SIAM J. OPTIM.
Vol. 33, No. 3, pp. 1587–1620


# RATIONAL GENERALIZED NASH EQUILIBRIUM PROBLEMS*

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**Abstract.** This paper studies generalized Nash equilibrium problems that are given by rational functions. The optimization problems are not assumed to be convex. Rational expressions for Lagrange multipliers and feasible extensions of KKT points are introduced to compute a generalized Nash equilibrium (GNE). We give a hierarchy of rational optimization problems to solve rational generalized Nash equilibrium problems. The existence and computation of feasible extensions are studied. The Moment-SOS relaxations are applied to solve the rational optimization problems. Under some general assumptions, we show that the proposed hierarchy can compute a GNE if it exists or detect its nonexistence. Numerical experiments are given to show the efficiency of the proposed method.

**Key words.** generalized Nash equilibrium, rational function, feasible extension, Lagrange multiplier expression, Moment-SOS relaxation

**MSC codes.** 90C23, 90C33, 91A10, 65K05

**DOI.** 10.1137/21M1456285

## 1. Introduction.

The generalized Nash equilibrium problem (GNEP) is a kind of game to find strategies for a group of players such that each player's objective cannot be further optimized for given strategies of other players. Suppose there are $N$ players and the $i$th player's strategy is the real vector $x_i \in \mathbb{R}^{n_i}$. We write that

$$x_i := (x_{i,1}, \ldots, x_{i,n_i}), \quad x := (x_1, \ldots, x_N).$$

Let $n := n_1 + \cdots + n_N$. When the $i$th player's strategy $x_i$ is focused, we also write that $x = (x_i, x_{-i})$, where

$$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N).$$

A strategy tuple $u := (u_1, \ldots, u_N)$ is said to be a generalized Nash equilibrium (GNE) if each $u_i$ is the optimizer for the $i$th player's optimization

$$\text{(1.1)} \qquad \mathrm{F}_i(u_{-i}) : \left\{ \begin{array}{cl} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, u_{-i}) \\ s.t. & x_i \in X_i(u_{-i}). \end{array} \right.$$

In the above, the $X_i(u_{-i})$ is the feasible set and $f_i(x_i, u_{-i})$ is the $i$th player's objective. They are parameterized by $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$. Each player's optimization is parameterized by the strategies of other players. We denote by $\mathcal{S}$

*Received by the editors November 1, 2021; accepted for publication (in revised form) February 13, 2023; published electronically July 27, 2023.
https://doi.org/10.1137/21M1456285
**Funding:** The first author's research was partially supported by NSF grant DMS-2110780.
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the set of all GNEs and denote by $\mathcal{S}_i(u_{-i})$ the set of minimizers for the optimization $\mathrm{F}_i(u_{-i})$. The entire feasible strategy set is

$$X := \{(x_1, \ldots, x_N) \,|\, x_i \in X_i(x_{-i})\,, i = 1, \ldots, N\}\,. \tag{1.2}$$

A strategy tuple $x = (x_1, \ldots, x_N)$ is said to be feasible if each $x_i \in X_i(x_{-i})$.

This paper studies *rational generalized Nash equilibrium problems* (rGNEPs); i.e., all the objectives and constraining functions are rational functions in $x$. We assume the $i$th player's feasible set is given as

$$X_i(x_{-i}) = \left\{ x_i \in \mathbb{R}^{n_i} \;\middle|\; \begin{array}{l} g_{i,j}(x_i, x_{-i}) = 0\,(j \in \mathcal{I}_0^{(i)}), \\ g_{i,j}(x_i, x_{-i}) \geq 0\,(j \in \mathcal{I}_1^{(i)}), \\ g_{i,j}(x_i, x_{-i}) > 0\,(j \in \mathcal{I}_2^{(i)}) \end{array} \right\}, \tag{1.3}$$

where $\mathcal{I}_0^{(i)}, \mathcal{I}_1^{(i)}, \mathcal{I}_2^{(i)}$ are respectively the labeling sets (possibly empty) for equality, weak inequality, and strict inequality constraints. For the rational function to be well defined, we assume all denominators are positive in the feasible set. If this is not the case, we can add strict inequality constraints for denominators. Rational functions frequently appear in GNEPs. When defining functions are polynomials, the GNEPs are studied in the recent work [40, 42, 43]. For convenience, rational functions are also called rational polynomials throughout the paper.

A special case of GNEPs is the *Nash equilibrium problem* (NEP): each feasible set $X_i(x_{-i})$ is independent of $x_{-i}$. When NEPs are defined by polynomials, a method is given in [42] to solve them. For GNEPs given by convex polynomials, how to solve them is studied in the recent work [43]. We refer the reader to [9, 12, 13, 15, 53] for related work.

One may reformulate rGNEPs equivalently as polynomial GNEPs by introducing new variables or changing the description of the feasible set. However, doing so may lose some useful properties. For instance, the convexity may be lost if we use polynomial reformulations. The following is such an example.

*Example* 1.1. Consider the 2-player rGNEP

$$\begin{array}{cll|cll} \min\limits_{x_1 \in \mathbb{R}^2} & \frac{2(x_{1,1})^2 + (x_{1,2})^2 + x_{1,1}x_{1,2} \cdot e^T x_2}{x_{1,1}} & & \min\limits_{x_2 \in \mathbb{R}^2} & \frac{2(x_{2,1})^2 + (x_{2,2})^2 - x_{2,1}x_{2,2} \cdot e^T x_1}{x_{2,1}} & \\ s.t. & x_{1,1} - \frac{x_{2,1}}{x_{1,2}} \geq 0, & & s.t. & 1 - e^T(x_2 - x_1) \geq 0, & \\ & x_{1,1} > 0,\, x_{1,2} > 0, & & & x_{2,1} - 1 \geq 0,\, x_{2,2} - 1 \geq 0. & \end{array} \tag{1.4}$$

In the above, $e = [1\ 1]^T$. In the domain $(x_1, x_2) > 0$, each player's optimization is convex in its strategy variable. We can equivalently express this GNEP as polynomial optimization

$$\begin{array}{cl|cl} \min\limits_{x_1 \in \mathbb{R}^3} & x_{1,3}[2(x_{1,1})^2 + (x_{1,2})^2 + x_{1,1}x_{1,2} \cdot \hat{e}^T x_2] & \min\limits_{x_2 \in \mathbb{R}^3} & x_{2,3}[2(x_{2,1})^2 + (x_{2,2})^2 - x_{2,1}x_{2,2} \cdot \hat{e}^T x_1] \\ s.t. & x_{1,1}x_{1,2} - x_{2,1} \geq 0, & s.t. & 1 - \hat{e}^T(x_2 - x_1) \geq 0, \\ & x_{1,1} > 0,\, x_{1,2} > 0, & & x_{2,1} - 1 \geq 0,\, x_{2,2} - 1 \geq 0, \\ & x_{1,1}x_{1,3} = 1, & & x_{2,1}x_{2,3} = 1, \end{array} \tag{1.5}$$

where $\hat{e} = [1\ 1\ 0]^T$. However, the two above optimization problems are not convex.

The GNEPs were originally introduced to model economic problems. They are now widely used in various fields, such as transportation, telecommunications, and machine learning. We refer the reader to [1, 5, 7, 25, 31, 46] for recent applications of GNEPs. It is typically difficult to solve GNEPs. The major challenge is due to interactions among different players' strategies on the objectives and feasible sets. The set of GNEs may be nonconvex, even for convex NEPs (see [42]). Convex GNEPs can be reformulated as variational inequality (VI) or quasi-variational inequality (QVI) problems [11, 32, 45]. A semidefinite relaxation method for convex GNEPs of polynomials is given in [43]. The penalty functions are used to solve GNEPs in [2, 14]. An augmented Lagrangian method is given in [24]. The Nikaido–Isoda function-related methods are given in [10, 52]. Newton-type methods are given in [12, 53]. An interior point method is given in [9]. Gauss–Seidel-type methods are studied in [16, 40]. Lemke's method is used to solve affine GNEPs [49]. An ADMM-type method for solving GNEPs in Hilbert spaces is given in [4]. Moreover, quasi-NEs for nonconvex GNEPs are studied in [8, 47]. We refer the reader to [13, 15, 17] for surveys on GNEPs.

**Contributions.** We study GNEPs that are given by rational functions. This is motivated by earlier work on polynomial NEPs [42] and convex GNEPs [43]. In various applications, people often face GNEPs given by rational functions. Even for polynomial GNEPs, the Lagrange multiplier expressions are usually given by rational functions instead of polynomial ones. This was observed in [43]. Mathematically, rGNEPs can be equivalently formulated as polynomial GNEPs by introducing new variables. However, such a reformulation usually destroys some nice properties (e.g., convexity may be lost; see Example 1.1). Moreover, solving the reformulated polynomial GNEPs is usually more computationally expensive. This can be observed in numerical experiments.

For convex GNEPs, each feasible KKT point is a GNE. For nonconvex GNEPs, a KKT point is typically not a GNE (see Example 3.1). When we solve nonconvex GNEPs, the earlier existing methods may not get a GNE or are not able to detect its nonexistence. There exists relatively little work for solving nonconvex GNEPs. In this paper, we propose a new approach for solving rGNEPs. The optimization problems are not assumed to be convex. Our new approach is based on a hierarchy of rational optimization problems. Our major contributions are the following:

- First, we introduce rational expressions for Lagrange multipliers of each player's optimization. These expressions can be used to give new constraints for GNEs.
- Second, we introduce the new concept of feasible extensions for some KKT points. More specifically, for a KKT point that is not a GNE, we extend it to the image of a rational function, such that the image is feasible on the KKT set. The feasible extension can be used to preclude KKT points that are not GNEs. For nonconvex rGNEPs, the usage of rational feasible extensions is important for computing a GNE (if it exists) or for detecting its nonexistence.
- Third, the Moment-SOS relaxations are used to solve rational optimization problems that are obtained from using Lagrange multiplier expressions and feasible extensions of some KKT points. Unlike polynomial optimization, a rational optimization problem may have strict inequalities. We study the properties of Moment-SOS relaxations for solving them.

The paper is organized as follows. Some preliminaries for moment and polynomial optimization are given in section 2. A hierarchy of rational optimization problems for solving the GNEP is proposed in section 3. Feasible extensions of KKT points are studied in section 4. We show how to solve rational optimization problems in section 5. Some numerical experiments are given in section 6. Some conclusions and discussions are given in section 7.

**2. Preliminaries.** This following notation is used throughout the paper. The symbol $\mathbb{N}$ denotes the set of nonnegative integers. The symbol $\mathbb{R}$ denotes the set of real numbers. For a positive integer $k$, denote the set $[k] := \{1, \ldots, k\}$. For a real number $t$, $\lceil t \rceil$ denotes the smallest integer not smaller than $t$. We use $e_i$ to denote the vector such that the $i$th entry is 1 and all others are zeros, and we use $e$ to denote the vector of all ones. For a vector $u$ in the Euclidean space, its Euclidean norm is denoted as $\|u\|$. By writing $A \succeq 0$ (resp., $A \succ 0$), we mean that the matrix $A$ is symmetric positive semidefinite (resp., positive definite). Let $\mathbb{R}[x]$ denote the ring of real polynomials in $x$ and $\mathbb{R}[x]_d$ denote the set of polynomials with degrees not bigger than $d$. For the $i$th player's strategy vector $x_i$, the notations $\mathbb{R}[x_i]$ and $\mathbb{R}[x_i]_d$ are defined similarly. For a polynomial $p \in \mathbb{R}[x]$, we write $p = 0$ to mean that $p$ is the identically zero polynomial, and $p \neq 0$ means that $p$ is not identically zero. The total degree of $p$ is denoted by $\deg(p)$, and its partial degree on $x_i$ is denoted by $\deg_{x_i}(p)$. For a function $f(x)$, the notation $\nabla_{x_i} f := (\frac{\partial f}{\partial x_{i,j}})_{j\in[n_i]}$ denotes its gradient with respect to $x_i$. For a set $X$, we use $cl(X)$ to denote its closure in the Euclidean topology. A property is said to hold *generically* if it holds for all points in the space of input data except a set of Lebesgue measure zero.

Let $z = (z_1, \ldots, z_l)$ stand for the vector $x$ or $x_i$. For a power $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we denote that $z^\alpha := z_1^{\alpha_1} \cdots z_l^{\alpha_l}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_l$. For a degree $d > 0$, we denote the power set $\mathbb{N}_d^l := \{\alpha \in \mathbb{N}^l : \ |\alpha| \le d\}$. We use $[z]_d$ to denote the vector of all monomials in $z$ whose degrees are at most $d$, ordered in the graded alphabetical ordering, i.e., $[z]_d := [1, z_1, \ldots, z_l, z_1^2, \ldots, z_l^d]^T$.

**2.1. Ideals and quadratic modules.** For a polynomial $p \in \mathbb{R}[x]$ and subsets $I, J \subseteq \mathbb{R}[x]$, define the product and Minkowski sum

$$p \cdot I := \{pq : q \in I\}, \quad I + J := \{a + b : a \in I, b \in J\}.$$

The subset $I$ is an ideal if $p \cdot I \subseteq I$ for all $p \in \mathbb{R}[x]$ and $I + I \subseteq I$. The ideal generated by a polynomial tuple $h = (h_1, \ldots, h_{m_1})$ is $\mathrm{Ideal}[h] := h_1 \cdot \mathbb{R}[x] + \cdots + h_{m_1} \cdot \mathbb{R}[x]$. For a degree $d$, the $d$th truncation of $\mathrm{Ideal}[h]$ is

$$\mathrm{Ideal}[h]_d := h_1 \cdot \mathbb{R}[x]_{d-\deg(h_1)} + \cdots + h_{m_1} \cdot \mathbb{R}[x]_{d-\deg(h_{m_1})}.$$

A polynomial $\sigma \in \mathbb{R}[x]$ is said to be a sum-of-squares (SOS) if $\sigma = p_1^2 + \cdots + p_k^2$ for some $p_i \in \mathbb{R}[x]$. We use $\Sigma[x]$ to denote the set of all SOS polynomials in $x$ and denote the truncation $\Sigma[x]_d := \Sigma[x] \cap \mathbb{R}[x]_d$. The quadratic module of a polynomial tuple $g = (g_1, \ldots, g_{m_2})$ is $\mathrm{Qmod}[g] := \Sigma[x] + g_1 \cdot \Sigma[x] + \cdots + g_{m_2} \cdot \Sigma[x]$. Similarly, the degree-$d$ truncation of $\mathrm{Qmod}[g]$ is

$$\mathrm{Qmod}[g]_d := \Sigma[x]_d + g_1 \cdot \Sigma[x]_{d-\deg(g_1)} + \cdots + g_{m_2} \cdot \Sigma[x]_{d-\deg(g_{m_2})}.$$

The polynomial tuples $h, g$ determine the basic closed semi-algebraic set

$$T := \{x \in \mathbb{R}^n : h_i(x) = 0 \, (i \in [m_1]), g_j(x) \ge 0 \, (j \in [m_2])\}. \tag{2.1}$$

Clearly, every polynomial in $\mathrm{Ideal}[h] + \mathrm{Qmod}[g]$ is nonnegative on the set $T$. We denote by $\mathscr{P}(T)$ the set of polynomials nonnegative on $T$ and denote the truncation $\mathscr{P}_d(T) := \mathscr{P}(T) \cap \mathbb{R}[x]_d$. Clearly, $\mathrm{Ideal}[h] + \mathrm{Qmod}[g] \subseteq \mathscr{P}(T)$. The sets $\mathscr{P}(T)$, $\mathscr{P}_d(T)$ are convex cones, and $\mathscr{P}_d(T)$ is the dual cone of the moment cone

$$\mathscr{R}_d(T) := \left\{ \sum_{i=1}^{M} \lambda_i [u_i]_d : u_i \in T, \lambda_i \ge 0, M \in \mathbb{N} \right\}.$$

When $T$ is compact, the cone $\mathscr{R}_d(T)$ is closed and it equals the dual cone of $\mathscr{P}_d(T)$.

The set $\text{Ideal}[h]+\text{Qmod}[g]$ is said to be *archimedean* if there exists $p\in\text{Ideal}[h]+\text{Qmod}[g]$ such that the inequality $p(x)\geq 0$ defines a compact set. If $\text{Ideal}[h]+\text{Qmod}[g]$ is archimedean, then $T$ is compact. Conversely, if $T$ is compact, say $T$ is contained in the ball $\|x\|^2\leq R$, then $\text{Ideal}[h]+\text{Qmod}[g,R-\|z\|^2]$ is archimedean. When $\text{Ideal}[h]+\text{Qmod}[g]$ is archimedean, if a polynomial $p>0$ on $T$, then $p\in\text{Ideal}[h]+\text{Qmod}[g]$. This conclusion is referenced as Putinar's Positivstellensatz [48].

