



# 3D TQFT and HOMFLYPT homology

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## Abstract

We describe a family of 3d topological B-models whose target spaces are Hilbert schemes of points in  $\mathbb{C}^2$ . The interfaces separating theories with different numbers of points correspond to braid strands. The Hilbert space of the picture of a closed braid is the HOMFLY-PT homology of the corresponding link.

**Keywords** Knot homology · TQFT · Matrix factorizations · Kapustin–Rozansky–Saulina models · Bow varieties

**Mathematics Subject Classification** 57R56 · 57K10

## 1 Introduction

In our previous papers, we proposed a construction of the HOMFLY-PT link homology which is based on an interaction between two quiver varieties: the cotangent bundle to the flag variety and the Hilbert scheme of points in  $\mathbb{C}^2$ . For  $r, n \geq 0$  denote  $\mathcal{M}_{n;r}$  the moduli space of  $n$   $U(r)$  instantons in  $\mathbb{C}^2$ . Equivalently,  $\mathcal{M}_{n;r}$  is the Nakajima quiver variety of the framed  $\tilde{A}_0$  quiver:

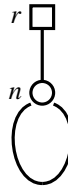
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(1.1)

Following [21, 22], we associate to  $\mathcal{M}_{n;r}$  the 2-category  $\check{\text{Cat}}(\mathcal{M}_{n;r})$ . We consider a special flag-related object  $\mathcal{F}_n$  in this category. Using our previous results [38], we construct a monoidal functor

$$\Phi : \mathfrak{Bt}_n \longrightarrow \text{Hom}(\mathcal{F}_n, \mathcal{F}_n), \quad \beta \mapsto \Phi(\beta),$$

where  $\mathfrak{Bt}_n$  is a braid group on  $n$  strands.

Our main knot theoretic result from [38, 40] can be recast into

**Theorem 1.1** *If  $\beta \in \mathfrak{Bt}_n$  is a braid, then in case of  $r = 1$  there is an isomorphism*

$$\mathcal{H}om_{\text{Hom}(\mathcal{F}_n, \mathcal{F}_n)}(\Phi(1_n), \Phi(\beta) \otimes \Lambda^\bullet V_n) \cong HHH(L(\beta)). \quad (1.2)$$

Here,  $1_n$  is the identity braid,  $V_n = \mathbb{C}^n$  is the defining representation of  $GL(n)$ ,  $L(\beta)$  is the link constructed by closing  $\beta$  and  $HHH(L(\beta))$  is its HOMFLY-PT homology.

In this paper, we consider this construction of link homology from a 3-category perspective. We outline a construction of a 3-category  $\check{\text{Cat}}(Q)$  for any  $\tilde{A}_n$ -type quiver  $Q$ . In the case  $Q = \tilde{A}_0$ , we provide a conjectural description of a part of 3-categorical structure by relating our setting to the theory of monoidal 2-categories of Kapranov and Voevodsky [23]; in Sect. 4.3, we construct the relevant monoidal 2-category  $\check{\text{Cat}}(Q)$ . That allows us to define the 3-category  $\check{\text{Cat}}(Q)$  as a subcategory of module 2-category over  $\check{\text{Cat}}(Q)$  in Sect. 4.2.

The 2-category  $\check{\text{Cat}}(\mathcal{M}_{n;r})$  emerges as the 2-category of morphisms between the objects  $\tilde{A}_0(n; r)$  and  $\tilde{A}_0(0; 0)$  that is  $\check{\text{Cat}}(\mathcal{M}_{n;r}) = \mathbb{H}om(\tilde{A}_0(n; r), \tilde{A}_0(0; 0))$ . The objects of the 3-category  $\check{\text{Cat}}(\tilde{A}_0)$  are  $\tilde{A}_0(n; r)$  or, equivalently, pairs of numbers  $(n; r)$ . For two pairs  $(n; r)$  and  $(n + k; r)$ , we define two ‘box’ functors:

$$\hat{\square}_k, \blacksquare_k : \check{\text{Cat}}(\mathcal{M}_{n;r}) \longrightarrow \check{\text{Cat}}(\mathcal{M}_{n+k;r}). \quad (1.3)$$

The functors represent generalized Lagrangian correspondences between  $\mathcal{M}_{n;r}$  and  $\mathcal{M}_{n+k;r}$ , and those originate from the edges of the second Nakajima quiver which is ‘transversal’ to the  $\tilde{A}_0$  quiver (1.1) in the sense of Gaiotto–Witten defects [12] in Kapustin–Witten theory [24], or brane arrangements in the superstring theory, see Sect. 2.

The quiver variety  $\mathcal{M}_{0;r}$  is just a point, hence its 2-category  $\check{\text{Cat}}(\mathcal{M}_{0;r})$  has a canonical object  $\mathcal{O}_\emptyset$ , and we show in Proposition 5.1 that  $\mathcal{F}_n$  is the result of applying  $\hat{\square}_1$   $n$  times to this object:

$$\mathcal{F}_n = \underbrace{\hat{\square}_1 \cdots \hat{\square}_1}_n \mathcal{O}_\emptyset \in \check{\text{Cat}}(\mathcal{M}_{n;r}).$$

Each individual  $\widehat{\square}_1$  corresponds to a braid strand, and an elementary braid  $\sigma_i$  which braids the  $i$ th and the  $(i + 1)$ st string, becomes a natural self-transformation of the composition:

$$\sigma_i \rightsquigarrow \widehat{\square}_1 \widehat{\square}_1 : \check{\text{Cat}}(\mathcal{M}_{i-1,r}) \longrightarrow \check{\text{Cat}}(\mathcal{M}_{i+1,r}).$$

When  $r = 0$  we define two more ‘box’ functors

$$\check{\square}_k, \blacksquare_k : \check{\text{Cat}}(\mathcal{M}_{n,0}) \longrightarrow \check{\text{Cat}}(\mathcal{M}_{n+k,0}). \quad (1.4)$$

These functors are related to Cherkis bow edges of the transversal ‘arrow-bow’ quiver. For an  $n$ -box,  $m$ -row Young diagram  $\lambda$  with rows  $\lambda_1 \geq \dots \geq \lambda_m \geq 0$ , we consider an object

$$\mathcal{S}_\lambda = \check{\square}_{\lambda_1} \cdots \check{\square}_{\lambda_m} \mathcal{O}_\emptyset \in \check{\text{Cat}}(\mathcal{M}_{n+k,0})$$

related to the equivariant Slodowy slice of a nilpotent matrix corresponding to  $\lambda$ . The category of morphisms between  $\mathcal{F}_n$  and  $\mathcal{S}_\lambda$  is the category of sheaves on (the S-dual of) the Nakajima quiver variety related to the weight space  $\lambda$  in the tensor product of  $n$  fundamental representations of  $\text{GL}(m)$ .

Most recently, the related categories were studied by Nakajima, Takayama [32] in the context of 3D mirror symmetry (see Sect. 6.2 for more discussion). The case of  $\mathcal{S}_{\frac{n}{2}, \frac{n}{2}}$  is related to the equivariant resolution of the Slodowy slice for the two-block nilpotent orbit in  $\mathfrak{gl}_{2n}$ . In [2], it was conjectured that the corresponding category can be used to construct the annular  $\text{GL}(2)$  link homology [3]. We explain a precise statement of the conjecture and its TQFT interpretation in Sect. 6.2.

The box functors have a natural interpretation as defects in 3d B-models considered by Kapustin–Rozansky–Saulina [21, 22]. This allows us to expand our results in three directions.

First, we provide details of the mathematical construction of some the structural morphisms of 3-category  $\check{\text{Cat}}(\mathcal{Q})$  for a linear quiver.

Second, we use basic TQFT axioms to give a physical interpretation of the Chern functor from our previous work [37] and thus provide an explanation for the appearance of the Hilbert scheme of points on  $\mathbb{C}^2$  in knot homology [40].

Third, we unify several homology theories of the closures of an affine braid  $\beta \in \mathfrak{Bt}_n^{\text{aff}}$  by interpreting them as the TQFT partitions functions on the annulus with external boundary consisting of  $n$  NS5 defects braided into the braid  $\beta$ . More precisely, by setting an appropriate configuration of the defects on the internal boundary of the annulus, we obtain: Hochschild homology of the Rouquier complex [20], the Hilbert scheme version of the triply braided homology [40] and the trace of the braid action on  $\mathfrak{sl}_2$  conformal blocks.

While exploring the categorical foundations of Kapustin–Rozansky–Saulina (KRS) theory [22], we avoid working with 3-category of KRS directly. Instead, we work with the de-looped version  $\check{\text{Cat}}(\mathcal{Q})$  of the 3-category. The 3-categorical nature of the 2-category  $\check{\text{Cat}}(\mathcal{Q})$  is reflected in the 2-categorical monoidal structure of  $\check{\text{Cat}}(\mathcal{Q})$ . The categorical details can be found in Sect. 4; in the rest of the introduction, we discuss

the last two directions that are concerned the most topologically motivated case TQFT that is based on the three category  $\check{\text{Cat}}(\hat{A}_0)$ .

### 1.1 Partition-function evaluation for HOMFLY-PT homology

The initial construction in [38] is based on the homomorphism from the braid group to a special category of matrix factorizations which appears as the category of endomorphism of the object  $\mathcal{F}_n \in \check{\text{Cat}}(\mathcal{M}_{n;1})$   $\text{MF}^{\text{st}} := \text{Hom}(\mathcal{F}_n, \mathcal{F}_n)$ :

$$\mathfrak{B}\tau_n \rightarrow \text{MF}^{\text{st}} = \text{MF}_{\text{GL}_n}((\mathfrak{gl}_n \times \mathbb{C}^n \times \text{T}^*\text{Fl} \times \text{T}^*\text{Fl})^{\text{st}}, W)^1, \quad W = \mu_1 - \mu_2, \quad (1.5)$$

where  $\text{Fl}$  is the flag variety,  $\mu$  is the moment map of the  $\text{GL}_n$  action on  $\text{T}^*\text{Fl}$  and the index ‘st’ stands for the stability condition.<sup>1</sup>

The object  $\mathcal{F}_n$  appears as a boundary condition for the KRS theory with the target

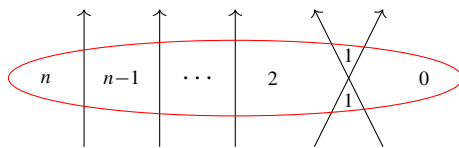
$$\mathcal{M}_{n;r} = (\text{T}^*\mathfrak{gl}_n \times \text{Hom}(\mathbb{C}^r, \mathbb{C}^n))^{\text{st}} // \text{GL}_n.$$

The cases  $r = 0$  and  $r = 1$  are the most relevant for the link homology. We refer to the first case as *unframed* and to the second case as *framed*. We choose the space-time of all our 3d TQFTs to be  $S^2 \times \mathbb{R}$ .

The total space-time  $S^2 \times \mathbb{R}$  can have ‘defect surfaces’ separating different models, that is, the connected components of the complement of defect surfaces are labeled by integers  $n$ . The defects can intersect along curves and curves can intersect at points. In the framed category, the defects have orientation and in both settings the curves of the intersection of the defects have signs.

The simplest case of the defect picture is when the defect is equal to  $\text{Def} = C \times \mathbb{R} \subset S^2 \times \mathbb{R}$  where  $C$  is an immersed curve (possibly with many connected components). In this case, we define the partition-function evaluation  $\mathbf{Z}^\bullet$  where  $\bullet$  is  $\emptyset$  in the unframed case and  $\bullet = \text{f}$  in the framed case. Our choices in the construction of the partition-function evaluation are driven by the topological applications [38] of the homomorphism (1.5):

**Theorem 1.2** *Let  $S^1 \times 0 \subset S^2 \times \{0\} \subset S^2 \times \mathbb{R}$  be an embedded circle such that it intersect the defect  $\text{Def} = C \times \mathbb{R}$  ( $C$  can have many connected components) transversally at  $2n$  points and the labels of the connected components of  $S^1 \setminus S^1 \cap \text{Def}$  are  $0, 1, 2, \dots, n, n-1, \dots, 2, 1$ , as for example, in the picture below<sup>2</sup>:*



<sup>1</sup> In the original paper we worked with the affine version of this category of matrix factorizations.

<sup>2</sup> On the picture the red circle is  $S^1$  and the black lines are the defect curve  $C$ .

then

$$Z(S^1) = MF_{GL_n}(\mathfrak{gl}_n \times T^*Fl \times T^*Fl, W).$$

Moreover, if the first  $n$  intersection points of  $S^1 \cap Def$  are oriented up and the rest down, as in the above picture, then

$$Z^f(S^1) = MF_{GL_n}((\mathfrak{gl}_n \times \mathbb{C}^n \times T^*Fl \times T^*Fl)^{st}, W). \quad (1.6)$$

Let us remark that in the last statement, we evaluated partition functions on the circle  $S^1$  and the values do not depend on the part of curve  $C$  that is away from  $S^1$ . On the other hand, the values of the partitions function on a disc  $D \subset S^2 \times \{0\}$ ,  $\partial D = S^1$  depends on the part of curve  $C$  that is inside  $D$ . We discuss values of the partition function on the discs below.

Let us use short-hand notation  $MF^{st}$  for  $MF_{GL_n}((\mathfrak{gl}_n \times \mathbb{C}^n \times T^*Fl \times T^*Fl)^{st}, W)$ . In [37], we constructed the pair of adjoint functors:

$$\begin{array}{ccc} & \xrightarrow{CH_{loc}^{st}} & \\ MF^{st} & & D^{per}(\text{Hilb}_n(\mathbb{C}^2)) \\ & \xleftarrow{HC_{loc}^{st}} & \end{array}, \quad (1.7)$$

where  $\text{Hilb}_n(\mathbb{C}^2)$  is the Hilbert scheme of  $n$  points on  $\mathbb{C}^2$ , while  $D^{per}(\text{Hilb}_n(\mathbb{C}^2))$  is the derived category of two-periodic  $\mathbb{T}_{q,t}$ -equivariant complexes on the Hilbert scheme. We also showed that

$$HHH(\beta) = \mathcal{H}om(CH_{loc}^{st}(\beta), \Lambda^\bullet \mathcal{B}). \quad (1.8)$$

Let us recall that  $\text{Hilb}_n(\mathbb{C}^2)$  is a manifold that parameterizes ideals  $I \subset \mathbb{C}[x, y]$  of codimension  $n$ . Respectively,  $\mathcal{B}$  is a rank  $n$  vector bundle with a fiber at  $I$  equal to the vector space dual to  $\mathbb{C}[x, y]/I$ . The vector bundle  $\mathcal{B}$  is called *tautological* vector bundle.

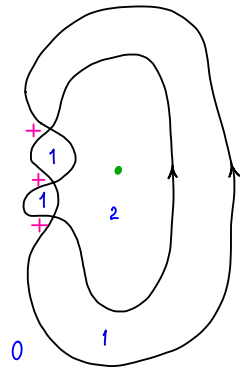
The TQFT picture gives a natural interpretation of the isomorphism (1.8) as the result of gluing the same disc in two different ways. The appearance of  $\text{Hilb}_n(\mathbb{C}^2)$  is due to the following:

**Theorem 1.3** *Let  $K = S^1 \times 0 \subset S^2 \times \mathbb{R}$  be an embedded circle which does not intersect the defect  $Def = C \times \mathbb{R}$  and lies in the connected component of the complement of the defect with the label  $n$  (see Fig. 4 for  $n = 2$  example) then*

$$Z^f(K) = D^{per}(\text{Hilb}_n(\mathbb{C}^2)).$$

Moreover, if  $D_\emptyset$  is a disc in the complement of the defect, such that  $\partial D_\emptyset = K$ , then

$$Z^f(D_\emptyset) = \mathcal{O} \in D^{per}(\text{Hilb}_n(\mathbb{C}^2)).$$

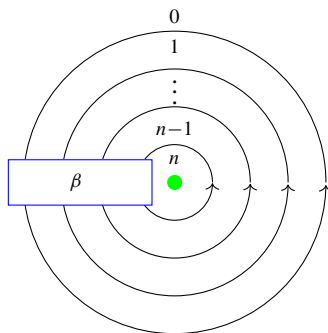
Fig. 1  $\mathbb{R}_\beta^2$  for  $\beta = \sigma_1^3$ 

To explain the appearance of the exterior powers of  $\mathcal{B}$ , we introduce a special line of defect in our theory:  $(0, 0) \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} \subset S^2 \times \mathbb{R}$ ,  $S^2 = \mathbb{R}^2 \cup \infty$ . We assume that this line of defect does not intersect the surface of defect. For a small disc  $D_{\text{taut}}$  transversally intersecting this line of defect, we have:

$$Z^f(D_{\text{taut}}) = \Lambda^\bullet \mathcal{B} \in D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2)). \quad (1.9)$$

The link homology emerges as the vector space associated with a disc which intersects defect surfaces and a defect line:

**Theorem 1.4** *Suppose that the curve  $C \subset \mathbb{R}^2 \setminus (0, 0) \subset \mathbb{R}^2 \cup \infty = S^2$  in the defect  $\text{Def} = C \times \mathbb{R}$  is the picture of the natural closure of the braid  $\beta \in \mathfrak{B}\mathfrak{r}_n$ , the closure goes around the line of defect. We also assume the following constraint on labeling of the connected components of  $S^2 \times \mathbb{R} \setminus \text{Def}$ . The connected component that contains the infinite point  $\infty$  is labeled by 0. The connected component that contains  $(0, 0)$  has label  $n$ . The labels decrease by 1 as we cross the surface of defect while moving along a ray that starts at  $(0, 0)$  and avoids singular points of  $\text{Def}$ . In the picture below*

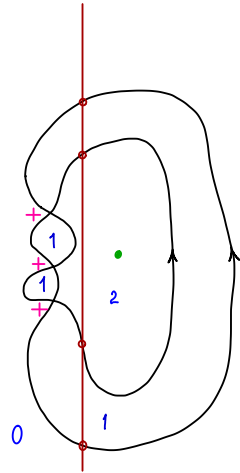


*the green dot represents a cross section of the defect line. Then*

$$Z^f(S^2) = HHH(\beta). \quad (1.10)$$

**Fig. 2** Plane  $\mathbb{R}^2_{\sigma_3}$  cut by

$\mathbb{R}_{0,1,2,1,0}$



Thus, the formulas (1.2) and (1.8) correspond to two ways to present  $S^2$  as a gluing of two  $D^2$  along their common boundary  $S^1$ . The formula (1.2) is given by cutting along  $S^1$  with  $S^1$  as in Theorem 1.2 and the formula (1.8) is given by cutting along  $S^1$  that is boundary of a tubular neighborhood of the line of defect. The Figs. 3 and 4 represent examples of these two cuts.

Let us denote by  $p_n$  a point in a connected component of  $S^2 \times \mathbb{R} \setminus \text{Def}$  that has label  $n$ . Note that Theorem 1.3 interprets  $D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2))$  as the category of endomorphisms of the identity functor in the two category  $Z^f(p_n)$ :

$$\mathbb{H}om(\mathbb{I}d, \mathbb{I}d) = D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2))$$

where  $\mathbb{I}d$  is the identity endomorphism of 2-category  $Z^f(p_n)$ . Thus, it is reasonable to call  $D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2))$  the Drinfeld center of 2-category  $Z^f(p_n)$ . Since it is not common to work with the Drinfeld centers of 2-categories, we spell out the expected property of such a center in Corollary 5.1.

## 1.2 Unified perspective on traces

The space  $\text{HHH}(\beta)$ ,  $\beta \in \mathfrak{B}\tau_n$  is triply graded, two gradings come from  $\mathbb{T}_{q,t}$ -action and the third is from the exponent in the exterior power  $\Lambda^\bullet \mathcal{B}$ . The graded dimension  $\mathcal{P}(\beta) = \dim_{q,t,a} \text{HHH}(\beta)$  is sometimes called *super-polynomial* of the closure of  $\beta$ , [8]. It is shown in [39] that  $\mathcal{P}(\beta)|_{t=-1}$  is equal to the HOMFLYPT polynomial [17].

Thus, equation (1.10) can be interpreted as a categorification of the Jones–Ocneanu trace [17] on  $\bigcup_n \mathfrak{B}\tau_n$ . Another construction of a categorification of the Jones–Ocneanu trace is the triply graded HOMFLYPT homology from the papers [18, 20]. In the last cited paper, Rouquier’s realization of braids inside Soergel bimodule is used. We denote the homology from the paper [18] by  $\text{HHH}_{\text{alg}}(\beta)$ ,  $\beta \in \mathfrak{B}\tau_n$ .

We have shown in [40] that  $\mathrm{HHH}(\beta) = \mathrm{HHH}_{\mathrm{alg}}(\beta)$  for any  $\beta \in \mathfrak{B}\tau_n$ . So far, we do not have a physical explanation for this equality but both categorifications of the Jones–Oceanu trace fit in our TQFT picture. In this section, we briefly discuss construction of  $\mathrm{HHH}_{\mathrm{alg}}$  and explain how we construct  $\mathrm{HHH}_{\mathrm{alg}}$  by a slight modification of the TQFT construction for  $\mathrm{HHH}$  from the previous section. We also explain how one can modify the TQFT construction to obtain other interesting traces on  $\bigcup_n \mathfrak{B}\tau_n$ .

Let us fix notations for the traces we plan to discuss. Soergel explained [43] a realization of the braid graphs  $\gamma \in \mathfrak{B}\tau_n^b$  inside the additive category of Soergel bimodules  $\mathrm{Sbim}_n$ . Thus, the Hochschild homology functor  $\mathrm{HH}_*$  yields a trace on  $\mathfrak{B}\tau_n^b$ :

$$\Phi^S : \mathfrak{B}\tau_n \rightarrow \mathrm{Sbim}_n, \quad \mathrm{HHH}_{\mathrm{alg}}(\gamma) = \mathrm{HH}_*(\Phi^S(\gamma)) = \mathrm{Hom}(\Phi^S(\beta), \Phi^S(I)).$$

The last example that we discuss is related to the space of conformal blocks. Let us denote by  $V_\lambda$  and  $L_\lambda$  the Verma module and its irreducible quotient for quantum group  $U_q(\mathfrak{sl}_2)$  and  $\lambda \in \mathbb{C} = \mathfrak{h}^*$ . In particular,  $L_1 = \mathbb{C}^2$  is the vector representation of  $U_q(\mathfrak{sl}_2)$  and  $L_\lambda$  is irreducible for not integer  $\lambda$  and  $L_\lambda = V_\lambda$ . The space of conformal on the cylinder is defined as

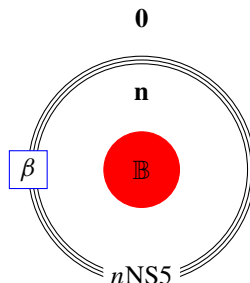
$$H_{1^n, \mu}^\lambda = \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(L_\lambda, L_1 \otimes \cdots \otimes L_1 \otimes L_\mu),$$

where we assume that  $\lambda, \mu$  are not integers.

