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REVIEW

Nonlocal elasticity and boundary condition paradoxes: a review

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Abstract Nonclassical continuum mechanics theories have seen a rise in implementation over the past several years due to the increased research into micro-/nanoelectromechanical systems (MEMS/NEMS), micro-/nanoresonators, carbon nanotubes (CNTs), etc. Typically, these systems exist in the range of several nanometers to the micro-scale. There are several available theories that can capture phenomena inherent to nanoscale structures. Of the available theories, researchers utilize Eringen's nonlocal theory most frequently because of its ease of implementation and seemingly accurate results for specific loading conditions and boundary conditions. Eringen's integral nonlocal theory, which leads to a set of integro-partial differential equations, is difficult to solve; therefore, the integral form was reduced to a set of singular partial differential equations using a Green-type attenuation function. However, a so-called paradox has arisen between the integral and differential formulations of Eringen's nonlocal elasticity. For certain boundary and loading conditions, instead of the expected softening effect inherent in nonlocal particle interactions, some researchers have found a stiffening effect. Still, others have found no variation from those results found using classical theories. As such, the discrepancies between the integral and differential forms have been the subject of debate for nearly two decades, with several proposed resolutions published in recent years. This paper serves to review and consolidate existing theories in nonlocal elasticity along with selected theories in nonclassical continuum mechanics, the utilization of Eringen's nonlocal elasticity in beams, shells, and plates, the existing discrepancies and proposed solutions, and recommendations for future work.

Keywords Nonclassical continuum mechanics · Integral and differential models · Nonlocal elasticity paradox · Boundary conditions · Nanostructures

Abbreviations

C-C	Clamped-clamped
CCCC	All clamped plate boundary conditions
CCFF	Two sides clamped, two sides free boundary conditions
C-F	Clamped-free (cantilever)
C-FP	Cantilever beam with concentrated load *P* at the free end
C-H	Clamped-hinged
CNC	Carbon nanocone
CNT	Carbon nanotube
CPT	Classical plate theory
DQM	Differential quadrature method
DWCNT	Double-walled carbon nanotube
EBT	Euler-Bernoulli beam theory

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FEM	Finite element method
FGM	Functionally graded material
FOPT	First-order plate theory
H-H	Hinged-hinged
HHCC	Two sides hinged, two sides clamped boundary conditions
HHHH	All hinged plate boundary conditions
HOPT	Higher-order plate theory
HSDT	Higher-order shear deformation theory
KPT	Kirchhoff's plate theory
MEMS	Micro-electromechanical system
MPT	Mindlin's plate theory
MWCNT	Multiwalled carbon nanotube
NEMS	Nanoelectromechanical system
SDM	Stress-driven model
SGM	Strain gradient method
SLGS	Single-layer graphene sheets
SSSD	Simply supported beam with uniformly distributed load
SWCNT	Single-walled carbon nanotube
TBT	Timoshenko beam theory
WRA	Weighted residual approach

Nomenclature

A	Cross-sectional area
C	Compliance tensor
E	Young's modulus
e_0	Material parameter
f	Body force
F	Distributed loading
I	Second area moment of inertia
L	Length
l_c	Nonlocal length scale parameter
M	Bending moment
m_0	Mass density
P	Concentrated point force
Q	Shear force
q	Axial force
T	Kinetic energy
U	Potential energy
u	Displacement field
V	Elastic domain
w	Transverse displacement
α	Regularizing parameter: gradient model
δ	Dirac delta
ε	Strain field
κ	Nonlocal parameter
λ	Elastic modulus
μ	Shear modulus
ξ	Volume fraction
ρ	Material density
σ	Nonlocal stress tensor
τ	Local stress tensor
ω	Frequency

Introduction

In recent years, the applications and modeling of nanoscale materials and structures, such as micro-electromechanical systems (MEMS), nanoelectromechanical systems (NEMS), nanoscale sensors, actuators, gyroscopes, carbon nanotubes (CNTs), and nanoplates have expanded significantly (Arash and Wang 2012; Askari et al. 2017; Eltaher et al. 2016). It is vital to develop accurate mathematical models that can capture additional degrees of freedom present at the nanoscale such as macro-/micro-rotations, translations, and deformations inherent to the nanoscale. As such, there are several theories in nonclassical continuum mechanics aimed at capturing these characteristics. In this effort, a brief review on the existing theories in nonclassical continuum mechanics is presented for the sake of demonstrating the complexity of these theories and the use of them from a structural mechanics, dynamics, and materials points of view. Then, one of the most widely utilized theories, Eringen's nonlocal elasticity, is discussed in detail. Eringen's work in the field of nonclassical continuum mechanics is extensive and should not be overlooked, spanning from the mid-1960s up until the mid-2000s (Eringen 1966; Eringen 1972; Eringen and Edelen 1972; Eringen 1974; Eringen 1976; Eringen 1977; Eringen 1983; Eringen 2002; Eringen 2006).

Eringen's nonlocal elasticity theory was initially proposed in the 1970s to account for the long-range interatomic interactions in nanostructures (Eringen 1972; Eringen 1974; Eringen 1976; Eringen 1977). In Eringen's nonlocal theory, the size-dependent phenomena are manifested through what is termed the nonlocal parameter, also named the small-scale or size-dependent parameter. Eringen's nonlocal theory relies on representing particles as a point mass subject only to translation, in which the stress of any point is influenced by the strain of all other points in the medium, and thus can be considered a strain-driven model. To further explain, any reference point in an elastic domaipdeln experiences a neighboring interaction with other nearby points. This theory exists in an integral form, which leads to a set of integro-partial differential equations that are difficult to solve. The integral form is referred to as strongly nonlocal, since it relies on an integral and thus includes all neighboring points in the medium. From a certain Green-type attenuation function (kernel function), Eringen (1983) converted the nonlocal integral theory into differential form. The differential nonlocal theory and other gradient-type theories are termed weakly nonlocal as they rely on derivatives instead of the integral.

The reduction to the differential model has not been accepted with full accuracy for certain boundary and loading conditions. For example, in the case of bounded domains that require the addition of constitutive boundary conditions to obtain a closed-form solution for a stress field induced by a given strain field, the differential theory is ill-posed and cannot be considered correct. It has been shown by researchers, in general, that increasing nonlocal effects in the model lead to an overall reduction in the effective stiffness of the system. This softening is accompanied by a decrease of the natural frequencies, a decrease in critical buckling loads, and an increase in the maximum deflection of the structure (Xu et al. 2016). However, there are reported discrepancies between the expected results of the integral and differential models, in which several researchers have found a hardening effect, different from expected softening effect, or no variations from those found using classical theories when modifying the nonlocal parameter (Peddieson et al. 2003).

The conversion of the integral model to the differential model was originally developed without full consideration of the boundary conditions, thus the so-called paradox becomes apparent in boundary-valued problems. Clarification is needed to determine if this model is valid for certain boundary conditions. There are cases in which there is a mathematically ill-posed problem, but the results are accepted in the literature (softening as expected). On the contrary, there are cases in which authors claim to resolve the ill-posed problem, but then obtain unexpected results (hardening). For example, when the differential form of Eringen's theory is applied to beams, hardening effects were observed, particularly for the bending, critical buckling loads, and the fundamental natural frequency of nanocantilever beams (Khodabakhshi and Reddy 2015; Murmu and Pradhan 2009a, b). Another inconsistency is in the effects of the nonlocal parameter for cantilever beams with concentrated forces (Peddieson et al. 2003).

Possible causes of the so-called paradox have been attributed to the common misrepresentation of the shear and moment boundary conditions at the free end. Additionally, one research effort (Romano and Barretta 2017a, b) recently discussed that Eringen's nonlocal elasticity theory, when applied to boundary-valued problems, is ill-posed. The existence of a solution implies a serious discrepancy between the constitutive equations and the governing equations of motion. Through the derivation of Eringen's nonlocal elasticity, the stress field changes from a local to a nonlocal one by means of the nonlocal parameter. However, when deriving the equations of motion for a system by, for example, the Hamilton's principle, the boundary conditions are not affected by the nonlocal parameter. Thus, the resulting system is composed of nonlocal equations of motion and local (size-independent) boundary conditions. While the boundary conditions are not affected by the nonlocal parameter in the differential model, clamped-clamped and simply supported–simply supported boundary conditions without the inclusion of uniformly distributed loads are generally considered well-defined for differential models in terms of their anticipated results. However, even these cases are considered ill-posed solutions because of the discrepancy between the constitutive and equilibrium equations.

Attempts have been made to resolve the existing resultant inconsistencies from different physical and mathematical points of view through the following methods:

- Laplace transformation of the integral model (Tuna and Kirca 2016a; b)
- Laplace transform of the differential model for nanocantilever beams with end forces (Özgür Yayli and Yerel Kandemir 2017)
- Weighted residual approach (WRA) in which the classical strain and kinetic energy are replaced into the nonlocal governing equations of motion (Xu et al. 2016)
- Two types of Rayleigh-Ritz method with polynomial and combination of polynomial and trigonometric as admissible functions (Fakher and Hosseini-Hashemi 2017)
- Normalized kernel employed in Eringen's integral nonlocal elasticity model which corresponds to a finite domain (Koutsoumaris et al. 2017)
- Iterative nonlocal residual approach in which the local field with an imposed nonlocal residual is utilized to solve the field equation (Shaat et al. 2017)

Other attempts were made in which the classical boundary conditions were modified to nonclassical boundary conditions, particularly in the representation of the shear and moment in nanocantilever beams (Lu et al. 2006; Wang et al. 2007; Reddy and Pang 2008; Barretta et al. 2016). It is noted that, in general, inconsistencies arise in both the differential and the integral models. For instance, when the integral form is applied to a cantilever beam subjected to a point load at the free end, it was shown that the nonlocal model predicts no nonlocal effects by a cantilever subjected to any combination of concentrated loads (Peddieson et al. 2003). Similar results are shown in the theory known as the two-phase integral model when simply supported–simply supported boundary conditions are considered (Khodabakhshi and Reddy 2015).