**2.2. Localizing and moment matrices.** For an integer $k\geq 0$, a real vector $y=(y_\alpha)_{\alpha\in\mathbb{N}^n_{2k}}$ is said to be a *truncated multi-sequence* (tms) of degree $2k$. For a polynomial $f=\sum_{\alpha\in\mathbb{N}^n_{2k}} f_\alpha x^\alpha$, define the operation

$$\langle f,y\rangle := \sum_{\alpha\in\mathbb{N}^n_{2k}} f_\alpha y_\alpha. \tag{2.2}$$

The operation $\langle f,y\rangle$ is bilinear in $f$ and $y$. For a polynomial $q\in\mathbb{R}[x]_{2t}$ ($t\leq k$) and a degree $s\leq k-\lceil \deg(q)/2\rceil$, the $k$th order *localizing matrix* of $q$ for $y$ is the symmetric matrix $L_q^{(k)}[y]$ such that (the $vec(a)$ denotes the coefficient vector of $a$)

$$\langle qa^2,y\rangle = vec(a)^T\big(L_q^{(k)}[y]\big)vec(a) \tag{2.3}$$

for all $a\in\mathbb{R}[x]_s$. When $q=1$ (the constant one polynomial), the localizing matrix $L_q^{(k)}[y]$ becomes the $k$th order *moment matrix* $M_k[y]:=L_1^{(k)}[y]$.

Localizing and moment matrices can be used to approximate the moment cone $\mathscr{R}_d(T)$ by semidefinite programming relaxations. They are useful for solving polynomial, matrix, and tensor optimization [22, 37, 38, 39]. We refer the reader to [26, 28, 30] for a general introduction to polynomial optimization and moment problems.

**2.3. Lagrange multiplier expressions.** The Karush–Kuhn–Tucker (KKT) conditions are useful for solving GNEPs and NEPs. We review optimality conditions for nonlinear optimization (see [3]). Frequently used constraint qualifications are the linear independence constraint qualification (LICQ) and the Mangasarian–Fromovitz constraint qualification (MFCQ). For strict inequality constraints, their associated Lagrange multipliers are zeros, and hence the KKT conditions only concern weak inequality constraints. For convenience of description, we write that $\mathcal{I}_0^{(i)}\cup\mathcal{I}_1^{(i)}=\{1,\ldots,m_i\}$ and $g_i=(g_{i,1},\ldots,g_{i,m_i})$. Under certain constraint qualifications, if $x_i\in X_i(x_{-i})$ is a minimizer of $\mathrm{F}_i(x_{-i})$, then there exists a Lagrange multiplier vector $\lambda_i:=(\lambda_{i,1},\ldots,\lambda_{i,m_i})$ such that

$$\begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j}\nabla_{x_i} g_{i,j}(x)=0, \\ \lambda_i\perp g_i(x),\ \lambda_{i,j}\geq 0\,(j\in\mathcal{I}_1^{(i)}). \end{cases} \tag{2.4}$$

In the above, $\lambda_i\perp g_i(x)$ means that $\lambda_i$ is perpendicular to $g_i(x)$. The system (2.4) gives the first order KKT conditions for $\mathrm{F}_i(x_{-i})$. Such $(x_i,\lambda_i)$ is called a critical pair. Under the constraint qualifications, every GNE satisfies (2.4).

Consider the $i$th player's optimization problem $\mathrm{F}_i(x_{-i})$. If there exists a rational vector function $\tau_i(x)$ such that $\lambda_i=\tau_i(x)$ for every critical pair $(x_i,\lambda_i)$ of $\mathrm{F}_i(x_{-i})$, then $\tau_i(x)$ is called a rational *Lagrange multiplier expression* (LME) for $\lambda_i$. As in (2.4), each critical pair $(x_i,\lambda_i)$ of the optimization $\mathrm{F}_i(x_{-i})$ satisfies

$$
\underbrace{\begin{bmatrix} \nabla_{x_i} g_{i,1}(x) & \nabla_{x_i} g_{i,2}(x) & \cdots & \nabla_{x_i} g_{i,m_i}(x) \\ g_{i,1}(x) & 0 & \cdots & 0 \\ 0 & g_{i,2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{i,m_i}(x) \end{bmatrix}}_{G_i(x)} \underbrace{\begin{bmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,m_i} \end{bmatrix}}_{\lambda_i} = \underbrace{\begin{bmatrix} \nabla_{x_i} f_i(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\hat{f}_i(x)}. \tag{2.5}
$$

If there exist a matrix polynomial $T_i(x)$ and a nonzero scalar polynomial $q_i(x)$ such that

$$
T_i(x)G_i(x) = q_i(x)I_{m_i},
$$

then (2.5) implies that $q_i(x)\lambda_i = T_i(x)\hat{f}_i(x)$. This gives the following rational LME:

$$
\tau_i(x) = T_i(x)\hat{f}_i(x)/q_i(x). \tag{2.6}
$$

At a point $u$, if $q_i(u) = 0$, then $T_i(u)\hat{f}_i(u) = 0$.

The rational expression (2.6) almost always exists. This can be shown as follows. Let $H_i(x) := G_i(x)^T G_i(x)$; then $H_i(x)$ is a matrix of rational functions and $H_i(x) \succeq 0$ on $X$. If the determinant $\det H_i(x)$ is not identically zero (this is the general case), then we have

$$
\operatorname{adj} H_i(x) \cdot H_i(x) \;=\; \det H_i(x) \cdot I_{m_i},
$$

where $\operatorname{adj} H_i(x)$ denotes the adjugate matrix of $H_i(x)$. Let $d_i(x)$ be the denominator of $\det H_i(x)$; then $T_i(x)G_i(x) = q_i(x) \cdot I_{m_i}$ for the selection

$$
T_i(x) = d_i(x) \cdot \operatorname{adj} H_i(x) \cdot G_i(x)^T, \quad q_i(x) = d_i(x) \cdot \det H_i(x). \tag{2.7}
$$

The above choices of $T_i(x)$ and $q_i(x)$ may not be computationally efficient. However, there often exist different options for $T_i(x)$ and $q_i(x)$ to make (2.6) hold. For computational efficiency, we prefer that $T_i(x)$ and $q_i(x)$ have low degrees. It is worth noting that once their degrees are given, the equation $T_i(x)G_i(x) = q_i(x) \cdot I_{m_i}$ is linear in the coefficients of $T_i(x)$ and $q_i(x)$. So we can obtain $T_i(x), q_i(x)$ by solving linear equations. The following is such an example.

*Example* 2.1. Let $x = (x_1, x_2), x_1 \in \mathbb{R}^1, x_2 \in \mathbb{R}^1$, and $g_2(x) = (1 - x_1 - x_2, x_2)$. We look for $T_2(x)$, $q_2(x)$ such that $T_2(x)G_2(x) = q_2(x) \cdot I_2$, where

$$
G_2(x) = \begin{bmatrix} -1 & 1 \\ 1 - x_1 - x_2 & 0 \\ 0 & x_2 \end{bmatrix}.
$$

We consider $q_2(x)$ and $T_2(x)$ having degree 1, i.e.,

$$
\begin{aligned} T_2(x) &= (a_{i,j} + b_{i,j}x_1 + c_{i,j}x_2)_{1\le i\le 2, 1\le j\le 3}, \\ q_2(x) &= a_0 + b_0x_1 + c_0x_2. \end{aligned}
$$

The equality $T_2(x)G_2(x) = q_2(x) \cdot I_2$ gives the equations

$$
\begin{aligned} & a_{1,1} = b_{1,1} = b_{1,2} = c_{1,2} = b_{2,2} = c_{2,2} = b_{1,3} = c_{1,3} = b_{2,3} = c_{2,3} = 0, \\ & a_0 = a_{2,1} = a_{1,2} = a_{2,2} = -b_{2,1} = -c_{2,1} = -b_0, \\ & a_{1,3} = -c_{1,1},\ c_0 = -c_{1,1} - a_{1,2},\ a_{2,3} = c_0 - c_{2,1}. \end{aligned}
$$

We can choose $a_0 = 1$ and $c_{1,1} = -1$ to obtain

$$
T_2(x) = \begin{bmatrix} -x_2 & 1 & 1 \\ 1 - x_1 - x_2 & 1 & 1 \end{bmatrix}, \quad q_2(x) = 1 - x_1.
$$

We refer the reader to [36, 43] for more details about LMEs.

**3. A hierarchy of optimization problems.** In this section, we propose a new approach for solving rGNEPs. It requires solving a hierarchy of rational optimization problems. They are obtained from LMEs and feasible extensions of KKT points that are not GNEs. Under some general assumptions, we prove that this hierarchy either returns a GNE or detects its nonexistence.

As shown in subsection 2.3, one can express Lagrange multipliers as rational functions on the KKT set. Recall the set $X$ as in (1.2). For the $i$th player's optimization $\mathrm{F}_i(x_{-i})$, we suppose that there is a tuple $\tau_i = (\tau_{i,j})_{j\in\mathcal{I}_0^{(i)}\cup\mathcal{I}_1^{(i)}}$ of rational functions in $x$, with denominators positive on $X$, such that

$$\lambda_{i,j} = \tau_{i,j}(x), \quad j \in \mathcal{I}_0^{(i)} \cup \mathcal{I}_1^{(i)}, \tag{3.1}$$

for each critical pair $(x_i, \lambda_i)$ of $\mathrm{F}_i(x_{-i})$. When $G_i(x)$ has full column rank on $X$, there exist LMEs satisfying (3.1) by [43, Proposition 3.6]. Note that the Lagrange multipliers are zero for strict inequality constraints. So, the KKT set is

$$\mathcal{K} := \left\{ x \in X \;\middle|\; \begin{array}{l} \nabla_{x_i} f_i = \sum\limits_{j\in\mathcal{I}_0^{(i)}\cup\mathcal{I}_1^{(i)}} \tau_{i,j}(x)\nabla_{x_i} g_{i,j}(x) \ (i\in[N]), \\ \tau_{i,j}(x) g_{i,j}(x) = 0,\, \tau_{i,j}(x) \ge 0 \ (i \in [N],\, j \in \mathcal{I}_1^{(i)}) \end{array} \right\}. \tag{3.2}$$

Not every point $u = (u_1, \ldots, u_N) \in \mathcal{K}$ is a GNE. How do we preclude non-GNEs in $\mathcal{K}$? We consider the case that $u$ is not a GNE. Then there exist $i \in [N]$ and a point $v_i \in X_i(u_{-i})$ such that

$$f_i(v_i, u_{-i}) - f_i(u_i, u_{-i}) < 0. \tag{3.3}$$

However, if $x := (x_1, \ldots, x_N)$ is a GNE and $v_i$ is also feasible for $\mathrm{F}_i(x_{-i})$, i.e., $v_i \in X_i(x_{-i})$, then $x$ must satisfy the inequality

$$f_i(v_i, x_{-i}) - f_i(x_i, x_{-i}) \ge 0. \tag{3.4}$$

That is, every GNE $x$ satisfies the constraint (3.4) if $v_i \in X_i(x_{-i})$. This is used to solve NEPs in [42]. However, unlike NEPs, the feasible set of $X_i(x_{-i})$ depends on $x_{-i}$. As a result, a point $v_i \in X_i(u_{-i})$ may not be feasible for $\mathrm{F}_i(x_{-i})$; i.e., it is possible that $v_i \notin X_i(x_{-i})$ for a GNE $x$. For such a case, the inequality (3.4) may not hold for any GNEs. In other words, it is possible that for every GNE $x^* = (x_i^*, x_{-i}^*)$, it may happen that $v_i \notin X_i(x_{-i}^*)$ and

$$f_i(v_i, x_{-i}^*) < f_i(x_i^*, x_{-i}^*) = \min_{x_i \in X(x_{-i}^*)} f_i(x_i, x_{-i}^*).$$

The following is such an example.

*Example* 3.1. Consider the 2-player GNEP

$$\begin{array}{cl|cl} \min\limits_{x_1\in\mathbb{R}^2} & (x_{1,1} - x_{1,2})x_{2,1}x_{2,2} - x_1^T x_1 & \min\limits_{x_2\in\mathbb{R}^2} & 3(x_{2,1} - x_{1,1})^2 + 2(x_{2,2} - x_{1,2})^2 \\ s.t. & 1 - e^T x \ge 0,\, x_1 \ge 0, & s.t. & 2 - e^T x \ge 0,\, x_2 \ge 0. \end{array}$$

It has only two GNEs $x^* = (x_1^*, x_2^*)$:

$$x_1^* = x_2^* = (0.5, 0) \quad \text{and} \quad x_1^* = x_2^* = (0, 0.5).$$

Consider the point $u = (u_1, u_2) \in \mathcal{K}$, with $u_1 = u_2 = (0, 0)$. The $u_1$ is not a minimizer of $\mathrm{F}_1(u_2)$, so $u$ is not a GNE. The optimizers of $\mathrm{F}_1(u_2)$ are $v_1 = (1, 0)$ and $(0, 1)$. One can check that for either GNE $x^*$, it holds that

$$v_1 \notin X_1(x_2^*), \qquad f_1(v_1, x_2^*) - f_1(x_1^*, x_2^*) = -0.75 < 0.$$

The inequality (3.4) does not hold for any GNE.

The above example shows that the constraint (3.4) may not hold for any GNE. However, if there is a function $p_i$ in $x$ such that

$$v_i = p_i(u), \qquad p_i(x) \in X_i(x_{-i}) \quad \text{for all} \quad x \in \mathcal{K}, \tag{3.5}$$

then the inequality

$$f_i(p_i(x), x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \tag{3.6}$$

separates GNEs and non-GNEs. This is because $f_i(x_i, x_{-i}) \leq f_i(p_i(x), x_{-i})$ for every GNE $x$, since $p_i(x) \in X_i(x_{-i})$. This motivates us to make the following assumption.

*Assumption* 3.2. For a given triple $(u, i, v_i)$, with $u \in \mathcal{K}$, $i \in [N]$ and $v_i \in \mathcal{S}_i(u_{-i})$, there exists a rational vector-valued function $p_i$ in $x := (x_1, \ldots, x_N)$ such that (3.5) holds.

The function $p_i$ satisfying (3.5) is called a *feasible extension* of $v_i$ at the point $u$. Feasible extensions are useful for solving bilevel optimization [41]. In section 4, we will discuss the existence and computation of such $p_i$.

**3.1. An algorithm for solving GNEPs.** Based on LMEs and feasible extensions, we propose the following algorithm for solving GNEPs.

*Algorithm* 3.3. For the given GNEP of (1.1), do the following:

Step 0 Find the Lagrange multiplier expressions as in (3.1). Let $\mathscr{U} := \mathcal{K}$ and $k := 0$. Choose a generic positive definite matrix $\Theta$ of length $n+1$.

Step 1 Solve the following optimization (note $[x]_1 = \begin{bmatrix}1 & x^T\end{bmatrix}^T$):

$$\left\{ \begin{array}{ll} \min & [x]_1^T \Theta [x]_1 \\ s.t. & x \in \mathscr{U}. \end{array} \right. \tag{3.7}$$

If (3.7) is infeasible, output that either (1.1) has no GNEs or there is no GNE in the set $\mathcal{K}$. Otherwise, solve it for a minimizer $u := (u_1, \ldots, u_N)$ if it exists.

Step 2 For each $i = 1, \ldots, N$, solve the following optimization:

$$\left\{ \begin{array}{rl} \delta_i := \min & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ s.t. & x_i \in X_i(u_{-i}) \end{array} \right. \tag{3.8}$$

for a minimizer $v_i$. Denote the label set

$$\mathcal{N} := \{i \in [N] : \delta_i < 0\}. \tag{3.9}$$

If $\mathcal{N} = \emptyset$, then $u$ is a GNE and stop; otherwise, go to Step 3.

Step 3 For every above triple $(u, i, v_i)$ with $i \in \mathcal{N}$, find a rational feasible extension $p_i$ satisfying (3.5). Then update the set $\mathscr{U}$ as

$$\mathscr{U} := \mathscr{U} \cap \left\{ x \in \mathbb{R}^n : f_i(p_i(x), x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \ \forall i \in \mathcal{N} \right\}. \tag{3.10}$$

Then, let $k := k+1$ and go to Step 1.

In Step 0, we can let $\Theta := R^T R$ for a generically generated square matrix $R$. Then, the objective $[x]_1^T \Theta [x]_1$ is generic, coercive, and strictly convex, and so the optimization problem (3.7) has a unique minimizer if it is feasible. This gives computational convenience for solving rational optimization with Moment-SOS relaxations (see Theorem 5.3). Note that Algorithm 3.3 is applicable for all choices of $\Theta$ (e.g.,

$\Theta = I_{n+1}$). But a generically selected positive definite $\Theta$ is usually preferable in computational practice. The optimization problem (3.7) may have constraints given by rational polynomials, or it may have strict inequality constraints. The optimization (3.8) may have both rational objective and rational constraints. They can be solved by Moment-SOS relaxations. The optimization problem (3.8) has a nonempty feasible set, since $u_i \in X_i(u_{-i})$. In applications, people usually assume (3.8) has a minimizer. For instance, this is the case if its feasible set is compact or if its objective is coercive. We discuss how to solve the appearing rational optimization problems in section 5.

