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Complexity of Gaussian Random Fields with Isotropic Increments

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Abstract: We study the energy landscape of a model of a single particle on a random potential, that is, we investigate the topology of level sets of smooth random fields on $\mathbb{R}^N$ of the form $X_N(x) + \frac{\mu}{2}\|x\|^2$, where X_N is a Gaussian process with isotropic increments. We derive asymptotic formulas for the mean number of critical points with critical values in an open set as the dimension N goes to infinity. In a companion paper, we provide the same analysis for the number of critical points with a given index.

1. Introduction

In this paper we provide asymptotics for the number of critical points of Gaussian random fields with isotropic increments (a.k.a. locally isotropic Gaussian random fields) in the high dimensional limit. The definition of locally isotropic fields was first formulated by Kolmogorov about 80 years ago [Kol41] for the application in statistical theory of turbulence; see [Yag57] for an account of background and early history.

The model is defined as follows. Let $B_N \subset \mathbb{R}^N$ be a sequence of subsets and let $H_N : B_N \subset \mathbb{R}^N \to \mathbb{R}$ be given by

$$H_N(x) = X_N(x) + \frac{\mu}{2}\|x\|^2, \tag{1.1}$$

where $\mu \in \mathbb{R}$, $\|x\|$ is the Euclidean norm of x, and X_N is a Gaussian random field that satisfies

$$\mathbb{E}[(X_N(x) - X_N(y))^2] = N D\Big(\frac{1}{N}\|x - y\|^2\Big), \quad x, y \in \mathbb{R}^N.$$

Here the function $D : \mathbb{R}_+ \to \mathbb{R}_+$ is called the correlator (or structure) function and $\mathbb{R}_+ = [0, \infty)$. It determines the law of X_N up to an additive shift by a Gaussian random

variable. Complete characterization of all correlators was given in the work of Yaglom [Yag57] (see also the general form of a positive definite kernel due to Schoenberg [Sch38]). In short, if D is the correlator function for all $N \in \mathbb{N}$, then X_N must belong to one of the following two classes (see also [Kli12, Theorem A.1]):

1. **Isotropic fields.** There exists a function $B : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathbb{E}[X_N(x)X_N(y)] = NB\left(\frac{1}{N}\|x - y\|^2\right)$$

where B has the representation

$$B(r) = c_0 + \int_{(0,\infty)} e^{-rt^2} \nu(\mathrm{d}t),$$

and $c_0 \in \mathbb{R}_+$ is a constant and ν is a finite measure on $(0, \infty)$. In this case,

$$D(r) = 2(B(0) - B(r)).$$

2. **Non-isotropic fields with isotropic increments.** The correlator D can be written as

$$D(r) = \int_{(0,\infty)} (1 - e^{-rt^2})\nu(\mathrm{d}t) + Ar, \quad r \in \mathbb{R}_+, \tag{1.2}$$

where $A \in \mathbb{R}_+$ is a constant and ν is a σ-finite measure with

$$\int_{(0,\infty)} \frac{t^2}{1 + t^2}\nu(\mathrm{d}t) < \infty.$$

See [Yag87, Section 25.3] for more details on locally isotropic fields. Case 1 is known as short-range correlation (SRC) processes and case 2 as long-range correlation (LRC) in the physics literature.

Here is a special example of $B(r)$ and $D(r)$, which we learned from Yan Fyodorov.

Example 1. We assume $c_0 = 0$ and $A = 0$. For fixed $\varepsilon > 0$ and $\gamma > 0$, let

$$\nu(\mathrm{d}x) = 2e^{-\varepsilon x^2} x^{2\gamma - 3}\mathrm{d}x.$$

The case $\gamma > 1$ corresponds to SRC while the case $0 < \gamma \leq 1$ is LRC field. Indeed, if $\gamma > 1$,

$$B(r) = \int_0^\infty 2e^{-rt^2}e^{-\varepsilon t^2}t^{2\gamma - 3}\mathrm{d}t = \frac{\Gamma(\gamma - 1)}{(r + \varepsilon)^{\gamma - 1}},$$

while if $0 < \gamma < 1$, using integration by parts,

$$D(r) = \int_0^\infty (e^{-\varepsilon y} - e^{-(r+\varepsilon)y})y^{\gamma - 2}\mathrm{d}y = \frac{\Gamma(\gamma)}{1 - \gamma}[(r + \varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}].$$

The case $\gamma = 1$ can be obtained by sending $\gamma \uparrow 1$ and using the dominated convergence theorem with the control function $f(y) = (e^{-\varepsilon y} - e^{-(r+\varepsilon)y})y^{-1}$ for $y \leq 1$ and $= (e^{-\varepsilon y} - e^{-(r+\varepsilon)y})y^{-1/2}$ for $y > 1$. Then if $\gamma = 1$, we have

$$D(r) = \log(1 + r/\varepsilon).$$

In the LRC case, we see that the long range covariance behaves like a high dimensional analog of fractional Brownian motions.

Remark 1.1. Observe that any Bernstein function vanishing at 0 is a structure function. This is a consequence of the Lévy–Khintchine representation of Bernstein functions; see e.g. the monograph [SSV12], which also contains a comprehensive list of complete Bernstein functions. Conversely, any structure function is a Bernstein function. It follows that any correlator function D must be concave, infinitely differentiable, and non-decreasing on $(0, +\infty)$. Moreover, we have $D'(r) \geq 0$, $D''(r) \leq 0$, $D'''(r) \geq 0$ for $r > 0$.

Remark 1.2. One should not confuse SRC/LRC with short-range/long-range dependence. SRC here refers to the fact that $\mathbb{E}(X_N(x)X_N(y))$ decays as $\|x - y\| \to \infty$ while for LRC it may not. Short-range dependence requires the autocovariance function to have exponential decay.

1.1. Previous results. The Hamiltonian (1.1) has been considered in many papers, from physics to mathematics, since the 1950s. In particular, the model was introduced by Mezard–Parisi [MG91] and Engel [Eng93] among others as a model for a classical particle confined to an impenetrable spherical box or a toy model describing elastic manifolds propagating in a random potential [Fyo04]. A nice historical account can be found in [FS07] which also contains the phase diagram ($T - \mu$ relation) for the model at positive temperature. At zero temperature, in the seminal paper [Fyo04], Fyodorov considered the case of isotropic fields (SRC) and computed the mean total number of critical points, finding a phase transition for different values of μ and $D''(0)$. In a subsequent and impressive paper, [FW07] computes the mean number of saddles and minima for SRC fields. This paper considered a more general model where $\mu\|u\|_2^2/2$ is replaced by $NU(\|u\|_2^2/N)$ for suitable confining potential U.

Still in the case of isotropic fields, [FN12] computed the mean number of minima and studied the phenomena of topology trivialization and the relation of this quantity with the Tracy–Widom distribution. More recently, [CS18] considered the mean number of critical points of a fixed index and for finite N. After the first version of our paper (which includes [AZ22]) appeared in arXiv, the paper [GK22] considered the phase transition of annealed instability index for the isotropic fields, which is more closely related to our companion paper [AZ22]. The reader is also invited to take a look at [BD07, YV18].

For a similar Hamiltonian defined on the N dimensional sphere, known as the spherical p-spin model, the rigorous study of the complexity of saddles and minima started in [ABAC13] and now has solid body of work including [ABA13, Sub17, BAMMN19]. For the physics predictions of this model, the reader should consult [CL04, MPV86] and the references therein.

All of the rigorous work above only considered isotropic Gaussian fields (SRC case) or spherical spin glasses. Compared to the SRC case, LRC fields allow more flexibility in modeling real world phenomena. Another aspect of the significance of LRC fields is that at low temperature they belong to Full-step Replica Symmetry Breaking (RSB), while SRC fields are 1-step RSB according to the physics literature. Therefore, LRC fields represent the most complex landscape like valleys within valleys within valleys and are certainly harder to study compared with SRC fields. The free energy of LRC Gaussian fields was studied in a sequence of two remarkable physics papers [FS07, FB08] via the replica method and then in a mathematical paper [Kli12]. The Hessian spectrum at the global minimum, which is closely related to landscape complexity, was investigated in [FLD18, FLD20] still using the replica approach for both SRC and LRC fields and the more general elastic manifold model. However, the lack of symmetry in LRC fields

imposes a difficult obstacle and no rigorous results on the complexity are currently known.

The main purpose of this article and its companion paper is to close this gap by providing a comprehensive rigorous study of the complexity of LRC Gaussian fields. In this first paper, we focus on the high dimensional limit of the expected number of critical points when the domain or value of the fields are constrained to any particular set. In the companion paper [AZ22], we will provide information on local minima and saddles with given indices.

A word of comment is needed here. One of the main differences between LRC and SRC fields is the fact that the variance of an LRC field may change from location to location and the gradient ∇H_N is no longer independent of H_N. The main novelty of our two papers is the development of techniques to overcome this difficulty. Another set of important techniques to deal with "non-invariant" fields was also recently developed in [BBM21b,BBM21a]; these do not seem to apply to the model we consider.

1.2. Main results. To state our results, let $B_N \subset \mathbb{R}^N$ and $E \subset \mathbb{R}$ be (a sequence of) Borel sets. We define

$$\mathrm{Crt}_N(E, B_N) = \#\{x \in B_N : \nabla H_N(x) = 0, \frac{1}{N} H_N(x) \in E\}.$$

Throughout the paper we will consider the following extra assumptions on X_N.

Assumption I (Smoothness). The function D is four times differentiable at 0 and it satisfies

$$0 < |D^{(4)}(0)| < \infty. \tag{1.3}$$

Remark 1.3. By Kolmogorov's criterion, Assumption I ensures that almost surely the field H_N is twice differentiable. Moreover, Assumption I guarantees $D'(0)$, $D''(0)$ and $D'''(0)$ exist and are non-zero. This implies that for $r > 0$

$$D(r) > 0, \quad D'(r) > 0, \quad D''(r) < 0, \quad D'''(r) > 0,$$

and in particular all these functions are strictly monotone. From here we also know that $D(r) \leq D'(0)r$ and when ν in the representation (1.2) is not a finite measure (or equivalently in case 2), $\lim_{r \to \infty} D(r) = \infty$.

Assumption II (Pinning). We have

$$X_N(0) = 0.$$

Remark 1.4. Random fields with isotropic increments are high dimensional generalizations of stochastic processes with stationary increments in dimension one. It is a common practice to assume such processes (like Brownian motion or Poisson processes) to start from 0. Therefore, Assumption II is a natural choice for studying random fields with isotropic increments. Note that only the trivial isotropic field ($X_N = 0$) satisfies Assumption II.

We first consider the average of the *total* number of critical points of H_N confined to domains that are appropriately scaled. Then we count the average number of critical points of H_N with a given fixed critical value. The following condition is only needed when the critical value is not restricted.

Assumption III (Domain growth). Let z_N be a standard N dimensional Gaussian random variable. There exist Ξ or Θ such that the sequence of sets B_N satisfies

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(z_N \in |\mu| B_N / \sqrt{D'(0)}) = -\Xi \leq 0, \qquad \mu \neq 0, \qquad (1.4)$$

$$\lim_{N\to\infty} \frac{1}{N} \log |B_N| = \Theta, \qquad \mu = 0. \qquad (1.5)$$

Remark 1.5. Assumption III serves to select domains in the right scale and it is less restrictive than the shell domains considered in Theorem 1.2 when the field values are also constrained. As seen in the proof of our main theorems, the reader could consider other sequence of sets B_N provided some knowledge of their volumes. Probably the most interesting choices of B_N are shells defined below due to rotational symmetry. Although this special case of Theorem 1.1 can be obtained from Theorem 1.2, the formula and proof for the former result are clearer, thus we state them separately. We hope this organization provides a gentle introduction to the reader to appreciate the latter result, where most of the novelty (and difficulty of the paper) resides.

Theorem 1.1. *Under Assumptions I, II, and III, we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(\mathbb{R}, B_N)$$

$$= \begin{cases} -\Xi, & |\mu| > \sqrt{-2D''(0)}, \\ -\log \frac{|\mu|}{\sqrt{-2D''(0)}} + \frac{\mu^2}{-4D''(0)} - \frac{1}{2} - \Xi, & 0 < |\mu| \leq \sqrt{-2D''(0)}, \\ \log \sqrt{-2D''(0)} - \frac{1}{2} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log[D'(0)] + \Theta, & \mu = 0. \end{cases}$$

Remark 1.6. If we let $J = \sqrt{-2D''(0)}$ and $\Xi = 0$ as in [Fyo04], the second case can be rewritten as

$$\Sigma_{\mu,D} = \frac{1}{2}\left(\frac{\mu^2}{J^2} - 1\right) - \log \frac{\mu}{J} \geq 0. \qquad (1.6)$$

which matches Fyodorov's result for isotropic Gaussian random fields.

Next, we state our main result on the number of critical points with values in an open set $E \subset \mathbb{R}$ and confined to a shell $B_N(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < \frac{\|x\|}{\sqrt{N}} < R_2\}$. This is a natural choice, as the isotropy assumption implies rotational invariance. To emphasize the dependence on R_1 and R_2, we also write

$$\mathrm{Crt}_N(E, (R_1, R_2)) = \mathrm{Crt}_N(E, B_N(R_1, R_2)).$$

We will assume the following technical assumption:

Assumption IV. (3.8) and (3.9) hold for $x \in \mathbb{R}^N \setminus \{0\}$.

This assumption is rather mild, and is satisfied by e.g. the so called Thorin–Bernstein functions. It is likely that Assumption IV is redundant, as we do not know any structure function violating this condition. At the time of this writing, we can only provide some sufficient conditions, but cannot verify it for all structure functions; see Lemma 3.2 and Remark 3.1 for more details.

Theorem 1.2. *Let $0 \leq R_1 < R_2 \leq \infty$ and E be an open set of $\mathbb{R}$. Assume Assumptions I, II, IV, and $|\mu| + \frac{1}{R_2} > 0$. Then*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \mathrm{Crt}_N(E, B_N(R_1, R_2)) = \frac{1}{2} \log[-4D''(0)] - \frac{1}{2} \log D'(0)$$

$$+ \frac{1}{2} + \sup_{(\rho, u, y) \in F} \psi_*(\rho, u, y)$$

where $F = \{(\rho, u, y) : y \in \mathbb{R}, \rho \in (R_1, R_2), u \in E\}$, and the function ψ_* is given explicitly in (5.2).

The condition $|\mu| + \frac{1}{R_2} > 0$ merely says $R_2 < \infty$ if $\mu = 0$, which is necessary to get non-trivial asymptotics as we saw in Theorem 1.1. Heuristically, if $\mu = 0$ then the field X_N has comparable number of critical points for each increment due to stationarity so that the mean total number would be infinity, while for $\mu \neq 0$, the condition $\nabla X_N(x) = -\mu x$ will be harder to satisfy "in an exponential way" as $\|x\|$ become larger since $\nabla X_N(x)$ are centered Gaussians with equal covariance (see Lemma A.1) so that the mean total number of critical points tends to be finite. In Example 2 at the end of Sect. 5, we provide details on how to recover Theorem 1.1 from Theorem 1.2 when B_N is a shell, which also provides some insight on the location of the majority of critical points. In Example 3, we consider the example $D(r) = \log(1 + \frac{r}{\varepsilon})$ and investigate the phase transition in confining ball radius.

Let us end this section with a brief description of the proofs, highlighting the main difference from previous results that also computed the mean number of critical points. Similar to many results in this area, we use the Kac–Rice formula as a starting point. Since our fields do not have constant variance and in particular H_N is correlated to ∇H_N, we are unable to trace a direct parallel to random matrix theory as done in [ABAC13, ABA13, Sub17] where the Hessian is distributed as a matrix from the Gaussian Orthogonal Ensemble (GOE) plus a scalar matrix. This small difference actually provides major obstacles. To go around this difficulty, we first find out the conditional distribution of the Hessian after some matrix manipulations. The GOE matrix appears as a summand of a principal submatrix which itself is correlated to the other element on diagonal. Then we estimate from above and below the conditional expectation of the Hessian given H_N. Matching upper and lower bounds only come after long and non-trivial calculations and careful asymptotic analysis.

The rest of the paper is organized as follows. In Sect. 2, we fix our notation and provide some preliminary facts before giving the proof of Theorem 1.1. We find the (conditional) distribution of the Hessian with some of the tools from random matrix theory in Sect. 3 and establish various results on exponential tightness in Sect. 4, both of which will serve as the starting point for computing complexity functions in this paper and the companion paper [AZ22]. We prove Theorem 1.2 in Sect. 5.

2. Preliminary Facts and Proof of Theorem 1.1

Throughout, we regard a vector to be a column vector. We write e.g. $C_{\mu, D}$ for a constant depending on μ and D which may vary from line to line. For $N \in \mathbb{N}$, let us denote $[N] = \{1, 2, \ldots, N\}$. For a vector $(y_1, \ldots, y_N) \in \mathbb{R}^N$, we write $L(y_1^N) = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$ for its empirical measure. Recall that an $N \times N$ matrix M in the Gaussian Orthogonal

Ensemble (GOE) is a symmetric matrix with centered Gaussian entries that satisfy

$$\mathbb{E}(M_{ij}) = 0, \quad \mathbb{E}(M_{ij}^2) = \frac{1 + \delta_{ij}}{2N}. \tag{2.1}$$

We will simply write GOE_N or $\mathrm{GOE}(N)$ for the matrix M. Denoting by $\lambda_1 \leq \cdots \leq \lambda_N$ the eigenvalues of M, we write $L_N = L(\lambda_1^N) = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}$ for its empirical spectral measure. From time to time, we may also use λ_k to denote the kth smallest eigenvalue of GOE_{N+1} or GOE_{N-1}. This should be clear from context and should not affect any results as we only care about the large N behavior eventually. For a closed set $F \subset \mathbb{R}$, we denote by $\mathcal{P}(F)$ the set of probability measures with support contained in F. We equip the space $\mathcal{P}(\mathbb{R})$ with the weak topology, which is compatible with the distance

$$d(\mu, \nu) := \sup \left\{ \left| \int f \mathrm{d}\mu - \int f \mathrm{d}\nu \right| : \|f\|_\infty \leq 1, \|f\|_L \leq 1 \right\}, \quad \mu, \nu \in \mathcal{P}(\mathbb{R}), \tag{2.2}$$

where $\|f\|_\infty$ and $\|f\|_L$ denote the L^∞ norm and Lipschitz constant of f, respectively. Let $B(\nu, \delta)$ denote the open ball in the space $\mathcal{P}(\mathbb{R})$ with center ν and radius δ w.r.t. to the distance d given in (2.2). Similarly, we write $B_K(\nu, \delta) = B_K(\nu, \delta) \cap \mathcal{P}([-K, K])$ for some constant $K > 0$. We denote by σ_{sc} the semicircle law scaled to have support $[-\sqrt{2}, \sqrt{2}]$.