Throughout this review, the existing methods for applying Eringen's nonlocal elasticity theory to static and dynamic systems, such as beams, shells, and plates, are consolidated. Then, the proposed solutions are categorized and discussed in detail based on their assumptions, formulations, and results. Great efforts have been made by these researchers to study these paradoxes from physical, theoretical, and mathematical points of view. In this review paper, we aim to provide the reader with a strong foundation for reviewing nonclassical continuum mechanics, nonlocal elasticity in nanostructures, and what has been termed by many researchers as the boundary condition paradox. Final comments are made on the possible limits of applicability and robustness of the proposed solutions.

Existing theories in nonclassical continuum mechanics

Classical continuum theories of elasticity state that a point mass is an appropriate representation of a particle within a bulk, macro-scale material. When acted upon by an external force, the point mass exhibits a macro-translation that is independent of the other particles in the medium. In nanomaterials, the size of the bulk material approaches the size of its internal structures, such that particles cannot be represented as a point mass, in which the deformation and rotation of the particle are neglected. Thus, it has been shown that classical continuum theories result in inaccurate representations of the static and dynamic behaviors of nanomaterials and nanostructures. When the external characteristic length is approximately equal to the internal characteristic length of the material, nonclassical continuum theories must be used to accurately model and predict the system response (Askari et al. 2017).

Size-dependent models have been of great interest to researchers because of the recent growth in the use of nanomaterials for MEMS and NEMS devices. Nonclassical continuum mechanics theories are used to describe the long-range and/or short-range interactions between material particles in a domain. These theories include Cosserat theory (Cosserat and Cosserat 1909), micro-structure theory (Mindlin 1963), second-strain gradient theory (Mindlin 1965), Eringen's nonlocal theory (Eringen 1972), first-strain gradient theory (Mindlin and Eshel 1968), asymmetric nonlocal theory (Demiray 1977), classical couple stress theory (Mindlin and Tiersten 1962; Toupin 1962), modified couple stress theory (Yang et al. 2002), consistent couple stress theory (Hadjesfandiari and Dargush 2011), second-order rotation gradient theory (Shaat and Abdelkefi 2016), and a general nonlocal theory (Shaat and Abdelkefi 2017). Each of these theories is categorized in terms of their assumptions on particle representation. The most encompassing theories represent material particles as volume elements that can experience micro-rotations, micro-deformations, and translations (Mindlin 1963; Mindlin 1965). However, these theories are extremely complex and computationally expensive when used to model dynamical systems. Other theories consider extremely rigid material particles that exhibit micro-rotation and translation only (Cosserat and Cosserat 1909; Mindlin and Tiersten 1962; Toupin 1962; Yang et al. 2002; Hadjesfandiari and Dargush 2011; Shaat and Abdelkefi 2016). Each theory has a specific number of additional constants required to encompass the structural behavior of the material. These constants are needed in addition to the classical material parameters to determine the characteristics and response of the nanostructure, such as Young's modulus, Poisson's ratio, or Lamé constants. Another consideration when choosing a nonlocal theory for a particular application is the micro-structure of the nanomaterial. Nanomaterials can have a crystalline, granular, or amorphous structure (Eringen 1972; Shaat and Abdelkefi 2017; Chen et al. 2004; Maranganti and Sharma 2007; Polyzos and Fotiadis 2012; Chen and Lee 2003). It is important to select the most appropriate theory accounting for the material's unique characteristics. For example, nanocrystalline diamond has extremely rigid crystals that are unlikely to undergo micro-deformations. Therefore, a theory that neglects the effects of micro-deformations may be considered. Table 1 outlines the nonclassical continuum theories listed above in terms of material particle representation, kinematic variables, number of additional parameters needed, and applicability for materials, respectively (Shaat and Abdelkefi 2016).

Though significant efforts were made to develop each model from a material science point of view, many problems may exist in applying these models to physical systems. For example, one of the greatest challenges with working with theories, such as the micro-structure theory (Mindlin 1963), which requires 16 additional parameters, is how to accurately determine the additional parameters and give them a physical meaning. In many of the higher-order theories, experimental data or atomistic simulations that would provide information about the macro-/micro-rotations, deformations, translations, and their gradients are needed. This information comes in the form of dispersion relations that may not always provide direct physical interpretation of the constants. In addition, the boundary conditions needed for higher-order models are in excess of the number of natural boundary conditions of the system. For this reason, many researchers have chosen to use Eringen's nonlocal elasticity theory because it requires only one additional parameter and when using the differential form, the resulting order of the equations of motion matches the number of available natural boundary conditions.

In addition to the models and theories presented in Table 1, it is important to mention that several other weakly nonlocal theories exist. Of the weakly nonlocal theories, separate from Eringen's, it appears that the most widely studied ones are gradient elasticity theories and their modifications. The bulk of studies considering gradient elasticity were conducted by Aifantis and his colleagues (Aifantis 2016; 2011a, b; Askes and Aifantis 2011; Aifantis 1999a; b; Gutkin and Aifantis 1999; Lazar et al. 2005; Askes et al. 2007; Aifantis 2009; Askes and Aifantis 2009; Forest and Aifantis 2010; Lurie et al. 2011; Aifantis 2011b; Mokios and Aifantis 2012; Xu et al. 2013; Sun and Aifantis 2014; Aifantis 2014; Xu et al. 2014; Yue et al. 2015; Lurie et al. 2017; De Domenico et al. 2018; Gurtin et al. 2010; Aifantis 1984; Aifantis 1987; Lazar et al. 2006). In these studies, Aifantis (2016) often discusses the need to resort to higher-order gradients of the key constitutive variables to model the evolution of deformation and fracture when homogenous material states become unstable and the governing equations lead to pathological or unphysical behavior. Additionally, Aifantis (2016) paid tribute to Eringen and discussed the similarities between Eringen's nonlocal theory and the gradient approach or strain gradient theory. Both rely on the introduction of a characteristic material parameter or internal length that may account for effects of underlying micro-structures in complex media or the interaction of bulk and surface points in nanoscale volumes. However, rather than Eringen's reduced constitutive equation that incorporates the Laplacian of stress in the linear expression of Hooke's Law (eliminating stress singularities), the gradient theory relies on a generalization of the classical Hooke's law by an extra term containing the Laplacian of strain (eliminating strain singularities) (Fleck and Hutchinson 2001).

Some other efforts related to gradient elasticity include discussions on Laplacian-based gradient elasticity theories (Askes and Aifantis 2011), gradient deformation models and various scales (Aifantis 1999a), and a strain gradient interpretation of size effects (Aifantis 1999b). In 2005, special classes of static theories of gradient elasticity, nonlocal elasticity, gradient micro-polar elasticity, and nonlocal micro-polar elasticity were discussed (Gutkin and Aifantis 1999). Other efforts in gradient elasticity focused on wave dispersions (Askes and Aifantis 2011; Askes et al. 2007; Askes and Aifantis 2009; De Domenico et al. 2018; Lazar et al. 2006).

Many of these theories are rather robust and have numerous applications on size effects and elimination of elastic singularities. A few other examples of gradient models include the Fleck-Hutchinson and the Gao-Nix-Huang strain gradient theories (Fleck and Hutchinson 2001; Gurtin and Anand 2009; Fleck and Hutchinson 2001; Gudmundson 2004; Aifantis and Willis 2005). Additionally, there are improved gradient theories that consider surface effects (Aifantis and Willis 2005; Voyiadjis and Al-Rub 2005; Al-Rub and Voyiadjis 2006; Zhu and Karihaloo 2008). Note that several of these studies were motivated by works on the thermodynamic basis of gradient theory (Gurtin et al. 2010) and the review chapter ion the internal length gradient material mechanics across scales and disciplines. A further detailed review on several other weakly nonlocal and strain gradient theories is out of the scope of this effort, but related works on these can be found in (Aifantis 2003).

Table 1 A summary of the available nonclassical continuum theories and their applicability for materials (Shaat and Abdelkefi 2016)

Theory	Material particle representation	Kinematic variables	No.	Applicability for material
Cosserat theory (Cosserat and Cosserat 1909)	Volume element has micro-rotations and macro--displacements.	Macro-strain, the gradient of the micro-rotation, and the difference between the micro-rotation and the macro-rotation.	4	- Single molecular crystal materials with nearly rigid molecules (Chen et al. 2004; Chen and Lee 2003). - Polycrystalline materials with rigid crystals. - Amorphous materials with short nonlocal-range effects.
Microstructure theory (Mindlin 1963)	Volume element has micro-deformations (micro--strains and micro-rotations) and macro-displacements.	Macro-strain, the gradient of the micro-deformation, and the difference between the micro-deformation and the macro--deformation.	16	- Single molecular crystal materials (Chen et al. 2004; Chen and Lee 2003). - Polycrystalline materials. - Amorphous materials with short nonlocal-range effects.
Second-strain gradient theory (Mindlin 1965)	Volume element with higher-order deformations, micro-strains, micro--rotations, and macro--displacements.	Macro-strain, the first-strain gradient, and the second-strain gradient.	16	- Single crystal materials that exhibit short-range acoustic phonons (Polyzos and Fotiadis 2012). - Amorphous materials with short nonlocal-range effects (Polyzos and Fotiadis 2012). - It may be applied for polycrystalline materials (Maranganti and Sharma 2007).
Eringen's nonlocal elasticity theory (Eringen 1972)	Mass point with only translational motions.	Macro-strain and the nonlocal residual at each point.	1	- Single crystal materials that exhibit long-range acoustic phonons (Chen et al. 2004; Chen and Lee 2003). - Amorphous materials (Maranganti and Sharma 2007).
First-strain gradient theory (Mindlin and Eshel 1968)	Volume element has micro-strains, micro-rotations, and macro-displacements.	Macro-strain and the first-strain gradient.	5	- Single crystal material that exhibit short-range acoustic phonons (Polyzos and Fotiadis 2012). - Amorphous materials with short nonlocal-range effects (Polyzos and Fotiadis 2012). - It may be applied for polycrystalline materials.
Asymmetric nonlocal elasticity theory (Demiray 1977)	Volume element with translational and rigid rotational motions.	Macro-strain, the nonlocal residual at each point, and the relative rotation.	2	- Diatomic single crystal materials that exhibit long-range acoustic and external optical phonons (Demiray 1977). - Amorphous materials.
Classical couple stress theory (Mindlin and Tiersten 1962; Toupin 1962)	Volume element has micro-rotations and macro--displacements.	Macro-strain and the rotation gradient.	2	- Single crystal material that exhibit short-range acoustic phonons. - Amorphous materials with short nonlocal-range effects. - Polycrystalline materials.