The  $R$ -matrix yields the braid group  $\mathfrak{B}\tau_n$  action on the last space. Let's denote the trace of the action by  $\mathrm{Tr}(\beta)[H_{1^n, \mu}^\lambda]$ . The space  $H_{1^n, \mu}^\lambda$  is trivial unless  $|\lambda - \mu| \leq n$ . On the other hand if  $\lambda - \mu = k$ ,  $|k| \leq n$  then the above mentioned trace does not depend on the value of  $\lambda$  as long as  $\lambda$  is not integer.

The vector space  $H_{1^n, \mu}^\lambda$  has two gradings that originate from the weight grading on  $L_\mu$  and the decomposition of  $L_1^{\otimes n}$  on the irreducible components. The braid action respects these gradings. We propose a construction of doubly graded vector space  $\mathrm{Tr}(\beta)[H_{1^n, \mu}^\lambda]$  that categorifies the above trace.

Now let us turn to TQFT interpretation of the traces. A pictorial presentation of the configuration of the defects that participate in the formula (1.10) is the annulus with two boundary circles:



(1.11)

here  $\beta \in \mathfrak{B}\tau_n$ , the numeric labels indicate the type TQFT in the connected component.

The picture is topologically equivalent to the pictures in Theorem (1.4) and for  $n = 3$  and  $\beta = \sigma_1^3$  to Fig. 4. Let us explain appearance of labels NS5 in the last



picture. In the Sect. 2, we explain that in a string-theoretic presentation of our TQFT, the defect  $\text{Def} = C \times \mathbb{R}$  is an  $S^1$ -reduction of the stack of  $n$  NS5 branes. Thus, in the picture, we have  $n$  NS5 defects following the outer boundary of the annulus and braid themselves into the braid  $\beta$ .

The red domain contains the inner circle boundary defect configuration  $\mathbb{B}$ . In the case treated by Eq. (1.10), the boundary configuration is the domain with the line defect that carries the vector bundle  $\Lambda^\bullet \mathcal{B}$ . The line defect is reflected by equation (1.9).

It turns out that by varying the configuration of defects  $\mathbb{B}$  in the center we can categorify several other interesting trace evaluations on the braid group as well as algebra of the braid graphs  $\mathfrak{B}\tau_n$ .<sup>3</sup>

The three constructions are partition-function evaluations of defect configuration as in the picture and the internal disc defect  $\mathbb{B}$  as in the table where  $\text{NS5}^{(k)}$ ,  $\text{D5}^{(k)}$  are

$\mathbb{B}$	$\Lambda^\bullet V_n$	$\text{NS5}^{(n)}$	$\text{D5}^{(k)} + \text{D5}^{(n-k)}$
$\beta$	$\mathfrak{B}\tau_n$	$\mathfrak{B}\tau_n^b$	$\mathfrak{B}\tau_n$
$r$	1	1	0
$Z$	$\text{HHH}(\beta)$	$\text{HHH}_{\text{alg}}(\beta)$	$\text{Tr}(\beta)[H_{1n, \lambda+k}^\lambda]$

NS5 and D5 defects of charge  $k$  discussed in Sect. 2.5.3. The first two columns of the table reflect mathematically proven statements and the last column seems to be new mathematical construction. We explain the mathematical details of the relations in the table in Sect. 6.

In Sect. 2, we discuss the physics theories that motivate our key results. In details, we discuss in Sect. 2.2 Kapustin–Witten theory with Gaiotto–Witten interfaces and give mathematical description of the interfaces in Sect. 2.3. In Sect. 2.4, we explain a reduction to 3d theory. Finally, in Sects. 2.6 and 2.7, we explain a string theoretic perspective and the relation with the Chern-Simons theory, respectively.

In Sect. 3, we recall some basics of the constructions in [21]. In the Sect. 4, we explain how our particular example of TQFT fits into the setting of [21]. In Sect. 5, we construct the partition-function  $Z^f$  and prove the results that we mentioned in the introduction. We also discuss the Drinfeld center subtleties in Sect. 5.3.

## 2 Physics background

### 2.1 The hierarchy of type-B categories

From the mathematical perspective, we work within a 4-category **BGL** which, according to Kapustin and Witten [24] represents the Galois side of the geometric Langlands duality. Let us review the pyramid of **BGL** from the bottom to the top.

<sup>3</sup> The elements of  $\mathfrak{B}\tau_n$  are related to the braid graphs by the MOY relations [27].

### 2.1.1 The category of matrix factorizations

At the first level lies the category of  $G$ -equivariant matrix factorizations  $\mathrm{MF}^G(\mathcal{X}; W)$ , where  $\mathcal{X}$  is a (affine) variety with an action of an algebraic group  $G$ , while  $W$  is a  $G$ -invariant function on  $\mathcal{X}$ :  $W \in \mathbb{C}[\mathcal{X}]^G$ . The category  $\mathrm{MF}^G(\mathcal{X}; W)$  is the category of boundary conditions of the gauged Landau-Ginzburg B-model with the target space  $\mathcal{X}$  and the *superpotential*  $W$ .

### 2.1.2 The equivariant 2-category of a symplectic variety

At the second level lies the 2-category  $\ddot{\mathbf{B}}^G(\mathbf{X}^{\mathrm{sym}})$  of a symplectic variety  $\mathbf{X}^{\mathrm{sym}}$  with a Hamiltonian action of  $G$ . If  $\mathbf{X}^{\mathrm{sym}}$  is a cotangent bundle

$$\mathbf{X}^{\mathrm{sym}} = T^*\mathcal{X}$$

and the Hamiltonian action comes from the action of  $G$  on  $\mathcal{X}$ , then an object of  $\ddot{\mathbf{B}}^G(\mathbf{X}^{\mathrm{sym}})$  is a pair  $(\mathcal{Z}; W)$ , where  $\mathcal{Z}$  is an *auxiliary* variety with an action of  $G$ , while  $W \in \mathbb{C}[\mathcal{Z} \times \mathcal{X}]^G$  in this context is the *action* describing a Lagrangian submanifold in classical mechanics and  $\mathcal{Z}$  is the source of *auxiliary variables*. Intuitively, the pair  $(\mathcal{Z}; W)$  represents a *Lagrangian object* in the Hamiltonian reduction  $\mathbf{X}^{\mathrm{sym}}//G$ , that is, a  $G$ -equivariant fibration  $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$  whose base is a Lagrangian subvariety  $\mathcal{L} \subset \mathbf{X}^{\mathrm{sym}}$ . Here,  $\tilde{\mathcal{L}} \subset T^*(\mathcal{Z} \times \mathcal{X})$  is the graph of  $dW$ .

The category of morphisms between two Lagrangian objects is the category of  $G$ -equivariant matrix factorizations

$$\mathrm{Hom}((\mathcal{Z}_1; W_1), (\mathcal{Z}_2; W_2)) = \mathrm{MF}^G(\mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{X}; W_2 - W_1). \quad (2.1)$$

For two varieties  $\mathbf{X}_1^{\mathrm{sym}}$  and  $\mathbf{X}_2^{\mathrm{sym}}$  a Lagrangian object  $(\mathcal{Z}_{12}; W_{12})$  in the 2-category  $\ddot{\mathbf{B}}^{G_1 \times G_2}(T^*(\mathcal{X}_1 \times \mathcal{X}_2))$  determines a *Lagrangian functor*

$$\ddot{\mathbf{B}}^{G_1}(T^*\mathcal{X}_1) \xrightarrow{(\mathcal{Z}_{12}; W_{12})} \ddot{\mathbf{B}}^{G_2}(T^*\mathcal{X}_2)$$

stemming from the Lagrangian correspondence between  $T^*\mathcal{X}_1$  and  $T^*\mathcal{X}_2$ .

The 2-category  $\ddot{\mathbf{B}}^G(\mathbf{X}^{\mathrm{sym}})$  is the category of boundary condition of the gauged 3d B-model with target  $\mathbf{X}^{\mathrm{sym}}$ .

### 2.1.3 The 3-category of $G$ -equivariant symplectic varieties

At the third level is a 3-category  $\ddot{\mathbf{B}}(G)$ . Its objects are symplectic varieties  $\mathbf{X}^{\mathrm{sym}}$  with a Hamiltonian action of  $G$ . The 2-category of morphisms between two such varieties is the 2-category of their product:

$$\mathrm{Hom}(\mathbf{X}_1^{\mathrm{sym}}, \mathbf{X}_2^{\mathrm{sym}}) = \ddot{\mathbf{B}}^G(\mathbf{X}_1^{\mathrm{sym}} \times \mathbf{X}_2^{\mathrm{sym}}).$$

Equivalently, a morphism is a (generalized) Lagrangian correspondence, and the morphisms are composed accordingly. The 3-category  $\mathbf{\ddot{B}}(G)$  is the category of the boundary conditions of the Kapustin–Witten theory at  $t = i$ , which is related to the Galois side of the geometric Langlands duality.

### 2.1.4 The 4-category of Lie groups of type A

Finally, at the top of the pyramid is the 4-category  $\mathbf{BGL}$  of the groups of type A:

$$\text{Obj}(\mathbf{BGL}) = \mathbb{Z}_{\geq 0}$$

that is an object  $n$  corresponds to the group  $\text{GL}(n)$ . The morphisms between two groups form the 3-category of their product:

$$\text{Hom}(\text{GL}(n_1), \text{GL}(n_2)) = \mathbf{\ddot{B}}(\text{GL}(n_1) \times \text{GL}(n_2)) \quad (2.2)$$

and the composition of morphisms  $\mathbf{X}_{1,2}^{\text{sym}}$  and  $\mathbf{X}_{2,3}^{\text{sym}}$  is their joint Hamiltonian reduction:

$$\mathbf{X}_{2,3}^{\text{sym}} \circ \mathbf{X}_{1,2}^{\text{sym}} = (\mathbf{X}_{1,2}^{\text{sym}} \times \mathbf{X}_{2,3}^{\text{sym}}) // \text{GL}(n_2).$$

The 4-category  $\mathbf{BGL}$  is the category of Gaiotto–Witten interfaces in the  $t = i$  Kapustin–Witten theory.

## 2.2 Kapustin–Witten theory with Gaiotto–Witten interfaces

The 4-category  $\mathbf{BGL}$  comes from the TQFT of Kapustin and Witten [24] combined with the interfaces of Gaiotto and Witten [12] and inspired by the string theory setup of Hanany and Witten [15].

Following [24], we consider a family of 4-dimensional (4d)  $N = 4$  supersymmetric Yang–Mills (SYM) theories with gauge groups  $\text{U}(n)$ ,  $n = 0, 1, \dots$ . The space-time is a 4d manifold  $M^4$  containing 3d (possibly mutually intersecting) submanifolds (interfaces) of two types: NS5 and D5. These interfaces separate  $M^4$  into cells, each cell being assigned a particular value of  $n$ . The interface submanifolds may contain their own 2d interfaces-submanifolds, splitting 3d interfaces into 3d cells, etc. Topologically, the whole construction is an example of a 4d ‘foam’ (a smooth  $CW$  complex) with a particular property that the total ambient space  $M^4$  is just a manifold.

Following [24], we consider the topological version of our SYM theory: we apply the  $\text{GL}$ -twist and choose the differential corresponding to  $t = i$ . As explained by Kapustin and Witten, the resulting theory corresponds to the Galois side of the Langlands duality. The gauge groups  $\text{U}(n)$  are effectively complexified to  $\text{GL}(n)$ .

Mathematically, a TQFT on a 4d manifold with interfaces is equivalent to a 4-category. Each 4d cell is colored by its object, a cell of a 3d interface is colored by a morphism between the adjoint cells, etc. In order to determine a category whose objects should be assigned to an  $m$ -dimensional cell, one has to compactify the full theory on its  $(3 - m)$ -dimensional link. The cell becomes a boundary of the resulting

$(m + 1)$ -dimensional TQFT and the category of the cells emerges as the category of its boundary conditions.

Consider a 3d interface in the Kapustin–Witten theory separating the gauge groups  $GL(n_1)$  and  $GL(n_2)$ . Consider the Kapustin–Witten theory at  $t = i$ . The link of a 3d interface is two points, and the compactification amounts to folding the 4d space across the interface resulting in a 4d half-space whose boundary is the 3d interface and whose gauge group is  $GL(n_1) \times GL(n_2)$ . According to Gaiotto–Witten, the boundary condition is described by a symplectic variety  $\mathbf{X}^{\text{sym}}$  with the Hamiltonian action of  $GL(n_1) \times GL(n_2)$ , that is, by an object of **BGL**.

Suppose that the 3d interface has a 2d interface inside, separating the boundary conditions  $\mathbf{X}_1^{\text{sym}}$  and  $\mathbf{X}_2^{\text{sym}}$ . The link of this interface is a semicircle ending at the boundary of the 4d halfspace, so compactifications results in a 3d half-space whose theory is the 3d gauged B-model with the target  $\mathbf{X}_1^{\text{sym}} \times \mathbf{X}_2^{\text{sym}}$  and the gauge group  $GL(n_1) \times GL(n_2)$ . The 2-category of the boundary conditions of this theory is (2.1) as explained (in the non-equivariant case) in [21, 22]. In particular, an object  $(\mathcal{L}; W)$  describes a combination of a Lagrangian boundary condition with an added 2d B-model at the boundary, while the category of matrix factorizations describing interfaces between two such boundary conditions is the category of boundary conditions of the 2d Landau–Ginzburg B-model emerging after the folding of the 3d halfspace carrying the 3d B-model.

### 2.3 Bow (D5) and arrow (NS5) interfaces

Gaiotto and Witten suggest the version of the 4-category **BGL** in which all morphisms (that is, 3d interfaces) are compositions of two types of elementary ones: the NS5 interface and the D5 interface. Recall that any interface in  $\mathbf{Hom}(GL(n_1), GL(n_2))$  is represented by a symplectic variety with the Hamiltonian action of  $GL(n_1) \times GL(n_2)$ . The NS5 and D5 varieties also emerge as edges in the Nakajima–Cherkis quiver varieties, the NS5 variety corresponding to an arrow edge and the D5 variety corresponding to a bow edge. Ignoring the stability conditions, these edge-related varieties have a form

$$\prod_{(i,j)} T^* \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) // \prod_i G_i = T^* \left( \prod_{(i,j)} \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) / \prod_i G_i \right).$$

In subsequent pictures, we mark only the arrows of  $\text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$  (ignoring the cotangent arrows) because only these arrows (together with the auxiliary ones marked as dashed) are the variables of the superpotential  $W$  in Lagrangian objects  $(\mathcal{L}; W)$ .

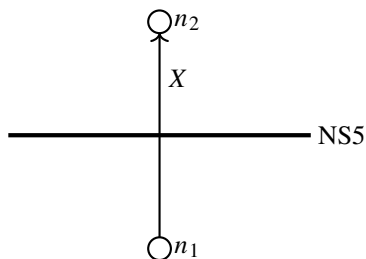
Ignoring the stability condition, the arrow variety is

$$\mathcal{A}_{n_1, n_2} = T^* \text{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2}), \quad (2.3)$$

the action of  $GL(n_1) \times GL(n_2)$  on  $\text{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2})$  being

$$(g_1, g_2) \cdot X = g_2 X g_1^{-1}.$$

Pictorially,

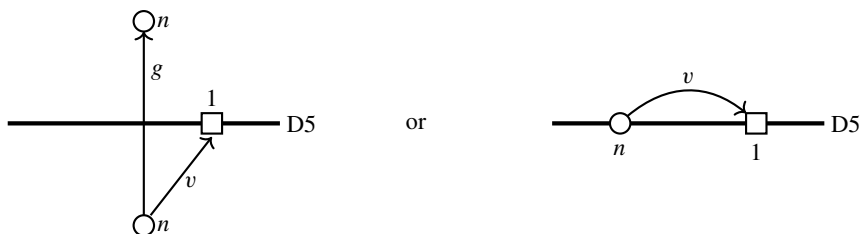


where  $X \in \text{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2})$ .

The description of the bow variety  $\mathcal{B}_{n_1, n_2}$  is more complicated, but the case of  $n_1 = n_2 = n$  is straightforward:

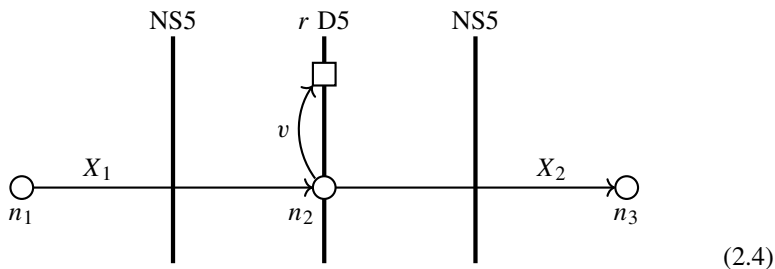
$$\mathcal{B}_{n,n} = T^*\text{GL}(n) \times T^*\mathbb{C}^n, \quad (g_1, g_2) \cdot (g, v) = (g_2 g g_1^{-1}, g_2 v)$$

(one can replace  $g_2 v$  with  $g_1 v$ ). The role of  $T^*\text{GL}(n)$  factor is to reduce the gauge group  $\text{GL}(n) \times \text{GL}(n)$  down to its diagonal subgroup  $\text{GL}(n)$ . Pictorially,



the second picture corresponding to the diagonal  $\text{GL}(n)$  equivariance.

With these choices, the composition of elementary morphisms in the category **BGL** produces a Nakajima quiver variety of type A:



## 2.4 From 4d to 3d

### 2.4.1 Compactification

We consider the Kapustin–Witten theory on the 4-manifold

$$M^4 = \mathbb{R}_0 \times \Sigma \times S_3^1, \quad (2.5)$$

where  $\Sigma$  is a 2-dimensional surface,  $\mathbb{R}_{i_1 \dots i_k}^k$  denotes  $\mathbb{R}^k$  with coordinates  $x_{i_1}, \dots, x_{i_k}$  and  $S_i^1$  denotes the circle  $S^1$  with the periodic coordinate  $x_i$ . In fact, we will be mostly interested in the case of  $\Sigma = \mathbb{R}_{12}^2$ , which is a local form of a general  $\Sigma$ .

We place simultaneously two arrangements of parallel elementary NS5 and D5 interfaces in  $M^4$ . The first *horizontal* arrangement consists of parallel NS5 and D5-interfaces, each interface sweeping a 3-dimensional subspace  $\mathbb{R}_0 \times \Sigma \times \{s_I\}$ , where  $s_I \in S_3^1$  is a point on the circle corresponding to  $x_3 = s_I$ . We assume that the values of  $n$  are the same on both sides of each D5-interface, but they can vary across NS5-interfaces.

An interface  $I$  of the second *vertical* arrangement has the form  $\mathbb{R}_0^1 \times \alpha_I \times S_3^1$ , where  $\alpha_I \subset \Sigma$  is a curve. The curves of different additional interfaces may intersect.

Fix a point  $b \in \mathbb{R}_0$  and a curve  $\beta \subset \Sigma$  possibly intersecting the curves  $\alpha_I$  of additional interfaces. Consider a 2d submanifold  $\mathcal{C}$  and a 3d submanifold  $\mathcal{N}$  of  $M^4$ :

$$\mathcal{C} = \{b\} \times \beta \times S_3^1, \quad \mathcal{N} = \{b\} \times \Sigma \times S_3^1.$$

We want to describe the category of  $\mathcal{C}$  and the vector space (that is, the Hilbert space) of  $\mathcal{N}$ . Since both manifolds have  $S_3^1$  as a factor, we can first compactify  $M^4$  on  $S_3^1$ , thus reducing it to  $M^3 = \mathbb{R}_0 \times \Sigma$ , which is split by vertical interfaces into domains. Each domain of  $M^3$  carries a 3d topological ‘Coulomb-twisted’  $N = 4$  SYM theory with matter fields.

The B-twisted Higgs branch of this theory is the Nakajima quiver variety  $\mathcal{M}_{\mathbf{n}, \mathbf{r}}(\tilde{A}_{m-1})$  whose affine quiver  $\tilde{A}_{m-1}$  and the dimensions at circles  $\mathbf{n}$  and at framing boxes  $\mathbf{r}$  are determined by the positions of the horizontal arrangement interfaces on  $S_3^1$  and the gauge groups  $\mathrm{GL}(n_i)$  between them in accordance with the picture (2.4):  $m$  is the number of NS5-interfaces in the basic arrangement, the dimensions of the circles are  $\mathbf{n} = (n_1, \dots, n_m)$  and the dimensions of the framing boxes  $\mathbf{r} = (r_1, \dots, r_m)$  equal the numbers of D5 interfaces between each pair of adjacent NS5 interfaces. If one of the numbers  $n_i$  is zero, then the affine quiver  $\tilde{A}_{m-1}$  becomes an  $A_{m-1}$  quiver.

Let  $Q$  be either the  $\tilde{A}_{m-1}$  or the  $A_{m-1}$  quiver. We denote by  $Q(\mathbf{r})$  the assignment of dimensions  $\mathbf{r}$  to the framing boxes and by  $Q(\mathbf{r}; \mathbf{n})$  the additional assignment of dimensions  $\mathbf{n}$  to the circles.

In mathematical language, the compactification on a circle  $S_3^1$  intersecting the vertical brane arrangement means that we reduce the 4-category **BGL** to the 3-category  $\check{\mathrm{Cat}}(Q(\mathbf{r}))$ . An object of  $\check{\mathrm{Cat}}(Q(\mathbf{r}))$  is  $Q(\mathbf{r}; \mathbf{n})$  and the 2-category of morphisms is the subcategory

$$\mathbb{H}\mathrm{om}(Q(\mathbf{r}; \mathbf{n}), Q(\mathbf{r}; \mathbf{n}')) \subset \check{\mathrm{B}}(\mathcal{M}_{\mathbf{n}, \mathbf{r}} \times \mathcal{M}_{\mathbf{n}', \mathbf{r}})$$

formed by Lagrangian functors corresponding to the NS5 branes of the vertical arrangement and described in Sect. 2.4.3.

## 2.4.2 Elementary intersections

From the 3d TQFT perspective, the original 3d vertical interfaces become 2d interfaces between quiver-related gauged 3d B-models. Hence a vertical interface is a Lagrangian functor between two  $\tilde{A}_{m-1}$  quiver varieties with different dimensions at circles. This Lagrangian functor is a combination of Lagrangian functors assigned to the intersections of the vertical interface with the horizontal interfaces.

There are two types of the vertical NS5 branes that appear in our discussion, see Sect. 2.6.3 for a string-theoretic interpretation of these two types of NS5 branes. The first type of branes we denote  $NS5$ , in the language of Sect. 2.6.3 these are branes that wrap cigar  $\mathbb{R}_{37}^2$ . A vertical NS5 brane of the second type wraps the cigar  $\mathbb{R}_{89}^2$  and we use notation  $NS5'$  for such branes.