If a GNE is a KKT point, i.e., it belongs to the set $\mathcal{K}$ as in (3.2), then it belongs to the set $\mathscr{U}$ in every loop. In other words, the update of $\mathscr{U}$ in Algorithm 3.3 does not preclude any GNEs. The set $\mathscr{U}$ stays nonempty if there is a GNE lying in $\mathcal{K}$.

In Algorithm 3.3, we need LMEs and feasible extensions. As shown in subsection 2.3, LMEs almost always exist. For standard constraints like box, simplex, or balls, explicit LMEs are given in (6.2)–(6.5). When denominators of LMEs vanish at some points, Algorithm 3.3 is still applicable, because denominators can be cancelled by multiplying their least common multiples. We refer the reader to Example 6.2 for such cases. The existence of a feasible extension is ensured if $\mathcal{K}$ is a finite set (see Theorem 4.2). There exist explicit expressions for many common constraints; see subsection 4.1. In summary, Algorithm 3.3 can be used for solving many rGNEPs.

**3.2. Convergence analysis.** We now study the convergence of Algorithm 3.3.

First, an interesting case is the convex rGNEP. A GNEP is said to be convex if every player's optimization problem is convex: for each fixed $x_{-i}$, the objective $f_i(x_i, x_{-i})$ is convex in $x_i$, the inequality constraining functions in (1.3) are concave in $x_i$, and all equality constraining functions are linear in $x_i$. Interestingly, the concavity of constraining functions can be weakened to the convexity of feasible sets under certain assumptions. As in [27], for given $x_{-i}$, the feasible set $X_i(x_{-i})$ is said to be *nondegenerate* if for every $j \in \mathcal{I}_0^{(i)} \cup \mathcal{I}_1^{(i)}$, the gradient $\nabla_{x_i} g_{i,j}(x) \neq 0$ for all $x_i \in X_i(x_{-i})$ such that $g_{i,j}(x) = 0$. The set $X_i(x_{-i})$ is said to satisfy *Slater's condition* if it contains a point that makes all inequalities strictly hold.

THEOREM 3.4. *Assume the Lagrange multipliers are expressed as in* (3.1) *with denominators positive on $X$. Suppose that each objective $f_i$ is convex in $x_i$, each $g_{i,j}$ is linear in $x_i$ for $j \in \mathcal{I}_0^{(i)}$, and each strategy set $X_i(x_{-i})$ is convex and nondegenerate and satisfies Slater's condition. Then, Algorithm* 3.3 *terminates at the initial loop $k = 0$, and it either returns a GNE or detects nonexistence of GNEs.*

*Proof.* Under the given assumptions, a feasible point is a minimizer of the optimization $\mathrm{F}_i(x_{-i})$ if and only if it is a KKT point. This is shown in [27]. Equivalently, a point is a GNE if and only if it belongs to the set $\mathcal{K}$. If there is a GNE, Algorithm 3.3 can get one in Step 2 for the initial loop $k = 0$, and then it terminates. If there is no GNE, the KKT point set $\mathcal{K}$ is empty, then Algorithm 3.3 terminates in Step 1 for the initial loop. ∎

We remark that if there exist a matrix function $T_i(x)$ and a scalar function $q_i(x)$ such that

$$T_i(x) G_i(x) = q_i(x) I_{m_i}$$

and $q_i(x) > 0$ on $X$ (see (2.5) for $G_i(x)$), then $X_i(x_{-i})$ must be nondegenerate. This can be implied by [43, Proposition 3.6]. Moreover, when each $g_{i,j}$ is linear in $x_i$ for $j \in \mathcal{I}_0^{(i)}$ and every $g_{i,j}$ is concave in $x_i$ for $j \in \mathcal{I}_1^{(i)}$, the $X_i(x_{-i})$ is nondegenerate when it satisfies Slater's condition [27]. When the nondegeneracy condition fails, a GNE may not be a KKT point, even under the convexity assumption and Slater's condition. The following is such an example.

*Example* 3.5. Consider the GNEP

$$\begin{array}{ll|ll} \min\limits_{x_1\in\mathbb{R}^2} & 2x_{1,1}+x_{1,2} & \min\limits_{x_2\in\mathbb{R}^2} & \|x_1+x_2\|^2 \\ s.t. & x_1^Tx_2\ge 0,\ x_{1,1}x_{1,2}\ge 0, & s.t. & x_{2,1}-1\ge 0,\ x_{2,2}-1\ge 0. \end{array} \tag{3.11}$$

In the above, all player's objectives and feasible sets are convex, and Slater's condition holds. The feasible set $X_1(x_2)$ is degenerate. The KKT system for this GNEP is

$$\left\{\begin{array}{c} e+e_1=x_2\lambda_{1,1}+(x_{1,1}e_2+x_{1,2}e_1)\lambda_{1,2}, \\ 2(x_1+x_2)=e_1\cdot\lambda_{2,1}+e_2\cdot\lambda_{2,2}, \\ \lambda_{1,1}\cdot x_1^Tx_2=0,\ \lambda_{1,2}\cdot x_{1,1}x_{1,2}=0, \\ \lambda_{2,1}\cdot(x_{2,1}-1)=0,\ \lambda_{2,2}\cdot(x_{2,2}-1)=0, \\ x_1^Tx_2\ge 0,\ x_{1,1}x_{1,2}\ge 0,\ x_{2,1}\ge 1,\ x_{2,2}\ge 1, \\ \lambda_{1,1}\ge 0,\ \lambda_{1,2}\ge 0,\ \lambda_{2,1}\ge 0,\ \lambda_{2,2}\ge 0. \end{array}\right. \tag{3.12}$$

One may check that (3.12) has no solutions, i.e., this convex GNEP does not have any KKT point. However, the first player's feasible set is degenerate at $x_1=(0,0)$, which corresponds to the unique GNE

$$x^*=(x_1^*,x_2^*),\quad x_1^*=(0,0),\ x_2^*=(1,1).$$

Since the feasible set is degenerate, there do not exist LMEs in the form of (2.6) that have denominators positive on $X$. However, if we choose

$$\left\{\begin{array}{lcl} T_1(x) & = & \begin{bmatrix} -x_{1,1}x_{1,2} & 0 & x_{1,2} & x_{1,2} \\ x_1^Tx_2 & 0 & -x_{2,1} & -x_{2,1} \end{bmatrix}, \\ T_2(x) & = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ q_1(x) & = & (x_{1,2})^2x_{2,2}, \\ q_2(x) & = & 1, \end{array}\right. \tag{3.13}$$

then $T_i(x)G_i(x)=q_i(x)I_{m_i}$ for each $i=1,2$, and (2.6) gives the LMEs

$$\lambda_{1,1}=\frac{-x_{1,1}}{x_{1,2}x_{2,2}}\frac{\partial f_1}{\partial x_{1,1}},\quad \lambda_{1,2}=\frac{x_1^Tx_2}{(x_{1,2})^2x_{2,2}}\frac{\partial f_1}{\partial x_{1,1}},$$

$$\lambda_{2,1}=\frac{\partial f_2}{\partial x_{2,1}},\quad \lambda_{2,2}=\frac{\partial f_2}{\partial x_{2,2}}.$$

The denominator $q_1$ has zeros on $X$. Interestingly, Algorithm 3.3 still finds the GNE in the initial loop (see Example 6.3(iv)).

Second, we prove that Algorithm 3.3 terminates within finitely many loops under a finiteness assumption on KKT points. Recall that $\mathcal{S}$ denotes the set of all GNEs. When the complement $\mathcal{K}\setminus\mathcal{S}$ is a finite set, Algorithm 3.3 must terminate within finitely many loops.

THEOREM 3.6. *Assume the Lagrange multipliers are expressed as in* (3.1). *Suppose Assumption* 3.2 *holds for every triple* $(u,i,v_i)$ *produced by Algorithm* 3.3. *If the complement set* $\mathcal{K}\setminus\mathcal{S}$ *is finite, then Algorithm* 3.3 *must terminate within finitely many loops, and it either returns a GNE or detects its nonexistence.*

*Proof.* When $\mathcal{K} \setminus \mathcal{S} = \emptyset$, the algorithm terminates in the initial loop $k = 0$. When $\mathcal{K} \setminus \mathcal{S} \neq \emptyset$ and some $u \in \mathcal{K} \setminus \mathcal{S}$ is the minimizer of (3.7), the set $\mathcal{N} \neq \emptyset$. For each $i \in \mathcal{N}$, there exists $v_i \in \mathcal{S}_i(u_{-i})$ such that

$$\delta_i = f_i(v_i, u_{-i}) - f_i(u_i, u_{-i}) < 0.$$

By Assumption 3.2, the set $\mathscr{U}$ is updated with the newly added constraints (for $i \in \mathcal{N}$)

$$f_i(p_i(x), x_{-i}) - f_i(x_i, x_{-i}) \geq 0.$$

The point $u$ does not belong to $\mathscr{U}$ for all future loops. The cardinality of the set $\mathscr{U}$ decreases at least by one after each loop. Note that $\mathscr{U} \subseteq \mathcal{K}$. Therefore, if $\mathcal{K} \setminus \mathcal{S}$ is a finite set, then Algorithm 3.3 must terminate within finitely many loops.

Next, suppose Algorithm 3.3 terminates with a minimizer $u$ in Step 2. Then, $\delta_i \geq 0$ for all $i$, so every $u_i$ is a minimizer of $\mathrm{F}_i(u_{-i})$; i.e., $u$ is a GNE. ∎

In Theorem 3.6, the set $\mathcal{K} \setminus \mathcal{S}$ being finite is a genericity assumption. For GNEPs given by generic polynomials, there are finitely many KKT points. This is shown in the recent work [44]. For GNEPs given by generic rational functions, this can be shown by an argument similar to that in [44, Theorem 3.1]. Moreover, we remark that the cardinality $|\mathcal{K} \setminus \mathcal{S}|$ is only an upper bound for the number of loops taken by Algorithm 3.3. This bound is certainly not sharp, because the inequality constraint (3.6) may preclude several (or even all) KKT points that are not GNEs. In our numerical experiments, Algorithm 3.3 often terminates within a few loops.

For some special problems, the KKT point set may be infinite. When the complement set $\mathcal{K} \setminus \mathcal{S}$ is infinite, Algorithm 3.3 may not be guaranteed to terminate within finitely many loops. However, we can prove its asymptotic convergence under certain assumptions. For each $i = 1, \ldots, N$, we define the $i$th player's value function

$$\nu_i(x_{-i}) := \inf_{x_i \in X_i(x_{-i})} f_i(x_i, x_{-i}). \tag{3.14}$$

The function $\nu_i(x_{-i})$ is continuous under certain conditions, e.g., under the restricted inf-compactness (RIC) condition (see [18, Definition 3.13]). A sequence of functions $\{\phi^{(k)}(x)\}$ is said to be *uniformly continuous* at a point $x^*$ if for each $\epsilon > 0$, there exists $\tau > 0$ such that $\|\phi^{(k)}(x) - \phi^{(k)}(x^*)\| < \epsilon$ for all $k$ and for all $x$ with $\|x - x^*\| < \tau$. The following is the asymptotic convergence result.

THEOREM 3.7. *For the GNEP* (1.1)*, suppose Lagrange multipliers can be expressed as in* (3.1) *and Assumption* 3.2 *holds for every triple* $(u, i, v_i)$ *produced by Algorithm* 3.3*. In the kth loop, let* $u^{(k)}$*,* $v_i^{(k)}$ *be the minimizers of* (3.7)*,* (3.8)*, respectively, and let* $p_i^{(k)}$ *be the feasible extension in Step* 3*. Suppose* $u^* := (u_1^*, \ldots, u_N^*)$ *is an accumulation point of the sequence* $\{u^{(k)}\}_{k=1}^{\infty}$*. If for each* $i = 1, \ldots, N$*,*

(i) *the strict inequality* $g_{i,j}(u^*) > 0$ *holds for all* $j \in \mathcal{I}_2^{(i)}$*, and*
(ii) *the value function* $\nu_i(x_{-i})$ *is continuous at* $u_{-i}^*$*, and*
(iii) *the sequence of feasible extensions* $\{p_i^{(k)}\}_{k=1}^{\infty}$ *is uniformly continuous at* $u^*$*, then* $u^*$ *is a GNE for* (1.1).

*Proof.* Up to the selection of a subsequence, we assume that $u^{(k)} \to u^*$ as $k \to \infty$, without loss of generality. The condition (i) implies that $u^* \in X$ and $u_i^* \in X_i(u_{-i}^*)$ for every $i$. We need to show that each $u_i^*$ is a minimizer for the optimization $\mathrm{F}_i(u_{-i}^*)$. By the definition of $\nu_i$ as in (3.14), this is equivalent to showing that

$$\nu_i(u_{-i}^*) - f_i(u^*) \geq 0, \quad i = 1, \ldots, N. \tag{3.15}$$

For convenience of notation, let $p_i^{(k)}(x) = x_i$ for each $i \notin \mathcal{N}$ in the $k$th loop. Since $u^{(k)}$ is feasible for (3.7) in all previous loops, we have that

$$f_i(p_i^{(k')}(u^{(k)}), u_{-i}^{(k)}) - f_i(u^{(k)}) \geq 0 \quad \text{for all} \quad k' \leq k.$$

As $k \to \infty$, the above implies that

$$f_i(p_i^{(k')}(u^*), u_{-i}^*) - f_i(u^*) \geq 0 \quad \text{for all} \quad k'.$$

Then, for every $i$ and for every $k \in \mathbb{N}$,

$$\begin{aligned} &\nu_i(u_{-i}^*) - f_i(u^*) \\ &= \big(\nu_i(u_{-i}^*) - f_i(p_i^{(k)}(u^*), u_{-i}^*)\big) + \big(f_i(p_i^{(k)}(u^*), u_{-i}^*) - f_i(u^*)\big) \\ &\geq \nu_i(u_{-i}^*) - f_i(p_i^{(k)}(u^*), u_{-i}^*). \end{aligned} \tag{3.16}$$

Note that $\nu_i(u_{-i}^{(k)}) = f_i(p_i^{(k)}(u^{(k)}), u_{-i}^{(k)})$ for all $k$ and for all $i \in \mathcal{N}$ in the $k$th loop. Indeed, this is clear by construction when $i \in \mathcal{N}$. For $i \notin \mathcal{N}$, we know $u_i^{(k)}$ is a minimizer for $F_i(u_{-i}^{(k)})$. Let $p_i^{(k)}(x) = x_i$, then

$$\nu_i(u_{-i}^{(k)}) = f_i(u_i^{(k)}, u_{-i}^{(k)}) = f_i(p_i^{(k)}(u^{(k)}), u_{-i}^{(k)}).$$

Under the continuity assumption of $\nu_i$ at $u_{-i}^*$, the convergence $u^{(k)} \to u^*$ implies that

$$\nu_i(u_{-i}^*) = \lim_{k\to\infty} \nu_i(u_{-i}^{(k)}) = \lim_{k\to\infty} f_i(p_i^{(k)}(u^{(k)}), u_{-i}^{(k)}).$$

Because $\{p_i^{(k)}\}_{k=1}^{\infty}$ is uniformly continuous at $u^*$, for every fixed $\epsilon > 0$, there exists $\tau > 0$ such that for all $k$ big enough, we have

$$\|u^* - u^{(k)}\| < \tau, \quad \|p_i^{(k)}(u^*) - p_i^{(k)}(u^{(k)})\| < \epsilon.$$

Since $f_i$ is rational and the denominator is positive on $X$, we have

$$f_i(p_i^{(k)}(u^*), u_{-i}^*) - f_i(p_i^{(k)}(u^{(k)}), u_{-i}^{(k)}) \to 0 \quad \text{as} \quad k \to \infty.$$

In view of inequality (3.16), we can conclude that $\nu_i(u_{-i}^*) - f_i(u^*) \geq 0$. This shows that $u^*$ is a GNE. ∎

When there are strict inequality constraints (i.e., $\mathcal{I}_2^{(i)} \neq \emptyset$), the RIC condition is more subtle to check, but it is still applicable. Please note that the strict inequality $g_{i,j}(x_i, x_{-i}) > 0$ is equivalent to

$$g_{i,j}(x_i, x_{-i}) \cdot (z_{i,j})^2 = 1$$

for a new variable $z_{i,j}$. Similarly, rational functions can be equivalently reformulated as polynomials by introducing new variables. Therefore, the value function $\nu_i(x_{-i})$ can be equivalently expressed as the optimal value of a polynomial optimization problem in a higher dimensional space with weak inequalities only. If the RIC holds for the new formulation, then one can show the continuity of $\nu_i(x_{-i})$. There exist some conveniently checkable conditions for RIC (see, e.g., [6, section 6.5.1]). For instance, this is the case if the feasible set is compact or the objective satisfies some growth conditions. However, checking RIC directly for the rational optimization with strict inequality constraints is typically difficult. This issue is outside the scope of this paper.