Table 1 (continued)

Theory	Material particle representation	Kinematic variables	No.	Applicability for material
Modified couple stress theory (Yang et al. 2002)	Volume element has micro-rotations and macro--displacements.	Macro-strain and the symmetric part of the rotation gradient.	1	- Single crystal material that exhibit short-range acoustic phonons. - Amorphous materials with short nonlocal-range effects. - Polycrystalline materials
Consistent couple stress theory (Hadjesfandiari and Dargush 2011)	Volume element has micro-rotations and macro--displacements.	Macro-strain and the skew-symmetric part of the rotation gradient.	1	- Single crystal material that exhibit short-range acoustic phonons. - Amorphous materials with short nonlocal-range effects. - Polycrystalline materials
Second-order rotation gradient theory (Shaat and Abdelkefi 2016)	Volume element with micro-displacement, micro-rotation and higher-order rotation gradients	Macro-strain, first-order rotation gradient, and second-order rotation gradient	11,8, or 5	- Amorphous materials of various types
General nonlocal theory (Shaat and Abdelkefi 2017)	Mass point with only translational motions.	Macro-strain and the nonlocal residual at each point.	2	- Single crystal materials that exhibit long-range acoustic phonons. - Amorphous materials.

Eringen's nonlocal elasticity theory

Eringen's integral nonlocal elasticity theory: assumptions and formulations

One of the most popular developed theories in nonclassical continuum mechanics is Eringen's nonlocal elasticity theory. In his model, Eringen proposed a point mass representation subject to translation only in the medium for the material particle. Furthermore, the stress experienced by one particle is influenced by the strain of every particle in the medium. This effect diminishes as the space between the particles increases. This relationship is expressed by Eringen (1983)'s nonlocal integral as:

$$\sigma_{ij}(x) = \int_V k\left(\left|x-\bar{x}\right|, \kappa\right)\tau_{ij}\left(\bar{x}\right)dV \tag{1}$$

where $\sigma_{ij}(x)$ denotes the stress at point x, τ_{ij} is the local stress at reference point $\bar{x}$, $k(|x-\bar{x}|, \kappa)$ represents the kernel function, and κ is the nonlocal parameter. The nonlocal parameter κ depends on the product of the material parameter, e_0, and the length scale parameter, a. The kernel function $k(|x-\bar{x}|, \kappa)$ is a decaying function that represents the long-range interactions between particles. The standard kernel function used is:

$$k\left(\left|x-\bar{x}\right|, \kappa\right) = \frac{1}{2\kappa}e^{-\frac{\left|x-\bar{x}\right|}{\kappa}} \tag{2}$$

The nonlocal material parameter, e_0, should be determined experimentally by matching the acoustic dispersion curves of a material. Determination of the nonlocal material parameter is discussed further when a general form of Eringen's nonlocal elasticity theory is reviewed in "Eringen's differential nonlocal theory: a conversion from the integral formulation." It is important to mention that the original integral formulation is often referred to as the strain-driven model with an exponential kernel, where the nonlocal stress is output as a convolution between the local response to the elastic *strain* and a scalar kernel dependent on the nonlocal parameter. Thus, nonlocal models are referred to as strain-driven when the source field is the elastic strain and referred to as stress-driven when the roles of the bending interaction and curvature fields are swapped with those of the original strain-driven integral model.

Eringen's differential nonlocal theory: a conversion from the integral formulation

In its integral form, for homogenous and isotropic bodies, Eringen's nonlocal theory leads to a set of integro-partial differential equations for the strain or displacement field that are generally difficult to solve. In certain cases, and for certain materials, the integro-partial differential equations may be reduced to a set of singular partial differential equations. It should be noted that this conversion is not valid for bounded domains where the constitutive boundary conditions must be added in order to gain closure of the constitutive problem of determining the stress field induced be a certain strain field. With this in mind, Eringen (1983) introduced the differential model for his nonlocal theory in 1983. In reducing the integral form of Eringen's nonlocal theory to the differential form, it was noted that the appropriate class of kernels should be used based on certain mathematical and physical conditions, i.e., for when a well-posed elastostatic problem is considered. As determined in the work of Eringen (Eringen 1972; Eringen 1974; Eringen 1976; Eringen 1977), the stress at the reference point x depends not only on the strain at the specified point but also on the strain of all other points in the domain. This is in accordance with the atomic theory of lattice dynamics and experimental observations on phonon dispersion. If the effects of the strain at all other points are neglected in the model, Eringen's nonlocal theory reduced to the classical theory of elasticity. The local stress, τ_{ij} is defined as (Eringen 1983):

$$\tau_{ij}\left(\overline{x}\right) = \lambda\varepsilon_{rr}\left(\overline{x}\right)\delta_{ij} + 2\mu\varepsilon_{ij}\left(\overline{x}\right) \quad (3)$$

where λ and μ are the material Lamé constants and $\varepsilon_{ij}(\overline{x}) = \frac{1}{2}\left(\frac{\partial u_i(\overline{x})}{\partial j} + \frac{\partial u_j(\overline{x})}{\partial i}\right)$, such that $\vec{u}\ (\overline{x})$ represents the displacement field of a material particle. When considering the attenuation function and the local stress tensor, the nonlocal stress field, σ_{ij}, is obtained. The nonlocal stress field is represented as a nonlocal field conjugate to the fundamental field of the infinitesimal strain measure. It can then be said that the equilibrium equation for a nonlocal elastic field can be written as:

$$\sigma_{ji,j}(x) + f_i(x) = \rho(x)\ddot{u}_i(x) \quad (4)$$

where $f_i(x)$ represents the body force and $\rho(x)$ denotes the mass density. In Eringen's nonlocal theory, the strain energy density depends both on the local strain and the diffused fractions of the strain, which depend on the attenuation function at neighboring points. Consequently, the local stress field is replaced with the corresponding nonlocal stress field. To get the Erignen's differential form, a linear differential operator for a Green's function type attenuation kernel is used which has the form of:

$l = (1-(e_0 a)^2\nabla^2)$, i.e., $lk(x-\overline{x}) = \delta(x-\overline{x})$ (5)

where ∇^2 represents the Laplace operator and δ is the Dirac delta function. Finally, the equilibrium equation for a nonlocal elastic continuum is expressed as:

$$\tau_{ji,j} + l\left(f_i - \rho\ddot{u}_i\right) = 0 \quad (6)$$

From Eq. (6), a set of singular differential equations are derived and are representative of the equilibrium equation using Eringen's differential nonlocal elasticity. Generally, this is a system whose solution exists and is easy to find. However, problems still exist in applying the natural boundary conditions of the system. Furthermore, the existence of solution does not guarantee a well-posed problem. In studies utilizing a differential form of nonlocal elasticity, it is important to apply boundary conditions that do not contradict the underlying physics of the system. At the same time, the boundary conditions must mathematically agree with the physical interpretation.

Determining the nonlocal parameter for various materials: a general nonlocal theory

After studying both Eringen's integral and differential nonlocal theories, Shaat and Abdelkefi (2017) made efforts to provide new insights on the limits of applicability of Eringen's nonlocal theory from a material science point of view. In doing so, a new general nonlocal theory was developed. In several works by Eringen related to nonlocal elasticity, a strong assumption was made for the rapid attenuation of interatomic interactions (Eringen 1972; Eringen 1983; Eringen 2002; Eringen 2006). In doing so, the same attenuation function was considered for all elastic moduli. Thus, all nonlocal material moduli are determined from the same attenuation function, multiplied by different local material moduli. This underlying assumption can lead to inaccurate results when considering materials whose longitudinal and transverse acoustic dispersions cannot simultaneously be fit with one nonlocal parameter. As such, two nonlocal parameters should be determined when these limits have been exceeded (Shaat and Abdelkefi 2017). To define the limits of Eringen's differential nonlocal theory and its use of only one nonlocal parameter, Shaat and Abdelkefi (2017) studied the ability to reflect the exact dispersion functions of different materials, as moduli. It was found that Eringen's nonlocal theory cannot simultaneously fit the dispersion curves of some materials, such as silicon, gold, and platinum. To address this issue, a general nonlocal theory was proposed that considers two different attenuation functions such that both the longitudinal and transverse (shear) acoustic dispersions are used to determine two nonlocal parameters. When using Eringen's assumption, the nonlocal material moduli are defined as:

$$k\left(\left|x-\overline{x}\right|\right)=\frac{\overline{\lambda}\left(\left|x-\overline{x}\right|\right)}{\lambda}+\frac{\overline{\mu}\left(\left|x-\overline{x}\right|\right)}{\mu} \tag{7}$$

Again, λ and μ are the conventional material Lamé constants, defined in classical continuum mechanics. $\overline{\lambda}(|x-\overline{x}|)$ and $\overline{\mu}(|x-\overline{x}|)$ are the modified material moduli for the nonlocal field. From here, for two material constants, two attenuations functions should be considered, leading to two different nonlocal parameters. In Eringen's nonlocal theory, it is known that the value of e_0 is found by matching the dispersions of the material. In cases when the limits of applicability of Eringen's nonlocal theory have been exceeded, two distinct attenuation functions should be considered. To replace Eq. (1), the general form is given as:

$$\sigma_{ij}\left(\overline{x}\right)=\int_V\left(\beta_1\left(\left|x-\overline{x}\right|\right)\lambda\varepsilon_{rr}(x)\delta_{ij}+\beta_2\left(\left|x-\overline{x}\right|\right)(2\mu)\varepsilon_{ij}(x)\right)dV \tag{8}$$

where $\beta_1(|x-\overline{x}|)$ and $\beta_2(|x-\overline{x}|)$ are Green's functions. Two differential operators are then derived as:

$$L_1\beta_1\left(\left|x-\overline{x}\right|\right)=\delta\left(\left|x-\overline{x}\right|\right) \text{ and } L_2\beta_2\left(\left|x-\overline{x}\right|\right)=\delta\left(\left|x-\overline{x}\right|\right) \tag{9}$$

