



On an Equichordal Property of a Pair of Convex Bodies

Dmitry Ryabogin¹

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Abstract

Let $d \geq 2$ and let K and L be two convex bodies in \mathbb{R}^d such that $L \subset \text{int } K$ and the boundary of L does not contain a segment. If K and L satisfy the $(d+1)$ -equichordal property, i.e., for any line l supporting the boundary of L and the points $\{\zeta_{\pm}\}$ of the intersection of the boundary of K with l ,

$$\text{dist}^{d+1}(L \cap l, \zeta_+) + \text{dist}^{d+1}(L \cap l, \zeta_-) = 2\sigma^{d+1}$$

holds, where the constant σ is independent of l , does it follow that K and L are concentric Euclidean balls? We prove that if K and L have C^2 -smooth boundaries and L is a body of revolution, then K and L are concentric Euclidean balls.

Keywords Convex and floating bodies · Equichordal property · Ulam's problems

Mathematics Subject Classification 52A20 · 52A38

1 Introduction

Let $d \geq 2$ and let K and L be two convex bodies in \mathbb{R}^d such that $L \subset \text{int } K$ and the boundary of L does not contain a segment. For any line l supporting L we consider two points ζ_{\pm} of the intersection of the boundary of K with l . Given $i \in \mathbb{R}$ we say that the bodies K and L satisfy the i -equichordal property if there exists a constant σ

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Dmitry Ryabogin
ryabogin@math.kent.edu

¹ Department of Mathematics, Kent State University, Kent, OH 44242, USA

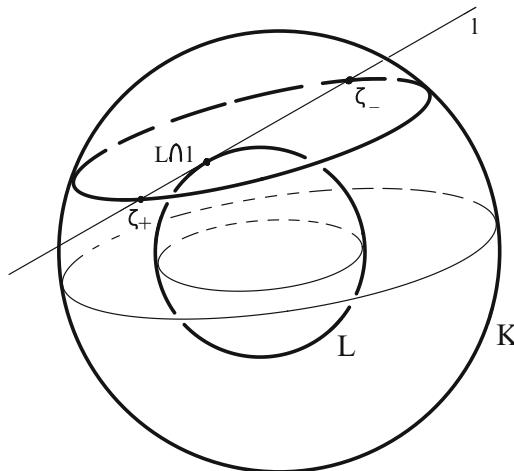


Fig. 1 We have $\text{dist}^i(L \cap l, \zeta_+) + \text{dist}^i(L \cap l, \zeta_-) = 2\sigma^i$

independent of l such that

$$\text{dist}^i(L \cap l, \zeta_+) + \text{dist}^i(L \cap l, \zeta_-) = 2\sigma^i \quad (1)$$

(see Fig. 1). If $i = 0$ we replace (1) with

$$\text{dist}(L \cap l, \zeta_+) \text{dist}(L \cap l, \zeta_-) = \sigma^2 \quad (2)$$

(cf. [5, p. 233]).

Problem 1.1 *Let $d \geq 2$ and $i \in \mathbb{R}$. Are two concentric Euclidean balls the only pair of bodies in \mathbb{R}^d satisfying the i -equichordal property?*

Similar questions to that of the problem above were raised in [12], [2], [4, A1, p. 9], and [15]; see also [11] and references therein. In particular, it is known that the answer to Problem 1.1 is affirmative for $d \geq 3$, provided L is a Euclidean ball [2].

We would also like to mention several results related to the connection between Problem 1.1 and Problem 19 of Ulam from the Scottish book, which asks *if a solid of uniform density which floats in water in every position is necessarily a sphere*, [6, p. 90], [4, A9, p. 19].

The plane counterexamples to Ulam's problem constructed in [1, 13, 14], show that for $d = 2$, $i = 1$, the answer to Problem 1.1 is negative, even under the additional assumption that for every line l supporting L , the point of tangency $L \cap l$ divides the chord $K \cap l$ into two parts of equal length. On the other hand, it is known [3] that, under this division assumption and under the assumption that l divides the boundary of K in constant ratio $\mu(\sigma)/(1 - \mu(\sigma))$ for $\mu = 1/3$, $\mu = 1/4$, $\mu = 1/5$, and $\mu = 2/5$, the answer to Problem 1.1 is affirmative; see also [7]. Additionally, if $d \geq 3$, and if for every line l supporting L the point $L \cap l$ divides the chord $K \cap l$ into two parts of equal length, then the answer to Problem 1.1 is affirmative, [8]. Finally, we remark that a

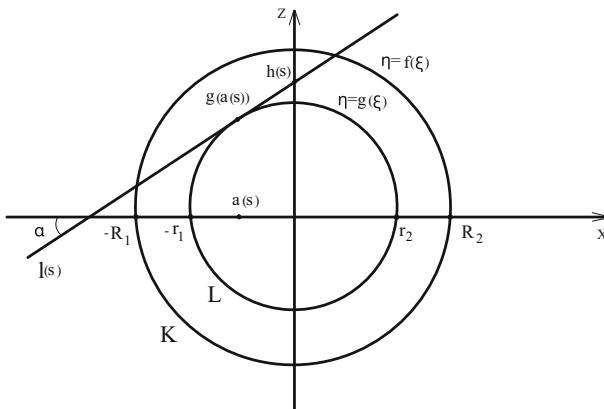


Fig. 2 $K \cap \{(x, y, z) : y = 0\}$ and $L \cap \{(x, y, z) : y = 0\}$ with their boundaries described by the graphs of functions $\eta = f(\xi)$ and $\eta = g(\xi)$

negative answer to Problem 1.1 in the case $i = d + 1, d \geq 3$, presents a possibility for a negative answer to Ulam's conjecture [9, 10]. In this paper we prove the following result.

Theorem 1.2 *Let $d \geq 3$ and let K and L be two convex bodies in \mathbb{R}^d of class C^2 satisfying the $(d + 1)$ -equichordal property. If L is a body of revolution, then K and L are concentric Euclidean balls.*

A similar result can be proved for general i -equichordal property, $i \in \mathbb{R}$. Since our interest in Problem 1.1 comes, partly, from its relation to the problem of Ulam, and since, in our opinion, the proof for $i \neq d + 1$ does not add to the ideas when L is a body of revolution, we restrict ourselves to the case $i = d + 1$.

Notation and basic definitions. Let $d \geq 2$. A convex body $K \subset \mathbb{R}^d$ is a convex compact set with a non-empty interior $\text{int } K$. We denote by $B^d(r)$ the Euclidean ball centered at the origin of radius $r > 0$. Given $\xi \in S^{d-1}$ we put $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ to be the subspace orthogonal to ξ , and $p \cdot \xi = p_1 \xi_1 + \cdots + p_d \xi_d$ is the usual inner product in \mathbb{R}^d . We say that a line l is a supporting line of a convex body L if $L \cap l \neq \emptyset$, but $\text{int } L \cap l = \emptyset$. Let $m \in \mathbb{N}$. We say that a convex body K in \mathbb{R}^d is of class C^m if for every point z on the boundary ∂K of $K \subset \mathbb{R}^d$ there exists a neighborhood U_z of z in \mathbb{R}^d such that $\partial K \cap U_z$ can be written as a graph of a function having all continuous partial derivatives up to the m -th order.

2 Auxiliary Statements, K and L are the Bodies of Revolution About the Same Axis in \mathbb{R}^3

First we introduce some convenient notation which helps to work with bodies of revolution. Let $K \subset \mathbb{R}^3$ be a body of revolution about the x -axis with C^3 boundary described by a function $\eta = f(\xi) \geq 0$ supported by the segment $[-R_1, R_2]$. Assume also that L is a body of revolution about the same axis, and its boundary is described

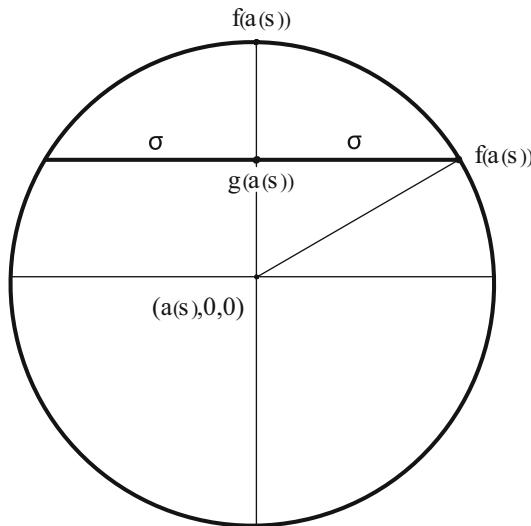


Fig. 3 The section $K \cap \{(x, y, z) : x = a(s)\}$. We have $g(a(s)) = \sqrt{f^2(a(s)) - \sigma^2}$

by the function $\eta = g(\xi) \geq 0$ supported by the segment $[-r_1, r_2] \subset (-R_1, R_2)$ (see Fig. 2).