Feasible extensions are sometimes given by polynomials. For such cases, a sufficient condition for condition (iii) of Theorem 3.7 to hold is that the degrees and coefficients of $\{p_i^{(k)}\}_{k=1}^{\infty}$ are uniformly bounded. As shown in subsection 4.1, when $F_i(x_{-i})$ has box, simplex, or ball constraints, feasible extensions have explicit expressions, and the corresponding polynomial function sequence $\{p_i^{(k)}\}_{k=1}^{\infty}$ has uniformly bounded degrees and coefficients. For rational feasible extensions, condition (iii) is harder to check, since it needs to be checked case by case.

We would like to remark that Theorems 3.4 and 3.6 only give sufficient conditions for Algorithm 3.3 to terminate within finitely many loops. But these conditions are not necessary. In other words, Algorithm 3.3 may still have finite convergence even if $|\mathcal{K} \setminus \mathcal{S}| = \infty$. This is because the positive definite matrix $\Theta$ is generically selected (so the optimization (3.7) has a unique minimizer) and feasible extensions may preclude several (or even all) KKT points that are not GNEs. We refer the reader to Example 6.1(i)–(ii) for such cases. When Algorithm 3.3 does not terminate within finitely many loops, Theorem 3.7 proves the asymptotic convergence under certain assumptions. We would like to remark that Algorithm 3.3 does not need to check whether these assumptions are satisfied or not, because it is self-verifying. By solving the optimization (3.8) for each player, we get a candidate GNE and then verify whether it is a true GNE or not. This does not require checking any other assumptions.

## 4. Feasible extensions of KKT points.

In this section, we discuss the existence and computation of feasible extensions $p_i$ required as in Assumption 3.2. They are important for solving GNEPs.

### 4.1. Some common cases.

The feasible extensions in Assumption 3.2 can be explicitly given for some common cases of optimization problems. Suppose the triple $(u, i, v_i)$ is given.

*Box constraints.* Suppose the feasible set of $\mathrm{F}_i(x_{-i})$ is

$$a(x_{-i}) \le A(x_{-i})x_i \le b(x_{-i}),$$

where $a, b \in \mathbb{R}[x_{-i}]^{m_i}$, $A \in \mathbb{R}[x_{-i}]^{m_i \times n_i}$. Suppose $A(x_{-i})$ has full row rank for all $x \in X$ and there is a matrix polynomial $B_0(x_{-i})$ such that

$$B(x_{-i}) := \begin{bmatrix} A(x_{-i})^T & B_0(x_{-i}) \end{bmatrix} \in \mathbb{R}[x_{-i}]^{n_i \times n_i}$$

is nonsingular for all $x \in X$. Let $\mu := (\mu_1, \ldots, \mu_{m_i})$ be the vector such that

$$\big(b_j(u_{-i}) - a_j(u_{-i})\big) \cdot \mu_j = b_j(u_{-i}) - (B(u_{-i})^T v_i)_j.$$

For the case $a_j(u_{-i}) = b_j(u_{-i})$, we just let $\mu_j = 0$. Since $v_i \in X_i(u_{-i})$, it is clear that each $\mu_j \in [0, 1]$. Then we choose $p_i$ as

$$p_i = B(x_{-i})^{-T} \hat{p}_i, \tag{4.1}$$

where $\hat{p}_i = (\hat{p}_{i,1}, \ldots, \hat{p}_{i,n_i})$ is defined by

$$\hat{p}_{i,j}(x) := \begin{cases} \mu_j a_j(x_{-i}) + (1 - \mu_j) b_j(x_{-i}), & 1 \le j \le m_i, \\ (B(x_{-i})^T x)_j, & m_i + 1 \le j \le n_i. \end{cases}$$

One can check that $p_i(u) = v_i$ and $p_i(x) \in X_i(x_{-i})$ for all $x \in \mathcal{K} \subseteq X$.

We would like to make some remarks about the existence of $B(x_{-i})$, which is nonsingular for all $x \in X$. When $A(x_{-i}) = A$ is independent of $x_{-i}$, such a constant matrix $B$ always exists. When $A(x_{-i})$ depends on $x_{-i}$, we may still have such a $B(x_{-i})$.

*Example* 4.1. Consider the 2-player GNEP with $x_1 \in \mathbb{R}^1, x_2 = (x_{2,1}, x_{2,2}) \in \mathbb{R}^2$. Suppose $X_1(x_2) = \{x_1 : (x_1)^2 \le \|x_2\|^2\}$ and $X_2(x_1)$ is given by the inequalities

$$0 \le \underbrace{\begin{bmatrix} x_1 & 1 + x_1 \end{bmatrix}}_{A(x_1)} \begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix} \le 3 - x_1.$$

The $A(x_1)$ has full row rank for all $x \in X$. We can construct

$$B(x_1) = \begin{bmatrix} x_1 & x_1 - 1 \\ 1 + x_1 & x_1 \end{bmatrix}$$

such that $\det(B(x_1)) = (x_1)^2 - ((x_1)^2 - 1) = 1$. Therefore, the matrix $B(x_1)$ is nonsingular for all $x_1 \in \mathbb{R}^1$.

*Simplex constraints.* Suppose the feasible set $X_i(x_{-i})$ is given as

$$d(x_{-i})^T x_i \le b(x_{-i}), \quad c_j(x_{-i}) x_{i,j} \ge a_j(x_{-i}), \, j \in [n_i].$$

In the above, $b \in \mathbb{R}[x_{-i}]$, $a = (a_1, \ldots, a_{n_i})$, $c = (c_1, \ldots, c_{n_i})$ and $d$ are vectors of polynomials in $x_{-i}$. Assume $c(x_{-i}), d(x_{-i}) > 0$ for all $x = (x_i, x_{-i}) \in X$. For convenience, use $\odot$ to denote the entrywise product, i.e.,

$$(c^{-1} \odot a)(x_{-i}) := \begin{bmatrix} c_1^{-1}(x_{-i}) a_1(x_{-i}) & \ldots & c_{n_i}^{-1}(x_{-i}) a_{n_i}(x_{-i}) \end{bmatrix}^T.$$

Let $\mu := (\mu_1, \ldots, \mu_{n_i})$ be a vector such that

$$\big((b - d^T c^{-1} \odot a)(u_{-i})\big) \cdot \mu_j = v_{i,j} - (c_j^{-1} a_j)(u_{-i}).$$

For the case that $b(u_{-i}) = (d^T c^{-1} \odot a)(u_{-i})$, just choose $\mu_j = 0$. For $v_i \in X_i(u_{-i})$, each $\mu_j \in [0,1]$. Then, we choose $p_i := (p_{i,1}, \ldots, p_{i,n_i})$ such that

$$p_{i,j}(x) = \mu_j \cdot \big((b - d^T c^{-1} \odot a)(x_{-i})\big) + (c_j^{-1} a_j)(x_{-i}). \tag{4.2}$$

One can check that $p_i(u) = v_i$ and $p_i(x) \in X_i(x_{-i})$ for all $x \in \mathcal{K} \subseteq X$.

*Ball constraints*. Suppose $X_i(x_{-i})$ is given as

$$\sum_{j=1}^{n_i} \big(a_j(x_{-i}) x_{i,j} - c_j(x_{-i})\big)^2 \le (R(x_{-i}))^2,$$

where $R \in \mathbb{R}[x_{-i}]$, and $a = (a_1, \ldots, a_{n_i})$, $c = (c_1, \ldots, c_{n_i})$ are vectors of rational functions in $x_{-i}$. Assume $a_j(x_{-i}) \neq 0$ on $X$. Let $\mu$ be such that

$$\|a(u_{-i}) \odot v_i - c(u_{-i})\| = \mu |R(u_{-i})|, \quad 0 \le \mu \le 1.$$

Then, choose scalars $(s_1, \ldots, s_{n_i})$ such that

$$\|a(u_{-i}) \odot v_i - c(u_{-i})\| \cdot s_j = a_j(u_{-i}) v_{i,j} - c_j(u_{-i}).$$

For the case $\|a(u_{-i}) \odot v_i - c(u_{-i})\| = 0$, just let $s_j = 1/\sqrt{n_i}$. Then we can choose $p_i := (p_{i,1}, \ldots, p_{i,n_i})$ as

$$p_{i,j}(x) := \big(c_j(x_{-i}) + s_j \cdot \mu \cdot R(x_{-i})\big) / a_j(x_{-i}). \tag{4.3}$$

One can verify that $p_i(u) = v_i$ and $p_i(x) \in X_i(x_{-i})$ for all $x \in \mathcal{K} \subseteq X$.

**4.2. The existence of feasible extensions.** The existence of rational feasible extensions in Assumption 3.2 can be shown under some assumptions. We consider the general case that the KKT set $\mathcal{K}$ as in (3.2) is finite. A polynomial feasible extension $p_i$ exists when $\mathcal{K}$ is finite.

THEOREM 4.2. *Assume $\mathcal{K}$ is a finite set. Then, for every triple $(u, i, v_i)$ with $u \in \mathcal{K}$, $i \in [N]$, and $v_i \in X_i(u_{-i})$, there must exist a feasible extension $p_i$ satisfying Assumption* 3.2. *Moreover, such a $p_i$ can be chosen as a polynomial vector function.*

*Proof.* Since the set $\mathcal{K}$ is finite, by polynomial interpolation, there must exist a real polynomial vector function $p_i$ such that

$$p_i(u) = v_i, \quad p_i(z) = z_i \quad \text{for all} \quad z := (z_1, \ldots, z_N) \in \mathcal{K} \setminus \{u\}. \tag{4.4}$$

Note that $\mathcal{K} \subseteq X$. For every $x = (x_1, \ldots, x_N) \in \mathcal{K} \setminus \{u\}$, we have $p_i(x) = x_i \in X_i(x_{-i})$. The polynomial function $p_i$ satisfies Assumption 3.2. ∎

When the set $\mathcal{K}$ is known, we can get a polynomial feasible extension $p_i$ as in Theorem 4.2 by polynomial interpolation. The following is such an example.

*Example* 4.3. Consider Example 3.1. There are four KKT points:

$$\begin{array}{ll} u_1^{(1)} = u_2^{(1)} = (0,0), & u_1^{(2)} = u_2^{(2)} = \left(\frac{\sqrt{17}-3}{4}, \frac{5-\sqrt{17}}{4}\right), \\ u_1^{(3)} = u_2^{(3)} = \left(\frac{1}{2}, 0\right), & u_1^{(4)} = u_2^{(4)} = \left(0, \frac{1}{2}\right). \end{array}$$

The $u^{(1)} = (u_1^{(1)}, u_2^{(1)})$ and $u^{(2)} = (u_1^{(2)}, u_2^{(2)})$ are not GNEs. For $u^{(1)}$, there are two minimizers for $\mathrm{F}_1(u_2^{(1)})$, which are $(1,0)$ and $(0,1)$. We can construct the feasible extension $p_1$ of $(1,0)$ at $u^{(1)}$ using polynomial interpolation. Consider a linear function $p_1$ such that

$$p_1 = (a_0 + a_1 x_{1,1} + a_2 x_{1,2} + a_3 x_{2,1} + a_4 x_{2,2},\, b_0 + b_1 x_{1,1} + b_2 x_{1,2} + b_3 x_{2,1} + b_4 x_{2,2}).$$

Equation (4.4) requires that

$$p_1(u_1^{(1)}, u_2^{(1)}) = (1,0), \quad p_1(u_1^{(k)}, u_2^{(k)}) = u_1^{(k)},\, k = 2,3,4.$$

This gives a linear system about coefficients of $p_1$:

$$\begin{array}{l} a_0 = 1,\, b_0 = 0, \\ a_0 + \frac{1}{2}a_1 + \frac{1}{2}a_3 = \frac{1}{2},\, b_0 + \frac{1}{2}b_1 + \frac{1}{2}b_3 = 0, \\ a_0 + \frac{1}{2}a_2 + \frac{1}{2}a_4 = 0,\, b_0 + \frac{1}{2}b_2 + \frac{1}{2}b_4 = \frac{1}{2}, \\ a_0 + \frac{\sqrt{17}-3}{4}a_1 + \frac{5-\sqrt{17}}{4}a_2 + \frac{\sqrt{17}-3}{4}a_3 + \frac{5-\sqrt{17}}{4}a_4 = \frac{\sqrt{17}-3}{4}, \\ b_0 + \frac{\sqrt{17}-3}{4}b_1 + \frac{5-\sqrt{17}}{4}b_2 + \frac{\sqrt{17}-3}{4}b_3 + \frac{5-\sqrt{17}}{4}b_4 = \frac{5-\sqrt{17}}{4}. \end{array}$$

The above linear system is consistent, and we get the feasible extension

$$p_1(x_1, x_2) = (1 - x_{1,1} - x_{1,2} - x_{2,2}, \quad x_{2,2}).$$

Similarly, we can also get the feasible extension of $(0,1)$ at $u^{(1)}$, which is

$$(x_{1,1}, \quad 1 - x_{2,1} - x_{2,2} - x_{1,1}).$$

At the point $u^{(2)}$, the minimizer of $\mathrm{F}_1(u_2^{(2)})$ is $\left(0, \frac{1}{2}\right)$. We apply polynomial interpolation again. The linear system in coefficients of $p_1$ is consistent for $\deg(p_1) = 2$. The following is a feasible extension:

$$\left(x_{2,1}(x_{2,1} - \tfrac{\sqrt{17}-3}{4})(x_{2,1} + \tfrac{3+\sqrt{17}}{2(5-\sqrt{17})}), \quad \tfrac{1}{2} - (x_{2,2} - \tfrac{1}{2})(x_{2,2} - \tfrac{5-\sqrt{17}}{4})(x_{2,2} + \tfrac{4}{5-\sqrt{17}})\right).$$

When the set $\mathcal{K}$ is not finite, Assumption 3.2 may still hold for some GNEPs. For instance, consider that there are no equality constraints, i.e., $\mathcal{I}_0^{(i)} = \emptyset$. Suppose $\mathcal{K}$ is compact and there exists a continuous map $\rho : \mathbb{R}^n \to \mathbb{R}^{n_i}$ such that $\rho(u) = v_i$ and $g_{i,j}(\rho(x), x_{-i}) > 0$ for all $x \in \mathcal{K}$ and for all $j \in \mathcal{I}_1^{(i)} \cup \mathcal{I}_2^{(i)}$. For every $\epsilon > 0$, one can approximate $\rho$ by a polynomial $p_i$ such that $\|p_i - \rho\| < \epsilon$ on $\mathcal{K}$. Therefore, for $\epsilon$ sufficiently small, $g_{i,j}(p_i(x), x_{-i}) > 0$ on $x \in \mathcal{K}$. Such a polynomial function $p_i$ is a feasible extension of $v_i$ at $u$.

**4.3. Computation of feasible extensions.** We discuss how to compute the rational feasible extension $p_i$ satisfying Assumption 3.2. For the set $\mathcal{K}$ as in (3.2), let $E_0$ denote the set of its equality constraining polynomials, and let $E_1$ denote the set of its (both weak and strict) inequality ones. Consider the set

$$\mathcal{K}_1 := \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} g(x) = 0 \, (g \in E_0), \\ g(x) \geq 0 \, (g \in E_1) \end{array} \right\}.$$

The set $\mathcal{K}$ may not be closed, but $\mathcal{K}_1$ is, and the closure of $\mathcal{K}$ is contained in $\mathcal{K}_1$. For a polynomial $p(x)$, if $p(x) \in X_i(x_{-i})$ for all $x \in \mathcal{K}_1$, then we also have $p(x) \in X_i(x_{-i})$ for all $\mathcal{K}$. Therefore, it is sufficient to get $p_i$ satisfying Assumption 3.2 with $\mathcal{K}$ replaced by $\mathcal{K}_1$.

Suppose the triple $(u, i, v_i)$ is given. First, choose a priori degree $l$, and choose a denominator $h$ that is positive on $\mathcal{K}$ (e.g., one may choose $h = 1$). Then, we consider the following feasibility problem in $(q, \mu)$:

$$\left\{ \begin{array}{l} q := (q_1, \ldots, q_{n_i}) \in (\mathbb{R}[x]_{2l})^{n_i}, \ \mu := (\mu_j)_{j \in \mathcal{I}_1^{(i)} \cup \mathcal{I}_2^{(i)}}, \\ q(u) = h(u) v_i, \ h \cdot g_{i,j}(q, x_{-i}) = 0 \, (j \in \mathcal{I}_0^{(i)}), \\ \mu_j \geq 0 \, (j \in \mathcal{I}_1^{(i)}), \ \mu_j > 0 \, (j \in \mathcal{I}_2^{(i)}), \\ h \cdot g_{i,j}(q, x_{-i}) - \mu_j \in \text{Ideal}[E_0]_{2l} + \text{Qmod}[E_1]_{2l}. \end{array} \right. \tag{4.5}$$

When all constraining polynomials $g_{i,j}$ are linear in $x_i$, the system (4.5) is convex in $(q, \mu)$, and it ensures that $p_i := q/h$ is a rational feasible extension satisfying Assumption 3.2. For such a case, a feasible pair $(q, \mu)$ for (4.5) can be obtained by solving a linear conic optimization problem.