The operators are given as:

$$L_1=\left(1-\varepsilon_1\nabla^2\right) \text{ and } L_2=\left(1-\varepsilon_2\nabla^2\right) \tag{10}$$

Then, by multiplying the nonlocal stress tensor by the two differential operators, the constitutive equations are obtained as:

$$\left(1-\varepsilon_1\nabla^2\right)\left(1-\varepsilon_2\nabla^2\right)\sigma_{ij}\left(\overline{x}\right)=\left(1-\varepsilon_2\nabla^2\right)\overline{\tau}_{ij}\left(\overline{x}\right)+\left(1-\varepsilon_1\nabla^2\right)\overline{\overline{\tau}}_{ij}\left(\overline{x}\right) \tag{11}$$

where $\overline{\tau_{ij}}=\lambda\varepsilon_{rr}\ \delta_{ij}$ and $\overline{\overline{\tau_{ij}}}=2\mu\varepsilon_{ij}$. Clearly, when $\varepsilon_1=\varepsilon_2$, the general nonlocal theory reduces to Eringen's nonlocal theory. To validate the proposed model, experimental dispersions were obtained from the literature for materials, such as diamond (Warren et al. 1965), silver (Bian et al. 2008), copper (Nilssom and Rolandson 1973), graphite (Jishi et al. 1993), gold (Lynn et al. 1973), silicon (Cochran 1973), and platinum (Dutton et al. 1972). Results showed that Eringen's nonlocal theory is valid for materials such as diamond, silver, and graphite, but fails for other materials. It should be mentioned that while this does not directly apply to the boundary condition paradox, the utilization of the general nonlocal theory will lead to more accurate identification of the nonlocal parameters. However, in considering this higher-order nonlocal model, physical and mathematical problems will arise. The number of required boundary conditions in this model will increase from four to six, and thus exceed the number of available natural boundary conditions. It should be noted that other efforts have also been made for the determination of the nonlocal parameter, particularly for CNTs and graphene sheets (Duan et al. 2007; Liang and Han 2014).

Eringen's nonlocal elasticity in shells, plates, and beams

The use of nonlocal elasticity theories has expanded significantly in recent years because of the extensive applications for the modeling of MEMS, NEMS, and CNTs. In particular, Eringen's nonlocal elasticity has extensively been applied to study the static and dynamic responses and behaviors of beams, shells, and plates. While the discrepancies and so-called paradox have most notably discussed in beams, it is important to note the extensive use of Eringen's nonlocal elasticity in other nanostructures. In each of the considered structures, the static bending, vibrations, buckling, and/or wave propagation were studied. Boundary conditions including clamped-clamped, clamped-hinged, hinged-hinged, and clamped-free were considered. For certain models, concentrated loads or distributed loads were also studied.

On the implementation of nonlocal elasticity to shells (CNTs), cones, plates, and graphene sheets

Most commonly, researchers have coupled the differential form of Eringen's nonlocal theory with Euler-Bernoulli beam theory (EBT) or Timoshenko beam theory (TBT) to study the static and dynamic responses of nanobeams and CNTs. While Eringen's nonlocal elasticity has been most extensively applied to these systems, it has also been applied to various shell and plate models. The shell and plate models are inherently more complex than beam models, due to their extra degrees of freedom. Thus, they were not studied extensively in the context of addressing the boundary conditions paradox for Eringen's nonlocal elasticity. However, they are discussed briefly in this review for the sake of demonstrating the vast use of Eringen's nonlocal theory.

Several works have been performed to study the wave propagation, vibrations, and buckling behaviors of the structures, most often CNTs, using shell models. These studies are outlined in Table 2. It should be noted that few, if any, researchers address the apparent paradox for shell models. Because of their increased complexity, it does not seem as though researchers considered the same boundary conditions that cause the issues discussed in the beam models. Thus, a paradox is not observed, and for the considered shell models, the increasing nonlocal parameter consistently led to a softening in the material, i.e., a decrease in the fundamental natural frequency, an increase deflection in the bending solutions, or a decrease in the critical buckling loads of the structure.

Shells are not the only complex structures that have been studied using Eringen's nonlocal elasticity. The differential form of Eringen's nonlocal theory has been used to model plates and graphene sheets. These works are summarized in Table 3. Several interesting works exist in the modeling of the nanoscale plates. The novelty in these works lies in the consideration of various types of plate models and different solution procedures for the governing equations of motion. Depending on the properties, size, and structures of the plates, models, such as the Kirchhoff's plate theory (KPT), classical plate theory (CPT), first-order plate theory (FOPT), Mindlin's plate theory (MPT), and higher-order plate theories (HOPT) have been considered in the modeling of nanoplates. As with the shell models, the differential form of Eringen's nonlocal elasticity was most commonly implemented. Boundary conditions including HHHH (all sides hinged), CCCC (all sides clamped), CCHH (two sides hinged, two sides clamped), and CCFF (two sides clamped, two sides free) were investigated. It is interesting to note that in the models of the plates with one or two free edge boundary conditions, the softening behavior was still consistently observed, unlike the paradoxical cantilever beam.

For the convenience of the reader, Table 4 is provided to summarize Eringen's nonlocal elasticity in shells and plates. The research efforts are categorized by the types of analyses performed, including wave propagation, vibrations, bending, and buckling of shells and plates.

Eringen's nonlocal elasticity in beam models: the so-called boundary condition paradox

The coupling of Eringen's differential nonlocal theory with Euler-Bernoulli beam theory or Timoshenko beam theory yields the simplest models for integrating Eringen's nonlocal elasticity. Due to its simplicity, the differential constitutive model has received great attention and has been used in dislocation mechanics, composite materials, damage and fracture mechanics, and other related fields. Most often, the differential form of Eringen's nonlocal elasticity has been used to study the static and dynamic responses of beams and CNTs considering small-scale effects (Ghavanloo and Fazelzadeh 2014; Sudak 2003; Wang 2005; Wang and Varadan 2006; Wang et al. 2006; Reddy 2007; Heireche et al. 2008; Murmu and Pradhan 2009a, b; Amara et al. 2010; Kiani 2013; Zhu and Dai 2012; Wang and Li 2014; Şimşek 2014; Zhang 2017; Thai 2012; Arash and Wang 2014). In addition, several reviews have been published regarding nonlocal elasticity in beams and CNTs and contain valuable and useful references therein (Arash and Wang 2012; Arash and Wang 2014; Askari et al. 2017; Eltaher et al. 2016).

While the above research efforts did not explicitly address the ill-posed nature of Eringen's nonlocal elasticity and the paradox, it is important to include them for the convenience of the reader. However, the goals of this effort are to highlight the discrepancies that arise for varying boundary conditions in nonlocal elasticity and to create a timeline of the progression of the proposed solution procedures. In addition, researchers who have addressed the underlying causes of the paradox have been included, most notably specifying the

Table 2 A summary of nonlocal elasticity in nanoshells and nanocones

Ref.	Keywords	Approach and findings
Zhang et al. (2004)	- Differential - Shell - H-H - Buckling	Developed a cylindrical *shell* model for the *axial buckling* of MWCNTs with *H-H boundary conditions*. They found *softening* in the beam when increasing the nonlocal parameter using the *differential* form.
Wang (2006)	- Differential - Flugge shell - Wave propagation	Proposed a nonlocal elastic *shell* model to study the small-scale effect in axisymmetric *wave propagation* in CNTs. Two coupled radial and longitudinal modes and one decoupled torsional mode were developed from the model.
Wang and Varadan (2007)	- Differential - Shell - Wave propagation - Vibrations	Investigated the small-scale effect of *wave dispersion* relations for different CNT wavenumbers in the longitudinal and circumferential directions. They were able to demonstrate the potential of a nonlocal shell theory to study the vibrations and phonon dispersion relations of CNTs.
Hu et al. (2008)	- Differential - Shell - Wave propagation	Investigated the transverse and torsional *wave propagations* in SWCNTs and DWCNTs and found that the *nonlocal elastic shell theory provided a better prediction of the dispersion relationships* than the classical shell theory for large enough wavenumbers.
Yang and Lim (2011)	- Differential - Shell - Wave propagation	Proposed an analytical nonlocal *shell* model to investigate the axisymmetric *wave propagation in CNTs*. They claim that their results confirm that the analytical shell model can predict the *stiffness* of nonlocal CNTs.
Ansari and Arash (2013)	- Differential - Flugge shell - H-H - Vibrations	Studied the *vibrations* of DWCNTs, modeled as *Flugge shells* and considering the *differential* form of Eringen's nonlocal theory. They applied different boundary conditions to the inner and outer tubes with overall *H-H boundary conditions.* They observed the *softening behavior.*
Fotouhi et al. (2013)	- Differential - Nanocone - H-H - Vibrations	Applied the *thin shell theory* and the *differential* form of Eringen's nonlocal elasticity to study the *vibrations* of nanocones with *H-H boundary conditions*. They observed the *softening behavior*.
Ghavanloo and Fazelzadeh 2014	- Differential - Shell - Vibrations - Legendre polynomials	Investigated the *axisymmetric vibrations* of spherical *shell-like nanostructures* using the *differential* form of Eringen's nonlocal elasticity. They presented a new prediction formula for the axisymmetric vibration of nanospherical membrane shell by employing Legendre and *Legendre polynomials*.
Ansari and Torabi (2016)	- Differential - Nanocone - Vibrations - Variational DQM - Softening	Employed the *variational DQM* to study the free *vibrations* of carbon nanocones (CNCs) embedded in an elastic foundation based on the *differential* form of Eringen's nonlocal elasticity. They investigated the effects of the nonlocal parameter, boundary conditions, semi-apex angle, and Winkler and Pasternak coefficients on the vibrational behavior and observed the *softening* behavior.