We will denote by H_s the plane parallel to the y -axis and containing the line $l(s) = \{(\xi, 0, s\xi + h(s)) : \xi \in \mathbb{R}\}$, where $l(s)$ is tangent to the graph of g at the corresponding point $(a(s), 0, g(a(s)))$, $s = \tan \alpha$ with $\alpha \in (-\pi/2, \pi/2)$ being the angle between the x -axis and $l(s)$, and $h(s)$ is the z -intercept of $l(s)$.

Let $s \in \mathbb{R}$ be fixed and let ℓ_s be the line parallel to the y -axis passing through $(a(s), 0, g(a(s)))$. Since the section $K \cap H_s$ is symmetric with respect to the line $l(s)$, the chord $G_s = K \cap \ell_s$ is divided by $(a(s), 0, g(a(s)))$ into two parts of equal length σ .

Let $s \in \mathbb{R}$ and $a(s) \in (-r_1, r_2)$. Since K is a body of revolution, the chord of length 2σ can be inscribed into a circle of radius $f(a(s))$ only provided

$$g(a(s)) = \sqrt{f^2(a(s)) - \sigma^2}$$

(see Fig. 3). Since $K \cap \{(x, y, z) \in \mathbb{R}^3 : x = -r_1, r_2\}$ are discs of radius σ , we have

$$f(-r_1) = f(r_2) = \sigma, \quad g(\xi) = \sqrt{f^2(\xi) - \sigma^2} \quad \forall \xi \in [-r_1, r_2]. \quad (3)$$

Translating the bodies if necessary, we can and do assume that $a(0) = 0$.

2.1 Some Results on the $(d+1)$ -Equichordal Plane Bodies Symmetric with Respect to the Axis

Let $P \subset \mathbb{R}^2$ be a convex body containing the origin in its interior with C^3 -smooth boundary. Assume also that P is symmetric with respect to the x -axis and it satisfies the $(d+1)$ -equichordal property with respect to the origin, i.e., there exists a constant σ such that

$$\forall \theta \in S^1, \quad \rho_P^{d+1}(\theta) + \rho_P^{d+1}(-\theta) = 2\sigma^{d+1},$$

for some $d \geq 3$. If the upper part of the boundary of P is described by a graph of a positive function ϕ on $[-\tau_1, \tau_2]$, then by the Pythagorean Theorem and the symmetry with respect to the x -axis, the function ϕ satisfies

$$(x^2 + \phi^2(x))^{(d+1)/2} + (y^2 + \phi^2(y))^{(d+1)/2} = 2\sigma^{d+1} \quad (4)$$

(see Fig. 4). Here $y \in [-\tau_1, 0]$, $\tau_1 > 0$, and $x \in [0, \tau_2]$, $\tau_2 > 0$, are such that

$$\frac{\phi(x)}{x} = \frac{\phi(y)}{|y|},$$

i.e.,

$$|y|^{d+1} = \frac{(2\sigma^{d+1} - (x^2 + \phi^2(x))^{(d+1)/2})x^{d+1}}{(x^2 + \phi^2(x))^{(d+1)/2}}. \quad (5)$$

To simplify the computations we will write

$$\phi^2(x) = \sigma^2 - x^2 + \chi(x), \quad x \in [-\tau_1, \tau_2], \quad (6)$$

where χ is a function we want to determine. By the symmetry of P with respect to the x -axis, we have $\phi(0) = \sigma$, hence, $\chi(0) = 0$. We rewrite (4) as

$$(\sigma^2 + \chi(x))^{(d+1)/2} + (\sigma^2 + \chi(y))^{(d+1)/2} = 2\sigma^{d+1}, \quad (7)$$

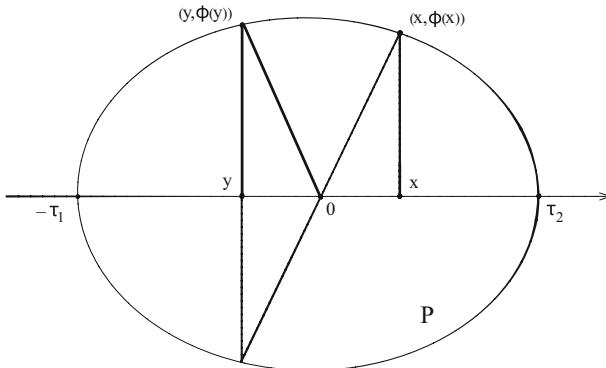


Fig. 4 The equichordiality of P

$y \in [-\tau_1, 0]$, $x \in [0, \tau_2]$. Let

$$q(z) = \frac{\chi(z)}{\sigma^2} = \frac{\phi^2(z) + z^2 - \sigma^2}{\sigma^2} = \frac{\phi^2(z) + z^2}{\sigma^2} - 1. \quad (8)$$

Then conditions (7) and (5) can be written as

$$(1 + q(x))^{(d+1)/2} + (1 + q(y))^{(d+1)/2} = 2, \quad (9)$$

$$|y|^{d+1} = \frac{(2\sigma^{d+1} - (\sigma^2 + \chi(x))^{(d+1)/2})x^{d+1}}{(\sigma^2 + \chi(x))^{(d+1)/2}} = \frac{(2 - (1 + q(x))^{(d+1)/2})x^{d+1}}{(1 + q(x))^{(d+1)/2}}. \quad (10)$$

Our first lemma is technical, but it is crucial for our further considerations.

Lemma 2.1 *Let χ be as in (7), where y is as in (10). Then*

$$2\sigma^2\chi''(0) + (d+1)(\chi'(0))^2 = 0. \quad (11)$$

Proof Since $q(0) = 0$, we can assume that there exists a neighborhood U_0 of the origin such that $|q(x)| < 1$ for all $x \in U_0$. Using (10) we see that for $y < 0$ and $x > 0$ we have

$$|y| = x \frac{(2 - (1 + q(x))^{(d+1)/2})^{1/(d+1)}}{(1 + q(x))^{1/2}} \quad \forall x \in U_0. \quad (12)$$

We will show first that

$$|y| = x \left(1 - \varepsilon_1 x + \left(-\varepsilon_2 + \frac{3-d}{4}\varepsilon_1^2 \right) x^2 + o(x^2) \right) \quad \forall x \in V_0, \quad (13)$$

where $V_0 \subset U_0$ is a neighborhood of the origin that will be chosen later and ε_j , $j = 1, 2$, are the Taylor coefficients of the decomposition of q near the origin,

$$q(x) = \varepsilon_1 x + \varepsilon_2 x^2 + o(x^2), \quad \varepsilon_j = \frac{q^{(j)}(0)}{j!} = \frac{\chi^{(j)}(0)}{\sigma^2 j!}, \quad x \in V_0. \quad (14)$$

To prove (13), we compute the first and second derivatives of the function

$$\frac{(2 - (1 + z)^{(d+1)/2})^{1/(d+1)}}{(1 + z)^{1/2}}.$$

Routine calculations show that they are equal to -1 and $(3 - d)/4$ respectively, and we can express y via x up to the terms of the second order,

$$|y| = 1 - q(x) + \frac{3 - d}{4} q^2(x) + o(q^2).$$

Now we will use (14) and the previous decompositions to obtain

$$|y| = x \left(1 - (\varepsilon_1 x + \varepsilon_2 x^2) + \frac{3 - d}{4} (\varepsilon_1 x + \varepsilon_2 x^2)^2 + o(x^2) \right).$$

