Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

RESEARCH ARTICLE

# Fully discrete error analysis of first-order low regularity integrators for the Allen-Cahn equation

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**Funding information**

T.-T.-P. Hoang's work is partially supported by U.S. National Science Foundation under grant number DMS-2041884. L. Ju's work is partially supported by U.S. National Science Foundation under grant number DMS-2109633.

**Abstract**

The Allen-Cahn equation satisfies the maximum bound principle, i.e., its solution is uniformly bounded for all time by a positive constant under appropriate initial and/or boundary conditions. It has been shown recently that the time-discrete solutions produced by low regularity integrators (LRIs) are likewise bounded in the infinity norm; however, the corresponding fully discrete error analysis is still lacking. This work is concerned with convergence analysis of the fully discrete numerical solutions to the Allen-Cahn equation obtained based on two first-order LRIs in time and the central finite difference method in space. By utilizing some fundamental properties of the fully discrete system and the Duhamel's principle, we prove optimal error estimates of the numerical solutions in time and space while the exact solution is only assumed to be continuous in time. Numerical results are presented to confirm such error estimates and show that the solution obtained by the proposed LRI schemes is more accurate than the classical exponential time differencing (ETD) scheme of the same order.

**KEYWORDS:**

Allen-Cahn equation, low regularity integrators, fully discrete error estimates

## 1 | INTRODUCTION

For an open and bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$) with Lipschitz boundary $\partial\Omega$ and a terminal time $T > 0$, we consider the following semilinear parabolic equation of Allen-Cahn type:

$$u_t = \varepsilon^2 \Delta u + f(u), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1}$$

where the parameter $\varepsilon > 0$ characterizes the width of diffuse interface, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function, and $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is the unknown function subject to the initial condition $u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x})$ for $\boldsymbol{x} \in \overline{\Omega}$ and the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n}(\boldsymbol{x}, t) = 0$ for $\boldsymbol{x} \in \partial\Omega$ and $0 \leq t \leq T$. Note that periodic and homogeneous Dirichlet boundary conditions also can be imposed instead as discussed later. Similar to[5,6], $f$ is first assumed to satisfy two conditions as follows:

(A1). $f(\beta) \leq 0 \leq f(-\beta)$ for some contant $\beta > 0$.

(A2). $f$ is continuously differentiable and nonconstant on $[-\beta, \beta]$.

Under these two assumptions, the exact solution of (1) preserves the maximum bound principle (MBP)[6] in the sense that if $\|u_0\|_{L^\infty} \leq \beta$ then $|u(\boldsymbol{x}, t)| \leq \beta$ for all $(\boldsymbol{x}, t) \in \overline{\Omega} \times (0, T]$. For the need of error analysis presented in this work, we also impose an additional assumption on $f$:

(A3). $f'$ is Lipschitz continuous on $[-\beta, \beta]$.

Examples of such a nonlinear function $f$ include the double-well potential case, $f(u) = u - u^3$, and the Flory-Huggins potential case, $f(u) = \frac{\theta}{2} \ln \frac{1-u}{1+u} + \theta_c u$ for $0 < \theta < \theta_c$, which are widely used in phase-field modeling to describe the motion of antiphase boundaries in crystalline solids.

To obtain accurate and stable approximate solutions to the model equation (1), various numerical schemes have been proposed, including the stablized semi-implicit method [31,9], fully implicit scheme [26], implicit-explicit scheme [27,8], semi-analytical Fourier spectral method [15], operator splitting scheme [16,28,4], exponential time differencing (ETD) schemes [3,13,33,11] and integrating factor Runge-Kutta (IFRK) methods [32,18,12,22]. Among these methods, exponential integrators [10,14] such as ETD and IFRK schemes have been shown to be more effective when solving problems with strong stiff terms. It should be noted that error analysis of these methods often requires sufficient regularity of the exact solution; for example, convergence of the first- and second-order ETD schemes is theoretically proved only when the exact solution belongs to $C^1$ and $C^2$ in time, respectively.

A new class of time discretization techniques, namely low regularity integrators (LRIs) [24], have recently been introduced for time integration of evolution problems, which requires weaker regularity assumptions for error analysis than existing methods. The so-called resonance based scheme was developed to control nonlinear terms at low regularity in dispersive equations such as the Korteweg-de Vries (KdV) [25,29,30], Schrödinger [20,23] and Klein–Gordon [1,2] equations. Since this approach requires periodic boundary conditions to represent the exact solution as Fourier series expansion, a general framework of LRIs [24] was proposed for evolution equations where various boundary conditions can be considered. LRIs are obtained by iteratively applying Duhamel's principle and also belong to the family of exponential integrators. The method was studied in [5] to solve the Allen-Cahn type equations in combination with the central finite difference approximation in space and the proposed first- and second-order LRI schemes are shown to preserve MBP and be energy stable. With a fixed spatial mesh, optimal temporal error estimates were derived for all the proposed schemes, given that the solution of the space-discrete problem is only assumed to be continuous in time. Note that the constants in all the obtained estimates are dependent on the spatial mesh size.

In this paper, we aim to remove such dependence on the spatial mesh size in the error estimates and carry out the fully discrete error analysis of the two first-order LRI schemes proposed in [5] when *the time step size and the spatial mesh size simultaneously go to zero*. In particular, we prove that, under assumptions (A1), (A2) and (A3), the fully discrete solution converges to the exact solution with first order in time and second order in space. To the best of our knowledge, this is the first time that a rigorous proof of the fully discrete convergence analysis of LRI schemes with optimal error estimates is obtained in both time and space. Numerical experiments are carried out to confirm these optimal error estimates and show that the LRI schemes are more accurate and efficient than the ETD ones of same order, especially when the interfacial parameter $\varepsilon$ is very small. We remark that in [24], convergence analysis of the first- and second-order LRI schemes for a class of evolution equations was only performed in the semi-discrete sense, i.e., without considering spatial discretization. Such a framework may not be generalizable to the fully discrete problems (after spatial discretization) as the analysis relies heavily on several assumptions of the continuous linear operator (cf. Assumptions 2.1 and 2.3 in [24]). These assumptions in general are not uniformly satisfied by the corresponding space-discrete operators. In [17], a semi-implicit LRI scheme was introduced for the Navier-Stokes equation with finite element approximations in space; first-order convergence in time of the fully discrete numerical solution was proved under the constraint that the spatial mesh size has the same order of magnitude as the time step size.

