mathcal{T}}_{k+k_*}$. It remains to realize that any function $w \in V_j$ is thus piecewise linear over $\widehat{\mathcal{T}}_{k+k_*}$ and vanishes outside $\widetilde{\omega}_j$. $\square$

We next exploit the L^2-stability of the *nodal basis* $\{\widehat{\phi}_q\}_{q \in \widehat{\mathcal{V}}}$ of $\widehat{V}_{k+k_*}$, where $\widehat{\mathcal{V}} = \widehat{\mathcal{V}}_{k+k_*}$ is the set of interior vertices of $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_{k+k_*}$. In fact, if $w = \sum_{q \in \widehat{\mathcal{V}}} w(q)\widehat{\phi}_q$, then

(7.8)
$$\|w\|_0^2 = \sum_{\tau \in \widehat{\mathcal{T}}} \|w\|_{0,\tau}^2 \simeq \sum_{\tau \in \widehat{\mathcal{T}}} |\tau| \sum_{q \in \tau} w(q)^2 = \sum_{q \in \widehat{\mathcal{V}}} w(q)^2 \sum_{\tau \ni q} |\tau| \simeq \sum_{q \in \widehat{\mathcal{V}}} w(q)^2 \|\widehat{\phi}_q\|_0^2.$$

Our goal now is to represent each function $\widehat{\phi}_q \in \widehat{V}_{k+k_*}$ in terms of functions of W_{k+k_*}, which in turn shows $\widehat{V}_{k+k_*} \subset W_{k+k_*}$ and thus (7.7). We start with a partition of $\widehat{\mathcal{V}}_{k+k_*}$,

$$\widehat{\mathcal{P}}_{k+k_*} := \{q \in \widehat{\mathcal{V}}_{k+k_*} : \widehat{\ell}(q) \leq k + k_* - 1\}, \quad \widehat{\mathcal{P}}^c_{k+k_*} := \widehat{\mathcal{V}}_{k+k_*} \setminus \widehat{\mathcal{P}}_{k+k_*},$$

where $\widehat{\ell}(q) \leq k + k_*$ is the level of q on $\widehat{\mathcal{T}}_{k+k_*}$. Consequently, $\widehat{\ell}(q) = k + k_*$ for all $q \in \widehat{\mathcal{P}}^c_{k+k_*}$ and the corresponding functions $\widehat{\phi}_q$ have all the same scaling due to shape regularity of $\mathbb{T}(\mathcal{T}_0)$. In Lemmas 7.3 and 7.4 we represent the functions $\widehat{\phi}_q$ in terms of W_{k+k_*}.

Lemma 7.3 (Nodal basis $\widehat{\phi}_q$ with $q \in \widehat{\mathcal{P}}_{k+k_*}$)**.** *For any $q \in \widehat{\mathcal{P}}_{k+k_*}$, there holds*

$$\widehat{\phi}_q = \phi_q \quad q \in \mathcal{P}_{k+k_*-1},$$

where $\mathcal{P}_k$ is defined in (7.1)*; hence, $\widehat{\phi}_q \in W_{k+k_*-1}$.*

Proof. Since $\widehat{\ell}(q) \leq k+k_*-1$, all elements $\tau \in \mathcal{R}(q)$ have generation $g(\tau) \leq k+k_*-1$. This implies that no further bisection is allowed in τ because all the bisections with generation less than or equal to $k + k_*$ have been incorporated in $\widehat{\mathcal{T}}_{k+k_*}$ by definition. Therefore, $\mathcal{R}(q)$ belongs to the finest grid $\mathcal{T}$ and $\ell(q) = \widehat{\ell}(q) \leq k+k_*-1$, whence $\widehat{\phi}_q \in W_{k+k_*-1}$. $\square$

Next, we consider a nodal basis function $\widehat{\phi}_q$ corresponding to $q \in \widehat{\mathcal{P}}^c_{k+k_*}$. There exists a bisection triplet T_{j_q} that contains q and $k \leq \ell_{j_q}(q) \leq k + k_*$, for otherwise $\ell_{j_q}(q) < k$ would violate Lemma 5.3 (levels of a vertex). We thus deduce

$$k - k_* \leq \ell_{j_q}(q) - k_* \leq g_{j_q} \leq \ell_{j_q}(q) \leq k + k_*. \tag{7.9}$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

In accordance with (5.6), we denote by $\phi_{j_q,q}$ the nodal basis function of V_{j_q} centered at q. We next show that $\widehat{\phi}_q$ can be obtained by a suitable modification of $\phi_{j_q,q}$ within W_{k+k_*}.

Lemma 7.4 (Nodal basis $\widehat{\phi}_q$ with $q \in \widehat{\mathcal{P}}^c_{k+k_*}$)**.** *For any $q \in \widehat{\mathcal{P}}^c_{k+k_*}$, let*

$$\mathcal{S}_q := \{j \in \mathcal{J}_{k+k_*} : \ j > j_q, \ \omega_j \cap \operatorname{supp} \phi_{j_q,q} \neq \varnothing\}$$

be the set of bisection indices $j > j_q$ such that $g_j \leq k + k_*$, ϕ_{j,p_j} be the function of V_j centered at the bisection vertex p_j and $\omega_j = \operatorname{supp} p_j$. Then there exist numbers $c_{j,q} \in (-1, 0]$ for $j \in \mathcal{S}_q$ such that the nodal basis function $\widehat{\phi}_q \in V_{k+k_*}$ associated with q can be written as

$$\widehat{\phi}_q = \phi_{j_q,q} + \sum_{j\in\mathcal{S}_q} c_{j,q}\phi_{j,p_j}, \tag{7.10}$$

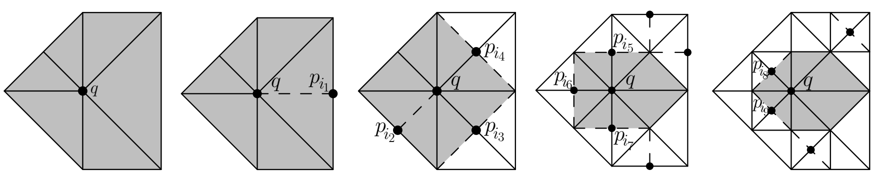
and the representation is L^2-stable, i.e.,

$$\|\widehat{\phi}_q\|_0^2 \simeq \|\phi_{j_q,q}\|_0^2 + \sum_{j\in\mathcal{S}_q} c_{j,q}^2 \|\phi_{j,p_j}\|_0^2. \tag{7.11}$$

Proof. The discussion leading to (7.9) yields $k \leq \ell_{j_q}(q) \leq k + k_*$ which, combined with (5.3), implies that all elements $\tau \in \mathcal{R}_{j_q}(q)$ have generation between $k-k_*$ and $k + k_*$. The idea now is to start from the patch $\mathcal{R}_{j_q}(q)$, the local conforming mesh associated with $\phi_{j_q,q}$, and successively refine it with compatible bisections in the spirit of the construction of $\widehat{\mathcal{T}}_j$ in (5.5) until we reach the level $k + k_*$; see Figure 7.1. To this end, let $\widehat{\mathcal{T}}_{k-k_*}(q) := \mathcal{R}_{j_q}(q)$ and consider the sequence of local auxiliary meshes

$$\widehat{\mathcal{T}}_j(q) := \widehat{\mathcal{T}}_{j-1}(q) + \{b_i \in \mathcal{B} : \ i \in \mathcal{S}_q, \ g_i = j\} \quad k - k_* + 1 \leq j \leq k + k_*,$$

which are conforming according to Lemma 5.4 (conformity of $\widehat{\mathcal{T}}_j$).