This gives (13). Next, we use (13) to obtain two relations that will lead to (11). We see that

$$x - |y| = \varepsilon_1 x^2 + o(x^2). \quad (15)$$

Since

$$y^2 = x^2(1 - 2\varepsilon_1 x) + o(x^3),$$

we also have

$$x^2 + y^2 = 2x^2 + o(x^2). \quad (16)$$

Using Taylor's decomposition,

$$(1 + z)^{(d+1)/2} = 1 + \frac{d+1}{2} z + \frac{d^2 - 1}{8} z^2 + o(z^2), \quad |z| < 1, \quad (17)$$

and applying it for $q(x)$ and $q(y)$, the sum of these and (9) results in

$$0 = \frac{d+1}{2} (q(x) + q(y)) + \frac{d^2 - 1}{8} (q^2(x) + q^2(y)) + o(q^2(x)) + o(q^2(y)).$$

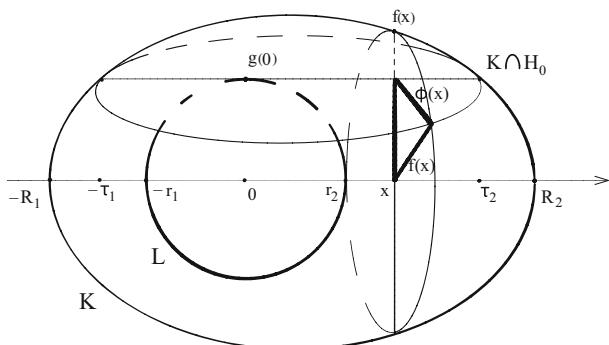


Fig. 5 We have $f^2(x) = \phi^2(x) + g^2(0) = \phi^2(x) + f^2(0) - \sigma^2$

This and (14) yield

$$\begin{aligned}
0 &= \frac{d+1}{2}(\varepsilon_1 x + \varepsilon_2 x^2 - \varepsilon_1 |y| + \varepsilon_2 y^2) \\
&\quad + \frac{d^2-1}{8}((\varepsilon_1 x + \varepsilon_2 x^2)^2 + (-\varepsilon_1 |y| + \varepsilon_2 y^2)^2) + o(x^2) \\
&= \frac{d+1}{2}(\varepsilon_1(x - |y|) + \varepsilon_2(x^2 + y^2)) + \frac{d^2-1}{8}\varepsilon_1^2(x^2 + y^2) + o(x^2) \\
&= \frac{d+1}{2}\varepsilon_1(x - |y|) + \left(\frac{d^2-1}{8}\varepsilon_1^2 + \frac{d+1}{2}\varepsilon_2\right)(x^2 + y^2) + o(x^2).
\end{aligned}$$

It remains to apply (15) and (16) to obtain

$$0 = \frac{d+1}{2} \varepsilon_1^2 x^2 + \left(\frac{d^2-1}{8} \varepsilon_1^2 + \frac{d+1}{2} \varepsilon_2 \right) 2x^2 + o(x^2).$$

Therefore,

$$\left(\frac{d+1}{2} + \frac{d^2-1}{4}\right)\varepsilon_1^2 + (d+1)\varepsilon_2 = 0,$$

or

$$(d+1)\varepsilon_1^2 + 4\varepsilon_2 = 0.$$

This gives the desired result by (8).

2.2 Auxiliary Formulas Describing the Boundary of the Horizontal Section

$$P = K \cap H_0$$

We use the notation from the previous subsection. If f describes the boundary of K , and ϕ describes the boundary of the horizontal section $K \cap H_0 - (0, 0, g(0))$, then

$$f^2(x) = \phi^2(x) + g^2(0) = \phi^2(x) + f^2(0) - \sigma^2 \quad \forall x \in [-\tau_1, \tau_2] \quad (18)$$

(see Fig. 5). Observe that if σ is sufficiently close to zero, then $[-\tau_1, \tau_2] \subset [-r_1, r_2]$. On the other hand, if σ is large enough, then $[-r_1, r_2] \subset [-\tau_1, \tau_2]$. The next lemma shows that in general we have only these possibilities.

Lemma 2.2 *We have $[-r_1, r_2] \subseteq [-\tau_1, \tau_2]$ or $[-\tau_1, \tau_2] \subseteq [-r_1, r_2]$.*

Proof Assume the contrary, we have

$$-r_1 < -\tau_1, \quad r_2 < \tau_2, \quad \text{or} \quad -\tau_1 < -r_1, \quad \tau_2 < r_2. \quad (19)$$

We will show that the first case in (19) is not possible, the proof that the second one is not possible either is similar.

To this end, consider the horizontal chord inscribed into ∂K and tangent to the graph of g at $(0, 0, g(0))$. We have $f(-\tau_1) = f(\tau_2) = g(0)$ and $f(r_2) > f(\tau_2)$, otherwise, the points $(R_2, 0, 0)$, $(\tau_2, 0, f(\tau_2))$ and $(r_2, 0, f(r_2))$ are on the boundary of K , which contradicts its convexity. On the other hand, by (3) we have $f(-r_1) = f(r_2) = \sigma$. Hence, $f(-r_1) > f(-\tau_1)$. This contradicts the convexity of K , for, the points $(-R_1, 0, 0)$, $(-r_1, 0, f(-r_1))$, and $(-\tau_1, 0, f(-\tau_1))$ must lie on its boundary. \square

Let

$$A = s \sqrt{f^2(a(0)) - a^2(s) + \chi(a(s)) - \sigma^2},$$

where for every $s \in \mathbb{R}$ we have $a(s) \in (-r_1, r_2)$.

Lemma 2.3 *Let $s \in \mathbb{R}$, $a(s) \in (-r_1, r_2) \cap (-\tau_1, \tau_2)$ be fixed, and let x, y be so small that $a(s) + x, a(s) - y \in (-r_1, r_2) \cap (-\tau_1, \tau_2)$. Then (7) and (9) hold for these x, y , with*

$$\chi_a(x) = -2(a + A)x + \chi(a + x) - \chi(a)$$

instead of χ , and $q_a(x) = \chi_a(x)/\sigma^2$ instead of q .

Proof Fix any $s \in \mathbb{R}$ and $a(s) \in (-r_1, r_2) \cap (-\tau_1, \tau_2)$. We can assume that the boundary of $K \cap H_s$ is described by a positive function ψ satisfying the $(d+1)$ -equichordal property (we pick $(a(s), 0, g(a(s)))$ as the origin in H_s),

$$\begin{aligned} & (x^2(1+s^2) + \psi^2(x\sqrt{1+s^2}))^{(d+1)/2} \\ & + (y^2(1+s^2) + \psi^2(y\sqrt{1+s^2}))^{(d+1)/2} = 2\sigma^{d+1}, \end{aligned} \quad (20)$$

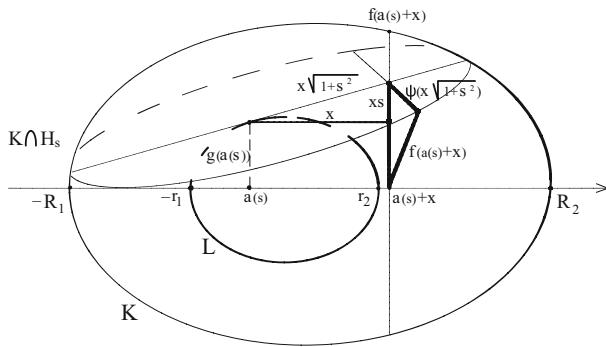


Fig. 6 We have $\psi^2(x\sqrt{1+s^2}) = f^2(a(s)+x) - (g(a(s))+xs)^2$

where

$$\frac{\psi(x\sqrt{1+s^2})}{x} = \frac{\psi(y\sqrt{1+s^2})}{|y|}.$$

This gives

$$|y|^{d+1} = \frac{(2\sigma^{d+1} - ((1+s^2)x^2 + \psi^2(x\sqrt{1+s^2}))^{(d+1/2)})x^{d+1}}{((1+s^2)x^2 + \psi^2(x\sqrt{1+s^2}))^{(d+1)/2}}, \quad (21)$$

and x and y are so small that the conditions of the lemma are satisfied.