The intersection of a vertical NS5 brane with a horizontal NS5 brane results in a 2d defect that is an object in the 2-category  $\tilde{\mathbf{B}}$  of the  $\tilde{A}_3$  quiver. This 2-category has two simplest objects:  $(\mathcal{Z}; W)$  and its Legendre dual  $(\tilde{\mathcal{Z}}; \tilde{W})$  which correspond to the intersection of a horizontal NS5 brane with a vertical  $NS5$  and  $NS5'$ , respectively:

$$(2.6)$$

The auxiliary variety and the superpotential of the first object are

$$\mathcal{Z} = \text{Hom}(\mathbb{C}^{n'_2}, \mathbb{C}^{n_1}), \quad W = \text{Tr} \psi (X' \varphi_1 - \varphi_2 X).$$

The Legendre dual object has auxiliary variety

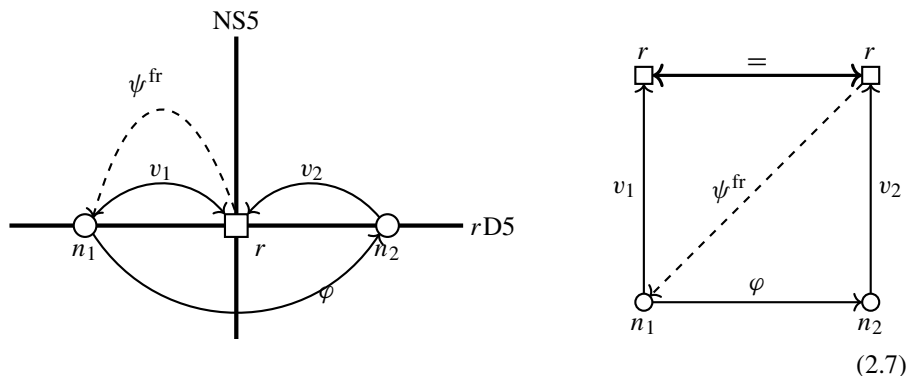
$$\tilde{\mathcal{Z}} = \mathcal{Z} \times \text{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1}) \times \text{Hom}(\mathbb{C}^{n'_2}, \mathbb{C}^{n'_1}),$$

while the superpotential becomes

$$\tilde{W} = W - \text{Tr} \tilde{Y} X - \text{Tr} \tilde{Y}' X'.$$

The intersection of an NS5 interface with a stack of  $r$  D5 interfaces results in a 2d defect from the category of the quiver  $A_2$  with the common framing space, the

corresponding object being  $(\mathcal{L}; W)$ :



the second picture omitting the branes and being more quiver-style,

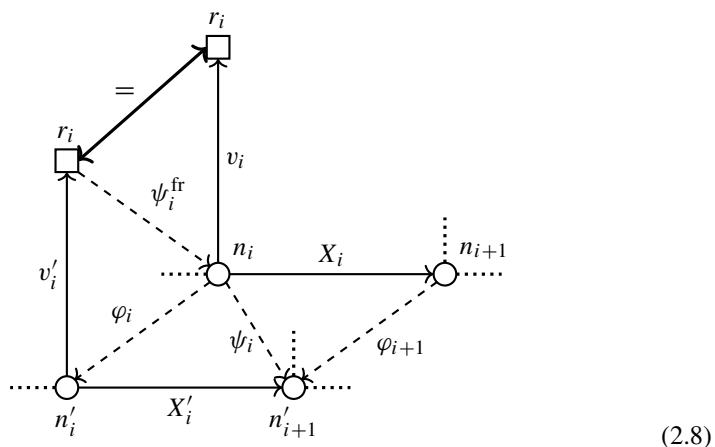
$$\mathcal{L} = \text{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1}), \quad W = \text{Tr } \psi^{\text{fr}}(v_2 \varphi - v_1).$$

Note that the spaces  $\mathbb{C}^r$  in the second picture of (2.7) are canonically identical.

### 2.4.3 A Lagrangian correspondence between two quiver varieties

Now we can describe a (formerly vertical) interface between two  $\tilde{A}_{m-1}$  quiver related domains in the 3d TQFT. We assume that both  $\tilde{A}_{m-1}$  quivers share the same dimensions at boxes while having generally different dimensions at circles.

In order to assign a pair  $(\mathcal{L}; W)$  to the interface, we put both quivers side by side. For each pair of matching circle to circle edges we put one of two squares (2.6) and for each pair of matching circle to box edges we put a square (2.7). Here is a portion of the resulting diagram with two such squares:



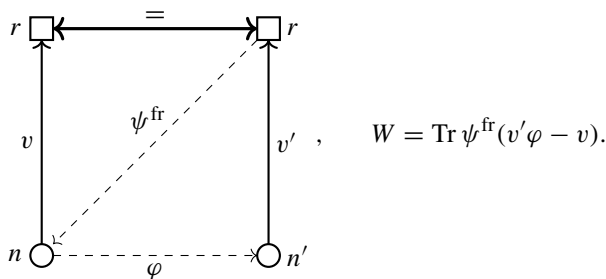


The auxiliary variety  $\mathcal{Z}$  is formed by all dashed arrows:  $\varphi_i$ ,  $\psi_i$  and  $\psi_i^{\text{fr}}$  as well as  $\tilde{Y}_i$  and  $\tilde{Y}'_i$ , while the superpotential  $W$  is the sum of all individual superpotentials.

## 2.5 Examples of inter-quiver Lagrangian correspondences

### 2.5.1 NS5 interfaces between Grassmannians

This example was communicated to us by Tudor Dimofte and it represents an interface between two  $A_1$  quivers, obtained from  $\tilde{A}_1$  quivers by setting  $n_2 = r_2 = 0$ . If we impose the Grassmannian stability conditions requiring that  $v$  and  $v'$  have highest ranks ( $n$  and  $n'$ ), then the corresponding quiver varieties are cotangent bundles  $T^*\text{Gr}_{n,r}$  and  $T^*\text{Gr}_{n',r}$ . If  $n \leq n'$ , then the first diagram of (3.10) yields a well-known Lagrangian functor:



whose Lagrangian correspondence is the conormal bundle to the subvariety of  $\text{Gr}_{n,r} \times \text{Gr}_{n',r}$  determined by the condition

$$\mathbb{C}^n \subset \mathbb{C}^{n'} \subset \mathbb{C}^r. \quad (2.9)$$

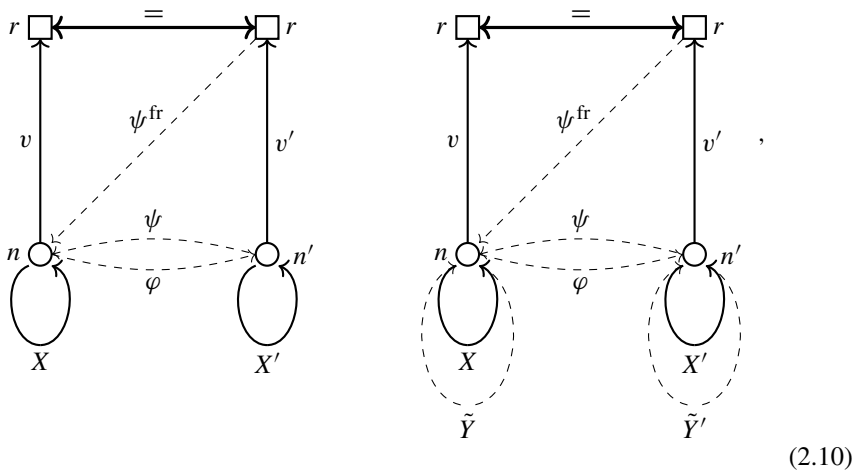
Indeed, the criticality condition for  $W$  with respect to  $\psi^{\text{fr}}$  implies the commutativity of the vertical and horizontal arrows, so  $v = v'\varphi$  and subdiagram of the arrows  $\varphi$  and  $v'$  determines the partial flag formed by the subspaces of both Grassmannians, as stated in (2.9).  $\mathbb{C}^n \subset \mathbb{C}^{n'} \subset \mathbb{C}^r$ . In addition, the criticality condition with respect to  $\varphi$  guarantees that the map  $\psi^{\text{fr}}$  represents the vector of the conormal bundle to the condition (2.9).

The superpotential and the space of maps  $\varphi$ , realizes NS5 defect of charge  $n - n'$ ,  $\text{NS5}^{(n-n')}$  in three category  $\check{\text{Cat}}(A_0)$ . We use the super-index in the notation of NS5 branes to indicate a difference of labels on two sides of the defect: the difference of labels on two sides of  $\text{NS5}^{(k)}$  is  $k$ .

### 2.5.2 NS5 interfaces between instanton moduli spaces

Now we consider the interfaces between two  $\tilde{A}_0$  quivers with  $n' = n + k$  and  $r = r'$  which yield the functors  $\hat{\square}_k$  and  $\hat{\blacksquare}_k$  of (1.3). The NS5 interfaces  $\text{NS5}^{(n-n')}$  of charge

$n - n'$  are described by diagrams



and the superpotentials are

$$W_1 = W(X, X') + W^{\text{fr}}, \quad W_2 = \text{Tr}(X\tilde{Y}) + \text{Tr}(X'\tilde{Y}') - W(\tilde{Y}, \tilde{Y}') + W^{\text{fr}},$$

where

$$W(X, X') = \text{Tr}(X'\varphi\psi) - \text{Tr}(X\psi\varphi), \quad W^{\text{fr}} = \text{Tr}\psi^{\text{fr}}(v'\varphi - v).$$

Note that the second Lagrangian object is the Legendre transform of the first one with respect to the loop arrows  $X, X'$ .

### 2.5.3 NS5 and D5 interfaces between commuting varieties

The following discussion is parallel to that of [42].

The commuting variety  $\mathcal{M}_n = T^*\mathfrak{gl}_n//\text{GL}(n)$  is the instanton moduli space for  $r = 0$ , the quiver diagram being a single loop:



For the family of these quivers, in addition to NS5 Lagrangian functions, we introduce a pair of D5-related Lagrangian functions, borrowing them from bow edges of Nakajima-Cherkis quiver varieties. The commuting variety  $\mathcal{M}_n$  is intimately related to the Hamiltonian reduction of a symplectic variety by the action of  $\text{GL}(n)$ , and the composition of NS5-related and D5-related Lagrangian functions results in a Nakajima-Cherkis quiver variety, this time related to the vertical arrangement of interfaces.

A symplectic variety  $\mathcal{Z}$  with the Hamiltonian action of  $\mathrm{GL}(n)$  and the moment map  $\mu: \mathcal{Z} \rightarrow \mathfrak{gl}(n)$  determines an object  $(\mathcal{Z}; \mathrm{Tr} X \mu)$  in the 2-category  $\mathbf{\check{B}}(\mathcal{M}_n)$ . Assuming that the action of  $\mathrm{GL}(n)$  is free, the category of morphisms between two such objects is

$$\mathrm{Hom}((\mathcal{Z}_1; \mathrm{Tr} X \mu_1), (\mathcal{Z}_2; \mathrm{Tr} X \mu_2)) = \mathrm{DCoh}((\mathcal{Z}_1 \times \mathcal{Z}_2) // \mathrm{GL}(n)).$$

This relation is due to the fact the category of matrix factorizations of a linear superpotential  $W = \mathrm{Tr} X \mu$  is equivalent to the category of coherent sheaves on  $\mu = 0$ .

Any symplectic variety  $\mathcal{Z}$  with the Hamiltonian action of  $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$  and momenta  $\mu_1, \mu_2$  determines a Lagrangian functor between  $\mathcal{M}_{n_1}$  and  $\mathcal{M}_{n_2}$ :

$$(\mathcal{Z}, W_{\mathrm{sym}}), \quad W_{\mathrm{sym}}(X_1, X_2) = \mathrm{Tr} X_2 \mu_2 - \mathrm{Tr} X_1 \mu_1, \quad (2.12)$$

and its Legendre transform

$$(\mathcal{Z} \times \mathrm{T}^* \mathfrak{gl}_n \times \mathrm{T}^* \mathfrak{gl}_n; \mathrm{Tr}(X_1 \tilde{Y}_1) + \mathrm{Tr}(X_2 \tilde{Y}_2) - W(\mathcal{Y}_1, \mathcal{Y}_2))$$

If the auxiliary variety  $\mathcal{Z}$  is the arrow edge variety (2.3):  $\mathcal{A}_{n_1, n_2} = \mathrm{T}^* \mathrm{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2})$ , then the Lagrangian functor (2.12) is that of (2.10) for  $r = 0$ .

A general bow edge variety has a more complicated form: ignoring the stability conditions, it is the critical locus of the pair

$$W_{\mathrm{bow}}(\tilde{Y}, \tilde{Y}') = \mathrm{Tr} \psi(\tilde{Y}' \varphi - \varphi \tilde{Y} + \sigma \rho).$$

Hence, its Lagrangian functor has the form

$$, \quad W = \text{Tr} X' \tilde{Y}' + \text{Tr} X \tilde{Y} - W_{\text{bow}}(\tilde{Y}, \tilde{Y}').$$

while its Legendre transform cousin has fewer arrows:

$$, \quad W = W_{\text{bow}}(X, X').$$

These two correspondences yield the functors  $\check{\square}_k$  and  $\check{\blacksquare}_k$ ,  $k = n - n'$  of (1.4) which realize the D5 interfaces  $\text{D5}^{(k)}$  of charge  $k$ . We use the super-index in the notation of D5 branes to indicate a difference of labels on two sides of the defect: the difference of labels on two sides of  $\text{D5}^{(k)}$  is  $k$ .

## 2.6 String theory perspective

### 2.6.1 IIB construction

The Galois version of the Kapustin–Witten model with two interface arrangements describes the physics of D3 branes stretched between two Hanany–Witten type NS5 and D5 brane arrangements in IIB string theory.

Consider the IIB string theory on  $M_{\text{IIB}}^{10} = \mathbb{R}_{0123}^4 \times \mathbb{R}_{4567}^4 \times \mathbb{R}_{89}^2$ , where  $\mathbb{R}_{i_1 \dots i_k}^k$  denotes  $\mathbb{R}^k$  with coordinates  $x_{i_1}, \dots, x_{i_k}$ . The twisting identifies the first two factors, so that the  $\text{SO}(4)$  rotation of  $\mathbb{R}_{0123}^4$  is accompanied by the matching rotation of  $\mathbb{R}_{4567}^4$ . D3-branes are stretched along  $\mathbb{R}_{0123}^4$ . Consider an orthogonal splitting  $\mathbb{R}_{0123}^4 = \mathbb{R}_{\text{br}}^3 \oplus \mathbb{R}^1$ , where  $\mathbb{R}_{\text{br}}^3$  subspace is tangent to an NS5 or a D5 brane, and the corresponding splitting  $\mathbb{R}_{4567}^4 = \mathbb{R}_{\text{br}}^3 \oplus \mathbb{R}^1$ . For this choice of the brane directions within the stack of D3-branes, the D3-transverse subspace  $\mathbb{R}_{456789}^6$  splits into the sum of the Coulomb

subspace  $\mathbb{R}_c^3 = \mathbb{R}_{br}^{3'}$  and the Higgs subspace  $\mathbb{R}_h^3 = \mathbb{R}^{1'} \times \mathbb{R}_{89}^2$ . Now an NS5-brane must be parallel to  $\mathbb{R}_{br}^3 \times \mathbb{R}_c^3$ , while a D5-brane must be parallel to  $\mathbb{R}_{br}^3 \times \mathbb{R}_h^3$ .

## 2.6.2 IIA construction

Replace  $\mathbb{R}_{0123}^4$  with  $M^4$  of (2.5). Now the IIB space-time is

$$M_{IIB}^{10} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times S_3^1 \times \mathbb{R}_{4567}^4 \times \mathbb{R}_{89}^2.$$

The horizontal Hanany–Witten brane arrangement corresponds to the choice  $\mathbb{R}_{br}^3 = \mathbb{R}_{012}^3$ , so its  $m$  NS5-branes stretch along  $\mathbb{R}_{012}^3 \times \mathbb{R}_{456}^3$  and its D5-branes stretch along  $\mathbb{R}_{012}^3 \times \mathbb{R}_{789}^3$ . The vertical Hanany–Witten arrangement has the form  $\mathbb{R}_{br}^3 = \mathbb{R}_0^1 \times \mathbb{R}_{dr}^1 \times S_3^1$ , where the ‘directional’ subspace  $\mathbb{R}_{dr}^1 \subset \mathbb{R}_{12}^2$  is tangent to the brane interface which is represented by its curve  $\alpha_I$  in the  $\mathbb{R}_{12}^2$  plane.

In string theory, the compactification of  $S_3^1$  within the KW theory is a result of performing  $T$ -duality on  $S_3^1$ . This duality turns the IIB theory into IIA theory and the stack of D3-branes turns into the stack of D2-branes stretched along  $\mathbb{R}_{012}^3$ . Now their movements are described by a 3d TQFT. Since the circle  $S_3^1$  intersects  $m$  NS5-branes of the horizontal arrangement, the  $T$ -duality turns their orthogonal space  $S_3^1 \times \mathbb{R}_{789}^3$  into the  $m$ -centered Taub-NUT space  $mTN$ , so the IIA space-time is

$$M_{IIA}^{10} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times \mathbb{R}_{456}^3 \times mTN.$$

D5 branes of the horizontal arrangement become D6-branes wrapping  $\mathbb{R}_{012}^3 \times mTN$ .

After the  $T$ -duality, the NS5 and D5 branes of the vertical HW arrangement turn into NS5 and D4-branes of IIA. They separate the stack of D2-branes into domains. In other words, they become the interfaces of the 3d B-model and hence they are described by Lagrangian correspondences. The form of the Lagrangian correspondence for an additional brane depends on which 2-cycle this brane wraps inside the Taub-NUT space  $mTN$ .

## 2.6.3 The case of a single NS5 brane

The commuting variety of Sect. 2.5.3 emerges when the horizontal arrangement consists of a single NS5 brane. If we ignore the metric, identifying the Taub-NUT space with  $\mathbb{R}_{3789}^4$ , then after the  $T$ -duality the IIA space-time becomes

$$M_{IIA}^{10} = \mathbb{R}_0 \times \mathbb{R}_{12}^2 \times \mathbb{R}_{456}^3 \times \mathbb{R}_{3789}^4 \cong \mathbb{R}^{10}.$$

Here, we use dotted indices to separate the new coordinates from the  $T$ -dual old ones. The manifold  $M_{IIA}^{10} \cong \mathbb{R}^{10}$  this time contains a single (formerly, vertical) Hanany–Witten-type arrangement of NS5 and D4 branes, and the Higgs branch of D2 branes stretched between them is the Nakajima quiver variety corresponding to the *vertical* brane arrangement. As explained in Sect. 2.5.3, this time the zero moment conditions at quiver circles are due to the criticality of the superpotential.

The origin of Legendre pairs of interfaces is now transparent. The Taub-NUT space  $\mathbb{R}_{3789}^4$  has two cigar subspaces:  $\mathbb{R}_{37}^2$  and  $\mathbb{R}_{89}^2$  that correspond to the arrow  $X$  in the commuting variety quiver diagram (2.11) and to its omitted symplectic dual arrow. The NS5 brane of the interface  $\widehat{\square}_k$  and the D5 brane of the interface  $\blacksquare_k$  wrap the cigar  $\mathbb{R}_{37}^2$ , while the NS5 brane of  $\blacksquare_k$  and the D5 brane of  $\widehat{\square}_k$  wrap the cigar  $\mathbb{R}_{89}^2$  (the latter branes are often denoted in string theory literature as NS5' and D5').

## 2.7 Link homology from a stack of D2 branes

### 2.7.1 A general setup

A stack of D3 branes sandwiched between two Hanany–Witten arrangements and a stack of D2 branes split by NS5 and D4 related interfaces into domains carrying quiver-based 3d B-models, appear in theories of link homologies. Generally, link homology is a special example of (categorified) Donaldson–Thomas invariants of a Calabi-Yau 3-fold  $CY^3$  in presence of special Lagrangian submanifolds  $L \subset CY^3$ . Hence, we begin with M-theory on a manifold of the form

$$M_M^{11} = CY^3 \times HK^2 \times \mathbb{R}_{\text{cob}}, \quad (2.14)$$

where  $HK^2$  is a (complex) 2-dimensional hyper-Kähler manifold and  $\mathbb{R}_{\text{cob}}$  is the cobordism time. Hanany–Witten brane arrangements come from M5 branes. Each M5 brane has a form  $L \times \mathcal{C} \times \mathbb{R}_{\text{cob}}$ , where  $L \subset CY^3$  is Lagrangian, while  $\mathcal{C} \subset HK^2$  is a complex curve.

Assume that  $CY^3$  is “small” (that is, all relevant physics happens within a small domain inside  $CY^3$ ). Then, link homology is (a part of) the space of states of BPS particles attached to M5-related defects in the effective 5-dimensional theory on  $HK^2 \times \mathbb{R}_{\text{cob}}$ . The BPS particles are M2 branes of the form

$$\Sigma \times \{p_{M2}\} \times \mathbb{R}_{\text{cob}}, \quad (2.15)$$

where  $\Sigma \subset CY^3$  is a holomorphic curve attached by its boundary components to the Lagrangian submanifolds  $L$ , while  $p \in HK^2$  is a point.

Our approach to the description of the BPS states involves three steps.

First, suppose that a group  $U(1)_{CY}$  acts on  $CY^3$  preserving the Calabi-Yau form. Taking a quotient by this action reduces M-theory on  $M_M^{11}$  to IIA string theory on

$$M_{\text{IIA}}^{10} = M^5 \times HK^2 \times \mathbb{R}_{\text{cob}}, \quad M^5 = CY^3/U(1)_{CY}.$$

The quotient may produce D6 branes of the form  $\mathcal{C}_{D6} \times HK^2 \times \mathbb{R}_{\text{cob}}$ , where  $\mathcal{C}_{D6} \subset M^5$  is the 2-dimensional surface of  $U(1)_{CY}$ -stable points. Depending on whether  $L$  is transverse to  $U(1)_{CY}$  orbits or contains them, each M5 brane becomes an NS5 brane or a D4 brane. If M2 branes are transverse to  $U(1)_{CY}$  orbits or invariant with respect to  $U(1)_{CY}$  action, then they become D2 branes, possibly lying inside D6 branes and with boundaries on NS5 and D4 branes.

Second, we place M5 branes inside  $CY^3$  in such a way that the resulting D2 branes assemble in a stack, whose collective movements and vibrations are described by a  $d = 3$ ,  $N = 4$  SYM theory. The boundaries of individual D2 branes ending on NS5 and D4 branes now transform into interfaces inside the stack of D2 branes, separating domains with (generally) different “stack thickness”.

Third, the SYM theory of the D2 stack may be twisted in such a way that a special scalar supercharge  $Q$  would allow us to turn the theory into a (possibly gauged) topological 3d B-model. If the NS5 and D4 interfaces are  $Q$ -invariant, then the space of BPS particle states becomes isomorphic to the homology of  $Q$ , that is to the space of states of this 3d topological B-model.