The rest of this paper is organized as follows. In Section 2, we briefly recall the fully discrete numerical schemes for the model equation (1) obtained with two first-order LRIs in time as in [5]. The error analysis of the corresponding numerical solutions, which is the main result of this work, is then carried out in Section 3. Numerical results are presented in Section 4, where the performance of the proposed LRI schemes is investigated and compared with that of ETD with simultaneously varying time step size and the spatial mesh size. Finally, some concluding remarks are given in Section 5.

## 2 | FULLY DISCRETE FIRST-ORDER LOW REGULARITY INTEGRATORS

Throughout the paper, we assume that the nonlinear function $f$ satisfies assumptions (A1) and (A2) and the initial data is bounded by $\beta$ in the absolute value, i.e., $\|u_0\|_{L^\infty} \leq \beta$. Suppose that the domain $\Omega$ is an interval, a rectangle, or a rectangular parallelepiped corresponding to $d = 1, 2$ and 3, respectively. For spatial discretization, we apply a uniform partition of $\Omega$ into square elements of length $h$, and let $N$ be the total number of grid nodes. Denote by $I^h$ the operator restricting a function over the grid points. Using the central finite difference method and the homogeneous Neumann boundary condition, we obtain the corresponding semi-discrete in space problem:

$$\frac{d\boldsymbol{u}_h}{dt} = \boldsymbol{A}_h \boldsymbol{u}_h + \boldsymbol{F}(\boldsymbol{u}_h), \quad \boldsymbol{u}_h(0) = I^h u_0 := (u_0(x_1), u_0(x_2), \dots, u_0(x_N))^T, \tag{2}$$

where $\boldsymbol{u}_h(t) = (u_1(t), u_2(t), \dots, u_N(t))^T$ is the approximation of $u(\cdot, t)$ over the grid points, $\boldsymbol{F}(\boldsymbol{u}_h) = (f(u_1), f(u_2), \dots, f(u_N))^T$, $\boldsymbol{A}_h = \varepsilon^2 \boldsymbol{D}_h$ with $\boldsymbol{D}_h = \lambda_h$ in one dimension, $\boldsymbol{D}_h = \boldsymbol{I} \otimes \lambda_h + \lambda_h \otimes \boldsymbol{I}$ in two dimensions, and

$$\boldsymbol{D}_h = \boldsymbol{I} \otimes \boldsymbol{I} \otimes \lambda_h + \boldsymbol{I} \otimes \lambda_h \otimes \boldsymbol{I} + \lambda_h \otimes \boldsymbol{I} \otimes \boldsymbol{I}$$

in three dimensions. Here $\otimes$ denotes the Kronecker product, $\boldsymbol{I}$ is the identity matrix of size $N$, and $\lambda_h$ is given by

$$\lambda_h = \frac{1}{h^2} \begin{bmatrix} -2 & 2 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 2 & -2 \end{bmatrix}_{N \times N}.$$

For temporal discretization, let us consider the uniform partition of the time interval $[0, T] : 0 = t_0 < t_1 < \cdots < t_M = T$, with the time step size $\Delta t = T/M$. Denote by $\boldsymbol{u}_h^m$ the approximation of $\boldsymbol{u}_h(t_m)$ with $\boldsymbol{u}_h^0 = \boldsymbol{u}_h(0)$. Applying the Duhamel's principle to (2) gives us

$$\boldsymbol{u}_h(t_m + s) = e^{s\boldsymbol{A}_h} \boldsymbol{u}_h(t_m) + \int_0^s e^{(s-\sigma)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m + \sigma)) d\sigma. \tag{3}$$

Setting $s = \Delta t$, we deduce that

$$\boldsymbol{u}_h(t_{m+1}) = e^{\Delta t \boldsymbol{A}_h} \boldsymbol{u}_h(t_m) + \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m + s)) ds. \tag{4}$$

By approximating $\boldsymbol{u}_h(t_m + s)$ in (4) with $e^{s\boldsymbol{A}_h} \boldsymbol{u}_h(t_m)$ and using one-point quadrature rules to compute the resulting integral, we obtain the following two first-order LRI schemes (the reader is referred to[5] for the details):

i) LRI1a scheme: $\quad \boldsymbol{u}_h^{m+1} = e^{\Delta t \boldsymbol{A}_h} (\boldsymbol{u}_h^m + \Delta t \boldsymbol{F}(\boldsymbol{u}_h^m)),$ (5)

ii) LRI1b scheme: $\quad \boldsymbol{u}_h^{m+1} = e^{\Delta t \boldsymbol{A}_h} \boldsymbol{u}_h^m + \Delta t \boldsymbol{F}(e^{\Delta t \boldsymbol{A}_h} \boldsymbol{u}_h^m).$ (6)

## 3 | OPTIMAL ERROR ESTIMATES

We assume that *the exact solution u of the model semilinear parabolic equation* (1) *belongs to* $C([0, T], C^4(\overline{\Omega}))$, and write $u(t)$ in place of $u(\cdot, t)$ for the sake of simplicity. Denote by $\boldsymbol{u}_h$ the solution of the space-discrete problem (2) and by $z_1 \lesssim z_2$ the relation $z_1 \le C z_2$ for some generic constant $C > 0$ independent of $h$ and $\Delta t$.

### 3.1 | Preliminaries

Let us first introduce some important lemmas. A key property of $\boldsymbol{A}_h$ is that its elements $a_{ij}$ satisfy

$$a_{ii} < 0, \quad a_{ii} + \sum_{j=1, j \neq i}^{N} |a_{ij}| \le 0, \quad i = 1, 2, \dots, N.$$

In other words, $\boldsymbol{A}_h$ is a diagonally dominant matrix with all diagonal entries negative. As a consequence, we obtain the following result from the conclusion of Lemma 3.3 in[12].

**Lemma 1.** For any $s > 0$, we have $\|e^{s\boldsymbol{A}_h}\|_\infty \le 1$.

Using this lemma, the semi-discrete solution of (2) is shown to preserve MBP as stated below (see[19] Theorem 2 for the detailed proof).

**Lemma 2.** There exists a unique solution $\boldsymbol{u}_h \in C([0,T],\mathbb{R}^N)$ to the semi-discrete problem (2) and

$$\|\boldsymbol{u}_h(t)\|_\infty \leq \beta, \quad \text{for all } t \in [0,T].$$

Moreover, we have the following result on the fully discrete solutions produced by the LRI1 schemes (cf.[5] Theorem 2).