(A) $\widehat{\mathcal{T}}_{k-2,q}$ (B) $\widehat{\mathcal{T}}_{k-1,q}$ (C) $\widehat{\mathcal{T}}_{k,q}$ (D) $\widehat{\mathcal{T}}_{k+1,q}$ (E) $\widehat{\mathcal{T}}_{k+2,q}$

FIGURE 7.1. Local auxiliary meshes $\widehat{\mathcal{T}}_{j,q}$ with $|j - k| \leq k_* = 2$. Index sets $\mathcal{S}_{k-1,q} = \{i_1\}$, $\mathcal{S}_{k,q} = \{i_2, i_3, i_4\}$, $\mathcal{S}_{k+1,q} = \{i_5, i_6, i_7\}$, $\mathcal{S}_{k+2,q} = \{i_8, i_9\}$ of compatible bisections to transition from $\widehat{\phi}_{j-1,q}$ to $\widehat{\phi}_{j,q}$. The support of $\widehat{\phi}_{j,q}$ is monotone decreasing as j increases and is plotted in grey.

We now consider the following recursive procedure: let $\widehat{\phi}_{k-k_*,q} := \phi_{j_q,q}$ and

$$\widehat{\phi}_{j,q} := \widehat{\phi}_{j-1,q} - \sum_{i\in\mathcal{S}_{j,q}} \widehat{\phi}_{j-1,q}(p_i)\phi_{i,p_i} \quad k - k_* + 1 \leq j \leq k + k_*, \tag{7.12}$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where p_i is the bisection node of $b_i \in \mathcal{B}$ and

$$\mathcal{S}_{j,q} := \Big\{ i \in \mathcal{J}_{k+k_*} : \ g_i = j, \ \omega_i \cap \operatorname{supp} \widehat{\phi}_{j-1,q} \neq \varnothing \Big\}.$$

Unless p_i belongs to the boundary of $\operatorname{supp} \widehat{\phi}_{j-1,q}$, the construction (7.12) always modifies $\widehat{\phi}_{j-1,q}$; compare Figure 7.1(b) with Figure 7.1(c)–7.1(e). In view of Lemma 5.2 (generation and patches), the sets $\mathring{\omega}_i$ for $i \in \mathcal{S}_{j,q}$ are disjoint, whence $\widehat{\phi}_{j,q}(p) = \delta_{pq}$ for all nodes p of $\widehat{\mathcal{T}}_j(q)$ and $\widehat{\phi}_{j,q}$ is the nodal basis function centered at q on $\widehat{\mathcal{T}}_j(q)$. Moreover,

$$\widehat{\phi}_{j,q} = \widehat{\phi}_{j-1,q} + \sum_{i \in \mathcal{S}_{j,q}} c_{i,q} \phi_{i,p_i}$$

with coefficients $c_{i,q} \in (-1, 0]$. Notice that $\|\widehat{\phi}_{j,q}\|_0 \simeq \|\widehat{\phi}_{j-1,q}\|_0 \simeq \|\phi_{i,p_i}\|_0$ due to the shape regularity, the scales of these functions being comparable yields

$$\|\widehat{\phi}_{j,q}\|_0^2 \simeq \|\widehat{\phi}_{j-1,q}\|_0^2 + \sum_{i \in \mathcal{S}_{j,q}} c_{i,q}^2 \|\phi_{i,p_i}\|_0^2.$$

Since k_* is uniformly bounded depending on shape regularity of $\mathbb{T}(\mathcal{T}_0)$, iterating these two expressions at most $2k_*$ times leads to (7.10) and (7.11), and concludes the proof. □

We are now in a position to exploit the representation of the nodal basis of $\widehat{V}_{k+k_*}$, given in Lemmas 7.3 and 7.4, to decompose functions in W_k. We do this next.

Corollary 7.2 (L^2-stable decomposition of W_k). *Given any $0 \leq k \leq \bar{J}$ consider the sets*

(7.13)

$$\mathcal{P}_{k+k_*} = \{ q \in \mathcal{P} : \ \ell(q) \leq k + k_* \}, \quad \mathcal{I}_{k+k_*} = \{ 0 \leq i \leq J : \ k - k_* \leq g_i \leq k + k_* \}.$$

Then, every function $w \in W_k$ admits an L^2-stable decomposition

$$w = \sum_{q \in \mathcal{P}_{k+k_*}} w_q + \sum_{j \in \mathcal{I}_{k+k_*}} w_j, \quad \|w\|_0^2 \simeq \sum_{q \in \mathcal{P}_{k+k_*}} \|w_q\|_0^2 + \sum_{j \in \mathcal{I}_{k+k_*}} \|w_j\|_0^2, \tag{7.14}$$

where $w_q \in V_q$ for all $q \in \mathcal{P}_{k+k_*}$ and $w_j \in V_j$ for all $j \in \mathcal{I}_{k+k_*}$.

Proof. Invoking Lemma 7.2 (geometric structure of W_k), we infer that $w \in \widehat{V}_{k+k_*}$, which yields the L^2-stable decomposition of w in terms of nodal basis of $\widehat{V}_{k+k_*}$

$$w = \sum_{q \in \widehat{\mathcal{V}}_{k+k_*}} w(q)\widehat{\phi}_q = \sum_{q \in \widehat{\mathcal{P}}_{k+k_*}} w(q)\widehat{\phi}_q + \sum_{q \in \widehat{\mathcal{P}}_{k+k_*}^c} w(q)\widehat{\phi}_q.$$

On the one hand, Lemma 7.3 (nodal basis $\widehat{\phi}_q$ with $q \in \widehat{\mathcal{P}}_{k+k_*}$) implies that $\widehat{\phi}_q = \phi_q$ and $\widehat{\mathcal{P}}_{k+k_*} \subset \mathcal{P}_{k+k_*}$; hence we simply take $w_q := w(q)\phi_q$. On the other hand, using the representation (7.10) of $\widehat{\phi}_q$ from Lemma 7.4 (nodal basis $\widehat{\phi}_q$ with $q \in \widehat{\mathcal{P}}_{k+k_*}^c$) and reordering, we arrive at

$$\sum_{q \in \widehat{\mathcal{P}}_{k+k_*}^c} w(q)\widehat{\phi}_q = \sum_{q \in \widehat{\mathcal{P}}_{k+k_*}^c} w(q) \Big(\phi_{j_q,q} + \sum_{j \in \mathcal{S}_q} c_{j,q} \phi_{j,p_j} \Big) = \sum_{j \in \mathcal{I}_{k+k_*}} w_j,$$

where

$$w_j := \sum_{q : j_q = j} w(q) \phi_{j,q} + \sum_{q : \mathcal{S}_q \ni j} w(q)\, c_{j,q} \phi_{j,p_j} \in V_j.$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