Table 3 A summary of nonlocal elasticity in nanoplates and graphene sheets

Ref.	Keywords	Approach and findings
Pradhan and Phadikar (2009)	- Differential - CPT - Graphene sheet - Vibrations	Used Navier's approach to study the small-scale effects of *vibration* on multilayer graphene sheets embedded in a matrix modeled using *differential* Eringen's non-local elasticity and *classical plate theory (CPT)*. They found that the nonlocal parameter has a significant effect on plate behavior.
Reddy (2010)	- Differential - EOMs - EBT, TBT, KPT, CPT, FOPT, HOPT, MPT	Derived the *differential* nonlocal equations of motion for *nanoplates* without considering specific *boundary conditions*. They also considered EBT, TBT, Kirchhoff's plate theory, classical plate theory, first-order plate theory, higher-order plate theory, and Mindlin plate theory .
Wang et al. (2011)	- Differential - Elastic plate theory - Graphene sheets - Vibrations - HHHH - Softening	Studied the mechanisms of the nonlocal effect on the *transverse vibrations* of 2D nanoplates using the *differential* form and the *elastic plate theory*. They found that the nonlocal effect stems from the distributed transverse force due to the curvature change and surface stress from the atom-atom interactions. For all *hinged boundary conditions*, they found nonlocal nanoplates to be *softer* than local ones.
Pradhan and Kumar (2011)	- Differential - Orthotropic SLGS- CPT - DQM - Vibrations - Softening	Performed a *vibration analysis* of orthotropic single-layered graphene sheets using *differential* nonlocal elasticity and classical plate theory. They varied the *boundary conditions*, considering different combinations of boundary conditions and found a *decrease* in the natural frequency for increasing nonlocal parameter.
Civalek and Akgöz (2013)	-Differential - Thin Plate Theory -Winkler-Pasternak - Vibrations - Softening	Performed a free *vibration analysis* of micro-scaled annular sector and sector shaped graphene on an elastic matrix using *differential* nonlocal elasticity. The elastic matrix was modeled via the *Winkler-Pasternak* elastic foundations. They utilized the *thin plate theory* and employed the discrete singular convolution method for numerical solutions.
Hosseini-Hashemi et al. (2013)	- Differential - Mindlin's plate theory - Vibration - Softening	Used the *differential* form of Eringen's nonlocal theory into *Mindlin plate theory* to consider the small-scale effects on the *free vibration* of rectangular nanoplates. They applied mixed *Levy-type boundary conditions* and found a *decrease* in the natural frequency for all boundary conditions and increasing values of the nonlocal parameter.
Liu et al. (2013)	- Differential - Kirchhoff's Plate Theory - HHHH - Vibrations - Softening	Studied the thermos-electro-mechanical *free vibration* of piezoelectric *nanoplates* using the *differential* nonlocal elasticity and Kirchhoff's plate theory. They assumed all simply supported boundary conditions, a biaxial force, external electric voltage, and a uniform temperature change. They found a *decrease* in the natural frequency for increasing values of the nonlocal parameter.
Nami and Janghorban (2014)	- Differential - Strain gradient theory - Kirchhoff's plate theory - HHHH - Vibrations	Investigated the *resonance behaviors* of functionally graded *micro-/nanoplates* using both the *differential* nonlocal elasticity and strain gradient theory on Kirchhoff's plate theory. *Simply supported* boundary conditions were considered and the effects of the gradient parameter, aspect ratio, and nonlocal parameter were studied.
Daneshmehr and Rajabpoor (2014)	- Differential - FG nanoplates - DQM - Buckling - Softening	Presented a higher-order plate theory (HOPT) for the *buckling* analysis of *FG nanoplates*. They implemented the generalized *DQM* to solve the buckling analysis. Their results were compared with those of classical plate theory and first-order shear deformation plate theory.

Table 3 (continued)

Ref.	Keywords	Approach and findings
Zang et al. (2014)	- Differential - Nanoplate - Surface effects - Wave propagation	Considered *surface effects* and differential nonlocal elasticity to study the *wave propagation* of *nanoplates* and found *softening* without addressing specific boundary conditions.
Jung and Han (2014)	- Differential - HSDT - Vibrations - Softening	Investigated the small-scale effect on the transient analysis of nanoscale *plates*. Applied the *differential* form of Eringen's nonlocal elasticity and a higher-order shear deformation plate theory. They concluded that the results of the nanoscale plate can be used as a benchmark test for transient analysis of the *dynamic response*.
Liang and Han (2014)	- Differential - MD simulations - Bending - Scaling parameter	Presented a model for the explicit expression of the *nonlocal scaling parameter* and obtained the exact closed form solution for the nonlocal scaling parameter for zigzag and armchair graphene sheets with *CCHH boundary conditions.* They verified their model with MD simulations.
Ke et al. (2015)	- Differential - MPT - Vibration - Softening	Investigated the thermos-electro-mechanical *vibration* of the rectangular piezoelectric nanoplate with various *boundary conditions*. They employed *Mindlin's plate theory* and the *differential* form of Eringen's nonlocal elasticity and examined the effects of the nonlocal parameter, boundary conditions, and aspect ratio on the natural frequencies and mode shapes of the nanoplate.
Pilafkan et al. (2017)	- Differential - CPT - DQM - Buckling - Softening	Studied the *biaxial buckling* behavior of a *SLGS* using the *differential* form of Eringen's nonlocal elasticity, *classical plate theory*, and a generalized *DQM*. By considering three types of *boundary conditions*, namely *HHHH*, *CCCC*, *HHCC*, they found a *decrease* in the critical buckling load for increasing values of the nonlocal parameter.
Faroughi et al. (2017)	- Two-phase - Bending - Free vibration - Finite element	Presented the Ritz formulation, for the *two-phase* integro-differential form of Eringen's nonlocal elasticity. The formulations were applied to study the *static bending* and the *free vibrations* of the Kirchoff plate model.
Sari et al. (2018)	- Differential - Kirchhoff's plate theory - FGM - Buckling - Softening	Studied the *buckling* behavior of *functionally graded* nanoplates with varying boundary conditions. They used *Kirchhoff's plate theory* and the *differential* form of Eringen's nonlocal elasticity and found a *decrease* in the critical buckling load for increasing values of the nonlocal parameter with all boundary conditions.

disagreement between the constitutive and equilibrium equations and the existence of a solution for Eringen's integral model. It should be mentioned that methods used by researchers include variations of the differential model by the derivation of variationally consistent boundary conditions and potential and kinetic energy approaches, a two-phase model, a stress-driven model, a gradient model, and iterative and finite element–based solutions of the integral model. A summary of nonlocal elasticity in beams when addressing the boundary condition paradox is presented in Table 5.

Paradoxes in nonlocal elasticity: existing proposed methods and solutions

After presenting a brief review of Eringen's nonlocal theory in beams, shells, and plates, the boundary condition paradox should be deeply discussed. As shown in Table 5, there are researchers who observed the paradox and researchers who proposed solutions to the paradox. Note that the proposed solutions are approached from different points of view: the goal of some researchers is to address the ill-posedness of Eringen's nonlocal elasticity, while others attempt to reach results of consistent

Table 4 Analysis and categorization of Eringen's nonlocal elasticity in shells and plates

Analysis	Shells and cones	Plates and graphene sheets
Wave propagation	- Wang et al. (2006) - Wang and Varadan (2007) - Hu et al. (2008) - Yang and Lim (2011)	- Zang et al. (2014)
Vibrations	- Wang and Varadan (2007) - Ansari and Arash (2013) - Fotouhi et al. (2013) - Ghavanloo and Fazelzadeh (2014) - Ansari and Torabi (2016)	- Pradhan and Phadikar (2009) - Wang et al. (2011) - Pradhan and Kumar (2011) - Civalek and Akgöz (2013) - Hosseini-Hashemi et al. (2013) - Liu et al. (2013) - Nami and Janghorban (2014) - Jung and Han (2014) - Ke et al. (2015)
Buckling	- Zhang et al. (2004)	- Daneshmehr and Rajabpoor (2014) - Pilafkan et al. (2017) - Sari et al. (2018)
Bending	–	- Reddy (2010) - Liang and Han (2014)

softening with the increasing of the nonlocal parameter. Furthermore, there are several subcategories of methods for addressing the paradox, but they can be broken up into two main groups, namely, integral methods and differential methods.

Before tackling the solution procedures, it should be noted that there have been several efforts focused on discussing nonlocal elasticity in general and different formulations of nonlocal elasticity in beam problems including (Lim 2010; Challamel et al. 2014; Demir and Civalek 2017; Romano et al. 2018; Oskouie et al. 2018).

Finite element and iterative procedures

The first class of solution procedures used to address paradoxes arising from Eringen's nonlocal elasticity is developed using the integral form. Several researchers have attempted to solve the integral by finite element–based or iterative procedures (Koutsoumaris et al. 2017; Polizzotto 2001; Tuna and Kirca 2017a, 2017b; Barretta and Marotti de Sciarra 2015; Norouzzadeh and Ansari 2017; Norouzzadeh et al. 2017). Other researchers have attempted to rigorously transform the integral formulation to the differential one with boundary conditions that admit the problem to be well-posed (Fernández-Sáez et al. 2016).

For example, in one work (Fernández-Sáez et al. 2016), they claimed that for a general loading case, the solution of the integral model can be found by adding the solutions of an associated differential problem and of two integral equations. Furthermore, they concluded that for a given value of the nonlocal parameter, the solutions of the integral equations may be stated in terms of two canonical functions and that the solution is valid for any loading case. Moreover, they reported that the integral and differential forms of Eringen's model are the same if they satisfy the mathematical condition given by (Polyanin and Manzhirov 2008). In that case, the integral model was claimed to be properly transformed to the differential counterpart. Applying these techniques, they were able to show that the cantilever paradox was "resolved" by using the integral formulation of Eringen's nonlocal elasticity. To be clear, solving the nonlocal integral model based on an exponential kernel function is still ill-posed.