**Lemma 3** (Discrete MBP)**.** The fully discrete solutions produced by the LRI1a scheme (5) or the LRI1b scheme (6) preserve the discrete MBP conditionally:

$$\|\boldsymbol{u}_h^m\|_\infty \leq \beta, \quad m = 0,1,\dots,M, \tag{7}$$

provided that $0 < \Delta t \leq \omega_0$, where $\omega_0 := -\left(\min_{|x|\leq\beta} f'(x)\right)^{-1} > 0$.

Next, let us state two versions of Gronwall's inequality for use in our error analysis later.

**Lemma 4.** (i) If a real-value function $\varphi$ is continuous on $[0,T]$ and there exist two constants $C_1, C_2 > 0$ such that

$$\varphi(t) \leq C_1 + C_2 \int_0^t \varphi(s)ds, \quad \text{for all } t \in [0,T],$$

then

$$\varphi(t) \leq C_1 e^{C_2 t}$$

for all $t \in [0,T]$.

(ii) If there exist two constants $C_1, C_2 > 0$ and a nonnegative sequence $\{z_n\}_{n=0}^\infty$ satisfying

$$z_{n+1} \leq (1+C_1)z_n + C_2, \quad \text{for all } n \geq 0,$$

then

$$z_n \leq (1+C_1)^n \left(z_0 + \frac{C_2}{C_1}\right) - \frac{C_2}{C_1}$$

for all $n \geq 1$.

It is well known that the central finite difference method is a second-order approximation[7]. Note that periodic boundary condition is considered in[7], but the consistency result still holds true for the homogeneous Neumann boundary condition.

**Lemma 5.** $\|\boldsymbol{D}_h \boldsymbol{I}^h u(t) - \boldsymbol{I}^h \Delta u(t)\|_\infty \lesssim h^2$ for all $t \in [0,T]$.

This consistency estimate leads to the convergence of the semi-discrete solution $\boldsymbol{u}_h$ of (2) to the exact solution $\boldsymbol{I}^h u$ of (1) as demonstrated in the lemma below.

**Lemma 6.** $\|\boldsymbol{u}_h(t) - \boldsymbol{I}^h u(t)\|_\infty \lesssim h^2$ for all $t \in [0,T]$.

*Proof.* Let $\boldsymbol{r}(t) = \boldsymbol{u}_h(t) - \boldsymbol{I}^h u(t)$, we observe that $\boldsymbol{r}(0) = 0$. From (1) and (2), we deduce that

$$\frac{d\boldsymbol{r}(t)}{dt} = \boldsymbol{A}_h \boldsymbol{r}(t) + \boldsymbol{\delta}(t), \quad \forall\, t \in [0,T], \tag{8}$$

where $\boldsymbol{\delta}(t) = \varepsilon^2(\boldsymbol{D}_h \boldsymbol{I}^h u(t) - \boldsymbol{I}^h \Delta u(t)) + \boldsymbol{F}(\boldsymbol{u}_h(t)) - \boldsymbol{F}(\boldsymbol{I}^h u(t))$. Using Lemmas 2, 5 and the Lipschitz continuity of $f$, we obtain

$$\begin{aligned} \|\boldsymbol{\delta}(t)\|_\infty &\leq \varepsilon^2 \|\boldsymbol{D}_h \boldsymbol{I}^h u(t) - \boldsymbol{I}^h \Delta u(t)\|_\infty + \|\boldsymbol{F}(\boldsymbol{u}_h(t)) - \boldsymbol{F}(\boldsymbol{I}^h u(t))\|_\infty \\ &\lesssim h^2 + \|\boldsymbol{r}(t)\|_\infty, \quad \forall\, t \in [0,T]. \end{aligned} \tag{9}$$

Applying the Duhamel's principle for (8), we get

$$\boldsymbol{r}(t) = e^{t\boldsymbol{A}_h}\boldsymbol{r}(0) + \int_0^t e^{(t-s)\boldsymbol{A}_h}\boldsymbol{\delta}(s)ds = \int_0^t e^{(t-s)\boldsymbol{A}_h}\boldsymbol{\delta}(s)ds, \quad \forall\, t \in [0,T].$$

Taking the infinity norm of both sides, along with using Lemma 1 and (9), we arrive at

$$\begin{aligned}\|\boldsymbol{r}(t)\|_\infty &\le \int_0^t \|e^{(t-s)\boldsymbol{A}_h}\|_\infty \|\boldsymbol{\delta}(s)\|_\infty ds \\ &\lesssim \int_0^t (h^2 + \|\boldsymbol{r}(s)\|_\infty) ds \lesssim h^2 + \int_0^t \|\boldsymbol{r}(s)\|_\infty ds, \quad \forall t \in [0,T]. \end{aligned} \tag{10}$$

Noting that $\|\boldsymbol{r}(t)\|_\infty$ is continuous, then the proof is completed by using Lemma 4-(i) and (10). □

Based on Lemma 6 and the triangle inequality (cf. Equation (12)), there remains to bound $\|\boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m)\|_\infty$ to obtain fully discrete error estimates for the LRI1a and LRI1b schemes. The most challenging part is to cancel out the factor $h^{-2}$ under the weak regularity in time of $u$ and $\boldsymbol{u}_h$ when taking the infinity norm of the matrix $\boldsymbol{A}_h$ ($\|\boldsymbol{A}_h\|_\infty \lesssim h^{-2}$, cf. Theorems 5 and 6 in[5]). This will be partly accomplished by the following lemma, which is one of the most crucial results in this paper.

**Lemma 7.** The following statements are true for all $s, t \in [0,T]$:

(i) $\|\boldsymbol{A}_h \boldsymbol{u}_h(t)\|_\infty \lesssim 1$;

(ii) $\|\boldsymbol{u}_h(t) - \boldsymbol{u}_h(s)\|_\infty \lesssim |t-s|$;

(iii) If $f$ additionally satisfies Assumption (A3), then $\|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{u}_h(t))\|_\infty \lesssim 1$.