By the Pythagorean theorem (see Fig. 6), the assumption $a(0) = 0$, and (18), we have

$$\begin{aligned} \psi^2(x\sqrt{1+s^2}) &= f^2(a(s)+x) - (g(a(s))+xs)^2 \\ &= f^2(0) - \sigma^2 + \phi^2(a(s)+x) - (\sqrt{f^2(a(s)) - \sigma^2} + sx)^2 \\ &= f^2(0) - \sigma^2 + \phi^2(a(s)+x) - (\sqrt{f^2(0) + \phi^2(a(s)) - 2\sigma^2} + sx)^2. \end{aligned}$$

Therefore, using (6) we have

$$\begin{aligned} x^2(1+s^2) + \psi^2(x\sqrt{1+s^2}) &= x^2 + f^2(0) - \sigma^2 + \phi^2(a(s)+x) - (f^2(0) + \phi^2(a(s)) - 2\sigma^2) \\ &\quad - 2xs\sqrt{f^2(a(s)) - \sigma^2} \\ &= x^2 + \phi^2(a(s)+x) + \sigma^2 \\ &\quad - \phi^2(a(s)) - 2xs\sqrt{f^2(a(s)) - \sigma^2} \\ &= x^2 + \phi^2(a(s)+x) + \sigma^2 - \phi^2(a(s)) \end{aligned}$$

$$\begin{aligned}
& -2xs\sqrt{f^2(a(0)) + \phi^2(a(s)) - 2\sigma^2} \\
& = \sigma^2 - 2a(s)x + \chi(a(s) + x) - \chi(a(s)) - 2xA.
\end{aligned}$$

Substituting the last expression into (20) with y defined by (21), we have

$$\begin{aligned}
& (\sigma^2 - 2a(s)x + \chi(a(s) + x) - \chi(a(s)) - 2xA)^{(d+1)/2} \\
& + (\sigma^2 + 2a(s)y + \chi(a(s) - y) - \chi(a(s)) + 2yA)^{(d+1)/2} = 2\sigma^{d+1},
\end{aligned}$$

where

$$|y|^{d+1} = x^{d+1} \frac{2\sigma^{d+1} - (\sigma^2 - 2(a+A)x + \chi(a+x) - \chi(a))^{(d+1)/2}}{(\sigma^2 - 2(a+A)x + \chi(a+x) - \chi(a))^{(d+1)/2}}.$$

This gives the desired result. \square

Corollary 2.4 *Let $s \in \mathbb{R}$ be fixed and such that $a = a(s) \in (-r_1, r_2) \cap (-\tau_1, \tau_2)$. Then*

$$2\sigma^2 \chi''(a) + (d+1)(\chi'(a) - 2(a+A))^2 = 0, \quad (22)$$

where A is as in the previous lemma.

Proof By the previous lemma, we have (9) and (12) with q_a instead of q and χ_a instead of χ , $\chi_a(0) = 0$. This gives (11) with χ_a instead of χ , which is the desired result. \square

2.3 Consequences of the Concavity of χ on $(-r_1, r_2) \cap (-\tau_1, \tau_2)$

Our next goal is to show that

$$\chi(a) = 0 \quad \forall a \in [-r_1, r_2] \cap [-\tau_1, \tau_2]. \quad (23)$$

The proof of (23) is contained in the following three statements.

Lemma 2.5 *Let $\lambda_1 > 0, \lambda_2 > 0$ be such that $[-\lambda_1, \lambda_2] \subseteq [-\tau_1, \tau_2]$ and*

$$(\sigma^2 + \chi(-\lambda_1))^{(d+1)/2} + (\sigma^2 + \chi(\lambda_2))^{(d+1)/2} = 2\sigma^{d+1}. \quad (24)$$

If $\chi \leq 0$ on $[-\lambda_1, \lambda_2]$, then $\chi = 0$ on $[-\lambda_1, \lambda_2]$. In particular, if $\chi \leq 0$ on $[-\tau_1, \tau_2]$, then $\tau_1 = \tau_2 = \sigma$.

Proof By (6), we have

$$0 \leq -\chi(x) \leq \sigma^2 - x^2 \leq \sigma^2, \quad x \in [-\lambda_1, \lambda_2].$$

By (24) we can assume that for all $x \in (0, \lambda_2]$ and for the corresponding $y \in [-\lambda_1, 0]$ we have the equality in (7). If $\chi(x) < 0$, then the left-hand side of this equality is strictly less than $2\sigma^{d+1}$. Hence, $\chi = 0$ on $[-\lambda_1, \lambda_2]$.

Assume now that $[-\lambda_1, \lambda_2] = [-\tau_1, \tau_2]$. Since

$$\tau_1^{(d+1)/2} + \tau_2^{(d+1)/2} = 2\sigma^{(d+1)/2},$$

by (6) we have two possibilities

$$\tau_1 \leq \sigma, \quad \tau_2 \geq \sigma, \quad \chi(\sigma) = \phi^2(\sigma) \geq 0, \quad \text{or} \quad (25)$$

$$\tau_1 \geq \sigma, \quad \tau_2 \leq \sigma, \quad \chi(-\sigma) = \phi^2(-\sigma) \geq 0. \quad (26)$$

We will consider case (25), the proof for (26) is similar. By (25), $\chi(\sigma) = 0$. Hence, $\phi(\sigma) = 0$ and $[0, \tau_2] = [0, \sigma]$, i.e., $\tau_2 = \sigma$. This gives $\tau_1 = \sigma$ and the lemma is proved. \square

Lemma 2.6 *Let χ satisfy (22), $\chi(0) = 0$, and let $\chi'(0) = 0$. Then $\chi = 0$ on $[-r_1, r_2] \cap [-\tau_1, \tau_2]$.*

Proof Using (22) we have $\chi''(a) \leq 0$ for all $a \in (-r_1, r_2) \cap (-\tau_1, \tau_2)$, i.e., χ is concave down on $(-r_1, r_2) \cap (-\tau_1, \tau_2)$. Then using the conditions of the lemma we get $\chi \leq 0$ on $[-r_1, r_2] \cap [-\tau_1, \tau_2]$. Now we apply Lemma 2.5.

If $[-\tau_1, \tau_2] \subset [-r_1, r_2]$ we put $\lambda_1 = \tau_1$, $\lambda_2 = \tau_2$. Let $[-r_1, r_2] \subset [-\tau_1, \tau_2]$. Consider the maximal segment $[-\lambda_1, \lambda_2] \subseteq [-r_1, r_2]$ for which (24) holds. We can assume that $\lambda_1 = r_1$ and $\lambda_2 \leq r_2$ (the proof in the case $\lambda_2 = r_2$, $-\lambda_1 \geq -r_1$ is similar). By Lemma 2.5 we have $\chi = 0$ on $[-\lambda_1, \lambda_2]$. Therefore, using (3), (6), and (18) we have

$$g^2(x) = f^2(x) - \sigma^2, \quad \phi^2(x) = \sigma^2 - x^2, \quad f^2(x) = \sigma^2 - x^2 + g^2(0),$$

for all $x \in [-\lambda_1, \lambda_2]$. We recall that $a(0) = 0$. Since for all $y \in [-\lambda_1, 0]$ we have

$$|(0, 0, g(0)) - (y, \phi(y), g(0))| = \sigma,$$

by the $(d+1)$ -equichordal property we also have

$$|(0, 0, g(0)) - (x, \phi(x), g(0))| = \sigma \quad \forall x \in [0, \lambda_2].$$

This gives $\lambda_2 = -\lambda_1$ and L must be a Euclidean ball, i.e., we can assume that $r_2 = \lambda_2$. \square

Lemma 2.7 *We have $\chi'(0) = 0$.*

Proof We recall that $\chi(0) = 0$. Assume the contrary, that $\chi'(0) \neq 0$. Let $\chi'(0) < 0$ (the proof for the case $\chi'(0) > 0$ is similar). By (22) we can assume that χ is concave down. Hence, there exists $\varepsilon > 0$ such $\chi > 0$ on $(-\varepsilon, 0)$ (we recall that $a(0) = 0$).

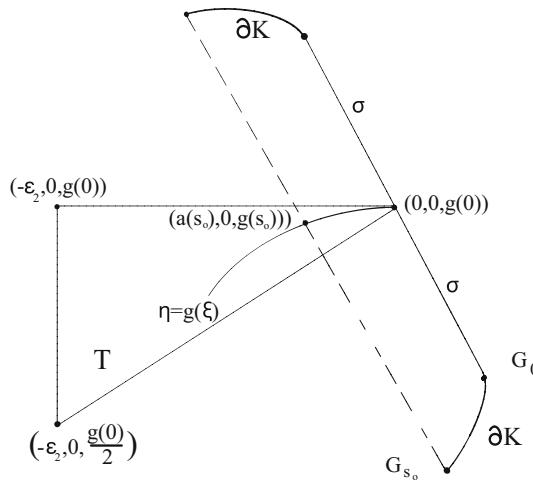


Fig. 7 The chord G_{s_o} intersects $\text{int } T$, but it is longer than 2σ

Let $Q = \{(x, y, z) : |y| \leq \sigma, x \leq 0\}$, and we recall that $G_s = K \cap \ell_s$ is a chord centered at $(a(s), 0, g(a(s)))$, parallel to the y -axis, and inscribed into ∂K (G_s is of length 2σ).