We will frequently use the following facts which are consequences of large deviations. Using the large deviation principle (LDP) of empirical measures of GOE matrices [BAG97], for any $\delta > 0$, there exists $c = c(\delta) > 0$ and $N_\delta > 0$ such that for all $N > N_\delta$,

$$\mathbb{P}(L(\lambda_1^N) \notin B(\sigma_{\mathrm{sc}}, \delta)) \leq e^{-cN^2}. \tag{2.3}$$

On the other hand, the LDP of the smallest eigenvalue of GOE matrices [BADG01] states that λ_1 satisfies an LDP with speed N and a good rate function

$$J_1(x) = \begin{cases} \int_x^{-\sqrt{2}} \sqrt{z^2 - 2} \mathrm{d}z, & x \leq -\sqrt{2}, \\ \infty, & x > -\sqrt{2}, \end{cases}$$
$$= \begin{cases} \frac{1}{2} \log 2 - \frac{1}{2} x \sqrt{x^2 - 2} - \log(-x + \sqrt{x^2 - 2}), & x \leq -\sqrt{2}, \\ \infty, & x > -\sqrt{2}. \end{cases} \tag{2.4}$$

In particular, writing $\lambda_N^* = \max_{i \in [N]} |\lambda_i|$ for the operator norm of an $N \times N$ GOE matrix, by [BADG01, Lemma 6.3], there exists $N_0 > 0$ and $K_0 > 0$ such that for $K > K_0$ and $N > N_0$,

$$\mathbb{P}(\lambda_N^* > K) \leq e^{-NK^2/9}. \tag{2.5}$$

This can also be seen directly from the LDP of λ_1, even though it was originally proved as a technical input for the LDP of λ_1. It follows that there exists an absolute constant $C > 0$ such that

$$\mathbb{E}[\lambda_N^{*\,k}] \leq C^k \tag{2.6}$$

for any $k \geq 0$ and $N > N_0$. For a probability measure ν on $\mathbb{R}$, let us define

$$\Psi(\nu, x) = \int_{\mathbb{R}} \log |x - t| \nu(\mathrm{d}t), \qquad \Psi_*(x) = \Psi(\sigma_{\mathrm{sc}}, x). \tag{2.7}$$

By calculation,

$$
\begin{aligned}
\Psi_*(x) &= \frac{1}{2}x^2 - \frac{1}{2} - \frac{1}{2}\log 2 - \int_{\sqrt{2}}^{|x|}\sqrt{y^2 - 2}\,dy\mathbf{1}\{|x| \geq \sqrt{2}\}\\
&= \begin{cases} \frac{1}{2}x^2 - \frac{1}{2} - \frac{1}{2}\log 2, & |x| \leq \sqrt{2},\\ \frac{1}{2}x^2 - \frac{1}{2} - \log 2 - \frac{1}{2}|x|\sqrt{x^2 - 2} + \log(|x| + \sqrt{x^2 - 2}), & |x| > \sqrt{2}. \end{cases}
\end{aligned}
\tag{2.8}
$$

Note that $\Psi_*(x) - \frac{x^2}{2} \leq -\frac{1}{2} - \frac{1}{2}\log 2$. Unless specified otherwise, we assume Assumptions I and II throughout.

Let us prove the result for the total number of critical points. The strategy we employ is well-known and similar to the one developed in [ABAC13]: We start by applying the Kac–Rice formula and we derive the asymptotics in high dimensions with the use of random matrix theory and large deviation principles. The proof is somewhat straightforward since we do not face the main obstacle of the next sections, i.e., the dependence of H_N and ∇H_N.

Proof of Theorem 1.1. Let E be a Borel subset of $\mathbb{R}$. By the Kac–Rice formula [AT07, Theorem 11.2.1],

$$
\mathbb{E}\mathrm{Crt}_N(E, B_N) = \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)|\mathbf{1}\{\tfrac{1}{N}H_N(x) \in E\}|\nabla H_N(x) = 0]p_{\nabla H_N(x)}(0)\mathrm{d}x,
\tag{2.9}
$$

where $p_{\nabla H_N(x)}(t)$ is the p.d.f. of $\nabla H_N(x)$ at t.

When $E = \mathbb{R}$, the restriction on the range of $H_N(x)$ disappears. By independence of ∇H_N and $\nabla^2 H_N$ (see Lemma A.1), the above formula simplifies to

$$
\mathbb{E}\mathrm{Crt}_N(\mathbb{R}, B_N) = \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)|]p_{\nabla H_N(x)}(0)\mathrm{d}x.
\tag{2.10}
$$

The following lemma is a random matrix computation.

Lemma 2.1. *Let M be an $N \times N$ GOE matrix and set*

$$
P = aM - \left(b + \frac{\sigma}{\sqrt{N}}Z\right)I,
$$

where Z is a standard Gaussian random variable independent of M, I is the identity matrix and $a, b, \sigma \in \mathbb{R}$. Then

$$
\begin{aligned}
\mathbb{E}|\det P| = {}&\frac{\Gamma(\frac{N+1}{2})(N + 1)a^{N+1}}{\sqrt{\pi}\sigma N^{\frac{N}{2}}e^{\frac{Nb^2}{2\sigma^2}}}\mathbb{E}\int \exp\left[\frac{(N+1)x^2}{2}\left(1 - \frac{a^2}{\sigma^2}\right)\right.\\
&\left. + \frac{\sqrt{N(N+1)}axb}{\sigma^2}\right]L_{N+1}(\mathrm{d}x).
\end{aligned}
$$

Proof. Use [ABAC13, Lemma 3.3] with $m = \frac{b}{a}$, $t = \frac{\sigma}{\sqrt{N}a}$, $G = \mathbb{R}$ and sum over the eigenvalues. $\square$

From Lemma A.1, $\nabla^2 H_N(x)$ and $\sqrt{-4D''(0)}M - (\sqrt{\frac{-2D''(0)}{N}}Z - \mu)I$ have the same distribution. Then with

$$m = -\mu/\sqrt{-4D''(0)},$$

from the Lemma above with $a = \sqrt{-4D''(0)}, b = -\mu, \sigma = \sqrt{-2D''(0)}$ we obtain

$$
\begin{aligned}
&\mathbb{E}|\det \nabla^2 H_N(x)| \\
&= \frac{\sqrt{2}[-4D''(0)]^{N/2}\Gamma(\frac{N+1}{2})(N+1)}{\sqrt{\pi}N^{N/2}e^{Nm^2}}\mathbb{E}\int e^{-\frac{1}{2}(N+1)w^2 + 2\sqrt{N(N+1)}mw}L_{N+1}(\mathrm{d}w).
\end{aligned}
$$

From Lemma B.1, we see that for the asymptotic analysis we can replace the above $\sqrt{N(N+1)}$ in the exponent by $N+1$, leaving us to compute asymptotics of

$$I_N = \mathbb{E}\int e^{(N+1)\phi(x)}L_{N+1}(\mathrm{d}x) = \frac{1}{N+1}\sum_{i=1}^{N+1}\mathbb{E}_{\mathrm{GOE}(N+1)}e^{(N+1)\phi(\lambda_i)},$$

where

$$\phi(x) = -\frac{1}{2}x^2 - \frac{\mu x}{\sqrt{-D''(0)}}.$$

This is obtained in the following Lemma.

Lemma 2.2. *If* $|\mu| \geq \sqrt{-2D''(0)}$ *then*

$$\lim_{N\to\infty}\frac{1}{N}\log I_N = \frac{\mu^2}{-4D''(0)} + \log\frac{|\mu|}{\sqrt{-2D''(0)}} + \frac{1}{2},$$

while if $|\mu| < \sqrt{-2D''(0)}$ *we have*

$$\lim_{N\to\infty}\frac{1}{N}\log I_N = \frac{\mu^2}{-2D''(0)}.$$

From Lemma A.1, we note that

$$\int_{B_N} p_{\nabla H_N(x)}(0)\mathrm{d}x = \begin{cases} \frac{1}{|\mu|^N}\mathbb{P}(z_N \in |\mu|B_N/\sqrt{D'(0)}), & \mu \neq 0, \\ \frac{1}{(2\pi)^{N/2}D'(0)^{N/2}}|B_N|, & \mu = 0, \end{cases}$$

where $|B_N|$ is the Lebesgue measure of B_N and z_N is a standard N dimensional Gaussian vector. It follows from (2.10) that

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\mathrm{Crt}_N(\mathbb{R},B_N) = \lim_{N\to\infty}\frac{1}{N}\left(\log C_N + \log I_N\right),$$

where

$$C_N = \begin{cases} \frac{\sqrt{2}[-4D''(0)]^{N/2}\Gamma(\frac{N+1}{2})(N+1)}{\sqrt{\pi}N^{N/2}e^{Nm^2}|\mu|^N}\mathbb{P}(z_N \in |\mu|B_N/\sqrt{D'(0)}), & \mu \neq 0, \\ \frac{\sqrt{2}[-4D''(0)]^{N/2}\Gamma(\frac{N+1}{2})(N+1)|B_N|}{\sqrt{\pi}N^{N/2}(2\pi)^{N/2}D'(0)^{N/2}}, & \mu = 0. \end{cases} \quad (2.11)$$

From Assumption III and Stirling's formula,

$$\lim_{N \to \infty} \frac{1}{N} \log C_N = \begin{cases} \log \frac{\sqrt{-2D''(0)}}{|\mu|} + \frac{\mu^2}{4D''(0)} - \frac{1}{2} - \Xi, & \mu \neq 0, \\ \log \sqrt{-2D''(0)} - \frac{1}{2} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log[D'(0)] + \Theta, & \mu = 0. \end{cases}$$

The above computation combined with Lemma 2.2 finishes the proof of the Theorem.
□

We finish this section with the proof of Lemma 2.2.

Proof of Lemma 2.2. The proof follows from the large deviation principle for the extreme eigenvalues of GOE. In short, in the case of small $|\mu|$, the maximum of ϕ is attained in the bulk while in the case of large $|\mu|$, the extreme eigenvalue (λ_1 for $\mu > 0$ and λ_{N+1} for $\mu < 0$) contributes to the asymptotics of I_N. We argue the first case $|\mu| \geq \sqrt{-2D''(0)}$. By symmetry, we only consider $\mu \geq \sqrt{-2D''(0)}$. Since $\phi(x)$ is bounded from above, by the LDP for λ_1 as in (2.4) and Varadhan's Lemma,

$$\sup_{x \in \mathbb{R}} \phi(x) - J_1(x) \leq \liminf_{N \to \infty} \frac{1}{N+1} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)}$$

$$\leq \limsup_{N \to \infty} \frac{1}{N+1} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)} \leq \sup_{x \in \mathbb{R}} \phi(x) - J_1(x).$$

(2.12)

Note that

$$\arg\max_x [\phi(x) - J_1(x)] = -\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu} \leq -\sqrt{2}.$$

Since $I_N \geq \frac{1}{N+1} \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)}$, it follows that

$$\liminf_{N \to \infty} \frac{1}{N} \log I_N \geq \liminf_{N \to \infty} \frac{1}{N} \log \left[\frac{1}{N+1} \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)} \right]$$

$$\geq \frac{\mu^2}{-4D''(0)} + \log \frac{\mu}{\sqrt{-2D''(0)}} + \frac{1}{2}.$$

On the other hand,

$$I_N \leq \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)} \mathbf{1}\{\lambda_1 \geq -\frac{\mu}{\sqrt{-D''(0)}}\} + e^{(N+1)\phi\left(-\frac{\mu}{\sqrt{-D''(0)}}\right)} \mathbb{P}\left(\lambda_1 < -\frac{\mu}{\sqrt{-D''(0)}}\right).$$

For an upper bound for the first term on the right-hand side, we have by (2.12),

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_1)}$$

$$= \phi\left(-\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu}\right) - J_1\left(-\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu}\right).$$

And for the second term, we find by (2.4)

$$
\limsup_{N\to\infty} \frac{1}{N} \log\left[e^{(N+1)\phi(-\frac{\mu}{\sqrt{-D''(0)}})} \mathbb{P}\left(\lambda_1 < -\frac{\mu}{\sqrt{-D''(0)}}\right)\right]
$$

$$
\leq \phi\left(-\frac{\mu}{\sqrt{-D''(0)}}\right) - J_1\left(-\frac{\mu}{\sqrt{-D''(0)}}\right)
$$

$$
\leq \phi\left(-\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu}\right) - J_1\left(-\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu}\right).
$$

It follows that

$$
\limsup_{N\to\infty} \frac{1}{N} \log I_N \leq \frac{\mu^2}{-4D''(0)} + \log\frac{\mu}{\sqrt{-2D''(0)}} + \frac{1}{2}.
$$

We have proved the claim.

For the second case $|\mu| < \sqrt{-2D''(0)}$, the maximum of $\phi(x)$ on $\mathbb{R}$ is achieved at $x = -\frac{\mu}{\sqrt{-D''(0)}} \in (-\sqrt{2}, \sqrt{2})$. Then for $\varepsilon > 0$ and N large enough,

$$
\mathbb{E}\int_{-\frac{\mu}{\sqrt{-D''(0)}}}^{-\frac{\mu}{\sqrt{-D''(0)}}+\varepsilon} e^{(N+1)\phi(-\frac{\mu}{\sqrt{-D''(0)}}+\varepsilon)} L_{N+1}(\mathrm{d}x) \leq I_N \leq e^{(N+1)\phi\left(-\frac{\mu}{\sqrt{-D''(0)}}\right)}.
$$

Since $\lim_{N\to\infty} \mathbb{E}L_{N+1}\left(-\frac{\mu}{\sqrt{-D''(0)}}, -\frac{\mu}{\sqrt{-D''(0)}}+\varepsilon\right) > 0$, it follows that

$$
\frac{\mu^2}{-2D''(0)} - \frac{\varepsilon^2}{2} \leq \liminf_{N\to\infty} \frac{1}{N} \log I_N \leq \limsup_{N\to\infty} \frac{1}{N} \log I_N \leq \frac{\mu^2}{-2D''(0)}.
$$

The claim follows by sending $\varepsilon \to 0+$.   □

3. Conditional Law of $\nabla^2 H_N$ with Constrained Critical Values

In this section, we provide the initial steps for computing complexity functions. Our main result is a relation between a conditional Hessian $\nabla^2 H_N$ and the GOE given in Proposition 3.3 which implies (3.20) in the Kac–Rice representation for structure functions D that satisfy Assumptions I, II and IV.

Recall the Kac–Rice formula (2.9). Note that $(H_N(x), \partial_i H_N(x), \partial_{kl} H_N(x))_{1\leq i\leq N, 1\leq k\leq l\leq N}$ is a Gaussian field. A natural way to proceed is through conditioning. From Lemma A.1, we have $\mathrm{Var}(H_N(x)) = N D(\frac{1}{N}\|x\|^2)$ and the means

$$
\mathbb{E}(H_N(x)) = \frac{\mu}{2}\|x\|^2, \quad \mathbb{E}(\nabla H_N(x)) = \mu x, \quad \mathbb{E}(\nabla^2 H_N(x)) = \mu I_N.
$$

Since H_N and $(\nabla H_N, \nabla^2 H_N)$ are not independent, our strategy is to first remove the dependence of H_N on ∇H_N by subtracting from H_N a linear transform of ∇H_N. Let $\Sigma_{01} = \mathrm{Cov}(H_N(x), \nabla H_N(x)) = D'(\frac{\|x\|^2}{N})x^{\mathsf{T}}$ and $\Sigma_{11} = \mathrm{Cov}(\nabla H_N(x)) = D'(0)I_N$. By the conditional distribution of Gaussian vectors, we know

$$
Y := \frac{1}{N}[H_N(x) - \Sigma_{01}\Sigma_{11}^{-1}\nabla H_N(x)] = \frac{H_N(x)}{N} - \frac{D'(\frac{\|x\|^2}{N})\sum_{i=1}^{N} x_i \partial_i H_N(x)}{N D'(0)}
$$

is independent from $\nabla H_N(x)$. Now we reduce the problem to understand the dependence of $\nabla^2 H_N$ on Y, which contains information for both of H_N and ∇H_N. Since $\nabla H_N(x)$ is independent from $\nabla^2 H_N(x)$, by conditioning, we may rewrite (2.9) as

$$
\begin{aligned}
&\mathbb{E}\mathrm{Crt}_N(E, B_N) \\
&= \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)|\mathbf{1}\{Y + \frac{1}{N}\Sigma_{01}\Sigma_{11}^{-1}\nabla H_N(x) \in E\}|\nabla H_N(x) = 0]p_{\nabla H_N(x)}(0)\mathrm{d}x \\
&= \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)|\mathbf{1}\{Y \in E\}]p_{\nabla H_N(x)}(0)\mathrm{d}x \\
&= \int_{B_N} \int_E \mathbb{E}(|\det \nabla^2 H_N(x)||Y = u)\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \, p_{\nabla H_N(x)}(0)\mathrm{d}u\mathrm{d}x, \qquad (3.1)
\end{aligned}
$$

where

$$
m_Y = \mathbb{E}(Y) = \frac{\mu\|x\|^2}{2N} - \frac{\mu D'(\frac{\|x\|^2}{N})\|x\|^2}{D'(0)N},
$$

$$
\sigma_Y^2 = \mathrm{Var}(Y) = \frac{1}{N}\left(D(\frac{\|x\|^2}{N}) - \frac{D'(\frac{\|x\|^2}{N})^2}{D'(0)}\frac{\|x\|^2}{N}\right).
$$

To proceed, we need the conditional distribution of $\nabla^2 H_N(x)$ given $Y = u$. A crucial difficulty arises here, however. Namely, one can check that the off-diagonal entries of $\nabla^2 H_N(x)$ given $Y = u$ may have negative covariance, for example,

$$
\mathrm{Cov}[(\partial_{ij}H_N(x), \partial_{kl}H_N(x))|Y = u] = -\frac{1}{N}\frac{\alpha x_i x_j}{N}\frac{\alpha x_k x_l}{N}, \ i \neq j, k \neq l, \{i, j\} \neq \{k, l\},
$$

for some α defined below, which prevents using GOE directly. This conditional covariance structure also motivates the introduction of rotation matrix $U(x)$ defined below, after close scrutiny and several guess and verify procedures.