This gives the decomposition (7.14). The L^2-stability (7.8) of $\{\widehat{\phi}_q\}_{q\in\widehat{\mathcal{V}}_{k+k_*}}$

$$\|w\|_0^2 \simeq \sum_{q\in\widehat{\mathcal{P}}_{k+k_*}} w(q)^2\|\widehat{\phi}_q\|_0^2 + \sum_{q\in\widehat{\mathcal{P}}^c_{k+k_*}} w(q)^2\|\widehat{\phi}_q\|_0^2,$$

in conjunction with (7.11), gives

$$\begin{aligned}\|w\|_0^2 &\simeq \sum_{q\in\widehat{\mathcal{P}}_{k+k_*}} \|w(q)\widehat{\phi}_q\|_0^2 + \sum_{q\in\widehat{\mathcal{P}}^c_{k+k_*}} w^2(q)\Big(\|\phi_{j_q,q}\|_0^2 + \sum_{j\in\mathcal{S}_q} c^2_{j,q}\|\phi_{j,p_j}\|_0^2\Big)\\ &= \sum_{q\in\mathcal{P}_{k+k_*}} \|w_q\|_0^2 + \sum_{j\in\mathcal{I}_{k+k_*}} \Big(\sum_{j_q=j} w^2(q)\|\phi_{j,q}\|_0^2 + \sum_{\mathcal{S}_q\ni j} w^2(q)c^2_{j,q}\|\phi_{j,p_j}\|_0^2\Big).\end{aligned}$$

To prove the L^2-stability in (7.14), it remains to show that the term in parentheses is equivalent to $\|w_j\|_0^2$ for any $j\in\mathcal{I}_{k+k_*}$, which in turn is a consequence of the number of summands being bounded uniformly. We first observe that the cardinality of $\{q : j_q = j\}$ is at most three because this corresponds to $q\in T_j$, the j-th bisection triplet. Finally, the cardinality of the set $\{q\in\widehat{\mathcal{P}}^c_{k+k_*} : j\in\mathcal{S}_q\cap\mathcal{I}_{k+k_*}\}$ is bounded uniformly by a constant that depends solely on shape regularity of $\mathbb{T}(\mathcal{T}_0)$. To see this, note that $\widehat{\ell}(q) = k+k_*$ yields $k\le g(\tau)\le k+k_*$ for all elements τ within $\operatorname{supp}\phi_{j_q,q}$ and $k-k_*\le g_j\le k+k_*$, whence the number of vertices q such that $\operatorname{supp}\phi_{j_q,q}\cap\omega_j\neq\varnothing$ is uniformly bounded as asserted. Hence

$$\|w_j\|_0^2 \simeq \sum_{j_q=j} w^2(q)\|\phi_{j,q}\|_0^2 + \sum_{\mathcal{S}_q\ni j} w^2(q)c^2_{j,q}\|\phi_{j,p_j}\|_0^2$$

yields the norm equivalence in (7.14) and finishes the proof. ☐

7.3. **Construction of stable decomposition.** We first construct a BPX preconditioner that hinges on the space decomposition of Section 7.1 and the nodal basis functions just discussed in Section 7.2. We next show that this preconditioner is equivalent to (5.9).

Theorem 7.1 (Stable decomposition on graded bisection grids)**.** *For every $v\in V$, there exist $v_p\in V_p$ with $p\in\mathcal{P}$, $v_{p,k}\in V_p$ with $p\in\mathcal{P}_{k+k_*}$, and $v_{j,k}\in V_j$ with $j\in\mathcal{I}_{k+k_*}$, such that*

$$v = \sum_{p\in\mathcal{P}} v_p + \sum_{k=0}^{\bar{J}}\Bigg(\sum_{q\in\mathcal{P}_{k+k_*}} v_{q,k} + \sum_{j\in\mathcal{I}_{k+k_*}} v_{j,k}\Bigg), \tag{7.15}$$

where $\mathcal{P}_{k+k_*}$ and $\mathcal{I}_{k+k_*}$ are given in (7.13)*, and there exists a constant c_0 independent of $s\in(0,1]$ and J such that*

$$\gamma^{-2s\bar{J}}\sum_{p\in\mathcal{P}}\|v_p\|_0^2 + \sum_{k=0}^{\bar{J}}\frac{\gamma^{-2sk}}{1-\gamma^{2s}}\Bigg(\sum_{p\in\mathcal{P}_{k+k_*}}\|v_{p,k}\|_0^2 + \sum_{j\in\mathcal{I}_{k+k_*}}\|v_{j,k}\|_0^2\Bigg) \le c_0|v|_s^2. \tag{7.16}$$

Proof. We construct the decomposition (7.15) in three steps.

Step 1 (Decomposition on W_k). Applying Corollary 7.1 (s-uniform decomposition on W_k), we observe that there exist $w_k\in W_k$, $k=0,1,\cdots,\bar{J}$ such that $v=$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$\sum_{k=0}^{\bar{J}} w_k$ and

$$\gamma^{-2s\bar{J}}\|w_{\bar{J}}\|_0^2 + \sum_{k=0}^{\bar{J}-1} \frac{\gamma^{-2sk}}{1-\gamma^{2s}}\|w_k\|_0^2 \lesssim |v|_s^2. \tag{7.17}$$

Step 2 (Finest scale). We let $\{\phi_p\}_{p\in\mathcal{P}}$ be the nodal basis of V and set $v_p := w_{\bar{J}}(p)\phi_p$; hence $w_{\bar{J}} = \sum_{p\in\mathcal{P}} v_p$. Applying the L^2-stability (7.8) to $\{\phi_p\}_{p\in\mathcal{P}}$ gives

$$\|w_{\bar{J}}\|_0^2 \simeq \sum_{p\in\mathcal{P}} \|v_p\|_0^2. \tag{7.18}$$

We also choose the finest scale of $v_{p,k}$ and $v_{j,k}$ to be $v_{q,\bar{J}} = 0$ and $v_{j,\bar{J}} = 0$.

Step 3 (Intermediate scales). By Corollary 7.2 (L^2-stable decomposition of W_k), we have the L^2-stable decomposition (7.14) of $w_k \in W_k$ for every $k = 0, \dots, \bar{J}-1$. Combining the stability bound (7.17) with (7.18) and (7.14), we deduce the stable decomposition (7.16).

□

In view of Theorem 7.1, we consider the BPX preconditioner

$$\widehat{B} := \gamma^{2s\bar{J}} \sum_{p\in\mathcal{P}} I_p Q_p + (1-\gamma^{2s}) \sum_{k=0}^{\bar{J}} \gamma^{2sk} \left(\sum_{p\in\mathcal{P}_{k+k_*}} I_p Q_p + \sum_{j\in\mathcal{I}_{k+k_*}} I_j Q_j \right). \tag{7.19}$$

Corollary 7.3 is a direct consequence of (7.16) and (3.4).