*Proof.* (i) Note that $\|\boldsymbol{A}_h \boldsymbol{I}^h u(t)\|_\infty \lesssim 1$ for all $t \in [0,T]$ because of Lemma 5 and the boundedness of $\Delta u$. By applying Lemma 6, we have

$$\begin{aligned}\|\boldsymbol{A}_h \boldsymbol{u}_h(t)\|_\infty &\le \|\boldsymbol{A}_h \boldsymbol{I}^h u(t)\|_\infty + \|\boldsymbol{A}_h(\boldsymbol{u}_h(t) - \boldsymbol{I}^h u(t))\|_\infty \\ &\lesssim 1 + \|\boldsymbol{A}_h\|_\infty \|\boldsymbol{u}_h(t) - \boldsymbol{I}^h u(t))\|_\infty \\ &\lesssim 1 + h^{-2}h^2 \lesssim 1, \quad \forall t \in [0,T].\end{aligned}$$

(ii) Suppose $s < t$ and $s, t \in [0,T]$. Using the Duhamel's principle for (2), we obtain

$$\boldsymbol{u}_h(t) = e^{(t-s)\boldsymbol{A}_h}\boldsymbol{u}_h(s) + \int_0^{t-s} e^{(t-s-\sigma)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(s+\sigma))\, d\sigma.$$

This gives us

$$\boldsymbol{u}_h(t) - \boldsymbol{u}_h(s) = [e^{(t-s)\boldsymbol{A}_h} - \boldsymbol{I}]\boldsymbol{u}_h(s) + \int_0^{t-s} e^{(t-s-\sigma)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(s+\sigma))\, d\sigma =: K_1 + K_2,$$

where $K_1$ and $K_2$ are given by

$$\begin{aligned} K_1 &= [e^{(t-s)\boldsymbol{A}_h} - \boldsymbol{I}]\boldsymbol{u}_h(s), \\ K_2 &= \int_0^{t-s} e^{(t-s-\sigma)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(s+\sigma)).\end{aligned}$$

By Lemmas 1 and 2 (the boundedness of $\boldsymbol{u}_h$), we have

$$\|K_2\|_\infty \le \int_0^{t-s} \|e^{(t-s-\sigma)\boldsymbol{A}_h}\|_\infty \|\boldsymbol{F}(\boldsymbol{u}_h(s+\sigma))\|_\infty\, d\sigma \lesssim t-s.$$

To bound $\|K_1\|_\infty$, using the fundamental theorem of calculus and the result of Part (i) yields

$$\|K_1\|_\infty = \left\| \int_0^{t-s} e^{\sigma \boldsymbol{A}_h} \boldsymbol{A}_h \boldsymbol{u}_h(s) d\sigma \right\|_\infty \le \int_0^{t-s} \|e^{\sigma \boldsymbol{A}_h}\|_\infty \|\boldsymbol{A}_h \boldsymbol{u}_h(s)\|_\infty\, d\sigma \lesssim t-s.$$

Thus, we conclude that

$$\|\boldsymbol{u}_h(t) - \boldsymbol{u}_h(s)\|_\infty \le \|K_1\|_\infty + \|K_2\|_\infty \lesssim t-s.$$

(iii) We first prove that $\|\boldsymbol{A}_h \boldsymbol{I}^h[f(u(t))]\|_\infty \lesssim 1$ for all $t \in [0,T]$. It suffices to consider the one dimensional case (the proofs in two and three dimensions can be done in a similar manner). Denote by $v(x,t) = f(u(x,t))$ then $v_x$ is Lipschitz continuous on $\Omega$ with respect to spacial variable because $u_x$ is Lipschitz continuous on the same domain and $f'$ is Lipschitz continuous on $[-\beta,\beta]$ (cf. Assumption (A3)). Moreover, based on the MBP and the regularity of $u$, one can show that this Lipschitz constant is independent of $t$. Since $u$ satisfies the homogeneous Neumann boundary condition, so is $v$, namely, $v_x(x_1,t) = v_x(x_N,t) = 0$. By the definition of $\boldsymbol{A}_h$ and $\boldsymbol{I}^h$, we see that

$$\boldsymbol{A}_h \boldsymbol{I}^h[f(u(t))] = \boldsymbol{A}_h \boldsymbol{I}^h v(t) = \frac{\varepsilon^2}{h^2}\begin{bmatrix} -2 & 2 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} v(x_1,t) \\ v(x_2,t) \\ \vdots \\ v(x_{N-1},t) \\ v(x_N,t) \end{bmatrix} =: \varepsilon^2 \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_{N-1}(t) \\ \alpha_N(t) \end{bmatrix}. \tag{11}$$

For any $j = 2,3,\dots,N-1$ and $t \in [0,T]$, we have

$$\begin{aligned}\alpha_j(t) &= \frac{v(x_{j+1},t) - 2v(x_j,t) + v(x_{j-1},t)}{h^2} \\ &= \frac{[v(x_{j+1},t) - v(x_j,t)] - [v(x_j,t) - v(x_{j-1},t)]}{h^2}.\end{aligned}$$

By the mean value theorem, there exist $\gamma_1 \in (x_j, x_{j+1})$ and $\gamma_2 \in (x_{j-1}, x_j)$ such that

$$v(x_{j+1},t) - v(x_j,t) = h v_x(\gamma_1,t), \quad \text{and} \quad v(x_j,t) - v(x_{j-1},t) = h v_x(\gamma_2,t).$$

This implies that

$$|\alpha_j(t)| = \frac{|v_x(\gamma_1,t) - v_x(\gamma_2,t)|}{h} \lesssim \frac{|\gamma_1 - \gamma_2|}{h} \lesssim 1, \quad j = 2,3,\dots,N-1.$$

where we have used the Lipschitz continuity of $v_x$ in the second last estimate. Now, when $j=1$ we have

$$\alpha_1(t) = \frac{2v(x_2,t) - 2v(x_1,t)}{h^2}.$$

By the mean value theorem, there exists $\gamma_3 \in (x_1,x_2)$ such that

$$|v(x_2,t) - v(x_1,t)| = h|v_x(\gamma_3,t)| = h|v_x(\gamma_3,t) - v_x(x_1,t)| \lesssim h|\gamma_3 - x_1| \le h^2,$$

This shows that $|\alpha_1(t)| \lesssim 1$. By the same argument, we also get $|\alpha_N(t)| \lesssim 1$. The above estimates give us $\|\boldsymbol{A}_h \boldsymbol{I}^h[f(u(t))]\|_\infty \lesssim 1$ for all $t \in [0,T]$ and therefore

$$\|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{I}^h u(t))\|_\infty = \|\boldsymbol{A}_h \boldsymbol{I}^h[f(u(t)]\|_\infty \lesssim 1, \quad t \in [0,T].$$

