Stability of Equilibria in Time-inconsistent Stopping Problems

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Abstract

We investigate the stability of equilibrium-induced optimal values with respect to reward functions f and transition kernels Q for time-inconsistent stopping problems under nonexponential discounting in discrete time. First, with locally uniform convergence of f and Q equipped with total variation distance, we show that the optimal value is semi-continuous with respect to (f, Q). We provide examples showing that continuity may fail in general, and the convergence for Q in total variation cannot be replaced by weak convergence. Next we show that with the uniform convergence of f and Q, the optimal value is continuous with respect to (f, Q) when we consider a relaxed limit over ε-equilibria. We also provide an example showing that for such continuity the uniform convergence of (f, Q) cannot be replaced by locally uniform convergence.

Keywords: Optimal Stopping, Time-inconsistency, Optimal equilibrium, ε-equilibria, Stability
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Contents

1 Introduction

Consider the optimal stopping problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_x[\delta(\tau) f(X_\tau)], \tag{1.1}$$

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where $X = (X_t)_{t=0,1,\ldots}$ is a time-homogeneous Markov process taking values in some state space $\mathbb{X}$, $\mathcal{T}$ is the set of all stopping times, δ is a discount function and f is a reward function. It is well known that when δ is not exponential, the problem (1.1) may be time-inconsistent. That is, a stopping strategy that is optimal from today's point of view may no longer be optimal from a future's perspective. A popular approach to address this time-inconsistency is to look for a subgame perfect Nash equilibrium instead of solving (1.1): a strategy such that once it is imposed over the planning horizon, the current self has no incentive to deviate from the strategy, given all future selves will follow it.

There have been a lot of works on equilibrium strategies for time-inconsistent control problems, and we refer to [4, 5, 18, 11] and the references therein. The development for theory of time-inconsistent stopping is more recent, and we refer to [14, 12, 13, 15, 17, 7, 6, 3, 21, 2, 1, 16]. Let us also mention the work [19] which analyzes a time-inconsitent Dynkin game, and [20] which considers a time-inconsistent controller-stopper problem. It is worth to mention that most of the papers on time-inconsistent control and stopping focus on the characterization of equilibria. A few exceptions include [16, 17, 14, 22] where the optimality and selection of equilibria are first analyzed in the presence of multiple equilibria. In particular, it is shown in settings of these papers that there exists an optimal equilibrium which pointwisely dominates all other equilibria in terms of the associated value functions; moreover, this optimal equilibrium is given by the intersection of all equilibria and thus is the smallest equilibrium.

The focus of this paper differs from those in the existing literature on time-inconsistent problems: we consider the stability of the smallest optimal equilibrium as well as the optimal values induced by these equilibria (or by the smallest optimal equilibrium). More specifically, we investigate the continuity of the optimal equilibrium and optimal value with respect to (w.r.t.) the reward function f and the transition kernel Q of the Markov process X. We assume the discount function is log-subadditive and we consider pure strategies (i.e., not mixed strategies). Our first main result, Theorem 3.1, states that, with the local convergence of f and Q which is equipped with the total variation distance, the smallest optimal equilibrium (in terms of inclusion) is lower semicontinuous, and the optimal value function is upper semicontinuous w.r.t. (f, Q). We provide examples showing that the exact continuity w.r.t. (f, Q) for either the optimal equilibrium or the optimal value function may fail. Moreover, we also construct an example in which the semi-continuity fails if the convergence of Q in total variation is changed to weak convergence. Let us emphasize that our first main result contrasts with the stability of the optimal value w.r.t. (f, Q) under time-consistent stopping (i.e., with exponential discounting): the continuity indeed holds for time-consistent stopping in our setup, as indicated in Remark 3.2.

In our second main result, Theorem 4.1, we recover the continuity (under a relaxation) of the optimal value function w.r.t. (f, Q) by relaxing the equilibrium concept and including ε-equilibria: Specifically, we show that as (f^n, Q^n) uniformly converges to (f, Q), it holds that $\lim_{\varepsilon\searrow 0}\lim_{n\to\infty} V_\varepsilon^{Q^n}(\cdot, f^n) = V_0^Q(\cdot, f)$, where $V_\varepsilon^{Q^n}(\cdot, f^n)$ is the optimal value induced by all ε-equilibria w.r.t. (f^n, Q^n). The two limits in ε and n cannot be changed due to the first main result; see Remark 4.2. To prove the second main result, we introduce the notion of pseudo ε-equilibrium which captures the idea of penalizing the possible deviation in the continuation region but not in the stopping region; see Definition 4.2. It turns out that pseudo ε-equilibria have better properties than ε-equilibria: One can embed the set of pseudo-ε-equilibria to pseudo equilibria corresponding to a perturbed reward function; see Lemma 4.5. A remarkable observation is that the smallest optimal pseudo equilibrium is actually the smallest optimal equilibrium; see Proposition 4.2. In Example 4.1, we demonstrate that the continuity in our second main result may fail if we replace the uniform convergence of (f, Q) with locally uniform convergence. In Proposition 4.1, however, we show that if the relaxation is over the pseudo ε equilibria, then the uniform convergence can be loosened.

Stability analysis is an important topic in control and optimization problems. For the stability of equilibria, the very recent works [9, 10] consider the stability of equilibria w.r.t. position/path (and time) in Nash games. Let us also mention the study in [8, Section 5] for the stability of mixed equilibrium strategies in time-inconsistent stopping problems w.r.t some "myopic adjustment" procedure which is used to produce a sequence of mixed strategies that may convergence to a mixed equilibrium. To the best of our knowledge, there is no literature so far studying the stability of equilibria w.r.t. the reward function and the dynamics of the underlying process for time-inconsistent (stopping) problems. In this regards, our paper provides very novel and conceptual contributions to the stability analysis in the topic of time-inconsistent problems. Our results also give a theoretical guidance for the numerical computation of optimal equilibrium values for time-inconsistent stopping. In reality the reward function f and transition kernel Q may not be fully known and may be obtained via estimation (Q can be obtained by statistical analysis of the state variables and reward function could be determined from a survey). A natural question is how the equilibrium-induced optimal value $V^Q(\cdot, f)$ is estimated based on the approximated (f^n, Q^n) for the reward function and transition kernel. Our results indicate that, when (f^n, Q^n) are close to (f, Q), using the optimal value $V^{Q^n}(\cdot, f^n)$ induced by perfect equilibria w.r.t. (f^n, Q^n) to estimate $V^Q(\cdot, f)$ can still lead to a large error. Instead, one should look for the value induced by all ε-equilibria w.r.t. (f^n, Q^n) to get a good estimation for $V^Q(\cdot, f)$.

The rest of the paper is organized as follows. The setup and main assumptions are introduced in Section 2, together with several preliminary lemmata. In Section 3, we present our first main result, the proof of which is given in Section 3.1. In Section 4, we provide the second main result by introducing (pseudo) ε-equilibria. The proof of this result is collected in Section 4.1. Appendix gathers the proofs of lemmata in Section 2.

2 Setup and preliminaries

Consider a measurable space $(\Omega, \mathcal{F})$ and let $X = (X_t)_{t=0,1,\dots}$ be a time-homogeneous Markov process in discrete time, taking values in some polish space $\mathbb{X}$. Let $\mathbb{F}$ be the filtration generated by X and $\mathcal{T}$ be the set of $\mathbb{F}$-stopping times. Denote $\mathcal{B}$ the class of Borel sets of $\mathbb{X}$, and $\mathbb{N} := \{0, 1, 2, \dots\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathbb{R}_+ := [0, \infty)$. Let $f : \mathbb{X} \to \mathbb{R}_+$ be a reward function that may be discontinuous. Denote $||f||_\infty := \sup_{x\in\mathbb{X}} |f(x)|$. Let $\delta : \mathbb{N} \mapsto [0, 1]$ be a discount function that is decreasing with $\delta(0) = 1$, $\delta(1) < 1$ and $\lim_{t\to\infty} \delta(t) = 0$. We further make the following assumption on the discount function $\delta(\cdot)$.

Assumption 2.1. *$\delta(\cdot)$ is log sub-additive, i.e.,*

$$\delta(t + s) \geq \delta(t)\delta(s), \quad \forall s, t \geq 0. \tag{2.1}$$

Remark 2.1. *Typical discount functions, including exponential, hyperbolic, generalized hyperbolic and pseudo-exponential discounting, satisfy Assumption 2.1.*

Given the transition kernel $Q(x, dy)$ for X and a stopping time τ, define

$$v^Q(x, \tau, f) := \mathbb{E}^Q_x[\delta(\tau)f(X_\tau)],$$

where $\mathbb{E}^Q_x$ is the expectation w.r.t. Q given $X_0 = x$. For $S \in \mathcal{B}$, denote

$$\rho(S) := \inf\{t \geq 1, X_t \in S\},$$

and

$$J^Q(x,S,f) := \mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})]\cdot 1_{\{x\notin S\}} + f(x)\cdot 1_{\{x\in S\}}, \quad \forall x\in\mathbb{X}.$$

We provide the definition of equilibria and optimal equilibria in the following.

Definition 2.1 (Equilibria and optimal equilibria)**.** *Fix a reward function f and a transition kernel Q. A Borel set $S\subset\mathbb{X}$ is called an equilibrium (w.r.t. f and Q) if*

$$\begin{cases} f(x)\leq \mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})], & \forall x\notin S,\\ f(x)\geq \mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})], & \forall x\in S. \end{cases} \tag{2.2}$$

Denote $\mathcal{E}^Q(f)$ the set of equilibria w.r.t. f and Q. $S\in\mathcal{E}^Q(f)$ is called an optimal equilibrium (w.r.t. f and Q), if for any $T\in\mathcal{E}^Q(f)$,

$$J^Q(x,S,f)\geq J^Q(x,T,f), \quad \forall x\in\mathbb{X}.$$

Let

$$V^Q(x,f) := \sup_{S\in\mathcal{E}^Q(f)} J^Q(x,S,f), \quad x\in\mathbb{X}, \tag{2.3}$$

which represents the optimal value generated over all equilibria. As indicated by results in [16] (also see Lemma 2.1) there exists an (smallest) optimal equilibrium and thus the supremum for $V^Q(x,f)$ is attained universally at this optimal equilibrium for all $x\in\mathbb{X}$. In this paper, we investigate the stability of $V^Q(x,f)$ w.r.t. the transition kernel Q and reward function f. To begin with, recall the total variation distance between two measures μ and ν,

$$||\mu-\nu||_{\mathrm{TV}} := \sup_{g\in B(\mathbb{X};[-1,1])}\left\{\int_{\mathbb{X}} g\,d\mu - \int_{\mathbb{X}} g\,d\nu\right\},$$

where $B(\mathbb{X};[-1,1])$ is the set of Borel measurable functions on $\mathbb{X}$ taking values in $[-1,1]$. We will use the following notions of convergence for f and Q for the stability analysis of $V^Q(x,f)$.

Definition 2.2. *Let $(f^n)_{n\in\overline{\mathbb{N}}}$ be a sequence of functions on $\mathbb{X}$. We say f^n converges to f^∞ locally uniformly if for any compact set $K\subset\mathbb{X}$,*

$$\lim_{n\to\infty}\sup_{x\in K}|f^n(x)-f^\infty(x)| = 0.$$

Recall that f^n converges to f^∞ uniformly if $\|f^n-f^\infty\|_\infty\to 0$ as $n\to\infty$.