### 2.7.2 Witten's setup

According to Witten, in order to relate the graded dimension of BPS states to the Chern-Simons-Witten (CSW) partition function, one has to make two specific choices in (2.14). First, one has to choose  $HK^2 = TN$ ,  $TN$  being the Taub-NUT space, that is  $TN = \mathbb{C}^2 = \mathbb{C}_x \times \mathbb{C}_y$  with a special ALF metric. Second, one has to choose  $CY^3 = T^*M^3$ , where  $M^3$  is a 3d manifold carrying the CSW theory. A Chern-Simons generating stack of  $N$  M5 branes wraps  $M^3 \times \mathbb{C}_y \times \mathbb{R}_{cob}$ . For each link component  $\mathcal{L} \subset M^3$  one introduces an M5 brane of the form  $N^\vee \mathcal{L} \times \mathbb{C}_x \times \mathbb{R}_{cob}$ , where  $N^\vee \mathcal{L} \subset T^*M^3$  is the conormal bundle of  $\mathcal{L}$ . One has to push these link-related M5 branes off the zero section  $M^3$ , then the BPS particles will be M2 branes of the form (2.15), where  $\Sigma$  is an annulus, one of its boundary components being attached to the stack of Chern-Simons M5 branes and the other being attached to the pushed-off link-related M5.

In order to implement our approach and following M. Aganagic, we choose  $M^3 = \mathbb{C}_{cyl}^* \times \mathbb{R}_{br}$ , where  $\mathbb{C}_{cyl}^*$  is a cylinder and  $\mathbb{R}_{br}$  is the braid time. A link component is locally a braid component of the form  $\{p_{br}\} \times \mathbb{R}_{br}$ ,  $p_{br} \in \mathbb{C}_{cyl}^*$ . Present  $T^*\mathbb{R}_{br}$  as a product:  $T^*\mathbb{R}_{br} = \mathbb{R}_{br} \times \mathbb{R}_{br}^\vee$ . Each M5 brane is represented by a point at the origin of  $\mathbb{R}_{br}^\vee$ , which is the zero section of  $T^*\mathbb{R}_{br}$ . We push all M5 branes off the zero section by giving each M5 brane its own position at  $\mathbb{R}_{br}^\vee$ . It is particularly convenient to keep Chern-Simons M5 branes to the left of the braid M5 branes. M2 branes are now stretched along  $\mathbb{R}_{br}^\vee$ , being sandwiched between these M5 branes. For example, the following diagram represents the case of a  $U(2)$  Chern-Simons theory and a two-strand braid:

$$(2.16)$$

Now we implement our approach. Present the cylinder  $\mathbb{C}_{cyl}^*$  as  $\mathbb{C}_{cyl}^* = S_{cyl}^1 \times \mathbb{R}_{cyl}$ . The group  $U(1)_{CY}$  rotates  $S_{cyl}^1$  and M-theory is reduced to IIA theory on

$$M_{IIA}^{10} = \mathbb{R}_{cyl} \times \mathbb{C}_{cyl}^\vee \times \mathbb{R}_{br} \times \mathbb{R}_{br}^\vee \times TN \times \mathbb{R}_{cob}.$$

Chern-Simons related M5 branes become D4 branes spanning  $\mathbb{R}_{\text{cyl}} \times \mathbb{R}_{\text{br}} \times \mathbb{C}_x \times \mathbb{R}_{\text{cob}}$ , braid related M5 branes become NS5 branes spanning  $\mathbb{C}_{\text{cyl}}^\vee \times \mathbb{R}_{\text{br}} \times \mathbb{C}_x \times \mathbb{R}_{\text{cob}}$  and M2 branes which represent the link homology related BPS particles of the 5d theory on  $\text{TN} \times \mathbb{R}_{\text{cob}}$  become D2 branes spanning  $\bullet\bullet \times \mathbb{R}_{\text{br}} \times \mathbb{R}_{\text{cob}}$ , where  $\bullet\bullet$  denotes the segments stretched along  $\mathbb{R}_{\text{br}}^\vee$ . The diagram (2.16) now becomes a familiar Hanany–Witten type arrangement:

$$\text{---} \bigcirc_{\text{D4}} \text{---} \text{D2} \text{---} \bigcirc_{\text{D4}} \text{---} \text{two D2} \text{---} \bigcirc_{\text{NS5}} \text{---} \text{D2} \text{---} \bigcirc_{\text{NS5}} \text{---} \mathbb{R}_{\text{br}}^\vee \quad (2.17)$$

Following the presentation  $\mathbb{C}_{\text{cyl}}^* = S_{\text{cyl}}^1 \times \mathbb{R}_{\text{br}}$ , we decompose  $\mathbb{C}_{\text{cyl}}^\vee = \mathbb{R}_{S^1}^\vee \times \mathbb{R}_{\text{cyl}}^\vee$ . We twist the 3d theory of D2 branes by identifying their worldspace  $\mathbb{R}_{\text{br}}^\vee \times \mathbb{R}_{\text{br}} \times \mathbb{R}_{\text{cob}}$  with  $\mathbb{R}_{\text{cyl}} \times \mathbb{R}_{\text{cyl}}^\vee \times \mathbb{R}_{S^1}^\vee$ . Now a domain of thickness  $n$  in the stack of D2 branes carries the 3d B-model whose target is the commuting variety  $\mathcal{M}_n = T^*\mathfrak{gl}_n // \text{GL}(n)$  of Sect. 2.5.3. These domains are separated by NS5 and D4 interfaces corresponding to the functors  $\widehat{\square}_k$  and  $\check{\square}_k$  respectively. If the sandwich (2.17) of NS5 and D4 branes is squeezed, then 3d B-models become a single 2d B-model whose target is the Nakajima quiver variety  $\mathcal{M}_Q$  of the corresponding bow-arrow quiver. The braiding of the braid strands  $\{p_{\text{br}}\} \times \mathbb{R}_{\text{br}}$  within  $M^3 = \mathbb{C}_{\text{cyl}}^* \times \mathbb{R}_{\text{br}}$  results in the action of the affine braid group on the category  $\text{DCoh}(\mathcal{M}_Q)$  introduced by Rina Anno [2].

This construction can be generalized, if we allow Chern-Simons-related and braid-related M5 branes to wrap both types of cigars:  $\mathbb{C}_x$  and  $\mathbb{C}_y$ . If  $N$  M5 branes wrap  $M^3 \times \mathbb{C}_y \times \mathbb{R}_{\text{cob}}$  while  $M$  M5 branes wrap  $M^3 \times \mathbb{C}_x \times \mathbb{R}_{\text{cob}}$ , then  $M^3$  carries the supergroup  $U(N|M)$  Witten-Chern-Simons theory. The  $\mathbb{C}_x$ -related D4 branes are denoted  $\text{D4}'$  and they correspond to the interfaces  $\blacksquare_k$ . The original braid M5 branes wrapped on  $\mathbb{C}_x$  cigar yield  $U(M|N)$  representations  $\mathbb{C}^{N|M}$ . If a braid-related M5 brane is wrapped on  $\mathbb{C}_y$  cigar, then it becomes an NS5' brane in the IIA string theory corresponding to an interface  $\check{\square}_k$  and producing the representation  $\mathbb{C}^{N|M}$ . Note that in the original setup the group  $U(N)$  appears as  $U(N|0)$  and a braid-related NS5 generates the odd version  $\mathbb{C}^{0|N}$  of the defining representation.

### 2.7.3 Vafa's setup

The following is a brief account of a part of [7]. If  $M^3 = S^3$ , then the corresponding Calabi–Yau manifold  $\text{CY}_{\text{dc}}^3 = T^*S^3$  is a deformed conifold. Assume that a link in  $S^3$  is a closed ‘tight’ braid winding along a ‘base unknot’. Then, according to Vafa, this M-theory setup with  $N$  Chern-Simons related M5 branes wrapping the zero section  $S^3 \subset T^*S^3$  can be equivalently replaced by a resolved conifold

$$\text{CY}_{\text{rc}}^3 = \left( \mathcal{O}(-1)_1 \otimes \mathcal{O}(-1)_2 \longrightarrow \mathbb{P}^1 \right).$$

The Chern-Simons-related M5 branes disappear, but the braid-related M5 branes remain as Lagrangian submanifolds of  $\text{CY}_{\text{rc}}^3$  in such a way that the BPS-generating



M2 branes emerge as a tight disk-shaped stack inside the fiber of  $\mathcal{O}(-1)_1$  at the “north pole” of  $\mathbb{P}^1$ .

We select the symmetry  $U(1)_{CY}$  of  $CY^3$  which rotates  $\mathbb{P}^1$  around the polar axis (so that 0 and  $\infty$  (that is, the north and south poles of  $\mathbb{P}^1$ ) remain fixed) while rotating the fibers of  $\mathcal{O}(-1)_1$  and  $\mathcal{O}(-1)_2$  in such a way that the fiber of  $\mathcal{O}(-1)_1$  at the north pole is fixed. As a result, the braid-related branes M5 become NS5, while a stack of M2 branes becomes a stack of D2 branes inside a D6 brane sweeping  $\mathcal{O}(-1)_1|_0 \times TN \times \mathbb{R}_{cob}$ . This stack carries the 3d B-model whose target is the Hilbert scheme of points on  $\mathbb{C}^2 = TN$  and it is separated into domains by braid-related NS5 branes.

The fiber of  $\mathcal{O}(-1)_2$  at the south pole of  $\mathbb{P}^1$  (that is,  $\mathcal{O}(-1)_2|_\infty$ ) is also stable under the action of  $U(1)_{CY}$ ; hence, the IIA picture includes another D6 brane  $\mathcal{O}(-1)_2|_\infty \times TN \times \mathbb{R}_{cob}$ . The M2 branes wrapping the zero section  $\mathbb{P}^1$  of  $CY^3_{rc}$  and attached to M2 branes of link homology become strings stretched between the south pole D6 and the center of the stack of D2 branes at north pole, creating the Wilson line carrying the exterior powers of the tautological bundle over the Hilbert scheme target.

### 3 Mathematical perspective

This section provides a motivation for the definition of our main working category. We do not provide details for the constructions in this section. The main 2-category that we work with appears in Sect. 3.4 and the details of the main working category are spelled out in this section.

In [21, 22], a 2-category  $\check{\text{Cat}}(\mathbf{Y})$  was associated with a symplectic variety  $\mathbf{Y}$ . This construction admits a generalization to the equivariant case, namely a 2-category  $\check{\text{Cat}}^G(\mathbf{Y})$  emerges when a group  $G$  has a Hamiltonian action on  $\mathbf{Y}$ . We present an three-categorical description of this category.

#### 3.1 Large 3-category $\check{\check{\text{Cat}}}_{\text{sym}}$ : a proposal

The objects of  $\text{Obj}(\check{\check{\text{Cat}}}_{\text{sym}})$  are holomorphic symplectic manifolds. The morphisms between two such manifolds  $\mathbf{X}, \mathbf{Y} \in \text{Obj}(\check{\check{\text{Cat}}}_{\text{sym}})$  form a 2-category  $\text{Hom}(\mathbf{X}, \mathbf{Y})$  whose objects are fibrations with Lagrangian bases:

$$(F, L, f: F \rightarrow L), \quad L \subset \mathbf{X} \times \mathbf{Y} \text{ is a Lagrangian subvariety.} \quad (3.1)$$

As hinted in [22], the fibration of  $F$  over the Lagrangian  $L$  is a categorical analog of a local system over the Lagrangian in the Fukaya category theory.

The composition of  $(F, L, f) \in \mathbb{H}\text{om}(\mathbf{X}, \mathbf{Y})$  and  $(G, L', g) \in \mathbb{H}\text{om}(\mathbf{Y}, \mathbf{W})$  is defined to be  $(H, L'', h)$  where

$$H := (F \times \mathbf{W}) \times_{\mathbf{Y}} (\mathbf{X} \times G),$$

while  $h: H \rightarrow \mathbf{X} \times \mathbf{W}$  is the natural projection and  $L'' := h(H)$ . The composition is not always defined since the  $L''$  is not always a Lagrangian.

The morphisms between the objects  $(F, L, f), (F', L', f') \in \mathbb{H}om(\mathbf{X}, \mathbf{Y})$  is defined as follows:

$$\mathrm{Hom}((F, L, f), (F', L', f')) := \mathrm{Qcoh}(F \times_{\mathbf{X} \times \mathbf{Y}} F').$$

The category of quasi-coherent sheaves can be enriched to  $(\infty, 1)$ -category. The study of the structure of these categories as well as  $(2, \infty)$ -category of such  $(\infty, 1)$ -categories is initiated in [11]. The abovementioned work is extremely technical and we do not attempt to use infinity categories in our work. Instead, we use the previous definition as a guiding principle.

Let us also remark that the category  $\check{\check{\check{\mathrm{Cat}}}}_{\mathrm{sym}}$  contains a final object  $pt$  which is just a point. Thus for every  $\mathbf{X} \in \mathrm{Obj}(\check{\check{\check{\mathrm{Cat}}}}_{\mathrm{sym}})$  there is a related 2-category

$$\check{\check{\check{\mathrm{Cat}}}(\mathbf{X})} := \mathbb{H}om(pt, \mathbf{X}).$$

### 3.2 Small 3-category: first steps

In this paper, we construct smaller category  $\check{\check{\check{\mathrm{Cat}}}}$  that has fewer objects and fewer morphisms. As a pay-back we get a mathematically satisfactory theory for two and three-morphisms in such categories, without use of heavy machinery the derived algebraic geometry mentioned above.

The objects of the category  $\check{\check{\check{\mathrm{Cat}}}}$  are smooth algebraic varieties. The 2-category of morphisms  $\mathbb{H}om(\mathcal{X}, \mathcal{Y})$  between two objects  $\mathbf{X}, \mathbf{Y} \in \mathrm{Obj}(\check{\check{\check{\mathrm{Cat}}}})$  has objects

$$(\mathcal{Z}, W), \quad \mathcal{Z} \text{ is algebraic manifold, } W \in \mathbb{C}[\mathbf{X} \times \mathcal{Z} \times \mathbf{Y}].$$

For  $\mathbf{X}, \mathbf{Y}, \mathbf{W} \in \mathrm{Obj}(\check{\check{\check{\mathrm{Cat}}}})$  and  $(\mathcal{Z}, W) \in \mathbb{H}om(\mathbf{X}, \mathbf{Y})$ ,  $(\mathcal{Z}', W') \in \mathbb{H}om(\mathbf{Y}, \mathbf{W})$  the composition is defined by:

$$(\mathcal{Z}', W') \circ (\mathcal{Z}, W) = (\mathcal{Z} \times \mathbf{Y} \times \mathcal{Z}', \pi_{xy}^*(W') - \pi_{yz}^*(W)) \in \mathbb{H}om(\mathbf{X}, \mathbf{W}),$$

where  $\pi_{xy}, \pi_{yz}$  are the projections from  $\mathbf{X} \times \mathcal{Z} \times \mathbf{Y} \times \mathcal{Z}' \times \mathbf{Y}$  onto  $\mathbf{X} \times \mathcal{Z} \times \mathbf{Y}$  and  $\mathbf{Y} \times \mathcal{Z}' \times \mathbf{Z}$ , respectively.

The category of morphisms  $\mathrm{Hom}((\mathcal{Z}, W), (\mathcal{Z}', W'))$  between the objects  $(\mathcal{Z}, W), (\mathcal{Z}', W') \in \mathrm{Hom}(\mathbf{X}, \mathbf{Y})$  is a triangulated one-category of matrix factorizations

$$\mathrm{Hom}((\mathcal{Z}, W), (\mathcal{Z}', W')) = \mathrm{MF}(\mathbf{X} \times \mathcal{Z} \times \mathcal{Z}' \times \mathbf{Y}, \pi'^*(W') - \pi^*(W)),$$

where  $\pi, \pi'$  are projections onto  $\mathbf{X} \times \mathcal{Z} \times \mathbf{Y}$  and  $\mathbf{X} \times \mathcal{Z}' \times \mathbf{Y}$ .

In the discussion above we indicated pull-backs in the formulas for the potentials. To lighten up the notations in the discussion below we omit the pull-backs in formulas for the potentials, since the pull-backs are clear from the context.

The objects of the homotopy category  $\mathrm{MF}(\mathcal{Z}, W)$ ,  $W \in \mathbb{C}[\mathcal{Z}]$  are pairs

$$(M, D), \quad M = M_0 \oplus M_1, \quad D \in \mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]}(M, M), \quad D^2 = W,$$

where  $D$  is  $\mathbb{Z}_2$ -graded morphism:  $D(M_i) = M_{i+1}$ . If we think of the matrix factorizations as two periodic curved complexes, then the homotopy is defined in exactly the same way as for usual complexes.

The standard convolution product  $\star$  (see the next section for the details) yields the monoidal structure:

$$\star : \mathrm{MF}(\mathcal{Z} \times \mathcal{Z}', W - W') \times \mathrm{MF}(\mathcal{Z}' \times \mathcal{Z}'', W' - W'') \rightarrow \mathrm{MF}(\mathcal{Z} \times \mathcal{Z}'', W - W'').$$

For two objects  $\mathcal{F} = (M, D)$ ,  $\mathcal{G} = (M', D') \in \mathrm{MF}(\mathcal{Z}, W)$ , the space of morphisms is defined by:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \{\phi \in \mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]}(M, M') \mid D \circ \phi = \phi \circ D'\}.$$

We also define  $\mathrm{D}^{\mathrm{per}}(\mathcal{Z})$  to be a derived category of the two-periodic complexes of  $\mathbb{C}[\mathcal{Z}]$ -modules. Given an element  $\mathcal{F} = (M, D) \in \mathrm{MF}(\mathcal{Z}, W)$ , by inverting the sign of the component of the differential from  $M_0$  to  $M_1$  we obtain an element of  $\mathrm{MF}(\mathcal{Z}, -W)$  which we denote  $\mathcal{F}^*$ . Since the potentials of the matrix factorizations add if take tensor product, we have:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}(\mathcal{F}, \mathcal{G}) = \mathcal{G} \otimes \mathcal{F}^* \in \mathrm{D}^{\mathrm{per}}(\mathcal{Z}).$$

There is a special class of invertible morphisms that we want to discuss separately. These morphisms provide an explicit realization of the Knörrer periodicity functor [19]. Namely, if  $V \rightarrow \mathcal{Z}$  is a finite rank vector bundle over  $\mathcal{Z}$  and  $V^*$  is its dual then there is a canonical bilinear function

$$Q_V \in \mathbb{C}[V \times V^*].$$

For any  $W \in \mathbb{C}[V]$ , there is a invertible functor [19]:

$$\begin{aligned} \mathrm{KN}_V : \mathrm{MF}(\mathcal{Z}, W) &\rightarrow \mathrm{MF}(V \times V^*, W + Q_V), \quad \mathrm{KN}_V(\mathcal{F}) = \mathcal{KN}_Q \otimes \mathcal{F}, \\ \mathcal{KN}_Q &= \left( \Lambda^\bullet V_\theta, D = \sum v_i \frac{\partial}{\partial \theta_i} + v_i^* \theta \right), \end{aligned}$$

where  $\Lambda^\bullet V_\theta$  is an exterior algebra with the generators  $\theta_i$ ,  $v_i^*$  and  $v_i$  are dual coordinates on the spaces  $V$  and  $V^*$ .

That is  $\mathcal{KN}_Q \in \mathrm{Hom}((\mathcal{Z}, W), (V \times V^*, W + Q_V))$ ,  $(\mathcal{Z}, W), (V \times V^*, W + Q_V) \in \mathbb{H}\mathrm{om}(\mathbf{X}, \mathbf{Y})$ . The functor  $\mathrm{KN}_V$  is an equivalence of categories [19]. To construct the inverse, we need a functor:

$$\mathrm{KN}_V^* : \mathrm{MF}(V \times V^*, W + Q_V) \rightarrow \mathrm{MF}(\mathcal{Z}, W), \quad \mathrm{KN}_V^*(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{KN}_Q).$$

The composition of functors  $\mathrm{KN}_V$  and  $\mathrm{KN}_V^*$  is the duality functor  $D$ . In general, for a matrix factorization  $\mathcal{F} \in \mathrm{MF}(\mathcal{Z}, W)$ , the dual matrix factorization  $D(\mathcal{F}) \in \mathrm{MF}(\mathcal{Z}, -W)$  is defined by

$$D(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathcal{Z}}).$$

Since,  $\mathrm{KN}_V^*$  and  $D$  are functors we can enlarge  $\mathrm{Hom}((V \times V^*, W + Q_V), (\mathcal{Z}, W))$  by including  $\mathrm{KN}_V^*$  and  $D$ . Thus, the Knörrer functor internally invertible in our category.

### 3.3 Relation between the 3-categories

There is a functor  $j_3: \ddot{\mathrm{Cat}} \rightarrow \ddot{\mathrm{Cat}}_{\mathrm{sym}}$  which acts on the objects as

$$\mathbf{X} \mapsto T^*\mathbf{X} = \mathbf{X}^{\mathrm{sym}}.$$

This functor explains an introduction of matrix factorizations in our construction, for more discussion of the role of matrix factorizations in context of 3D TQFT see the original [21].

The embedding at the level of morphism is based on describing Lagrangian submanifolds by generating functions [4]. Let us recall the basic facts. Given a (complex) manifold  $\mathcal{Z}$  and function  $W: \mathbf{X} \times \mathcal{Z} \rightarrow \mathbb{C}$  we define a subvariety  $F_W \subset T^*\mathbf{X} \times \mathcal{Z}$  by the equations

$$\partial_{z_i} W(x, p, z) = 0, \quad \partial_{x_i} W(x, p, z) = p_i,$$

where  $z_i$  are local coordinates along  $\mathcal{Z}$ ,  $x_i$  are local coordinates along  $\mathbf{X}$  and  $p_i$  are the coordinates on the cotangent space that are dual to the coordinates  $x_i$ . As shown in [4], the image  $L_w$  of  $F_w$  in  $T^*\mathbf{X}$  under the natural projection  $\pi$  is a generically Lagrangian subvariety.

Thus, the functor  $j_3$  at the level of homomorphisms between the objects is defined as

$$(\mathcal{Z}, W) \mapsto (F_W, \pi, L_W).$$

The real problem arises at the level of morphisms between the morphisms of objects. It is tempting to say that we have a functor

$$\mathrm{MF}(\mathbf{X} \times \mathcal{Z} \times \mathcal{Z}' \times \mathbf{Y}, W' - W) \longrightarrow \mathrm{DG}_{\mathrm{coh}}^{\mathrm{per}}(F_w \times_{T^*(\mathbf{X} \times \mathbf{Y})} F_{w'}).$$

It is not clear to the authors how one could construct such a functor in a canonical way. One option here is to use the functor  $\mathcal{H}om$ , if we fix some element  $\mathcal{F} \in \mathrm{MF}(\mathbf{X} \times \mathcal{Z} \times \mathcal{Z}' \times \mathbf{Y}, W' - W)$ ; then we obtain a functor

$$\mathcal{H}om(\mathcal{F}, \cdot): \mathrm{MF}(\mathbf{X} \times \mathcal{Z} \times \mathcal{Z}' \times \mathbf{Y}, W' - W) \rightarrow \mathrm{DG}_{\mathrm{coh}}^{\mathrm{per}}(\mathbf{X} \times \mathcal{Z} \times \mathcal{Z}' \times \mathbf{Y}).$$

However, it is not clear whether we can make a choice of  $\mathcal{F}$  canonical.