Finally,

$$\begin{aligned}\|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{u}_h(t))\|_\infty &\le \|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{I}^h u(t))\|_\infty + \|\boldsymbol{A}_h[\boldsymbol{F}(\boldsymbol{u}_h(t)) - \boldsymbol{F}(\boldsymbol{I}^h u(t))]\|_\infty \\ &\lesssim 1 + \|\boldsymbol{A}_h\|_\infty \|\boldsymbol{F}(\boldsymbol{u}_h(t)) - \boldsymbol{F}(\boldsymbol{I}^h u(t))\|_\infty \\ &\lesssim 1 + \|\boldsymbol{A}_h\|_\infty \|\boldsymbol{u}_h(t) - \boldsymbol{I}^h u(t)\|_\infty \\ &\lesssim 1 + h^{-2}h^2 \lesssim 1, \quad t \in [0,T],\end{aligned}$$

where we have used Lemmas 2, 6 and the Lipschitz continuity of $f$. □

*Remark 1.* The analysis presented above can be extended to the periodic and the homogeneous Dirichlet boundary conditions. For the periodic boundary condition, Lemma 7 remains valid. For the homogeneous Dirichlet boundary condition, we need to assume an extra condition on the nonlinear function $f$, namely $f(0) = 0$, to guarantee that $\boldsymbol{A}_h$ acting on $f(u)$ in (11) is bounded in the infinity norm, so that the result (iii) in Lemma 7 still holds true. Indeed, in this case, $\boldsymbol{A}_h$ is given by

$$\boldsymbol{A}_h = \frac{\varepsilon^2}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}_{N\times N}.$$

Denote by $x_0$ and $x_{N+1}$ two endpoints of $\overline{\Omega}$ (instead of $x_1$ and $x_N$ as in the homogeneous Neumann boundary condition case). By assuming $f(0) = 0$, we have $v(x_0, t) = v(x_{N+1}, t) = 0$. Therefore,

$$\alpha_1(t) = \frac{v(x_2,t) - 2v(x_1,t)}{h^2} = \frac{v(x_2,t) - 2v(x_1,t) + v(x_0,t)}{h^2},$$

and

$$\alpha_N(t) = \frac{v(x_{N-1},t) - 2v(x_N,t)}{h^2} = \frac{v(x_{N-1},t) - 2v(x_N,t) + v(x_{N+1},t)}{h^2}.$$

Thus $|\alpha_1(t)| \lesssim 1$ and $|\alpha_N(t)| \lesssim 1$ by the same arguments as above. This leads to $\|\boldsymbol{A}_h \boldsymbol{I}^h[f(u(t))]\|_\infty \lesssim 1$ for all $t \in [0, T]$ and hence (iii) in Lemma 7 holds true. It should be noted that both the double-well and Flory-Huggins potential functions satisfy the extra condition $f(0) = 0$.

## 3.2 | Main results

From now on, $f$ is assumed to satisfy additionally Assumption (A3) so that the result (iii) in Lemma 7 holds. For any vector $\boldsymbol{v} = (v_1, v_2, \dots, v_N)^T \in \mathbb{R}^N$, we denote by

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}(\boldsymbol{v}) = \mathrm{diag}(f'(v_1), f'(v_2), \dots, f'(v_N)).$$

Then we have the following optimal error estimates for the fully discrete LRI1a scheme (5).

**Theorem 1** (Error estimate of LRI1a)**.** Let $\{\boldsymbol{u}_h^m\}$ be generated by the LRI1a scheme (5) with $\boldsymbol{u}_h^0 = \boldsymbol{u}_h(0)$, then

$$\|\boldsymbol{u}_h^m - \boldsymbol{I}^h u(t_m)\|_\infty \lesssim h^2 + \Delta t, \quad \forall\, m = 0, 1, \dots, M,$$

provided that $0 < \Delta t \le \omega_0$.

*Proof.* By the triangle inequality, we can see that

$$\|\boldsymbol{u}_h^m - \boldsymbol{I}^h u(t_m)\|_\infty \le \|\boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m)\|_\infty + \|\boldsymbol{u}_h(t_m) - \boldsymbol{I}^h u(t_m)\|_\infty. \tag{12}$$

The second term $\|\boldsymbol{u}_h(t_m) - \boldsymbol{I}^h u(t_m)\|_\infty \lesssim h^2$ due to Lemma 6, thus it is sufficient to show that

$$\|\boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m)\|_\infty \lesssim \Delta t. \tag{13}$$

Denote by $\boldsymbol{e}_m = \boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m)$, we first observe that $\boldsymbol{e}_0 = 0$. Subtracting (4) from (5), we have

$$\boldsymbol{e}_{m+1} = e^{\Delta t \boldsymbol{A}_h} \boldsymbol{e}_m + \Delta t e^{\Delta t \boldsymbol{A}_h} (\boldsymbol{F}(\boldsymbol{u}_h^m) - \boldsymbol{F}(\boldsymbol{u}_h(t_m))) + \boldsymbol{R}_m^{(1)},$$

where

$$\boldsymbol{R}_m^{(1)} = \Delta t e^{\Delta t \boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m + s)) ds.$$

Using Lemmas 1, 2 and 3 (the boundedness of $\boldsymbol{u}_h$ and $\boldsymbol{u}_h^m$) and the Lipschitz continuity of $f$, we obtain

$$\begin{aligned} \|\boldsymbol{e}_{m+1}\|_\infty &\le \|\boldsymbol{e}_m\|_\infty + \Delta t \|\boldsymbol{F}(\boldsymbol{u}_h^m) - \boldsymbol{F}(\boldsymbol{u}_h(t_m))\|_\infty + \|\boldsymbol{R}_m^{(1)}\|_\infty \\ &\le (1 + C_1 \Delta t)\|\boldsymbol{e}_m\|_\infty + \|\boldsymbol{R}_m^{(1)}\|_\infty, \end{aligned} \tag{14}$$

where $C_1 = \max_{|x| \le \beta} |f'(x)|$. Rewrite $\boldsymbol{R}_m^{(1)}$ as follows

$$\begin{aligned} \boldsymbol{R}_m^{(1)} &= \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h} [e^{s\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_m + s))] ds \\ &= \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h} [e^{s\boldsymbol{A}_h} \boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(e^{s\boldsymbol{A}_h} \boldsymbol{u}_h(t_m))] ds + \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h} [\boldsymbol{F}(e^{s\boldsymbol{A}_h} \boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_m + s))] ds \\ &=: I_1 + I_2, \end{aligned}$$

where $I_1$ and $I_2$ are given by

$$I_1 = \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h}[e^{s\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m))]ds,$$

$$I_2 = \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h}[\boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_m + s))]ds.$$