Definition 2.3. *Let $(Q^n)_{n\in\overline{\mathbb{N}}}$ be a sequence of transition kernels. We say Q^n converges to Q^∞ locally uniformly in total variation, if for any compact set $K\subset\mathbb{X}$,*

$$\lim_{n\to\infty}\sup_{x\in K}||Q^n(x,\cdot)-Q^\infty(x,\cdot)||_{TV} = 0.$$

We say Q^n converges to Q^∞ uniformly in total variation, if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{X}}||Q^n(x,\cdot)-Q^\infty(x,\cdot)||_{TV} = 0.$$

Remark 2.2. *When $\mathbb{X}$ is countable and under the discrete topology, locally uniform convergence of $(Q^n(x,y))_{n\in\overline{\mathbb{N}}}$ in total variation is the same as the pointwise weak convergence.*

When $\mathbb{X}$ is uncountable (e.g., the process under Q^n is a time-discretized diffusion), then the locally uniform convergence of $(Q^n(x,y))_{n\in\overline{\mathbb{N}}}$ in total variation can be implied by the following condition: There exist a reference measure μ such that any $Q^n(x,\cdot)$ has a probability density $q^n(x,\cdot)$ w.r.t. μ, i.e., $Q^n(x,dy)=q^n(x,y)\mu(dy)$, and for any compact set $K\subset\mathbb{X}$,

$$\lim_{n\to\infty}\sup_{x\in K}\int_{\mathbb{X}}|q^n(x,y)-q^\infty(x,y)|d\mu(y) = 0.$$

Now we present three lemmata that will be used in later sections, and their proofs are collected in Appendix A. The first lemma is an analogue of [3, Theorem 2.5] for the discrete-time setting, which provides the existence of an optimal equilibrium, as well as an iterative approach for its construction. To this end, define

$$S^*(f,Q) := \cap_{S\in\mathcal{E}^Q(f)} S. \tag{2.4}$$

We have the following.

Lemma 2.1. *Let Assumption 2.1 hold. Suppose f is bounded and non-negative, and Q is a transition kernel. Define $S_0 = \emptyset$ and for $k = 1, 2, \dots,$*

$$S_{k+1} := S_k \cup \left\{ x \in \mathbb{X} \setminus S_k : \ f(x) > \sup_{1\le\tau\le\rho(S_k)} v^Q(x,\tau,f) \right\}.$$

Then $\cup_{k\in\mathbb{N}} S_k = S^*(f,Q)$. Moreover, $S^*(f,Q)$ is an optimal equilibrium, and thus

$$V^Q(x,f) = J^Q(x, S^*(f,Q), f), \quad \forall x \in \mathbb{X}.$$

Remark 2.3. *Lemma 2.1 indicates that there exists a "smallest" equilibrium, which is also an optimal one. The supremum for $V^Q(x,f)$ is achieved by the same equilibrium $S^*(f,Q)$. Moreover, If the discount function is exponential, i.e., when the stopping problem (1.1) is time-consistent, a similar discussion as that in [3] would show that $S^*(f,Q)$ and $V^Q(x,f)$ would coincide with the optimal stopping region and value respectively in the classical sense.*

Lemma 2.2. *Let $(Q^n)_{n\in\overline{\mathbb{N}}}$ be transition kernels.*

(a) Suppose Q^n converges to Q^∞ locally uniformly in total variation. Then for any $x \in \mathbb{X}$ and $T \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{g\in B(\mathbb{X}^T;[-1,1])} \left| \mathbb{E}_x^{Q^n} g(X_1, X_2, \dots, X_T) - \mathbb{E}_x^{Q^\infty} g(X_1, X_2, \dots, X_T) \right| = 0.$$

(b) In addition to the condition in part (a), assume that, for any compact set K and $\varepsilon > 0$, there exists a compact set K' such that $\sup_{x\in K} Q^\infty(x, K') \ge 1 - \varepsilon$. Then for any compact set K and $T \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{x\in K, g\in B(\mathbb{X}^T;[-1,1])} \left| \mathbb{E}_x^{Q^n} g(X_1, X_2, \dots, X_T) - \mathbb{E}_x^{Q^\infty} g(X_1, X_2, \dots, X_T) \right| = 0.$$

(c) Suppose Q^n converges to Q^∞ uniformly in total variation. Then for any $T \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{x\in \mathbb{X}, g\in B(\mathbb{X}^T;[-1,1])} \left| \mathbb{E}_x^{Q^n} g(X_1, X_2, \dots, X_T) - \mathbb{E}_x^{Q^\infty} g(X_1, X_2, \dots, X_T) \right| = 0.$$

Remark 2.4. *Suppose under Q^∞,*

$$X_{t+1} = h(X_t, \xi_t),$$

where $\xi_0, \xi_1, \dots$ are i.i.d. random variables taking values in $\mathbb{R}^d$ and $h : \mathbb{X} \times \mathbb{R}^d \mapsto \mathbb{X}$ is continuous. Then the additional assumption in Lemma 2.2(b) is satisfied. Indeed, fix compact set $K \subset \mathbb{X}$ and $\varepsilon > 0$. There exists constant $C > 0$ such that $\mathbb{P}(|\xi_0| \le C) \ge 1-\varepsilon$. Let $C' := \sup_{(x,y)\in K\times\overline{B_C}} |h(x,y)| < \infty$ and $K' := \overline{B_{C'}} \subset \mathbb{X}$, where B_r is the ball centered at zero with radius r. Then $\sup_{x\in K} Q^\infty(x, K') \ge \mathbb{P}(|\xi_0| \le C) \ge 1 - \varepsilon$.

Lemma 2.3. *Let $(Q^n)_{n\in\overline{\mathbb{N}}}$ be transition kernels, and $(f^n)_{m\in\overline{\mathbb{N}}}$ be non-negative reward functions such that $\sup_{n\in\overline{\mathbb{N}}}\|f^n\|_\infty<\infty$. Suppose Assumption 2.1 holds.*

(a) Suppose Q^n converges to Q^∞ locally uniformly in total variation and f^n converges to f^∞ locally uniformly. Then

$$\lim_{n\to\infty}\sup_{\tau\in\mathcal{T}}|v^{Q^n}(x,\tau,f^n)-v^{Q^\infty}(x,\tau,f^\infty)|=0,\quad \forall x\in\mathbb{X}.$$

(b) In addition to the conditions in part (a), assume that for any compact set K and $\varepsilon>0$, there exists a compact set K' such that $\sup_{x\in K}Q^\infty(x,K')\geq 1-\varepsilon$. Then for any compact set K,

$$\lim_{n\to\infty}\sup_{x\in K,\tau\in\mathcal{T}}|v^{Q^n}(x,\tau,f^n)-v^{Q^\infty}(x,\tau,f^\infty)|=0.$$

(c) Suppose Q^n converges to Q^∞ uniformly in total variation and $\|f^n-f^\infty\|_\infty\to 0$. Then

$$\lim_{n\to\infty}\sup_{x\in\mathbb{X},\tau\in\mathcal{T}}|v^{Q^n}(x,\tau,f^n)-v^{Q^\infty}(x,\tau,f^\infty)|=0.$$

3 Semi-contintuity of the smallest optimal equilibrium and its associated value

In this section, we present the first main result: the semi-continuity of $V^Q(x,f)$ and $S^*(f^\infty,Q^\infty)$ w.r.t. f and Q. The proof is collected in Section 3.1. Examples for discontinuity are also provided.

Theorem 3.1. *Suppose Assumption 2.1 holds. Let $(Q^n)_{n\in\overline{\mathbb{N}}}$ be transition kernels, and $(f^n)_{n\in\overline{\mathbb{N}}}$ be non-negative reward functions with $\sup_{n\in\overline{\mathbb{N}}}\|f^n\|_\infty<\infty$. Suppose Q^n converges to Q^∞ locally uniformly in total variation, and f^n converges to f^∞ locally uniformly. Then*

$$S^*(f^\infty,Q^\infty)\subset\liminf_{n\to\infty}S^*(f^n,Q^n), \tag{3.1}$$

and

$$V^{Q^\infty}(x,f^\infty)\geq\limsup_{n\to\infty}V^{Q^n}(x,f^n),\quad\forall x\in\mathbb{X}. \tag{3.2}$$

Remark 3.1. *We also have the semi-continuity in terms of the equilibria sets: under the conditions in Theorem 3.1,*

$$\limsup_{n\to\infty}\mathcal{E}^{Q^n}(f^n)\subset\mathcal{E}^{Q^\infty}(f^\infty).$$

Indeed, for $S\in\limsup_{n\to\infty}\mathcal{E}^{Q^n}(f^n)$, there exists a subsequence $(n_k)_k$ such that $S\in\mathcal{E}^{Q^{n_k}}(f^{n_k})$, and thus

$$\begin{cases} f^{n_k}(x)\leq\mathbb{E}^{Q^{n_k}}[\delta(\rho(S)f^{n_k}(X_{\rho(S)}))],\quad\forall x\notin S;\\ f^{n_k}(x)\geq\mathbb{E}^{Q^{n_k}}[\delta(\rho(S)f^{n_k}(X_{\rho(S)}))],\quad\forall x\in S.\end{cases}$$

By Lemma 2.3(a), letting $k\to\infty$ we can conclude that $S\in\mathcal{E}^{Q^\infty}(f^\infty)$.

Remark 3.2. *If δ is exponential, i.e., $\delta(t+s)=\delta(t)\delta(s)$ for any $s,t\geq 0$, then by a similar discussion as that in [3], we have that*

$$V^{Q^n}(x,f^n)=\sup_{\tau\in\mathcal{T}}\mathbb{E}_x^{Q^n}[\delta(\tau)f^n(X_\tau)]. \tag{3.3}$$

By Lemma 2.3(a), (3.3) implies that

$$\lim_{n\to\infty} V^{Q^n}(x, f^n) = V^{Q^\infty}(x, f^\infty), \quad \forall x \in \mathbb{X}, \tag{3.4}$$

which is the continuity of the optimal value function. However, we still only have the semi-continuity for the "smallest" optimal stopping region $S^*(f^n, Q^n)$.

We now present three examples of discontinuity. The first two examples show that the strict inequalities in (3.1) and (3.2) can happen. Example 3.1 is for discontinuity w.r.t. the transition kernel, and Example 3.2 is for discontinuity w.r.t. the reward function. Then we provide a discontinuity example under weak convergence of transition kernels, which indicates that the locally uniform convergence in total variation for transition kernels is the right assumption.

Example 3.1. *Let $\mathbb{X} = \{a, b, c\} \subset \mathbb{R}$ with $c < b < a$, $\delta(1) = 1/2$ and $\delta(2) = 1/3$. Define*

$$\begin{cases} Q^n : Q^n(c, b) = 1, \quad Q^n(b, a) = p_n = 1 - \frac{1}{n}, \quad Q^n(b, b) = \frac{1}{n}, \quad \forall n \in \overline{\mathbb{N}}, \\ f(a) = 2, \quad f(b) = 1, \quad f(c) = \frac{1}{2}, \end{cases}$$

where $\frac{1}{\infty} := 0$, and with a bit of abuse of notation $Q(x, y) := \mathbb{P}(X_1 = y | X_0 = x)$. It is easy to check that Q^n converges to Q^∞ uniformly in total variation.