Laplace transform of the integral model

In an effort to solve Eringen's integral model, a few researchers applied the Laplace transform to the

Table 5 A summary of nonlocal elasticity in nanobeams addressing the boundary condition paradox

Ref.	Keywords	Approach and findings
Peddieson et al. (2003)	- Differential - EBT - C-F - Bending - Concentrated load	Utilized the *differential* form of Eringen with *EBT* to study the *deflection* of *cantilever with concentrated load*. The results show that the nonlocal findings are *equal* to the local ones for these loading and boundary conditions.
Wang et al. (2007)	- Differential - TBT - Cantilever - Vibration	Studied the *free vibration* of nanobeams using *TBT* and the *differential* form of Eringen's nonlocal elasticity. They observed the *hardening* paradox on the *cantilever* beam.
Wang and Liew (2007)	- Differential - EBT, TBT - Bending	Studied the scale effect on the *static deformation* of nanobeams using *EBT* and *TBT* and the *differential* form of Eringen's nonlocal elasticity. They observed the paradox of the nonlocal effect disappearing for certain concentrated forces.
Challamel and Wang 2008	- Gradient - EBT - Vibration - Regularizing parameter	Overcame the boundary condition paradox with a *gradient* elastic model that combines the local and nonlocal curvatures in the constitutive elastic relation. This model contains two small-scale parameters. At a critical value, a transition is made from the softening to hardening behavior.
Pisano et al. (2009)	- Two-phase - FEM	Implemented a nonlocal finite element, theorized by Polizzotto [65], similar to a two-phase model. The goal of the effort was to inquire about the computational issues to establish a basis for further developments.
Zhang et al. (2009)	- Differential - MD simulations - TBT - C-F - Vibrations	Studied the flexural *vibration* of SWCNTs through *TBT*. They used Eringen's *differential* nonlocal elasticity to perform a vibration analysis of SWCNTs compared to molecular dynamics simulations based on second-generation reactive empirical bond order potential. They observed the *hardening paradox* of the *cantilever beam*.
Murmu and Pradhan (2009)	- Differential - C-F - Vibrations	Employed the differential form of Eringen's nonlocal elasticity to study the vibrations of nanocantilever beams. They found that the softening/hardening phenomena depended on the aspect ratio of the beam.
Zhang et al. (2010)	- Hybrid/gradient - Curvature - EBT - H-H, C-C, - C-F - Bending, buckling, vibrations	Applied the *hybrid/gradient nonlocal beam model* to the vibration, bending, and buckling of *EBT* beams with *H-H*, *C-C*, and *C-F boundary conditions*. By including the nonlocal and local *curvatures* in the strain energy functional, they claim that the hybrid nonlocal beam theory could *overcome the paradoxes* of Eringen's nonlocal theory.
Yang and Lim (2012)	- Differential - TBT - Nonlocal BCs - Buckling - Thermal effects	Investigated the *thermal buckling* of nanocolumns using *TBT* and *differential* nonlocal elasticity. They considered higher-order *nonlocal boundary conditions* and found that the buckling load increases with increasing nanoscale effects, i.e., they found *hardening behavior*.
Zhu and Dai (2012)	- Neumann's series - Two-phase - Bar - Tension	Utilized Neumann's series as an *iterative method* to solve the *two-phase* integral for a nonlocal elastic bar in *tension*.

Table 5 (continued)

Ref.	Keywords	Approach and findings
Marotti de Sciarra (2014)	- Integral FEM - EBT - Bending - Concentrated load	Consistently developed a high-order nonlocal EBT to investigate the bending of beams with various boundary and loading conditions. They show that they overcame the paradox of the cantilever nanobeam with the intermediate applied force by demonstrating that the nonlocal effect was present both to the left and the right of the applied force.
Khodabakshi and Reddy (2015)	- Two-phase - EBT - Bending, vibration - C-F, H-H	Applied the *two-phase* model to investigate the boundary condition *paradox*. They found *softening* for all boundary and loading conditions except for a beam with H-H boundary conditions and a uniformly applied load. In that case, they found hardening. It is possible that the hardening can be attributed to a numerical issue.
Wang et al. (2016)	- Two-phase - EBT - Cantilever - Bending	Performed an analytical study to analyze the static bending of nonlocal EBT using the two-phase local/nonlocal model. They perform a rigorous reduction method to transform the integral form to a differential form with mixed boundary conditions. Exact solutions were obtained for all sets of boundary conditions, namely C-C, C-F, C-H, H-H, C-FP and they found consistent softening.
Xu et al. (2016)	- WRA - TBT, EBT - Nonlocal BCs - C-F - Vibrations - Buckling	Attempted to address the *cantilever paradox* by applying a weighted residual approach (*WRA*) using *EBT* and *TBT* to determine the nonclassical force resultants *and nonclassical boundary conditions*. They observed the *softening* phenomenon when studying the *vibrations* and *buckling* of the nanostructures.
Eptaimeros et al. (2016)	- Two-phase - FEM - Cantilever	Applied the two-phase local/nonlocal model to study nanbeams with various boundary conditions. They overcame the hardening paradox of the cantilever beam.
Tuna and Kirca (2016a, b)	- Laplace - Bending, buckling, vibration - Softening	Applied the Laplace transform to Eringen's integral model to obtain exact solutions for the bending, buckling, and vibrations of nanobeams. Their model was criticized and discussed for its mathematical formulations.
Fernández-Sáez et al. (2016)	- Integral - EBT - Bending	For a general loading case, they obtained the solution of the integral model by adding those of an associated differential problem and two integral equations. They were able to rigorously transform the integral to the differential with considering mixed boundary conditions.
Romano and Barretta 2017a, b	- Stress-driven - Bending - BCs - Hardening	Developed the stress-driven model to overcome the ill-posedness of Eringen's strain-driven integral elasticity. They investigated the classes of beam problems that have shown paradoxes, including H-H with a uniform load, cantilever with end point load, and the C-C beam with a uniform load. They found that though their model is said to be well-posed, increasing the nonlocal parameter leads to an increase in the stiffness of the structure.
Zhu et al. (2017)	- Two-phase - EBT - Reduction method - Mixed BCs - Buckling - Softening	Used a *reduction method* to reduce integro-differential equation into a fourth order differential equation with mixed boundary conditions for the *two-phase* integral. They found *softening* when increasing the nonlocal parameter for *all boundary conditions*. They also found that they nonlocal effect can be first-order or second-order depending on the boundary conditions.
Xu et al. (2017)	- Nonlocal strain gradient - EBT - WRA - C-C, C-H, C-F	Investigated the size effects on rods with *nonlocal strain gradient* elasticity theory. They derived variationally consistent boundary conditions using the weighted residual approach. In their model, they have two material length parameters can show either stiffening or softening for three sets of boundary conditions.

Table 5 (continued)

Ref.	Keywords	Approach and findings
Tuna and Kirca 2017a, 2017b	- Integral - FEM - EBT - Bending, buckling, vibration	Performed a finite element–based procedure to solve Eringen's integral nonlocal elasticity using EBT. They studied the bending, buckling, and vibrations of nanobeams and found that increasing the nonlocal parameter leads to the softening phenomena for all boundary conditions.
Fernández-Sáez and Zaera (2017)	- Two-phase - EBT - Vibrations	Utilized the two-phase model to address the boundary condition paradox. They used EBT and studied the vibrations. Their results showed that they overcame the boundary condition paradox.
Barretta et al. (2018)	- Stress-driven - Vibration - Hardening	Applied the stress-driven model to study the vibrations of nanorods and found that for all boundary conditions, increasing the nonlocal parameter leads to an increase in the stiffness of the nanostructure.
Faroughi et al. (2020)	- General nonlocal - Reddy beam - Wave propagation	Studied the *wave propagation* of two-dimensional functionally graded nanobeams using the *general nonlocal theory* and *Reddy's beam model.*
Apuzzo et al. (2020)	- Two-phase - Free vibration	Defined a *two-phase* model through convex combination of local/nonlocal phases through a mixture parameter and determined the closed-form solution for *free vibrations* of a nanobeam.
Cao and Niu (2020)	- Buckling - Shear - Unified	Presented a new exact analytical solution of buckling of sandwich beams. They considered the shear deformation of the face-sheet for the first time in their theoretical framework.
Rahmani et al. (2020)	- General nonlocal - Reddy beam - Vibration	Performed a comprehensive vibrational analysis of bi-directional functionally graded rotated nanobeam. Modeled the beam using the *general nonlocal theory* which depends on two nonlocal parameters. The results reveal both *softening* and *hardening* depending on the material.
Ceballes and Abdelkefi (2020)	- General nonlocal - EBT - Buckling - Vibration	Applied the *general nonlocal theory* to an axially loaded nanobeam and determined logical boundary conditions using *weighted residual approach*. Determined the critical buckling loads and the pre- and post-buckling dynamic response for clamped-clamped, clamped--hinged, and hinged-hinged boundary conditions.

equations of motion for the nanobeam. Özgür Yayli and Yerel Kandemir (2017) performed a bending analysis of a cantilever with end forces by Laplace transform of the integral model. Most notably though, Tuna and Kirca (2016a, b) aimed to derive the closed-form analytical solutions for the bending, vibrations, and buckling of

Table 6 Validation of Tuna and Kirca (2016b)'s Laplace transform model with Wang et al. (2016)'s two-phase model

	$\frac{e_0 a}{L} = 0.05$		$\frac{e_0 a}{L} = 1.0$		$\frac{e_0 a}{L} = 10.0$	
Reference	Wang et al. (2016)	Tuna and Kirca (2016b)	Wang et al. (2016)	Tuna and Kirca (2016b)	Wang et al. (2016)	Tuna and Kirca (2016b)
SSSD1	0.01333	0.01333	0.13802	0.13802	12.513	12.513
SSSD2	0.04290	0.04290	0.54170	0.54170	50.041	50.041
C-FP	0.38583	0.38583	2.33333	2.33333	110.3333	110.3333

Table 7 Comparison of the model developed by Xu et al. (2016) (1) with the first three natural frequencies of a cantilever beam presented by Lu et al. (2006) (2), Tuna and Kirca (2016b) (3), and Challamel et al. (2014) (4)

	$\frac{e_0a}{L}=0$				$\frac{e_0a}{L}=0.1$			$\frac{e_0a}{L}=0.2$				
Ref.	1	2	3	4	1	2	3	4	1	2	3	4
1st	1.8751	1.8751	1.8751	1.8751	1.8792	1.7084	1.8530	1.8539	1.8919	1.5759	1.7958	1.7961
2nd	4.6941	4.6940	4.6940	4.6940	4.5474	4.1779	4.3688	4.3745	4.1924	3.7038	3.8117	3.8178
3rd	7.8548	7.8548	7.8548	7.8548	7.1459	6.6080	6.8222	6.8186	6.0674	5.5470	5.6076	5.6019

Euler-Bernoulli and Timoshenko beams with various loading and boundary conditions. To solve the integral, the Fredholm type governing equations for all systems were transformed to Volterra integral equations of the second kind. Next, the Laplace transform was applied to the transformed Volterra equations. The inverse Laplace transform was used to determine the static and dynamic responses of the nanobeams.