### 3.4 Equivariant version of the 3-category $\check{\check{\check{\text{Cat}}}}$

If  $\mathbf{X}$  is an affine variety with an action of  $G$  and  $T^*\mathbf{X}$ , then an object in the 2-category  $\check{\check{\check{\text{Cat}}}^G}(T^*\mathbf{X})$  is a pair  $(\mathcal{Z}, W)$ , where the auxiliary variety  $\mathcal{Z}$  is an affine variety with the action of  $G$ , while  $W$  is a super-potential:  $W \in \mathbb{C}[\mathbf{X} \times \mathcal{Z}]^G$ .

Let us introduce an equivariant version of 3-category  $\check{\check{\check{\text{Cat}}}}$  that has as objects pairs:

$$\text{Obj}(\check{\check{\check{\text{Cat}}}}) = \{(\mathbf{X}, G) | G \text{ is an algebraic group acting on } \mathbf{X}\}.$$

Respectively, for  $\mathbf{X}$  with action of  $G$  and  $\mathbf{Y}$  with action of  $H$  the space of morphisms  $\mathbb{H}\text{om}((\mathbf{X}, G), (\mathbf{Y}, H))$  consists of the pairs  $(\mathcal{Z}, W)$ ,  $W \in \mathbb{C}[\mathbf{X} \times \mathcal{Z} \times \mathbf{Y}]^{G \times H}$  and  $\mathcal{Z}$  has an action of  $G \times H$ .

For a pair  $(\mathcal{Z}^1, W^1), (\mathcal{Z}^2, W^2) \in \mathbb{H}\text{om}(\mathbf{X}, \mathbf{Y})$ , we define the category of morphisms as category of  $G \times H$ -equivariant matrix factorizations.

$$\text{Hom}((\mathcal{Z}^1, W^1), (\mathcal{Z}^2, W^2)) = \text{MF}_{G \times H}(\mathbf{X} \times \mathcal{Z}^1 \times \mathcal{Z}^2 \times \mathbf{Y}, W^2 - W^1).$$

In most of the cases, we omit the group from the notation; on the other hand if we want to emphasize the presence of the group action, we use notation  $(\mathbf{X}, G)$  for the object with  $G$ -action. Similarly, sometimes we include the group in the notation of the composition of morphisms, that is  $(\mathbf{X}, G), (\mathbf{Y}, H), (\mathbf{W}, K) \in \check{\check{\check{\text{Cat}}}}$  and  $(\mathcal{Z}_{xy}, W_{xy}) \in \mathbb{H}\text{om}((\mathbf{X}, G), (\mathbf{Y}, H)), (\mathcal{Z}_{yw}, W_{yx}) \in \mathbb{H}\text{om}((\mathbf{Y}, H), (\mathbf{W}, K))$  then

$$(\mathcal{Z}_{xw}, W_{xw}) = (\mathcal{Z}_{yw}, W_{yw}) \circ_H (\mathcal{Z}_{xy}, W_{xy}) = (\mathcal{Z}_{yw} \times \mathbf{Y} \times \mathcal{Z}_{xy} / H, W_{yw} - W_{xy}) \quad (3.2)$$

Furthermore, for

$$\mathcal{F}_{xy}^{12} \in \text{Hom}((\mathcal{Z}_{xy}^1, W_{xy}^1), (\mathcal{Z}_{xy}^2, W_{xy}^2)), \quad \mathcal{F}_{yw}^{12} \in \text{Hom}((\mathcal{Z}_{yw}^1, W_{yw}^1), (\mathcal{Z}_{yw}^2, W_{yw}^2)) \quad (3.3)$$

the vertical composition is defined by

$$\mathcal{F}_{xw}^{12} = \mathcal{F}_{yw}^{12} \circ_H \mathcal{F}_{xy}^{12} := (\pi_{YW}^*(\mathcal{F}_{yw}^{12}) \otimes \pi_{XY}^*(\mathcal{F}_{xy}^{12}))^{H^2},$$

where  $\pi_{yw}, \pi_{xy}$  are the projections from  $(\mathbf{X} \times \mathcal{Z}_{yw}^1 \times \mathcal{Z}_{yw}^2 \times \mathbf{Y}) \times (\mathbf{Y} \times \mathcal{Z}_{xy}^1 \times \mathcal{Z}_{xy}^2 \times \mathbf{W})$  to the first and the last factors, respectively and

$$(\mathcal{Z}_{xy}^i, W_{xy}^i) \in \mathbb{H}\text{om}((\mathbf{X}, G), (\mathbf{Y}, H)), \quad (\mathcal{Z}_{yw}^i, W_{yw}^i) \in \mathbb{H}\text{om}((\mathbf{Y}, H), (\mathbf{W}, K)). \quad (3.4)$$

Similarly, we define the horizontal composition for  $\mathcal{F}_{xy}^{12}$  as before and

$$\mathcal{F}_{xy}^{23} \in \text{Hom}((\mathcal{Z}_{xy}^2, W_{xy}^2), (\mathcal{Z}_{xy}^3, W_{xy}^3)), \quad (3.5)$$

here we assume (3.4). In more details, we have

$$\mathcal{F}_{xy}^{13} = \mathcal{F}_{xy}^{23} \star \mathcal{F}_{xy}^{12} = \pi_{13*}(\pi_{12}^*(\mathcal{F}_{xy}^{23}) \otimes \pi_{23}^*(\mathcal{F}_{xy}^{12})),$$

where  $\pi_{ij} : \mathbf{X} \times \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3 \times \mathbf{Y} \rightarrow \mathbf{X} \times \mathcal{Z}_i \times \mathcal{Z}_j \times \mathbf{Y}$ . In particular,

$$\mathring{\text{Cat}}^G(T^*\mathbf{X}) = \mathbb{H}\text{om}((X, G), (p, \{1\})).$$

We often omit the group super-script when the group is clear from the context. Finally, let us observe that the vertical composition satisfies the exchange rule:

**Proposition 3.1** *For any  $(X, G), (Y, H), (W, K) \in \mathring{\text{Cat}}$  and  $(\mathcal{Z}_{xy}, W_{xy}^i), (\mathcal{Z}_{yw}, W_{yw}^i)$  as in (3.4) and  $\mathcal{F}_{xy}^{12}, \mathcal{F}_{yw}^{12}$  as in (3.3),  $\mathcal{F}_{xy}^{23}$  as in (3.5) and*

$$\mathcal{F}_{yw}^{23} \in \text{Hom}((\mathcal{Z}_{yw}^2, W_{yw}^2), (\mathcal{Z}_{yw}^3, W_{yw}^3))$$

*there is a three-isomorphism:*

$$E(\mathcal{Z}_{xy}^\bullet; \mathcal{Z}_{yw}^\bullet; \mathcal{F}_{xy}^{\bullet*}; \mathcal{F}_{yw}^{\bullet*}) : (\mathcal{F}_{yw}^{12} \circ \mathcal{F}_{xy}^{12}) \star (\mathcal{F}_{yw}^{23} \circ \mathcal{F}_{xy}^{23}) \rightarrow (\mathcal{F}_{yw}^{12} \star \mathcal{F}_{yw}^{23}) \circ (\mathcal{F}_{xy}^{12} \star \mathcal{F}_{xy}^{23}).$$

Both sides of the last equation are functors between one-categories and the three-isomorphism  $E$  is a natural transformation of between these functors.

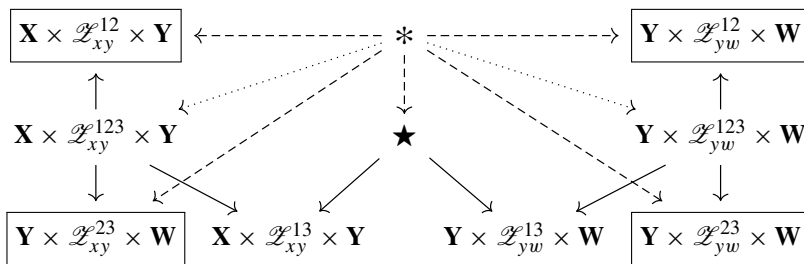
**Proof** In the proof below, we use the short hand notation  $\mathcal{Z}_{\bullet\bullet}^{i_1\dots i_k} = \mathcal{Z}_{\bullet\bullet}^{i_1} \times \dots \times \mathcal{Z}_{\bullet\bullet}^{i_k}$ . First, we observe that  $(\mathcal{F}_{yw}^{12} \circ \mathcal{F}_{xy}^{12}) \star (\mathcal{F}_{yw}^{23} \circ \mathcal{F}_{xy}^{23})$  is computed with the push-forwards and pull-backs along the maps in the diagram and taking  $H^2$ -quotient:

$$\begin{array}{ccccc}
 & & \mathbf{X} \times \mathcal{Z}_{xy}^{123} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{123} \times \mathbf{W} & & \\
 & \swarrow & \downarrow & \searrow & \\
 \boxed{\mathbf{X} \times \mathcal{Z}_{xy}^{12} \times \mathbf{Y}} & & & & \boxed{\mathbf{X} \times \mathcal{Z}_{xy}^{23} \times \mathbf{Y}} \\
 \uparrow & \swarrow & \downarrow & \searrow & \uparrow \\
 \mathbf{X} \times \mathcal{Z}_{xy}^{12} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{12} \times \mathbf{W} & & \mathbf{X} \times \mathcal{Z}_{xy}^{13} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{13} \times \mathbf{W} & & \mathbf{X} \times \mathcal{Z}_{xy}^{23} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{23} \times \mathbf{W} \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \boxed{\mathbf{Y} \times \mathcal{Z}_{yw}^{12} \times \mathbf{W}} & & & & \boxed{\mathbf{Y} \times \mathcal{Z}_{yw}^{23} \times \mathbf{W}}
 \end{array}$$

here all the maps are the natural projections. In more details, the matrix factorizations  $\mathcal{F}_{\bullet\bullet}^{ij}$  are the matrix factorizations on the spaces in the boxes. Thus, to obtain  $(\mathcal{F}_{yw}^{12} \circ \mathcal{F}_{xy}^{12}) \star (\mathcal{F}_{yw}^{23} \circ \mathcal{F}_{xy}^{23})$ , we pull-back and push-forward these matrix factorizations along the solid arrow until we reach the space in the center of the bottom row.

The dashed arrow maps indicate the natural projections that make the diagram commute. By functoriality of the pull-back, we can obtain  $(\mathcal{F}_{yw}^{12} \circ \mathcal{F}_{xy}^{12}) \star (\mathcal{F}_{yw}^{23} \circ \mathcal{F}_{xy}^{23})$  by first pulling back along the dashed arrow and then pushing forward along the central down arrow.

Similarly, we draw the diagram for the maps that participate in the construction of  $(\mathcal{F}_{yw}^{12} \star \mathcal{F}_{yw}^{23}) \circ (\mathcal{F}_{xy}^{12} \star \mathcal{F}_{xy}^{23})$ :



here the maps are the natural projections and

$$\star = \mathbf{X} \times \mathcal{Z}_{xy}^{13} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{13} \times \mathbf{W}.$$

We obtain  $(\mathcal{F}_{yw}^{12} \star \mathcal{F}_{yw}^{23}) \circ (\mathcal{F}_{xy}^{12} \star \mathcal{F}_{xy}^{23})$  by the sequence of pull-backs and push-forwards that starts from the boxed spaces and ends at  $\star$ . By setting

$$\ast = \mathbf{X} \times \mathcal{Z}_{xy}^{123} \times \mathbf{Y}^2 \times \mathcal{Z}_{yw}^{123} \times \mathbf{W}$$

we can complete the diagram to the commuting diagram where the dashed arrows are the natural projections.

The dashed arrow maps are the same maps as in the previous diagram. Thus to complete our proof we need to show that  $(\mathcal{F}_{yw}^{12} \star \mathcal{F}_{yw}^{23}) \circ (\mathcal{F}_{xy}^{12} \star \mathcal{F}_{xy}^{23})$  can be obtained by the pull-back along the dashed arrows followed with the push-forward along the vertical dashed arrow.

The last statement follows from a repeated application the base change relation. Indeed, first we apply base change to the commuting squares that have as sides the vertical dashed arrow and the dotted arrows. Finally, we apply the functoriality of the pull-backs since the composition of the pull-backs along the vertical arrow and the dotted arrows are equal to the pull-back along the corresponding non-vertical dashed arrows.

Each base change in the construction is facilitated by a particular invertible morphism between the matrix factorizations. The composition of these morphisms yields the three-morphism  $E(\mathcal{Z}_{xy}^\bullet, \mathcal{Z}_{yw}^\bullet; \mathcal{F}_{xy}^\bullet, \mathcal{F}_{yw}^\bullet)$ . The base change morphism are natural transformations. Hence,  $E$  is a natural transformation.  $\square$

In the cases when the first group of arguments of  $E(\dots)$  is clear, we abbreviate the notation as  $E(\mathcal{F}_{xy}^{12}, \mathcal{F}_{xy}^{23}; \mathcal{F}_{yw}^{12}, \mathcal{F}_{yw}^{23})$ .

Let us also discuss the exchange relation for two-morphisms and three-morphisms. Let us fix the objects  $(\mathbf{X}, G)$ ,  $(\mathbf{Y}, K)$  and let

$$(\mathcal{Z}^i, W^i) \in \mathbb{H}\text{om}(\mathbf{X}, \mathbf{Y}), \quad \mathcal{F}^{ii+1}, \mathcal{G}^{ii+1}, \mathcal{H}^{ii+1} \in \text{Hom}((\mathcal{Z}^i, W^i), (\mathcal{Z}^{i+1}, W^{i+1})). \quad (3.6)$$

Respectively, we define  $\mathcal{F}^{i+2} = \mathcal{F}^{i+1} \star \mathcal{F}^{i+1, i+2}$ ,  $\mathcal{G}^{i+2} = \mathcal{G}^{i+1} \star \mathcal{G}^{i+1, i+2}$ .

Let us also fix notation for the three-morphisms

$$u_{fg}^{ii+1} \in \mathcal{H}om(\mathcal{F}^{i+1}, \mathcal{G}^{ii+1}), \quad u_{gh}^{ii+1} \in \mathcal{H}om(\mathcal{G}^{ii+1}, \mathcal{H}^{ii+1}). \quad (3.7)$$

We denote by  $\bullet$  for the composition of morphisms in the category of matrix factorizations, for example  $u_{gh}^{12} \bullet u_{fg}^{12} \in \mathcal{H}om(\mathcal{F}^{12}, \mathcal{H}^{12})$ .

On the other hand, we can define the horizontal composition  $u_{fg}^{23} \star u_{fg}^{12} \in \mathcal{H}om(\mathcal{F}^{13}, \mathcal{G}^{13})$  by

$$\begin{aligned} u_{fg}^{23} \star u_{fg}^{12} &= \pi_{13*}(\mathcal{G}^{12} \otimes \pi_{23}^*(u_{fg}^{23}) \bullet \pi_{12}^*(u_{fg}^{12}) \otimes \mathcal{F}^{23}), \\ \pi_{12}^*(u_{fg}^{12}) \otimes \mathcal{F}^{23} &= \pi_{12}^*(u_{fg}^{12}) \otimes \text{Id}_{\pi_{23}^*(\mathcal{F}^{23})} \in \mathcal{H}om(\pi_{12}^*(\mathcal{F}^{12}) \\ &\quad \otimes \pi_{23}^*(\mathcal{F}^{23}), \pi_{12}^*(\mathcal{G}^{12}) \otimes \pi_{23}^*(\mathcal{F}^{23})), \\ \mathcal{G}^{12} \otimes \pi_{23}^*(u_{fg}^{23}) &= \text{Id}_{\pi_{12}^*(\mathcal{G}^{12})} \otimes \pi_{23}^*(u_{fg}^{23}) \in \mathcal{H}om(\pi_{12}^*(\mathcal{G}^{12}) \\ &\quad \otimes \pi_{23}^*(\mathcal{F}^{23}), \pi_{12}^*(\mathcal{G}^{12}) \otimes \pi_{23}^*(\mathcal{G}^{23})). \end{aligned} \quad (3.8)$$

The exchange rule in this setting is given by

**Proposition 3.2** *In notations of (3.6), (3.7), (3.8), we have*

$$(u_{gh}^{23} \star u_{gh}^{12}) \bullet (u_{fg}^{23} \star u_{fg}^{12}) = (u_{gh}^{23} \bullet u_{fg}^{23}) \star (u_{gh}^{12} \bullet u_{fg}^{12}).$$

**Proof** To show that the morphisms of the matrix factorizations are equal, it is enough to check that the morphisms are equal as morphisms of stalks at a point  $z \in \mathbf{X} \times \mathcal{Z}^1 \times \mathcal{Z}^3 \times \mathbf{Y}$ . The convolution  $\star$  intertwines  $i_{(x, z_1, y)}^* \times i_{(x, z_3, y)}^*$  with  $i_z^*$  where  $z = (x, z_1, z_3, y)$ . Thus, it is enough to show the statement in the case  $\mathbf{X} = \mathbf{Y} = \mathcal{Z}_1 = \mathcal{Z}_3 = \mathbf{p}$ . In this case, the statement of the theorem is equivalent to the functoriality of the push-forward.

Indeed, in this situation  $\mathcal{F}^{12}, \mathcal{G}^{12}, \mathcal{H}^{12} \in \text{MF}(\mathcal{Z}^2, W)$ ,  $\mathcal{F}^{23}, \mathcal{G}^{23}, \mathcal{H}^{23} \in \text{MF}(\mathcal{Z}^2, -W)$ . Respectively, the convolutions  $\mathcal{F}^{12} \star \mathcal{F}^{23} = H^*(\mathcal{F}^{12} \otimes \mathcal{F}^{23})$ ,  $\mathcal{G}^{12} \star \mathcal{G}^{23} = H^*(\mathcal{G}^{12} \otimes \mathcal{G}^{23})$ ,  $\mathcal{H}^{12} \star \mathcal{H}^{23} = H^*(\mathcal{H}^{12} \otimes \mathcal{H}^{23})$  are the derived push-forwards  $R\pi_*$  to the point,  $R\pi_* : \mathcal{Z} \rightarrow \mathbf{p}$ .

Since the derived push-forward is a functor, the convolutions of the endomorphism are defined by means of functoriality  $u_{gh}^{23} \star u_{gh}^{12} = R\pi_*(u_{gh}^{23} \otimes u_{gh}^{12})$ ,  $u_{fg}^{23} \star u_{fg}^{12} = R\pi_*(u_{fg}^{23} \otimes u_{fg}^{12})$ ,  $(u_{gh}^{23} \bullet u_{fg}^{23}) \star (u_{gh}^{12} \bullet u_{fg}^{12}) = R\pi_*((u_{gh}^{23} \bullet u_{fg}^{23}) \otimes (u_{gh}^{12} \bullet u_{fg}^{12}))$ . Thus, the statement of the proposition is equivalent to the relation:

$$R\pi_*(u_{gh}^{23} \otimes u_{gh}^{12}) \bullet R\pi_*(u_{fg}^{23} \otimes u_{fg}^{12}) = R\pi_*((u_{gh}^{23} \bullet u_{fg}^{23}) \otimes (u_{gh}^{12} \bullet u_{fg}^{12}))$$

which follows from the functoriality of  $R\pi_*$ .  $\square$

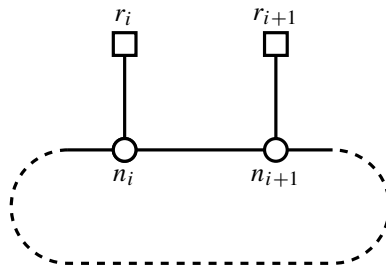


### 3.5 Categories of linear quivers and NS5 interfaces

In this paper, we consider the subcategories  $\check{\check{\check{\text{Cat}}}(\tilde{A}_m)}$  (resp.  $\check{\check{\check{\text{Cat}}}(A_m)}$ ) of  $\check{\check{\check{\text{Cat}}}}$  formed by Nakajima quiver varieties of (Dynkin diagram) type  $\tilde{A}_m$  (resp.  $A_m$ ) and describe some generalized Lagrangian correspondences connecting them. These correspondences are of two distinct types: NS5 and D4 (or D5). We will describe mostly the NS5 correspondences (1.3) for  $m = 1$ . Their auxiliary varieties originate from arrows similar to the ones appearing in Nakajima quivers, which could be expected given that both come from NS5 branes in the super-string theory. We will also conjecture the D4 Lagrangian correspondences connecting the so-called commuting varieties, that is, Nakajima varieties of the  $\tilde{A}_0$  quiver without framing (1.4).

#### 3.5.1 NS5 interfaces between $\tilde{A}_{m-1}$ quiver varieties

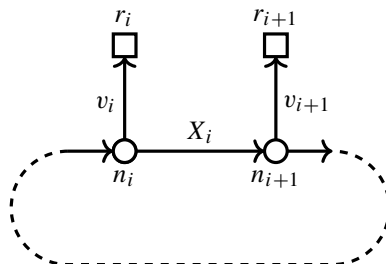
The  $\tilde{A}_{m-1}$  quiver has the form



where  $i = 1, \dots, m$ . It is ‘colored’ by the numbers  $\mathbf{n} = (n_1, \dots, n_m)$  and  $\mathbf{r} = (r_1, \dots, r_m)$ . We denote this quiver by  $\tilde{A}_m(\mathbf{n}; \mathbf{r})$ . An edge between the  $i$ th and  $(i+1)$ st circles corresponds to the  $i$ th basic, in the sense of Sect. 2, NS5 interface, the number  $n_i$  refers to the gauge group  $U(n_i)$  between the  $(i-1)$ st and  $i$ th NS5-interfaces, and there are  $r_i$  D5-interfaces there.

**Remark 3.1** A ‘linear’  $A_m$  quiver appears as a particular case of the  $\tilde{A}_{m-1}$  quiver if one sets  $n_m = r_m = 0$ .