By symmetry with respect to the xz -plane the ends of $G_s, s \geq 0$, must belong to $\partial Q \cap \partial K$. We will show that for some small $s > 0$ this is not true and, by this, will obtain a contradiction. To this end, let $0 < \varepsilon_1 < \varepsilon$ be so small that for $\xi \in (-\varepsilon_1, 0)$ we have

$$\phi^2(\xi) = \sigma^2 + \chi'(0)\xi + o(\xi) > \sigma^2 \quad \forall \xi \in (-\varepsilon_1, 0), \quad (27)$$

where $o(\xi)$ is the remainder from the Taylor decomposition of ϕ . This shows that the points on the curve $\gamma_- = \{(\xi, -\phi(\xi), g(0)) : \xi \in (-\varepsilon_1, 0)\} \subset \partial K$ do not belong to Q . By the symmetry of $K \cap H_0$ with respect to the line $K \cap H_0 \cap \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$, the points on the curve $\gamma_+ = \{(\xi, \phi(\xi), g(0)) : \xi \in (-\varepsilon_1, 0)\} \subset \partial K$ do not belong to Q either. Define the plane set

$$\begin{aligned} \mathcal{B} &= \text{convhull}(\gamma_-, \gamma_+) \\ &= \{(\xi, y, g(a(0))) \in \mathbb{R}^3 : -\varepsilon_1 < \xi < 0, -\phi(\xi) \leq y \leq \phi(\xi)\} \subset K \cap H_0, \end{aligned}$$

and let $\mathcal{A} = \text{convhull}(K \cap Q, \mathcal{B})$. By convexity of K we have $\mathcal{A} \subset K$. We claim that for some $s > 0$ small enough, the ends of G_s are not on $\partial Q \cap \partial K$, which is a contradiction.

Indeed, let $0 < \varepsilon_2 < \varepsilon_1$. Consider a triangle T with vertices $(0, 0, g(0))$, $(-\varepsilon_2, 0, g(0))$, $(-\varepsilon_2, 0, -g(0)/2)$, and let $G_{(x, 0, z)}$ be a chord inscribed into ∂K , parallel to the y -axis and passing through $(x, 0, z) \in \text{int } T$. If ε_2 is small enough and $(x, 0, z) \in \text{int } T$, then the ends of $G_{(x, 0, z)}$ are not in Q and by (27) its length exceeds 2σ . Since g is concave, we can find $s_o = s_o(\varepsilon_2) > 0$ so small that $a(s_o) \in (-\varepsilon_2/2, 0)$,

and $(a(s_o), 0, g(a(s_o))) \in \text{int } T$. In other words, the chord G_{s_o} intersects $\text{int } T$ (see Fig. 7). But as we noticed above, the length of G_{s_o} exceeds 2σ , a contradiction. The proof in the case $\chi'(0) < 0$ is complete.

The case $\chi'(0) > 0$ can be proved similarly, one has only to consider $a(s_o) > 0$ for which $\chi(a(s_o)) > 0$ and $s < 0$, and to take $Q = \{(x, y, z) : |y| \leq \sigma, x \geq 0\}$. The lemma is proved. \square

2.4 Conclusions

Let K and L be two bodies of revolution about the x_1 -axis in \mathbb{R}^3 satisfying the conditions of Theorem 1.2. We recall that $a(0) = 0$, and by (23) we know that $\chi = 0$ on $[-r_1, r_2] \cap [-\tau_1, \tau_2]$. This means that $\phi^2(x) = \sigma^2 - x^2$ for all $x \in [-r_1, r_2] \cap [-\tau_1, \tau_2]$, and (18) yields

$$f^2(x) = \phi^2(x) + f^2(0) - \sigma^2 = f^2(0) - x^2 \quad (28)$$

for all $x \in [-r_1, r_2] \cap [-\tau_1, \tau_2]$. Moreover, (3) and (28) yield

$$g(x) = \sqrt{f^2(0) - x^2 - \sigma^2} \quad (29)$$

for all $x \in [-r_1, r_2] \cap [-\tau_1, \tau_2]$, and

$$g(-r_1) = g(r_2) = 0, \quad r_1 = r_2 = \sqrt{f^2(0) - \sigma^2},$$

provided $[-r_1, r_2] \subseteq [-\tau_1, \tau_2]$.

3 Auxiliary Statements, the Versions of Theorem 1 of Barker and Larman [2, pp. 83–84]

Lemma 2.1 *Let $K \subset \mathbb{R}^3$ and $L \subset \mathbb{R}^3$ be two convex bodies of revolution about the x -axis. Assume as above that their boundaries are described by f and g and satisfy (28) and (29). Then K and L are concentric Euclidean balls of radii $f(0)$ and $g(0)$.*

Proof Let Π be the xz -plane, and let $K \cap \Pi$ and $L \cap \Pi$ be the corresponding sections. Observe that since K and L are the bodies of revolution, the sets $K \cap \Pi$ and $L \cap \Pi$ are symmetric with respect to the x -axis.

We will set up a certain 2-dimensional *sweeping procedure* in which the ends of the chords, that are tangent to the circular part of $\partial L \cap \Pi$ and inscribed into $\partial K \cap \Pi$, will sweep out the corresponding circular arcs on $\partial K \cap \Pi$. Then, we will show that these arcs comprise $\partial K \cap \Pi$, thus concluding that $K \cap \Pi$ and $L \cap \Pi$ are concentric discs.

Case 1: $[-r_1, r_2] \subseteq [-\tau_1, \tau_2]$. As we just mentioned, $L \cap \Pi$ is a disc of radius $g(0) = \sqrt{f^2(0) - \sigma^2} = r_1 = r_2$. We will show that $K \cap \Pi$ is a disc of radius $f(0)$.

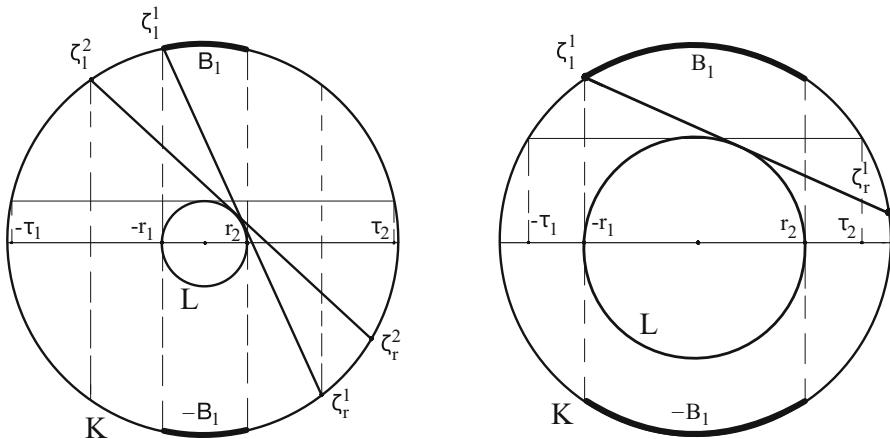


Fig. 8 The first steps in the sweeping procedure in the case $[-r_1, r_2] \subseteq [-\tau_1, \tau_2]$. On the left ζ_r^1 is below the x -axis, and on the right it is above the x -axis

Let $\mathcal{J}_1 = [\zeta_l^1, \zeta_r^1]$ be the chord inscribed into $\partial K \cap \Pi$ and tangent to $\partial L \cap \Pi$ at $(a_1, g(a_1))$, and such that its left end is $\zeta_l^1 = (-g(0), f(-g(0)))$, and right end is $\zeta_r^1 = (b_1, d_1)$. We have two possibilities, $d_1 = f(b_1) > 0$ or $d_1 = -f(b_1) < 0$ (ζ_r^1 is below or above the x -axis, see Fig. 8). Consider the arcs

$$\mathcal{B}_1 = \{(a, f(a)) : a \in [-g(0), g(0)]\}, \quad \beta_1 = \{(a, g(a)) : a \in [-g(0), g(0)]\},$$

of concentric circles, and let $\mathcal{J}^1(b) = [\zeta_l^1(b), \zeta_r^1(b)]$ be the chord inscribed into $\partial K \cap \Pi$ and tangent to $\partial L \cap \Pi$ at $(b, g(b)) \in \beta_1$ for $b \in [a_1, g(0)]$. Since the distance between $(b, g(b))$ and $\zeta_l^1(b)$ is σ , $\mathcal{J}^1(b)$ is divided by $(b, g(b))$ into two parts of equal length. Hence, while the left end of $\mathcal{J}^1(b)$ is sweeping out \mathcal{B}_1 by moving from $(-g(0), f(-g(0)))$ to $(g(0), f(g(0)))$, its right end must move along the arc of the circle of radius $f(0)$ (from (b_1, d_1) to $(g(0), -f(g(0)))$) joining $-\mathcal{B}_1$ from the right at $(g(0), -f(g(0)))$.