In order to estimate $\|I_2\|_\infty$, we use Lemmas 1 and 2, the Lipschitz continuity and the boundedness of $f$, the Duhamel's formula (3) to get

$$\begin{aligned} \|I_2\|_\infty &\lesssim \int_0^{\Delta t} \|e^{(\Delta t - s)\boldsymbol{A}_h}\|_\infty \|\boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_m + s))\|_\infty ds \\ &\lesssim \int_0^{\Delta t} \|e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m) - \boldsymbol{u}_h(t_m + s)\|_\infty ds = \int_0^{\Delta t} \left\| \int_0^s e^{(s-\sigma)\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m + \sigma))d\sigma \right\|_\infty ds \\ &\leq \int_0^{\Delta t} \left( \int_0^s \|e^{(s-\sigma)\boldsymbol{A}_h}\|_\infty \|\boldsymbol{F}(\boldsymbol{u}_h(t_m + \sigma))\|_\infty d\sigma \right) ds \lesssim \int_0^{\Delta t} s ds \lesssim \Delta t^2. \end{aligned} \tag{15}$$

On the other hand, define $\boldsymbol{G}(s) = e^{s\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m))$, clearly $\boldsymbol{G}(0) = 0$. Therefore,

$$\boldsymbol{G}(s) = \boldsymbol{G}(s) - \boldsymbol{G}(0) = \int_0^s \boldsymbol{G}'(\sigma)d\sigma,$$

where

$$\boldsymbol{G}'(\sigma) = e^{\sigma\boldsymbol{A}_h}\boldsymbol{A}_h\boldsymbol{F}(\boldsymbol{u}_h(t_m)) - \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h(t_m))e^{\sigma\boldsymbol{A}_h}\boldsymbol{A}_h\boldsymbol{u}_h(t_m).$$

Since $\|\boldsymbol{A}_h\boldsymbol{F}(\boldsymbol{u}_h(t_m))\|_\infty \lesssim 1$ and $\|\boldsymbol{A}_h\boldsymbol{u}_h(t_m)\|_\infty \lesssim 1$ according to Lemma 7, we have $\|\boldsymbol{G}'(\sigma)\|_\infty \lesssim 1$. Hence,

$$\|\boldsymbol{G}(s)\|_\infty \leq \int_0^s \|\boldsymbol{G}'(\sigma)\|_\infty d\sigma \lesssim s,$$

which implies

$$\|I_1\|_\infty \leq \int_0^{\Delta t} \|e^{(\Delta t - s)\boldsymbol{A}_h}\|_\infty \|\boldsymbol{G}(s)\|_\infty ds \lesssim \int_0^{\Delta t} s ds \lesssim \Delta t^2. \tag{16}$$

Combining (15) and (16) gives us $\|\boldsymbol{R}_m^{(1)}\|_\infty \lesssim \Delta t^2$, or $\|\boldsymbol{R}_m^{(1)}\|_\infty \leq C_2\Delta t^2$ for some $C_2 > 0$. Thus, from (14) we further deduce that

$$\|\boldsymbol{e}_{m+1}\|_\infty \leq (1 + C_1\Delta t)\|\boldsymbol{e}_m\|_\infty + C_2\Delta t^2.$$

The inequality (13) is then obtained by using the discrete Gronwall's inequality in Lemma 4-(ii), and the proof is completed. □

Following a similar idea as in the proof of the above theorem, we also establish the optimal error estimate for the fully discrete LRI1b scheme.

**Theorem 2** (Error estimate of LRI1b). Let $\{\boldsymbol{u}_h^m\}$ be generated by the LRI1b scheme (6) with $\boldsymbol{u}_h^0 = \boldsymbol{u}_h(0)$, then

$$\|\boldsymbol{u}_h^m - \boldsymbol{I}^h u(t_m)\|_\infty \lesssim h^2 + \Delta t, \quad \forall\, m = 0, 1, \dots, M,$$

provided that $0 < \Delta t \leq \omega_0$.

*Proof.* As done in Theorem 1, the main step is again to prove $\|\boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m)\|_\infty \lesssim \Delta t$. Subtracting (4) from (6) yields

$$\boldsymbol{e}_{m+1} = e^{\Delta t\boldsymbol{A}_h}\boldsymbol{e}_m + \Delta t[\boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h^m) - \boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m))] + \boldsymbol{R}_m^{(2)},$$

where $\boldsymbol{R}_m^{(2)} = \Delta t \boldsymbol{F}(e^{\Delta t \boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \int_0^{\Delta t} e^{(\Delta t - s)\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m+s))ds$. Thus we deduce that

$$\begin{aligned}\|\boldsymbol{e}_{m+1}\|_\infty &\le \|\boldsymbol{e}_m\|_\infty + \Delta t\|\boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h^m) - \boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m))\|_\infty + \|\boldsymbol{R}_m^{(2)}\|_\infty \\ &\le \|\boldsymbol{e}_m\|_\infty + C_1\Delta t\|e^{\Delta t\boldsymbol{A}_h}(\boldsymbol{u}_h^m - \boldsymbol{u}_h(t_m))\|_\infty + \|\boldsymbol{R}_m^{(2)}\|_\infty \\ &\le (1 + C_1\Delta t)\|\boldsymbol{e}_m\|_\infty + \|\boldsymbol{R}_m^{(2)}\|_\infty,\end{aligned} \tag{17}$$

where $C_1$ is the same constant as in (14). We then rewrite $\boldsymbol{R}_m^{(2)}$ as follows:

$$\begin{aligned}\boldsymbol{R}_m^{(2)} &= \Delta t[\boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1}))] + \int_0^{\Delta t} \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1})) - e^{(\Delta t - s)\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m+s))ds \\ &= \Delta t[\boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1}))] + \int_0^{\Delta t} \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1})) - \boldsymbol{F}(\boldsymbol{u}_h(t_m+s))ds \\ &\quad + \int_0^{\Delta t} (\boldsymbol{I} - e^{(\Delta t - s)\boldsymbol{A}_h})\boldsymbol{F}(\boldsymbol{u}_h(t_m+s)))ds \\ &=: J_1 + J_2 + J_3,\end{aligned}$$

where $J_1, J_2$ and $J_3$ are given by

$$\begin{aligned}J_1 &= \Delta t[\boldsymbol{F}(e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m)) - \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1}))], \\ J_2 &= \int_0^{\Delta t} \boldsymbol{F}(\boldsymbol{u}_h(t_{m+1})) - \boldsymbol{F}(\boldsymbol{u}_h(t_m+s))ds, \\ J_3 &= \int_0^{\Delta t} (\boldsymbol{I} - e^{(\Delta t - s)\boldsymbol{A}_h})\boldsymbol{F}(\boldsymbol{u}_h(t_m+s)))ds.\end{aligned}$$