In order to describe additional interfaces as Lagrangian correspondences, we have to present the quiver varieties as cotangent bundles. Hence, we choose the orientation of quiver edges (for example, clockwise and from circles to boxes) and denote the corresponding maps as  $X_i$  and  $v_i$ :



The quiver variety  $\mathbf{X}_{\mathbf{n}; \mathbf{r}}^{\text{sym}}$  is the stable part of the cotangent bundle

$$\mathbf{X}_{\mathbf{n}; \mathbf{r}}^{\text{sym}} = (\mathbf{T}^* \mathcal{X}_{\mathbf{n}; \mathbf{r}})^{\text{st}}, \quad \mathcal{X}_{\mathbf{n}; \mathbf{r}} = \left( \prod_{i=1}^m \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}}) \right) \times \left( \prod_{i=1}^m \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{r_i}) \right).$$

The stable locus is defined by means the Hamiltonian action of the group

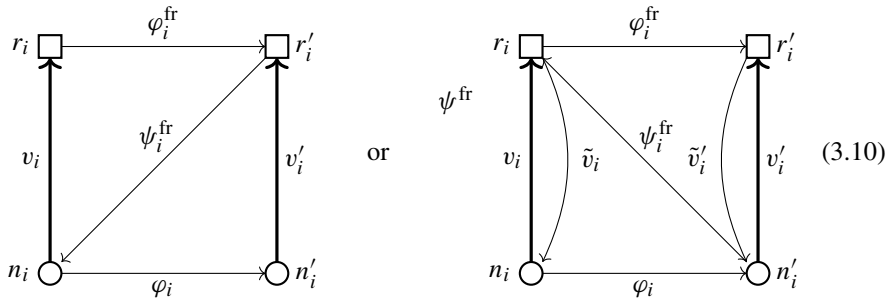
$$G_{\mathbf{n}} = \prod_{i=1}^m \text{GL}(n_i).$$

In more details, given a character of  $\chi = (\chi_1, \dots, \chi_m)$  of  $G_{\mathbf{n}}$  the  $\chi$ -stable locus of  $\mathbf{T}^* \mathcal{X}_{\mathbf{n}; \mathbf{r}}$  is the projection of the GIT stable locus of  $\mathbf{T}^* \mathcal{X}_{\mathbf{n}; \mathbf{r}} \times \mathbb{C}_{\chi_1}^* \times \dots \times \mathbb{C}_{\chi_m}^*$ , see for example [30, Section 3.1]. The  $\chi$ -stable condition can also be described by explicit linear algebra construction as explained in [30, Section 3.1], we also use the linear algebra construction in Proposition 5.1 below. If  $\mathbf{r}$  is zero, we do not impose any stability constraints.

Consider two affine quivers  $Q = \tilde{A}_m(\mathbf{n}; \mathbf{r})$  and  $Q' = \tilde{A}_m(\mathbf{n}'; \mathbf{r}')$  with the same  $m$  but, generally, different dimensions:  $(\mathbf{n}; \mathbf{r})$  and  $(\mathbf{n}'; \mathbf{r}')$ . The NS5-interface auxiliary variety  $\mathcal{Z}$  has an arrow description, that is, it is based on the spaces of linear maps. First of all, we introduce the maps  $\varphi_i \in \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n'_i})$  and  $\varphi_i^{\text{fr}} \in \text{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r'_i})$ . We fix the values of all  $\varphi_i^{\text{fr}}$  and require that they all have the highest rank. Then for each ‘horizontal square’

$$(3.9)$$

and for each ‘vertical square’



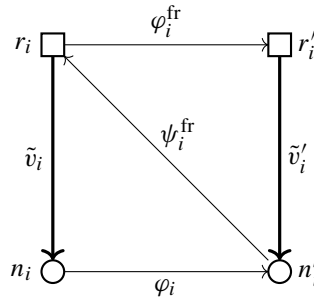
Two columns of the formulas (3.9) and (3.10) are Legendre dual to each other as we explain in Sect. 2.4.3. In the same section, we explain these two types of interfaces correspond to the interface between a horizontal NS5 brane and vertical NS5 and NS5' brane. Figure (2.8) explains how the vertical and horizontal squares a glued together.

In more details, we choose either the first diagonal (on the left) or the second diagonal (on the right) and add the corresponding linear maps  $\psi_i$ ,  $\psi_i^{\text{fr}}$  and, in case of the second diagonal choice we also add maps  $\tilde{Y}_i$ ,  $\tilde{Y}'_i$ ,  $\tilde{v}_i$  and  $\tilde{v}'_i$ . The variety  $\mathcal{Z}$  is formed by all these maps, see the figure (2.8), including  $\varphi_i$ , while the superpotential  $W$  is the sum of superpotentials  $W_{\text{square}}$  of individual squares. The latter being the sums of traces along the boundaries of triangles and digons. The formula below list these sums for the squares from the formulas (3.9) and (3.10):

$$W_{\text{square}} = \begin{cases} \text{Tr}(\psi_i X'_i \varphi_i) - \text{Tr}(\psi_i \varphi_{i+1} X_i), \\ (\text{Tr}(X'_i \tilde{Y}'_i) - \text{Tr}(\tilde{Y}'_i \varphi_{i+1} \psi_i)) - (\text{Tr}(\tilde{Y}_i X_i) - \text{Tr}(\tilde{Y}_i \psi_i \varphi_i)) \\ \text{Tr}(\psi_i^{\text{fr}} v'_i \varphi_i) - \text{Tr}(\psi_i^{\text{fr}} \varphi_i^{\text{fr}} v_i) \\ (\text{Tr}(\tilde{v}'_i v'_i) - \text{Tr}(\psi_i^{\text{fr}} \tilde{v}'_i \varphi_i^{\text{fr}})) - (\text{Tr}(\tilde{v}_i v_i) - \text{Tr}(\psi_i^{\text{fr}} \varphi_i v_i)) \end{cases}$$

We get more interface squares by swapping the quivers  $Q$  and  $Q'$  in the diagrams (3.9) and (3.10), that is, reversing the directions of horizontal arrows and adjusting the choices of diagonals.

Not all of these choices are compatible with stability conditions on the arrows of the quivers, and some of these choices are Legendre-equivalent if we replace the spaces at the quiver vertices by their dual spaces and, at the same time, reverse the directions of all arrows. For example, the generalized Lagrangian correspondence of the second diagram of (3.10) can be equivalently presented, if we reverse the directions of vertical arrows:



In our previous work [35, 37, 38] and in Sect. 5, we work with the last choice of the direction of circle-square edge. In the rest of this section, we work with convention of Formulas (3.9), (3.10) since this choice is closer the standard quiver description of Grassmannians.

**Remark 3.2** If  $\mathbf{r}' = \mathbf{r}$  while  $\mathbf{n} \leq \mathbf{n}'$ , that is,  $n_i \leq n'_i$  for all  $i$ , and one requires all horizontal arrow maps to have highest rank, then the first correspondences of (3.9) and (3.10) describe the ‘standard’ interface which has a simple symplectic interpretation. Vertical arrow maps describe representations of  $Q$  and  $Q'$ , the horizontal arrows establish the representation of  $Q$  as a subrepresentation of  $Q'$ , and one takes the conormal bundle to this arrangement within the product of two full quiver varieties. These correspondences are also known under the name *Hecke correspondences*. The theory of Hecke correspondences was developed by Nakajima, please consult [14] for a survey and further references.

We consider several special cases the special cases of NS5 interfaces between Grassmannians in Sect. 2.5.1. More generally, the NS5 interfaces are discussed in Sect. 2.5.2. Also we discuss D5 and NS5 interfaces between the commuting varieties as described in Sect. 2.5.3.

## 4 Categorical aspects of the theory

In this section, we discuss the categorical setting for the proposed TQFT. In particular, we explain how de-looping procedure allows us to state a precise mathematical statement that underlies a construction of the TQFT. The delooping procedure conjecturally produces a monoidal 2-category  $\check{\mathcal{C}}\mathbf{at}(Q)$  for a quiver  $Q$ .

### 4.1 Delooping

In the previous section discussed the general setting for a construction of the 3-category  $\check{\mathcal{C}}\mathbf{at}$  as well as the simplest equivariant version of the 3-category  $\check{\mathcal{C}}\mathbf{at}(Q)$  for a linear quiver  $Q$ . It is natural to conjecture that  $\check{\mathcal{C}}\mathbf{at}$  and  $\check{\mathcal{C}}\mathbf{at}(Q)$  are weak 3-categories [10]. However, checking the conditions for the structural morphisms of a weak 3-category is a rather challenging task. We choose a different path for constructing  $\check{\mathcal{C}}\mathbf{at}(Q)$ . First,

we define a delooped monoidal 2-category  $\check{\check{\mathcal{C}at}}(Q)$ , then we define a large 3-category  $\check{\check{\mathcal{C}at}}(Q)$  as category of module 2-categories over  $\check{\mathcal{C}at}(B)$ . The 3-category  $\check{\check{\mathcal{C}at}}(Q)$  is a subcategory inside  $\check{\mathcal{C}at}(Q)$ .

To construct 2-category  $\check{\mathcal{C}at}(Q)$ , we first reduce the 3-category  $\check{\check{\mathcal{C}at}}(Q)$  to a 3-category with one object  $\mathbb{O}_Q = \oplus_{\mathbf{n}, \mathbf{r}} Q(\mathbf{n}; \mathbf{r})$ . Respectively, the category of the endomorphisms  $\mathbb{H}om(\mathbb{O}_Q, \mathbb{O}_Q)$  of  $\mathbb{O}_Q$  has objects that are collections of elements

$$(\mathcal{Z}_{\mathbf{n}, \mathbf{r}}, W_{\mathbf{n}, \mathbf{r}}) \in \mathbb{H}om(Q(\mathbf{n}; \mathbf{r}), Q(\phi(\mathbf{n}); \psi(\mathbf{r}))), \quad \mathbf{n}, \mathbf{r} \in \mathbb{Z}_+^Q, \quad \phi, \psi : \mathbb{Z}_+^Q \rightarrow \mathbb{Z}_+^Q.$$

The composition law for the morphisms in  $\check{\mathcal{C}at}(Q)$  yields a monoidal structure on  $\mathbb{H}om(\mathbb{O}_Q, \mathbb{O}_Q)$ .

Now we declare that the 2-category  $\check{\mathcal{C}at}(Q)$  has elements of  $\mathbb{H}om(\mathbb{O}_Q, \mathbb{O}_Q)$  as objects and the rest of the structure is inherited from the 3-category  $\check{\check{\mathcal{C}at}}(Q)$ . Thus, we state our main categorical statement:

**Theorem 4.1** *The delooped 2-category  $\check{\mathcal{C}at}(Q)$  has a structure of a monoidal 2-category (without a unit) in the sense of Kapranov–Voevodsky [5, 23].*

The essence of the monoidal 2-category is its monoidal structure and the corresponding coherence conditions that are imposed by the compatibility with the monoidal structure. If we ignore the monoidal structure, we essentially put ourself in the standard setting similar to the one considered in [6].

**Proposition 4.1** *The vertical composition  $\star$  and the horizontal composition  $\bullet$  of  $\mathcal{H}om$  functors define a structure of a weak 2-category on  $\check{\mathcal{C}at}(Q)$ .*

**Proof** It is enough to show that  $\mathbb{H}om(Q(\mathbf{n}, \mathbf{r}), Q(\mathbf{n}', \mathbf{r}'))$  is a weak 2-category for a fixed choice  $\mathbf{n}, \mathbf{r}, \mathbf{n}', \mathbf{r}'$ . For a given  $(\mathcal{Z}, W) \in \mathbb{H}om(Q(\mathbf{n}, \mathbf{r}), Q(\mathbf{n}', \mathbf{r}'))$ , there is a canonical choice of the convolution unit:

$$\begin{aligned} \mathbf{1}^{\mathcal{Z}} &\in \text{Hom}((\mathcal{Z}, W), (\mathcal{Z}, W)) \\ &= \text{MF}_{G(\mathbf{n}) \times G(\mathbf{n}')} (Q(\mathbf{n}, \mathbf{r}) \times \mathcal{Z}^2 \times Q(\mathbf{n}', \mathbf{r}'), W^{(1)} - W^{(2)}), \end{aligned}$$

where  $W^{(1)} = \pi_i^*(W)$  is the pull-back of the potential  $W$  along two natural projections to  $Q(\mathbf{n}, \mathbf{r}) \times \mathcal{Z} \times Q(\mathbf{n}', \mathbf{r}')$ . The unit is given by the push-forward of the structure sheaf along the inclusion  $1 \times \Delta \times 1$  of  $Q(\mathbf{n}, \mathbf{r}) \times \mathcal{Z} \times Q(\mathbf{n}', \mathbf{r}')$  into  $Q(\mathbf{n}, \mathbf{r}) \times \mathcal{Z}^2 \times Q(\mathbf{n}', \mathbf{r}')$ , here  $\Delta$  is the diagonal inclusion.

To show that  $\mathbf{1}^{\mathcal{Z}}$  is the convolution unit we need to use the base change transformation relations in the commuting diagram of maps:

$$\begin{array}{ccccc} \mathcal{Z}^3 & \xrightarrow{\Delta \times 1 \times 1} & \mathcal{Z}^4 & \xleftarrow{\pi_{12} \times \pi_{23}} & \mathcal{Z}^3 \\ \downarrow \pi_{23} & & \downarrow \pi_{124} & & \downarrow \pi_{13} \\ \mathcal{Z}^2 & \xrightarrow{\Delta \times 1} & \mathcal{Z}^3 & & \mathcal{Z}^2 \end{array},$$

here  $\pi_{i_1 \dots i_s}$  is the projection on the corresponding product of copies of  $\mathcal{Z}$ . Thus, we have

$$\begin{aligned} \mathbf{1}^{\mathcal{Z}} \star \mathcal{F} &= \pi_{13*} \circ (\pi_{12} \times \pi_{23})^* \circ (\Delta \times 1 \times 1)_* \circ \pi_{23}^* (\mathcal{F}) \\ &= \pi_{13*} \circ (\pi_{12} \times \pi_{23})^* \circ \pi_{124}^* \circ (\Delta \times 1)_* (\mathcal{F}) \\ &= \pi_{13*} \circ (\Delta \times 1)_* (\mathcal{F}) = \mathcal{F}. \end{aligned}$$

Here, the second equality follows from the base-change along the commuting square and the last two equalities follow from  $\pi_{124} \circ \pi_{12} \times \pi_{23} = 1$  and  $\pi_{13} \circ (\Delta \times 1) = 1$ . Since we use the base-change, we obtain an invertible natural transformation  $U_l(\mathcal{Z})$  between the functor of the left multiplication by  $\mathbf{1}^{\mathcal{Z}}$  and the identity functor. Similarly, one defines the natural transformation  $U_r(\mathcal{Z})$  for the right multiplication.

The first group of axioms of the weak 2-category that is concerned with the interaction of the vertical composition  $\star$  and horizontal composition  $\circ$  is exactly the interchange relation from Proposition 3.1.

Our 2-category is weak because in the standard proof of the associativity of the convolution product  $\star$  one has to use the base change relation. Thus, the base change natural transformations yield the associator natural transformation:

$$A(\mathcal{F}, \mathcal{G}, \mathcal{H}) : (\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \rightarrow \mathcal{F} \star (\mathcal{G} \star \mathcal{H}).$$

The associator in a weak two category satisfies the pentagon axiom and the unitors  $U_l, U_r$  satisfy the triangle axioms see for example [9]. The construction of the associator and the unitors is in terms of base-change relations. Thus, we can use the standard argument for convolution of coherent sheaves to show the axioms in our setting.

We refer to the original paper [23] and subsequent paper [5] for a streamlined treatment of the (semi-strict) monoidal 2-category. In our case, the composition operation  $\circ$  is strictly associative since the equivariant pull-backs in the definition of the composition in (3.2).

Thus, the only nontrivial structural two-morphism  $R = R(\mathcal{F}, \mathcal{G})$  of our 2-category is the two-isomorphism  $\circ_{\mathcal{F}, \mathcal{G}}$  that relates the morphisms of pairs of objects  $\mathcal{F} \in \text{Hom}(\mathcal{Z}, \mathcal{Z}'), \mathcal{G} \in \text{Hom}(\mathcal{X}, \mathcal{X}')$  as in the diagram below:

$$\begin{array}{ccc} \mathcal{Z} \circ \mathcal{X} & \xrightarrow{1_{\mathcal{Z}} \circ \mathcal{G}} & \mathcal{Z} \circ \mathcal{X}' \\ \mathcal{F} \circ 1_{\mathcal{X}} \downarrow & \swarrow R & \downarrow \mathcal{F} \circ 1_{\mathcal{X}'} \\ \mathcal{Z}' \circ \mathcal{X} & \xrightarrow{1_{\mathcal{Z}'} \circ \mathcal{G}} & \mathcal{Z}' \circ \mathcal{X}' \end{array}$$

We define  $R(\mathcal{F}, \mathcal{G})$  as composition of exchange morphisms from Proposition 3.1:

$$\begin{aligned} R(\mathcal{F}, \mathcal{G}) &= E(\mathcal{Z}, \mathcal{Z}', \mathcal{Z}'; \mathcal{X}, \mathcal{X}', \mathcal{X}'; \mathbf{1}_{\mathcal{X}\mathcal{Y}}^{\mathcal{Z}}, \mathcal{F}, \mathcal{G}, \mathbf{1}_{\mathcal{Y}\mathcal{Z}}^{\mathcal{X}'}) \\ &\quad \circ E^{-1}(\mathcal{Z}, \mathcal{Z}', \mathcal{Z}'; \mathcal{X}, \mathcal{X}, \mathcal{X}'; \mathcal{F}, \mathbf{1}_{\mathcal{X}\mathcal{Y}}^{\mathcal{Z}'}, \mathbf{1}_{\mathcal{Y}\mathcal{Z}}^{\mathcal{X}}, \mathcal{G}), \end{aligned}$$

where  $\mathbf{1}_{\star, \star}^{\mathcal{Y}}$  stands for the identity element in  $\text{Hom}(\mathcal{Y}, \mathcal{Y})$ . □

**Proof of Theorem 4.1** The 2-category  $\ddot{\text{Cat}}(Q)$  is almost semi-strict. Indeed, the convolution  $\circ$  is strictly associative on the objects. In more details, in notations [23]  $\circ$  corresponds to the composition  $\otimes$ . The structural one-morphisms that appear in the definition of the monoidal of [23] are  $a_{A,B,C} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ , in notations of [23]. The analogous one-morphisms in our setting are the identity morphisms. Respectively, the related analogs of two-morphisms  $a_{A,B,C,D}$ ,  $a_{u,A,B}$ ,  $a_{A,u,B}$ ,  $a_{A,B,u}$  are also identity in our setting.

The analogs of the structural two-morphisms  $\otimes_{u,u',B}$ ,  $\otimes_{A,v,v'}$ ,  $\otimes_{u,v}$  in our setting are expressed in terms of the exchange natural-transformation  $E$  from Proposition 3.1:

$$\begin{aligned}\otimes_{u,u',B} \quad R(\mathcal{F}, \mathcal{G}; \mathcal{X}) &= E(\mathcal{Z}, \mathcal{Z}', \mathcal{Z}'', \mathcal{X}, \mathcal{X}; \mathcal{F}, \mathcal{G}; \mathbf{1}_{xy}^{\mathcal{X}}, \mathbf{1}_{yw}^{\mathcal{X}}) \\ \otimes_{A,v,v'} \quad R(\mathcal{Z}; \mathcal{F}, \mathcal{G}) &= E(\mathcal{Z}, \mathcal{Z}, \mathcal{Z}; \mathcal{X}, \mathcal{X}', \mathcal{X}''; \mathbf{1}_{xy}^{\mathcal{Z}}, \mathbf{1}_{yw}^{\mathcal{Z}}; \mathcal{F}, \mathcal{G}) \\ \otimes_{u,v} \quad R(\mathcal{F}, \mathcal{G}).\end{aligned}$$

where we set  $u = \mathcal{F}$ ,  $u' = \mathcal{G}$ ,  $B = \mathcal{X}$  for the first relation,  $A = \mathcal{Z}$ ,  $v = \mathcal{F}$ ,  $v' = \mathcal{G}$  for the second relation and  $u = \mathcal{F}$ ,  $v = \mathcal{G}$  for the last relation. We do not repeat definitions of the Kapranov-Voevodsky tensor products from the left column of the last three formulas, see the original text for the definition. Our formulas give a definition of the operators  $R(\mathcal{F}, \mathcal{G}; \mathcal{X})$ ,  $R(\mathcal{Z}; \mathcal{F}, \mathcal{G})$ ,  $R(\mathcal{F}, \mathcal{G})$  as well as reference to the relevant operation from [23]. We also translate Kapranov-Voevodsky axioms to our language in our argument below.

The rest non-trivial two-morphisms in our setting are structural morphisms of the 2-category from Proposition 4.1. One can apply the method of MacLane strictness theorem to turn the 2-categories of 1-morphisms into the strict 2-categories, thus we suppress associator and unitor two-morphism in the discussion below.

The axioms that need to be checked for the structural two-morphisms  $R(\bullet, \bullet; \bullet)$ ,  $R(\bullet, \bullet)$ , are on the pages 221 and 222 of [23]. Let us first explain how one can prove the relations on the page 222 of [23] which are also the essential relations of the semi-strict monoidal 2-category (see Lemma 4 in [5]). The rest of the axioms can be check by a similar computation.

The relation from the top of the page 221 of [23] as well as relation (viii) in Lemma 4 of [5] can be read of the commuting diagram:

$$\begin{array}{ccccc}\mathcal{X} \circ \mathcal{Z} & \xrightarrow{(F \star F) \circ \mathbf{1}^{\mathcal{Z}}} & \mathcal{X}'' \circ \mathcal{Z} & & \\ & \searrow F \circ \mathbf{1}^{\mathcal{Z}} & \swarrow F \circ \mathbf{1}^{\mathcal{Z}} & & \\ & \mathcal{X}' \circ \mathcal{Z} & & & \\ \mathbf{1}^{\mathcal{X}} \circ \mathcal{G} \downarrow & & & & \downarrow \mathbf{1}^{\mathcal{X}''} \circ \mathcal{G} \\ \mathcal{X} \circ \mathcal{Z}' & \xrightarrow{(F \star F) \circ \mathbf{1}^{\mathcal{Z}'}} & \mathcal{X}'' \circ \mathcal{Z}' & & \\ & \searrow F \circ \mathbf{1}^{\mathcal{Z}'} & \swarrow F \circ \mathbf{1}^{\mathcal{Z}'} & & \\ & \mathcal{X}' \circ \mathcal{Z}' & & & \end{array}$$

here  $\mathcal{F} \in \text{Hom}(\mathcal{X}, \mathcal{X}')$ ,  $\mathcal{F}' \in \text{Hom}(\mathcal{X}', \mathcal{X}'')$ ,  $\mathcal{G} \in \text{Hom}(\mathcal{Z}, \mathcal{Z}')$  and  $\mathcal{Z}, \mathcal{Z}' \in \mathbb{H}\text{om}(\mathbf{Y}, \mathbf{Z})$ ,  $\mathcal{X}, \mathcal{X}', \mathcal{X}'' \in \mathbb{H}\text{om}(\mathbf{X}, \mathbf{Y})$ , we omit spaces  $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$  from our notations.