To overcome this difficulty, let us define

$$
\alpha = \alpha(\|x\|^2/N) = \frac{2D''(\|x\|^2/N)}{\sqrt{D(\frac{\|x\|^2}{N}) - \frac{D'(\|x\|^2/N)^2}{D'(0)}\frac{\|x\|^2}{N}}},
$$

$$
\beta = \beta(\|x\|^2/N) = \frac{D'(\|x\|^2/N) - D'(0)}{\sqrt{D(\frac{\|x\|^2}{N}) - \frac{D'(\|x\|^2/N)^2}{D'(0)}\frac{\|x\|^2}{N}}}. \qquad (3.2)
$$

Note that $\alpha \leq 0$ and $\beta \leq 0$. One should think of α and β as $O(1)$ quantities. Let us define $A = A_N = U(x)\nabla^2 H_N(x)U(x)^\mathsf{T}$ where $U(x)$ is an $N \times N$ orthogonal matrix such that

$$
U(\frac{\alpha x x^\mathsf{T}}{N} + \beta I_N)U^\mathsf{T} = \begin{pmatrix} \frac{\alpha\|x\|^2}{N} + \beta & 0 & \cdots & 0 \\ 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta \end{pmatrix}. \qquad (3.3)
$$

In other words, we have for $U = (u_{ij})$,

$$\sum_{k,l} u_{ik}(\frac{\alpha x_k x_l}{N} + \beta\delta_{kl})u_{jl} = \alpha\delta_{i1}\delta_{j1}\frac{\|x\|^2}{N} + \beta\delta_{ij}. \tag{3.4}$$

Indeed, such a $U(x)$ can be found by imposing the first row to be $\frac{x^\mathsf{T}}{\|x\|}$ for $x \neq 0$; and if $x = 0$, $U(x)$ can be arbitrary orthogonal matrix. It follows that $\mathbb{E}(A) = \mu I_N$, and by Lemma A.1,

$$\begin{aligned}
\mathrm{Cov}(A_{ij}, A_{i'j'}) &= \sum_{k,l,k',l'} u_{ik}u_{jl}u_{i'k'}u_{j'l'}\mathrm{Cov}(\partial_{kl}H_N(x), \partial_{k'l'}H_N(x)) \\
&= \frac{-2D''(0)}{N}(\delta_{ij}\delta_{i'j'} + \delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j}),
\end{aligned}$$

$$\mathrm{Cov}(A_{ij}, \partial_l H_N(x)) = \sum_{a,b} u_{ia}u_{jb}\mathrm{Cov}(\partial_{ab}H_N(x), \partial_l H_N(x)) = 0,$$

$$\begin{aligned}
\mathrm{Cov}(A_{ij}, H_N(x)) &= \sum_{a,b} u_{ia}u_{jb}(\frac{2D''(\|x\|^2/N)x_a x_b}{N} + [D'(\|x\|^2/N) - D'(0)]\delta_{ab}) \\
&= \frac{2D''(\|x\|^2/N)\delta_{i1}\delta_{j1}\|x\|^2}{N} + [D'(\|x\|^2/N) - D'(0)]\delta_{ij}.
\end{aligned}$$

Since A and $\nabla^2 H_N(x)$ have the same eigenvalues, by (3.1),

$$\mathbb{E}\mathrm{Crt}_N(E, B_N) = \int_{B_N}\int_E \mathbb{E}(|\det A||Y = u)\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{(u - m_Y)^2}{2\sigma_Y^2}}p_{\nabla H_N(x)}(0)\mathrm{d}u\mathrm{d}x. \tag{3.5}$$

We need the conditional distribution of A given $Y = u$. Note that

$$\mathrm{Cov}(A_{ij}, Y) = \mathrm{Cov}(A_{ij}, \frac{H_N}{N}) = \frac{2D''(\|x\|^2/N)\delta_{i1}\delta_{j1}\|x\|^2}{N^2} + \frac{[D'(\|x\|^2/N) - D'(0)]\delta_{ij}}{N}.$$

Then conditioning on $Y = u$ we have

$$\mathbb{E}(A_{ij}|Y = u) = \mathbb{E}(A_{ij}) + \mathrm{Cov}(A_{ij}, Y)\sigma_Y^{-2}(u - \mathbb{E}(Y))$$

$$= \mu\delta_{ij} + \frac{(\frac{2D''(\frac{\|x\|^2}{N})\delta_{i1}\delta_{j1}\|x\|^2}{N} + [D'(\frac{\|x\|^2}{N}) - D'(0)]\delta_{ij})(u - \frac{\mu\|x\|^2}{2N} + \frac{\mu D'(\frac{\|x\|^2}{N})\|x\|^2}{D'(0)N})}{D(\frac{\|x\|^2}{N}) - \frac{D'(\frac{\|x\|^2}{N})^2\|x\|^2}{D'(0)N}},$$

$$m_{A|u} := \mathbb{E}(A|Y = u) = \mu I_N + \frac{u - \frac{\mu\|x\|^2}{2N} + \frac{\mu D'(\frac{\|x\|^2}{N})\|x\|^2}{D'(0)N}}{D(\frac{\|x\|^2}{N}) - \frac{D'(\frac{\|x\|^2}{N})^2\|x\|^2}{D'(0)N}}$$

$$\times \begin{pmatrix} \frac{2D''(\frac{\|x\|^2}{N})\|x\|^2}{N} + D'(\frac{\|x\|^2}{N}) - D'(0) & 0 \\ 0 & [D'(\frac{\|x\|^2}{N}) - D'(0)]I_{N-1} \end{pmatrix}, \tag{3.6}$$

$$\mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}|Y = u] = \mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}] - \mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}, Y]\sigma_Y^{-2}\mathrm{Cov}[Y, (A_{ij}, A_{i'j'})^\mathsf{T}]$$

$$= \mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}] - \frac{1}{N}$$

$$\times \begin{pmatrix} (\frac{\alpha\delta_{i1}\delta_{j1}\|x\|^2}{N} + \beta\delta_{ij})^2 & (\frac{\alpha\delta_{i1}\delta_{j1}\|x\|^2}{N} + \beta\delta_{ij})(\frac{\alpha\delta_{i'1}\delta_{j'1}\|x\|^2}{N} + \beta\delta_{i'j'}) \\ (\frac{\alpha\delta_{i1}\delta_{j1}\|x\|^2}{N} + \beta\delta_{ij})(\frac{\alpha\delta_{i'1}\delta_{j'1}\|x\|^2}{N} + \beta\delta_{i'j'}) & (\frac{\alpha\delta_{i'1}\delta_{j'1}\|x\|^2}{N} + \beta\delta_{i'j'})^2 \end{pmatrix},$$

where $\mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}]$ denotes the 2×2 covariance matrix of A_{ij} and $A_{i'j'}$ while $\mathrm{Cov}[(A_{ij}, A_{i'j'})^\mathsf{T}, Y]$ denotes the 2×1 covariance matrix of $(A_{ij}, A_{i'j'})^\mathsf{T}$ and Y. From here we see conditioning on $Y = u$,

$$\mathrm{Cov}[(A_{ij}, A_{i'j'})|Y = u]$$
$$= \frac{-2D''(0)(\delta_{ij}\delta_{i'j'} + \delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j})}{N} - \frac{1}{N}(\frac{\alpha\delta_{i1}\delta_{j1}\|x\|^2}{N} + \beta\delta_{ij})(\frac{\alpha\delta_{i'1}\delta_{j'1}\|x\|^2}{N} + \beta\delta_{i'j'})$$

$$= \begin{cases} \frac{-6D''(0)}{N} - \frac{1}{N}(\frac{\alpha\|x\|^2}{N} + \beta)^2, & i = j = i' = j' = 1, \\ \frac{-2D''(0)}{N} - \frac{1}{N}(\frac{\alpha\|x\|^2}{N} + \beta)\beta, & i = j = 1 \neq i' = j', \text{ or } i' = j' = 1 \neq i = j, \\ \frac{-6D''(0)}{N} - \frac{\beta^2}{N}, & i = j = i' = j' \neq 1, \\ \frac{-2D''(0)}{N} - \frac{\beta^2}{N}, & 1 \neq i = j \neq i' = j' \neq 1, \\ \frac{-2D''(0)}{N}, & i = i' \neq j = j', \text{ or } i = j' \neq j = i', \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, one can find the above conditional covariances using spherical coordinates, which could avoid the matrix function $U(x)$. Observe that the conditional variances are positive by definition. In order to draw connection with GOE, we first have to check that all the quantities $\mathrm{Cov}[(A_{ij}, A_{i'j'})|Y = u]$ are positive. Note that α and β depend on $\|x\|^2$ and N through $\|x\|^2/N$. Let us write $\rho = \rho_N(x) = \frac{\|x\|}{\sqrt{N}}$ so that $\alpha = \alpha(\rho^2)$ and $\beta = \beta(\rho^2)$.

Lemma 3.1. *We have* $\lim_{\rho \to 0+} \frac{D(\rho^2)}{\rho^4} - \frac{D'(\rho^2)^2}{D'(0)\rho^2} = -\frac{3}{2}D''(0)$ *and*

$$\lim_{\rho \to 0+} \beta(\rho^2)^2 = -\frac{2}{3}D''(0), \quad \lim_{\rho \to 0+} \alpha(\rho^2)\beta(\rho^2)\rho^2 = -\frac{4}{3}D''(0), \quad \lim_{\rho \to 0+} [\alpha(\rho^2)\rho^2]^2$$
$$= -\frac{8}{3}D''(0).$$

Proof. Using l'Hospital's rule together with $D(0) = 0$,

$$\lim_{\rho \to 0+} \frac{D(\rho^2)}{\rho^4} - \frac{D'(\rho^2)^2}{D'(0)\rho^2} = \lim_{\rho \to 0+} \frac{D'(\rho^2)\rho^2 - D(\rho^2)}{\rho^4} - \frac{2D'(\rho^2)D''(\rho^2)}{D'(0)} = -\frac{3}{2}D''(0).$$

It follows that

$$\lim_{\rho \to 0+} \beta(\rho^2)^2 = \lim_{\rho \to 0+} \frac{[\frac{D'(\rho^2) - D'(0)}{\rho}]^2}{\frac{D(\rho^2)}{\rho^4} - \frac{D'(\rho^2)^2}{D'(0)\rho^2}} = -\frac{2}{3}D''(0),$$

$$\lim_{\rho \to 0+} \alpha(\rho^2)\beta(\rho^2)\rho^2 = \lim_{\rho \to 0+} \frac{[2D''(\rho^2)]\frac{D'(\rho^2) - D'(0)}{\rho^2}}{\frac{D(\rho^2)}{\rho^4} - \frac{D'(\rho^2)^2}{D'(0)\rho^2}} = -\frac{4}{3}D''(0),$$

$$\lim_{\rho \to 0+} [\alpha(\rho^2)\rho^2]^2 = \lim_{\rho \to 0+} \frac{[2D''(\rho^2)]^2}{\frac{D(\rho^2)}{\rho^4} - \frac{D'(\rho^2)^2}{D'(0)\rho^2}} = -\frac{8}{3}D''(0). \qquad \square$$

In light of Lemma 3.1, we make the following observation. Following [SSV12, Theorem 8.2], a function $f : (0, \infty) \to (0, \infty)$ is a Thorin–Bernstein function if and only if $\lim_{x \to 0+} f(x)$ exists and its derivative has a representation

$$f'(x) = \frac{a}{x} + b + \int_{(0,\infty)} \frac{1}{x+t} \sigma(\mathrm{d}t), \tag{3.7}$$

where $a, b \geq 0$ and σ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} \frac{1}{1+t} \sigma(\mathrm{d}t) < \infty$. In particular, the functions $D(r) = \log(1 + r/\varepsilon)$ and $D(r) = (r + \varepsilon)^\gamma - \varepsilon^\gamma$ are Thorin–Bernstein functions. Recall the definitions of α and β as in (3.2). The proof of the following analytical result is deferred to Appendix Sect. B.

Lemma 3.2. *For any $x \in \mathbb{R}^N \setminus \{0\}$, we have*

$$-2D''(0) > \left(\frac{\alpha \|x\|^2}{N} + \beta \right) \beta, \tag{3.8}$$

$$-4D''(0) > \left(\frac{\alpha \|x\|^2}{N} + \beta \right) \frac{\alpha \|x\|^2}{N}, \tag{3.9}$$

provided any one of the following conditions holds:

1. For all $x \neq 0$,

$$\beta^2 \leq -\frac{2}{3} D''(0). \tag{3.10}$$

2. For all $y \geq 0$,

$$2D'(0)D''(y)[D(y) - D'(y)y] + D'(y)[D'(y) - D'(0)]^2 \geq 0. \tag{3.11}$$

3. For all $y \geq 0$

$$\frac{D'(y)y}{D'(0)} - \frac{D'(y) - D'(0)}{D''(0)} \geq 0. \tag{3.12}$$

4. For all $y \geq 0$,

$$-\frac{D'(y)}{D''(y)} + \frac{D'(0)}{D''(0)} \geq y. \tag{3.13}$$

5. For all $y \geq 0$,

$$\frac{-D''(y)^2 + D'''(y)D'(y)}{D''(y)^2} \geq 1. \tag{3.14}$$

6. D is a Thorin–Bernstein function with $a = 0$ in (3.7).

From now on, we always assume (3.8) and (3.9) (i.e. Assumption IV), which implies that σ_1, σ_2 below are well defined and $\text{Cov}[(A_{ij}, A_{i'j'})|Y = u] \geq 0$ for all i, i', j, j'. Recalling (3.6), let us write

$$m_1 = m_1(\rho, u) = \mu + \frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})(2D''(\rho^2)\rho^2 + D'(\rho^2) - D'(0))}{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}},$$

$$m_2 = m_2(\rho, u) = \mu + \frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})(D'(\rho^2) - D'(0))}{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}},$$

$$\sigma_1 = \sigma_1(\rho) = \sqrt{\frac{-4D''(0) - (\alpha\rho^2 + \beta)\alpha\rho^2}{N}}, \quad \sigma_2 = \sigma_2(\rho) = \sqrt{\frac{-2D''(0) - (\alpha\rho^2 + \beta)\beta}{N}},$$

$$m_Y = m_Y(\rho) = \frac{\mu\rho^2}{2} - \frac{\mu D'(\rho^2)\rho^2}{D'(0)}, \quad \sigma_Y = \sigma_Y(\rho) = \sqrt{\frac{1}{N}\left(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}\right)},$$

$$\alpha = \alpha(\rho^2) = \frac{2D''(\rho^2)}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}}, \quad \beta = \beta(\rho^2) = \frac{D'(\rho^2) - D'(0)}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}}, \qquad (3.15)$$

where $\rho = \frac{\|x\|}{\sqrt{N}}$. From time to time, we also use the following change of variable

$$v = \frac{u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)}}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}} = \frac{u - m_Y}{\sqrt{N}\sigma_Y} \qquad (3.16)$$

so that

$$m_1 = \mu + v(\alpha\rho^2 + \beta), \quad m_2 = \mu + v\beta. \qquad (3.17)$$

Let

$$G = G(u) = \begin{pmatrix} z_1' & \xi^\mathsf{T} \\ \xi & \sqrt{-4D''(0)}(\sqrt{\frac{N-1}{N}}\text{GOE}_{N-1} - z_3'I_{N-1}) \end{pmatrix} =: \begin{pmatrix} z_1' & \xi^\mathsf{T} \\ \xi & G_{**} \end{pmatrix}, \quad (3.18)$$

where with z_1, z_2, z_3 being independent standard Gaussian random variables,

$$z_1' = \sigma_1 z_1 - \sigma_2 z_2 + m_1, \quad z_3' = \frac{1}{\sqrt{-4D''(0)}}\left(\sigma_2 z_2 + \frac{\sqrt{\alpha\beta}\rho}{\sqrt{N}}z_3 - m_2\right), \qquad (3.19)$$

and ξ is a centered column Gaussian vector with covariance matrix $\frac{-2D''(0)}{N}I_{N-1}$ which is independent from z_1, z_2, z_3 and the GOE matrix GOE_{N-1}. The above discussion yields our main result of this section.

Proposition 3.3. *Assume Assumptions I, II and IV. Then we have in distribution*

$$(U\nabla^2 H_N U^\mathsf{T}|Y = u) \overset{d}{=} G.$$

To connect with (3.5), we have

$$\mathbb{E}(|\det A||Y = u) = \int |\det a| p_{A|Y}(a|u)\mathrm{d}a = \mathbb{E}(|\det G|). \qquad (3.20)$$

Remark 3.1. Here we explain the necessity for Assumption IV. Note that as a (conditional) variance, we have

$$N\text{Var}(A_{11}|Y = u) = -6D''(0) - (\alpha\rho^2 + \beta)^2 \geq 0, \tag{3.21}$$

which implies that

$$|\alpha^2\rho^4| \leq -6D''(0), \quad \beta^2 \leq -6D''(0). \tag{3.22}$$

Since $\text{Cov}(G_{ii}, G_{jj}) = \frac{1}{N}(-2D''(0) - \beta^2)$ for $i \neq j, i, j > 1$, we need to introduce the independent Gaussian variable z_3' such that

$$N[-4D''(0)]\text{Var}(z_3') = -2D''(0) - \beta^2 > 0, \quad \text{for } \rho > 0. \tag{3.23}$$

The condition (3.22) and assumption (3.23) are sufficient in technical proofs in this paper. However, we need the stronger assumptions (3.8) and (3.9) for the matrix model (3.18), which is the cornerstone for the rest of the paper and the companion one [AZ22]. Indeed, since $\text{Cov}(G_{11}, G_{ii}) = \frac{1}{N}[-2D''(0) - (\alpha\rho^2 + \beta)\beta]$ for $i > 1$, to extract the dependence between G_{11} and G_{ii}, we need (3.8) to introduce z_2 and decompose G_{11} and G_{ii}. For G_{ii}, the remaining variance

$$\text{Var}(G_{ii}) - \sigma_2^2 = \frac{1}{N}[-4D''(0) + \alpha\beta\rho^2]$$

can be explained by a GOE diagonal term and $\sqrt{\alpha\beta\rho^2}z_3$ since $\alpha\beta\rho^2 \geq 0$ is known. Then we can write z_3' as in (3.19). But to decompose G_{11}, we need to assume $\text{Var}(G_{11}) - \sigma_2^2 > 0$ to write G_{11} as z_1' in (3.19), and this assumption is exactly (3.9).