Corollary 7.3 (Uniform bound for $\lambda_{\min}(\widehat{B}A)$)**.** *The preconditioner $\widehat{B}$ in* (7.19) *satisfies*

$$\lambda_{\min}(\widehat{B}A) \geq c_0^{-1}.$$

We are now ready to prove the main result of this section, namely that B in (5.9) is a robust preconditioner for A on graded bisection grids. To this end, we need to show that $\widehat{B}$ in (7.19) is spectrally equivalent to B.

Theorem 7.2 (Uniform preconditioning on graded bisection grids)**.** *Let Ω be a bounded Lipschitz domain and $s \in [0,1]$. Assume the extended patch $\widetilde{S}_\tau$ defined in* (2.12) *is Lipschitz for every $\tau \in \mathcal{T}_j$ with a uniform Lipschitz constant. Let V be the space of continuous piecewise linear finite elements over a graded bisection grid $\mathcal{T}$, and consider the space decomposition* (5.8)*. The corresponding BPX preconditioner B in* (5.9)*, namely*

$$B = \sum_{p\in\mathcal{P}} I_p h_p^{2s} Q_p + (1-\widetilde{\gamma}^s) \sum_{j=0}^{J} I_j h_j^{2s} Q_j,$$

is spectrally equivalent to $\widehat{B}$ in (7.19)*, whence $\lambda_{\min}(BA) \gtrsim c_0^{-1}$. Therefore, the condition number of BA satisfies*

$$\operatorname{cond}(BA) \lesssim c_0 c_1,$$

where the constants c_0 and c_1, given in (7.16) *and* (6.1)*, are independent of s and mesh-parameters except for the shape-regularity constant.*

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. We show that the ratio $\frac{(Bv,v)}{(\widehat{B}v,v)}$ is bounded below and above by constants independent of s and J for all $v \in V$. We first observe that for $p \in \mathcal{P}$ with level $\ell(p)$, we have $h_p \simeq \gamma^{\ell(p)}$. Then,

$$h_p^{2s} \simeq \gamma^{2s\ell(p)} = \gamma^{2s(\bar{J}+1)} + (1-\gamma^{2s}) \sum_{k=\ell(p)}^{\bar{J}} \gamma^{2sk},$$

whence $B_1 := \sum_{p\in\mathcal{P}} I_p h_p^{2s} Q_p$ and $v_p = Q_p v$ satisfy

$$(B_1 v, v) \simeq \gamma^{2s\bar{J}} \sum_{p\in\mathcal{P}} \|v_p\|_0^2 + (1-\gamma^{2s}) \sum_{p\in\mathcal{P}} \sum_{k=\ell(p)}^{\bar{J}} \gamma^{2sk} \|v_p\|_0^2.$$

The rightmost sum can be further decomposed as follows:

$$\begin{aligned}
\sum_{p\in\mathcal{P}} \sum_{k=\ell(p)}^{\bar{J}} \gamma^{2sk} \|v_p\|_0^2 &= \sum_{j=0}^{\bar{J}} \sum_{\ell(p)=j} \sum_{k=j}^{\bar{J}} \gamma^{2sk} \|v_p\|_0^2 \\
&= \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{\ell(p)\le k} \|v_p\|_0^2 \le \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{\ell(p)\le k+k_*} \|v_p\|_0^2 \\
&= \gamma^{-2sk_*} \sum_{k=0}^{\bar{J}} \gamma^{2s(k+k_*)} \sum_{\ell(p)\le k+k_*} \|v_p\|_0^2 \\
&\le \gamma^{-2sk_*} \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{\ell(p)\le k} \|v_p\|_0^2.
\end{aligned}$$

Since $\gamma^{-2sk_*} \simeq 1$, there exist equivalence constants independent of s and J such that

$$(B_1 v, v) \simeq \gamma^{2s\bar{J}} \sum_{p\in\mathcal{P}} \|v_p\|_0^2 + (1-\gamma^{2s}) \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{p\in\mathcal{P}_{k+k_*}} \|v_p\|_0^2. \tag{7.20}$$

We now consider the bisection triplets T_j and 3-dimensional spaces V_j, for which $h_j \simeq \gamma^{g_j}$. We let $\widehat{B}_2 := \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{j\in\mathcal{I}_{k+k_*}} I_j Q_j$, $B_2 := \sum_{j=0}^{J} I_j h_j^{2s} Q_j$ and $v_j := Q_j v$, to write

$$\begin{aligned}
(\widehat{B}_2 v, v) &= \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{k-k_*\le g_j\le k+k_*} \|v_j\|_0^2 \\
&= \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{i=-k_*}^{i=k_*} \gamma^{2si} \sum_{g_j=k} \|v_j\|_0^2 \\
&\simeq \sum_{k=0}^{\bar{J}} \gamma^{2sk} \sum_{g_j=k} \|v_j\|_0^2 = \sum_{j=1}^{J} \gamma^{2sg_j} \|v_j\|_0^2 \simeq (B_2 v, v),
\end{aligned} \tag{7.21}$$

because $\sum_{i=-k_*}^{i=k_*} \gamma^{2si} \simeq 1$ due to the fact that k_* is a fixed integer depending solely on shape regularity of $\mathbb{T}(\mathcal{T}_0)$. Combining (7.20) and (7.21) we obtain

$$(Bv, v) = (B_1 v, v) + (1-\widetilde{\gamma}^s)(B_2 v, v) \simeq (\widehat{B}v, v) \quad \forall v \in V,$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

whence the operators B and $\widehat{B}$ are spectrally equivalent. Invoking Corollary 7.3 (uniform bound for $\lambda_{\min}(\widehat{B}A)$), we readily deduce $\lambda_{\min}(BA) \gtrsim c_0^{-1}$. We finally recall that $\lambda_{\max}(BA) \lesssim c_1$, according to Proposition 6.1 (boundedness), to infer the desired uniform bound $\operatorname{cond}(BA) = \lambda_{\max}(BA)\lambda_{\min}(BA)^{-1} \lesssim c_0 c_1$. □

8. Numerical experiments

This section presents some experiments with conjugate gradient and BPX preconditioners (4.2) for quasi-uniform grids and (5.9) for graded bisection grids, whose main difference with BPX for the classical Laplacian is the scaling factor $1-\widetilde{\gamma}^s$ for coarse levels. Therefore, if $N = \dim V$ denotes the number of degrees of freedom of the finest space $V = \mathbb{V}(\mathcal{T}_J)$, the computational cost for applying (4.2) and (5.9) is CN with a modest constant C and is comparable with the classical Laplacian. However, a key difference is that the stiffness matrix A is dense and a matrix-vector product requires N^2 operations. The effect of BPX is thus to limit such matrix-vector products to a fixed number regardless of s and J. However, to reduce the total computational cost to log-linear in N requires further sparsification of A.