Note that any equilibrium must contain the global maximum of the reward function. By computation

$$J^{Q^\infty}(b, \{a\}, f) = 1 = f(b) \quad \text{and} \quad J^{Q^\infty}(c, \{a\}, f) = 2/3 > f(c),$$

which imply that $S^*(f, Q^\infty) = \{a\}$. Moreover, since for $n < \infty$,

$$f(b) > J^{Q^n}(b, \{a\}, f) = J^{Q^n}(b, \{a, c\}, f),$$

any equilibrium w.r.t. Q^n for $n < \infty$ must contain $\{a, b\}$. As $f(c) > \delta(1)f(b)$, $\{a, b\}$ is not equilibrium w.r.t. Q^n for $n < \infty$. Consequently, $\mathcal{E}^{Q^n}(f) = \{\mathbb{X}\}$ for $n < \infty$. Hence,

$$S^*(f, Q^\infty) = \{c\} \subsetneqq \mathbb{X} = S^*(f, Q^n), \quad \forall n < \infty,$$

and

$$V^{Q^n}(c, f) = f(c) < J^{Q^\infty}(c, \{a\}) = V^{Q^\infty}(c, f).$$

Example 3.2. *Let $\mathbb{X} = \{a, b, c\} \subset \mathbb{R}$ with $c < b < a$, $\delta(1) = 1/2$ and $\delta(2) = 1/3$. Define*

$$\begin{cases} Q(c, b) = 1, \quad Q(b, a) = 1, \quad Q(a, a) = 1, \\ f^n(a) = 2, \quad f^n(b) = 1 + \frac{1}{n}, \quad f^n(c) = \frac{1}{2} + (1 + \delta(1))\frac{1}{n}, \quad \forall n \in \overline{\mathbb{N}}. \end{cases}$$

Obviously, $\|f^n - f^\infty\|_\infty \to \infty$.

We can compute that

$$J^Q(b, \{a\}, f^\infty) = 1 = f^\infty(b), \quad J^Q(c, \{a\}, f^\infty) = 2/3 > f^\infty(c),$$

and thus $\hat{S}^\infty = \{a\}$. Meanwhile,

$$J^Q(b, \{a\}, f^n) = J^Q(b, \{a, c\}, f^n) = 1 < f^n(b),$$

so neither $\{a\}$ nor $\{a, c\}$ belongs to $\mathcal{E}^Q(f^n)$ for $n < \infty$. By

$$f^n(c) = \frac{1}{2} + (1 + \delta(1))\frac{1}{n} > \frac{1}{2} + \delta(1)\frac{1}{n} = \delta(1)f^n(b),$$

$\{a,b\}$ is not equilibrium for all f^n for $n<\infty$. Therefore, $\mathbb{X}$ is the only equilibrium w.r.t. f^n for $n<\infty$. Hence,

$$S^*(f^\infty, Q)=\{c\}\subsetneqq \mathbb{X}=S^*(f,Q^n),\quad \forall\, n<\infty,$$

and

$$\limsup_{n\to\infty} V^Q(c,f^n)=\limsup_{n\to\infty} f^n(c)=\frac{1}{2}<\frac{2}{3}=V^Q(c,f^\infty).$$

When $\mathbb{X}$ is finite, convergence locally uniformly in total variation is equivalent to weak convergence. When $\mathbb{X}$ is not finite, we provide below an example showing that the semi-continuity in Theorem 3.1 fails when only weak convergence is assumed. Hence, weak convergence is too weak to establish the semi-continuity in Theorem 3.1.

Example 3.3. *Let $\mathbb{X}=\{y,x_\infty,x_1,x_2,...\}\subset\mathbb{R}$, where $0\le x_n\nearrow x_\infty$ and $y=\frac{x_\infty}{\delta(2)}+1$. Let $f(x)=x$. Define for $n<\infty$,*

$$Q^n:\begin{cases}Q^n(x_i,x_n)=1, & \textit{for}\quad i\neq n,\\ Q^n(x_\infty,x_n)=1, Q^n(x_n,y)=1, Q^n(y,y)=1,\end{cases}\quad \textit{and}\quad Q^\infty:\begin{cases}Q^\infty(x_i,x_\infty)=1, & \textit{for}\quad\forall i,\\ Q^\infty(x_\infty,x_\infty)=1, Q^\infty(y,y)=1.\end{cases}$$

It can be shown that $Q^n(z,\cdot)$ weakly converges to $Q^\infty(z,\cdot)$ for any $z\in\mathbb{X}$. However, since $Q^n(x_1,\{x_\infty\})=0$ for $n<\infty$ while $Q^\infty(x_1,\{x_\infty\})=1$, the locally uniform convergence in total variation fails.

For $n<\infty$, since $y>\frac{x_\infty}{\delta(2)}$, we have that

$$\mathbb{E}^{Q^n}_{x_i}[\delta(\rho(\{y\})f(X_{\rho(\{y\})}))]=\begin{cases}\delta(2)y, & i\in\overline{\mathbb{N}}\setminus\{n\}\\ \delta(1)y, & i=n\end{cases}>x_\infty\ge x_i.$$

This implies $S^*(f,Q^n)=\{y\}$ for $n<\infty$. On the other hand, denote

$$S_1:=\left\{x\in\mathbb{X}:\ f(x)>\sup_{1\le\tau} v^{Q^\infty}(x,\tau,f)\right\}.$$

Obviously, $\{x,y\}\subset S_1$. By Lemma 2.1, we have that $\{x,y\}\subset S^*(f,Q^\infty)$. Hence,

$$\limsup_{n\to\infty} S^*(f,Q^n)\subsetneqq S^*(f,Q^\infty)\quad \textit{and}\quad V^{Q^\infty}(x_\infty,f)=x_\infty<\liminf_{n\to\infty} V^{Q^\infty}(x_\infty,f)=\delta(2)y.$$

3.1 Proof of Theorem 3.1

Proof of Theorem 3.1. For $n\in\overline{\mathbb{N}}$, define $S^n_0=\emptyset$ and

$$S^n_{k+1}:=S^n_k\cup\left\{x\in\mathbb{X}\setminus S^n_k:\ f(x)>\sup_{1\le\tau\le\rho(S^n_k)} v^{Q^n}(x,\tau,f^n)\right\}.\tag{3.5}$$

By Lemma 2.1, $S^*(f^n,Q^n)=\cup_k S^n_k=\lim_{k\to\infty}S^n_k$, $\forall n\in\overline{\mathbb{N}}$. We show by induction that

$$S^\infty_k\subset\liminf_{n\to\infty} S^n_k,\quad k=0,1,\dots,\tag{3.6}$$

which in particular implies that $S^*(f^\infty,Q^\infty)\subset\liminf_{n\to\infty}S^*(f^n,Q^n)$.

Obviously, (3.6) holds for $k=0$. Suppose it holds for $k=i$ and consider the case $k=i+1$. Take $x\in S^\infty_{i+1}$. If $x\in S^\infty_i$, then by induction hypothesis

$$x\in\liminf_{n\to\infty} S^n_i\subset\liminf_{n\to\infty} S^n_{i+1}.$$

Now assume $x \notin S_i^\infty$. Then

$$\alpha := f^\infty(x) - \sup_{1 \le \tau \le \rho(S_i^\infty)} v^{Q^\infty}(x, \tau, f^\infty) > 0. \tag{3.7}$$

Denote the probability measure $\mathbb{P}^n$ induced by Q^n. By induction hypothesis,

$$\rho(S_i^\infty) \ge \rho\left(\bigcup_{1 \le n < \infty}\left(\bigcap_{n \le j < \infty,} S_i^j\right)\right) = \lim_{n\to\infty} \rho\left(\bigcap_{n \le j < \infty} S_i^j\right), \quad \mathbb{P}_x^\infty - \text{a.s..}$$

Therefore, there exists $N \in \mathbb{N}$ such that for any $n \ge N$,

$$\mathbb{P}_x^{Q^\infty}\left[\rho(S_i^n) > \rho(S_i^\infty)\right] \le \mathbb{P}_x^{Q^\infty}\left[\rho\left(\bigcap_{n \le j < \infty} S_i^j\right) > \rho(S_i^\infty)\right] < \frac{\alpha}{2M}, \tag{3.8}$$

where $M := \sup_{n \in \overline{\mathbb{N}}} \|f^n\|_\infty < \infty$. Then for any τ' with $1 \le \tau' \le \rho(S_i^n)$, we have that

$$v^{Q^\infty}(x, \tau', f^\infty) \le v^{Q^\infty}(x, \tau' \wedge \rho(S_i^\infty), f^\infty) + \frac{\alpha}{2} \le \sup_{1 \le \tau \le \rho(S_i^\infty)} v^{Q^\infty}(x, \tau, f^\infty) + \frac{\alpha}{2},$$

and thus

$$\sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^\infty}(x, \tau, f^\infty) \le \sup_{1 \le \tau \le \rho(S_i^\infty)} v^{Q^\infty}(x, \tau, f^\infty) + \frac{\alpha}{2}, \quad \forall n \ge N.$$

This together with (3.7) implies that

$$f^\infty(x) - \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^\infty}(x, \tau, f^\infty) \ge \frac{\alpha}{2} > 0. \tag{3.9}$$

By Lemma 2.3 part (a), for n large enough, we have that

$$\begin{aligned}\left|\sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^\infty}(x, \tau, f^\infty) - \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^n}(x, \tau, f^n)\right| &\le \sup_{1 \le \tau \le \rho(S_i^n)} \left|v^{Q^\infty}(x, \tau, f^\infty) - v^{Q^n}(x, \tau, f^n)\right| \\ &\le \sup_{\tau \in \mathcal{T}} \left|v^{Q^\infty}(x, \tau, f^\infty) - v^{Q^n}(x, \tau, f^n)\right| < \frac{\alpha}{3}.\end{aligned} \tag{3.10}$$

Meanwhile, we can choose N' such that for all $n \ge N'$ (3.10) holds and

$$|f^n(x) - f^\infty(x)| \le \frac{\alpha}{12}. \tag{3.11}$$

Thus, for all $n \ge \max\{N, N'\}$, combine (3.9), (3.10) and (3.11),

$$\begin{aligned} f^n(x) - \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^n}(x, \tau, f^n) =& f^n(x) - f^\infty(x) + f^\infty(x) - \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^\infty}(x, \tau, f^\infty) \\ &+ \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^\infty}(x, \tau, f^\infty) - \sup_{1 \le \tau \le \rho(S_i^n)} v^{Q^n}(x, \tau, f^n) \\ \ge& -\frac{\alpha}{12} + \frac{\alpha}{2} - \frac{\alpha}{3} > 0. \end{aligned}$$

Consequently, for n large enough, no matter x is in S_i^n or not, we always have $x \in S_{i+1}^n$, and thus $x \in \liminf_{n\to\infty} S_{i+1}^n$. By the arbitrariness of x, (3.6) holds for $k = i+1$. We have proved (3.1).