*Example* 4.4. Consider the following 2-player GNEP:

$$\begin{array}{cl|cl} \min\limits_{x_1 \in \mathbb{R}^2} & \frac{(x_{2,1} + x_{2,2} - 2x_{1,1})(x_{1,1})^2 + 2x_{1,2}}{x_{2,1}} & \min\limits_{x_2 \in \mathbb{R}^2} & \frac{x_{2,1} - (x_{2,2})^2}{x_{2,2} + x_{1,1} + x_{1,2}} \\ s.t. & 2x_{1,1}x_{2,1} - x_{1,2}x_{2,2} \geq 0, & s.t. & 2x_{2,1}x_{2,2} - 1 \geq 0, \\ & x_{2,1}x_{2,2} - x_{1,1}x_{2,1} \geq 0, & & 1 - x_{2,2} \geq 0, \\ & 2x_{1,2}x_{2,2} - 1 \geq 0, & & 2 - x_{2,1} \geq 0, \\ & 2 - x_{1,2}x_{2,2} \geq 0, & & x_{2,1} \geq 0. \end{array} \tag{4.6}$$

Consider the triple $(u, 1, v_1)$ for $u = (u_1, u_2)$ with

$$u_1 = (0.5, 0.5), \quad u_2 = (0.5, 1), \quad v_1 = (1, 0.5).$$

For $l = 2$ and $h = x_{2,1}x_{2,2}$, a feasible $q$ given by (4.5) is $(x_{2,2}, x_{2,1})/2$. Let $p_1 = \frac{1}{2x_{2,1}x_{2,2}}(x_{2,2}, x_{2,1})$. Then, we have each $h \cdot g_{1,j}(p_1, x_2) \in \text{Ideal}[E_0]_{2l} + \text{Qmod}[E_1]_{2l}$:

$$\begin{array}{l} h \cdot g_{1,1}(p_1, x_2) = 0.25 + 0.25(2x_{2,1}x_{2,2} - 1), \\ h \cdot g_{1,2}(p_1, x_2) = (x_{2,1}x_{2,2} - 0.5)^2 + 0.25(2x_{2,1}x_{2,2} - 1), \\ h \cdot g_{1,3}(p_1, x_2) = 0, \quad h \cdot g_{1,4}(p_1, x_2) = 0.75 + 0.75(2x_{2,1}x_{2,2} - 1). \end{array}$$

For the triple $(u, i, v_i)$, when some constraining polynomials $g_{i,j}$ are nonlinear in $x_i$, the system (4.5) may not be convex in $(q, \mu)$. For such cases, it is not clear how to obtain feasible extensions in a computationally efficient way. The existence of such a $p_i$ is guaranteed when $\mathcal{K}$ is a finite set. This is shown in Theorem 4.2. When $\mathcal{K}$ is fully known, we can get the $p_i$ by polynomial interpolation. For other cases, it is not clear to us how to compute such a $p_i$ efficiently.

**5. Rational optimization problems.** This section discusses how to solve the rational optimization problems appearing in Algorithm 3.3.

**5.1. Rational polynomial optimization.** A general rational polynomial optimization problem is

$$\begin{cases} \min & A(x) := \frac{a_1(x)}{a_2(x)} \\ s.t. & x \in K, \end{cases} \tag{5.1}$$

where $a_1, a_2 \in \mathbb{R}[x]$ and $K \subseteq \mathbb{R}^n$ is a semialgebraic set. We assume the denominator $a_2(x) > 0$ on $K$; otherwise, one can minimize $A(x)$ over two subsets $K \cap \{x : a_2(x) > 0\}$ and $K \cap \{x : -a_2(x) > 0\}$ separately. Moment-SOS relaxations can be applied to solve (5.1). We refer the reader to [21, 23, 33] for related work. Please note that Lagrange multipliers are zeros for strict inequality constraints. So the KKT system does not need to consider strict inequality constraints. However, the strict inequalities are still used in the Moment-SOS relaxations, because they are relaxed to weak inequality constraints.

The rational optimization problems in Algorithm 3.3 may have strict inequalities. So we consider the case that $K$ is given as

$$K = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} p(x) = 0\,(p \in \Psi_0), \\ q(x) \ge 0\,(q \in \Psi_1), \\ q(x) > 0\,(q \in \Psi_2) \end{array} \right\}, \tag{5.2}$$

where $\Psi_0$, $\Psi_1$, and $\Psi_2$ are finite sets of constraining polynomials in $x$. Since $a_2(x) > 0$ on $K$, we have $A(x) \ge \gamma$ on $K$ if and only if $a_1(x) - \gamma a_2(x) \ge 0$ on $K$, or equivalently $a_1 - \gamma a_2 \in \mathscr{P}_d(K)$ for the degree

$$d := \max\{\deg(a_1), \deg(a_2)\}.$$

The rational optimization problem (5.1) is then equivalent to

$$\begin{cases} \gamma^* := \max & \gamma \\ \qquad s.t. & a_1(x) - \gamma a_2(x) \in \mathscr{P}_d(K). \end{cases} \tag{5.3}$$

Denote the weak inequality set

$$K_1 := \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} p(x) = 0\,(p \in \Psi_0), \\ q(x) \ge 0\,(q \in \Psi_1 \cup \Psi_2) \end{array} \right\}. \tag{5.4}$$

Note that $K_1$ is closed and $\mathrm{cl}(K) \subseteq K_1$. We consider the moment optimization problem

$$\begin{cases} \min & \langle a_1, w \rangle \\ s.t. & \langle a_2, w \rangle = 1,\, w \in \mathscr{R}_d(K_1). \end{cases} \tag{5.5}$$

It is a moment reformulation for the optimization

$$
\left\{\begin{array}{rl} a^* := \min & A(x) \\ s.t. & x \in K_1. \end{array}\right. \tag{5.6}
$$

Note that (5.6) is a relaxation of (5.1). It is worthy to observe that if a minimizer of (5.6) lies in the set $K$, then it is also a minimizer of (5.1).

We apply Moment-SOS relaxations to solve (5.5). Let

$$
d_0 := \max\left\{\lceil d/2\rceil,\, \lceil \deg(g)/2\rceil\,(g \in \Psi_0 \cup \Psi_1 \cup \Psi_2)\right\}. \tag{5.7}
$$

For an integer $k \ge d_0$, the $k$th order SOS relaxation for (5.3) is

$$
\left\{\begin{array}{rl} \gamma^{(k)} := \max & \gamma \\ s.t. & a_1(x) - \gamma a_2(x) \in \operatorname{Ideal}[\Psi_0]_{2k} + \operatorname{Qmod}[\Psi_1 \cup \Psi_2]_{2k}. \end{array}\right. \tag{5.8}
$$

The dual optimization of (5.8) is the $k$th order moment relaxation

$$
\left\{\begin{array}{rl} a^{(k)} := \min & \langle a_1, y\rangle \\ s.t. & L_p^{(k)}[y] = 0\,(p \in \Psi_0), \\ & L_q^{(k)}[y] \succeq 0\,(q \in \Psi_1 \cup \Psi_2), \\ & \langle a_2, y\rangle = 1,\, M_k[y] \succeq 0,\, y \in \mathbb{R}^{\mathbb{N}^n_{2k}}. \end{array}\right. \tag{5.9}
$$

Since (5.9) is a relaxation of (5.5), if (5.9) is infeasible, then (5.1) is also infeasible.

The following is the Moment-SOS algorithm for solving (5.1). It can be conveniently implemented with the software `GloptiPoly 3` [21].

*Algorithm* 5.1. For the rational optimization problem (5.1), let $k := d_0$.

Step 1 Solve the $k$th order moment relaxation (5.9). If it is infeasible, then (5.1) has no feasible points and stop. Otherwise, solve it for the optimal value $a^{(k)}$ and a minimizer $y^*$ if they exist. Let $t := d_0$ and go to Step 2.

Step 2 Check whether or not there is an order $t \in [d_0, k]$ such that

$$
r := \operatorname{rank} M_t[y^*] = \operatorname{rank} M_{t-d_0}[y^*]. \tag{5.10}
$$

Step 3 If (5.10) fails, let $k := k+1$ and go to Step 1; if (5.10) holds, find points $z_1, \dots, z_r \in K_1$ and scalars $\mu_1, \dots, \mu_r > 0$ such that

$$
y^*|_{2t} = \mu_1 [z_1]_{2t} + \cdots + \mu_r [z_r]_{2t}. \tag{5.11}
$$

Step 4 Output each $z_i \in K$ with $a_2(z_i) > 0$ as a minimizer of (5.1).

In Step 2, the rank condition (5.10) is called *flat truncation*. It is sufficient and almost necessary for checking convergence of the Moment-SOS hierarchy (see [34]). Once (5.10) is met, the moment relaxation (5.9) is tight for solving (5.5), and the decomposition (5.11) can be computed by the Schur decomposition [20]. This is also implemented in the software `GloptiPoly 3` [21]. When $\operatorname{Ideal}[\Psi_0] + \operatorname{Qmod}[\Psi_1 \cup \Psi_2]$ is archimedean, one can show that $a^{(k)} \to a^*$ as $k \to \infty$ (see [35]). The following is the justification for the conclusion in Step 4.

THEOREM 5.2. *Assume $a_2 \ge 0$ on $K_1$. Suppose $y^*$ is a minimizer of* (5.9) *and it satisfies* (5.10) *for some order $t \in [d_0, k]$. Then, each $z_i$ in* (5.11)*, such that $a_2(z_i) > 0$ and $z_i \in K$, is a minimizer of* (5.1).

*Proof.* Under the rank condition (5.10), the decomposition (5.11) holds for some points $z_1,\dots,z_r \in K_1$ (see [20, 34]). The constraint $\langle a_2, y^*\rangle = 1$ implies that

$$1 = \langle a_2, y^*\rangle = \mu_1 a_2(z_1) + \cdots + \mu_r a_2(z_r).$$

Since $a_2 \geq 0$ on $K_1$, we know all $a_2(z_j) \geq 0$. Let $J_1 := \{j : a_2(z_j) > 0\}$ and $J_2 := \{j : a_2(z_j) = 0\}$; then

$$\langle a_1, y^*\rangle = \sum_{j\in J_1} \mu_j a_2(z_j) A(z_j) + \sum_{j\in J_2} \mu_j a_1(z_j).$$

Note that $\sum_{j\in J_1} \mu_j a_2(z_j) = 1$ and each $[z_j]_{2k} \in \mathscr{R}_{2k}(K_1)$. For all nonnegative scalars $\nu_j \geq 0$, $j \in J_1 \cup J_2$, such that $\sum_{j\in J_1} \nu_j a_2(z_j) = 1$, the tms

$$z(\nu) := \nu_1 [z_1]_{2k} + \cdots + \nu_r [z_r]_{2k}$$

is a feasible point for the moment relaxation (5.9). Therefore, the optimality of $y^*$ implies that $A(z_j) = a^{(k)}$ for all $j \in J_1$. Since $a^{(k)} \leq a^*$ and each $z_j \in K_1$, we have $A(z_j) \geq a^*$. Hence, $A(z_j) = a^*$ for all $j \in J_1$. Note that (5.5) is a relaxation of (5.6). So each $z_j$ $(j \in J_1)$ is a minimizer of (5.6). Therefore, every $z_i \in K$ satisfying $a_2(z_i) > 0$ is a minimizer of (5.1). ◻

In the decomposition (5.11), it is possible that no $z_i$ belongs to the set $K$. This is because the feasible set $K$ may not be closed, due to strict inequality constraints. For such a case, the optimal value of (5.1) may not be achievable. If we obtain a minimizer $y^*$ of (5.9) such that $\operatorname{rank} M_k[y^*]$ is maximum and (5.10) is satisfied, then we can get all minimizers of (5.6). Moreover, if (5.6) has infinitely many minimizers, the rank condition (5.10) cannot be satisfied easily. We refer the reader to [30, 34] for this fact. When primal-dual interior point methods are used to solve (5.9), a minimizer $y^*$ with $\operatorname{rank} M_k[y^*]$ maximum is often returned. Therefore, if (5.6) has finitely many minimizers and primal-dual interior point methods are used, then some points $z_i$ (5.11) must belong to the set $K$. This means that we can typically find all minimizers of (5.1) and (5.6), even if there are strict inequality constraints. However, if the optimal value of (5.1) is not achievable, then no $z_i$ in (5.11) belongs to $K$. We refer the reader to [21, 23, 33] for the work on solving rational optimization problems.

**5.2. The optimization for all players.** The rational optimization problem in Step 2 of Algorithm 3.3 is

$$\left\{ \begin{array}{ll} \min & \theta(x) := [x]_1^T \Theta [x]_1 \\ s.t. & x \in \mathscr{U}, \end{array} \right. \tag{5.12}$$

where $\Theta$ is a generic positive definite matrix. The feasible set $\mathscr{U}$ can be expressed as in the form (5.2), with polynomial equalities and weak/strict inequalities, for some polynomial sets $\Psi_0, \Psi_1, \Psi_2$. That is, (5.12) can be expressed in the form of (5.1), with denominators being 1. Denote the corresponding set

$$\mathscr{U}_1 = \{x \in \mathbb{R}^n \,|\, p(x) = 0\, (p \in \Psi_0),\, q(x) \geq 0\, (q \in \Psi_1 \cup \Psi_2)\}. \tag{5.13}$$

Since $\Theta$ is positive definite, the objective $\theta$ is coercive and strictly convex. When $\Theta$ is also generic, the function $\theta$ has a unique minimizer $u^*$ on the set $\mathscr{U}_1$ if it is nonempty. Suppose $y^*$ is a minimizer of the $k$th order moment relaxation of (5.12). Then, in Algorithm 5.1, the rank condition (5.10) is reduced to

$$\operatorname{rank} M_t[y^*] = 1$$

for some order $t \in [d_0, k]$ and the decomposition (5.11) is equivalent to $y^*|_{2t} = \mu_1 [z_1]_{2t}$ for some $z_1 \in \mathscr{U}_1$. Algorithm 5.1 can be applied to solve (5.12). The following are some special properties of Moment-SOS relaxations for (5.12).

THEOREM 5.3. *Assume $\Theta$ is a generic positive definite matrix.*

(i) *If the set $\mathscr{U}_1$ is empty and Ideal$[\Psi_0]$ + Qmod$[\Psi_1 \cup \Psi_2]$ is archimedean, then the moment relaxation for* (5.12) *must be infeasible when the order $k$ is big enough.*

(ii) *Suppose $\mathscr{U}_1 \neq \emptyset$ and Ideal$[\Psi_0]$ + Qmod$[\Psi_1 \cup \Psi_2]$ is archimedean. Let $u^{(k)} := (y^{(k)}_{e_1}, \dots, y^{(k)}_{e_n})$, where $y^{(k)}$ is the minimizer of the kth order moment relaxation of* (5.12). *Then, $u^{(k)}$ converges to the unique minimizer of $\theta$ on $\mathscr{U}_1$.*

(iii) *Suppose the real zero set of $\Psi_0$ is finite. If $\mathscr{U}_1 \neq \emptyset$, then we must have* $\operatorname{rank} M_t[y^*] = 1$ *for some $t \in [d_0, k]$, when $k$ is sufficiently large.*

*Proof.* (i) When $\mathscr{U}_1 = \emptyset$, the constant $-1$ can be viewed as a positive polynomial on $\mathscr{U}_1$. Since Ideal$[\Psi_0]$ + Qmod$[\Psi_1 \cup \Psi_2]$ is archimedean, we have $-1 \in$ Ideal$[\Psi_0]_{2k}$ + Qmod$[\Psi_1 \cup \Psi_2]_{2k}$ for $k$ big enough, by Putinar's Positivstellensatz. For such $k$, the corresponding SOS relaxation (5.8) is unbounded from above, and hence the corresponding moment relaxation must be infeasible.

(ii) When $\mathscr{U}_1 \neq \emptyset$, the objective $\theta$ has a unique minimizer $u^*$ on $\mathscr{U}_1$. The convergence of $u^{(k)}$ is implied by [34, Theorem 3.3] (see also [50]).