4. Exponential Tightness

The purpose of this section is to prove several exponential tightness results so that our future analysis will be reduced to the compact setting. Let $E \subset \mathbb{R}$ be a Borel set. Hereafter, for simplicity, let us assume B_N is a shell $B_N(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < \frac{\|x\|}{\sqrt{N}} < R_2\}, 0 \leq R_1 < R_2 \leq \infty$. Recall that in this case we write $\text{Crt}_N(E, (R_1, R_2)) = \text{Crt}_N(E, B_N(R_1, R_2))$. Using spherical coordinates and writing $\rho = \frac{\|x\|}{\sqrt{N}}$, by the Kac–Rice formula we have

$$\mathbb{E}\text{Crt}_N(E, (R_1, R_2)) = \int_{B_N} \int_E \mathbb{E}[|\det A||Y = u] \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} p_{\nabla H_N(x)}(0) \mathrm{d}u\mathrm{d}x$$

$$= S_{N-1}N^{(N-1)/2} \int_{R_1}^{R_2} \int_E \mathbb{E}[|\det G|] \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2}D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}u\mathrm{d}\rho. \tag{4.1}$$

Here $S_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of $N - 1$ dimensional unit sphere, G depends on u implicitly. Using the Stirling formula, we have

$$\lim_{N\to\infty} \frac{1}{N} \log(S_{N-1}N^{\frac{N-1}{2}}) = \frac{1}{2}\log(2\pi) + \frac{1}{2}. \tag{4.2}$$

Recall the representation (3.18). Let $\lambda_1 \leq \cdots \leq \lambda_{N-1}$ be the eigenvalues of GOE_{N-1}. The eigenvalues of G_{**} can be represented as $\{\sqrt{-4D''(0)}((\frac{N-1}{N})^{1/2}\lambda_i -$

$z'_3)\}_{i=1}^{N-1}$. By the representation, we may find a random orthogonal matrix V which is independent of the unordered eigenvalues $\tilde{\lambda}_j$, $j = 1, \ldots, N-1$ and z'_3, such that

$$G_{**} = \sqrt{-4D''(0)} V^{\mathsf{T}} \begin{pmatrix} (\frac{N-1}{N})^{1/2}\tilde{\lambda}_1 - z'_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\frac{N-1}{N})^{1/2}\tilde{\lambda}_{N-1} - z'_3 \end{pmatrix} V. \quad (4.3)$$

By the rotational invariance of Gaussian measures, $V\xi$ is a centered Gaussian vector with covariance matrix $\frac{-2D''(0)}{N} I_{N-1}$ that is independent of z'_3 and $\tilde{\lambda}_j$'s. We can rewrite $V\xi \overset{d}{=} \sqrt{\frac{-2D''(0)}{N}} Z$, where $Z = (Z_1, \ldots, Z_{N-1})$ is an $N-1$ dimensional standard Gaussian random vector. Using the determinant formula for block matrices or the Schur complement formula,

$$\det G = \det(G_{**})(z'_1 - \xi^{\mathsf{T}} G_{**}^{-1}\xi) \overset{d}{=} [-4D''(0)]^{(N-1)/2} z'_1 \prod_{j=1}^{N-1} ((\frac{N-1}{N})^{1/2}\lambda_j - z'_3)$$

$$- \frac{[-4D''(0)]^{N/2}}{2N} \sum_{k=1}^{N-1} Z_k^2 \prod_{j \neq k}^{N-1} ((\frac{N-1}{N})^{1/2}\lambda_j - z'_3). \quad (4.4)$$

It follows from (4.1) that

$$\mathbb{E}\mathrm{Crt}_N(E, (R_1, R_2)) = S_{N-1} N^{(N-1)/2} \int_{R_1}^{R_2} \int_E \mathbb{E}\Big(\Big|[-4D''(0)]^{(N-1)/2} z'_1 \prod_{j=1}^{N-1} ((\frac{N-1}{N})^{1/2}\lambda_j - z'_3)$$

$$- \frac{[-4D''(0)]^{N/2}}{2N} \sum_{k=1}^{N-1} Z_k^2 \prod_{j \neq k}^{N-1} ((\frac{N-1}{N})^{1/2}\lambda_j - z'_3)\Big|\Big) \frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} du d\rho$$

$$\leq S_{N-1} N^{(N-1)/2} [I_1(E, (R_1, R_2)) + I_2(E, (R_1, R_2))], \quad (4.5)$$

where

$$I_1(E, (R_1, R_2)) = [-4D''(0)]^{\frac{N-1}{2}} \int_{R_1}^{R_2} \int_E \mathbb{E}\Big[|z'_1| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z'_3|\Big]$$

$$\frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} du d\rho,$$

$$I_2(E, (R_1, R_2)) = \frac{[-4D''(0)]^{\frac{N}{2}}}{2N} \sum_{i=1}^{N-1} \int_{R_1}^{R_2} \int_E \mathbb{E}\Big[Z_i^2 \prod_{j \neq i}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_j - z'_3|\Big]$$

$$\frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} du d\rho. \quad (4.6)$$

In the following we will employ hard analysis to derive various estimates that would reduce the problem to the compact setting.

Lemma 4.1. *For any $\rho > 0$, $u \in \mathbb{R}$, we have*

$$\frac{1}{D'(0) - D'(\rho^2)} \leq \frac{C_D(1 + \rho^2)}{\rho^2}, \quad \frac{1}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2 \rho^2}{D'(0)}}} \leq \frac{C_D(1 + \rho^2)}{\rho^2},$$

$$|m_i| \leq |\mu| + C_D \left| \frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right| (1 + \rho^2), \quad i = 1, 2. \tag{4.7}$$

Proof. Since $\lim_{\rho \to 0+} \frac{D'(\rho^2) - D'(0)}{\rho^2} = D''(0)$ and $D'(\rho^2)$ is strictly decreasing to 0 as ρ^2 tends to ∞, we have the first assertion. By (3.22), we have

$$\frac{1}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2 \rho^2}{D'(0)}}} \leq \frac{\sqrt{-6D''(0)}}{D'(0) - D'(\rho^2)} \leq \frac{C_D(1 + \rho^2)}{\rho^2},$$

which together with (3.22) again implies

$$|m_1| \leq |\mu| + \left| \frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right| \frac{C_D \rho^2}{D'(0) - D'(\rho^2)}$$

$$\leq |\mu| + C_D \left| \frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right| (1 + \rho^2),$$

$$|m_2| \leq |\mu| + \left| \frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right| \frac{C_D \rho^2}{D'(0) - D'(\rho^2)}$$

$$\leq |\mu| + C_D \left| \frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right| (1 + \rho^2). \qquad \square$$

Recall $z_1' = \sigma_1 z_1 - \sigma_2 z_2 + m_1$, $z_3' = (\sigma_2 z_2 + \frac{\rho \sqrt{\alpha\beta} z_3}{\sqrt{N}} - m_2)/\sqrt{-4D''(0)}$. Note that the conditional distribution of z_1' given $z_3' = y$ is given by

$$z_1' | z_3' = y \sim N \left(\bar{\mathsf{a}}, \frac{\mathsf{b}^2}{N} \right), \tag{4.8}$$

where

$$\bar{\mathsf{a}} = m_1 - \frac{\sigma_2^2 (\sqrt{-4D''(0)} y + m_2)}{\sigma_2^2 + \frac{\alpha\beta\rho^2}{N}}$$

$$= \frac{-2D''(0)\alpha\rho^2 (u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})}{(-2D''(0) - \beta^2)\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2 \rho^2}{D'(0)}}} + \frac{\alpha\beta\rho^2 \mu}{-2D''(0) - \beta^2}$$

$$- \frac{(-2D''(0) - \beta^2 - \alpha\beta\rho^2)\sqrt{-4D''(0)} y}{-2D''(0) - \beta^2},$$

$$\frac{\mathsf{b}^2}{N} = \sigma_1^2 + \sigma_2^2 - \frac{\sigma_2^4}{\sigma_2^2 + \frac{\alpha\beta\rho^2}{N}} = \frac{-4D''(0)}{N} + \frac{2D''(0)\alpha^2\rho^4}{N(-2D''(0) - \beta^2)}.$$

Note that if z is a standard Gaussian r.v. and Φ is the c.d.f. of z, then we have for $a \in \mathbb{R}, b > 0$,

$$\sqrt{\frac{2}{\pi}} b \leq \mathbb{E}|a + bz| = \frac{\sqrt{2}b}{\sqrt{\pi}} e^{-\frac{a^2}{2b^2}} + a(2\Phi(\frac{a}{b}) - 1) \leq \sqrt{\frac{2}{\pi}} b + |a|. \qquad (4.9)$$

Lemma 4.2. *Suppose $\mu \neq 0$. Then*

$$\limsup_{T \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N([-T, T]^c, (0, \infty)) = -\infty,$$

$$\limsup_{R \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(\mathbb{R}, (R, \infty)) = -\infty,$$

$$\limsup_{\varepsilon \to 0+} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(\mathbb{R}, (0, \varepsilon)) = -\infty.$$

Proof. (1) Note that $\mathsf{b}^2 \leq -4D''(0)$ and that

$$\mathbb{E}[|z_3'|^{N-1}] \leq C^{N-1}\left[m_2^{N-1} + \left(\frac{-2D''(0) - \beta^2}{-4ND''(0)}\right)^{\frac{N-1}{2}}\right] \leq C^{N-1}(1 + m_2^{N-1}). \quad (4.10)$$

We write $m_u = |\mu| + C_D|\frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}|(1 + \rho^2)$. Using the conditional distribution (4.8), (4.9), (2.6), Lemma 4.1 and the elementary fact $m_u \leq \max\{1, m_u^N\}$,

$$\mathbb{E}\left[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\right]$$

$$= \int_{\mathbb{R}} \mathbb{E}\left[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - y| \Big| z_3' = y\right] \frac{\sqrt{-4ND''(0)} \exp\{-\frac{N(\sqrt{-4D''(0)}y + m_2)^2}{2(-2D''(0) - \beta^2)}\}}{\sqrt{2\pi(-2D''(0) - \beta^2)}} dy$$

$$\leq \int_{\mathbb{R}} \left(\frac{\sqrt{2}\mathsf{b}}{\sqrt{\pi N}} + |\bar{a}|\right)\mathbb{E}(\lambda_{N-1}^* + |y|)^{N-1} \frac{\sqrt{-4ND''(0)} \exp\{-\frac{N(\sqrt{-4D''(0)}y + m_2)^2}{2(-2D''(0) - \beta^2)}\}}{\sqrt{2\pi(-2D''(0) - \beta^2)}} dy$$

$$\leq C^{N-1}\mathbb{E}[(\mathsf{b} + |m_1| + |m_2| + \sqrt{-4D''(0)}|z_3'|)(\lambda_{N-1}^{*N-1} + |z_3'|^{N-1})]$$

$$\leq C_D^N(1 + m_u^N),$$

where λ_{N-1}^* is the operator norm of GOE_{N-1}. Similarly,

$$\mathbb{E}\left[Z_i^2 \prod_{j \neq i, 1 \leq j \leq N-1} |(\frac{N-1}{N})^{1/2}\lambda_j - z_3'|\right] \leq \mathbb{E}(\lambda_{N-1}^* + |z_3'|)^{N-2} \leq C^N(1 + |m_2|^{N-2}).$$

Since $D(r) \leq D'(0)r$, we have $D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)} \leq D'(0)\rho^2$. Together with Lemma 4.1, we obtain after a change of variable $u = \rho^2 s$,

$$\mathbb{E}\mathrm{Crt}_N([-T,T]^c, (0,\infty))$$

$$\leq C_D^N S_{N-1} \int_{\mathbb{R}_+} \int_{[-T,T]^c} (1+m_u^N) \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2}D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} du d\rho$$

$$\leq C_{\mu,D}^N S_{N-1} \int_{\mathbb{R}_+} \int_{[-T/\rho^2, T/\rho^2]^c} \left[1 + (1+\rho^{2N})|s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}|^N\right]$$

$$\frac{\sqrt{N}}{\sqrt{2\pi}\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}} \exp\left(-\frac{N\rho^4(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2}{2(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)})}\right) \frac{1}{(2\pi)^{N/2}D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N+1} ds d\rho$$

$$\leq \frac{C_{\mu,D}^N S_{N-1}\sqrt{N}}{(2\pi)^{\frac{N+1}{2}} D'(0)^{N/2}} \left(\int_0^\infty \int_{\sqrt{T/s}}^\infty + \int_{-\infty}^0 \int_{\sqrt{-T/s}}^\infty \right)[1 + (1+\rho^{2N})(|s|+|\mu|)^N]$$

$$\frac{(1+\rho^2)}{\rho^2} \exp\left(-\frac{N[(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2 + \mu^2]\rho^2}{2D'(0)}\right)\rho^{N+1} d\rho ds.$$

We need to find a good lower bound for $(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2$. To save space, let

$$f(s,\rho^2) = [1 + (1+\rho^{2N})(|s|+|\mu|)^N](\rho^{N-1}+\rho^{N+1}) \exp\left(-\frac{N[(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2 + \mu^2]\rho^2}{2D'(0)}\right).$$

We will use the estimate $\int_x^\infty e^{-\frac{y^2}{2\sigma^2}} dy \leq \frac{\sigma^2}{x} e^{-\frac{x^2}{2\sigma^2}}$ repeatedly in the following.

Case 1: $s > 0$. If $s > |\mu|$, since $|\frac{1}{2} - \frac{D'(\rho^2)}{D'(0)}| \leq \frac{1}{2}$, we have

$$\left(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}\right)^2 \geq \left(s - \left|\frac{1}{2} - \frac{D'(\rho^2)}{D'(0)}\right| |\mu|\right)^2 \geq \frac{s^2}{4}.$$

Then

$$\int_{|\mu|}^\infty \int_{\sqrt{T/s}}^\infty f(s,\rho^2) d\rho ds$$

$$\leq \int_{|\mu|}^\infty \int_{\sqrt{T/s}}^\infty [1 + (s+|\mu|)^N + (s+|\mu|)^N\rho^{2N}](\rho^{N-1}+\rho^{N+1})e^{-\frac{N[\frac{s^2}{4}+\mu^2]\rho^2}{2D'(0)}} d\rho ds$$

$$\leq C_{\mu,D}\left(\frac{8D'(0)}{5\mu^2}\right)^{N+1} \int_{|\mu|}^\infty \int_{\sqrt{\frac{T}{2D'(0)}(\frac{s}{4}+\frac{\mu^2}{s})}}^\infty \left(\frac{2D'(0)}{\frac{s^2}{4}+\mu^2}\right)^{(N-1)/2} (1+(s+|\mu|)^N)r^{3N+1}e^{-Nr^2} dr ds$$

$$\leq \frac{C_{\mu,D}^N}{N\sqrt{T}} \int_{|\mu|}^\infty \frac{1+(s+|\mu|)^N}{(s^2+4\mu^2)^{(N-1)/2}} e^{-\frac{NT}{4D'(0)}(\frac{s}{4}+\frac{\mu^2}{s})} ds$$

$$\leq \frac{C_{\mu,D}^N}{N\sqrt{T}} \int_{|\mu|}^\infty \frac{1}{s^2} e^{-\frac{NTs}{32D'(0)}} ds \leq \frac{C_{\mu,D}^N}{N\sqrt{T}} e^{-\frac{|\mu|NT}{32D'(0)}}.$$

Here we have used the fact that $\sqrt{\frac{T}{2D'(0)}(\frac{s}{4} + \frac{\mu^2}{s})} \geq |\mu|\sqrt{\frac{T}{2D'(0)}}$ so that we can always choose T large to guarantee $r > 1$ and $r^4 \leq e^{r^2/2}$.

If $s \leq |\mu|$, using the trivial bound $(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2 \geq 0$, we have

$$\int_0^{|\mu|} \int_{\sqrt{T/s}}^\infty f(s, \rho^2) \mathrm{d}\rho \mathrm{d}s$$

$$\leq \int_0^{|\mu|} \int_{\sqrt{T/|\mu|}}^\infty [1 + (s + |\mu|)^N + (s + |\mu|)^N \rho^{2N}](\rho^{N-1} + \rho^{N+1}) e^{-\frac{N\mu^2 \rho^2}{2D'(0)}} \mathrm{d}\rho \mathrm{d}s$$

$$\leq C_D \int_0^{|\mu|} \int_{\sqrt{\frac{|\mu| T}{2D'(0)}}}^\infty \left[\left(\frac{2D'(0)}{\mu^2} \right)^{3N/2} + 1 \right] (1 + (s + |\mu|)^N) r^{3N+1} e^{-Nr^2} \mathrm{d}r \mathrm{d}s$$

$$\leq \frac{C_{\mu,D}^N}{N\sqrt{T}} e^{-\frac{|\mu| NT}{4D'(0)}}.$$

Case 2: $s < 0$. After change of variable $s \to -s$, we can proceed in the same way as the case $s > 0$ and find

$$\int_{-\infty}^0 \int_{\sqrt{-T/s}}^\infty f(s, \rho^2) \mathrm{d}\rho \mathrm{d}s = \int_0^\infty \int_{\sqrt{T/s}}^\infty f(-s, \rho^2) \mathrm{d}\rho \mathrm{d}s$$

$$= \left(\int_0^{|\mu|} \int_{\sqrt{T/s}}^\infty + \int_{|\mu|}^\infty \int_{\sqrt{T/s}}^\infty \right) f(-s, \rho^2) \mathrm{d}\rho \mathrm{d}s$$

$$\leq \frac{C_{\mu,D}^N}{N\sqrt{T}} \left(e^{-|\mu| NT/[32D'(0)]} + e^{-|\mu| NT/[4D'(0)]} \right).$$

Putting things together, we see that

$$\mathbb{E}\mathrm{Crt}_N([-T, T]^c, (0, \infty)) \leq \frac{C_{\mu,D}^N}{N\sqrt{T}} \left(e^{-|\mu| NT/[32D'(0)]} + e^{-|\mu| NT/[4D'(0)]} \right).$$

From here the first assertion follows.