There are two dotted paths that connect  $\mathcal{X} \circ \mathcal{Z}$  to  $\mathcal{X}'' \circ \mathcal{Z}'$ . The relation that we are after is the equality of two compositions of two-morphisms that relate these two paths. In more details, the first sequences of two-morphisms is as follows:

$$\begin{aligned} & (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X} \circ \mathcal{Z}' \rightarrow \mathcal{X}' \circ \mathcal{Z}' \rightarrow \mathcal{X}' \circ \mathcal{Z}'') \\ & \xrightarrow{R(\mathcal{F}, \mathcal{F}'; \mathcal{Z}')} (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X} \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}'), \\ & (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X} \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}') \\ & \xrightarrow{R(\mathcal{F} \star \mathcal{F}', \mathcal{G})} (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}'' \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}''). \end{aligned}$$

The second sequence of two-morphisms is

$$\begin{aligned} & (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X} \circ \mathcal{Z}' \rightarrow \mathcal{X}' \circ \mathcal{Z}' \rightarrow \mathcal{X}' \circ \mathcal{Z}'') \\ & \xrightarrow{R(\mathcal{F}, \mathcal{G})} (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}'), \\ & (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}') \\ & \xrightarrow{R(\mathcal{G}, \mathcal{F}')} (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z} \rightarrow \mathcal{X}'' \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}'), \\ & (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}' \circ \mathcal{Z} \rightarrow \mathcal{X}'' \circ \mathcal{Z} \rightarrow \mathcal{X}'' \circ \mathcal{Z}') \\ & \xrightarrow{R(\mathcal{F}, \mathcal{F}'; \mathcal{Z})} (\mathcal{X} \circ \mathcal{Z} \rightarrow \mathcal{X}'' \circ \mathcal{Z}' \rightarrow \mathcal{X}'' \circ \mathcal{Z}'). \end{aligned}$$

The two morphisms in the sequences are compositions of the base-change two-morphisms of some geometric maps. Let us discuss sequences of two-morphism in details. There is a sequence of maps  $g_{i+1/2;k}$ ,  $k = 1, \dots, r$  between spaces  $S_{i;k}$ ,  $g_{i+1/2;k} : S_{i+(-\epsilon_{i;k}+1)/2} \rightarrow S_{i+(\epsilon_{i;k}+1)/2}$ ,  $\epsilon_{i;k} = \pm 1$  and the functors  $F_k = f_1 \circ \dots \circ f_{m_k}$  where  $f_i = g_{i+1/2;k*}$  if  $\epsilon_{i;k} = 1$  and  $f_i = g_{i+1/2;k}^*$  if  $\epsilon_{i;k} = -1$  such that

- (1)  $F_1 : (\mathcal{F}, \mathcal{F}', \mathcal{G}) \mapsto ((\mathbf{1}^{\mathcal{X}} \circ \mathcal{G}) \star (\mathcal{F} \circ \mathbf{1}^{\mathcal{Z}'})) \star \mathcal{F}' \circ \mathbf{1}^{\mathcal{Z}'}$ ,
- (2) There is  $s > 1$  such that  $F_s : (\mathcal{F}, \mathcal{F}', \mathcal{G}) \mapsto ((\mathbf{1}^{\mathcal{X}} \circ \mathcal{G}) \star (\mathcal{F} \star \mathcal{F}' \circ \mathbf{1}^{\mathcal{Z}'}))$ ,
- (3)  $F_r : (\mathcal{F}, \mathcal{F}', \mathcal{G}) \mapsto ((\mathcal{F} \star \mathcal{F}' \circ \mathbf{1}^{\mathcal{Z}'}) \star (\mathbf{1}^{\mathcal{X}} \circ \mathcal{G}))$ ,
- (4)  $F_{k+1}$  is obtained from  $F_k$  either by the base-change morphism or composition of some the push-forwards or or pull-backs in the sequence  $f_{i;k}$ .
- (5) the corresponding sequence of natural transformations connecting  $F_1$  with  $F_s$  is  $R(\mathcal{F}, \mathcal{F}'; \mathcal{Z})$  and the sequence connecting  $F_s$  and  $F_r$  is  $R(\mathcal{F} \star \mathcal{F}', \mathcal{G})$ .

Similarly, there is a sequence of maps  $h_{i+1/2;k}$ ,  $\epsilon'_{i;k} = \pm 1$  and spaces  $S'_{i;k}$ ,  $k = 1, \dots, l$  that realizes the sequence of the two-morphisms  $R(\mathcal{F}, \mathcal{G})$ ,  $R(\mathcal{G}, \mathcal{F}')$ ,  $R(\mathcal{F}, \mathcal{F}'; \mathcal{Z})$ . Moreover,  $h_{i+1/2;k} = g_{i+1/2;k'}$ ,  $\epsilon'_{i;k} = \epsilon_{i;k'}$  for  $k = k' = 1$  and  $k = r, k' = l$ .

Thus, we can paste the sequence of the maps  $h_{i+1/2;k}$  and the sequence  $g_{i+1/2;k}$  along the sequences  $g_{i+1/2;1}$  and  $g_{i+1/2;r}$ . One half of the pasted diagram yields the compositions of the natural transformations  $R(\mathcal{F}, \mathcal{F}'; \mathcal{Z})$  and  $R(\mathcal{F} \star \mathcal{F}', \mathcal{G})$  and the other half the composition of  $R(\mathcal{F}, \mathcal{G})$ ,  $R(\mathcal{G}, \mathcal{F}')$ ,  $R(\mathcal{F}, \mathcal{F}'; \mathcal{Z})$ . The commutativity of the diagram of maps implies the equality of two compositions of the natural transformations.



The next group of the relations on the page 221 of [23] are equivalent to the naturality of  $R(\mathcal{F}, \mathcal{G})$ , in our case the naturality is automatic since  $R(\mathcal{F}, \mathcal{G})$  is a composition of natural transformations of base change. The last group of relation relation on that page follows immediately from the exchange relation from Proposition 3.2.

The first group of relations on the page 222 of [23] is equivalent to naturality of  $R(\mathcal{F}, \mathcal{G}; \mathcal{Z})$ . Finally, the last relation on this page is based on the commuting diagram:

$$\begin{array}{ccccc}
 & & \mathcal{X}''' \circ \mathcal{Z} & & \\
 & \nearrow^{F \star F' \star F'' \circ 1} & \uparrow & \nwarrow^{F'' \circ 1} & \\
 \mathcal{X} \circ \mathcal{Z} & \xrightarrow{\quad F \star F' \circ 1 \quad} & & \mathcal{X}'' \circ \mathcal{Z} & \\
 & \nwarrow_{F \circ 1} & \downarrow & \nearrow_{F \circ 1} & \\
 & & \mathcal{X}' \circ \mathcal{Z} & & 
 \end{array}$$

There are two compositions of the dotted arrow that start at  $\mathcal{X} \circ \mathcal{Z}$  and end at  $\mathcal{X}''' \circ \mathcal{Z}$ . Respectively, there are two ways to connect these compositions by the two-morphisms which suppose to result into the same two-morphism:

$$R(\mathcal{F}, \mathcal{F}' \star \mathcal{F}''; \mathcal{Z}) \circ R(\mathcal{F}', \mathcal{F}'', \mathcal{Z}) = R(\mathcal{F} \star \mathcal{F}', \mathcal{F}''; \mathcal{Z}) \circ R(\mathcal{F}, \mathcal{F}'; \mathcal{Z}).$$

The last equality follows from the pasting argument as in the case discussed at the beginning of the proof.  $\square$

## 4.2 Module category construction of $\check{\check{\check{\text{Cat}}}}(Q)$

Let us define  $\check{\check{\check{\text{Cat}}}}(Q)$  to be a 3-category of 2-categories with the two-monoidal action of  $\check{\check{\check{\text{Cat}}}}(Q)$ . This 3-category is too big and the 3-category  $\check{\check{\check{\text{Cat}}}}(Q)$  is a sub 3-category inside  $\check{\check{\check{\text{Cat}}}}(Q)$ . Indeed, the product of 2-categories

$$\mathbb{M}_{\mathbf{n}, \mathbf{r}} = \prod_{\mathbf{n}', \mathbf{r}'} \mathbb{H}\text{om}(Q(\mathbf{n}, \mathbf{r}), Q(\mathbf{n}', \mathbf{r}'))$$

is a 2-category with the monoidal action of  $\check{\check{\check{\text{Cat}}}}(Q)$ , that is an object in  $\check{\check{\check{\text{Cat}}}}(Q)$ .

Respectively, the 2-category  $\mathbb{H}\text{om}(Q(\mathbf{n}, \mathbf{r}), Q(\mathbf{m}, \mathbf{s}))$  is a sub 2-category of 2-category of morphisms from  $\mathbb{M}_{\mathbf{n}, \mathbf{r}}$  to  $\mathbb{M}_{\mathbf{m}, \mathbf{s}}$ . Thus, we define the 3-category  $\check{\check{\check{\text{Cat}}}}(Q)$  to be the 3-category with objects  $\mathbb{M}_{\mathbf{n}, \mathbf{r}}$  and the morphisms between these objects as above.

## 4.3 The case $Q = \tilde{A}_0$

In the case of general quiver  $Q$  it is not immediately clear how to construct a unit in the monoidal 2-category  $\check{\check{\check{\text{Cat}}}}(Q)$ . However, in the case  $Q = \tilde{A}_0$ , there is a natural

candidate for the unit.

$$\mathbb{L}_{\text{id}} = \{\mathbb{L}_{\text{id}}^{n,r}\}_{n,r \geq 0}, \quad \mathbb{L}_{\text{id}}^{n,r} \in \mathbb{H}\text{om}(\tilde{A}_0(n; r), \tilde{A}_0(n; r)),$$

the one-morphism  $\mathbb{L}_{\text{id}}^{n,r}$  is defined in the next section where we provide a proof for

**Theorem 4.2** *The 2-category  $\ddot{\text{Cat}}(Q)$ ,  $Q = \tilde{A}_0$  is a monoidal 2-category with a unit  $\mathbb{L}_{\text{id}}$ .*

In the notations of the previous section, the 3-category  $\ddot{\text{Cat}}_{\text{gl}}$  is a subcategory of  $\ddot{\text{Cat}}(\tilde{A}_0)$  with objects  $\tilde{A}_0(n, 0)$ ,  $n \geq 0$ . We also define the framed version of the 3-category  $\ddot{\text{Cat}}_{\text{gl}}^f$  in the second half of this section. The later category is a sub-category of  $\ddot{\text{Cat}}(\tilde{A}_0)$  with objects  $\tilde{A}_0(n, 1)$ . Implicitly, this category was studied in [33, 36, 38] where several results in theory of knot homology were derived. We spell out the details of the categories  $\ddot{\text{Cat}}_{\text{gl}}$  and  $\ddot{\text{Cat}}_{\text{gl}}^f$  and state some conjectures about these categories.

#### 4.3.1 Objects and morphisms

The objects of  $\ddot{\text{Cat}}_{\text{gl}}$  are labeled by  $\mathbb{Z}_{\geq 0}$ :

$$\text{Obj } \ddot{\text{Cat}}_{\text{gl}} = \{\mathbf{n} | \mathbf{n} \in \mathbb{Z}_{\geq 0}\}.$$

The objects in the 2-category of morphisms are pairs  $(Z, w)$ , where  $Z$  is an algebraic variety with an action of  $\text{GL}_n \times \text{GL}_m$ :

$$\text{Obj } \mathbb{H}\text{om}(\mathbf{n}, \mathbf{m}) = \{(Z, w), \quad w \in \mathbb{C}[\mathfrak{gl}_n \times Z \times \mathfrak{gl}_m]^{\text{GL}_n \times \text{GL}_m}\}.$$

The composition of morphism  $(Z, w) \in \mathbb{H}\text{om}(\mathbf{n}, \mathbf{m})$ ,  $(Z', w') \in \mathbb{H}\text{om}(\mathbf{m}, \mathbf{k})$  is defined as

$$(Z, w) \circ (Z', w') = (Z \times \mathfrak{gl}_m \times Z' / {}_+ \text{GL}_m, w' + w) \in \mathbb{H}\text{om}(\mathbf{n}, \mathbf{k}).$$

Here, the quotient is defined via GIT theory as follows. Suppose that  $X$  is a variety with a  $\text{GL}_m$ -action. A character  $\chi$  of  $\text{GL}_m$  determines the trivial line bundle  $L_\chi$  with  $\text{GL}_m$ -equivariant structure defined by  $\chi$ . Recall that a point  $x \in X$  is semistable (with respect to  $L_\chi$ ) if there is  $m > 0$  and  $s \in \Gamma(X, L_\chi)^{\text{GL}_m}$  such that  $s(x) \neq 0$ . Denote

$$X / {}_\chi \text{GL}_m := X^{ss} / \text{GL}_m.$$

Since the group of characters of  $\text{GL}_m$  is generated by  $\det$ , we introduce short-hand notations:

$$X / {}_\pm \text{GL}_m := X / {}_{\det^{\pm 1}} \text{GL}_m, \quad X / {}_0 \text{GL}_m := X / {}_{\det^0} \text{GL}_m.$$

Two-morphism are the objects of the corresponding category of equivariant matrix factorizations. Given  $(Z, w), (Z', w') \in \mathbb{H}\text{om}(\mathbf{n}, \mathbf{m})$  we define:

$$\text{Hom}((Z, w), (Z', w')) = \text{MF}_{\text{GL}_n \times \text{GL}_m}(\mathfrak{gl}_n \times Z \times Z' \times \mathfrak{gl}_m, w - w').$$

The group  $\text{GL}_n \times \text{GL}_m$  is reductive; hence, the equivariance of the matrix factorization  $(M, D)$  is equivalent to the condition that the group action on  $M$  commutes with the differential  $D$ .

The space of morphisms  $\mathcal{H}\text{om}(\cdot, \cdot)$  between the equivariant matrix factorizations is defined to be the space of morphisms between the underlying matrix factorizations that commute the group action.

### 4.3.2 Framed version of the 3-category

We enlarge slightly our category to include the framing. The objects of the new 3-category  $\check{\check{\check{\text{Cat}}}}_{\text{gl}}^f$  are again labeled by the positive integers:

$$\text{Obj}(\check{\check{\check{\text{Cat}}}}_{\text{gl}}^f) = \{\mathbf{n}^f \mid n \in \mathbb{Z}_{\geq 0}\}.$$

For the space of morphisms, we have

$$\mathbb{H}\text{om}(\mathbf{n}^f, \mathbf{m}^f) = (Z, w), \quad w \in \mathbb{C}[V_n \times \mathfrak{gl}_n \times Z \times \mathfrak{gl}_m \times V_m]^{\text{GL}_n \times \text{GL}_m}$$

here  $Z$  is an algebraic variety with an action of  $\text{GL}_n \times \text{GL}_m$  and  $V_n = \mathbb{C}^n$ ,  $V_m = \mathbb{C}^m$  with the standard  $\text{GL}_n$  and  $\text{GL}_m$  actions.

The rest of definitions are identical to the constructions from the previous subsection, after we replace  $\mathfrak{gl}_n$  with  $V_n \times \mathfrak{gl}_n$ . For brevity, we introduce the following shorthand notation:

$$\mathfrak{gl}_n^f := \mathfrak{gl}_n \times V_n.$$

In general, many definitions in our paper are parallel in framed and unframed cases, in the cases when the definitions are parallel we use  $\bullet$  notation to indicate that  $\bullet$  could be "f" or empty set.

First of all, note that the category  $\mathbb{H}\text{om}(\mathbf{n}^\bullet, \mathbf{m}^\bullet)$  has a natural monoidal structure. Given  $\mathcal{F} \in \text{Hom}((Z, w), (Z', w'))$  and  $\mathcal{G} \in \text{Hom}((Z', w'), (Z'', w''))$  we define  $\mathcal{F} \star \mathcal{G} \in \text{Hom}((Z, w), (Z'', w''))$

$$\mathcal{F} \star \mathcal{G} := \pi_*(\mathcal{F} \otimes \mathcal{G}), \quad \pi : \mathfrak{gl}_n \times Z \times Z' \times Z'' \times \mathfrak{gl}_m \rightarrow \mathfrak{gl}_n \times Z \times Z'' \times \mathfrak{gl}_m$$

**Remark 4.1** The 3-categories  $\check{\check{\check{\text{Cat}}}}_{\text{gl}}, \check{\check{\check{\text{Cat}}}}_{\text{gl}}^f$  can be realized in terms of 4-category **BGL** by assigning to  $\mathbf{n}$  the morphisms

$$\mathfrak{gl}_n \in \mathbf{Hom}(\text{GL}(n), \text{GL}(0)), \quad \mathfrak{gl}_n \times V_n \in \mathbf{Hom}(\text{GL}(n), \text{GL}(0)),$$

respectively. That is  $\ddot{\text{Cat}}_{\text{gl}} = \ddot{\text{Cat}}(\tilde{A}_0(0))$ ,  $\ddot{\text{Cat}}_{\text{gl}}^f = \ddot{\text{Cat}}(\tilde{A}_0(1))$  in notations of Sect. 2.4.1.

#### 4.4 Monoidal properties of $\ddot{\text{Cat}}_{\text{gl}}^\bullet$

The objects of  $\ddot{\text{Cat}}_{\text{gl}}^\bullet$  are labeled by the positive numbers and it seems natural to use the addition for the monoidal structure on the level of objects. It is much harder to extend this monoidal structure to 1, 2, 3-morphisms. Nevertheless, we conjecture that it is possible to construct such extension. Below, we provide detailed motivation for the

**Conjecture 4.1** *The 3-category  $\ddot{\text{Cat}}_{\text{gl}}^\bullet$  can be enhanced to a monoidal 3-category.*

Let us introduce the auxiliary object that facilitates the extension of monoidal structure to level of 1-morphisms. The object is the pair

$$\begin{aligned} \mathbb{I}_{n,k;n+k} &= (T^*\text{GL}_{n+k}, W_{n,k;n+k}), \\ W_{n,k;n+k}(X, z, X') &= \text{Tr}(X\mu_{\text{gl}_{n+k}}(z)) - \text{Tr}(X'\mu_{\mathfrak{p}_{n,k}}) \\ &\in \mathbb{C}[\mathfrak{gl}_{n+k} \times T^*\text{GL}_{n+k} \times \mathfrak{p}_{n,k}]^{\text{GL}_{n+k} \times P_{n,k}}, \end{aligned}$$

here  $P_{n,k} \subset \text{GL}_{n+k}$  is the standard parabolic subgroup with the Levi group  $\text{GL}_n \times \text{GL}_k$ , respectively  $\mathfrak{p}_{n,k} = \text{Lie}(P_{n,k})$  and  $\mu_{\text{gl}_{n+k}}$  and  $\mu_{\mathfrak{p}_{n,k}}$  are the moment maps for the left and right actions of the corresponding groups.

Given two objects  $\mathbb{O}_1 \in \mathbb{H}\text{om}(\mathbf{m}_1^\bullet, \mathbf{n}^\bullet)$ ,  $\mathbb{O}_2 \in \mathbb{H}\text{om}(\mathbf{m}_2^\bullet, \mathbf{k}^\bullet)$  we define the induced object

$$\mathbb{O}_1 \oplus \mathbb{O}_2 := \mathbb{O}_1 \times \mathbb{O}_2 \circ_{P_{n,k}} \mathbb{I}_{n,k;n+k} \in \mathbb{H}\text{om}(\mathbf{m}_1^\bullet \times \mathbf{m}_2^\bullet, (\mathbf{n} + \mathbf{k})^\bullet).$$

Thus, defined operation extends to the level of two-morphisms. Moreover, the exchange relation from Proposition 3.1 implies that this operation is monoidal with respect to the monoidal structure  $\star$ . In the case  $m_1 = m_2$ , the above operation was studied in [38] where it is called induction functor and it was used to construct the braid realization discussed in the next section.

We expect that the object  $\mathbb{I}_{n,k;n+k}$  will play an important role in the foam extension of our TQFT, this element is the element one needs for defining a value of the partition function on the configurations that contain three planes of defect intersecting along the line of defect. Also this element interacts well with NS5 defects discussed in the next section:

$$\mathbb{I}_{l|n} \times \mathbb{O} \circ_{P_{n,k}} \mathbb{I}_{n,k;n+k} = \mathbb{I}_{\text{id}}^l \times \mathbb{O} \circ_{P_{n,k}} \mathbb{I}_{l,k;l+k} \circ_{\text{GL}_{n+k}} \mathbb{I}_{l+k|n+k}, \quad l \leq n,$$

where  $\mathbb{O} \in \mathbb{H}\text{om}(\mathbf{m}^\bullet, \mathbf{k}^\bullet)$  is any one-morphism and  $\mathbb{I}_{l|n} \in \mathbb{H}\text{om}(\mathbf{l}^\bullet, \mathbf{n}^\bullet)$  is NS5 type one-morphism discussed in the next section.

## 5 Defects and knot invariants

In this section, we explain how we interpret the results of [38] in terms of 3-categories  $\ddot{\text{Cat}}_{\text{gl}}$  and  $\ddot{\text{Cat}}_{\text{gl}}^f$ . In particular, we make a connection with the theory of foams and provide an explanation for the Chern character construction. We also construct a partition functions  $Z^f$  for the KRS theories the targets  $\text{Hilb}_n(\mathbb{C}^2)$ ,  $n \in \mathbb{Z}_{\geq 0}$  with NS5 type defects in  $S^2 \times \mathbb{R}$ . In particular, we show that for a circle  $S^1$  inside of a connected component of the defect complement  $S^2 \times \mathbb{R} \setminus \text{Def}$ , the partition function yields  $Z^f(S^1) = D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2))$  where  $n$  is the label of the connected component.

Recall the standard setup of the 3D topological field theory with defects. A 3D QTFT is characterized by its partition-function evaluation  $Z$ . The partition function is an assignment:

closed connected three-manifold  $X \mapsto Z(X) \in \mathbb{C}$ ,

closed connected surface  $S \mapsto Z(S) \in \text{Vect}$ ,

three-manifold  $X$  with boundary  $\partial X = \bigcup_i S_i \mapsto Z(X) \in Z(\partial X) = \bigotimes_{i=1}^m Z(S_i)$ ,

closed connected curve  $C \mapsto Z(C) \in \text{Cat}$ ,

surface with boundary  $S \mapsto Z(S) \in Z(\partial S) = \bigotimes_{i=1}^k Z(C_i)$ ,

point  $p \mapsto Z(p) \in \ddot{\text{Cat}}_{\text{gl}}^\bullet$ ,

interval  $I = [b, e] \mapsto Z(I) \in \text{Hom}(Z(b), Z(e))$ .

This collection of data behaves naturally under the gluing operation. For example, suppose that a three-manifold  $X$  without a boundary is cut into two pieces over a surface  $S$ :

$$X = X_1 \cup X_2, \quad \partial(X_1) = S, \quad \partial(X_2) = S^\vee.$$

Then,  $S$  and  $S^\vee$  have opposite orientations, hence  $Z(S)$  and  $Z(S^\vee)$  are dual vector spaces and the partition function  $Z(X)$  is a pairing between their elements:

$$Z(X) = Z(X_1) \cdot Z(X_2),$$

More generally, the formalism of TQFT provides a method for computing values of  $Z(Y)$  by cutting  $Y$  into pieces, evaluating  $Z$  on the pieces and pairing them in a standard way. More details can be found in [25]. To define a 3D TQFT, we need to include into the domain of  $Z$  also manifolds with corners and work with more subtle setting of  $(\infty, k)$ -categories. We postpone the discussion of such extension to our future publication [34].