Now let $\varepsilon > 0$ and $x \notin S^*(f^\infty, Q^\infty)$. Following the argument in (3.8), we can show that there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$\mathbb{P}_x^{Q^\infty}\left[\rho(S^*(f^n, Q^n)) > \rho(S^*(f^\infty, Q^\infty))\right] < \frac{\varepsilon}{2M}. \tag{3.12}$$

Then there exists $N' > N$ such that for any $n > N'$,

$$\begin{aligned} v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty))) \geq & v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty) \cup S^*(f^n, Q^n))) \geq v^{Q^\infty}(x, \rho(S^*(f^n, Q^n))) - \frac{\varepsilon}{2} \\ \geq & v^{Q^n}(x, \rho(S^*(f^n, Q^n))) - \varepsilon, \end{aligned}$$

where the first inequality follows from [17, Lemma 3.1] (or Lemma 4.2), the second inequality follows from (3.12), the third inequality follows from Lemma 2.3 part (a). As a result,

$$v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty))) \geq \limsup_{n\to\infty} v^{Q^\infty}(x, \rho(S^*(f^n, Q^n))) - \varepsilon.$$

By the arbitrariness of ε, we have (3.2) holds. □

4 Continuity under a relaxed limit

As shown in the previous section, $V^Q(x, f)$ is not continuous w.r.t. Q or f in general. To achieve the stability, we need to relax the equilibrium set over which we take supremum.

Definition 4.1. *Fix a reward function f and a transition kernel Q. Take $\varepsilon \geq 0$. A Borel set S is called an ε-equilibrium (w.r.t. f and Q), if*

$$\begin{cases} f(x) \leq \mathbb{E}_x^Q[\delta(\rho(S)) f(X_{\rho(S)})] + \varepsilon, & \forall x \notin S, \\ f(x) + \varepsilon \geq \mathbb{E}_x^Q[\delta(\rho(S)) f(X_{\rho(S)})], & \forall x \in S. \end{cases} \tag{4.1}$$

Define

$$\mathcal{E}^Q(f, \varepsilon) := \{S \text{ is an } \varepsilon\text{-equilibrium w.r.t. } f \text{ and } Q\}.$$

When $\varepsilon = 0$, we still call S an equilibrium and may use the notation $\mathcal{E}^Q(f)$ instead of $\mathcal{E}^Q(f, 0)$.

We also need the following notion of pseudo ε-equilibria, which loosens the criterion of ε-equilibrium by giving up the condition in (4.1) when $x \in S$.

Definition 4.2. *Fix a reward function f and a transition kernel Q. Take $\varepsilon \geq 0$. A Borel set $S \subset \mathbb{X}$ is called a pseudo ε-equilibrium (w.r.t. f and Q), if*

$$f(x) \leq \mathbb{E}_x^Q[\delta(\rho(S)) f(X_{\rho(S)})] + \varepsilon, \quad \forall x \notin S. \tag{4.2}$$

Define

$$\mathcal{G}^Q(f, \varepsilon) := \{S \text{ is a pseudo } \varepsilon\text{-equilibrium w.r.t. } f \text{ and } Q\}.$$

When $\varepsilon = 0$, we simply call S is a pseudo equilibrium, and write $\mathcal{G}^Q(f)$ short for $\mathcal{G}^Q(f, 0)$. We say $S \in \mathcal{G}^Q(f)$ is an optimal pseudo equilibrium (w.r.t. f and Q), if for any $T \in \mathcal{G}^Q(f)$,

$$J(x, S, f) \geq J(x, T, f), \quad \forall x \in \mathbb{X}.$$

Remark 4.1. *There are three (pure) equilibrium concepts in the continuous-time setting, including mild equilibria, weak equilibria, and strong equilibria (see [2, 1]). The definition of mild equilibria is simply replacing $\rho(S) := \{t \geq 1, X_t \in S\}$ with $\rho(S) := \{t > 0, X_t \in S\}$ in Definition 2.1. [2, 1] provide a detailed comparison for these different types of equilibria, and show that under certain conditions an optimal mild equilibrium is also weak and strong. In the discrete-time setup, the notions of mild, weak, and strong equilibria are all equivalent.*

In continuous time when the process is regular enough, e.g., if X is a one-dimensional diffusion determined by a SDE: $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ with $|\sigma| > 0$, we have $\rho(S) = 0$, $\mathbb{P}^x$-a.s. and $f(x) = J^Q(x, S, f)$ for all $x \in S$, and thus the condition for $x \in S$ in the definition of mild equilibrium trivially holds. In this case, any pseudo equilibrium is automatically a mild equilibrium.

Now define

$$W_{\varepsilon}^{Q}(x,f) := \sup_{S\in\mathcal{G}^{Q}(f,\varepsilon)} J^{Q}(x,S,f); \quad V_{\varepsilon}^{Q}(x,f) := \sup_{S\in\mathcal{E}^{Q}(f,\varepsilon)} J^{Q}(x,S,f). \tag{4.3}$$

When $\varepsilon = 0$ we write $W^{Q}(x,f)$ instead of $W_0^{Q}(x,f)$, and we keep using the notation $V^{Q}(x,f)$ in (2.3) instead of $V_0^{Q}(x,f)$.

Pseudo ε-equilibria have better properties than ε-equilibria. As we will see in Lemma 4.5 below one can embed the set of pseudo-ε-equilibria to pseudo equilibria corresponding to a perturbed reward function. We will also observe that the smallest optimal pseudo equilibrium is actually the smallest optimal equilibrium in Proposition 4.2. These two results form the backbone of the proof of the second main result which we state below. The proof of this result is provided in Section 4.1.

Theorem 4.1. *Suppose Assumption 2.1 holds. Let $(Q^n)_{n\in\overline{\mathbb{N}}}$ be transition kernels, and $(f^n)_{n\in\overline{\mathbb{N}}}$ be bounded and non-negative reward functions. Suppose Q^n converges to Q^∞ uniformly in total variation, and $\|f^n - f^\infty\|_\infty \to 0$. Then*

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\left(\liminf_{n\to\infty} V_{\varepsilon}^{Q^n}(x,f^n)\right) = \lim_{\varepsilon\searrow 0}\left(\liminf_{n\to\infty} W_{\varepsilon}^{Q^n}(x,f^n)\right)\\
=&\lim_{\varepsilon\searrow 0}\left(\limsup_{n\to\infty} V_{\varepsilon}^{Q^n}(x,f^n)\right) = \lim_{\varepsilon\searrow 0}\left(\limsup_{n\to\infty} W_{\varepsilon}^{Q^n}(x,f^n)\right)\\
=&V^{Q^\infty}(x,f^\infty), \quad \forall x\in\mathbb{X}.
\end{aligned}$$

Letting $f^n = f$ and $Q^n = Q$ for $n\in\overline{\mathbb{N}}$ in Theorem 4.1, we achieve the following corollary, which shows that $V^{Q}(x,f)$ is indeed the limit of the supremum value over all ε-equilibria as $\varepsilon\searrow 0$.

Corollary 4.1. *Suppose Assumption 2.1 holds. Given a bounded reward function $f \geq 0$ and a transition kernel Q, we have that*

$$\lim_{\varepsilon\searrow 0} V_{\varepsilon}^{Q}(x,f) = \lim_{\varepsilon\searrow 0} W_{\varepsilon}^{Q}(x,f) = V^{Q}(x,f), \quad \forall x\in\mathbb{X}.$$

Remark 4.2. *Combining Theorem 3.1 and Corollary 4.1, we have*

$$\limsup_{n\to\infty}\left(\lim_{\varepsilon\searrow 0} V_{\varepsilon}^{Q^n}(x,f^n)\right) = \limsup_{n\to\infty} V^{Q^n}(x,f^n) \leq V^{Q^\infty}(x,f^\infty), \quad \forall x\in\mathbb{X}.$$

Recall that the strict inequality above can be achieved as shown in Examples 3.1 and 3.2. Hence, together with Theorem 4.1, we see that the order of taking $\varepsilon\searrow 0$ and taking $n\to\infty$ cannot be exchanged.

Moreover, the main results in this paper provide a guideline for numerical approximation for $V^{Q^\infty}(x,f^\infty)$: With good approximations of the transition kernel Q^∞ and reward function f^∞, taking supremum only over equilibria may not provide good estimation for the target optimal value. Instead, one should take supremum over all ε-equilibria.

Remark 4.3. *Analogous to Remark 3.1, if the same conditions in Theorem 4.1 hold, then*

$$\lim_{\varepsilon\searrow 0}\left(\liminf_{n\to\infty}\mathcal{E}_{\varepsilon}^{Q^n}(f^n)\right) = \lim_{\varepsilon\searrow 0}\left(\limsup_{n\to\infty}\mathcal{E}_{\varepsilon}^{Q^n}(f^n)\right) = \mathcal{E}^{Q^\infty}(f^\infty).$$

Proof. By a similar argument as in Remark 3.1, we can show that

$$\lim_{\varepsilon\searrow 0}\left(\limsup_{n\to\infty}\mathcal{E}_{\varepsilon}^{Q^n}(f^n)\right) \subset \mathcal{E}^{Q^\infty}(f^\infty).$$

It remains to show that

$$\mathcal{E}^{Q^\infty}(f^\infty) \subset \lim_{\varepsilon \searrow 0} \left(\liminf_{n\to\infty} \mathcal{E}_\varepsilon^{Q^n}(f^n) \right). \tag{4.4}$$

For $S \in \mathcal{E}^{Q^\infty}(f^\infty)$, we have

$$\begin{cases} f^\infty(x) \le \mathbb{E}^{Q^\infty}[\delta(\rho(S) f^\infty(X_{\rho(S)}))], & \forall x \notin S; \\ f^\infty(x) \ge \mathbb{E}^{Q^\infty}[\delta(\rho(S) f^\infty(X_{\rho(S)}))], & \forall x \in S. \end{cases}$$

Then for any $\varepsilon > 0$, Lemma 2.3 implies that, for n big enough,

$$\begin{cases} f^n(x) - \varepsilon \le \mathbb{E}^{Q^n}[\delta(\rho(S) f^n(X_{\rho(S)}))], & \forall x \notin S; \\ f^n(x) + \varepsilon \ge \mathbb{E}^{Q^n}[\delta(\rho(S) f^n(X_{\rho(S)}))], & \forall x \in S. \end{cases}.$$

Consequently, $S \in \liminf_{n\to\infty} \mathcal{E}_\varepsilon^{Q^n}(f^n)$ for any $\varepsilon > 0$, which implies (4.4). $\square$

The following example shows that the continuity result in Theorem 4.1 may fail if the convergence of $(Q_n)_{n\in\mathbb{N}}$ in total variation is only assumed to be locally uniform instead of uniform.