Let $d_1 = f(b_1) > 0$ (ζ_r^1 is above the x -axis, see the right part of Fig. 8). Then, the right end of $\mathcal{J}^1(b)$ for $b \in [a_1, g(0)]$ sweeps out the circular part of $\partial K \cap \Pi$ containing the one joining $(f(0), 0)$ with $(g(0), -f(g(0)))$. By the aforementioned symmetry of $\partial K \cap \Pi$ with respect to the x -axis, we see that the part of $\partial K \cap \Pi$ lying in the right half-plane is circular. Since the above procedure is symmetric with respect to the z -axis (we could start with the chord \mathcal{J}_1 tangent to $\partial L \cap \Pi$ at $(-a_1, g(-a_1))$ and follow the sweeping arc joining $(-b_1, d_1)$ to $(-g(0), -f(-g(0)))$), we conclude that $\partial K \cap \Pi$ is a circle of radius $f(0)$.

Now let $d_1 = -f(b_1) < 0$ (see the left part of Fig. 8). By the symmetry, four points $(\pm b_1, \pm f(b_1))$ are on $\partial K \cap \Pi$ and we recall that $f(x) = \sqrt{f^2(0) - x^2}$ for $x \in [-b_1, b_1]$. We will repeat the above procedure for the chord \mathcal{J}_2 inscribed into $\partial K \cap \Pi$ and tangent to $\partial L \cap \Pi$ at $(a_2, g(a_2))$, $0 < a_2 < a_1$, where $\mathcal{J}_2 = [\zeta_l^2, \zeta_r^2]$, $\zeta_l^2 = (-b_1, f(-b_1))$, $\zeta_r^2 = (b_2, d_2)$, $b_2 > b_1$, and the arcs $\mathcal{B}_2 = \{(a, f(a)) : a \in [-b_1, b_1]\}$, β_2 .

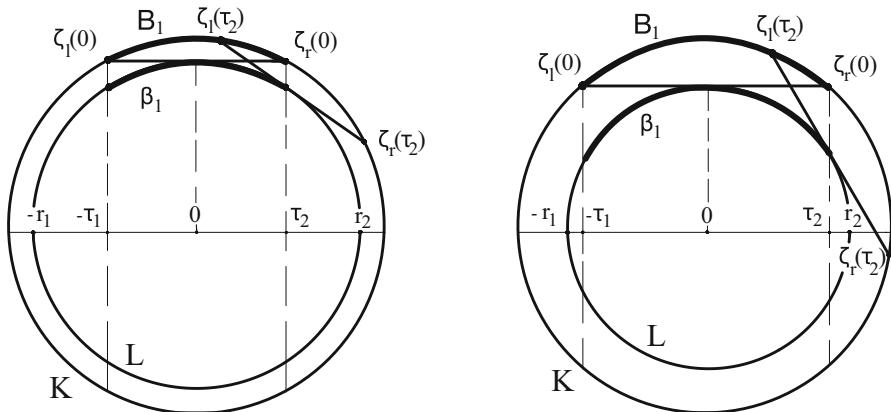


Fig. 9 The first steps in the sweeping procedure in the case $[-\tau_1, \tau_2] \subseteq [-r_1, r_2]$. On the left $\zeta_r(\tau_2)$ is above the x -axis, and on the right it is below the x -axis

We have two possibilities again, $d_2 = f(b_2) > 0$, and $d_2 = -f(b_2) < 0$. If $d_2 = f(b_2) > 0$, arguing as above, we see that the part of $\partial K \cap \Pi$ lying in the right half-plane is circular, and by the symmetry, $\partial K \cap \Pi$ is a circle. If $d_2 = -f(b_2) < 0$, taking into account that $(\pm b_2, \pm f(b_2))$, are on $\partial K \cap \Pi$ and $f(x) = \sqrt{f^2(0) - x^2}$ for $x \in [-b_2, b_2]$, we repeat the procedure again, producing the chords $J_3 = [\zeta_l^3, \zeta_r^3]$, $0 < a_3 < a_2$, and etc.

If for some $j \geq 3$ we have $d_j = f(b_j) > 0$, we finish as above. If, on the other hand, $d_j = -f(b_j) < 0$ for $j = 3, 4, \dots$, we produce a sequence of segments $\{[-b_j, b_j]\}_{j=1}^{\infty}$ such that $[-b_j, b_j] \subset [-b_{j+1}, b_{j+1}]$, and such that $f(x) = \sqrt{f^2(0) - x^2}$ for $x \in [-b, b]$, $b = \lim_{j \rightarrow \infty} b_j$.

We can also assume that $d_j = -f(b_j) < -g(0)$ for all $j = 3, 4, \dots$. Indeed, since the points $(\pm b_j, \pm f(b_j))$ must be on $\partial K \cap \Pi$, then the condition $-f(b_j) \geq -g(0)$ for some $j \geq 3$ implies that the chord with its left end at $(-b_j, f(-b_j))$ must have a positive second coordinate for its right end, so $d_{j+1} = f(b_{j+1}) > 0$.

We claim that $\partial K \cap \Pi$ is a circle. Indeed, let $b < f(0)$ (otherwise, we are done). If $-f(b) \geq -g(0)$, then the points $(\pm b, \pm f(b))$ must be on $\partial K \cap \Pi$. Hence, the chord with its left end at $(-b, f(-b))$ must have $(b, f(b))$ for its right end, $f(b) > 0$, and we are done. Finally, let $-f(b) < -g(0)$ and let

$$b = \sup \{x \in [0, f(0)] : f(x) = \sqrt{f^2(0) - x^2} \text{ on } [0, b]\}.$$

Then $-f(b) \geq -g(0)$, otherwise $(\pm b, \pm f(b))$ are on $\partial K \cap \Pi$, and we can repeat the procedure, contradicting the definition of b . This finishes the proof of Case 1.

Case 2: $[-\tau_1, \tau_2] \subsetneq [-r_1, r_2]$. Let $J_a = [\zeta_l(a), \zeta_r(a)]$ be a chord inscribed into $\partial K \cap \Pi$ and tangent to $\partial L \cap \Pi$ at the point $(a, g(a))$, for some $a \in (-\tau_1, \tau_2)$ (see Fig. 9).

Consider the arcs of concentric circles $B_1 = \{(a, f(a)) : a \in [-\tau_1, \tau_2]\}$ and $\beta_1 = \{(a, g(a)) : a \in [-\tau_1, \tau_2]\}$ and observe that for any $b \in [0, \tau_2]$ the distance between $(b, g(b))$ and $\zeta_l(b)$ is σ and J_b is divided by $(b, g(b))$ into two parts of equal length. Hence, while the left end $\zeta_l(b)$ for $b \in [0, \tau_2]$ is sweeping out the part of

\mathcal{B}_1 by moving from $\zeta_l(\tau_2)$ to $\zeta_l(0) = (-\tau_1, f(\tau_1))$, the right end $\zeta_r(b)$ must move along the arc of a circle of radius $f(0)$ joining \mathcal{B}_1 from the right at $\zeta_r(0)$. If we denote the coordinates of $\zeta_r(\tau_2)$ by (v_1, d_1) , we have (28) and (29) for all x in the interval $[-\tau_1, v_1]$ strictly containing $[-\tau_1, \tau_2]$. As in Case 1 we have two possibilities, $d_1 = f(v_1) < 0$ or $d_1 = -f(v_1) < 0$.