For reference, the deflections of cantilever beams with end point loading (Eq. 12), cantilever beams with uniformly distributed loading (Eq. 13), and simply supported beams with uniformly distributed loading (Eq. 14) are given as (Tuna and Kirca 2016a):

$$u_z^E(x)=\frac{P}{EI}\left(-\frac{x^3}{6}+\frac{Lx^2}{2}\right)+\frac{Pe_0a}{EI}(e_0ax+Lx) \quad (12)$$

$$u_z^E(x)=\frac{q}{EI}\left(\frac{x^4}{24}-\frac{Lx^3}{6}+\frac{L^2x^2}{4}\right)+\frac{qe_0a}{2EI}\left(-e_0ax^2+2e_0aLx+L^2x\right) \quad (13)$$

$$u_z^E(x)=\frac{q}{EI}\left(\frac{x^4}{24}-\frac{Lx^3}{12}+\frac{L^3x}{24}\right)+\frac{q(e_0a)^2}{2EI}\left(-x^2+Lx\right) \quad (14)$$

In their work, they found that applying the Laplace transform technique to the integral model produced a softening behavior for all cases, as predicted in the original form of Eringen's integral model. They validated their model with Wang et al. (2016) through presenting the deflection of the midpoint of the simply supported beam with a uniformly distributed load (SSSD1), the rotation of the endpoint of the simply supported beam with a uniformly distributed load (SSSD2), and the deflection of the endpoint of a cantilever beam with a concentrated load (C-FP). The validation is shown in Table 6.

While the results are consistent with the work of Wang et al. (2016), Romano and Barretta 2016have commented on the validity of the Laplace transform model. It is shown that in the cantilever case with the uniformly distributed loading, the slope field generated from the deflection equation yields a nonphysical, nonzero deflection at the clamped end of the beam. The same can be said regarding the bending moment for the simply supported beam with uniformly distributed loading. Tuna and Kirca (2017a, b) countered the argument of Romano and Barretta (2016) by stating that their model is valid only when $x > 0$, and thus showed the generated slope field and bending moment at $x = 0$ need not satisfy the physical conditions.

Weighted residual approach of the differential model

In the work of Xu et al. (2016), they aimed to solve the free vibration cantilever hardening by considering a modified version of the differential constitutive equations rather than the integral model. Starting from the differential form of Eringen's nonlocal theory, they applied the weighted residual approach to the nonlocal equations of motion established by Reddy and Pang (2008). It should be noted that the weighted residual approach was used by Zhang et al. (2013) to model nonlocal simply supported Timoshenko beams. After using the nonclassical shear and moment equations of Reddy and Pang (2008), they applied the Hamilton's principle to find the kinetic and potential energies of the system. The boundary conditions that were found using the weighted residual approach differ from those used by Reddy and Pang (2008) in that the inertial term vanishes from the moment equation.

They validated their results with several other researchers, as shown in Table 7.

Two-phase model for Eringen's nonlocal elasticity

The two-phase model for Eringen's nonlocal elasticity was originally proposed by Eringen (1972, 1987). Since then, several researchers have utilized this model as it overcomes the ill-posedness of Eringen's original integral formulations (Khodabakhshi and Reddy 2015; Eptaimeros et al. 2016; Altan 1989; Benvenuti and Simone 2013). This model is given by:

$$\sigma_{ij}(x) = \xi_1 \tau_{ij}(x) + \xi_2 \int_V k\left(\left|x-\bar{x}\right|, \kappa\right) \tau_{ij}\left(\bar{x}\right) dV \quad (15)$$

where ξ_1 and ξ_2 are two nonnegative parameters that represent the volume fraction of the local and nonlocal constituents of the material or structure, respectively, whose sum must be equal to one. It should be noted that the case of $\xi_2 = 0$ corresponds to a pure local model and the case of $\xi_2 = 1$ corresponds to the fully nonlocal integral model. In the latter case, it can be shown that Eringen (1983)'s original integral model is recovered. Then, it is important to consider the linear integral equation of the second kind, and the boundary conditions that will lead to the well-posed solutions as:

$$y(x) + A \int_a^b e^{\lambda(x-s)} y(s) ds = f(x) \quad (16)$$

$$y''(x) + \lambda(2A-\lambda) y(x) = f''(x) - \lambda^2 f(x) \quad (17)$$

$$y'(a) + \lambda y(a) = f'(a) + \lambda f(a) \quad (18)$$

$$y'(b) - \lambda y(b) = f'(b) - \lambda f(b) \quad (19)$$

It should be noted that the two-phase local/nonlocal model does admit a well-posed solution but tends towards an ill-posed formulation as the local constituent in the model approaches 0. Moreover, any techniques that have been applied to solve the original integral formulation should also be applied to solve the nonlocal part of the two-phase model. For future works, it would be interesting to transform the two-phase local/nonlocal integral model to a differential one with variationally consistent boundary conditions to see if it is possible to reduce computational time while maintaining the integrity of the model.

Hybrid nonlocal elasticity/gradient model

In the work of Zhang et al. (2010), the hybrid nonlocal approach was applied to EBT to study the vibrations, buckling, and bending of nanobeams. To overcome the vanishing nonlocal effect of the cantilever nanobeam with a concentrated load and the stiffening effect of Eringen's nonlocal theory on certain classes of beam bending problems, the researchers used a hybrid nonlocal beam model, originally proposed by Challamel and Wang (2008) and has been discussed again in Challamel et al. (2016). In this model, both the local and nonlocal curvatures are included in the strain energy functional to ensure the nonlocal parameter does not vanish from the deflection expression. It should be noted that the hybrid nonlocal model is also a two-phase nonlocal model with a modified exponential kernel.

The resulting equations of motion for the hybrid/gradient nonlocal model are of 6th order and thus require two additional nonclassical boundary conditions, namely, that the slope of the nonlocal curvatures must be equal to zero at both ends of the beam. To illustrate, the equations of motion for the Euler-Bernoulli beam are presented as (Zhang et al. 2010):

$$EI(\alpha l_c)^2 \frac{d^6 w}{dx^6} - \left(EI - l_c^2 P\right) \frac{d^4 w}{dx^4} - \left(l_c^2 \rho A \omega^2 + P\right) \frac{d^2 w}{dx^2} + \rho A \omega^2 w + q = 0 \quad (20)$$

Table 8 The stress-driven model and strain gradient model for increasing nonlocal parameters

$\frac{e_0 a}{l}$	Mode	Stress-driven model	Strain gradient model
0	I	1	1
	II	1	1
0.05	I	1.05795	1.00555
	II	1.10594	1.04988
0.1	I	1.13089	1.01972
	II	1.31355	1.17707

Table 9 A summary of the bending moments and shear forces for different proposed differential solutions to the boundary condition paradox

	Bending moments (M)	Shear forces (V)
Lu et al. (2006)	$-EIw'' + m_0(e_oa)^2\ddot{w} + P(e_oa)^2w''$	$-EIw''' + m_0(e_oa)^2\ddot{w}' + P(e_oa)^2w'''$
Reddy (2007)	$-EIw'' + m_0(e_oa)^2\ddot{w} + P(e_oa)^2w''$	$-EIw''' + m_0(e_oa)^2\ddot{w}' + P(e_oa)^2w'''-Pw'$
Xu et al. (2016)	$-EIw''+P(e_oa)^2w''$	$-EIw''' + m_0(e_oa)^2\ddot{w}' + P(e_oa)^2w'''-Pw'$
Challamel and Wang 2008	$EI\left(w''-(\alpha l_c)^2w^{(iv)}\right) + l_c^2\left(m_0\omega^2w + q-Pw''\right)$	$EI\left(w'''-(\alpha l_c)^2w^{(v)}\right) + l_c^2\left(m_0\omega^2w' + q-Pw'''\right)$
Barretta et al. (2016)	$EIw''-\eta\alpha(EIw'''-c\alpha q+\eta^2q')+\eta^2q$	N/A

$$M = EI\left(\frac{d^2w}{dx^2}-(\alpha l_c)^2\frac{d^4w}{dx^4}\right) + l_c^2\left(\rho A\omega^2w + q-P\frac{d^2w}{dx^2}\right) \quad (21)$$

$$Q = EI\left(\frac{d^3w}{dx^3}-(\alpha l_c)^2\frac{d^5w}{dx^5}\right) + l_c^2\left(\rho A\omega^2\frac{dw}{dx}-P\frac{d^3w}{dx^3}\right) \quad (22)$$

where l_c is the nonlocal length scale parameter, α is the nondimensional regularizing factor, q is the applied axial load, and σ_0 is the initially supplied compressive stress. It should be noted that if $\alpha = 0$ then the Eringen's nonlocal theory is recovered and $\alpha = 1$ corresponds to the classical local elasticity theory.

The hybrid nonlocal model produces the expected, nonparadoxical, softening results for cantilever beams with a uniformly distributed load, for $0 < \alpha < 1$. However, if α is taken to be greater than 1, hardening behavior is observed. This phenomenon was also observed for the case of a clamped-clamped beam subjected to a uniformly distributed load. This resolves another paradox that arises from the use of Eringen's nonlocal theory. It was observed by Wang et al. (2008) that, when using the Eringen's differential model, the deflection of a clamped-clamped beam under a uniformly distributed load is the same as the deflection predicted with local elasticity theories.