Example 4.1. *Let $\mathbb{X} = \{y, x_0, x_1, x_2, ...\} \subset \mathbb{R}$. Define*

$$Q^n : \begin{cases} Q^n(x_i, x_{i+1}) = \frac{1}{2}, Q^n(x_i, y) = \frac{1}{2}, & 0 \le i < n, \\ Q^n(x_i, y) = 1, & i > n \\ Q^n(x_n, x_n) = 1, Q^n(y, y) = 1. \end{cases};$$

$$Q^\infty : \begin{cases} Q^\infty(x_i, x_{i+1}) = \frac{1}{2}, Q^\infty(x_i, y) = \frac{1}{2}, & \forall i \ge 0, \\ Q^\infty(y, y) = 1. \end{cases}$$

One can easily see that Q^n converges to Q^∞ locally uniformly, but not uniformly. Let $f(x_i) = 1$ for $i \in \mathbb{N}$, $f(y) = 2.99$, and $\delta(k) = \frac{1}{1+k}$ for $k \in \mathbb{N}$.

We have $\frac{1}{2}\delta(1)(1 + f(y)) = \frac{3.99}{4} < 1$, and

$$\sum_{k=1}^{\infty} \delta(k) \left(\frac{1}{2}\right)^k f(y) > \sum_{k=1}^{3} \delta(k) \left(\frac{1}{2}\right)^k f(y) = 2.99 \left(\frac{1}{4} + \frac{1}{12} + \frac{1}{32}\right) > 1.$$

That is,

$$\frac{1}{2}\delta(1)(1 + f(y)) < 1 < \sum_{k=1}^{\infty} \delta(k) \left(\frac{1}{2}\right)^k f(y). \tag{4.5}$$

Take ε with $0 < \varepsilon < 1 - \frac{1}{2}\delta(1)(1 + f(y))$. For any $n < \infty$ and $S \in \mathcal{E}^{Q^n}(f, \varepsilon)$, it is easy to check that $y, x_n \in S$. For any $i \le n$, if $x_i \in S$, then by the first inequality in (4.5), $x_{i-1} \in S$. Hence, for any $n < \infty$,

$$\{x_0, x_1, ..., x_n\} \subset S, \quad \forall S \in \mathcal{E}^{Q^n}(f, \varepsilon).$$

As the above holds for any ε with $0 < \varepsilon < 1 - \frac{1}{2}\delta(1)(1 + f(y))$, we have that

$$\limsup_{n\to\infty} V_\varepsilon^n(x_0) = f(x_0), \quad \forall n < \infty,$$

which leads to

$$\limsup_{\varepsilon \searrow 0} \limsup_{n\to\infty} V_\varepsilon^n(x_0) = f(x_0). \tag{4.6}$$

On the other hand, the second inequality in (4.5) indicates $J^\infty(x_i,\{y\}) > f(x_i)$ for any $i\in\mathbb{N}$. This together with $J^\infty(y,\{y\}) < f(y)$ implies that

$$\hat{S}^\infty = \{y\} \quad \textit{and} \quad V^\infty(x_0) = J^\infty(x_0,\{y\}) = \sum_{k=1}^{\infty}\delta(k)\left(\frac{1}{2}\right)^k f(y).$$

Then by (4.6) and the second inequality in (4.5),

$$\limsup_{\varepsilon\searrow 0}\limsup_{n\to\infty} V_\varepsilon^n(x_0) < V^\infty(x_0).$$

However, if we use $W_\varepsilon^{Q^n}(.,f^n)$ (instead of $V_\varepsilon^{Q^n}(.,f^n)$) to approximate $V^{Q^\infty}(.,f^\infty)$, then we can weaken the uniform convergence in total variation condition to locally uniform convergence as shown in the following proposition.

Proposition 4.1. *Suppose the conditions for $(f^n)_{n\in\overline{\mathbb{N}}}$ and δ in Theorem 4.1 hold, and Q^n converges to Q^∞ locally uniformly in total variation. Assume that for any compact set K and $\varepsilon>0$, there exists a compact set K' such that $\sup_{x\in K} Q^\infty(x,K')\geq 1-\varepsilon$. Then*

$$\lim_{\varepsilon\searrow 0}\left(\liminf_{n\to\infty} W_\varepsilon^{Q^n}(x,f^n)\right) = \lim_{\varepsilon\searrow 0}\left(\limsup_{n\to\infty} W_\varepsilon^{Q^n}(x,f^n)\right) = V^{Q^\infty}(x,f^\infty), \quad \forall x\in\mathbb{X}.$$

The proof of Proposition 4.1 is presented in Section 4.1

4.1 Proofs of Theorem 4.1 and Proposition 4.1

To prepare for the proofs of Theorem 4.1 and Proposition 4.1, we first provide some auxiliary results for (pseudo) ε-equilibria.

Lemma 4.1. *Fix a bounded reward function f and a transition kernel Q. We have that*

$$\mathcal{E}^Q(f,\varepsilon)\subset\mathcal{G}^Q(f,\varepsilon), \quad \forall\varepsilon\geq 0,$$

and

$$V_\varepsilon^Q(x,f)\leq W_\varepsilon^Q(x,f), \quad \forall x\in\mathbb{X},\forall\varepsilon\geq 0.$$

Proof. The result directly follows from Definitions 4.1 and 4.2. □

Lemma 4.2. *Let Assumption 2.1 hold. Let $f\geq 0$ be a bounded reward function and Q be a transition kernel.*

(a) Given $S,T\in\mathcal{G}^Q(f)$, we have that $S\cap T\in\mathcal{G}^Q(f)$.

(b) Let $S,R\in\mathcal{B}$ such that $S\in\mathcal{G}^Q(f)$ and $R\supset S$. Then

$$J^Q(x,S,f)\geq J^Q(x,R,f), \quad \forall x\in\mathbb{X}.$$

Proof. Part (a): We can use the same argument as that in the proof of [16, lemma 4.1] to get that

$$J(x,S\cap T)\geq J(x,S)\vee J(x,T)\geq f(x), \quad \forall\, x\notin S\cap T,$$

which implies $S\cap T\in\mathcal{G}^Q(f)$.

Part (b): Notice that $J^Q(x,S,f)=f(x)=J^Q(x,R,f)$, for all $x\in S$. For $x\notin S$, same discussion in the proof of [17, Lemma 3.1] (or [14, Lemma 4.1]) can be applied to reach that

$$J^Q(x,S,f)\geq J^Q(x,R,f).$$

□

Define

$$S_*(f,Q) := \cap_{s\in\mathcal{G}^Q(f)} S.$$

Recall the smallest optimal equilibrium, $S^*(f,Q) = \cap_{s\in\mathcal{E}^Q(f)} S$ defined in (2.4). The following proposition shows that $S_*(f,Q)$ is optimal among all pseudo equilibria and also coincides with $S^*(f,Q)$.

Proposition 4.2. *Let Assumption 2.1 hold. Given a bounded reward function $f \geq 0$ and a transition kernel Q, we have that*

$$S_*(f,Q) = S^*(f,Q) \quad \textit{and} \quad W^Q(x,f) = J^Q(x, S_*(f,Q), f) = V^Q(x,f), \ \forall x \in \mathbb{X}.$$

Proof. By Lemma 4.1, $\mathcal{E}^Q(f) \subset \mathcal{G}^Q(f)$ and thus $S_*(f,Q) \subset S^*(f,Q)$. We show $S^*(f,Q) \subset S_*(f,Q)$ by the iterative construction for $S^*(f,Q)$. Recall $S^*(f,Q) = \cup_{n\in\mathbb{N}} S_n$ in Lemma 2.1, where $(S_n)_{n\in\mathbb{N}}$ is an increasing sequence defined as $S_0 = \emptyset$, and

$$S_{n+1} := \{x \in \mathbb{X} \setminus S_n : f(x) > \sup_{S: S_n \subset S \subset \mathbb{X}\setminus\{x\}} J^Q(x,S,f)\}, \quad n \in \mathbb{N}.$$

For any $R \in \mathcal{G}^Q(f)$, we prove by induction that

$$S_n \subset R, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

We have $S_0 = \emptyset \subset R$. Suppose $S_n \subset R$, then for any $x \notin R$,

$$f(x) \leq J^Q(x,R,f) \leq \sup_{S: S_n \subset S \subset \mathbb{X}\setminus\{x\}} J^Q(x,S,f),$$

and thus $x \notin S_{n+1}$. Therefore, $S_{n+1} \subset R$.

By (4.7), $S^*(f,Q) = \cup_{n\geq 0} S_n \subset R$ for any $R \in \mathcal{G}^Q(f)$, which implies $S^*(f,Q) \subset S_*(f,Q)$. Hence, $S_*(f,Q) = S^*(f,Q)$. Moroever, for any $S \in \mathcal{G}^Q(f)$, by Lemma 4.2 part (b),

$$J^Q(x, S_*(f,Q), f) \geq J^Q(x,S,f), \quad \forall x \in \mathbb{X},$$

so $J^Q(., S_*(f,Q), f) = W^Q(., f)$. Together with Lemma 2.1, we have that

$$W^Q(x,f) = J^Q(x, S_*(f,Q), f) = J^Q(x, S^*(f,Q), f) = V^Q(x,f), \quad \forall x \in \mathbb{X}.$$

□

Lemma 4.3. *Suppose Assumption 2.1 holds. For any $0 \leq \varepsilon_1 \leq \varepsilon_2$, we have that*

$$\mathcal{G}^Q((f-\varepsilon_1)\vee 0) \subset \mathcal{G}^Q((f-\varepsilon_2)\vee 0). \tag{4.8}$$

Therefore,

$$S_*((f-\varepsilon_1)\vee 0, Q) \supseteq S_*((f-\varepsilon_2)\vee 0, Q). \tag{4.9}$$

Proof. Let $S \in \mathcal{G}^Q(f-\varepsilon_1)$. For any $x \notin S$,

$$\begin{aligned}
&|J^Q(x,S,(f-\varepsilon_1)\vee 0) - J^Q(x,S,(f-\varepsilon_2)\vee 0)| \\
=&\mathbb{E}^x\left[\delta(\rho(S))\left(\left(\left(f\left(X_{\rho(S)}\right)-\varepsilon_1\right)\vee 0\right) - \left(\left(f\left(X_{\rho(S)}\right)-\varepsilon_2\right)\vee 0\right)\right)\right] \\
\leq&\mathbb{E}^x[\delta(\rho(S))(\varepsilon_2-\varepsilon_1)] \leq \varepsilon_2 - \varepsilon_1.
\end{aligned}$$

If $f(x) \geq \varepsilon_2$, then

$$\begin{aligned}
&J^Q(x,S,(f-\varepsilon_2)\vee 0) \geq J^Q(x,S,(f-\varepsilon_1)\vee 0) - (\varepsilon_2-\varepsilon_1) \\
\geq& f(x) - \varepsilon_1 - (\varepsilon_2 - \varepsilon_1) = f(x) - \varepsilon_2,
\end{aligned}$$

where the second inequality follows that $S \in \mathcal{G}^Q(f-\varepsilon_1)$. If $f(x) < \varepsilon_2$, then $J^Q(x,S,(f-\varepsilon_2)\vee 0) \geq 0 = (f(x)-\varepsilon_2)\vee 0$. Hence, $S \in \mathcal{G}^Q(f-\varepsilon_2)$. □