(iii) When the real zero set of $\Psi_0$ is finite and $\mathscr{U}_1 \neq \emptyset$, the conclusion can be implied by [29, Proposition 4.6] (see also [30]). ∎

**5.3. Checking generalized Nash equilibria.** Once we get a minimizer $u$ of (5.12), we need to check whether it is a GNE or not. For each $i = 1, \dots, N$, we need to solve the rational optimization problem

$$\left\{ \begin{array}{rl} \delta_i := \min & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ s.t. & x_i \in X_i(u_{-i}), \end{array} \right. \tag{5.14}$$

where $f_i, X_i(u_{-i})$ are given in (1.1). Assume the KKT conditions hold and the Lagrange multipliers can be expressed as in (3.1), then (5.14) is equivalent to

$$\left\{ \begin{array}{rl} \min & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ s.t. & \nabla_{x_i} f_i(x_i, u_{-i}) = \sum\limits_{j \in \mathcal{I}_0^{(i)} \cup \mathcal{I}_1^{(i)}} \tau_{i,j}(x_i, u_{-i}) \nabla_{x_i} g_{i,j}(x_i, u_{-i}), \\ & \tau_{i,j}(x_i, u_{-i}) g_{i,j}(x_i, u_{-i}) = 0,\ \tau_{i,j}(x_i, u_{-i}) \geq 0 \, (j \in \mathcal{I}_1^{(i)}), \\ & x_i \in X_i(u_{-i}). \end{array} \right. \tag{5.15}$$

We can equivalently express the feasible set of (5.15) in the form

$$Y_i(u_{-i}) = \left\{ x_i \in \mathbb{R}^{n_i} \;\middle|\; \begin{array}{l} p(x_i) = 0 \, (p \in \Psi_{i,0}), \\ q(x_i) \geq 0 \, (q \in \Psi_{i,1}), \\ q(x_i) > 0 \, (q \in \Psi_{i,2}) \end{array} \right\} \tag{5.16}$$

for three sets $\Psi_{i,0}, \Psi_{i,1}, \Psi_{i,2}$ of polynomials in $x_i$. In computational practice, we need to assume (5.15) is solvable, i.e., the solution set of (5.15) is nonempty. As in subsection 5.1, we can apply a similar version of Algorithm 5.1 to solve the rational optimization problem (5.15). Conclusions simlar to those in Theorem 5.3 hold for the corresponding Moment-SOS relaxations. A difference is that all rational functions for (5.14) are only in the variable $x_i$ instead of $x$. It may have several different minimizers, so the rank in (5.10) may be bigger than one. Generally, the optimization (5.15) is easier to solve than (5.12).

**6. Numerical experiments.** This section gives numerical experiments for Algorithm 3.3 to solve rGNEPs. The rational optimization problems are solved by Moment-SOS relaxations, which are implemented with the software `GloptiPoly 3` [21]. The semidefinite programs for the Moment-SOS relaxations are solved by SeDuMi [51]. The computation is implemented in MATLAB R2018a, in a Laptop with CPU 8th Generation Intel Core i5-8250U and RAM 16 GB. For cleanness of presentation, only four decimal digits are displayed for computational results. The accuracy for a point $u$ to be a GNE is measured by the quantity

$$\delta := \min\{\delta_1, \dots, \delta_N\}, \tag{6.1}$$

where $\delta_i$ is the optimal value of (3.8). The point $u$ is a GNE if and only if $\delta = 0$. Due to numerical issues, $u$ can be viewed as a GNE if $\delta$ is nearly zero (e.g., $\delta \geq -10^{-6}$). For cleanness of presentation, we do not list the constraining functions $g_{i,j}$ explicitly. Instead, they are ordered row by row, from top to bottom; in each row, they are ordered from left to right. If there is an inequality like $a(x) \leq b(x)$, then the corresponding constraining function is $b(x) - a(x)$.

To implement Algorithm 3.3, we need rational LMEs. This is reviewed in subsection 2.3. More details can be found in [36]. For some standard constraints (e.g., box, simplex, or balls), we can have LMEs explicitly given as follows.

(i) Consider the box constraints $a(x_{-i}) \leq x_i \leq b(x_{-i})$, where $a = (a_1, \dots, a_{n_i})$, $b = (b_1, \dots, b_{n_i})$. The LME is, for $j = 1, \dots, n_i$,

$$\lambda_{i,2j-1} = \frac{b_j(x_{-i}) - x_{i,j}}{b_j(x_{-i}) - a_j(x_{-i})} \cdot \frac{\partial f_i}{\partial x_{i,j}}, \quad \lambda_{i,2j} = \frac{a_j(x_{-i}) - x_{i,j}}{b_j(x_{-i}) - a_j(x_{-i})} \cdot \frac{\partial f_i}{\partial x_{i,j}}. \tag{6.2}$$

(ii) Consider the simplex constraints $u(x_{-i}) \geq e^T x_i$, $x_i \geq l(x_{-i})$, where $l$ is a vector function in $x_{-i}$. The LME is $\lambda_i = (\lambda_{i,1}, \hat{\lambda}_i)$, with

$$\lambda_{i,1} = -\frac{(x_i - l(x_{-i}))^T \nabla_{x_i} f_i}{u(x_{-i}) - e^T l(x_{-i})}, \quad \hat{\lambda}_i = \nabla_{x_i} f_i + \lambda_{i,1} \cdot e. \tag{6.3}$$

(iii) Consider the ball-type constraint $r(x_{-i}) \leq \|x_i - c\|^2 \leq R(x_{-i})$, where $c = (c_1, \dots, c_{n_i})$ is a constant vector. The LME is

$$\lambda_i = \left( \frac{R(x_{-i}) \frac{\partial f_i}{\partial x_{i,1}} - (x_{i,1} - c_1) \cdot (x_i - c)^T \nabla_{x_i} f_i}{2(x_{i,1} - c_1)(R(x_{-i}) - r(x_{-i}))}, \frac{r(x_{-i}) \frac{\partial f_i}{\partial x_{i,1}} - (x_{i,1} - c_1) \cdot (x_i - c)^T \nabla_{x_i} f_i}{2(x_{i,1} - c_1)(R(x_{-i}) - r(x_{-i}))} \right). \tag{6.4}$$

For the special case that $r(x_{-i}) = 0$, the LME is reduced to

$$\lambda_i = (c - x_i)^T \nabla_{x_i} f_i / \big(2R(x_{-i})\big). \tag{6.5}$$

**6.1. Some fractional quadratic GNEPs.** First, we consider rGNEPs with fractional quadratic objectives and standard constraints (e.g., box, simplex, or balls). These GNEPs often appear in various applications. We give details for applying Algorithm 3.3 in such problems.

*Example* 6.1. (i) Consider the 2-player rGNEP

$$\begin{array}{cll|cll} \min\limits_{x_1 \in \mathbb{R}^2} & \frac{-(x_{1,1})^2 - x_{2,1}x_{1,1}}{x_{1,2}x_{2,2}+1} & & \min\limits_{x_2 \in \mathbb{R}^2} & \frac{3x_{2,1}x_{2,2} - 2x_{2,2}}{x_{1,2}x_{2,2}+1} \\ s.t. & (x_{2,1})^2 - x_1^T x_1 \geq 0, & & s.t. & 0.5 \leq x_{2,1} \leq 1, \\ & & & & 0 \leq x_{2,2} \leq x_{1,1}. \end{array} \tag{6.6}$$

The LME for the first player is in the form of (6.5), and the LMEs for the second player are given as in (6.2). Precisely, we have $\lambda_1 = \frac{-x_1^T \nabla_{x_1} f_1}{2(x_{2,1})^2}$ and

$$\lambda_2 = \left( (2-2x_{2,1})\frac{\partial f_2}{\partial x_{2,1}}, \quad (1-2x_{2,1})\frac{\partial f_2}{\partial x_{2,1}}, \quad \frac{x_{1,1}-x_{2,2}}{x_{1,1}}\frac{\partial f_2}{\partial x_{2,2}}, \quad -\frac{x_{2,2}}{x_{1,1}}\frac{\partial f_2}{\partial x_{2,2}} \right).$$

By applying Algorithm 3.3, we get

| | |
|---|---|
| $k=0$ | $u_1^{(0)} = u_2^{(0)} = (0.6667, 0.0000)$,<br>$\delta_1 = -3.6732 \cdot 10^{-7}$, $\quad \delta_2 = -0.3333$,<br>$v_2^{(0)} = (0.5000, 0.6667)$, $\quad p_2^{(0)}(x) = (0.5, x_{1,1})$. |
| $k=1$ | $u_1^{(1)} = (0.4930, -0.0835)$, $\quad u_2^{(1)} = (0.5000, 0.4930)$,<br>$\delta_1 = -4.3101 \cdot 10^{-7}$, $\quad \delta_2 = -8.9324 \cdot 10^{-9}$.<br>A GNE is returned in 4.22 seconds. |

In the above, $u^{(k)} = (u_1^{(k)}, u_2^{(k)})$, $v_i^{(k)}$ denote the minimizers of (3.7)–(3.8) in the $k$th loop. The $p_2^{(0)}(x)$ is the feasible extension of $v_2^{(0)}$ at $u^{(0)}$, which is given as in (4.1). Interestingly, (6.6) has infinitely many non-GNE KKT points, because one can check that $(t,0,t,0) \in \mathcal{K} \setminus \mathcal{S}$ for every $t \in [\frac{2}{3}, 1]$. However, Algorithm 3.3 still had finite convergence, as verified in computational practice. It implies that the upper bound $|\mathcal{K} \setminus \mathcal{S}|$ given in Theorem 4.2 is not sharp. In addition, we would like to remark that finite convergence is guaranteed by the use of feasible extension $p_2(x) = (0.5, x_{1,1})$. Since

$$f_2(x_1, p_2(x)) - f_2(t,0,t,0) = -0.5t < 0 \quad \forall t \in [2/3,\, 1],$$

then the whole set $\{(t,0,t,0) : t \in [\frac{2}{3}, 1]\}$ can be precluded by (3.6).
(ii) For the GNEP (6.6), if the first player's objective function is changed to

$$\frac{-(x_{1,1})^2 + x_{2,1}x_{1,1}}{x_{1,2}x_{2,2}+1},$$

then Algorithm 3.3 produces the following computational results:

| | |
|---|---|
| $k=0$ | $u_1^{(0)} = (0.3333,\, -0.3049)$, $\quad u_2^{(0)} = (0.6667,\, 0.0000)$,<br>$\delta_1 = -1.0000$, $\quad \delta_2 = -0.1856$,<br>$v_1^{(0)} = (-0.6667,\, 0.0000)$, $\quad p_1^{(0)}(x) = (-x_{2,1},\, 0)$,<br>$v_2^{(0)} = (0.5000,\, 0.3333)$, $\quad p_2^{(0)}(x) = (0.5,\, x_{1,1})$. |
| $k=1$ | Nonexistence of GNEs is detected in 5.56 seconds. |

Similar to (i), there are infinitely many non-GNE KKT points, which are $(\alpha, \beta, 2\alpha, 0)$ with

$$\alpha \in [1/3,\, 1/2], \quad \beta \in [-\sqrt{3}\alpha,\, \sqrt{3}\alpha].$$

However, Algorithm 3.3 successfully detected the nonexistence of GNEs at the loop $k=1$.

*Example* 6.2. Consider the rGNEP with jointly simplex constraints

$$(6.7) \qquad \begin{array}{cc|cc} \min\limits_{x_1 \in \mathbb{R}^{n_1}} & \frac{x^TA_1x+x^Ta_1+c_1}{x^TB_1x+x^Tb_1+d_1} & \min\limits_{x_2 \in \mathbb{R}^{n_2}} & \frac{x^TA_2x+x^Ta_2+c_2}{x^TB_2x+x^Tb_2+d_2} \\ s.t. & x_1 \in \Delta_1(x_2), & s.t. & x_2 \in \Delta_2(x_1). \end{array}$$

Table 1
*Numerical results for Example* 6.2.

| $k$ | $(u_1^{(k)}, u_2^{(k)})$ | $\delta^{(k)}$ |
|---|---|---|
| 0 | $(0.0000, 0.5000)$, $(0.0000, 0.0000)$ | $-0.1429$ |
| 1 | $(0.0000, 0.0000)$, $(0.0000, 0.0354)$ | $-0.4425$ |
| 2 | $(0.0000, 0.4831)$, $(0.5169, 0.0000)$ | $-0.2476$ |
| 3 | $(0.2910, 0.1089)$, $(0.6001, 0.0000)$ | $-0.0583$ |
| 4 | $(0.0000, 0.2742)$, $(0.7258, 0.0000)$ | $-1.14 \cdot 10^{-7}$ |

In the above, for each $i = 1, 2$, $A_i, B_i \in \mathbb{R}^{n\times n}$, $a_i, b_i \in \mathbb{R}^n$, $c_i, d_i \in \mathbb{R}$, and

$$\Delta_i(x_{-i}) := \left\{x_i \in \mathbb{R}^{n_i} : 1 - e^T x \geq 0,\, x_{i,1} \geq 0, \ldots, x_{i,n_i} \geq 0\right\}.$$

For both players, we use LMEs as given in (6.3), of which denominators have zeros in the feasible set $X = \{x \in \mathbb{R}^n_+ : 1 - e^T x \geq 0\}$. Precisely, they vanish when $e^T x_{-i} = 1$, $i = 1, 2$. Moreover, the set of complex KKT points for (6.7) has a positive dimension (see [44]) for all $A_i, B_i, a_i, b_i, c_i$, and $d_i$. Indeed, for all $t \in [0, 1]$, the pair of $x_1 = (0, t, 0, \ldots, 0)$ and $x_2 = (1 - t, 0, \ldots, 0)$ is a complex KKT point, because the active constraint gradients $e, e_2, e_3, \ldots, e_n$ span the entire space.

For instance, let $n_1 = n_2 = 2$ and

$$A_1 = \begin{bmatrix} 3 & 2 & -1 & 3 \\ 2 & 0 & -2 & 0 \\ -1 & -2 & 0 & -2 \\ 3 & 0 & -2 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -2 & 3 & 1 \\ 0 & 3 & -4 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 4 & 0 & 2 & -2 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 3 & -1 \\ -2 & -1 & -1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 1 & 2 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 3 & 2 & 0 & 4 \end{bmatrix},$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 0.5 \\ 1 \\ -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 0 \\ -0.5 \\ 1 \end{bmatrix},$$

$$c_1 = 3, \quad c_2 = -2, \quad d_1 = 3.5, \quad d_2 = 3.$$

By a symbolic computation, one can check that the pair of $x_1 = (0, t)$ and $x_2 = (1 - t, 0)$ is a KKT point for all $t \in [0, \beta]$, where $\beta \approx 0.4831$ is the unique real zero of $\beta^3 + \frac{1}{3}\beta^2 + \frac{97}{48}\beta - \frac{7}{6} = 0$. Apply Algorithm 3.3 to the GNEP (6.7). The computational results are displayed in Table 1. In the $k$th loop, the $u^{(k)} = (u_1^{(k)}, u_2^{(k)})$ denotes the minimizer of (3.7), and $\delta^{(k)}$ is the accuracy for $u^{(k)}$ computed as in (6.1). Each feasible extension is selected in form of (4.2). We got a GNE at the loop $k = 4$ with $\delta = -1.14 \cdot 10^{-7}$. It took around 16.81 seconds.

**6.2. Some explicit examples.** In the following, we present some explicit examples of rGNEPs. For cleanness of presentation, we only report computational results at the last loop for Algorithm 3.3.

*Example* 6.3. (i) Consider the GNEP in (1.4). The LME for the first player is

$$\lambda_1 = \left(\frac{x_{1,2} x_1^T \nabla_{x_1} f_1}{2x_{2,1}}, 0, 0\right).$$

For the second player, the LME is given by (6.3). Each LME has a positive denominator on $X$. Algorithm 3.3 terminated at the initial loop $k=0$. The computed GNE is $u=(u_1,u_2)$, with

$$u_1=(1.3561, 0.7374), \quad u_2=(1.0000, 1.0468), \quad \delta=-3.44\cdot 10^{-8}. \tag{6.8}$$

It took around 8.36 seconds.

Consider its equivalent polynomials reformulation (1.5). For the first player, the LME is

$$\lambda_1 = \left(\frac{x_{1,2}}{x_{2,1}}\frac{\partial f_1}{\partial x_{1,2}},\, 0,\, 0,\, x_{1,3}\cdot\frac{\partial f_1}{\partial x_{1,3}}\right).$$

For the second player, the LME is

$$\begin{aligned}
\lambda_{2,1} &= \left(\tfrac{\partial f_2}{\partial x_{2,1}} - (x_{2,3})^2\tfrac{\partial f_2}{\partial x_{2,3}}\right)\tfrac{1-x_{2,1}}{x_{1,1}+x_{1,2}-1} + \tfrac{\partial f_2}{\partial x_{2,2}}\tfrac{1-x_{2,2}}{x_{1,1}+x_{1,2}-1},\\
\lambda_{2,2} &= \left(\tfrac{\partial f_2}{\partial x_{2,1}} - (x_{2,3})^2\tfrac{\partial f_2}{\partial x_{2,3}}\right)+\lambda_{2,1}, \quad \lambda_{2,3}=\tfrac{\partial f_2}{\partial x_{2,2}}+\lambda_{2,1}, \quad \lambda_{2,4}=x_{2,3}\cdot\tfrac{\partial f_2}{\partial x_{2,3}}.
\end{aligned}$$

Each LME has a positive denominator on $X$. Algorithm 3.3 also terminated at the initial loop $k=0$. The computed GNE is $\hat{u}=(\hat{u}_1,\hat{u}_2)$ with

$$\hat{u}_1=(1.3561, 0.7374, 0.7374), \quad \hat{u}_2=(1.0000, 1.0468, 1.0000), \quad \delta=-2.70\cdot 10^{-8}.$$

The result is consistent with that in (6.8). But the computation took around 264.42 seconds. It is much more efficient to solve the original rational GNEP.