(2) The last two claims follow somewhat different strategy. By conditioning and Young's inequality,

$$\mathbb{E}\left[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2} \lambda_i - z_3'| \right]$$

$$\leq C^{N-1} \mathbb{E}[(\mathsf{b} + |m_1| + |m_2| + \sqrt{-4D''(0)}|z_3'|)({\lambda_{N-1}^*}^{N-1} + |z_3'|^{N-1})]$$

$$\leq C_D^N (1 + |m_1|^N + |m_2|^N).$$

Using Lemma 4.1, (3.22) together with the change of variable formulas (3.16) and (3.17),

$$\mathbb{E}\mathrm{Crt}_N(\mathbb{R}, (R, \infty))$$

$$\leq C_D^N S_{N-1} \int_R^\infty \int_{\mathbb{R}} (1 + |m_1|^N + |m_2|^N) \frac{e^{-\frac{(u - m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi} \sigma_Y} \frac{e^{-\frac{N\mu^2 \rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho$$

$$\leq C_{\mu,D}^N S_{N-1} \int_R^\infty \int_{\mathbb{R}} [1 + |v|^N (\alpha\rho^2 + \beta)^N] \frac{e^{-\frac{Nv^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{N\mu^2 \rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} \mathrm{d}v \mathrm{d}\rho$$

$$\leq C_{\mu,D}^N S_{N-1} \int_R^\infty e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}\rho$$

$$\leq \frac{C_{\mu,D}^N S_{N-1}}{NR} e^{-\frac{N\mu^2 R^2}{4D'(0)}}$$

for R large enough. Similarly,

$$\mathbb{E}\mathrm{Crt}_N(\mathbb{R}, (0, \varepsilon)) \leq C_{\mu,D}^N S_{N-1} \int_0^\varepsilon e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}\rho \leq \frac{C_{\mu,D}^N S_{N-1} \varepsilon^N}{N}.$$

This completes the proof. $\quad\square$

We remark that we have actually proved the following stronger results with heavier notations from (4.6):

$$\limsup_{T\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log[I_1([-T, T]^c, (0, \infty)) + I_2([-T, T]^c, (0, \infty))] = -\infty,$$

$$\limsup_{R\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log[I_1(\mathbb{R}, (R, \infty)) + I_2(\mathbb{R}, (R, \infty))] = -\infty,$$

$$\limsup_{\varepsilon\to 0+} \limsup_{N\to\infty} \frac{1}{N} \log[I_1(\mathbb{R}, (0, \varepsilon)) + I_2(\mathbb{R}, (0, \varepsilon))] = -\infty.$$

The third claim also holds for $\mu = 0$ with the same argument. If $\mu = 0$, observing the complexity function in Sect. 2, it is reasonable to require $R_2 < \infty$.

Lemma 4.3. *Let $\mu = 0$ and $R < \infty$. Then*

$$\limsup_{T\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N([-T, T]^c, [0, R)) = -\infty.$$

Proof. The argument follows that of Lemma 4.2 and is actually much easier. Indeed, we find

$\mathbb{E}\mathrm{Crt}_N([-T, T]^c, (0, R))$

$$\leq C_D^N S_{N-1} \int_0^R \int_{[-T,T]^c} (1 + m_u^N) \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} \mathrm{d}u\mathrm{d}\rho$$

$$\leq \frac{C_D^N S_{N-1}\sqrt{N}}{(2\pi)^{\frac{N+1}{2}} D'(0)^{N/2}} \int_0^R \left(\int_T^\infty + \int_{-\infty}^{-T} \right) [1 + \rho^N + (1 + \rho^{2N})|u|^N] \frac{(1+\rho^2)}{\rho^2} e^{-\frac{Nu^2}{2D'(0)\rho^2}} \rho^{N-1} \mathrm{d}u\mathrm{d}\rho$$

$$\leq \frac{C_{R,D}^N S_{N-1}\sqrt{N}}{T} e^{-\frac{NT^2}{4D'(0)R^2}}.$$

The proof is complete. $\quad\square$

We need the following fact.

Lemma 4.4. *Suppose $|\mu| + \frac{1}{R} > 0$. Then for any $a > 0, c > 0, b, d \in \mathbb{R}$ satisfying $aN + b < cN + d$, there exist constants $C_{\mu,D,a,b,c,d} > 0, N_0 > 0$ such that for all $N > N_0$,*

$$\int_0^R \int_{-\infty}^\infty (1 + |s|^{aN+b}) \exp\left(-\frac{N(s^2 + \mu^2)\rho^2}{2D'(0)} \right) \rho^{cN+d} \mathrm{d}s\mathrm{d}\rho \leq C_{\mu,R,D,a,b,c,d}^N.$$

Proof. If $\mu \neq 0$, changing the order of integration yields

$$\int_0^\infty \int_{\mathbb{R}} (1 + |s|^{aN+b}) \exp\left(-\frac{N(s^2 + \mu^2)\rho^2}{2D'(0)} \right) \rho^{cN+d} \mathrm{d}s \mathrm{d}\rho$$

$$= \int_{-\infty}^\infty \int_0^\infty \left(\frac{D'(0)}{s^2 + \mu^2} \right)^{\frac{cN+d+1}{2}} (1 + |s|^{aN+b}) r^{cN+d} e^{-Nr^2/2} \mathrm{d}r \mathrm{d}s$$

$$\leq C_{D,c,d}^N \int_{-\infty}^\infty \frac{1 + |s|^{aN+b}}{(s^2 + \mu^2)^{\frac{cN+d+1}{2}}} \mathrm{d}s \leq C_{\mu,D,c,d}^N,$$

where in the last step we used the assumption $aN + b < cN + d$. If $\mu = 0$, then $R < \infty$ and we have

$$\int_0^R \int_{-\infty}^\infty (1 + |s|^{aN+b}) \exp\left(-\frac{Ns^2\rho^2}{2D'(0)} \right) \rho^{cN+d} \mathrm{d}s \mathrm{d}\rho \leq C_{a,b,D}^N \int_0^R (1 + \rho^{-aN-b}) \rho^{cN+d} \mathrm{d}\rho,$$

which completes the proof. $\square$

To save space, for an event Δ that may depend on the eigenvalues of GOE and other Gaussian random variables in question, let us write

$$I_2(E, (R_1, R_2), \Delta) = \frac{[-4D''(0)]^{\frac{N}{2}}}{2N} \sum_{i=1}^{N-1} \int_{R_1}^{R_2} \int_E \mathbb{E}\left[Z_i^2 \prod_{j \neq i} |(\frac{N-1}{N})^{1/2}\lambda_j - z_3'|\mathbf{1}_\Delta \right]$$

$$\frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho.$$

Lemma 4.5. *Suppose* $|\mu| + \frac{1}{R_2} > 0$. *Then*

$$\limsup_{K \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log I_2(E, (R_1, R_2), \{\lambda_{N-1}^* > K\}) = -\infty,$$

$$\limsup_{K \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log I_2(E, (R_1, R_2), \{|z_3' - \mathbb{E}(z_3')| > K\}) = -\infty.$$

Proof. Using (2.5) and choosing K large so that $2t < e^{t^2/18}$ for $t \geq K$,

$$\mathbb{E}[(\lambda_{N-1}^*)^{N-2}\mathbf{1}\{\lambda_{N-1}^* > K\}]$$

$$= \int_0^K Kt^{K-1}\mathbb{P}(\lambda_{N-1}^* \geq K)\mathrm{d}t + \int_K^\infty (N-2)t^{N-3}\mathbb{P}(\lambda_{N-1}^* > t)\mathrm{d}t$$

$$\leq K^K e^{-(N-1)K^2/9} + \int_K^\infty e^{-(N-1)t^2/18}\mathrm{d}t \leq 2e^{-(N-1)K^2/18}. \tag{4.11}$$

If $\mu \neq 0$, using (4.10) and Lemma 4.1, we obtain

$$I_2(E, (R_1, R_2), \{\lambda_{N-1}^* > K\}) \leq C_D^N \int_0^\infty \int_{\mathbb{R}} \mathbb{E}[((\lambda_{N-1}^*)^{N-2} + z_3'^{N-2})\mathbf{1}\{\lambda_{N-1}^* > K\}]$$

$$\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho$$

$$\leq C_{\mu,D}^N e^{-(N-1)K^2/18} \int_0^\infty \int_{\mathbb{R}} \left[1 + |\frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}|^{N-2}(1 + \rho^{2(N-2)})\right]$$

$$\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho$$

$$\leq C_{\mu,D}^N e^{-(N-1)K^2/18} \int_0^\infty \int_{\mathbb{R}} \left[1 + s^{N-2}(1 + \rho^{2(N-1)})\right] \exp\left(-\frac{N(s^2 + \mu^2)\rho^2}{2D'(0)}\right) \rho^{N-1} \mathrm{d}s \mathrm{d}\rho.$$

Here in the last step we used the observation $(1 + \rho^{2(N-2)})(1 + \rho^2) \leq 4(1 + \rho^{2(N-1)})$. The assertion then follows from Lemma 4.4. Similarly, note that

$$\mathbb{P}(|z_3' - \mathbb{E}(z_3')| > K) \leq 2e^{-\frac{N(-4D''(0))K^2}{2(-2D''(0)-\beta^2)}} \leq 2e^{-NK^2}.$$

It follows that for K large enough,

$$\mathbb{E}(|z_3' - \mathbb{E}(z_3')|^{N-2}\mathbf{1}\{|z_3' - \mathbb{E}(z_3')| > K\}) \leq 4e^{-NK^2/2}.$$

From here we deduce that

$$I_2(E, (R_1, R_2), \{|z_3' - \mathbb{E}(z_3')| > K\}) \leq C_D^N \int_0^\infty \int_{\mathbb{R}} \mathbb{E}[((\lambda_{N-1}^*)^{N-2} + |\mathbb{E}(z_3')|^{N-2}$$

$$+ |z_3' - \mathbb{E}(z_3')|^{N-2})\mathbf{1}\{|z_3' - \mathbb{E}(z_3')| > K\}] \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho$$

$$\leq C_{\mu,D}^N e^{-NK^2/2} \int_0^\infty \int_{\mathbb{R}} \left[1 + |\frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}|^{N-2}(1 + \rho^{2(N-2)})\right]$$

$$\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho.$$

The rest of argument is the same as above. The case $\mu = 0$ and $R_2 < \infty$ follows the same steps and is omitted. $\square$

Lemma 4.6. *Suppose* $|\mu| + \frac{1}{R_2} > 0$. *Then for any* $\delta > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log I_2(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \notin B(\sigma_{sc}, \delta)\}) = -\infty.$$

Proof. We only argue for the harder case $\mu \neq 0$. Using (4.10), the Cauchy–Schwarz inequality and (2.3), we have

$$\mathbb{E}\left[\prod_{i=1,i\neq j}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\mathbf{1}\{L(\lambda_1^{N-1}) \notin B(\sigma_{sc}, \delta)\}\right]$$

$$\leq C^N \mathbb{E}[((\lambda_{N-1}^*)^{N-2} + z_3'^{N-2})\mathbf{1}\{L(\lambda_1^{N-1}) \notin B(\sigma_{sc}, \delta)\}]$$

$$\leq C^N [\mathbb{E}((\lambda_{N-1}^*)^{2(N-2)} + z_3'^{2(N-2)})]^{1/2}\mathbb{P}(L(\lambda_1^{N-1}) \notin B(\sigma_{sc}, \delta))^{1/2}$$

$$\leq C_{\mu,D}^N \left[1 + \left|\frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)}\right|^{N-2}(1 + \rho^{2(N-2)})\right] e^{-\frac{1}{2}c(N-1)^2}.$$

Together with Lemma 4.4, we deduce that

$$
\begin{aligned}
&I_2(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \notin B(\sigma_{\mathrm{sc}}, \delta)\}) \\
&\leq C_{\mu,D}^N e^{-cN^2} \int_{R_1}^{R_2} \int_{E/\rho^2} \left[1 + \left| s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)} \right|^{N-2} (1 + \rho^{2(N-1)}) \right] \\
&\quad \exp\left(-\frac{N[(s - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})^2 + \mu^2]\rho^2}{2D'(0)} \right) \rho^{N-1} \mathrm{d}s\mathrm{d}\rho \\
&\leq C_{\mu,D}^N e^{-cN^2} \int_0^\infty \int_{\mathbb{R}} \left[1 + |v|^{N-2}(1 + \rho^{2(N-1)}) \right] \exp\left(-\frac{N[v^2 + \mu^2]\rho^2}{2D'(0)} \right) \rho^{N-1} \mathrm{d}v\mathrm{d}\rho \\
&\leq C_{\mu,D}^N e^{-cN^2}.
\end{aligned}
$$

From here the assertion follows.   □

For an event Δ, let us write

$$
\begin{aligned}
I_1(E, (R_1, R_2), \Delta) &= [-4D''(0)]^{\frac{N-1}{2}} \int_{R_1}^{R_2} \int_E \mathbb{E}\left[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\mathbf{1}_\Delta \right] \\
&\quad \frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2}D'(0)^{N/2}} \rho^{N-1} \mathrm{d}u\mathrm{d}\rho.
\end{aligned}
$$

The argument in this part shares the same spirit as that for I_2.

Lemma 4.7. *Suppose* $|\mu| + \frac{1}{R_2} > 0$. *Then we have*

$$
\limsup_{K\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log I_1(E, (R_1, R_2), \{\lambda_{N-1}^* > K\}) = -\infty,
$$

$$
\limsup_{K\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log I_1(E, (R_1, R_2), \{|z_3' - \mathbb{E}(z_3')| > K\}) = -\infty.
$$

Proof. The argument is similar to that of Lemma 4.5. As there, we only provide details for the case $\mu \neq 0$. Note that $\mathsf{b}^2 \leq -4D''(0)$. By (4.11), (4.8), (4.9), Young's inequality and conditioning, we find

$$
\begin{aligned}
&\mathbb{E}\left[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\mathbf{1}\{\lambda_{N-1}^* > K\} \right] \\
&\leq C^N \mathbb{E}\left[\left(\frac{\sqrt{2}\mathsf{b}}{\sqrt{\pi N}} + |\bar{\mathsf{a}}| \right)((\lambda_{N-1}^*)^{N-1} + |z_3'|^{N-1})\mathbf{1}\{\lambda_{N-1}^* > K\} \right] \\
&\leq C^N \mathbb{E}[(\mathsf{b} + |m_1| + |m_2| + \sqrt{-4D''(0)}|z_3'|)((\lambda_{N-1}^*)^{N-1} + |z_3'|^{N-1})\mathbf{1}\{\lambda_{N-1}^* > K\}] \\
&\leq C_D^N e^{-(N-1)K^2/18}(1 + |m_1|^N + |m_2|^N).
\end{aligned}
$$

Using (3.22) and the change of variable formulas (3.16) and (3.17),

$$I_1(E, (R_1, R_2), \{\lambda_{N-1}^* > K\})$$

$$\leq C_D^N e^{-(N-1)K^2/18} \int_0^\infty \int_{\mathbb{R}} (1 + |m_1|^N + |m_2|^N) \frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2}D'(0)^{N/2}} \rho^{N-1} du d\rho$$

$$\leq C_{\mu,D}^N e^{-(N-1)K^2/18} \int_0^\infty \int_{\mathbb{R}} [1 + |v|^N(\alpha\rho^2 + \beta)^N] \frac{e^{-\frac{Nv^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2}D'(0)^{N/2}} \rho^{N-1} dv d\rho$$

$$\leq C_{\mu,D}^N e^{-(N-1)K^2/18} \int_0^\infty e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} d\rho$$

$$\leq C_{\mu,D}^N e^{-(N-1)K^2/18}.$$

From here the first assertion follows. The argument for the second one is in the same fashion after observing $|z_3'| \leq |z_3' - \mathbb{E}(z_3')| + |\mathbb{E}(z_3')|$ and

$$\mathbb{E}\Big[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\mathbf{1}\{|z_3' - \mathbb{E}(z_3')| > K\}\Big]$$

$$\leq C^N \mathbb{E}[(\mathsf{b} + |m_1| + |m_2| + \sqrt{-4D''(0)}|z_3'|)((\lambda_{N-1}^*)^{N-1} + |z_3'|^{N-1})\mathbf{1}\{|z_3' - \mathbb{E}(z_3')| > K\}]$$

$$\leq C_D^N e^{-NK^2/2}(1 + |m_1|^N + |m_2|^N). \qquad \square$$

Lemma 4.8. *Let $\delta > 0$. Suppose $|\mu| + \frac{1}{R_2} > 0$. Then we have*

$$\limsup_{N \to \infty} \frac{1}{N} \log I_1(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \notin B(\sigma_{\mathrm{sc}}, \delta)\}) = -\infty.$$

Proof. The proof is similar to that of Lemma 4.6 and we only provide the difference for the case $\mu \neq 0$. Conditioning as in the proof of Lemma 4.7, using Young's inequality, the Cauchy–Schwarz inequality and (2.3), we find

$$\mathbb{E}\Big[|z_1'| \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_i - z_3'|\mathbf{1}\{L(\lambda_1^{N-1}) \notin B(\sigma_{\mathrm{sc}}, \delta)\}\Big]$$

$$\leq C_D^N (1 + |m_1|^{2N} + |m_2|^{2N})^{1/2} e^{-cN^2}.$$

The rest of argument follows verbatim that of Lemma 4.7. $\square$

5. Proof of Theorem 1.2

For a probability measure ν defined on $\mathbb{R}$, recall the functions $\Psi(\nu, x)$ and $\Psi_*(x)$ as in (2.7). Let us define

$$\psi(v, \rho, u, y) = \Psi(v, y) - \frac{(u - m_Y)^2}{2\left(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}\right)}$$

$$- \frac{-2D''(0)}{-2D''(0) - \beta^2}\left(y + \frac{m_2}{\sqrt{-4D''(0)}}\right)^2$$

$$- \frac{\mu^2\rho^2}{2D'(0)} + \log\rho, \tag{5.1}$$

$$\psi_*(\rho, u, y) = \psi(\sigma_{\rm sc}, \rho, u, y).$$

Recalling the notations (3.15), $\psi_*(\rho, u, y)$ can be written explicitly as

$$\psi_*(\rho, u, y) = \Psi_*(y) - \frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})^2}{2(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)})}$$

$$- \frac{\mu^2\rho^2}{2D'(0)} + \log\rho - \frac{-2D''(0)}{-2D''(0) - \frac{[D'(\rho^2) - D'(0)]^2}{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}}$$

$$\times \left(y + \frac{1}{\sqrt{-4D''(0)}}\left[\mu + \frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})(D'(\rho^2) - D'(0))}{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}\right]\right)^2. \tag{5.2}$$

Lemma 5.1. *For any u and y fixed, we have $\lim_{\rho\to 0+}\psi_*(\rho, u, y) = -\infty$. For any ρ and u fixed, we have $\lim_{|y|\to\infty}\psi_*(\rho, u, y) = -\infty$.*

Proof. From Lemma 3.1, we know $D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)} \sim -\frac{3}{2}D''(0)\rho^4$ as $\rho \to 0+$. For any $\varepsilon > 0$ and $\rho \in (0, \varepsilon)$, we may find c_ε such that

$$\psi_*(\rho, u, y) - \Psi_*(y) \le -\frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})^2}{2(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)})} - \frac{\mu^2\rho^2}{2D'(0)} + \log\rho$$

$$\le -\frac{(\frac{u}{\rho^2} - \frac{\mu}{2} + \frac{\mu D'(\rho^2)}{D'(0)})}{-3c_\varepsilon D''(0)} - \frac{\mu^2\rho^2}{2D'(0)} + \log\rho.$$

The right-hand side clearly tends to $-\infty$ as $\rho \to 0+$.