Lemma 4.4. *Suppose Assumption 2.1 holds. Given a bounded reward function $f \geq 0$ and a transition kernel Q, we have that*

$$S^*((f-\varepsilon)\vee 0, Q) = S_*((f-\varepsilon)\vee 0, Q) \uparrow S_*(f,Q) = S^*(f,Q), \quad \text{as } \varepsilon \searrow 0, \tag{4.10}$$

and

$$\lim_{\varepsilon\searrow 0} V^Q(x,(f-\varepsilon)\vee 0) = V^Q(x,f), \quad \forall x \in \mathbb{X}. \tag{4.11}$$

Proof. We first consider (4.10). By Lemma 4.3, $S_*((f-\varepsilon)\vee 0, Q)$ increases as $\varepsilon \searrow 0$, so

$$S' := \cup_{\varepsilon>0} S_*((f-\varepsilon)\vee 0, Q) \subset S_*(f,Q).$$

Given $x \notin S'$,

$$\begin{aligned}\mathbb{E}^x[\delta(\rho(S'))f(X_{\rho(S')})] =& \lim_{\varepsilon\searrow 0} E^x[\delta(\rho(S_*((f-\varepsilon)\vee 0,Q)))((f(X_{\rho(S_*((f-\varepsilon)\vee 0,Q))})-\varepsilon)\vee 0)]\\ =& \lim_{\varepsilon\searrow 0} J^Q(x, S_*((f-\varepsilon)\vee 0,Q),(f-\varepsilon)\vee 0) \geq \lim_{\varepsilon\searrow 0}(f(x)-\varepsilon)\vee 0\\ =& f(x),\end{aligned}$$

where the second line follows that $x \notin S_*((f-\varepsilon)\vee 0, Q)$. Hence, $S' \in \mathcal{G}^Q(f)$ and $S_*(f,Q) \subset S'$, which implies $S' = S_*(Q,f)$. Then by Proposition 4.2,

$$S^*((f-\varepsilon)\vee 0, Q) = S_*((f-\varepsilon)\vee 0, Q) \uparrow S_*(f,Q) = S^*(f,Q), \quad \text{as } \varepsilon \searrow 0.$$

Now we prove (4.11). By (4.10), for $x \in S^*(f,Q)$, $x \in S^*((f-\varepsilon)\vee 0, Q)$ for ε small enough, and thus

$$\lim_{\varepsilon\searrow 0} V^Q(x,(f-\varepsilon)\vee 0) = \lim_{\varepsilon\searrow 0}(f(x)-\varepsilon)\vee 0 = f(x) = V^Q(x,f), \quad \forall x \in S_*(f,Q).$$

For $x \notin S_*(f,Q)$, by (4.10), $\rho(S^*(f-\varepsilon)\vee 0, Q) \to \rho(S^*(f,Q))$ a.s. and $(f-\varepsilon)\vee 0 \to f$ as $\varepsilon \searrow 0$. Then by Dominated Convergence Theorem,

$$\begin{aligned}\lim_{\varepsilon\searrow 0} V^Q(x,(f-\varepsilon)\vee 0) =& \lim_{\varepsilon\searrow 0} E^x[\delta(\rho(S^*((f-\varepsilon)\vee 0,Q)))((f(X_{\rho(S^*((f-\varepsilon)\vee 0,Q))})-\varepsilon)\vee 0)]\\ =& \mathbb{E}^x[\delta(\rho(S^*(f,Q)))f(X_{\rho(S^*(f,Q))})] = V^Q(x,f), \quad x \notin S_*(f,Q).\end{aligned}$$

which completes the proof of (4.11). $\square$

Lemma 4.5. *Suppose Assumption 2.1 holds. Let $f \geq 0$ be a bounded reward function and Q be a transition kernel. Then for any $\varepsilon > 0$, we have that*

$$\mathcal{G}^Q(f) \subset \mathcal{G}^Q((f-\varepsilon)\vee 0) \subset \mathcal{G}^Q(f,\varepsilon) \subset \mathcal{G}^Q\left(\left(f - \frac{\varepsilon}{1-\delta(1)}\right)\vee 0\right).$$

Proof. $\mathcal{G}^Q(f) \subset \mathcal{G}^Q((f-\varepsilon)\vee 0)$ follows from Lemma 4.3.

Let $S \in \mathcal{G}^Q((f-\varepsilon)\vee 0)$. For any $x \notin S$, if $f(x) \geq \varepsilon$, then

$$\mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})] \geq \mathbb{E}_x^Q[\delta(\rho(S))((f(X_{\rho(S)})-\varepsilon)\vee 0)] \geq (f(x)-\varepsilon)\vee 0 = f(x)-\varepsilon.$$

If $f(x) < \varepsilon$, obviously, $\mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})] \geq 0 > f(x)-\varepsilon$. So $S \in \mathcal{G}^Q(f,\varepsilon)$.

Let $S \in \mathcal{G}^Q(f, \varepsilon)$. Take $x \notin S$. If $f(x) \geq \frac{\varepsilon}{1-\delta(1)}$, then by $\rho(S) \geq 1$ we have that

$$\begin{aligned}\mathbb{E}_x^Q\left[\delta(\rho(S))\left(\left(f(X_{\rho(S)}) - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right)\right] \geq & \mathbb{E}_x^Q[\delta(\rho(S))f(X_{\rho(S)})] - \delta(1) \cdot \frac{\varepsilon}{1-\delta(1)} \\ \geq & f(x) - \varepsilon - \frac{\delta(1)\varepsilon}{1-\delta(1)} = f(x) - \frac{\varepsilon}{1-\delta(1)},\end{aligned}$$

where the second line follows from $S \in \mathcal{G}^Q(f, \varepsilon)$. If $f(x) < \frac{\varepsilon}{1-\delta(1)}$, then

$$\mathbb{E}_x^Q\left[\delta(\rho(S))\left(\left(f(X_{\rho(S)}) - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right)\right] \geq 0 = \left(f(x) - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0.$$

Hence, $S \in \mathcal{G}^Q((f - \frac{\varepsilon}{1-\delta(1)}) \vee 0)$.

□

Proof of Theorem 4.1. The proof is a combination of the following two steps.

Step 1. We first prove, under assumptions in Theorem 4.1, that

$$V^{Q^\infty}(x, f^\infty) \leq \liminf_{n\to\infty} V_\varepsilon^{Q^n}(x, f^n) \leq \liminf_{n\to\infty} W_\varepsilon^{Q^n}(x, f^n), \quad \forall \varepsilon > 0. \tag{4.12}$$

Let $\varepsilon > 0$. Applying Lemma 2.3(c) with $\tau = \rho(S^*(f^\infty, Q^\infty))$, there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{X}} |v^{Q^n}(x, \rho(S^*(f^\infty, Q^\infty)), f^n) - v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty)), f^\infty)| \leq \varepsilon.$$

Then

$$\begin{aligned}v^{Q^n}(x, \rho(S^*(f^\infty, Q^\infty)), f^n) \geq v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty)), f^\infty) - \varepsilon \geq f(x) - \varepsilon, \quad \forall x \notin S^*(f^\infty, Q^\infty),\\ v^{Q^n}(x, \rho(S^*(f^\infty, Q^\infty)), f^n) \leq v^{Q^\infty}(x, \rho(S^*(f^\infty, Q^\infty)), f^\infty) + \varepsilon \leq f(x) + \varepsilon, \quad \forall x \in S^*(f^\infty, Q^\infty).\end{aligned}$$

Hence, $S^*(f^\infty, Q^\infty) \in \mathcal{E}_\varepsilon^{Q^n}(f^n)$ for all $n \geq N$.

Now take $x \in \mathbb{X}$. For $n \geq N$, by Definition 4.2 and (4.3),

$$V_\varepsilon^{Q^n}(x, f^n) \geq J^{Q^n}(x, S^*(f^\infty, Q^\infty), f^n),$$

which leads to

$$\liminf_{n\to\infty} V_\varepsilon^{Q^n}(x, f^n) \geq \liminf_{n\to\infty} J^{Q_n}(x, S^*(f^\infty, Q^\infty), f^n) = V^{Q^\infty}(x, f^\infty),$$

where the second (in)equality follows from Lemma 2.3(a). By Lemma 4.1, $W_\varepsilon^{Q^n}(x, f^n) \geq V_\varepsilon^{Q^n}(x, f^n)$, and Step 1 is completed.

Step 2. Now we show, under the same assumptions in Theorem 3.1 (which are weaker than the assumptions in Theorem 4.1), that

$$\lim_{\varepsilon \searrow 0}\left(\limsup_{n\to\infty} V_\varepsilon^{Q^n}(x, f^n)\right) \leq \lim_{\varepsilon \searrow 0}\left(\limsup_{n\to\infty} W_\varepsilon^{Q^n}(x, f^n)\right) \leq V^{Q^\infty}(x, f^\infty), \quad \forall x \in \mathbb{X}. \tag{4.13}$$

By Theorem 3.1 and Proposition 4.2, for any $\varepsilon \geq 0$,

$$\begin{aligned}\limsup_{n\to\infty} V^{Q^n}\left(x, \left(f^n - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) =& \limsup_{n\to\infty} W^{Q^n}\left(x, \left(f^n - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) \\ \leq & V^{Q^\infty}\left(x, \left(f^\infty - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right), \quad \forall x \in \mathbb{X}.\end{aligned} \tag{4.14}$$

Meanwhile, for $n \in \overline{\mathbb{N}}$,

$$
\begin{aligned}
V_{\varepsilon}^{Q^n}(x, f^n) \leq & W_{\varepsilon}^{Q^n}(x, f^n) \\
\leq & W^{Q^n}\left(x, \left(f^n - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) + \frac{\varepsilon}{1-\delta(1)} \\
= & V^{Q^n}\left(x, \left(f^n - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) + \frac{\varepsilon}{1-\delta(1)}, \quad \forall x \in \mathbb{X}.
\end{aligned} \tag{4.15}
$$

where the first line follows from Lemma 4.1, the second line follows from $\mathcal{G}^{Q^n}(f^n, \varepsilon) \subset \mathcal{G}^{Q_n}((f^n - \frac{\varepsilon}{1-\delta(1)}) \vee 0)$ implied by Lemma 4.5, and the last line follows from Proposition 4.2. By (4.14) and (4.15), for any $\varepsilon \geq 0$ and $x \in \mathbb{X}$,

$$
\begin{aligned}
\limsup_{n\to\infty} V_{\varepsilon}^{Q^n}(x, f^n) \leq \limsup_{n\to\infty} W_{\varepsilon}^{Q^n}(x, f^n) \leq & \limsup_{n\to\infty} V^{Q^n}\left(x, \left(f^n - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) + \frac{\varepsilon}{1-\delta(1)} \\
\leq & V^{Q^\infty}\left(x, \left(f^\infty - \frac{\varepsilon}{1-\delta(1)}\right) \vee 0\right) + \frac{\varepsilon}{1-\delta(1)},
\end{aligned}
$$

Then (4.13) follows by setting $Q = Q^\infty$ in (4.11).