In the sequel, we consider the Dirichlet integral fractional Laplacian (1.1) in $\Omega = (-1,1)^2$ with various fractional powers and examine robustness of the BPX preconditioners.

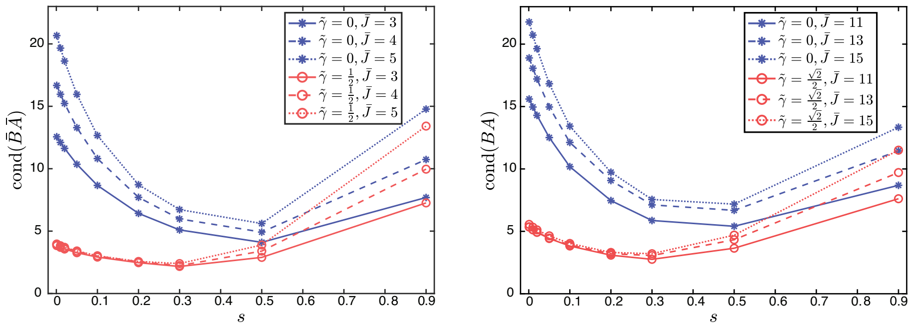
8.1. **Uniform grids.** We first perform computations on a family of nested, uniformly refined meshes. Table 2 lists the condition numbers obtained upon applying the standard BPX preconditioner (4.4) (i.e. $\widetilde{\gamma} = 0$) and the s-uniform BPX preconditioner (4.2) (i.e. $\widetilde{\gamma} > 0$). Limited by computational capacity, the largest $\bar{J}$ we take in our computations is 6, which corresponds to number of degrees of freedom $N = 16129$. In Figure 8.1(a), we plot the condition numbers vs. s for both standard and s-uniform BPX preconditioners for quasi-uniform grids. Even though this is a small-scale problem, the s-uniform BPX preconditioner (4.2) performs better than the standard one, especially when the fractional power s is small.

Table 2. Condition numbers $\operatorname{cond}(\overline{BA})$: Non-preconditioned system ($\kappa(\overline{A})$), standard BPX preconditioner ($\widetilde{\gamma} = 0$), and s-uniform BPX preconditioner ($\widetilde{\gamma} = \frac{1}{2}$)

$\bar{J}$	$h_{\bar{J}}$	N	$s=0.9$			$s=0.5$			$s=0.1$		
			$\kappa(\overline{A})$	$\tilde{\gamma}=0$	$\tilde{\gamma}=\frac{1}{2}$	$\kappa(\overline{A})$	$\tilde{\gamma}=0$	$\tilde{\gamma}=\frac{1}{2}$	$\kappa(\overline{A})$	$\tilde{\gamma}=0$	$\tilde{\gamma}=\frac{1}{2}$
1	2^{-1}	9	4.68	2.95	2.98	1.83	2.22	1.63	1.90	3.25	1.99
2	2^{-2}	49	17.07	5.21	4.90	3.49	3.20	2.35	2.44	6.10	2.62
3	2^{-3}	225	59.94	7.69	7.26	6.96	4.10	2.90	2.68	8.66	2.92
4	2^{-4}	961	209.12	10.74	9.97	13.94	4.92	3.40	2.75	10.81	3.00
5	2^{-5}	3969	728.66	14.78	13.42	27.93	5.61	3.89	2.77	12.66	3.03
6	2^{-6}	16129	2538.1	20.44	18.04	55.93	6.22	4.37	2.78	14.28	3.03

8.2. **Graded bisection grids.** We next consider graded bisection grids. As described in Remark 1, the graded grids are required to obtain better convergence rates. In order to obtain the mesh grading (2.11) of [1] when using bisection grids,

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use



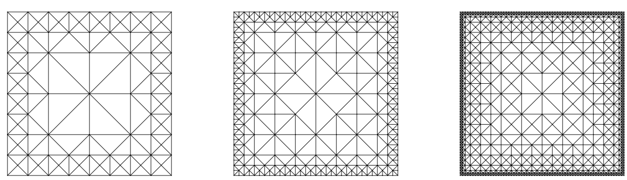
(A) quasi-uniform grids (B) graded bisection grids

FIGURE 8.1. Condition numbers vs. s for the standard BPX preconditioner ($\widetilde{\gamma} = 0$) and the s-uniform BPX preconditioners ($\widetilde{\gamma} = 1/2$ and $\widetilde{\gamma} = \sqrt{2}/2$)

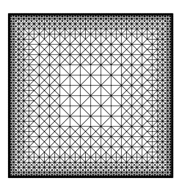
we consider the following strategy. Given an element $\tau \in \mathcal{T}$, let x_τ be its barycenter. Our strategy is based on choosing a number $\theta > 1$ and marking those elements τ such that

$$|\tau| > \theta N^{-1} \log N \cdot d(x_\tau, \partial\Omega)^{2(\mu-1)/\mu}. \tag{8.1}$$

We use the newest vertex bisection algorithm. Figure 8.2 displays graded bisection grids obtained with (8.1) and $\theta = 4$, $\mu = 2$, the latter being optimal for $d = 2$ [1].



(A) $\bar{J} = 6$ (B) $\bar{J} = 9$ (C) $\bar{J} = 12$



(D) $\bar{J} = 15$

FIGURE 8.2. Graded bisection grids on $(-1, 1)^2$, using strategy (8.1) with $\theta = 4$ and $\mu = 2$

We report condition numbers cond(BA) over graded bisection grids in Table 3. Note that the condition number $\kappa(A)$ of A could be relatively large for small s due to the factor $h_{\max} h_{\min}^{-1}$ in (1.6); this could be cured by diagonal scaling B as

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

documented in Table 3. The latter also shows that the s-uniform BPX preconditioner (5.9) performs well for a wide range of s. This is further confirmed by Figure 8.1(b). We also observe that diagonal scaling outperforms the preconditioner (5.9) for $s = 0.1$ and gives rise to condition numbers that seem to be independent of N. Such a phenomenon is not expected from (1.6), probably due to the grids not being refined enough since $N^{2s/d} \approx 2.496$ for the finest test ($\bar{J} = 18$). Another possibility is that formula (1.6) is not sharp for these special graded bisection grids. Note that (1.6) would also suggest that $\kappa(A) \lesssim N^{-1}$ for all s, because $h_{\min} \approx h_{\max}^2 \approx N$ and $d = 2$, but Table 3 shows sensitivity of $\kappa(A)$ with respect to s.