By applying the Duhamel's principle, we have

$$\begin{aligned}\|J_1\|_\infty &\lesssim \Delta t\|e^{\Delta t\boldsymbol{A}_h}\boldsymbol{u}_h(t_m) - \boldsymbol{u}_h(t_{m+1})\|_\infty \\ &= \Delta t\left\|\int_0^{\Delta t} e^{(\Delta t - \sigma)\boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h(t_m+\sigma))d\sigma\right\|_\infty \lesssim \Delta t^2.\end{aligned} \tag{18}$$

For $\|J_2\|_\infty$, we use the Lipschitz continuity of $f$ and $\boldsymbol{u}_h$ mentioned in Lemma 7 to obtain

$$\begin{aligned}\|J_2\|_\infty &\le \int_0^{\Delta t} \|\boldsymbol{F}(\boldsymbol{u}_h(t_{m+1})) - \boldsymbol{F}(\boldsymbol{u}_h(t_m+s))\|_\infty ds \\ &\lesssim \int_0^{\Delta t} \|\boldsymbol{u}_h(t_{m+1}) - \boldsymbol{u}_h(t_m+s)\|_\infty ds \\ &\lesssim \int_0^{\Delta t} [t_{m+1} - (t_m+s)]ds = \int_0^{\Delta t} (\Delta t - s)ds \lesssim \Delta t^2.\end{aligned} \tag{19}$$

Note that

$$J_3 = -\int_0^{\Delta t}\left(\int_0^{\Delta t - s} e^{\sigma\boldsymbol{A}_h}\boldsymbol{A}_h\boldsymbol{F}(\boldsymbol{u}_h(t_m+s))d\sigma\right)ds.$$

Now using the fact that $\|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{u}_h(t))\|_\infty \lesssim 1$ for all $t \in [0, T]$ (cf. Lemma 7), we deduce that

$$\begin{aligned}
\|J_3\|_\infty &\leq \int_0^{\Delta t} \left( \int_0^{\Delta t - s} \|e^{\sigma \boldsymbol{A}_h}\|_\infty \|\boldsymbol{A}_h \boldsymbol{F}(\boldsymbol{u}_h(t_m + s))\|_\infty d\sigma \right) ds \\
&\lesssim \int_0^{\Delta t} (\Delta t - s) ds \lesssim \Delta t^2. \qquad (20)
\end{aligned}$$

From (18), (19) and (20), we obtain $\|\boldsymbol{R}_m^{(2)}\|_\infty \lesssim \Delta t^2$, or $\|\boldsymbol{R}_m^{(2)}\|_\infty \leq C_3 \Delta t^2$ for some $C_3 > 0$. Thus it is implied from (17) that

$$\|\boldsymbol{e}_{m+1}\|_\infty \leq (1 + C_1 \Delta t)\|\boldsymbol{e}_m\|_\infty + C_3 \Delta t^2,$$

which completes the proof after applying the discrete Gronwall's inequality in Lemma 4-(ii). □

## 4 | NUMERICAL RESULTS

We verify the error estimates in both space and time of the proposed first-order LRI schemes and compare their performance with the first-order ETD (ETD1) scheme. The benchmark test[21] with a traveling wave solution is considered, where the nonlinear function is $f(u) = u - u^3$ corresponding to the double-well potential function and the spatial domain is $\Omega = (-0.5, 0.5)^2$. Homogeneous Neumann conditions are imposed on the boundary and the initial condition takes the following form

$$u_0(x, y) = \frac{1}{2}\left(1 - \tanh\left(\frac{x}{2\sqrt{2}\varepsilon}\right)\right), \quad (x, y) \in \overline{\Omega}.$$

An approximate exact solution (for $\varepsilon \ll 1$) to the initial-boundary value problem is then given by

$$u(x, y, t) = \frac{1}{2}\left(1 - \tanh\left(\frac{x - st}{2\sqrt{2}\varepsilon}\right)\right), \quad (x, y) \in \overline{\Omega},\ t \geq 0,$$

where $s = 3\varepsilon/\sqrt{2}$ denotes the speed of the traveling wave. We fix $T = 1/4s$ and vary both $h$ and $\Delta t$ simultaneously so that the ratio between $h$ and $\Delta t$ is fixed. The $L^2$ and $L^\infty$ errors of ETD1, LRI1a and LRI1b schemes are computed at the final time $T$ for different values of $\varepsilon \in \{0.01, 0.005\}$. The results are reported in Table 1 (for $L^2$ errors) and Table 2 (for $L^\infty$ errors) with corresponding convergence rates. We first observe that all methods, particularly LRI1a and LRI1b, have the overall first-order convergence as proved in the previous section. The errors produced by LRI1a and LRI1b are smaller than those by ETD1, meaning that the proposed LRI schemes are more accurate than ETD1. Moreover, the LRI schemes have better convergence rates than the ETD scheme, especially when $\varepsilon$ is sufficiently small. More comparisons regarding the second-order ETD and LRI schemes can by found in[5].