Since $\frac{-2D''(0)}{-2D''(0) - \beta^2} \ge 1$, from the definition it is clear to see $\lim_{|y|\to\infty}\psi_*(\rho, u, y) = -\infty$ for fixed ρ and u. $\quad\square$

Let $[\![\ell]\!] = \{i_1, \ldots, i_\ell\} \subset [N - 1]$. For any 1-Lipschitz function f, we have

$$\left|\frac{1}{N-1}\sum_{j=1}^{N-1}f(\lambda_j) - \frac{1}{N-1-\ell}\sum_{j\in[N-1]\setminus[\![\ell]\!]}f(\lambda_j)\right|$$

$$\le \frac{1}{(N-1)(N-1-\ell)}\sum_{j\in[N-1]\setminus[\![\ell]\!]}\left|(N-1-\ell)f(\lambda_j) + \sum_{i\in[\![\ell]\!]}f(\lambda_i) - (N-1)f(\lambda_j)\right|$$

$$\le \frac{\ell}{N-1}\max_{i,j}|\lambda_i - \lambda_j|. \tag{5.3}$$

5.1. Upper bound.

Proposition 5.2. *Suppose $\bar{E}$ is compact and $0 \leq R_1 < R_2 < \infty$. Under Assumptions I, II and IV, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log I_2(E, (R_1, R_2)) \leq \frac{1}{2} \log[-4D''(0)] - \frac{1}{2} \log D'(0)$$
$$- \frac{1}{2} \log(2\pi) + \sup_{(\rho,u,y)\in F} \psi_*(\rho, u, y),$$

where $F = \{(\rho, u, y) : y \in \mathbb{R}, \rho \in (R_1, R_2), u \in \bar{E}\}$ and $\psi_*(\rho, u, y)$ is given as in (5.2).

Proof. Since

$$I_2(E, (R_1, R_2)) = I_2(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \in B_K(\sigma_{\text{sc}}, \delta), |z_3' - \mathbb{E}(z_3')| \leq K\})$$
$$+ I_2(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \notin B_K(\sigma_{\text{sc}}, \delta)\} \cup \{|z_3' - \mathbb{E}(z_3')| > K\}),$$

by Lemmas 4.5 and 4.6, we can always choose K large enough so that the second term is exponentially negligible as $N \to \infty$, provided the first term yields a finite quantity in the limit. We only need to consider the first term.

Using (5.3), if $L(\lambda_{i=1}^{N-1}) \in B_K(\sigma_{\text{sc}}, \delta)$, we may choose N large enough so that $L((\frac{N-1}{N})^{1/2}\lambda_{j=1,j\neq i}^{N-1}) \in B_K(\sigma_{\text{sc}}, 2\delta)$. Then for any $i \in [N-1]$,

$$\prod_{j=1,j\neq i}^{N-1} |(\frac{N-1}{N})^{1/2}\lambda_j - z_3'| \mathbf{1}\{L(\lambda_{i=1}^{N-1}) \in B_K(\sigma_{\text{sc}}, \delta)\} \leq e^{(N-2)\sup_{\nu\in B_K(\sigma_{\text{sc}},2\delta)}\Psi(\nu,z_3')}.$$

$$(5.4)$$

By Lemma 3.1 and (3.23), we have $c_{D,R_2} := \inf_{R_1<\rho<R_2} -2D''(0) - \beta^2 > 0$ where $\beta = \beta(\rho^2)$ was defined in (3.2). It follows that

$$\frac{\sqrt{-4ND''(0)}}{\sqrt{2\pi(-2D''(0) - \beta^2)}} \exp\left(-\frac{-2ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{-2D''(0) - \beta^2}\right)$$
$$\leq \frac{\sqrt{-4ND''(0)}}{\sqrt{2\pi c_{D,R_2}}} \exp\left(-\frac{-2ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{-2D''(0) - \beta^2}\right).$$

Let

$$F(\delta) = \left\{(\nu, \rho, u, y) : \nu \in B_K(\sigma_{\text{sc}}, \delta), y \in \left[-\frac{m_2}{\sqrt{-4D''(0)}} - K, -\frac{m_2}{\sqrt{-4D''(0)}} + K\right],\right.$$
$$\left.\rho \in (R_1, R_2), u \in \bar{E}\right\}.$$

$$(5.5)$$

Using $\rho^2 \leq R_2^2$ and the fact that all summands of $\psi(v, \rho, u, y)$ in (5.1) are bounded from above on $F(\delta)$, we deduce from Lemma 4.1

$$I_2(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \in B_K(\sigma_{\mathrm{sc}}, \delta), |z_3' - \mathbb{E}(z_3')| \leq K\})$$

$$\leq [-4D''(0)]^{N/2} \int_{R_1}^{R_2} \int_E \mathbb{E}\Big[e^{(N-2)\sup_{v \in B_K(\sigma_{\mathrm{sc}}, 2\delta)} \Psi(v, z_3')} \mathbf{1}\{|z_3' - \mathbb{E}(z_3')| \leq K\}\Big]$$

$$\frac{\sqrt{N} e^{-\frac{N(u - m_Y)^2}{2\left[D(\rho^2) - \frac{D'(\rho^2)^2 \rho^2}{D'(0)}\right]}}}{\sqrt{2\pi(D(\rho^2) - \frac{D'(\rho^2)^2 \rho^2}{D'(0)})}} \frac{e^{-\frac{N\mu^2 \rho^2}{2D'(0)}}}{(2\pi)^{N/2} D'(0)^{N/2}} \rho^{N-1} \mathrm{d}u \mathrm{d}\rho$$

$$\leq \frac{C_{D, R_2, |E|} N[-4D''(0)]^{\frac{N+1}{2}}}{(2\pi)^{\frac{N+2}{2}} D'(0)^{\frac{N}{2}}} \exp\Big[(N-3) \sup_{(v, \rho, u, y) \in F(2\delta)} \psi(v, \rho, u, y)\Big],$$

where $|E|$ is the Lebesgue measure of E. Since $\psi(v, \rho, u, y)$ is an upper semi-continuous function on $F(2\delta)$ and attains its maximum on the closure $\overline{F(2\delta)}$, we have

$$\limsup_{\delta \to 0+} \sup_{(v, \rho, u, y) \in F(2\delta)} \psi(v, \rho, u, y) \leq \sup_{(\rho, u, y) \in F(0)} \psi_*(\rho, u, y).$$

By Lemmas 4.5 and 5.1, the continuous function $\psi_*(\rho, u, y)$ attains its maximum in $\bar{F}$ at some point (ρ_*, u_*, y_*) with $\rho_* > 0$. Therefore we may choose K large enough in the beginning so that

$$\sup_{(\rho, u, y) \in F(0)} \psi_*(\rho, u, y) = \psi_*(\rho_*, u_*, y_*).$$

This justifies that $\sup_{(\rho, u, y) \in F} \psi_*(\rho, u, y) > -\infty$ and the proof is complete. $\square$

Proposition 5.3. *Suppose $\bar{E}$ is compact and $0 \leq R_1 < R_2 < \infty$. Under Assumptions I, II and IV, we have*

$$\limsup_{N \to \infty} \frac{1}{N} \log I_1(E, (R_1, R_2)) \leq \frac{1}{2} \log[-4D''(0)] - \frac{1}{2} \log D'(0)$$

$$- \frac{1}{2} \log(2\pi) + \sup_{(\rho, u, y) \in F} \psi_*(\rho, u, y),$$

where $F = \{(\rho, u, y) : y \in \mathbb{R}, \rho \in (R_1, R_2), u \in \bar{E}\}$ and $\psi_*(\rho, u, y)$ is given as in (5.2).

Proof. By the remark after Lemma 4.2, we know

$$\limsup_{N \to \infty} \frac{1}{N} \log I_1(E, (0, R_2)) = \limsup_{N \to \infty} \frac{1}{N} \log I_1(E, (\varepsilon, R_2))$$

by choosing $\varepsilon > 0$ small enough. Hence, we may assume $R_1 > 0$. Similar to the proof of Proposition 5.2, since

$$I_1(E, (R_1, R_2)) = I_1(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \in B_K(\sigma_{\mathrm{sc}}, \delta), |z_3' - \mathbb{E}(z_3')| \leq K\})$$

$$+ I_1(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \notin B_K(\sigma_{\mathrm{sc}}, \delta)\} \cup \{|z_3' - \mathbb{E}(z_3')| > K\}),$$

thanks to Lemmas 4.7 and 4.8, by choosing K large enough, it suffices to consider the first term. Since $0 < R_1 < R_2 < \infty$, using continuity of functions in question, conditioning with (4.8) and Lemma 4.1 for σ_Y,

$$I_1(E, (R_1, R_2), \{L(\lambda_1^{N-1}) \in B_K(\sigma_{sc}, \delta), |z_3' - \mathbb{E}(z_3')| \le K\})$$

$$\le [-4D''(0)]^{\frac{N-1}{2}} \sup_{R_1 \le \rho \le R_2, u \in \bar{E}, |y + \frac{m_2}{\sqrt{-4D''(0)}}| \le K} (\mathsf{b} + |m_1| + |m_2| + \sqrt{-4D''(0)}|y|) \int_{R_1}^{R_2} \int_E$$

$$\mathbb{E}\left[e^{(N-1)\sup_{v \in B_K(\sigma_{sc}, \delta)} \Psi(v, z_3')} \mathbf{1}\{|z_3' - \mathbb{E}(z_3')| \le K\}\right] \frac{e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}}}{\sqrt{2\pi}\sigma_Y} \frac{e^{-\frac{N\mu^2\rho^2}{2D'(0)}}}{(2\pi)^{N/2}D'(0)^{N/2}} \rho^{N-1} du d\rho$$

$$\le \frac{C_{R_1, R_2, D, K, \bar{E}}[-4D''(0)]^{\frac{N}{2}}}{(2\pi)^{\frac{N+2}{2}}D'(0)^{\frac{N}{2}}} \exp\left[(N-3) \sup_{(v, \rho, u, y) \in F(\delta)} \psi(v, \rho, u, y)\right],$$

where $F(\delta)$ is given as in (5.5) and the supremum of $|m_1| + |m_2|$ may depend on R_1. The assertion follows from the upper semi-continuity of $\psi(v, \rho, u, y)$ on $F(\delta)$ by sending $N \to \infty$ and $\delta \to 0+$. $\square$

5.2. Lower bound.

Proposition 5.4. *Suppose E is an open set and $0 \le R_1 < R_2 < \infty$. Under Assumptions I, II and IV, we have*

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(E, (R_1, R_2)) \ge \frac{1}{2} + \frac{1}{2} \log[-4D''(0)]$$

$$- \frac{1}{2} \log D'(0) + \sup_{(\rho, u, y) \in F} \psi_*(\rho, u, y),$$

where $F = \{(\rho, u, y) : y \in \mathbb{R}, \rho \in (R_1, R_2), u \in \bar{E}\}$ and $\psi_*(\rho, u, y)$ is given as in (5.2).

Proof. Using (4.9) and (4.8), we know

$$\mathbb{E}\left[|z_1' - h(z_3')||z_3' = y\right] \ge \sqrt{\frac{2}{\pi}}\left[\frac{-4D''(0)}{N} + \frac{2D''(0)\alpha^2\rho^4}{N(-2D''(0) - \beta^2)}\right]^{1/2}, \qquad (5.6)$$

where $h(z_3')$ only depends on z_3'. By conditioning, using (4.4) and (3.18),

$$\mathbb{E}(|\det G|) = \mathbb{E}(|\det G_{**}||z_1' - \xi^\mathsf{T}G_{**}^{-1}\xi|)$$

$$= [-4D''(0)]^{\frac{N-1}{2}} \mathbb{E}[|\det((\frac{N-1}{N})^{1/2}\mathrm{GOE}_{N-1} - z_3'I_{N-1})|\mathbb{E}(|z_1' - \xi^\mathsf{T}G_{**}^{-1}\xi||\mathrm{GOE}_{N-1}, \xi, z_3')]$$

$$\ge [-4D''(0)]^{\frac{N-1}{2}} \sqrt{\frac{2}{\pi}}\left[\frac{-4D''(0)}{N} + \frac{2D''(0)\alpha^2\rho^4}{N(-2D''(0) - \beta^2)}\right]^{1/2}$$

$$\frac{\sqrt{N(-4D''(0))}}{\sqrt{2\pi(-2D''(0) - \beta^2)}} \int_{\mathbb{R}^{N-1}} \prod_{i=1}^{N-1} \int_{\mathbb{R}} |(\frac{N-1}{N})^{1/2}x_i - y| \exp\left[-\frac{-4ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right] dy$$

$$p_{\mathrm{GOE}}(x_1, \ldots, x_{N-1}) \prod_{i=1}^{N-1} dx_i$$

where $p_{\mathrm{GOE}}(x_1, \ldots, x_{N-1})$ is the joint density of the unordered eigenvalues of GOE.

Without loss of generality we assume E is non-empty. Choose (ρ_*, u_*, y_*) as that in the proof of Proposition 5.2; i.e., it is a maximum of $\psi_*(\rho, u, y)$ on $[R_1, R_2] \times \bar{E} \times \mathbb{R}$. If there are multiple points for ψ_* to attain its maximum, we just choose one to be (ρ_*, u_*, y_*). Recall that $\rho_* > 0$. Then $(\rho_* - \delta', \rho_* + \delta') \cap [R_1, R_2]$ and $(u_* - \delta', u_* + \delta') \cap \bar{E}$ must be non-empty for any $\delta' > 0$. If ρ_* and u_* are both interior points, we choose $\delta' > 0$ small enough so that $(\rho_* - \delta', \rho_* + \delta') \subset (R_1, R_2)$ and $(u_* - \delta', u_* + \delta') \subset E$. If either ρ_* or u_* is a boundary point, by abuse of notation we still write $(\rho_* - \delta', \rho_* + \delta')$ and $(u_* - \delta', u_* + \delta')$ with the understanding that one endpoint should be replaced by ρ_* or u_* so that we always have $(\rho_* - \delta', \rho_* + \delta') \subset (R_1, R_2)$ and $(u_* - \delta', u_* + \delta') \subset E$. Using (3.23), the right-hand side of (5.6) attains strictly positive minimum for $\rho \in [\rho_* - \delta', \rho_* + \delta']$. By restricting to small intervals, we find

$$
\int_{R_1}^{R_2} \int_E \mathbb{E}(|\det G|) \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} du d\rho
$$

$$
\geq [-4D''(0)]^{\frac{N-1}{2}} \sqrt{\frac{2}{\pi}} \int_{\rho_*-\delta'}^{\rho_*+\delta'} \int_{u_*-\delta'}^{u_*+\delta'} \int_{y_*-\delta_1}^{y_*+\delta_1}
$$

$$
\left[\frac{-4D''(0)}{N} + \frac{2D''(0)\alpha^2\rho^4}{N(-2D''(0) - \beta^2)}\right]^{1/2} \frac{\sqrt{N(-4D''(0))}}{\sqrt{2\pi(-2D''(0) - \beta^2)}}
$$

$$
\int_{\mathbb{R}^{N-1}} \prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2} x_i - y| \exp\left[-\frac{-4ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right]
$$

$$
p_{\text{GOE}}(x_1, \ldots, x_{N-1}) \prod_{i=1}^{N-1} dx_i \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(u-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} dy du d\rho
$$

$$
=: \mathcal{E}(\delta', \delta_1),
$$

where $\delta_1 > 0$ will be specified in the following. We consider two cases.