TABLE 3. Condition numbers cond(BA): Non-preconditioned system ($\kappa(A)$), diagonal scaling (diag), standard BPX preconditioner ($\tilde{\gamma} = 0$), and s-uniform BPX preconditioner (5.9) with $\tilde{\gamma} = \sqrt{2}/2$

$\bar{J}$	N	$s = 0.9$				$s = 0.5$				$s = 0.1$			
		$\kappa(A)$	diag	$\tilde{\gamma} = 0$	$\frac{\sqrt{2}}{2}$	$\kappa(A)$	diag	$\tilde{\gamma} = 0$	$\frac{\sqrt{2}}{2}$	$\kappa(A)$	diag	$\tilde{\gamma} = 0$	$\frac{\sqrt{2}}{2}$
7	61	9.14	8.38	4.64	4.55	6.21	2.55	2.88	2.42	15.87	2.76	5.54	3.20
8	153	14.22	12.83	5.95	5.67	9.15	3.21	3.76	2.94	31.52	2.81	7.13	3.75
9	161	15.63	14.90	6.89	6.27	9.29	3.44	4.21	3.18	29.82	2.74	8.06	3.67
10	369	24.69	22.84	8.06	7.03	13.75	4.36	5.03	3.51	60.72	2.78	9.09	3.78
11	405	31.30	30.42	8.69	7.61	13.56	4.91	5.41	3.65	55.66	2.71	10.18	3.83
12	853	40.48	37.52	10.12	8.72	26.53	5.88	6.31	4.14	193.58	2.77	10.80	3.86
13	973	60.90	54.95	11.45	9.71	20.59	6.77	6.68	4.33	114.42	2.71	12.12	3.91
14	1921	74.38	69.93	12.31	10.58	37.76	7.00	7.12	4.72	361.86	2.77	12.52	4.07
15	2265	111.31	100.94	13.34	11.50	30.13	9.44	7.18	4.70	213.77	2.72	13.43	4.03
16	4269	140.96	133.64	14.65	12.50	53.59	11.19	7.70	5.10	675.54	2.75	14.14	4.16
17	5157	206.53	187.62	16.13	13.76	43.56	13.33	7.69	5.04	407.64	2.72	14.79	4.10
18	9397	274.17	250.31	18.04	15.19	79.67	15.79	8.29	5.42	1390.3	2.75	15.53	4.17

9. SPECTRAL AND CENSORED LAPLACIANS

The spectral and censored Laplacians are useful variants of (1.3) in practice. We finally show that our preconditioners (4.2) and (5.9) are effective for these two operators as well because of their spectral equivalence to the integral fractional Laplacian.

We recall that the eigenpairs of the Laplacian $-\Delta$ with homogeneous Dirichlet condition on $\partial\Omega$ are denoted by $\{\widehat{\lambda}_k, \widehat{\varphi}_k\}_{k=1}^{\infty}$ and consider the space

$$\widehat{H}^s(\Omega) := \Big\{ v = \sum_{k=1}^{\infty} v_k \widehat{\varphi}_k \in L^2(\Omega) : \quad |v|^2_{\widehat{H}^s(\Omega)} = \sum_{k=1}^{\infty} \widehat{\lambda}_k^s v_k^2 < \infty \Big\},$$

which coincides with $\widetilde{H}^s(\Omega)$ and has equivalent norms according to (2.7). However, these norms induce different fractional operators. Minima of the functional $v \mapsto \frac{1}{2}|v|^2_{\widehat{H}^s(\Omega)} - \int_\Omega fv$ are weak solutions of the *spectral* fractional Laplacian in Ω with homogeneous Dirichlet condition for $0 < s < 1$, whose eigenpairs are $(\widehat{\lambda}_k^s, \widehat{\varphi}_k)_{k=1}^{\infty}$.

In contrast, let us consider the eigenvalue problem for the integral fractional Laplacian (1.1) with homogeneous Dirichlet condition,

$$\begin{cases} (-\Delta)^s u_k^{(s)} &= \mu_k^{(s)} u_k^{(s)} \quad \text{in } \Omega, \\ \qquad u_k^{(s)} &= 0 \qquad\quad\;\; \text{in } \Omega^c. \end{cases}$$

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

It is well-known that there exists an infinite sequence of eigenvalues $0 < \mu_1^{(s)} < \mu_2^{(s)} \leq \ldots$ with $\mu_k^{(s)} \to \infty$, and the following equivalence is derived in [22]

$$C(\Omega)\widehat{\lambda}_k^s \leq \mu_k^{(s)} \leq \widehat{\lambda}_k^s, \quad k \in \mathbb{N}.$$

There is yet a third family of fractional Sobolev spaces, namely $H_0^s(\Omega)$, which are the completion of $C_0^\infty(\Omega)$ with the L^2-norm plus the usual Gagliardo H^s-seminorm

$$|v|_{H^s(\Omega)}^2 = C(d,s) \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}} dxdy. \tag{9.1}$$

If Ω is Lipschitz, it turns out that $H_0^s(\Omega) = \widetilde{H}^s(\Omega)$ for all $0 < s < 1$ such that $s \neq \frac{1}{2}$; in the latter case $\widetilde{H}^{\frac{1}{2}}(\Omega) = H_{00}^{\frac{1}{2}}(\Omega)$ is the so-called Lions-Magenes space [38]. The seminorm (9.1) is a norm equivalent to $|\cdot|_s$ for $s \in (\frac{1}{2}, 1)$ but not for $s \in (0, \frac{1}{2}]$; note that $1 \in H_0^s(\Omega)$ and $|1|_{H^s(\Omega)} = 0$ for $s \in (0, \frac{1}{2}]$. Functions in $H_0^s(\Omega)$ for $s \in (\frac{1}{2}, 1)$ admit a trace on $\partial\Omega$ and minima of the functional $v \mapsto \frac{1}{2}|v|_{H^s(\Omega)}^2 - \int_\Omega fv$ are weak solutions of the *censored* fractional Laplacian, which reads as (1.1) but with integration over Ω instead of $\mathbb{R}^d$. In addition,

$$|v|_{H^s(\Omega)} \leq |v|_{\widetilde{H}^s(\Omega)} \leq C|v|_{H^s(\Omega)}, \quad v \in \widetilde{H}^s(\Omega) = H_0^s(\Omega), \; s > 1/2, \tag{9.2}$$

holds with a constant C that scales as $(s-1/2)^{-1}$. Indeed, splitting the integration to compute $\widetilde{H}^s(\Omega)$ above, one readily finds that

$$\begin{aligned} |v|_{\widetilde{H}^s(\Omega)}^2 &= |v|_{H^s(\Omega)}^2 + 2C(d,s) \int_\Omega \int_{\Omega^c} \frac{|u(x)|^2}{|x-y|^{d+2s}} \, dydx \\ &\simeq |v|_{H^s(\Omega)}^2 + \frac{C(d,s)}{s} \int_\Omega \frac{|u(x)|^2}{d(x,\partial\Omega)^{2s}} \, dx, \end{aligned}$$

and it is therefore necessary to bound the last integral in the right hand side in terms of the $H^s(\Omega)$-seminorm. Such is the purpose of the *Hardy inequality* (cf. [35, Theorem 1.4.4.4]), for which the optimal constant is of order $(s-1/2)^{-1}$ [9].