## 5 | CONCLUSION

In this paper we have carried out the fully discrete error analysis of the numerical schemes for solving the Allen-Cahn type semilinear parabolic equation, which use the two first-order LRI schemes, LRI1a and LRI1b, for time integration and the central finite difference approximations for spatial discretization. Optimal error estimates in both time and space are successfully derived under the low regularity assumption that the exact solution belongs to $C([0, T], C^4(\overline{\Omega}))$ (i.e., the solution is only continuous in time) and that the nonlinear function satisfies an extra assumption, namely (A3), in addition to the standard assumptions (A1) and (A2) in the literature. The proofs rely on the MBP property of the exact solution and the LRI numerical solutions, Duhamel's formula and a technical lemma, Lemma 7. The results are valid for the well-known double-well and Flory-Huggins potential functions and for various types of boundary conditions such as the homogeneous Dirichlet, the homogeneous Neumann and the periodic ones. The numerical results in this paper and also in[5] confirmed these error estimates and the outperformance of the LRI schemes compared to ETD in terms of accuracy and efficiency. Extension of the presented analysis to the second-order or even higher-order LRI schemes is still an open question and will be the subject of our future study. In particular, the second-order

**Table 1** $L^2$ errors and convergence rates by the first-order ETD and LRI schemes

| $\varepsilon$ | $h$ | $\Delta t$ | ETD1 | | LRI1a | | LRI1b | |
|---|---|---|---|---|---|---|---|---|
| | | | $L^2$ error | Rate | $L^2$ error | Rate | $L^2$ error | Rate |
| 0.01 | 1/64 | $T/32$ | 1.00e-01 | | 7.36e-02 | | 6.65e-02 | |
| | 1/128 | $T/64$ | 6.03e-02 | 0.73 | 4.40e-02 | 0.74 | 4.01e-02 | 0.73 |
| | 1/256 | $T/128$ | 3.31e-02 | 0.86 | 2.41e-02 | 0.87 | 2.21e-02 | 0.86 |
| | 1/512 | $T/256$ | 1.73e-02 | 0.93 | 1.26e-02 | 0.94 | 1.16e-02 | 0.93 |
| | 1/1024 | $T/512$ | 8.87e-03 | 0.97 | 6.44e-03 | 0.97 | 5.91e-03 | 0.97 |
| | 1/2048 | $T/1024$ | 4.48e-03 | 0.98 | 3.25e-03 | 0.98 | 2.99e-03 | 0.98 |
| 0.005 | 1/64 | $T/32$ | 1.63e-01 | | 1.13e-01 | | 1.03e-01 | |
| | 1/128 | $T/64$ | 1.27e-01 | 0.36 | 8.99e-02 | 0.33 | 8.54e-02 | 0.27 |
| | 1/256 | $T/128$ | 8.32e-02 | 0.61 | 5.84e-02 | 0.62 | 5.60e-02 | 0.61 |
| | 1/512 | $T/256$ | 4.76e-02 | 0.80 | 3.32e-02 | 0.82 | 3.18e-02 | 0.81 |
| | 1/1024 | $T/512$ | 2.53e-02 | 0.91 | 1.76e-02 | 0.91 | 1.69e-02 | 0.91 |
| | 1/2048 | $T/1024$ | 1.30e-02 | 0.96 | 9.07e-03 | 0.96 | 8.69e-03 | 0.96 |

**Table 2** $L^\infty$ errors and convergence rates by the first-order ETD and LRI schemes

| $\varepsilon$ | $h$ | $\Delta t$ | ETD1 | | LRI1a | | LRI1b | |
|---|---|---|---|---|---|---|---|---|
| | | | $L^\infty$ error | Rate | $L^\infty$ error | Rate | $L^\infty$ error | Rate |
| 0.01 | 1/64 | $T/32$ | 5.02e-01 | | 3.73e-01 | | 3.39e-01 | |
| | 1/128 | $T/64$ | 3.14e-01 | 0.68 | 2.30e-01 | 0.69 | 2.13e-01 | 0.67 |
| | 1/256 | $T/128$ | 1.74e-01 | 0.85 | 1.27e-01 | 0.86 | 1.17e-01 | 0.86 |
| | 1/512 | $T/256$ | 9.10e-02 | 0.93 | 6.65e-02 | 0.93 | 6.16e-02 | 0.93 |
| | 1/1024 | $T/512$ | 4.65e-02 | 0.97 | 3.40e-02 | 0.97 | 3.14e-02 | 0.97 |
| | 1/2048 | $T/1024$ | 2.35e-02 | 0.99 | 1.72e-02 | 0.98 | 1.59e-02 | 0.99 |
| 0.005 | 1/64 | $T/32$ | 8.60e-01 | | 6.95e-01 | | 6.54e-01 | |
| | 1/128 | $T/64$ | 7.79e-01 | 0.14 | 5.88e-01 | 0.24 | 5.78e-01 | 0.18 |
| | 1/256 | $T/128$ | 5.67e-01 | 0.46 | 4.11e-01 | 0.52 | 3.97e-01 | 0.54 |
| | 1/512 | $T/256$ | 3.42e-01 | 0.73 | 2.42e-01 | 0.76 | 2.34e-01 | 0.77 |
| | 1/1024 | $T/512$ | 1.85e-01 | 0.88 | 1.30e-01 | 0.90 | 1.25e-01 | 0.90 |
| | 1/2048 | $T/1024$ | 9.57e-02 | 0.95 | 6.69e-02 | 0.96 | 6.43e-02 | 0.96 |

LRI (LRI2) scheme is given by[5]:

$$\boldsymbol{u}_h^{m+1} = e^{\Delta t \boldsymbol{A}_h}\boldsymbol{u}_h^m + \frac{\Delta t}{2}\left[e^{\Delta t \boldsymbol{A}_h}\boldsymbol{F}(\boldsymbol{u}_h^m) + \boldsymbol{F}(e^{\Delta t \boldsymbol{A}_h}\boldsymbol{u}_h^m)\right] + \frac{\Delta t^2}{2} e^{\Delta t \boldsymbol{A}_h}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}(\boldsymbol{u}_h^m)\boldsymbol{F}(\boldsymbol{u}_h^m).$$

Proving the error estimates for this scheme is challenging as one needs to obtain the bounds $\|\boldsymbol{A}_h^2\boldsymbol{u}_h(t)\|_\infty \lesssim 1$ and $\|\boldsymbol{A}_h^2\boldsymbol{F}(\boldsymbol{u}_h(t))\|_\infty \lesssim 1$ (similarly to Lemma 7-(i), (iii) with $\boldsymbol{A}_h$ replaced by $\boldsymbol{A}_h^2$) as well as to control $\boldsymbol{A}_h^2\boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h)$ for any $s \in [0, \Delta t]$ by some constant independent of the spatial mesh size $h$. Note that the matrix exponential $e^{s\boldsymbol{A}_h}$ is dense, which makes it difficult to estimate the value of the nonlinear function $\boldsymbol{F}(e^{s\boldsymbol{A}_h}\boldsymbol{u}_h)$.

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