□

Proof of Proposition 4.1. Step 1. Let $\varepsilon > 0$. We first prove that for any $x \in \mathbb{X} \setminus S^*(f^\infty, Q^\infty)$, there exists a set S_x and $N \in \mathbb{N}$ such that

$$
S_x \in \mathcal{G}^{Q^n}(f^n, \varepsilon), \quad J^{Q^n}(x, S^*(f^\infty, Q^\infty), f^n) \leq J^{Q^n}(x, S_x, f^n) + \varepsilon \quad \text{and} \quad \forall n \geq N.
$$

Fix $x \notin S^*(f^\infty, Q^\infty)$. As $\sup_{n\in\overline{\mathbb{N}}} \|f^n\|_\infty =: M < \infty$, we can take $T \in \mathbb{N}$ such that $\delta(T)M < \varepsilon/2$. Then we apply the same discussion as (A.5) to find a compact set K and $N_1 \in \mathbb{N}$ (that may depend on x) such that

$$
2M\left(1 - \mathbb{P}_x^{Q^n}(X_t \in K,\ t = 0, \dots, T)\right) = 2M \cdot \mathbb{P}_x^{Q^n}(\rho(\mathbb{X} \setminus K) \leq T) < \varepsilon/2, \quad \forall N_1 \leq n \leq \infty. \tag{4.16}
$$

By Lemma 2.3(b),

$$
\lim_{n\to\infty} \sup_{y \in (K \setminus S^*(f^\infty, Q^\infty))} |J^{Q^\infty}(y, S^*(f^\infty, Q^\infty), f^\infty) - J^{Q^n}(y, S^*(f^\infty, Q^\infty), f^n)| = 0.
$$

This together with the locally uniform convergence of $(f^n)_{n\in\mathbb{N}}$, we can find $N_2 \in \mathbb{N}$ (that may depend on x) such that, for all $n \geq N_2$, $\sup_{y\in K} |f^n(y) - f^\infty(y)| < \frac{\varepsilon}{2}$ and

$$
J^{Q^\infty}(y, S^*(f^\infty, Q^\infty)) - \frac{\varepsilon}{2} \leq J^{Q^n}(y, S^*(f^\infty, Q^\infty), f^\infty), \quad \forall y \in (K \setminus S^*(f^\infty, Q^\infty)).
$$

This imply that for all $n \geq N_2$,

$$
\begin{aligned}
f^n(y) - \varepsilon \leq & f^\infty(y) - \frac{\varepsilon}{2} \leq J^{Q^\infty}(y, S^*(f^\infty, Q^\infty), f^\infty) - \frac{\varepsilon}{2} \\
\leq & J^{Q^n}(y, S^*(f^\infty, Q^\infty), f^n), \quad \forall y \in (K \setminus S^*(f^\infty, Q^\infty)).
\end{aligned} \tag{4.17}
$$

Let

$$
S_x := S^*(f^\infty, Q^\infty) \cup (\mathbb{X} \setminus K).
$$

By (4.17), $S_x \in \mathcal{G}_\varepsilon^{Q^n}(f^n)$ for $n \geq N_2$. Moreover, for any $n \geq N := N_1 \vee N_2$,

$$
\begin{aligned}
&|J^{Q^n}(x, S^*(f^\infty, Q^\infty), f^n) - J^{Q^n}(x, S_x, f^n)| \\
\leq &\mathbb{E}_x^{Q^n}[|\delta(\rho(S^*(f^\infty, Q^\infty)))f^n(X_{\rho(S^*(f^\infty,Q^\infty))}) - \delta(\rho(S_x))f^n(X_{\rho(S_x)})| \cdot 1_{\{X_{\rho(S_x)} \notin S^*(f^\infty,Q^\infty), \rho(S_x) \geq T\}}] \\
&+ \mathbb{E}_x^{Q^n}[|\delta(\rho(S^*(f^\infty, Q^\infty)))f^n(X_{\rho(S^*(f^\infty,Q^\infty))}) - \delta(\rho(S_x))f^n(X_{\rho(S_x)})| \cdot 1_{\{X_{\rho(S_x)} \notin S^*(f^\infty,Q^\infty), \rho(S_x) < T\}}] \\
\leq &2M\delta(T) + 2M \cdot \mathbb{P}_x^{Q^n}(\rho(\mathbb{X} \setminus K) \leq T) \\
< &\varepsilon,
\end{aligned}
$$

where the last line follows from (4.16) and $\delta(T)M < \varepsilon/2$. Step 1 is completed.

Step 2. For any $x \notin S^*(f^\infty, Q^\infty)$, we can find $N' \in \mathbb{N}$ (which may depend on x) such that

$$|J^{Q^\infty}(x, S^*(f^\infty, Q^\infty), f^\infty) - J^{Q^n}(x, S^*(f^\infty, Q^\infty), f^\infty)| < \frac{\varepsilon}{2}, \quad \forall n \geq N'.$$

Then from Step 1,

$$
\begin{aligned}
V^{Q^\infty}(x, f^\infty) = &J^{Q^\infty}(x, S^*(f^\infty, Q^\infty), f^\infty) \leq J^{Q^n}(x, S^*(f^\infty, Q^\infty), f^n) + \varepsilon \\
\leq &J^{Q^n}(x, S_x, f^n) + 2\varepsilon \leq W_\varepsilon^{Q^n}(f^n) + 2\varepsilon, \quad \forall n \geq N \vee N'.
\end{aligned}
$$

Letting $n \to \infty$ then $\varepsilon \searrow 0$, we have that

$$V^{Q^\infty}(x, f^\infty) \leq \lim_{\varepsilon \searrow 0} \left(\liminf_{n\to\infty} W_\varepsilon^{Q^n}(x, f^n) \right), \quad \forall x \in \mathbb{X}.$$

Then the rest follows from Step 2 in the proof of Theorem 4.1. □

A Proofs of the lemmata in Section 2

Proof of Lemma 2.1. Set $S_\infty := \cup_{k \in \mathbb{N}} S_k$. One can easily check that same arguments for S_∞ in the proof of Theorem 2.5 in [3] is applicable for S_∞.[1] More specifically, Lemmas 2.10, 2.12, 2.13, and the contradiction discussion in the first part of the proof for Theorem 2.5 in [3] can be applied, and one can obtain an inequality similar as that in [3, Theorem 2.5] as follows:

$$J^Q(y, R, f) - J^Q(y_\infty, S^*(f, Q), f) \leq \mathbb{E}_y^Q[\delta(\rho(R))]\alpha \leq \delta(1)\alpha < \alpha,$$

where the first inequality appears in the proof of [3, Theorem 2.5], and the second inequality follows our time discrete setting. Hence, the same contradiction is reached as that in first part of the proof for [3, Theorem 2.5], and we have the following:

(i) $S_\infty \subset R, \quad \forall R \in \mathcal{E}^Q(f)$;

(ii) For any $S \in \mathcal{E}^Q(f)$ and $T \in \mathcal{B}$ with $S \subset T$,

$$J^Q(x, S, f) \geq J^Q(x, T, f), \quad \forall x \in \mathbb{X}.$$

(iii) S_∞ is an equilibrium.

By (i) and (iii), $S_\infty = \cap_{S \in \mathcal{E}^Q(f)} S = S^*(Q, f)$. Then (ii) implies that $J^Q(x, S^*(Q, f), f) \geq J^Q(x, S, f)$ for any $S \in \mathcal{E}^Q(f)$. As a result, $S^*(Q, f)$ is an optimal equilibrium and $V^Q(x, f) = J^Q(x, S^*(Q, f), f)$. □

[1]The process X is a continuous-time Markov chain in [3], while in this paper X is a discrete-time Markov process.

Proof of Lemma 2.2. Denote

$$Q^n_T(x,\cdot) := Q^n(x,dx_1)\otimes Q^n(x_1,dx_2)\ldots\otimes Q^n(x_{k-1},dx_k),\quad x\in\mathbb{X}, n\in\overline{\mathbb{N}}.$$

Part (a): Let $\varepsilon>0$. For any $x\in\mathbb{X}$ and compact set $K_0\subset\mathbb{X}$ we have that

$$\begin{aligned}
&Q^n_T(x,(K_0)^T) = \int_{K_0} Q^n(x,dx_1)\int_{K_0} Q^n(x_1,dx_2)\ldots\int_{K_0} Q^n(x_{T-1},dx_T)\\
\geq&\int_{K_0} Q^n(x,dx_1)\ldots\int_{K_0} Q^n(x_{T-2},dx_{T-1})\int_{K_0} Q^\infty(x_{T-1},dx_T) - \sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}}\\
\geq&\int_{K_0} Q^n(x,dx_1)\ldots\int_{K_0} Q^n(x_{T-3},dx_{T-2})\int_{K_0} Q^\infty(x_{T-2},dx_{T-1})\int_{K_0} Q^\infty(x_{T-1},dx_T)\\
&-2\sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}}\\
\ldots&\\
\geq&\int_{K_0} Q^\infty(x,dx_1)\ldots\int_{K_0} Q^\infty(x_{T-1},dx_T) - T\sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}}\\
=&Q^\infty_T(x,(K_0)^T) - T\sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}}.
\end{aligned}\tag{A.1}$$

Exchanging Q^∞, Q^n in the above inequality and combining with (A.1), we have

$$|Q^n_T(x,(K_0)^T) - Q^\infty_T(x,(K_0)^T)| \leq T\sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}},\quad \forall x\in\mathbb{X}.\tag{A.2}$$

There exists compact subset K' (that may depend on x) such that

$$Q^\infty_T(x,(K')^T)\geq 1-\varepsilon/2.\tag{A.3}$$

By (A.2) with $K_0=K'$, there exists $N\in\mathbb{N}$ (that may depend on K') such that

$$T\cdot\sup_{y\in K'}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}}\leq\varepsilon/2,\quad\forall n\geq N,\tag{A.4}$$

This together with (A.3) implies that

$$Q^n_T(x,(K')^T)\geq 1-\varepsilon,\quad \forall N\leq n\leq\infty.\tag{A.5}$$

Hence, for any $g\in B(\mathbb{X}^T;[-1,1])$,

$$\left|\mathbb{E}^{Q^n}_x\left[g(X_1,X_2,\ldots,X_T)\right] - \mathbb{E}^{Q^n}_x\left[g(X_1,X_2,\ldots,X_T)\cdot 1_{\{X_t\in K', t=1,\ldots,T\}}\right]\right|\leq\varepsilon,\quad\forall n\geq N.\tag{A.6}$$

Using a similar argument as that for (A.2), we can show that for any compact set $K_0\subset\mathbb{X}$,

$$\begin{aligned}
&\left|\mathbb{E}^{Q^n}_x\left[g(X_1,X_2,\ldots,X_T)\cdot 1_{\{X_t\in K_0,t=1,\ldots,T\}}\right] - \mathbb{E}^{Q^\infty}_x\left[g(X_1,X_2,\ldots,X_T)\cdot 1_{\{X_t\in K_0,t=1,\ldots,T\}}\right]\right|\\
\leq&T\sup_{y\in K_0}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}},\quad\forall x\in\mathbb{X}.
\end{aligned}\tag{A.7}$$

By (A.7) with $K_0=K'$ and (A.6),

$$\begin{aligned}
&\sup_{g\in B(\mathbb{X}^T;[-1,1])}|\mathbb{E}^{Q^n}_x g(X_1,X_2,\ldots,X_T) - \mathbb{E}^{Q^\infty}_x g(X_1,X_2,\ldots,X_T)|\\
&\leq 2\varepsilon + T\sup_{y\in K'}||Q^n(y,.)-Q^\infty(y,.)||_{\mathrm{TV}},\quad\forall n\geq N.
\end{aligned}\tag{A.8}$$

Then the result follows by sending $n \to \infty$ and then $\varepsilon \to 0$.