In spite of their spectral equivalence, the inner products that give rise to the integral, spectral and censored fractional Laplacians are different and yield a strikingly different boundary behavior [10]. For a right-hand side $f \in L^\infty(\Omega)$, the boundary behavior of solutions u of the three operators is as follows: for the integral Laplacian, u is roughly like

$$u \simeq d(\cdot, \partial\Omega)^s, \tag{9.3}$$

whereas for the spectral Laplacian u behaves like

$$u \simeq d(\cdot, \partial\Omega)^{\min\{2s,1\}},$$

except for $s = \frac{1}{2}$ that requires an additional factor $|\log d(\cdot, \partial\Omega)|$, and for the censored Laplacian with $s \in (\frac{1}{2}, 1)$ the function u is quite singular at the boundary [5]

$$u \simeq d(\cdot, \partial\Omega)^{2s-1}.$$

We finally conclude that, in view of (2.7), the BPX preconditioners (4.2) for quasi-uniform meshes and (5.9) for graded bisection meshes are effective for the spectral and censored Laplacians, but the performance for the latter deteriorates as $s \to \frac{1}{2}$.

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

References

[1] G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: regularity of solutions and finite element approximations*, SIAM J. Numer. Anal. **55** (2017), no. 2, 472–495, DOI 10.1137/15M1033952. MR3620141

[2] M. Ainsworth and C. Glusa, *Aspects of an adaptive finite element method for the fractional Laplacian: a priori and a posteriori error estimates, efficient implementation and multigrid solver*, Comput. Methods Appl. Mech. Engrg. **327** (2017), 4–35, DOI 10.1016/j.cma.2017.08.019. MR3725761

[3] M. Ainsworth and C. Glusa, *Towards an efficient finite element method for the integral fractional Laplacian on polygonal domains*, Contemporary Computational Mathematics-A Celebration of the 80th Birthday of Ian Sloan, Springer, 2018, pp. 17–57.

[4] M. Ainsworth, W. McLean, and T. Tran, *The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling*, SIAM J. Numer. Anal. **36** (1999), no. 6, 1901–1932, DOI 10.1137/S0036142997330809. MR1712149

[5] A. Audrito, J.-C. Felipe-Navarro, and X. Ros-Oton, *The Neumann problem for the fractional Laplacian: regularity up to the boundary*, arXiv Preprint, `arXiv:2006.10026`, 2020.

[6] T. Bærland, M. Kuchta, and K.-A. Mardal, *Multigrid methods for discrete fractional Sobolev spaces*, SIAM J. Sci. Comput. **41** (2019), no. 2, A948–A972, DOI 10.1137/18M1191488. MR3934111

[7] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resour. Res. **36** (2000), no. 6, 1403–1412.

[8] J. Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564

[9] K. Bogdan and B. Dyda, *The best constant in a fractional Hardy inequality*, Math. Nachr. **284** (2011), no. 5-6, 629–638, DOI 10.1002/mana.200810109. MR2663757

[10] M. Bonforte, A. Figalli, and J. L. Vázquez, *Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Paper No. 57, 34, DOI 10.1007/s00526-018-1321-2. MR3773806

[11] A. Bonito, J. P. Borthagaray, R. H. Nochetto, E. Otárola, and A. J. Salgado, *Numerical methods for fractional diffusion*, Comput. Vis. Sci. **19** (2018), no. 5-6, 19–46, DOI 10.1007/s00791-018-0289-y. MR3893441

[12] F. Bornemann and H. Yserentant, *A basic norm equivalence for the theory of multilevel methods*, Numer. Math. **64** (1993), no. 4, 455–476, DOI 10.1007/BF01388699. MR1213412

[13] J. P. Borthagaray, D. Leykekhman, and R. H. Nochetto, *Local energy estimates for the fractional Laplacian*, SIAM J. Numer. Anal. **59** (2021), no. 4, 1918–1947, DOI 10.1137/20M1335509. MR4283703

[14] J. P. Borthagaray and R. H. Nochetto, *Besov regularity for the Dirichlet integral fractional Laplacian in Lipschitz domains*, J. Funct. Anal. **284** (2023), no. 6, Paper No. 109829, 33, DOI 10.1016/j.jfa.2022.109829. MR4530901

[15] J. P. Borthagaray and R. H. Nochetto, *Constructive approximation on graded meshes for the integral fractional Laplacian*, arXiv Preprint, `arXiv:2109.00451`, 2021.

[16] J. P. Borthagaray, R. H. Nochetto, and A. J. Salgado, *Weighted Sobolev regularity and rate of approximation of the obstacle problem for the integral fractional Laplacian*, Math. Models Methods Appl. Sci. **29** (2019), no. 14, 2679–2717, DOI 10.1142/S021820251950057X. MR4053241

[17] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces*, Optimal Control and Partial Differential Equations, IOS, Amsterdam, 2001, pp. 439–455. MR3586796

[18] J. H. Bramble, J. E. Pasciak, and J. Xu, *Parallel multilevel preconditioners*, Math. Comp. **55** (1990), no. 191, 1–22, DOI 10.2307/2008789. MR1023042

[19] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008, DOI 10.1007/978-0-387-75934-0. MR2373954

[20] S. N. Chandler-Wilde, D. P. Hewett, and A. Moiola, *Interpolation of Hilbert and Sobolev spaces: quantitative estimates and counterexamples*, Mathematika **61** (2015), no. 2, 414–443, DOI 10.1112/S0025579314000278. MR3343061

[21] L. Chen, R. H. Nochetto, and J. Xu, *Optimal multilevel methods for graded bisection grids*, Numer. Math. **120** (2012), no. 1, 1–34, DOI 10.1007/s00211-011-0401-4. MR2885595

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[22] Z.-Q. Chen and R. Song, *Two-sided eigenvalue estimates for subordinate processes in domains*, J. Funct. Anal. **226** (2005), no. 1, 90–113, DOI 10.1016/j.jfa.2005.05.004. MR2158176

[23] P. Ciarlet Jr., *Analysis of the Scott-Zhang interpolation in the fractional order Sobolev spaces*, J. Numer. Math. **21** (2013), no. 3, 173–180, DOI 10.1515/jnum-2013-0007. MR3118443

[24] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004. MR2042661

[25] W. Dahmen and A. Kunoth, *Multilevel preconditioning*, Numer. Math. **63** (1992), no. 3, 315–344, DOI 10.1007/BF01385864. MR1186345

[26] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573, DOI 10.1016/j.bulsci.2011.12.004. MR2944369

[27] M. Dryja and O. B. Widlund, *Multilevel additive methods for elliptic finite element problems*, Parallel Algorithms for Partial Differential Equations (Kiel, 1990), Notes Numer. Fluid Mech., vol. 31, Friedr. Vieweg, Braunschweig, 1991, pp. 58–69. MR1167868

[28] Q. Du, L. Tian, and X. Zhao, *A convergent adaptive finite element algorithm for nonlocal diffusion and peridynamic models*, SIAM J. Numer. Anal. **51** (2013), no. 2, 1211–1234, DOI 10.1137/120871638. MR3045653

[29] L. C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010, DOI 10.1090/gsm/019. MR2597943

[30] M. Faustmann, J. M. Melenk, and D. Praetorius, *Quasi-optimal convergence rate for an adaptive method for the integral fractional Laplacian*, Math. Comp. **90** (2021), no. 330, 1557–1587, DOI 10.1090/mcom/3603. MR4273109