(ii) For the GNEP in (1.4), if objective functions are changed to

$$f_1(x)=\frac{(x_{1,2})^2+x_{1,1}x_{1,2}(e^Tx_2)}{x_{1,1}}, \quad f_2(x)=\frac{(x_{2,2})^2-x_{2,1}x_{2,2}(e^Tx_1)}{x_{2,1}}, \tag{6.9}$$

then there is no GNE. This was detected by Algorithm 3.3 at the initial loop $k=0$. It took about 5.47 seconds.

Like in (i), we also consider the equivalent polynomial GNEP with the updated objective. By applying Algorithm 3.3, we detected the nonexistence of GNEs at the initial loop $k=0$. It took around 19.61 seconds.

(iii) Consider the GNEP in Example 3.1. We use the LMEs as in (6.3) and the feasible extension as in (4.2). By Algorithm 3.3, we got the GNE $u=(u_1,u_2)$ at the loop $k=1$ with

$$u_1=(0.0000, 0.5000), \quad u_2=(0.0000, 0.5000), \quad \delta=-4.47\cdot 10^{-8}.$$

It took around 3.28 seconds.

(iv) Consider the GNEP in Example 3.5. We use the LMEs as in (3.13). Since, for each $i$, the feasible set $X_i(x_{-i})$ is independent to $x_{-i}$, we apply the trivial feasible extension $p_i(x)=x_i$. By Algorithm 3.3, we got the GNE $u=(u_1,u_2)$ in the initial loop with

$$u_1=(0.0000, 0.0000), \quad u_2=(1.0000, 1.0000), \quad \delta=-5.45\cdot 10^{-9}.$$

It took around 2.03 seconds.

(v) Consider the GNEP in Example 4.4. For the first player's optimization, we have the following rational LMEs:

$$\begin{aligned}
\lambda_{1,1} &= \tfrac{x_{2,2}-x_{1,1}}{x_{2,2}(2x_{2,1}-x_{1,2})}\cdot\tfrac{\partial f_1}{\partial x_{1,1}}, & \lambda_{1,2} &= \tfrac{x_{1,2}x_{2,2}-2x_{1,1}x_{2,1}}{x_{2,1}x_{2,2}(2x_{2,1}-x_{1,2})}\cdot\tfrac{\partial f_1}{\partial x_{1,1}},\\
\lambda_{1,3} &= \tfrac{2-x_{1,2}x_{2,2}}{3x_{2,2}}\left(\tfrac{\partial f_1}{\partial x_{1,2}}+\tfrac{x_{2,2}-x_{1,1}}{2x_{2,1}-x_{1,2}}\cdot\tfrac{\partial f_1}{\partial x_{1,1}}\right), & \lambda_{1,4} &= \tfrac{1-2x_{1,2}x_{2,2}}{2-x_{1,2}x_{2,2}}\lambda_{1,3}.
\end{aligned}$$

For the second player's optimization, we have the following rational LMEs:

$$\begin{array}{ll} \lambda_{2,1} = \frac{1-x_{2,2}}{2x_{2,1}-1} \cdot \frac{\partial f_2}{\partial x_{2,2}}, & \lambda_{2,2} = \frac{1-2x_{2,1}x_{2,2}}{2x_{2,1}-1} \cdot \frac{\partial f_2}{\partial x_{2,2}}, \\ \lambda_{2,3} = \frac{1}{2}(\lambda_{2,1} - x_{2,1}\frac{\partial f_2}{\partial x_{2,1}}), & \lambda_{2,4} = \frac{1}{2}\left((2-x_{2,1}) \cdot \frac{\partial f_2}{\partial x_{2,1}} + (1-4x_{2,2})\lambda_{2,1}\right). \end{array}$$

We apply the feasible extension as in Example 4.4. Algorithm 3.3 terminated at the loop $k=1$. We got the GNE $u=(u_1,u_2)$ with

$$u_1 = (1.0000, 0.5000), \quad u_2 = (0.5000, 1.0000), \quad \delta = -1.82 \cdot 10^{-8}.$$

It took around 22.73 seconds.

*Example* 6.4. Consider the 2-player GNEP with the optimization

$$\begin{array}{cl|cl} \min\limits_{x_1 \in \mathbb{R}^3} & x_1^T(x_1+x_2) + x_{1,1} - x_{1,2} - x_{1,3} & \min\limits_{x_2 \in \mathbb{R}^3} & e^T x_2 + \sum_{j=1}^3 x_{1,j}(x_{2,j})^2 \\ s.t. & 1 + (e^T x_2)^2 - x_{1,1}x_{1,2}x_{1,3} \ge 0, & s.t. & (e^T x_1)^2 - x_2^T x_2 \ge 0. \end{array}$$

For the first player's optimization, we have the LME and the feasible extension

$$\lambda_1 = -\frac{x_1^T \nabla_{x_1} f_1}{3+3(e^T x_2)^2}, \quad p_1(x) = \left(v_{1,1}, v_{1,2}, \frac{1+(e^T x_2)^2}{1+(e^T u_2)^2} \cdot v_{1,3}\right).$$

For the second player, we have the LME as in (6.5) and the feasible extension as in (4.3). Algorithm 3.3 terminated at the loop $k=0$. We got the GNE $u=(u_1,u_2)$ with

$$u_1 = (0.3090, 0.8090, 0.8090), \quad u_2 = (-1.6180, -0.6180, -0.6180),$$

and the accuracy parameter $\delta = -2.77 \cdot 10^{-8}$. It took around 5.16 seconds.

*Example* 6.5. (i) Consider the 3-player GNEP

$$\begin{array}{ll} \mathrm{F}_1(x_2,x_3): & \left\{ \begin{array}{cl} \min\limits_{x_1 \in \mathbb{R}^2} & \|x_1 - \frac{1}{2}(x_2+x_3)\|^2 \\ s.t. & x_{1,1}x_{1,2} - x_3^T x_3 - 1 = 0,\ x_{1,1} \ge 0,\ x_{1,2} \ge 0, \end{array} \right. \\ \mathrm{F}_2(x_1,x_3): & \left\{ \begin{array}{cl} \min\limits_{x_2 \in \mathbb{R}^2} & x_2^T(x_1+x_3) + (x_{2,1})^3 - 3(x_{2,2})^2 \\ s.t. & (x_{1,2})^2 - \|x_{1,1} \cdot x_2\|^2 = 0, \end{array} \right. \\ \mathrm{F}_3(x_1,x_2): & \left\{ \begin{array}{cl} \min\limits_{x_3 \in \mathbb{R}^2} & x_3^T(x_1+x_2+x_3-e) \\ s.t. & x_1^T x_1 - e^T x_3 \ge 0,\ x_{3,1} - 0.1 \ge 0,\ x_{3,2} - 0.1 \ge 0. \end{array} \right. \end{array}$$

The LMEs for $\mathrm{F}_1(x_2,x_3)$ and $\mathrm{F}_2(x_1,x_3)$ are

$$\begin{array}{ll} \lambda_{1,1} = \frac{x_1^T \nabla_{x_1} f_1}{2+2x_3^T x_3}, & \lambda_{1,2} = \frac{\partial f_1}{\partial x_{1,1}} - x_{1,2}\lambda_{1,1}, \\ \lambda_{1,3} = \frac{\partial f_1}{\partial x_{1,2}} - x_{1,1}\lambda_{1,1}, & \lambda_2 = \frac{-x_2^T \nabla_{x_2} f_2}{2(x_{1,2})^2}. \end{array}$$

We use the LME as in (6.3) for $\mathrm{F}_3(x_1,x_2)$. The first two players have the feasible extension

$$p_1(x) = \left(v_{1,1}, \frac{1+x_3^T x_3}{v_{1,1}}\right), \quad p_2(x) = \frac{u_{1,1}x_{1,2}}{u_{1,2}x_{1,1}} \cdot (v_{2,1}, v_{2,2}).$$

For the third player, the feasible extension is given in (4.2). Algorithm 3.3 terminated at the initial loop $k=0$. We got the GNE $u=(u_1,u_2,u_3)$ with

$$u_1 = (1.1401, 1.0461), \quad u_2 = (-0.1743, -0.9009), \quad u_3 = (0.1000, 0.4274),$$

and $\delta = -6.19 \cdot 10^{-8}$. It took around 10.58 seconds.

(ii) It is interesting to note that if the third player's objective is changed to

$$x_3^T(x_1+x_2-e)+(x_{3,1})^2-(x_{3,2})^2,$$

then there is no GNE. This was detected by Algorithm 3.3 at the loop $k=1$. It took around 19.16 seconds.

We remark that Algorithm 3.3 can be generalized to compute more (or even all) GNEs. This can be done with the approach in [42]. Suppose a GNE $u$ is already known. Select a small scalar $\zeta>0$, and solve the maximization problem

$$\left\{\begin{array}{rl}\rho:=\max & [x]_1^T\Theta[x]_1\\ s.t. & x\in\mathscr{U},\ [x]_1^T\Theta[x]_1\le[u]_1^T\Theta[u]_1+\zeta.\end{array}\right. \tag{6.10}$$

If $\rho>[u]_1^T\Theta[u]_1$, then let $\zeta:=\zeta/2$ and solve (6.10) again. Repeat this until $\zeta$ is small enough to make $\rho=[u]_1^T\Theta[u]_1$. When $u$ is an isolated KKT point and $\Theta$ is generically positive definite, such a $\zeta$ always exists. This can be proved similarly to that in [42]. Once such $\zeta$ is found, we add the new inequality $[x]_1^T\Theta[x]_1\ge[u]_1^T\Theta[u]_1+\zeta$ to (3.7). Then, Algorithm 3.3 can be applied to get a new GNE if it exists. It is worth noting that if the optimization (3.7) is infeasible with the newly added constraints, then there are no other GNEs. By repeating this process, we can get all GNEs if there are finitely many ones. We refer the reader to [42] for more details. The following is such an example.

*Example* 6.6. Consider the 2-player GNEP

$$\begin{array}{cc|cc}\min\limits_{x_1\in\mathbb{R}^2} & \frac{x_{2,2}(x_{1,1})^2+x_{2,1}(x_{1,2})^2+x_{1,1}x_{1,2}}{(x_{1,1})^2+1} & \min\limits_{x_2\in\mathbb{R}^2} & \frac{x_{1,2}(x_{2,1})^2+x_{1,1}(x_{2,2})^2+x_{2,1}x_{2,2}}{(x_{2,1})^2+1}\\ s.t. & (1-e^Tx_2)^2\le\|x_1\|^2\le 1, & s.t. & (1-e^Tx_1)^2\le\|x_2\|^2\le 1.\end{array}$$

We use the LMEs as in (6.4). For both $i=1,2$, the feasible extension is

$$p_i(x)=\frac{v_i}{\|v_i\|}-\left(\frac{v_i}{\|v_i\|}-v_i\right)\frac{e^Tx_{-i}}{e^Tu_{-i}}.$$

Following the above process, we got two GNEs $u=(u_1,u_2)$ with

| | | |
|---|---|---|
| $u_1=(0.9250,-0.3799)$, | $u_2=(0.9250,-0.3799)$, | $\delta=-9.06\cdot10^{-8}$; |
| $u_1=(-0.2700,0.9629)$, | $u_2=(-0.2700,0.9629)$, | $\delta=-2.67\cdot10^{-7}$. |

It took around 29.80 seconds to get both of them. Since each rational LME has a positive denominator on $X$, we obtained all GNEs for this problem.

**6.3. Some examples in applications.** We give some examples arising from applications. The first one is an NEP with rational objectives.

*Example* 6.7. Consider the NEP for the electricity market problem [7, 14]. Suppose there are $N$ generating companies. For each $i\in[N]$, the $i$th company possesses $n_i$ generating units, where the $j$th generating unit has $x_{i,j}$ power generation. Assume each $x_{i,j}\ge 0$ and is bounded by the maximum capacity $E_{i,j}\ge 0$. Denote $\varphi_i=(\varphi_{i,1},\dots,\varphi_{i,n_i})$, where each $\varphi_{i,j}$ is the cost of the generating unit $x_{i,j}$:

$$\varphi_{i,j}(x):=a_{i,j}\cdot(x_{i,j})^3-b_{i,j}\cdot(x_{i,j})^2+c_{i,j}x_{i,j}.$$

The electricity price is given by $\phi(x):=\frac{B}{A+e^Tx}$. The aim of each company is to maximize its profits. The $i$th player's optimization problem is

$$\mathrm{F}_i(x_{-i}):\left\{\begin{array}{rl}\min & e^T\varphi_i(x)-\phi(x)\cdot e^Tx_i\\ s.t. & x_{i,j}\ge 0,\quad E_{i,j}-x_{i,j}\ge 0\,(j\in[n_i]).\end{array}\right.$$

TABLE 2
*Numerical results of Example* 6.8.

| $N$ | $u=(u_1,\dots,u_N)$ | $\delta$ | time |
|---|---|---|---|
| 10 | $u_i=0.2250\ (i=1,\dots,10)$ | $-1.05\cdot 10^{-9}$ | 11.16 |
| 11 | $u_i=0.2066\ (i=1,\dots,11)$ | $-4.75\cdot 10^{-9}$ | 24.36 |
| 12 | $u_i=\begin{cases}0.1883 & (i=1,\dots,9)\\ L_i & (i=10,\dots,12)\end{cases}$ | $-1.93\cdot 10^{-8}$ | 45.38 |
| 13 | $u_i=\begin{cases}0.1647 & (i=1,\dots,7)\\ L_i & (i=8,\dots,13)\end{cases}$ | $-4.83\cdot 10^{-8}$ | 70.81 |
| 14 | $u_i=\begin{cases}0.1282 & (i=1,2,3)\\ L_i & (i=4,\dots,14)\end{cases}$ | $-1.02\cdot 10^{-7}$ | 97.00 |

The objectives are rational functions in strategies. The LME in (6.2) is applicable with box constraints. Since this is an NEP, we can apply the trivial feasible extension $p_i(x)=x_i$ for each $i\in[N]$. We choose the following parameters:

| | | | | | |
|---|---|---|---|---|---|
| $N=3$, | $n_1=1$, | $n_2=2$, | $n_3=3$, | $A=0.5$, | $B=20$, |
| $a_{1,1}=0.7$, | $a_{2,1}=0.75$, | $a_{2,2}=0.65$, | $a_{3,1}=0.66$, | $a_{3,2}=0.7$, | $a_{3,3}=0.8$, |
| $b_{1,1}=0.8$, | $b_{2,1}=0.75$, | $b_{2,2}=0.65$, | $b_{3,1}=0.66$, | $b_{3,2}=0.95$, | $b_{3,3}=0.5$, |
| $c_{1,1}=2$, | $c_{2,1}=1.25$, | $c_{2,2}=1$, | $c_{3,1}=2.25$, | $c_{3,2}=3$, | $c_{3,3}=3$, |
| $E_{1,1}=2$, | $E_{2,1}=2.5$, | $E_{2,2}=1.5$, | $E_{3,1}=1.2$, | $E_{3,2}=1.8$, | $E_{3,3}=1.6$. |

Algorithm 3.3 terminated at the loop $k=0$. We got the GNE $u=(u_1,u_2,u_3)$, where

$$u_1=1.1432,\quad u_2=(1.0549,1.1771),\quad u_3=(0.8917,0.6439,0.0000),$$

and $\delta=-1.70\cdot 10^{-8}$. It took about 7.98 seconds.

*Example* 6.8. Consider the GNEP for internet switching [12, 25]. Assume there are $N$ users, and the maximum capacity of the buffer is $B$. Let $x_i$ denote the amount of $i$th user's "packets" in the buffer, which has a positive lower bound $L_i$. Suppose the buffer is managed with the *"drop-tail" policy*: if the buffer is full, further packets will be lost and resent. Suppose $\frac{x_i}{e^Tx}$ is the transmission rate of the $i$th user, $\frac{e^Tx}{B}$ is the congestion level of the buffer, and $1-\frac{e^Tx}{B}$ measures the decrease in the utility of the $i$th user as the congestion level increases. The $i$th user's optimization problem is

$$(6.11)\qquad \begin{cases} \min\limits_{x_i\in\mathbb{R}^1} & f_i(x)=-\frac{x_i}{e^Tx}(1-\frac{e^Tx}{B})\\ s.t. & x_i-L_i\geq 0,\ B-e^Tx\geq 0. \end{cases}$$

We apply the LME as in (6.3) and solve the GNEP for $N=10,\dots,14$, with parameters $B=2.5$ and $L_i=0.09+0.01i$ for each $i\in[N]$. Algorithm 3.3 terminated at the initial loop $k=0$ for each case. The numerical results are shown in Table 2. In the table, $u=(u_1,\dots,u_N)$ and $\delta$ denote, respectively, the GNE and the accuracy parameter, and "time" is the CPU time in seconds.

**6.4. Comparison with other methods.** We compare our method (i.e., Algorithm 3.3) with some existing methods for solving GNEPs, such as the interior point method (IPM) based on the KKT system [9], the quasi-variational inequality method (QVI) in [19], the augmented-Lagrangian method (ALM) in [24], and the Gauss–Seidel method (GSM) in [40]. For Example 6.6, we only compare for finding one GNE. For Example 6.8, we compare for $N=10$.