Stress-driven integral model

Recently, the stress-driven integral elasticity was proposed as a solution to Eringen's strain-driven integral elasticity by Romano and Barretta (2017a, b). Also, free vibration analysis of Euler-Bernoulli nanobeams was performed by Apuzzo et al. (2017) who used the novel stress-driven nonlocal integral model. Then, the

Table 10 A summary of stress-driven and strain-driven integral and differential forms of Eringen's nonlocal elasticity

Model	Formulation
Eringen's integral nonlocal elasticity: strain-driven model (Eringen 1983)	$\sigma_{ij}(x) = \int_V k(\lvert x-x\rvert, \kappa)\tau_{ij}(x)dV$
Modified Eringen's integral nonlocal elasticity: stress-driven model (Romano and Barretta 2017b)	$\tau_{ij}(x) = \int_V k(\lvert x-x\rvert, \kappa)\sigma_{ij}(x)dV$
Two-phase/hybrid integral nonlocal elasticity: strain-driven model (Wang et al. 2016)	$\sigma_{ij}(x) = \xi_1\tau_{ij}(x) + \xi_2\int_V k(\lvert x-x\rvert, \kappa)\tau_{ij}(x)dV$
Eringen's differential nonlocal elasticity: strain-driven model (Eringen 1983)	$(1-(e_0a)^2\nabla^2)\sigma_{ij}=\tau_{ij}$
Two-phase/hybrid differential nonlocal elasticity (Zhang et al. 2010)	$(1-(l_c)^2\nabla^2)\sigma_{ij}=(1-(\alpha l_c)^2\nabla^2)\tau_{ij}$

Table 11 Boundary condition paradoxes in nonlocal elasticity and their proposed solutions

Solution procedure	Vibrations	Buckling	Bending	Other
Integral	- Pisano et al. (2009) - Eptaimeros et al. (2016) - Tuna and Kirca (2017a, b)	- Tuna and Kirca 2017a, b	- Tuna and Kirca 2017a, b	
Differential	- Peddieson et al. (2003) - Wang et al. (2007) - Wang and Liew (2007) - Zhang et al. (2009) - Wang and Li (2014) - Zhang (2017)	- Wang et al. (2006) - Yang and Lim (2011)		- Wang (2005)
Laplace	- Tuna and Kirca (2016b)	- Tuna and Kirca (2016b)	- Tuna and Kirca (2016b) - Romano and Barretta (2016) - Tuna and Kirca (2017a, b) - Özgür Yayli and Yerel Kandemir (2017)	
Stress-driven	- Romano and Barretta (2017a, b) - Barretta et al. (2018)			
Two-phase	- Benvenuti and Simone (2013) - Khodabakhshi and Reddy (2015) - Eptaimeros et al. (2016) - Fernández-Sáez and Zaera (2017)	- Zhu et al. (2017)	- Wang et al. (2016)	- Altan (1989) - Pisano and Fuschi (2003) - Zhu and Dai (2012)
Gradient	- Challamel and Wang (2008) - Zhang et al. (2010)	- Zhang et al. (2010)	- Zhang et al. (2010)	
WRA—variationally consistent BCs	- Xu et al. (2016)	- Xu et al. (2016)		
Nonlocal strain gradient	- Xu et al. (2017)			
Other				- Arash and Wang (2012) - Eltaher et al. (2016) - Romano et al. (2017)

longitudinal vibrations of nanorods were studied using the stress-driven model by Barretta et al. (2018) after discussions on the underlying causes of the unavoidable paradox in Eringen's strain-driven form. Specifically, Romano et al. (2017) addressed the paradox in nonlocal nanobeams, in which the contradiction between the equilibrium and nonlocal constitutive conditions was considered responsible for the inability to use the differential constitutive formulation to solve the paradox. Furthermore, it was stated that for the ill-posed form of Eringen's strain-driven integral elasticity, the existence of a solution to the integro-partial differential must lead to the paradox. Most recently, a paradoxical behavior that appears when analyzing the uniform temperature rise effects on dynamic responses of functionally graded (FG) nanobeams based on Eringen's differential model was overcome by Mahmoudpour et al. (2018).

In order to obviate the ill-posedness of Eringen's strain-driven integral model, Romano et al. (2017) applied a direct solution procedure in order to capture the size-dependent phenomena in nanobeams, noting that the elastostatic problem of nanobeams formulated by Eringen's strain-driven integral model admits either a unique solution or no solution at all. This is dependent upon whether the bending interaction field fulfills the constitutive boundary conditions. Thus, they formulated the stress-driven model by interchanging the roll of the stress and strain in Eringen's integral formulation as:

$$\varepsilon(x) = \int_V k\left(\left|x-\overline{x}\right|, \kappa\right) C\left(\overline{x}\right) \sigma_{ij}\left(\overline{x}\right) dV \qquad (23)$$

where $C(\overline{x})$ is representative of the compliance tensor, the inverse of the stiffness tensor found in Eringen's strain-driven integral elasticity. Though the roles of the stress and strain were interchanged, it is important to understand that the stress-driven and strain-driven models are not the inverses of each other and will not yield similar structural models. In fact, the results of the stress-driven model are rather analogous to the strain gradient model used by Polizzotto (2003), as can be seen in Table 8. It should be noted that both the stress-driven and the strain-driven model will produce the hardening effect.

It was shown that the stress-driven model and strain gradient model yield similar results for low values of the length scale parameter and that the discrepancy between the two increases as the length scale parameter increases. However, there exist major differences in the basic assumptions for the formulation of each model. In considering the strain gradient model, the response of the system relies upon both the strain field and the first-order strain gradients. For the stress-driven model, the elastic strain field is calculated as the integral convolution of the stress field and a kernel function, as shown in Eq. 23. Comparing the models, it is shown that for both the strain gradient and the stress-driven model, a stiffer elastic response was found for increasing values of the nonlocal parameter for any prescribed kinematic boundary conditions. Finally, they concluded that the boundary values of the nonlocal curvature trend to one half as the nonlocal parameter approaches zero due to the cancelation of the kernel components that are outside the interval of integration (Barretta et al. 2018).

While the stress-driven nonlocal model is an effective tool for analyzing the response of nanostructures, the consistent hardening behavior does not align with the predicted softening behavior of Eringen's strain-driven integral elasticity.

Summary of proposed solutions and modification of Eringen's nonlocal elasticity

In the previous sections, several proposed methods were addressed which aimed at resolving the discrepancies between nanoscale systems with varying boundary conditions and loading conditions. In this section, several methods are consolidated in Tables 9, 10, and 11. In Table 9, the bending moments and shear forces are presented for each of the proposed differential solution procedures. It can easily be seen that these procedures can lead to seriously different resultants. As such, the methods should be used with caution and should be considered from mathematical and physical points of view.

In Table 10, the original formulations for different versions of Eringen's nonlocal elasticity are shown. These formulations include Eringen's integral nonlocal elasticity (strain-driven model), the sress-driven model presented by Romano and Barreta, the integral formulation of the hybrid or two-phase model, the differential form of Eringen's strain-driven theory, and a differential form of the two-phase model.

To consolidate the proposed solutions to the boundary condition paradox, Table 11 is presented. It follows from Table 11 that in recent years, there have been several researchers tackling the problem from different points of view. After considering the many different analyses that can be performed using nonlocal elasticity in beams, shells, and plates, it is vital to find a final, agreed upon solution whose mathematical solution is well-posed and whose results are consistent with one another. The works of the researchers below have provided an excellent benchmark for addressing this issue.

Conclusions

Several theories in nonclassical continuum mechanics were addressed and reviewed based upon the characteristics of the material. After presenting the formulation of Eringen's integral nonlocal theory, Eringen's differential nonlocal theory, and a general nonlocal theory, the use of Eringen's theory in various nanostructures was considered. The list of works in the field of nanomechanics and nonlocal elasticity are extensive, so an apology is made to those authors whose works were overlooked in this review. After a review of the existing nonclassical theories and the vast utilization of Eringen's nonlocal elasticity were discussed, existing paradoxes in Eringen's nonlocal elasticity and their possible solutions were studied in detail. Several concluding remarks are made:

- The available theories in nonclassical continuum mechanics are extensive, accounting for combinations of the rotations, deformations, and translations of considered elements. The appropriate theory should be selected based on the material structure and inherent material properties. Furthermore, many of the higher-order theories in nonclassical continuum mechanics have not yet been applied to physical systems because of the difficulty in derivations of natural boundary conditions and in the physical interpretation of each of the newly introduced parameters into the equations of motion.
- Of the available theories, Eringen's strain-driven differential nonlocal elasticity theory has been used most extensively for its seemingly accurate results for certain boundary and loading conditions since its original formulations. Specifically, the strain-driven differential form has been used to study the wave propagations, vibrations, bending, and buckling of size-dependent beams, shells, cones, plates, and graphene sheets or to model cracks or impurities in materials.
- In certain classes of beam problems, a so-called paradox has arisen in which discrepancies between the results of Eringen's integral nonlocal theory, Eringen's differential nonlocal theory, which should predict a softening phenomenon for increasing nonlocal parameters, and classical (local) theories, in which the size-dependent phenomena are not considered.
- Some of the most notable and interesting cases that implement Eringen's nonlocal elasticity include the hardening of nanocantilever beams, the nonlocal effect for nanocantilevers with concentrated point forces, and the nonlocal effect in simply supported beams with uniform loads.
- In recent years, there have been several efforts focused on defining the underlying cause of the paradox. It has been shown that the solution to Eringen's integral strain-driven model with the exponential kernel is ill-posed and that the existence of a solution implies a discrepancy between the constitutive equations and equilibrium equations, i.e., it is claimed that the boundary conditions are not variationally consistent.
- Not all proposed solutions are addressed with reference to the ill-posedness of Eringen's nonlocal elasticity. Furthermore, some well-posed proposed solutions do not lead to the expected softening results.
- Researchers have proposed solutions to the paradox by attempting to solve the original integral or differential formulations with iterative, finite element–based, and Laplace transform methods. However, it should me mentioned that although the so-called paradox can be solved by changing some assumptions, such as relaxing the kinematics or adding in length scales, if any numerical method is based on the same underlying assumptions as Eringen's model, their results should of course be identical to the analytical ones, that being said, the two-phase local/nonlocal model, the stress-driven model, the gradient model, and use of the weighted residual approach to obtain variationally consistent boundary conditions from potential and kinetic energy points of view.
- The proposed methods for solving paradoxes arising from Eringen's nonlocal elasticity theory produce significantly different results. Thus, the existence of a paradox at all is evidence that the constitutive boundary conditions are in contrast with the equilibrium boundary conditions. Thus, this confirms an ill-posed elastostatic problem where the stress field generated from the convolution is incapable of fulfilling the equilibrium conditions.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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