Case 1: $y_* \notin [-\sqrt{2}, \sqrt{2}]$. In this case, there exists $\varepsilon_1 > 0$ small enough so that $y_* \notin [-\sqrt{2} - 3\varepsilon_1, \sqrt{2} + 3\varepsilon_1]$. We can choose δ_1 small enough so that $y_* + \delta_1 < -\sqrt{2} - 2\varepsilon_1$ if $y_* < -\sqrt{2}$ or $y_* - \delta_1 > \sqrt{2} + 2\varepsilon_1$ if $y_* > \sqrt{2}$. According to our choice, if $x \in (y_* - \delta_1, y_* + \delta_1)$, then $x \notin [-\sqrt{2} - 2\varepsilon_1, \sqrt{2} + 2\varepsilon_1]$. With these considerations in mind, by restricting the empirical measure of GOE eigenvalues to $B_{\sqrt{2}+\varepsilon_1}(\sigma_{\text{sc}}, \delta)$ first, we find

$$
\mathcal{E}(\delta', \delta_1) \geq [-4D''(0)]^{\frac{N-1}{2}} \sqrt{\frac{2}{\pi}} \mathbb{P}(L((\frac{N-1}{N})^{1/2} \lambda_1^{N-1}) \in B_{\sqrt{2}+\varepsilon_1}(\sigma_{\text{sc}}, \delta))
$$

$$
\int_{\rho_*-\delta'}^{\rho_*+\delta'} \int_{u_*-\delta'}^{u_*+\delta'} \int_{y_*-\delta_1}^{y_*+\delta_1} e^{(N-1)\inf_{v \in B_{\sqrt{2}+\varepsilon_1}(\sigma_{\text{sc}}, \delta)} \Psi(v, y)} \exp\left[-\frac{-4ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right]
$$

$$
\left[\frac{-4D''(0)}{N} + \frac{2D''(0)\alpha^2\rho^4}{N(-2D''(0) - \beta^2)}\right]^{1/2} \frac{\sqrt{N(-4D''(0))}}{\sqrt{2\pi(-2D''(0) - \beta^2)}}
$$

$$
\frac{\sqrt{N}(2\pi)^{-(N+1)/2} D'(0)^{-N/2}}{\sqrt{D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}}} \exp\left(-\frac{N(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})^2}{2(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)})}\right) e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} dy du d\rho.
$$

Since $\Psi(v, y)$ is continuous in $\mathcal{P}[-\sqrt{2} - \varepsilon_1, \sqrt{2} + \varepsilon_1] \times (-\sqrt{2} - 2\varepsilon_1, \sqrt{2} + 2\varepsilon_1)^c$, we have

$$\lim_{\delta \to 0+} \inf_{v \in B_{\sqrt{2}+\varepsilon_1}(\sigma_{\mathrm{sc}}, \delta)} \Psi(v, y) = \Psi_*(y)$$

for all $y \in [y_* - \delta_1, y_* + \delta_1]$. By Wigner's semicircle law with the distance (2.2) and the LDP of the largest eigenvalue of GOE, we have

$$\liminf_{N \to \infty} \mathbb{P}(L((\frac{N-1}{N})^{1/2}\lambda_1^{N-1}) \in B_{\sqrt{2}+\varepsilon_1}(\sigma_{\mathrm{sc}}, \delta))$$

$$\geq \liminf_{N \to \infty} [\mathbb{P}(L((\frac{N-1}{N})^{1/2}\lambda_1^{N-1}) \in B(\sigma_{\mathrm{sc}}, \delta))$$

$$- \mathbb{P}(\max_{i=1,\dots,N-1} |(\frac{N-1}{N})^{1/2}\lambda_i| > \sqrt{2} + \varepsilon_1)] = 1.$$

Recall the function ψ as in (5.1). Since the functions in question are all continuous and thus attain strictly positive minimum in $\rho \in [\rho_* - \delta', \rho_* + \delta']$, $u \in [u_* - \delta', u_* + \delta']$, $y \in [y_* - \delta_1, y_* + \delta_1]$, using (4.2) and (4.1) we deduce that

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(E, (R_1, R_2)) \geq \liminf_{\substack{\delta' \to 0+, \\ \delta_1 \to 0+}} \liminf_{N \to \infty} \frac{1}{N} \log \mathcal{E}(\delta', \delta_1) + \frac{1}{2} + \frac{1}{2} \log(2\pi)$$

$$\geq \frac{1}{2} + \frac{1}{2} \log[-4D''(0)] - \frac{1}{2} \log D'(0)$$

$$+ \liminf_{\substack{\delta \to 0+, \delta' \to 0+, \\ \delta_1 \to 0+}} \inf_{\substack{\rho \in [\rho_* - \delta', \rho_* + \delta'], \\ u \in [u_* - \delta', u_* + \delta'], y \in [y_* - \delta_1, y_* + \delta_1]}} [\psi_*(\rho, u, y) - \Psi_*(y) + \inf_{v \in B_{\sqrt{2}+\varepsilon_1}(\sigma_{\mathrm{sc}}, \delta)} \Psi(v, y)]$$

$$= \frac{1}{2} + \frac{1}{2} \log[-4D''(0)] - \frac{1}{2} \log D'(0) + \psi_*(\rho_*, u_*, y_*). \tag{5.7}$$

Case 2: $y_* \in [-\sqrt{2}, \sqrt{2}]$. In this case, we can choose $\delta_1 > 0$ small such that $G(\delta_1) := (y_* - \delta_1, y_* + \delta_1) \cap (-\sqrt{2}, \sqrt{2}) \neq \emptyset$. Choosing K large we find

$$\int_{G(\delta_1)} \mathbb{E}[e^{(N-1)\Psi(L((\frac{N-1}{N})^{1/2}\lambda_1^{N-1}), y)}] \exp\left[-\frac{-4ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right] dy$$

$$\geq \frac{1}{Z'_{N-1}} \int_{G(\delta_1)} \int_{[-(\frac{N}{N-1})^{1/2}K, (\frac{N}{N-1})^{1/2}K]^{N-1}} \exp\left[-\frac{-4ND''(0)(y + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right]$$

$$\prod_{i=1}^{N-1} |(\frac{N-1}{N})^{1/2}x_i - y| \prod_{1 \leq i < j \leq N-1} |x_i - x_j| e^{-\frac{N-1}{2}\sum_{i=1}^{N-1} x_i^2} \prod_{i=1}^{N-1} dx_i dy$$

$$\overset{(\frac{N-1}{N})^{1/2}x_i \mapsto x_i}{=\!=\!=\!=\!=} \frac{1}{Z'_{N-1}} \left(\frac{N}{N-1}\right)^{\frac{N(N-1)}{4}} \int_{x_N \in G(\delta_1)} \exp\left[-\frac{-4ND''(0)(x_N + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right]$$

$$\int_{[-K,K]^{N-1}} \prod_{1 \leq i < j \leq N} |x_i - x_j| e^{-\frac{N}{2}\sum_{i=1}^{N} x_i^2} e^{\frac{N}{2}x_N^2} \prod_{i=1}^{N} dx_i$$

$$\geq \frac{Z'_N}{Z'_{N-1}} \frac{1}{Z'_N} \left(\frac{N}{N-1}\right)^{\frac{N(N-1)}{4}} \exp\left[N \min_{x \in G(\delta_1)} \left(\frac{x^2}{2} - \frac{-4D''(0)(x + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right)\right]$$

$$\int_{x_N \in G(\delta_1)} \int_{[-K,K]^{N-1}} \prod_{1 \leq i < j \leq N} |x_i - x_j| e^{-\frac{N}{2}\sum_{i=1}^{N} x_i^2} \prod_{i=1}^{N} dx_i$$

$$= \frac{Z'_N}{Z'_{N-1}} \left(\frac{N}{N-1}\right)^{\frac{N(N-1)}{4}} \exp\left[N \min_{x \in G(\delta_1)} \left(\frac{x^2}{2} - \frac{-4ND''(0)(x + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)}\right)\right]$$

$$\mathbb{E}\left[\frac{1}{N}\#\{i \in [N] : \tilde{\lambda}_i^N \in G(\delta_1)\}\mathbf{1}\{\max_{i=1,\dots,N} |\tilde{\lambda}_i^N| \le K\}\right].$$

Here $Z'_N = N! Z_N$ is the normalizing constant for the p.d.f. of unordered eigenvalues of GOE_N matrix. By Stirling's formula,

$$\lim_{N \to \infty} \frac{1}{N} \log\left[\frac{Z'_N}{Z'_{N-1}}\left(\frac{N}{N-1}\right)^{\frac{N(N-1)}{4}}\right] = -\frac{1}{2} - \frac{1}{2}\log 2.$$

From Wigner's semicircle law we deduce

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}\left[\frac{1}{N}\#\{i \in [N] : \tilde{\lambda}_i^N \in G(\delta_1)\}\mathbf{1}\{\max_{i=1,\dots,N} |\lambda_i| \le K\}\right]$$

$$= \lim_{N \to \infty} \frac{1}{N} \log \sigma_{\mathrm{sc}}[G(\delta_1)] = 0.$$

Since the functions in question are all continuous and thus attains strictly positive minimum in $\rho \in [\rho_* - \delta', \rho_* + \delta']$, $u \in [u_* - \delta', u_* + \delta']$, $y \in [y_* - \delta_1, y_* + \delta_1]$, using (4.2) and (4.1) we deduce that

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(E, (R_1, R_2)) \ge \liminf_{\substack{\delta' \to 0+, \\ \delta_1 \to 0+}} \liminf_{N \to \infty} \frac{1}{N} \log \mathcal{E}(\delta', \delta_1) + \frac{1}{2} + \frac{1}{2}\log(2\pi)$$

$$\ge \frac{1}{2} + \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) - \frac{1}{2} - \frac{1}{2}\log 2 + \liminf_{\substack{\delta' \to 0+, \\ \delta_1 \to 0+}} \inf_{\substack{\rho \in [\rho_* - \delta', \rho_* + \delta'], \\ u \in [u_* - \delta', u_* + \delta'], x \in G(\delta_1)}}$$

$$\left[\frac{x^2}{2} - \frac{-4ND''(0)(x + \frac{m_2}{\sqrt{-4D''(0)}})^2}{2(-2D''(0) - \beta^2)} - \frac{(u - \frac{\mu\rho^2}{2} + \frac{\mu D'(\rho^2)\rho^2}{D'(0)})^2}{2(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)})} - \frac{\mu^2\rho^2}{2D'(0)} + \log \rho\right]$$

$$= \frac{1}{2} + \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \psi_*(\rho_*, u_*, y_*). \tag{5.8}$$

Here in the last step, we used the fact (2.8) that $\Psi_*(y_*) = \frac{1}{2}y_*^2 - \frac{1}{2} - \frac{1}{2}\log 2$ as $y_* \in [-\sqrt{2}, \sqrt{2}]$. $\quad\square$

Proof of Theorem 1.2. If $\bar{E}$ is compact and $0 \le R_1 < R_2 < \infty$, the assertion follows from (4.2), (4.5), Propositions 5.2, 5.3 and 5.4.

Suppose $\bar{E}$ is not compact or $R_2 = \infty$. Thanks to Lemmas 5.1 and 4.2, we may choose $R < \infty$ and $T < \infty$ large enough such that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(E, (R_1, R_2)) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(E \cap (-T, T), (R_1, R_2) \cap [0, R])$$

$$= \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2} + \sup_{\substack{y \in \mathbb{R}, R_1 < \rho < R \wedge R_2, u \in \bar{E} \cap [-T,T],}} \psi_*(\rho, u, y)$$

$$= \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2} + \sup_{(\rho, u, y) \in F} \psi_*(\rho, u, y),$$

which completes the proof. $\quad\square$

We finish this section by showing how to recover Theorem 1.1 from Theorem 1.2 when the domain of field is confined in a shell.

Example 2. Let $0 \le R_1 < R_2 \le \infty$ and $E = \mathbb{R}$. This removes restriction on the range of the random field. Let $J = \sqrt{-2D''(0)}$. Using (3.15) and (3.16), we rewrite

$$\psi_*(\rho, u, y) = \Psi_*(y) - \frac{J^2}{J^2 - \beta^2}\left(y + \frac{\mu}{\sqrt{2}J} + \frac{\beta v}{\sqrt{2}J}\right)^2 - \frac{v^2}{2} - \frac{\mu^2\rho^2}{2D'(0)} + \log\rho.$$

From (2.8), we calculate

$$\partial_y\psi_* = \frac{-(\beta^2 + J^2)y - \sqrt{2}J(\mu + \beta v)}{J^2 - \beta^2} - \text{sgn}(y)\sqrt{y^2 - 2}\mathbf{1}\{|y| > \sqrt{2}\},$$

$$\partial_{yy}\psi_* = -\frac{J^2 + \beta^2}{J^2 - \beta^2} - \frac{|y|}{\sqrt{y^2 - 2}}\mathbf{1}\{|y| > \sqrt{2}\},$$

$$\partial_v\psi_* = \frac{-J^2v - \beta(\sqrt{2}Jy + \mu)}{J^2 - \beta^2}, \quad \partial_{yv}\psi_* = -\frac{\sqrt{2}J\beta}{J^2 - \beta^2}, \quad \partial_{vv}\psi_* = -\frac{J^2}{J^2 - \beta^2}.$$

Using the relation $\partial_v\psi_* = 0$ we find

$$v = -\frac{\beta(\sqrt{2}Jy + \mu)}{J^2}, \quad \sqrt{2}Jy + \mu + \beta v = \frac{(\sqrt{2}Jy + \mu)(J^2 - \beta^2)}{J^2}. \qquad (5.9)$$

Together with (2.8), we can eliminate v and simplify

$$\psi_*(\rho, u, y) = -\frac{1}{2}y^2 - \frac{1}{2} - \frac{1}{2}\log 2 - J_1(-|y|)\mathbf{1}\{|y| > \sqrt{2}\}$$
$$- \frac{\sqrt{2}\mu y}{J} - \frac{\mu^2}{2J^2} - \frac{\mu^2\rho^2}{2D'(0)} + \log\rho. \qquad (5.10)$$

Case 1: $\mu \ne 0$. Solving $\partial_y\psi_* = 0$, $\partial_v\psi_* = 0$ gives (after removing an extraneous solution)

$$\begin{cases} y = -\frac{\sqrt{2}\mu}{J}, \quad v = \frac{\mu\beta}{J^2}, & |\mu| \le J, \\ y = -\frac{1}{\sqrt{2}}\left(\frac{\mu}{J} + \frac{J}{\mu}\right), \quad v = \frac{\beta}{\mu}, & |\mu| > J. \end{cases}$$

From (3.23) we know $J^2 - \beta^2 > 0$ for $\rho > 0$. By the second derivative test, this critical point is the unique global maximum. Moreover, plugging in the critical point reveals that

$$\Psi_*(y) - \frac{J^2}{J^2 - \beta^2}\left(y + \frac{\mu}{\sqrt{2}J} + \frac{\beta v}{\sqrt{2}J}\right)^2 - \frac{v^2}{2}$$

does not depend on ρ. As a result, we choose ρ by optimizing $-\frac{\mu^2\rho^2}{2D'(0)} + \log\rho$. Let us consider $R_1 < \sqrt{D'(0)}/|\mu|$ only; the other case is similar. Choose

$$\rho_* = \begin{cases} \frac{\sqrt{D'(0)}}{|\mu|}, & \text{if } R_2 > \frac{\sqrt{D'(0)}}{|\mu|}, \\ R_2, & \text{otherwise.} \end{cases} \qquad (5.11)$$

If $|\mu| \leq \sqrt{-2D''(0)}$, we take $y_* = -\mu/\sqrt{-D''(0)}$, and

$$u_* = \frac{\mu[D'(\rho_*^2) - D'(0)]}{-2D''(0)} + \frac{\mu\rho_*^2}{2} - \frac{\mu D'(\rho_*^2)\rho_*^2}{D'(0)}. \tag{5.12}$$

Then we find

$$\psi_*(\rho_*, u_*, y_*) = \begin{cases} \frac{\mu^2}{-4D''(0)} - 1 - \frac{1}{2}\log 2 + \frac{1}{2}\log D'(0) - \log|\mu|, & \text{if } R_2 > \frac{\sqrt{D'(0)}}{|\mu|}, \\ \frac{\mu^2}{-4D''(0)} - \frac{1}{2} - \frac{1}{2}\log 2 + \log R_2 - \frac{\mu^2 R_2^2}{2D'(0)}, & \text{otherwise.} \end{cases} \tag{5.13}$$

If $|\mu| > \sqrt{-2D''(0)}$, we take $y_* = -\frac{\mu}{\sqrt{-4D''(0)}} - \frac{\sqrt{-D''(0)}}{\mu}$,

$$u_* = \frac{D'(\rho_*^2) - D'(0)}{\mu} + \frac{\mu\rho_*^2}{2} - \frac{\mu D'(\rho_*^2)\rho_*^2}{D'(0)}. \tag{5.14}$$

Then we find

$$\psi_*(\rho_*, u_*, y_*)$$
$$= \begin{cases} -\frac{1}{2}\log 2 - \log\sqrt{-2D''(0)} - \frac{1}{2} + \frac{1}{2}\log D'(0), & \text{if } R_2 > \frac{\sqrt{D'(0)}}{|\mu|}, \\ -\frac{1}{2}\log 2 - \log\sqrt{-2D''(0)} + \log|\mu| + \log R_2 - \frac{\mu^2 R_2^2}{2D'(0)}, & \text{otherwise.} \end{cases} \tag{5.15}$$

Note that if $R_2 > \frac{\sqrt{D'(0)}}{|\mu|}$ then the complxity function is 0 as in Theorem 1.1. Since $B_N = \{x \in \mathbb{R}^N : \sqrt{N}R_1 < \|x\| < \sqrt{N}R_2\}$, using Cramer's theorem for the chi-square distribution, we have

$$-\Xi = \begin{cases} -\frac{\mu^2 R_2^2}{2D'(0)} + \frac{1}{2} + \log R_2 + \log|\mu| - \frac{1}{2}\log D'(0), & \text{if } R_2 < \frac{\sqrt{D'(0)}}{|\mu|}, \\ 0, & \text{otherwise.} \end{cases}$$

where Ξ is defined as in (1.4).

Case 2: $\mu = 0$. We have to assume $R_2 < \infty$. Then the above computations show that ψ_* is optimized at $y_* = u_* = 0$ and $\rho_* = R_2$ which gives

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(\mathbb{R}, (R_1, R_2)) = \frac{1}{2}\log[-2D''(0)] - \frac{1}{2}\log D'(0) + \log R_2.$$

In addition, $\Theta = \lim_{N \to \infty} \frac{1}{N} \log |B_N| = \log R_2 + \frac{1}{2}\log(2\pi) + \frac{1}{2}$.

Our results here match all the three cases in Theorem 1.1. Therefore, this example explains the seemingly very different forms of the three phases, whose origin is hard to understand without the general Theorem 1.2. Moreover, this suggests that the critical points x around the norm $\|x\| = \sqrt{N}\rho_*$ and the value $H_N(x) = u_*$ dominate all other places.

Finally, let us consider an example with a special structure function.