If $d_1 = -f(v_1) < 0$, we stop the procedure and see that the parts of $\partial K \cap \Pi$ and $\partial L \cap \Pi$, located in the right half-plane are concentric circles.

Let $d_1 = f(v_1) > 0$. Denote by $\alpha \in (0, \pi/2)$ the angle between the tangent line to $\partial L \cap \Pi$ passing through $(R_2, 0)$ and the x_1 -axis (we recall that $(R_2, 0)$ is the point of intersection of ∂K with the x -axis). If α_1 is the angle between the line containing \mathcal{J}_{τ_2} and the x_1 -axis, then $v_1 - \tau_2 = \sigma \cos \alpha_1$, and by convexity $\alpha_1 < \alpha$. We repeat the process with the larger arcs of concentric circles $\mathcal{B}_2 = \{(a, f(a)) : a \in [-\tau_1, v_1]\}$ and $\beta_2 = \{(a, g(a)) : a \in [-\tau_1, v_1]\}$ instead of \mathcal{B}_1 and β_1 . As above we have two possibilities $d_2 = f(v_2) > 0$ or $d_2 = -f(v_2) < 0$ for the corresponding right end (v_2, d_2) of the chord \mathcal{J}_{v_1} . If $d_2 = -f(v_2) \leq 0$, we stop. If $d_2 = f(v_2) > 0$ we repeat, observing that $v_2 - v_1 = \sigma \cos \alpha_2$ for the angle α_2 between the line containing \mathcal{J}_{v_1} and the x_1 -axis, $\alpha_2 < \alpha$. Proceeding this way, we will construct the corresponding arcs \mathcal{B}_j and β_j , $j = 3, \dots, m$. If for some j we have $d_j = -f(v_j) < 0$ for the corresponding right end of the chord $\mathcal{J}_{v_{j-1}}$, we will stop. Otherwise, we will proceed with $d_j = f(v_j) > 0$ for all $j = 3, \dots, m$, and the corresponding angles $\alpha_j < \alpha$. Since $v_j - v_{j-1} = \sigma \cos \alpha_j \geq \sigma \cos \alpha$ for $j = 2, \dots, m$, we will have

$$v_m = \tau_2 + (v_1 - \tau_2) + \dots + v_m - v_{m-1} \geq \tau_2 + m\sigma \cos \alpha \geq R_2,$$

provided m is large enough. We have proved that the parts of $\partial K \cap \Pi$ and $\partial L \cap \Pi$, located in the right half-plane are concentric circles.

Similarly, while the right end $\zeta_r(b)$ for $b \in [-\tau_1, 0]$ is sweeping out the part of \mathcal{B}_1 by moving from $\zeta_r(-\tau_1)$ to $\zeta_r(0) = (\tau_2, f(\tau_2))$, the left end $\zeta_l(b)$ must move along the arc of a circle of radius $f(0)$ joining \mathcal{B}_1 from the left at $\zeta_l(0)$. If we denote the coordinates of $\zeta_l(-\tau_1)$ by $(-u_1, \mathfrak{d}_1)$, we have (28) and (29) for all x in the interval $[-u_1, \tau_2]$ strictly containing $[-\tau_1, \tau_2]$. This gives (28) and (29) for all $x \in [-u_1, \tau_2]$. Considering two cases $\mathfrak{d}_2 = f(\tau_2) > 0$ or $\mathfrak{d}_2 = -f(\tau_2) < 0$, we can repeat the argument above to obtain that $\partial K \cap \Pi$ and $\partial L \cap \Pi$ are concentric discs. \square

Lemma 2.2 *Let K and L be two convex bodies in \mathbb{R}^3 satisfying the conditions of Theorem 1.2. If L is a body of revolution, then K is also a body of revolution with the same axis of rotation.*

Proof We assume that the x -axis is the axis of rotation of L . We will set up a 3-dimensional sweeping procedure rotating the cones that are tangent to ∂L with vertices on ∂K .

Let W_x be a plane parallel to the yz -plane and passing through $(x, 0, 0)$, $x \in \mathbb{R}$, and let $M(x) \subset W_x$ be a circle centered at $(x, 0, 0)$. We will show that for every x such that $(x, y, z) \in \text{int } K$, the generators of the sweeping cones cut out the circles $M(x) \subset \partial K$, thus proving that K is a body of revolution about the x -axis.

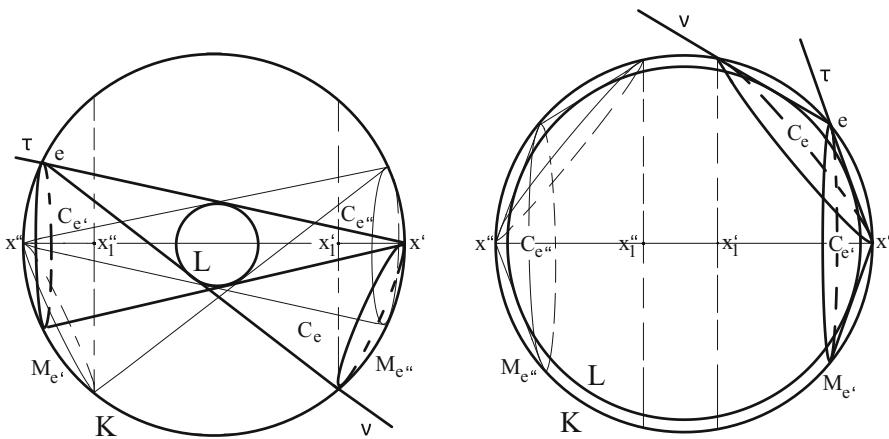


Fig. 10 The sweeping cones in the 3-dimensional procedure. On the left part we have $M_{e'}$ is left to $M_{e''}$ and on the right part $M_{e'}$ is right to $M_{e''}$

Let $e' = (x', 0, 0)$, $e'' = (x'', 0, 0)$, be two points of the intersection of the x -axis with ∂K , $x' > 0$, $x'' < 0$. To set up the procedure, we will make several auxiliary remarks and observations.

By the $(d+1)$ -equatorial property of K and L , for every ray τ emanating from e' and tangent to ∂L we have

$$|e' - \partial L \cap \tau|^{d+1} + |\partial L \cap \tau - \partial K \cap \tau|^{d+1} = 2\sigma^{d+1}. \quad (30)$$

Since $|e' - \partial L \cap \tau|$ is constant independent of τ , by (30) we see that the same is true for $|\partial L \cap \tau - \partial K \cap \tau|$. Therefore, for all rays τ emanating from e' and tangent to ∂L , all the chords $K \cap \tau$ have the same length. Since L is the body of revolution, for any rotation $\Phi = \Phi_\varphi$ by the angle $\varphi \in (0, 2\pi)$ around the x -axis, the points $\{\partial L \cap \Phi_\varphi \tau : \varphi \in [0, 2\pi]\}$ form a circle centered on the x -axis. By similarity of triangles, the ends $\{\partial K \cap \Phi_\varphi \tau \neq e' : \varphi \in [0, 2\pi]\}$ of the chords $K \cap \Phi_\varphi \tau$ form a circle $M_{e'} = \partial K \cap C_{e'}$ centered on the x -axis, where $C_{e'}$ is the cone tangent to ∂L with the vertex at e' (see Fig. 10).

Now we take any point $e \in M_{e'} \subset \partial K$ and repeat a similar argument for the cone C_e tangent to ∂L with the vertex at e . Observe that for any ray v generating C_e , the ends $\{\partial K \cap \Phi_\varphi v \neq \Phi_\varphi e : \varphi \in [0, 2\pi]\}$ of the chords $K \cap \Phi_\varphi v$ form a circle $M_{e,v}(e') \subset \partial K$ with the center on the x -axis and which is parallel to $M_{e'}$ (see Fig. 10).