Part (b): For any $x \in K$, the same discussion from (A.4) to (A.7) can be applied. Notice that now the compact set K' in (A.3) does not depend on x and the integer N in (A.4) only depends on K'. Hence, (A.8) is now rewritten as

$$\begin{aligned}
&\sup_{x\in K, g\in B(\mathbb{X}^T;[-1,1])} \left|\mathbb{E}_x^{Q^n} g(X_1, X_2, \dots, X_T) - \mathbb{E}_x^{Q^\infty} g(X_1, X_2, \dots, X_T)\right| \\
\le& 2\varepsilon + T \sup_{y\in K'} ||Q^n(y,.) - Q^\infty(y,.)||_{\mathrm{TV}}, \quad \forall n \ge N.
\end{aligned}$$

Part (c): The same argument from part (a) can be applied and in this case N is independent of x. Then we can extend (A.8) to

$$\begin{aligned}
&\sup_{x\in \mathbb{X}, g\in B(\mathbb{X}^T;[-1,1])} \left|\mathbb{E}_x^{Q^n} g(X_1, X_2, \dots, X_T) - \mathbb{E}_x^{Q^\infty} g(X_1, X_2, \dots, X_T)\right| \\
\le& 2\varepsilon + T \sup_{y\in \mathbb{X}} ||Q^n(y,.) - Q^\infty(y,.)||_{\mathrm{TV}}, \quad \forall\, n \ge N.
\end{aligned}$$

□

Proof of Lemma 2.3. Part (a): Let $\varepsilon > 0$. As $M := \sup_{n\in\overline{\mathbb{N}}} \|f^n\|_\infty < \infty$, there exists $T \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{X}, n\in\overline{\mathbb{N}}, \tau\in\mathcal{T}} \left|v^{Q^n}(x,\tau,f^n) - \mathbb{E}_x^{Q^n}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right]\right| < \varepsilon/4. \tag{A.9}$$

Take $x \in \mathbb{X}$. By Lemma 2.2(a), there exists $N \in \mathbb{N}$ (that may depend on x) such that

$$\sup_{m\in\overline{\mathbb{N}}, \tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^n}\left[\delta(\tau) f^m(X_\tau) 1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^m(X_\tau) 1_{\{\tau\le T\}}\right]\right| \le \varepsilon/4, \quad \forall n \ge N. \tag{A.10}$$

By the locally uniform convergence of $(f^n)_{n\in\overline{\mathbb{N}}}$, we can first choose a compact set K' (that may depend on x) then choose $N' \in \mathbb{N}$ (that may depend on K') such that

$$Q_T^\infty(x,(K')^T) \ge 1 - \frac{\varepsilon}{16M} \quad \text{and} \quad \sup_{y\in K'} |f^n(y) - f^\infty(y)| \le \frac{\varepsilon}{8}, \ \forall n \ge N'.$$

Then

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^\infty(X_\tau) 1_{\{\tau\le T\}}\right]\right| \\
\le& \sup_{\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T \text{ and } X_t\in K', 1\le t\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^\infty(X_\tau) 1_{\{\tau\le T \text{ and } X_t\in K', 1\le t\le T\}}\right]\right| \\
&+ 2\cdot M \cdot \frac{\varepsilon}{16M} \\
\le& \sup_{y\in K'} |f^n(y) - f^\infty(y)| + \frac{\varepsilon}{8} \le \frac{\varepsilon}{4}, \quad \forall n \ge N'.
\end{aligned} \tag{A.11}$$

Therefore, by (A.9)-(A.11), for all $n \ge N \vee N'$,

$$\begin{aligned}
\sup_{\tau\in\mathcal{T}} \left|v^{Q^n}(x,\tau,f^n) - v^{Q^\infty}(x,\tau,f^\infty)\right| \le& \sup_{\tau\in\mathcal{T}} \left|v^{Q^n}(x,\tau,f^n) - \mathbb{E}_x^{Q^n}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right]\right| \\
&+ \sup_{\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^n}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right]\right| \\
&+ \sup_{\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^n(X_\tau) 1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^\infty(X_\tau) 1_{\{\tau\le T\}}\right]\right| \\
&+ \sup_{\tau\in\mathcal{T}} \left|v^{Q^\infty}(x,\tau,f^\infty) - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau) f^\infty(X_\tau) 1_{\{\tau\le T\}}\right]\right| \\
\le& \varepsilon.
\end{aligned} \tag{A.12}$$

Part (b): Fix a compact set K. By Lemma 2.2(b), we can apply the steps through (A.10)—(A.12) by replacing all $\sup_{\tau\in\mathcal{T}}$ (respectively, $\sup_{m\in\overline{\mathbb{N}},\tau\in\mathcal{T}}$) with $\sup_{x\in K,\tau\in\mathcal{T}}$ (respectively, $\sup_{x\in K,m\in\overline{\mathbb{N}},\tau\in\mathcal{T}}$). Notice that, by assumption on Q^∞, the constants N, K' in this case only depend on K instead of x. Hence, the result follows.

Part (c): By Lemma 2.2(c), there exists $N > 0$ such that

$$\sup_{x\in\mathbb{X},m\in\overline{\mathbb{N}},\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^n}\left[\delta(\tau)f^m(X_\tau)1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau)f^m(X_\tau)1_{\{\tau\le T\}}\right]\right| \le \varepsilon/4, \quad \forall n\ge N. \tag{A.13}$$

In addition, choose $N'\in\mathbb{N}$ such that $\|f^n - f^\infty\|_\infty < \frac{\varepsilon}{4}$ for any $n\ge N'$, Then

$$\sup_{x\in\mathbb{X},\tau\in\mathcal{T}} \left|\mathbb{E}_x^{Q^\infty}\left[\delta(\tau)f^n(X_\tau)1_{\{\tau\le T\}}\right] - \mathbb{E}_x^{Q^\infty}\left[\delta(\tau)f^\infty(X_\tau)1_{\{\tau\le T\}}\right]\right| \le \|f^n - f^\infty\|_\infty < \frac{\varepsilon}{4}, \quad \forall n\ge N'. \tag{A.14}$$

Combining (A.9), (A.13) and (A.14), and replacing "$\sup_{\tau\in\mathcal{T}}$" with "$\sup_{x\in\mathbb{X},\tau\in\mathcal{T}}$" in (A.12), we achieve the desired result. □

References

[1] Erhan Bayraktar, Zhenhua Wang, and Zhou Zhou. Equilibria of Time-inconsistent Stopping for One-dimensional Diffusion Processes. *arXiv e-prints*, page arXiv:2201.07659, January 2022.

[2] Erhan Bayraktar, Jingjie Zhang, and Zhou Zhou. Time consistent stopping for the mean-standard deviation problem—the discrete time case. *SIAM J. Financial Math.*, 10(3):667–697, 2019.

[3] Erhan Bayraktar, Jingjie Zhang, and Zhou Zhou. Equilibrium concepts for time-inconsistent stopping problems in continuous time. *Math. Finance*, 31(1):508–530, 2021.

[4] Tomas Björk, Mariana Khapko, and Agatha Murgoci. On time-inconsistent stochastic control in continuous time. *Finance Stoch.*, 21(2):331–360, 2017.

[5] Tomas Björk, Mariana Khapko, and Agatha Murgoci. *Time-inconsistent control theory with finance applications.* Springer Finance. Springer, Cham, [2021] ©2021.

[6] Sören Christensen and Kristoffer Lindensjö. On finding equilibrium stopping times for time-inconsistent Markovian problems. *SIAM J. Control Optim.*, 56(6):4228–4255, 2018.

[7] Sören Christensen and Kristoffer Lindensjö. On time-inconsistent stopping problems and mixed strategy stopping times. *Stochastic Process. Appl.*, 130(5):2886–2917, 2020.

[8] Sören Christensen and Kristoffer Lindensjö. Time-inconsistent stopping, myopic adjustment and equilibrium stability: with a mean-variance application. In *Stochastic modeling and control*, volume 122 of *Banach Center Publ.*, pages 53–76. Polish Acad. Sci. Inst. Math., Warsaw, 2020.

[9] Zachary Feinstein. Continuity and sensitivity analysis of parameterized Nash games. *arXiv preprint arXiv:2007.04388*, 2020.

[10] Zachary Feinstein, Birgit Rudloff, and Jianfeng Zhang. Dynamic set values for nonzero-sum games with multiple equilibriums. *Math. Oper. Res.*, 47(1):616–642, 2022.

[11] Xue Dong He and Zhao Li Jiang. On the equilibrium strategies for time-inconsistent problems in continuous time. *SIAM J. Control Optim.*, 59(5):3860–3886, 2021.

[12] Yu-Jui Huang and Adrien Nguyen-Huu. Time-consistent stopping under decreasing impatience. *Finance Stoch.*, 22(1):69–95, 2018.

[13] Yu-Jui Huang, Adrien Nguyen-Huu, and Xun Yu Zhou. General stopping behaviors of naïve and noncommitted sophisticated agents, with application to probability distortion. *Math. Finance*, 30(1):310–340, 2020.

[14] Yu-Jui Huang and Zhenhua Wang. Optimal equilibria for multidimensional time-inconsistent stopping problems. *SIAM J. Control Optim.*, 59(2):1705–1729, 2021.

[15] Yu-Jui Huang and Xiang Yu. Optimal stopping under model ambiguity: a time-consistent equilibrium approach. *Math. Finance*, 31(3):979–1012, 2021.

[16] Yu-Jui Huang and Zhou Zhou. The optimal equilibrium for time-inconsistent stopping problems—the discrete-time case. *SIAM J. Control Optim.*, 57(1):590–609, 2019.

[17] Yu-Jui Huang and Zhou Zhou. Optimal equilibria for time-inconsistent stopping problems in continuous time. *Math. Finance*, 30(3):1103–1134, 2020.

[18] Yu-Jui Huang and Zhou Zhou. Strong and weak equilibria for time-inconsistent stochastic control in continuous time. *Math. Oper. Res.*, 46(2):428–451, 2021.

[19] Yu-Jui Huang and Zhou Zhou. A time-inconsistent Dynkin game: from intra-personal to inter-personal equilibria. *Finance Stoch.*, 26(2):301–334, 2022.

[20] Zongxia Liang and Fengyi Yuan. Weak equilibriums for time-inconsistent stopping control problems. *arXiv preprint arXiv:2105.06607*, 2021.

[21] Ken Seng Tan, Wei Wei, and Xun Yu Zhou. Failure of smooth pasting principle and nonexistence of equilibrium stopping rules under time-inconsistency. *SIAM J. Control Optim.*, 59(6):4136–4154, 2021.

[22] Zhou Zhou and Zhuo Jin. Optimal equilibrium barrier strategies for time-inconsistent dividend problems in discrete time. *Insurance Math. Econom.*, 94:100–108, 2020.