Example 3. Suppose $D(r) = \log(1 + \frac{r}{\varepsilon})$ for some $\varepsilon > 0$. This is regarded to be the boundary case of LRC fields in physics [FS07, FB08, FLD18]. Following the previous example, we have

$$\psi_*(\rho, u, y) = \Psi_*(y) - \frac{2}{2 - \varepsilon^2 \beta^2}\left(y + \frac{\mu\varepsilon}{2} + \frac{\beta v \varepsilon}{2}\right)^2 - \frac{v^2}{2} - \frac{\mu^2 \varepsilon \rho^2}{2} + \log \rho,$$

where

$$\beta = \frac{-\rho^2}{\varepsilon\sqrt{(\varepsilon + \rho^2)^2 \log(1 + \frac{\rho^2}{\varepsilon}) - \varepsilon\rho^2}}, \quad v = \frac{(\varepsilon + \rho^2)(u - \frac{\mu\rho^2}{2}) + \mu\varepsilon\rho^2}{\sqrt{(\varepsilon + \rho^2)^2 \log(1 + \frac{\rho^2}{\varepsilon}) - \varepsilon\rho^2}}.$$

As in the physics literature, let us assume $\mu > 0$, $R_1 = 0$ and we do not restrict the field value. Since we know the complexity function becomes trivial if $\mu > \sqrt{-2D''(0)} = \frac{\sqrt{2}}{\varepsilon}$ whenever $R_2 > \frac{1}{\mu\sqrt{\varepsilon}}$. Let us consider $\mu \leq \frac{\sqrt{2}}{\varepsilon}$, $R_2 \leq \frac{1}{\mu\sqrt{\varepsilon}}$ and investigate when the complexity function changes sign. From (5.13) together with the optimum $y_* = -\mu\varepsilon$, $v_* = \frac{\mu\varepsilon^2\beta}{2}$, $\rho_* = R_2$, we find

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[\mathrm{Crt}_N(\mathbb{R}, (0, R_2))] = \frac{1}{2}\log 2 - \frac{1}{2}\log \varepsilon + \frac{\mu^2\varepsilon^2}{4} + \log R_2 - \frac{\varepsilon\mu^2 R_2^2}{2}$$

$$=: \Phi_\varepsilon(R_2).$$

Note that $\Phi_\varepsilon(R_2)$ is strictly increasing for $R_2 \leq \frac{1}{\mu\sqrt{\varepsilon}}$ and $\Phi_\varepsilon(\frac{1}{\mu\sqrt{\varepsilon}}) \geq 0$ with equality exactly when $\varepsilon = \frac{\sqrt{2}}{\mu}$. On the other hand, $\lim_{R_2 \downarrow 0} \Phi_\varepsilon(R_2) = -\infty$ for any fixed $\varepsilon > 0$. Hence, there is a unique R_2^* such that $\Phi_\varepsilon(R_2^*) = 0$.

It is not hard to see that the above argument still holds for any structure function D considered in this paper. In conclusion, we find the phase transition of complexity in terms of confining ball radius.

A. Covariance Function and Its Derivatives

Let $D_N(r) = D(r/N)$. For $x, y \in \mathbb{R}^N$, let $\varphi(x, y) = \frac{1}{2}(D_N(\|x\|^2) + D_N(\|y\|^2) - D_N(\|x - y\|^2))$. Under $X_N(0) = 0$, isotropic increments imply that $\mathbb{E}X_N(x) = 0$; see [Yag87, p.439]. We have

$$\mathrm{Cov}[H_N(x), H_N(y)] = \mathrm{Cov}[X_N(x), X_N(y)] = \mathbb{E}[X_N(x)X_N(y)] = \varphi(x, y).$$

Lemma A.1. *Assume Assumptions I and II. Then for $x \in \mathbb{R}^N$,*

$$\mathbb{E}(\nabla H_N(x)) = \mu x, \quad \mathbb{E}(\nabla^2 H_N(x)) = \mu I_N,$$

$$\mathrm{Cov}[H_N(x), \partial_i H_N(x)] = D'\left(\frac{\|x\|^2}{N}\right) x_i,$$

$$\mathrm{Cov}[\partial_i H_N(x), \partial_j H_N(x)] = D'(0)\delta_{ij},$$

$$\mathrm{Cov}[H_N(x), \partial_{ij} H_N(x)] = 2D''\left(\frac{\|x\|^2}{N}\right)\frac{x_i x_j}{N} + \left[D'\left(\frac{\|x\|^2}{N}\right) - D'(0)\right]\delta_{ij}$$

$$\mathrm{Cov}[\partial_k H_N(x), \partial_{ij} H_N(x)] = 0,$$

$$\mathrm{Cov}[\partial_{lk} H_N(x), \partial_{ij} H_N(x)] = -2D''(0)[\delta_{jl}\delta_{ik} + \delta_{il}\delta_{kj} + \delta_{kl}\delta_{ij}]/N,$$

where δ_{ij} are the Kronecker delta function.

Proof. By [AT07, Theorem 1.4.2], $X_N(x)$ is smooth. We can differentiate inside expectation as in [AT07, (5.5.4)] to find the expectations and

$$
\begin{aligned}
\mathbb{E}[X_N(x)\partial_i X_N(y)]/N &= \partial_{y_i}\mathbb{E}(X_N(x)X_N(y))/N \\
&= D_N'(\|y\|^2)y_i + D_N'(\|x-y\|^2)(x_i - y_i), \\
\mathbb{E}[\partial_i X_N(x)\partial_j X_N(y)]/N &= \partial_{x_i}[D_N'(\|x-y\|^2)(x_j - y_j)] \\
&= 2D_N''(\|x-y\|^2)(x_i - y_i)(x_j - y_j) + D_N'(\|x-y\|^2)\delta_{ij}, \\
\mathbb{E}[X_N(x)\partial_{ij} X_N(y)]/N &= \partial_{y_i}[D_N'(\|y\|^2)y_j + D_N'(\|x-y\|^2)(x_j - y_j)] \\
&= 2D_N''(\|y\|^2)y_i y_j + D_N'(\|y\|^2)\delta_{ij} \\
&\quad - 2D_N''(\|x-y\|^2)(x_i - y_i)(x_j - y_j) - D_N'(\|x-y\|^2)\delta_{ij}, \\
\mathbb{E}[\partial_k X_N(x)\partial_{ij} X_N(y)]/N &= -4D_N'''(\|x-y\|^2)(x_k - y_k)(x_i - y_i)(x_j - y_j) \\
&\quad - 2D_N''(\|x-y\|^2)(x_j - y_j)\delta_{ki} - 2D_N''(\|x-y\|^2)(x_i - y_i)\delta_{kj} - 2D_N''(\|x-y\|^2)(x_k - y_k)\delta_{ij}, \\
\mathbb{E}[\partial_{lk} X_N(x)\partial_{ij} X_N(y)]/N &= -8D_N^{(4)}(\|x-y\|^2)(x_l - y_l)(x_k - y_k)(x_i - y_i)(x_j - y_j) \\
&\quad - 4D_N'''(\|x-y\|^2)[(x_i - y_i)(x_j - y_j)\delta_{kl} + (x_k - y_k)(x_j - y_j)\delta_{il} + (x_k - y_k)(x_i - y_i)\delta_{jl} \\
&\quad + (x_l - y_l)(x_j - y_j)\delta_{ki} + (x_l - y_l)(x_i - y_i)\delta_{kj} + (x_l - y_l)(x_k - y_k)\delta_{ij}] \\
&\quad - 2D_N''(\|x-y\|^2)[\delta_{jl}\delta_{ik} + \delta_{il}\delta_{kj} + \delta_{kl}\delta_{ij}].
\end{aligned}
$$

Substituting $x = y$,

$$
\begin{aligned}
\mathbb{E}[X_N(x)\partial_i X_N(x)]/N &= D_N'(\|x\|^2)x_i, \\
\mathbb{E}[\partial_i X_N(x)\partial_j X_N(x)]/N &= D_N'(0)\delta_{ij}, \\
\mathbb{E}[X_N(x)\partial_{ij} X_N(x)]/N &= 2D_N''(\|x\|^2)x_i x_j + D_N'(\|x\|^2)\delta_{ij} - D_N'(0)\delta_{ij} \\
\mathbb{E}[\partial_k X_N(x)\partial_{ij} X_N(x)]/N &= 0, \\
\mathbb{E}[\partial_{lk} X_N(x)\partial_{ij} X_N(x)]/N &= -2D_N''(0)[\delta_{jl}\delta_{ik} + \delta_{il}\delta_{kj} + \delta_{kl}\delta_{ij}].
\end{aligned}
$$

Then we note that $D_N'(r) = D'(r/N)/N$ and $D_N''(r) = D''(r/N)/N^2$. $\square$

B. Auxiliary Lemmas

For the integral $\mathbb{E}\int_{\mathbb{R}} \exp\left(-\frac{1}{2}(N+1)x^2 - \frac{\sqrt{N(N+1)}\mu x}{\sqrt{-D''(0)}}\right)L_{N+1}(dx)$, we have the following elementary fact which is used in Sect. 2.

Lemma B.1. *Let ν_N be probability measures on $\mathbb{R}$ and $\mu \neq 0$. Suppose*

$$\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) > -\infty.$$

Then we have

$$\lim_{N\to\infty} \frac{1}{N}\left(\log \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{\sqrt{N(N+1)}\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) - \log \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) \right) = 0.$$

Proof. Let

$$a_N = \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x),$$

$$b_N = \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{\sqrt{N(N+1)}\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x),$$

$$c_N = \int_{\mathbb{R}} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x).$$

We claim $\lim_{N\to\infty} \frac{1}{N} \log \frac{a_N}{c_N} = 0$. Indeed, note that

$$a_N = \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)(x+\frac{\mu}{\sqrt{-D''(0)}})^2 + \frac{(N+1)\mu^2}{-2D''(0)}} \nu_{N+1}(\mathrm{d}x) \leq e^{\frac{\mu^2}{-2D''(0)}} c_N.$$

By Jensen's inequality,

$$\frac{\mu^2}{2D''(0)} \leq \log \frac{c_N}{a_N} \leq \log \frac{a_N^{N/(N+1)}}{a_N} = -\frac{1}{N+1}\log a_N.$$

Then the claim follows from the assumption that $\lim_{N\to\infty} \frac{1}{N}\log a_N > -\infty$. From the elementary inequality $a \wedge b \leq (a+b)/2 \leq a \vee b$, we have $\lim_{N\to\infty} \frac{1}{N}(\log(a_N+c_N) - \log a_N) = 0$. It remains to prove that

$$\lim_{N\to\infty} \frac{1}{N}(\log(a_N+c_N) - \log b_N) = 0.$$

Note that

$$b_N \leq \int_{-\infty}^0 e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) + \int_0^\infty e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) \leq a_N + c_N.$$

Let t be a large constant (independent of N) such that

$$\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}} e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) > -\frac{t^2}{8}$$

and that

$$\int_{|x|>t} e^{-\frac{1}{2}(N+1)x^2 - \frac{(N+1)\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) \leq e^{-(N+1)t^2/4}.$$

It follows that

$$\int_{|x|>t} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) \le e^{-Nt^2/4},$$

and since $\frac{1}{N} \log \frac{a_N}{c_N} \to 0$ as $N \to \infty$,

$$\lim_{N \to \infty} \frac{1}{N} \log \int_{-t}^{t} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x)$$

$$= \lim_{N \to \infty} \frac{1}{N} \log \int_{-\infty}^{\infty} (1 - \mathbf{1}\{|x| > t\}) e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x)$$

$$= \lim_{N \to \infty} \frac{1}{N} \log \int_{-\infty}^{\infty} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x).$$

Note that

$$b_N \ge e^{-\frac{t^2}{2}} \int_{-t}^{0} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) + e^{-\frac{t^2}{2} - \frac{\mu t}{\sqrt{-D''(0)}}} \int_{0}^{t} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x)$$

$$\ge e^{-\frac{t^2}{2} - \frac{\mu t}{\sqrt{-D''(0)}}} \int_{-t}^{t} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x).$$

Since

$$\lim_{N \to \infty} \frac{1}{N} \left(\log(a_N + c_N) - \log \int_{-t}^{t} e^{-\frac{1}{2}Nx^2 - \frac{N\mu x}{\sqrt{-D''(0)}}} \nu_{N+1}(\mathrm{d}x) \right) = 0,$$

we have $\lim_{N \to \infty} \frac{1}{N}(\log(a_N + c_N) - \log b_N) = 0$. $\square$

The following discussion is about Assumption IV.

Proof of Lemma 3.2. 1. Since $y \mapsto D'(y)$ is a strictly decreasing convex function and $D'''(y) > 0$ for any $y > 0$, $|D''(y)| < \frac{D'(0) - D'(y)}{y}$. By assumption,

$$(\alpha \rho^2)^2 = \frac{4D''(\rho^2)^2 \rho^4}{D(\rho^2) - \frac{\rho^2 D'(\rho^2)^2}{D'(0)}} \le -\frac{8D''(\rho^2)^2 D''(0)}{3[D'(\rho^2) - D'(0)]^2/\rho^4} < -\frac{8}{3} D''(0).$$

It follows that

$$(\alpha \rho^2 + \beta)\beta < \sqrt{-\frac{2}{3}D''(0)}\sqrt{-\frac{8}{3}D''(0)} - \frac{2}{3}D''(0) = -2D''(0),$$

$$(\alpha \rho^2 + \beta)\alpha \rho^2 < -\frac{8}{3}D''(0) + \sqrt{-\frac{2}{3}D''(0)}\sqrt{-\frac{8}{3}D''(0)} = -4D''(0).$$

2. We verify (3.10). If (3.11) holds, then $y \mapsto \beta(y)^2$ is a decreasing function and (3.10) follows from Lemma 3.1.

3. By item 1, it suffices to check (3.10). Consider the function

$$f(y) = -D''(0)[D'(0)D(y) - D'(y)^2 y] - \frac{3}{2}D'(0)[D'(y) - D'(0)]^2.$$

Condition (3.10) is equivalent to $f(y) \geq 0$. Note that $f(0) = 0$ and that

$$f'(y) = [D'(0) - D'(y)][D'(0)D''(y) - D''(0)D'(y)] + 2D''(y)(D''(0)D'(y)y$$
$$- D'(0)[D'(y) - D'(0)]).$$

By convexity, $\frac{D'(y) - D'(0)}{y} \leq D''(y) \leq 0$. If (3.12) holds, $D''(0)D'(y)y - D'(0)[D'(y) - D'(0)] \leq 0$ and

$$\frac{D'(y)}{D'(0)} - \frac{D''(y)}{D''(0)} \geq 0.$$

Then (3.10) follows from here since $D'(0) \geq D'(y)$ and we have $f'(y) \geq 0$.

4. By Cauchy's mean value theorem, condition (3.12) is equivalent to (3.13).

5. Direct calculation yields

$$\frac{d}{dy} \frac{D'(y)}{-D''(y)} = \frac{-D''(y)^2 + D'''(y)D'(y)}{D''(y)^2}.$$

Then (3.14) implies (3.13).

6. By the representation (3.7) of Thorin–Bernstein functions, we have

$$D''(x) = -\int_{(0,\infty)} \frac{1}{(x+t)^2} \sigma(dt), \qquad D'''(x) = \int_{(0,\infty)} \frac{2}{(x+t)^3} \sigma(dt).$$

By the Cauchy–Schwarz inequality, we have

$$2D''(x)^2 \leq D'(x)D'''(x).$$

It follows that $\frac{d}{dy} \frac{D'(y)}{-D''(y)} \geq 1$ and (3.14) holds. $\square$

If $A = 0$ in the representation (1.2), using the Cauchy–Schwarz inequality, we can see

$$\frac{d}{dy} \frac{D'(y)}{-D''(y)} = \frac{-D''(y)^2 + D'''(y)D'(y)}{D''(y)^2} \geq 0,$$

compared with (3.14). It is easy to check that for any $\varepsilon > 0, 0 < \gamma < 1$, our major examples $D(r) = \log(1 + r/\varepsilon)$ and $D(r) = (r + \varepsilon)^\gamma - \varepsilon^\gamma$ satisfy (3.13). With more work, one can check that these functions satisfy (3.11).

On the other hand, according to [SSV12, p. 332],

$$D(x) = \frac{\sqrt{x} \sinh^2(\sqrt{x})}{\sinh(2\sqrt{x})}$$

is a complete Bernstein function which is not Thorin–Bernstein. One can check (at least numerically) that it violates (3.13) but still verifies (3.10). We suspect that (3.8) and (3.9) always hold for any structure function D. The following shows that this is the case at least in a neighborhood of 0.

Lemma B.2. *Assume $A = 0$ in* (1.2). *We have*

$$\lim_{y \to 0+} \frac{d}{dy}[\alpha(y)y + \beta(y)]\beta(y) < 0,$$

$$\lim_{y \to 0+} \frac{d}{dy}[\alpha(y)y + \beta(y)]\alpha(y)y < 0.$$

Consequently, there exists $\delta > 0$ such that $-2D''(0) > [\alpha(y)y + \beta(y)]\beta(y)$ and $-4D''(0) > [\alpha(y)y + \beta(y)]\alpha(y)y$ for $y \in (0, \delta)$.

Proof. We only prove the first inequality as the second is similar. Write

$$(\alpha y + \beta)\beta = \frac{[2D''(y) + \frac{D'(y)-D'(0)}{y}]\frac{D'(y)-D'(0)}{y}}{\frac{D(y)}{y^2} - \frac{D'(y)^2}{D'(0)y}} =: \frac{T}{B}.$$

Since $[(\alpha y + \beta)\beta]' = \frac{T'B-B'T}{B^2}$ and $\lim_{y\to 0+} B = -\frac{3}{2}D''(0) \neq 0$, it suffices to show that $\lim_{y\to 0+} T'B - B'T < 0$. By calculation, we have $\lim_{y\to 0+} T = 3D''(0)^2$ and

$$T' = [2D'''(y) + \frac{D''(y)y - D'(y) + D'(0)}{y^2}]\frac{D'(y) - D'(0)}{y}$$
$$+ [2D''(y) + \frac{D'(y) - D'(0)}{y}]\frac{D''(y)y - D'(y) + D'(0)}{y^2},$$
$$B' = \frac{D'(0)D'(y)y - 2D'(0)D(y) - 2D'(y)D''(y)y^2 + yD'(y)^2}{D'(0)y^3}.$$

After some tedious computation, we find $\lim_{y\to 0+} T' = 4D'''(0)D''(0)$ and $\lim_{y\to 0+} B' = -\frac{5}{6}D'''(0) - \frac{D''(0)^2}{D'(0)}$. Then

$$\lim_{y\to 0+} T'B - B'T = D''(0)^2\Big[\frac{3D''(0)^2}{D'(0)} - \frac{7}{2}D'''(0)\Big].$$

By the Cauchy–Schwarz inequality,

$$D''(0)^2 = \Big(\int_0^\infty t^4\nu(dt)\Big)^2 \leq \int_0^\infty t^2\nu(dt)\int_0^\infty t^6\nu(dt) = D'(0)D'''(0).$$

From here the conclusion follows. □

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