Indeed, let $e \in M_{e'}$ and let v be any ray generating C_e . By rotation invariance of the length, $|\Phi(K \cap v)| = |K \cap v|$, and by the rotation invariance of L , $|e - \partial L \cap v| = |\Phi(e) - \partial L \cap \Phi(v)|$. Since (30) holds with e , v , and $\Phi(e)$, $\Phi(v)$, instead of e' , τ , and since for $\varphi \in [0, 2\pi]$ the points $\Phi_\varphi(e)$ and $\partial L \cap \Phi_\varphi(v)$, “move along” the circles centered on the x -axis, we see that the ends $\{\partial K \cap \Phi_\varphi v \neq \Phi_\varphi e : \varphi \in [0, 2\pi]\}$ of

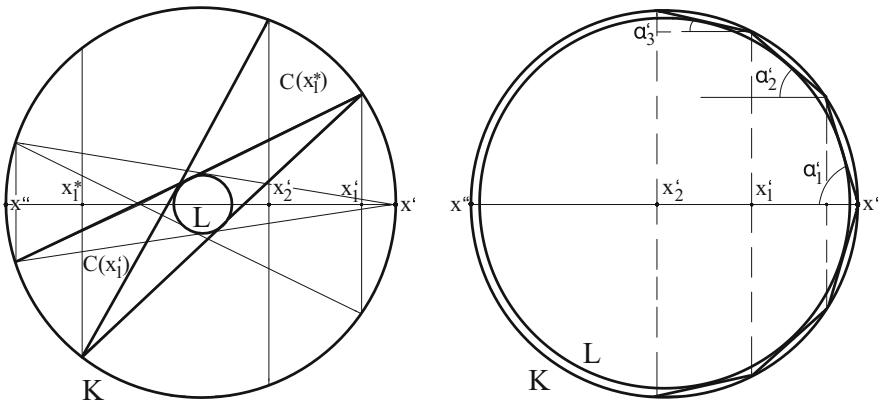


Fig. 11 Sections of the sweeping cones by the xz -plane and the construction of x'_1 and x'_2

the chords $K \cap \Phi_{\varphi} v$ form a circle $M_{e,v}(e')$ parallel to $M_{e'}$ and centered on the x -axis. This proves the observation. We can repeat the same argument with e'' instead of e' .

Now we are ready to make the first step of our procedure. Let

$$\begin{aligned} x'_1 &= \inf \{x : M(x) = M_{e,v}(e') \text{ with } v \text{ generating } C_e, e \in M_{e'}\}, \\ x''_1 &= \sup \{x : M(x) = M_{e,v}(e'') \text{ with } v \text{ generating } C_e, e \in M_{e''}\}. \end{aligned}$$

Observe that $x'_1 < x'$ and $x'' < x''_1$. We will consider two cases, $M_{e'}$ is right to $M_{e''}$ and $M_{e'}$ is left to $M_{e''}$ (see Fig. 10). In both cases, by the above observations we have $M(x) \subset \partial K$ for all $x \in [x'_1, x'] \cup [x'', x''_1]$, i.e., the sets $\{(x, y, z) \in K : x \in [x'_1, x']\}$ and $\{(x, y, z) \in K : x \in [x'', x''_1]\}$ are the bodies of revolution about the x -axis.

Let $M_{e'}$ be right to $M_{e''}$. We repeat the above argument for the generators of the cone C_e , with e belonging to the circles $M(x'_1)$ and $M(x''_1)$. This gives $M(x) \subset \partial K$ for all $x \in [x'_2, x'_1]$ for some $x'_2 \in [x'', x'_1]$ (see the right part of Fig. 11), and, similarly, $M(x) \subset \partial K$ for all $x \in [x''_1, x''_2]$ for some $x''_2 \in (x''_1, x']$, and etc.

We claim that after $m \in \mathbb{N}$ steps we have $M(x) \subset \partial K$ for all $x \in [x'', x']$, i.e., K is a body of revolution. In fact, since the lengths of all chords tangent to ∂L and inscribed into ∂K exceed or equal to

$$2^{1/(d+1)}\sigma = \min \{x + y : x^{d+1} + y^{d+1} = 2\sigma^{d+1} \text{ and } x \geq 0, y \geq 0\},$$

we have $x'_j - x'_{j+1} \geq 2^{1/(d+1)}\sigma \cos \alpha'_j$. Here by convexity $\alpha'_{j+1} < \alpha'_j < \pi/2$ for $j = 0, 1, \dots, m$, $x'_0 = x'$, $x''_0 = x''$ (see the right part of Fig. 11). Similarly, $x''_{j+1} - x''_j \geq 2^{1/(d+1)}\sigma \cos \alpha''_j$ for the corresponding α''_j . Hence, for sufficiently large m we have

$$\sum_{j=0}^m ((x'_j - x'_{j+1}) + (x''_{j+1} - x''_j)) \geq \sum_{j=0}^m 2^{1/(d+1)}\sigma (\cos \alpha'_j + \cos \alpha''_j) \geq x' - x'',$$

and the claim is proved.

It remains to consider the case where $M_{e'}$ is left to $M_{e''}$. As above, we will run the procedure that starts at e' and follows the cones C_e with vertices e at $M(x'_1), M(x'_2), \dots, M(x'_m)$. This time, however, each point x'_j , $j = 2, \dots, m$, will be constructed slightly differently: for the cones $C(x'_1) = C_e(x'_1)$, tangent to ∂L with $e \in M(x'_1)$, define

$$x_1^* = \sup \{x : (x, y, z) \in (C_e(x'_1) \cap \partial K) \setminus M(x'_1) \text{ for } e \in M(x'_1)\};$$

in its turn, for the cones $C(x_1^*) = C_e(x_1^*)$, tangent to ∂L with $e \in M(x_1^*)$, let

$$x_2' = \inf \{x : (x, y, z) \in (C_e(x_1^*) \cap \partial K) \setminus M(x_1^*) \text{ for } e \in M(x_1^*)\}$$

(see the left part of Fig. 11). Observe that $x_2' < x_1'$, and for all $x \in [x_2', x_1']$ we have $M(x) \subset \partial K$.

We can repeat the construction with the corresponding x_j^* and x'_j , $j = 2, \dots, m$, to see that $M(x) \subset \partial K$ for $x \in [x_m', x']$. Let r_1 and r_2 be such that $\{x : (x, y, z) \in L\} = [-r_1, r_2]$. If $0 < \inf_{j \geq 2} x'_j \leq r_2$, we stop the procedure. For, considering the cone C_e tangent to ∂L with $e \in M(\inf_{j \geq 2} x'_j)$, we see that the parts of K and L in $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$ are bodies of revolution.

Assume now that $\inf_{j \geq 2} x'_j > r_2$, and let

$$\gamma = \inf \{x : M(y) \subset \partial K \forall y \geq x\}, \quad 0 \leq \gamma \leq \inf_{j \geq 2} x'_j$$

(without loss of generality we can assume that $\gamma > r_2$, otherwise we finish as above). If $\sup_{j \geq 2} x_j^* \geq -r_1$, we stop. In this case, considering the cone C_e tangent to ∂L with $e \in M(\sup_{j \geq 2} x_j^*)$, we see that the part of K in $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$ is a body of revolution. Finally, the case $\sup_{j \geq 2} x_j^* < -r_1$ is impossible, for we could continue the procedure, which contradicts the definition of γ .

Thus, the parts of K and L in $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$ are bodies of revolution. The analogous argument for $\{(x, y, z) \in \mathbb{R}^3 : x \leq 0\}$ corresponding to e'' follows similarly. \square

4 Proof of Theorem 1.2

Let L be a body of revolution about the x_1 -axis and let W be any 3-dimensional subspace containing the x_1 -axis. If $d \geq 4$, we will consider $K \cap W$, $L \cap W$, where without loss of generality we assume that $W = \{x \in \mathbb{R}^d : x_4 = \dots = x_d = 0\}$.

By Lemma 2.2 we know that $K \cap W$ and $L \cap W$ are bodies of revolution about the x_1 -axis. It follows that, by Lemmas 2.6 and 2.7, we have (28) and (29). Hence, by Lemma 2.1, $K \cap W$ and $L \cap W$ are the concentric Euclidean balls.

Let now Π be any 2-dimensional subspace of \mathbb{R}^d , and let e_1 be the first coordinate vector. If $e_1 \notin \Pi$, let $W_\Pi = \text{span}(\Pi, e_1)$, and if $e_1 \in \Pi$ let W_Π be any 3-dimensional

subspace containing Π . In both cases, by the above, $K \cap W_\Pi$ and $L \cap W_\Pi$ are the concentric Euclidean balls. Hence, $K \cap \Pi$ and $L \cap \Pi$ are the concentric discs. Since Π was chosen arbitrarily, the application of [5, Corollary 7.1.4, p. 272] finishes the proof of Theorem 1.2.

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