[31] M. Faustmann, J. M. Melenk, and M. Parvizi, *On the stability of Scott-Zhang type operators and application to multilevel preconditioning in fractional diffusion*, ESAIM Math. Model. Numer. Anal. **55** (2021), no. 2, 595–625, DOI 10.1051/m2an/2020079. MR4238777

[32] H. Gimperlein and J. Stocek, *Space-time adaptive finite elements for nonlocal parabolic variational inequalities*, Comput. Methods Appl. Mech. Engrg. **352** (2019), 137–171, DOI 10.1016/j.cma.2019.04.019. MR3948751

[33] H. Gimperlein, J. Stocek, and C. Urzúa-Torres, *Optimal operator preconditioning for pseudodifferential boundary problems*, Numer. Math. **148** (2021), no. 1, 1–41, DOI 10.1007/s00211-021-01193-9. MR4265897

[34] M. Griebel and P. Oswald, *On the abstract theory of additive and multiplicative Schwarz algorithms*, Numer. Math. **70** (1995), no. 2, 163–180, DOI 10.1007/s002110050115. MR1324736

[35] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683]; With a foreword by Susanne C. Brenner, DOI 10.1137/1.9781611972030.ch1. MR3396210

[36] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math. **268** (2015), 478–528, DOI 10.1016/j.aim.2014.09.018. MR3276603

[37] M. Karkulik and J. M. Melenk, *$\mathcal{H}$-matrix approximability of inverses of discretizations of the fractional Laplacian*, Adv. Comput. Math. **45** (2019), no. 5-6, 2893–2919, DOI 10.1007/s10444-019-09718-5. MR4047021

[38] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth. MR0350177

[39] V. Maz′ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), no. 2, 230–238, DOI 10.1006/jfan.2002.3955. MR1940355

[40] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000. MR1742312

[41] S. V. Nepomnyaschikh, *Decomposition and fictitious domains methods for elliptic boundary value problems*, Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991), SIAM, Philadelphia, PA, 1992, pp. 62–72. MR1189564

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[42] R. H. Nochetto, K. G. Siebert, and A. Veeser, *Theory of adaptive finite element methods: an introduction*, Multiscale, Nonlinear and Adaptive Approximation, Springer, Berlin, 2009, pp. 409–542, DOI 10.1007/978-3-642-03413-8_12. MR2648380

[43] R. H. Nochetto and A. Veeser, *Primer of adaptive finite element methods*, Multiscale and Adaptivity: Modeling, Numerics and Applications, Lecture Notes in Math., vol. 2040, Springer, Heidelberg, 2012, pp. 125–225, DOI 10.1007/978-3-642-24079-9. MR3076038

[44] R. H. Nochetto, T. von Petersdorff, and C.-S. Zhang, *A posteriori error analysis for a class of integral equations and variational inequalities*, Numer. Math. **116** (2010), no. 3, 519–552, DOI 10.1007/s00211-010-0310-y. MR2684296

[45] P. Oswald, *Norm equivalencies and multilevel Schwarz preconditioning for variational problems*, Friedrich-Schiller-Univ., 1992.

[46] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary* (English, with English and French summaries), J. Math. Pures Appl. (9) **101** (2014), no. 3, 275–302, DOI 10.1016/j.matpur.2013.06.003. MR3168912

[47] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), no. 190, 483–493, DOI 10.2307/2008497. MR1011446

[48] D. W. Sims, E. J. Southall, N. E. Humphries, G. C. Hays, C. J. A. Bradshaw, J. W. Pitchford, A. James, M. Z. Ahmed, A. S. Brierley, M. A. Hindell, et al., *Scaling laws of marine predator search behaviour*, Nature **451** (2008), no. 7182, 1098–1102.

[49] R. Stevenson and R. van Venetië, *Uniform preconditioners for problems of positive order*, Comput. Math. Appl. **79** (2020), no. 12, 3516–3530, DOI 10.1016/j.camwa.2020.02.009. MR4094780

[50] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Lecture Notes of the Unione Matematica Italiana, vol. 3, Springer, Berlin; UMI, Bologna, 2007. MR2328004

[51] E. Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA **49** (2009), 33–44. MR2584076

[52] J. Wu and H. Zheng, *Uniform convergence of multigrid methods for adaptive meshes*, Appl. Numer. Math. **113** (2017), 109–123, DOI 10.1016/j.apnum.2016.11.005. MR3588590

[53] J. Xu, *Theory of multilevel methods*, ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–Cornell University. MR2637710

[54] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Rev. **34** (1992), no. 4, 581–613, DOI 10.1137/1034116. MR1193013

[55] J. Xu, *The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids* (English, with English and German summaries), Computing **56** (1996), no. 3, 215–235, DOI 10.1007/BF02238513. International GAMM-Workshop on Multi-level Methods (Meisdorf, 1994). MR1393008

[56] J. Xu, *An introduction to multigrid convergence theory*, Iterative Methods in Scientific Computing (Hong Kong, 1995), Springer, Singapore, 1997, pp. 169–241. MR1661962

[57] J. Xu, L. Chen, and R. H. Nochetto, *Optimal multilevel methods for $H(\mathrm{grad})$, $H(\mathrm{curl})$, and $H(\mathrm{div})$ systems on graded and unstructured grids*, Multiscale, Nonlinear and Adaptive Approximation, Springer, Berlin, 2009, pp. 599–659, DOI 10.1007/978-3-642-03413-8_14. MR2648382

[58] J. Xu and L. Zikatanov, *The method of alternating projections and the method of subspace corrections in Hilbert space*, J. Amer. Math. Soc. **15** (2002), no. 3, 573–597, DOI 10.1090/S0894-0347-02-00398-3. MR1896233

[59] K. Yosida, *Functional Analysis*, Die Grundlehren der mathematischen Wissenschaften, Band 123, Academic Press, Inc., New York; Springer-Verlag, Berlin, 1965. MR0180824

[60] X. Zhang, *Multilevel Schwarz methods*, Numer. Math. **63** (1992), no. 4, 521–539, DOI 10.1007/BF01385873. MR1189535

[61] X. Zhao, X. Hu, W. Cai, and G. E. Karniadakis, *Adaptive finite element method for fractional differential equations using hierarchical matrices*, Comput. Methods Appl. Mech. Engrg. **325** (2017), 56–76, DOI 10.1016/j.cma.2017.06.017. MR3693419

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Departamento de Matemática y Estadística del Litoral, Universidad de la República, Salto, Uruguay

Current address: Instituto de Matemática y Estadística Rafael Laguardia, Universidad de la República, Montevideo, Uruguay

Email address: jpborthagaray@fing.edu.uy

Department of Mathematics, University of Maryland, College Park, Maryland 20742; and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

Email address: rhn@math.umd.edu

School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

Email address: snwu@math.pku.edu.cn

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Email address: xu@math.psu.edu

Licensed to Univ of Maryland, College Park. Prepared on Tue Aug 22 19:55:31 EDT 2023 for download from IP 129.2.19.102.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use