For a computed tuple $u := (u_1, \dots, u_N)$, we use the quantity

$$\kappa := \max\left\{ \max_{i\in[N], j\in\mathcal{I}_1^{(i)}\cup\mathcal{I}_2^{(i)}} \{-g_{i,j}(u)\}, \max_{i\in[N], j\in\mathcal{I}_0^{(i)}} \{|g_{i,j}(u)|\} \right\}$$

to measure the feasibility violation. Note that $u$ is feasible if and only if $\kappa \le 0$ and $g_{i,j}(u) > 0$ for every $j \in \mathcal{I}_2^{(i)}$. For these methods, we use the following stopping criterion: for each generated iterate $u$, if its feasibility violation $\kappa < 10^{-6}$, then we compute the accuracy parameter $\delta$ for verifying GNEs. If $\delta > -10^{-6}$, then we stop the iteration.

For the above methods, the parameters are the same as in [9, 24, 40]. The full penalization is used for the augmented-Lagrangian method, and a Levenberg–Marquardt-type method (see [24, Algorithm 24]) is used to solve penalized subproblems. For the Gauss–Seidel method, the normalization parameters are updated as (4.3) in [40], and the Moment-SOS relaxations are used to solve each player's optimization problems. For the QVI method, the Moment-SOS relaxations are used to compute projections. We let 1000 be the maximum number of iterations for all the above methods. For initial points, we use $(0, 1, 1, 0)$ for Examples 6.1(i)–(ii), $(1, 1, 1, 1)$ for Examples 6.3(i), (ii), (iv), (v), $(\sqrt{2}, \sqrt{2}, 1, 1, 1, 1)$ for Example 6.5, $(0, 1, 0, 1)$ for Example 6.6, $0.25 \cdot (1, \cdots, 1)$ for Example 6.8, and the zero vectors for other examples. If the maximum number of iterations is reached but the stopping criterion is not met, we still solve (3.8) to check whether the latest iterating point is a GNE or not. For the QVI, the produced sequence is said to converge if the projection residue is sufficiently small. For the ALM and IPM, the produced sequence is considered to converge if the last iterate satisfies the KKT conditions up to a small round-off error (say, $10^{-6}$). The numerical results are shown in Table 3. The "$u$" column lists the most recent update by each method, "time" gives the total CPU time (in seconds), and the "$\max\{|\delta|, \kappa\}$" measures the feasibility violation and the accuracy of being GNEs. For all methods in the table, if the produced sequence is convergent, but the quantity $\max\{|\delta|, \kappa\}$ is not close to zero (e.g., $\le 10^{-6}$), then the method converges to a KKT point that is not a GNE.

The comparisons are summarized as follows:

- The ALM failed to get a GNE for Examples 6.3(i), (ii), (iv), 6.4, and 6.5(ii), because the penalization subproblems could not be solved accurately. It converged to non-GNE KKT points for Examples 6.1(ii), 6.2, 6.3(iii), 6.5(i) and 6.7. It did not converge for Examples 6.1(i), 6.3(v), and 6.6 when the maximum penalty parameter $10^{12}$ was reached.
- The IPM failed to get a GNE for Examples 6.3(iv), 6.4, and 6.5(ii), because the step length was too small to efficiently decrease the violation of KKT conditions. It converged to non-GNE KKT points for Examples 6.1(i)–(ii), 6.3(iii), and 6.7. It did not converge for Examples 6.1(i)–(ii), 6.3(ii), (v), and 6.7, because the Newton-type directions did not satisfy sufficient descent conditions.
- The QVI converged to non-GNE points for Examples 6.1(i) 6.2, and 6.5(i). It did not converge for Examples 6.1(ii), 6.3(ii), (iv), 6.5(ii), and 6.6, since the projection could not be computed successfully.
- The GSM failed to find a GNE for Examples 6.1(ii), 6.3(ii), (iv), 6.4, 6.5(ii), and 6.6, because some sub-optimization problems could not be solved successfully. It terminated at the maximum iteration number for Example 6.3(iii) but did not meet the stopping criterion.

TABLE 3
*Comparison with some existing methods.*

| Algorithm | $u$ | time | $\max\{\lvert\delta\rvert, \kappa\}$ |
|---|---|---|---|
| **Example 6.1(i)** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | (0.8911,-0.0000,0.8910,0.0000) | 298.10 | 0.22 |
| GSM | (0.4930,-0.0835,0.5000,0.4930) | 3.12 | $1.33 \cdot 10^{-8}$ |
| Alg. 3.3 | (0.4930,-0.0835,0.5000,0.4930) | 4.22 | $4.31 \cdot 10^{-7}$ |
| **Example 6.1(ii)** | | | |
| ALM | (0.5000,0.8660,1.0000,0.0000) | 63.81 | 2.25 |
| IPM | not convergent | | |
| QVI | not convergent | | |
| GSM | not convergent | | |
| Alg. 3.3 | nonexistence of GNEs detected | 5.56 | |
| **Example 6.2** | | | |
| ALM | (0.0000,0.1931,0.2889,0.0000) | 47.51 | 0.21 |
| IPM | (0.0000,0.1931,0.2889,0.0000) | 17.00 | 0.21 |
| QVI | (0.0000,0.0000,0.0000,0.0354) | 441.52 | 0.44 |
| GSM | (0.0000,0.0000,1.0000,0.0000) | 0.59 | $8.08 \cdot 10^{-8}$ |
| Alg. 3.3 | (0.0000,0.2742,0.7258,0.0000) | 16.81 | $1.14 \cdot 10^{-7}$ |
| **Example 6.3(i)** | | | |
| ALM | not convergent | | |
| IPM | (1.3561,0.7374,1.0000,1.0468) | 2.39 | $1.93 \cdot 10^{-7}$ |
| QVI | (1.3562,0.7375,1.0000,1.0469) | 2753.26 | $1.34 \cdot 10^{-4}$ |
| GSM | (1.3558,0.7376,1.0000,1.0466) | 3.47 | $2.60 \cdot 10^{-9}$ |
| Alg. 3.3 | (1.3561,0.7374,1.0000,1.0468) | 8.36 | $3.44 \cdot 10^{-8}$ |
| **Example 6.3(ii)** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | not convergent | | |
| GSM | not convergent | | |
| Algorithm 3.3 | nonexistence of GNEs detected | 5.47 | |
| **Example 6.3 (iii)** | | | |
| ALM | (0,0,0,0) | 49.34 | 1.00 |
| IPM | (0.2808,0.2192,0.2808,0.2192) | 12.98 | 0.16 |
| QVI | (0.0000,0.4999,0.0001,0.4999) | 616.29 | $5.35 \cdot 10^{-5}$ |
| GSM | (0.0000,0.4995,0.0000,0.4995) | 110.79 | $8.58 \cdot 10^{-4}$ |
| Alg. 3.3 | (0.0000,0.5000,0.0000,0.5000) | 3.28 | $4.47 \cdot 10^{-8}$ |
| **Example 6.3(iv)** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | not convergent | | |
| GSM | not convergent | | |
| Alg. 3.3 | (0.0000,0.0000,1.0000,1.0000) | 2.03 | $5.45 \quad 10^{-9}$ |
| **Example 6.3(v)** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | (1.0000,0.5000,0.5000,1.0000) | 490.93 | $9.51 \cdot 10^{-5}$ |
| GSM | (1.0000,0.5000,0.5000,1.0000) | 1.80 | $2.31 \cdot 10^{-10}$ |
| Alg. 3.3 | (1.0000,0.5000,0.5000,1.0000) | 22.73 | $1.82 \cdot 10^{-8}$ |
| **Example 6.4** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | $(0.3094, 0.8090, 0.8090, -1.6172, -0.6180, -0.6180)$ | 21.46 | $5.63 \cdot 10^{-7}$ |
| GSM | not convergent | | |
| Alg. 3.3 | $(0.3090, 0.8090, 0.8090, -1.6180, -0.6180, -0.6180)$ | 5.16 | $2.77 \cdot 10^{-8}$ |

TABLE 3
*Continued.*

| | | | |
|---|---|---|---|
| **Example 6.5(i)** | | | |
| ALM | $(0.7774, 1.3629, -0.2227, 1.7389, 0.2226, 0.1000)$ | 75.92 | 5.10 |
| IPM | $(1.1401, 1.0461, -0.1743, -0.9009, 0.1000, 0.4274)$ | 0.86 | $8.24 \cdot 10^{-7}$ |
| QVI | $(0.7775, 1.3628, -0.2227, 1.7386, 0.2227, 0.1000)$ | 192.73 | 5.10 |
| GSM | $(1.1403, 1.0463, -0.1743, -0.9009, 0.1000, 0.4273)$ | 6.28 | $1.88 \cdot 10^{-8}$ |
| Alg. 3.3 | $(1.1401, 1.0461, -0.1743\ -0.9009, 0.1000, 0.4274)$ | 10.58 | $6.19 \cdot 10^{-8}$ |
| **Example 6.5(ii)** | | | |
| ALM | not convergent | | |
| IPM | not convergent | | |
| QVI | not convergent | | |
| GSM | not convergent | | |
| Alg. 3.3 | nonexistence of GNEs detected | 19.16 | |
| **Example 6.6** | | | |
| ALM | not convergent | | |
| IPM | $(0.2665, 0.3184, 0.2665, 0.3184)$ | 11.22 | 0.27 |
| QVI | not convergent | | |
| GSM | not convergent | | |
| Alg. 3.3 | $(0.9250, -0.3799, 0.9250, -0.3799)$ | 2.78 | $9.06 \cdot 10^{-8}$ |
| **Example 6.7** | | | |
| ALM | $(1.1652, 1.0601, 1.1822, 0.9952, 0.0577, 0.2332)$ | 94.36 | 0.10 |
| IPM | not convergent | | |
| QVI | $(1.1432, 1.0549, 1.1770, 0.8916, 0.6440, 0.0001)$ | 523.06 | $2.35 \cdot 10^{-5}$ |
| GSM | $(1.1446, 1.0551, 1.1772, 0.8917, 0.6431, 0.0000)$ | 4.22 | $9.16 \cdot 10^{-7}$ |
| Alg. 3.3 | $(1.1432, 1.0549, 1.1771, 0.8917, 0.6439, 0.0000)$ | 7.98 | $1.70 \cdot 10^{-8}$ |
| **Example 6.8** | | | |
| ALM | $(0.2250, 0.2250, 0.2250, 0.2250, 0.2250, 0.2250, 0.2250\ 0.2250, 0.2250, 0.2250)$ | 3.06 | $5.28 \cdot 10^{-12}$ |
| IPM | $(0.2245, 0.2245, 0.2246, 0.2246, 0.2246, 0.2246, 0.2247\ 0.2251, 0.2260, 0.2275)$ | 10.89 | $5.13 \cdot 10^{-7}$ |
| QVI | $(0.2254, 0.2254, 0.2254, 0.2254, 0.2254, 0.2253, 0.2253\ 0.2253, 0.2252, 0.2251)$ | 9.10 | $4.59 \cdot 10^{-7}$ |
| GSM | $(0.2236, 0.2250, 0.2262, 0.2270, 0.2271, 0.2266, 0.2256\ 0.2245, 0.2236, 0.2232)$ | 21.33 | $8.59 \cdot 10^{-7}$ |
| Alg. 3.3 | $(0.2250, 0.2250, 0.2250, 0.2250, 0.2250, 0.2250, 0.2250\ 0.2250, 0.2250, 0.2250)$ | 11.16 | $1.05 \cdot 10^{-9}$ |

**6.5. About strict inequality constraints.** For rGNEPs, rational LMEs are used to get the KKT set. For strict inequality constraints, their Lagrange multipliers are always zeros. In Algorithm 3.3, the set $\mathcal{K}$ is as in (3.2), where the LMEs are zeros for strict inequalities. For each rational optimization problem, its feasible set is relaxed from (5.2) to (5.4), and then we solve it by Algorithm 5.1. Strict inequalities give open sets. When there are finitely many KKT points (this is the generic case), there does not exist a sequence of feasible KKT points that converge to the boundary given by strict inequality constraints. For some special cases, the KKT set may be infinite and there possibly exists a sequence of feasible KKT points converging to the boundary of strict inequality constraints. If this case happens, the limit may not be a GNE. The following is such an example.

*Example* 6.9. Consider the following GNEP:

$$\text{(6.12)}\qquad \begin{array}{cl} \min\limits_{x_1\in\mathbb{R}^1} & x_1x_2 \\ s.t. & x_1\geq 0,\, 1-x_1\geq 0, \end{array} \;\Bigg|\; \begin{array}{cl} \min\limits_{x_2\in\mathbb{R}^1} & \frac{-(x_2)^2}{1-(x_1)^2} \\ s.t. & x_2\geq 0,\, 1-(x_1)^2-(x_2)^2>0. \end{array}$$

The second player has a strict inequality constraint. The Lagrange multiplier vectors can be expressed as

$$\lambda_1 = (x_2 - x_1x_2, \quad -x_1x_2), \qquad \lambda_2 = \left(\frac{-2x_2}{1-(x_1)^2}, \quad 0\right).$$

The denominators of $\lambda_2$ and the second player's objective are positive in the feasible set but not positive on the boundary of its closure. The KKT set $\mathcal{K}$ is

$$\mathcal{K} = \left\{ (x_1,x_2) \;\middle|\; \begin{array}{c} x_1(x_2-x_1x_2)=0,\, -x_1x_2(1-x_1)=0, \\ 0\leq x_1\leq 1,\, x_2-x_1x_2\geq 0,\, -x_1x_2\geq 0, \\ x_2\cdot\frac{-2x_2}{1-(x_1)^2}=0, \\ x_2\geq 0,\, (x_1)^2+(x_2)^2<1,\, \frac{-2x_2}{1-(x_1)^2}\geq 0. \end{array} \right\}.$$

One can see that $\mathcal{K} = \{(x_1,x_2): 0\leq x_1<1, x_2=0\}$. After the cancellation for the denominator and relaxing $(x_1)^2+(x_2)^2<1$ to the weak inequality $(x_1)^2+(x_2)^2\leq 1$, the set $\mathcal{K}$ is changed to

$$\mathcal{K}_1 = \left\{ (x_1,x_2) \;\middle|\; \begin{array}{c} x_1(x_2-x_1x_2)=0,\, -x_1x_2(1-x_1)=0, \\ 0\leq x_1\leq 1,\, x_2-x_1x_2\geq 0,\, -x_1x_2\geq 0, \\ x_2\cdot(-2x_2)=0, \\ x_2\geq 0,\, (x_1)^2+(x_2)^2\leq 1,\, -2x_2\geq 0. \end{array} \right\}.$$

Then one can check that $\mathcal{K}_1 = \{(x_1,x_2): 0\leq x_1\leq 1, x_2=0\}$, i.e.,

$$\mathcal{K}_1 = [0,1]\times\{0\} \quad\text{and}\quad \mathcal{K}_1\setminus\mathcal{K} = \{(1,0)\}.$$

When we apply the algorithm to compute GNEs, we got the candidate $\hat{x} = (1,0)$, which is not feasible for (6.12) but lies on the boundary. The second player's objective is not well defined at $\hat{x}$. The candidate $\hat{x} = (1,0)$ is not a GNE. Indeed, this GNEP does not have any GNE.

**7. Conclusions and discussions.** This paper studies how to solve GNEPs given by rational functions. LMEs and feasible extensions are introduced to compute GNEs. We propose a hierarchy of rational optimization problems to solve GNEPs. This is given in Algorithm 3.3. The Moment-SOS relaxations are used to solve the appearing rational optimization problems. Under some general assumptions, we show that Algorithm 3.3 can get a GNE if it exists or detect its nonexistence.

The feasible extension is a major technique used in this paper. Its purpose is to preclude KKT points that are not GNEs. This technique was originally introduced for solving bilevel optimization in the work [41]. However, their properties are quite different for GNEPs and bilevel optimization. For instance, a generic polynomial GNEP has finitely many KKT points, which is implied by the recent work [44, Theorem 3.1]. It guarantees the existence of feasible extensions for generic rGNEPs, which is shown in Theorem 4.2. So, Algorithm 3.3 has finite convergence for general cases. However, for general polynomial bilevel optimization, the KKT set (for the lower level optimization) is usually not finite. There do not exist results on the existence of feasible extensions. Moreover, the work [41] only considers polynomial extensions. In this paper, we consider more general feasible extensions that are given by rational functions. It greatly broadens the usage of feasible extensions for solving GNEPs. For instance, we gave explicit rational feasible extensions in (4.3) for ball constraints parameterized by the polynomial $a_j(x_{-i})$. For this kind of constraint, polynomial extensions as in [41] usually do not exist.

There exists much interesting future work to do with feasible extensions. For instance, are there sufficient conditions weaker than those in Theorem 4.2 for the existence of feasible extensions? If they exist, how can we find them efficiently? These questions are mostly open.

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