Often TQFTs may be defined not only on smooth manifolds but also on ‘smooth’ CW-complexes. In particular, a TQFT may have defects (coming from lower-dimensional cells). Topologically, defects are unions of embedded surfaces and curves.

Surfaces and curves may intersect. The cuts must be transverse to the defects. All other properties of TQFT without defects are preserved.

The full categories  $\mathring{\mathcal{C}\mathring{a}\mathring{t}}_{\mathfrak{gl}}^\bullet$  will be used to construct 3D TQFT and we discuss the construction in the forthcoming publication [34] where we construct the corresponding maps  $Z$  and  $Z^f$ .

In this note, we concentrate on  $\mathbb{R}$ -invariant part of the TQFT. That is from now on all defects are invariant with respect to shift along last coordinate in  $S^2 \times \mathbb{R}$  and thus we can recover a configuration of defects from the two-dimensional cut which is an intersection with  $S^2 \times \{0\}$ .

We present  $S^2$  as  $\mathbb{R}^2$  compactified by a point  $\infty$  at infinity and we can assume that the defects do not contain  $\infty$ . The two-dimensional cut is  $\mathbb{R}^2 \cup \infty$ . Studying two-dimensional slice is equivalent to restricting ourselves to the defects of the form  $C \times \mathbb{R}$  where  $C$  is a curve in  $\mathbb{R}^2$ .

The surfaces of defect intersect our fixed  $\mathbb{R}^2$  along the union of oriented curves. Suppose that the curves on the plane lies in an annulus and their union is a projection of the closure of a braid. By assigning a sign to each intersection, we obtain an interpretation of the union of curves as a projection of a link in  $\mathbb{R}^3$  presented as a closure  $L(\beta)$  of a braid  $\beta \in \mathfrak{B}\mathfrak{r}_n$ . Denote the plane with defect  $\mathbb{R}_\beta^2$ .

From the results of [38], it follows that our TQFT provides an isotopy invariant of  $L(\beta)$ , namely the vector space  $Z^f(\mathbb{R}_\beta^2)$ , assigned to  $\mathbb{R}_\beta^2$ . We show that this invariant categorifies HOMFLYPT polynomial and coincides with previously defined Khovanov-Rozansky invariant [40].

Now let us give details of the  $\mathbb{R}^2$ -sliced TQFT. A small neighborhood of the plane contains two-dimensional surfaces of defect that are products of the defect curves in  $\mathbb{R}^2$  with an interval. The surfaces of defect divide  $\mathbb{R}^3$  into connected components and each connected component has an integer marking. We choose the markings in such a way that if we move along the (oriented) intersection of a defect surface with  $\mathbb{R}^2$  and the marking on the left is  $k$ , then the marking on the right is  $k + 1$ .

We assume that the curves of intersection are compact, thus their union is contained in a large disc. The marking of the disc complement ('the infinite marking') determines all other markings.

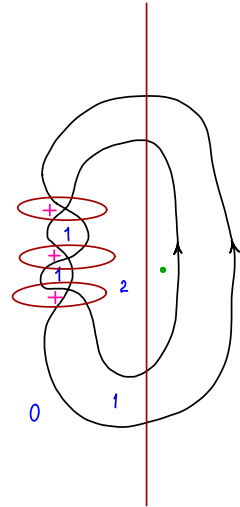
In this paper, we only consider the theories with the infinite marking equal to 0. Hence, if the intersection of the surface defect and  $\mathbb{R}^2$  is the braid closure then all markings are positive. Thus, the picture of the braid  $\beta$  determines the marking of our theory uniquely. Slightly abusing notation, we denote such data as  $\mathbb{R}_\beta^2$ . The figure is a slice  $\mathbb{R}_\beta^2$  for  $\beta = \sigma_1^3$ . The closure of  $L(\beta)$  is a trefoil and we explain below how we can compute the homology of this knot (Fig. 1).

## 5.1 Values on points and defect trivial intervals

There is a canonical way to upgrade the marking of  $\mathbb{R}_\beta^2$  to a categorical marking. Denote by  $p_n$  a point lying inside a region marked by  $n$ . To a pair of points  $(p_{n_1}, p_{n_2})$ , we assign a 2-category

$$Z^\bullet(p_{n_1}, p_{n_2}) = \mathbb{H}\text{om}(\mathbf{n}_1^\bullet, \mathbf{n}_2^\bullet). \quad (5.1)$$

**Fig. 3** Decomposing  $\mathbb{R}_\beta^2$  on two half-planes



Recall that an object of this 2-category is a pair  $(Z, W)$ , where  $Z$  is a variety with a  $(\mathrm{GL}(n_1) \times \mathrm{GL}(n_2))$ -action and  $W \in \mathbb{C}[\mathfrak{gl}_{n_1} \times \mathfrak{gl}_{n_2} \times Z]^{\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)}$ .

In accordance with the assignment (5.1), to an interval  $I$  connecting  $\mathbf{p}_{n_1}$  and  $\mathbf{p}_{n_2}$  (and possibly crossing defect surfaces) we assign an object  $Z^\bullet(I)$  of  $\mathbb{H}\mathrm{om}(\mathbf{n}^\bullet, \mathbf{n}^\bullet)$ , so that if  $I$  is the result of gluing together the intervals  $I_1$  and  $I_2$  over the common middle point, then the object of  $I$  is the composition:

$$Z^\bullet(I) = Z^\bullet(I_2) \circ Z^\bullet(I_1).$$

This relation implies that if the interval  $I$  lies within a single region marked by  $\mathbf{p}_n$ , then the corresponding object  $Z^\bullet(I)$  is the identity with respect to the monoidal structure (i.e., the composition) of the category  $\mathbb{H}\mathrm{om}(\mathbf{n}^\bullet, \mathbf{n}^\bullet)$ .

Theorem 4.2 states that the following pair is the identity object of the ‘unframed’ 2-category  $\mathbb{H}\mathrm{om}(\mathbf{n}, \mathbf{n})$ :

$$\mathbb{L}_{\mathrm{id}}^{n,0} = L_{\mathrm{id}}^n = \left( T^*\mathrm{GL}_n, W_{\mathrm{id}} := \mathrm{Tr} \phi(X - \mathrm{Ad}_g(X')) \right)$$

in which  $(g, \phi) \in \mathrm{GL}(n) \times \mathfrak{gl}_n \cong T^*\mathrm{GL}(n)$ , while  $(X, X') \in \mathfrak{gl}_n \times \mathfrak{gl}_n$ , and the action of  $\mathrm{GL}(n) \times \mathrm{GL}(n)$  on the total variety  $\mathfrak{gl}_n \times \mathfrak{gl}_n \times T^*\mathrm{GL}(n)$  is given by the following formula:

$$(a, b) \cdot (X, X', g, \phi) = (\mathrm{Ad}_a X, \mathrm{Ad}_b X', agb^{-1}, \mathrm{Ad}_a \phi)$$

The identity object in the framed 2-category  $\mathbb{H}\mathrm{om}(\mathbf{n}^f, \mathbf{n}^f)$  is the case  $r = 1$  of

$$\mathbb{L}_{\mathrm{id}}^{n,r} = (T^*\mathrm{GL}_n \times \mathrm{Hom}(V_n, \mathbb{C}^r), W_{\mathrm{id}}^f := W_{\mathrm{id}}(X, g, \phi, X') + \mathrm{Tr}(w \cdot (v - gv'))),$$

where  $(X, v), (X, v') \in \mathfrak{gl}_n \times \text{Hom}(\mathbb{C}^r, V_n)$ , while  $(g, \phi, w) \in T^*\text{GL}(n) \times \text{Hom}(V_n, \mathbb{C}^r)$  and the action of  $\text{GL}(n) \times \text{GL}(n)$  on framing related variables is

$$(a, b) \cdot (v, v', w) = (av, bv', aw).$$

**Proof of Theorem 4.2** We need to define the left and right unitizing 1-morphisms

$$\begin{aligned} l(\mathcal{X}) : \mathcal{X} &\rightarrow \mathbb{L}_{\text{id}}^{n;r} \circ \mathcal{X}, \quad \mathcal{X} \in \mathbb{H}\text{om}(\tilde{A}_0(n, r), \tilde{A}_0(n', r')), \\ r(\mathcal{X}) : \mathcal{X} &\rightarrow \mathcal{X} \circ \mathbb{L}_{\text{id}}^{n;r}, \quad \mathcal{X} \in \mathbb{H}\text{om}(\tilde{A}_0(n', r'), \tilde{A}_0(n, r)). \end{aligned}$$

We take the left unitizer to be the tensor product of matrix factorizations

$$l(\mathcal{X}) := \mathbf{1}_{\mathcal{X}} \circ \mathcal{KN}, \quad \mathcal{KN} = [\phi, X - \text{Ad}_g(X')] \otimes [w, v - gv'],$$

where we use the notation from the definition of the unit object as well as the standard notations for the Koszul matrix factorization. The right unitizer is defined by the identical formula.

The matrix factorization  $\mathcal{KN}|_{g=1}$  is the matrix factorization that yields the Knörrer's equivalence of categories [19]. The composition  $\circ$  in the product  $\mathbb{O} \circ \mathbb{L}_{\text{id}}$  involves taking quotient by the  $\text{GL}_n$ -action. Since the left and right actions on  $\text{GL}_n$  on  $T^*\text{GL}_n$  is free, we can trade the  $\text{GL}_n$ -quotient for the gauge fixing  $g = 1$ . Thus, we are in the setting of the Knörrer periodicity. Respectively, the inverse to the Knörrer equivalence is composition of  $\text{KN}^*$  and the duality functor.

The only non-trivial structural two-morphisms that involves the unit object are

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathcal{F}} & \mathcal{X}' \\ l(\mathcal{X}) \downarrow \swarrow l(\mathcal{F}) & & \downarrow l(\mathcal{X}') \\ \mathbb{L} \circ \mathcal{X} & \xrightarrow{\mathbb{L} \circ \mathcal{F}} & \mathbb{L} \circ \mathcal{X}' \end{array} \quad , \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathcal{F}} & \mathcal{X}' \\ r(\mathcal{X}) \downarrow \swarrow r(\mathcal{F}) & & \downarrow r(\mathcal{X}') \\ \mathcal{X} \circ \mathbb{L} & \xrightarrow{\mathcal{F} \circ \mathbb{L}} & \mathcal{X}' \circ \mathbb{L} \end{array}$$

which are defined by

$$l(\mathcal{F}) = R(\mathcal{KN}, \mathcal{F}), \quad r(\mathcal{F}) = R(\mathcal{F}, \mathcal{KN}).$$

Thus, these structural two-morphisms are compositions of the base-change natural transformations. The relevant conditions for the structural morphism related to the unit object of monoidal 2-category are listed on the page 224 of [23]. Since the structural two-morphisms in our setting are of geometric origin one prove the conditions by the pasting method used in theorem  $\square$

It follows from the axioms of the monoidal 2-category (see for example [5]) that the category of endomorphisms of unit object is a universal Drinfeld center, in the following sense.



**Corollary 5.1** *The monoidal category  $\text{Hom}(\mathbb{L}_{id}^{n;r}, \mathbb{L}_{id}^{n';r})$  is a braided monoidal category. Moreover for any  $\mathcal{X} \in \mathbb{H}\text{om}(\tilde{A}_0(n; r), \tilde{A}_0(n'; r'))$ , there is a monoidal functor to the Drinfeld center of  $\text{Hom}(\mathcal{X}, \mathcal{X})$ :*

$$HC[\mathcal{X}] : \text{Hom}(\mathbb{L}_{id}^{n;r}, \mathbb{L}_{id}^{n';r}) \rightarrow Z(\text{Hom}(\mathcal{X}, \mathcal{X}), \star).$$

In the special case of  $n' = 0$  and  $r = r' = 1$  and special choice of  $\mathcal{X}$ , this functor was studied in [37]. The exact connection is explained in the next subsections. In this work, we also define a right adjoint functor to the HC. We expect that the adjoint functor exists in more general setting:

**Conjecture 5.1** *For any  $\mathcal{X} \in \mathbb{H}\text{om}(\tilde{A}_0(n; r), \tilde{A}_0(n'; r'))$  there is a functor*

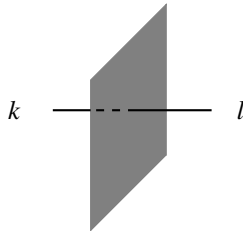
$$CH[\mathcal{X}] : \text{Hom}(\mathcal{X}, \mathcal{X}) \rightarrow \text{Hom}(\mathbb{L}_{id}^{n;r}, \mathbb{L}_{id}^{n';r})$$

*that is the right adjoint to  $HC[\mathcal{X}]$  and is a categorical trace*

$$CH[\mathcal{X}](\mathcal{F} \star \mathcal{G}) = CH[\mathcal{X}](\mathcal{G} \star \mathcal{F}).$$

## 5.2 Value on the intervals intersecting defects

In this section, we discuss the intersection of small interval with the three-dimensional trace of NS5-brane, as discussed in Sect. 2.5.3. We assume that  $I$  is a small interval intersecting a surface of defect in a smooth point. The surface of defect separates the regions with labels  $k$  and  $l$  as on the figure below:



Let us assume that the curve of defect separates regions with marking  $k$  and  $l$ . Denote by  $(\phi_{kl}, \phi_{lk})$  the coordinates on  $T^*\text{Hom}(V_k, V_l)$ , where  $\phi_{kl} \in \text{Hom}(V_k, V_l)$  and  $\phi_{lk} \in \text{Hom}(V_l, V_k)$ . Also denote by  $v_l$  and  $v_k$  the coordinates on  $V_l$  and  $V_k$  and denote by  $X_l$  and  $X_k$  the coordinates on  $\mathfrak{gl}_l$  and  $\mathfrak{gl}_k$ . Now the object assigned to the interval  $\tilde{I}$  in the unframed category is a pair

$$\begin{aligned} Z(\tilde{I}) &= \mathbb{I}_{k|l} = (T^*\text{Hom}(V_k, V_l), \\ W_{k,l} &:= \text{Tr}(X_k \phi_{kl} \phi_{lk}) - \text{Tr}(X_l \phi_{lk} \phi_{kl})). \end{aligned}$$

In the context of Sect. 3.5  $\mathbb{I}_{k|l}$  corresponds to the NS5 defect of charge  $k - l$  with no framing i.e.  $r = 0$ .

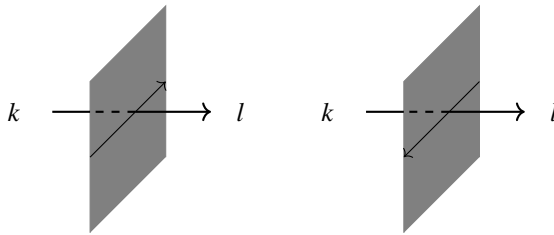
In case of framed 2-categories, if the interval starts at  $k$  and ends in  $l$  and the shortest path from the head of the vector  $\vec{I}$  to the head of the vector of direction of the defect goes clockwise, then we choose

$$\begin{aligned} Z^f(\vec{I}) &= \mathbb{I}_{k|l}^f = (\mathrm{T}^*\mathrm{Hom}(V_k, V_l) \times V_k^*, \\ W_{k,l}^f &:= \mathrm{Tr}(X_k \phi_{kl} \phi_{lk}) - \mathrm{Tr}(X_l \phi_{lk} \phi_{kl}) + \mathrm{Tr}(\psi(v_k - \phi_{kl} v_l)). \end{aligned}$$

In case of opposite orientation, we set

$$\begin{aligned} Z^f(\vec{I}) &= (\mathrm{T}^*\mathrm{Hom}(V_k, V_l) \times V_l^*, \\ W_{k,l}^f &:= \mathrm{Tr}(X_k \phi_{kl} \phi_{lk}) - \mathrm{Tr}(X_l \phi_{lk} \phi_{kl}) + \mathrm{Tr}(\psi(v_l - \phi_{lk} v_k)). \end{aligned}$$

The three-dimensional picture of these two cases could be visualized as two cases below:



Two pictures above present a local picture of the intersection in the case when all defects (the grey wall on the picture) are invariant with respect to the vertical translation.

To match the objects  $\mathbb{I}_{k|l}^f$  with the description of NS5 defects of charge  $k - l$  with  $r$  from Sect. 2.5.2, we need to set  $\varphi^{\mathrm{fr}} = 1$  and  $\psi = \psi^{\mathrm{fr}}$ . The element  $Z^\bullet(\vec{I})$  is an element of the 2-category  $\mathbb{H}\mathrm{om}(\mathbf{k}^\bullet, \mathbf{l}^\bullet)$  thus the composition construction allow us to interpret an element  $Z^\bullet(\vec{I})$  as morphism from  $\mathbb{H}\mathrm{om}(\mathbf{k}^\bullet, \mathbf{0}^\bullet)$  to  $\mathbb{H}\mathrm{om}(\mathbf{l}^\bullet, \mathbf{0}^\bullet)$ .

Let us denote the intervals as above by  $\vec{I}_{k \uparrow l}$  and  $\vec{I}_{k \downarrow l}$ . More generally we denote by

$$\vec{I}_{k_1 \uparrow k_2 \uparrow \dots \uparrow k_l}, \quad \vec{I}_{k_1 \downarrow k_2 \downarrow \dots \downarrow k_l}$$

the interval that connects the connected components with the labels  $k_1$  and  $k_l$  and traverses the domains with the labels  $k_2, \dots, k_l$  in between with the orientation of the intersections as indicated by the arrows. We also allow a mixture of down/up orientations of the intersections.

According to the definition of our TQFT, we have:

$$Z(\vec{I}_{k_1 \uparrow k_2 \dots \uparrow k_l}) = Z(\vec{I}_{k_1 \uparrow k_2}) \circ Z(\vec{I}_{k_2 \uparrow k_3}) \circ \dots \circ Z(\vec{I}_{k_{l-1} \uparrow k_l}). \quad (5.2)$$

The GIT quotient in the definition of the composition can be made explicit in many important cases:

**Proposition 5.1** *For any  $n \geq 0$ , we have:*

$$\begin{aligned} Z^f(\vec{I}_{0\uparrow 1\uparrow \dots n}) &= (T^*Fl_n \times V_n, w), \quad Z^f(\vec{I}_{0\downarrow 1\downarrow \dots n}) = (T^*Fl_n, w), \\ Z(\vec{I}_{0|1|\dots n}) &= (T^*Fl_n, w), \quad w = \mu \cdot X \in \mathbb{C}[\mathfrak{gl}_n \times T^*Fl_n]^{GL_n}, \end{aligned}$$

where  $\mu : T^*Fl_n \rightarrow \mathfrak{gl}_n^*$  is the moment map and  $X$  are the coordinates on  $\mathfrak{gl}_n$ .

**Proof** Let us first prove the last equation, the other equations are analogous and it will be indicated at the end of the proof how one needs to modify the proof to get the first two formulas. We proceed by induction on  $n$ . Thus, we need to compute the composition:

$$Z(\vec{I}_{0|1|\dots|n-1}) \circ Z(\vec{I}_{n-1|n}).$$

It is convenient to think of  $T^*Fl_n$  as  $B_n$ -quotient because the trace map gives a natural pairing on  $\mathfrak{gl}_n$  thus we can think of  $\mu$  as a map  $T^*Fl_n \rightarrow \mathfrak{gl}_n$ :

$$T^*Fl_n = GL_n \times \mathfrak{n}_n / B_n, \quad \mu(g, Y) = \text{Ad}_g Y,$$

where  $g$  and  $Y$  are the coordinates on  $GL_n$  and  $\mathfrak{n}_n$ .

In these notations, the composition in question is the pair of the GIT quotient space and a potential:

$$\begin{aligned} T^*Fl_{n-1} \times T^*\text{Hom}(V_{n-1}, V_n) / {}_+GL_{n-1}, w_{n-1,n} \\ = \text{Tr}(X' \text{Ad}_g Y') + \text{Tr}(X' \phi \psi) - \text{Tr}(X'' \psi \phi), \end{aligned}$$

where  $X' \in \mathfrak{gl}_{n-1}$ ,  $X'' \in \mathfrak{gl}_n$ ,  $g, Y$  are the coordinates along  $T^*Fl_{n-1}$  and  $\psi \in \text{Hom}(V_n, V_{n-1})$ ,  $\phi \in \text{Hom}(V_{n-1}, V_n)$  are the coordinates along  $T^*\text{Hom}(V_{n-1}, V_n)$ .

The GIT quotient in last formula could be made very explicit, and we choose to describe the quotient by constructing explicit charts in the quotients. Then, we show that in each chart, we can apply the Knorrer periodicity to simplify the potential.

The GIT stable locus consists of points where  $\phi$  is injective. Thus, we can assume that there is  $k$  such that

$$\det(\phi_{\widehat{k}}) \neq 0,$$

where  $\phi_{\widehat{k}}$  is  $\phi$  with  $k$ th row removed. Let us denote the locus where the last inequality holds by  $U_k$ . It is clear that the quotient is covered by the charts  $U_k/GL_{n-1}$  and we can analyze the potential in each chart.

To simplify notations let us consider the case  $k = n$ . The natural slice to the  $GL_{n-1}$ -action is the closed subset of elements constrained by:

$$\phi_{ij} = \delta_{i,j}, \quad 1 \leq i \leq j \leq n-1.$$

Let us also denote the last row of  $\phi$  by  $v$  and the matrix of the first  $n - 1$  columns of  $\psi$  by  $\tilde{\psi}$  and the last row of  $\psi$  by  $\psi'$ . Then, the potential  $w_{n-1,n}$  becomes:

$$\mathrm{Tr}(X' \mathrm{Ad}_g Y') + \mathrm{Tr}(X' \tilde{\psi}) + \mathrm{Tr}(v X' \psi') - \mathrm{Tr}(X'' \phi \psi). \quad (5.3)$$

The sum of the first three terms is quadratic and we can apply Knorrer reduction. The reduction forces the following vanishing of the coordinates:

$$X' = 0, \quad \mathrm{Ad}_g Y' + \tilde{\psi} + \psi' v = 0.$$

Thus, the new coordinates on the Knorrer reduced space are  $X'', Y', v, \psi'$ , and in these coordinates, we have:

$$\phi \psi = \begin{bmatrix} -\psi' v - \mathrm{Ad}_g Y' & \psi' \\ -v \psi' v - v \mathrm{Ad}_g Y' & v \psi' \end{bmatrix}$$

Thus, a direct computation shows that the last term of (5.3) is equal to

$$\mathrm{Tr}(X'' \mathrm{Ad}_h Y), \quad \text{with } Y = \begin{bmatrix} Y & g^{-1} \psi' \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} g & 0 \\ v g & 1 \end{bmatrix}.$$

Hence, we proved the last formula in the chart  $U_n$  and the computations in other charts are analogous. The argument in the framed case is basically the same.  $\square$

The last proposition computes the object  $\mathcal{F}_n$  from the introduction since  $Z^f(\vec{I}_{0 \uparrow 1 \uparrow \dots n}) = \underbrace{\widehat{\square}_1 \cdots \widehat{\square}_1}_n \mathcal{O}.$

### 5.3 The categories of closed curves

The choice of defect-related objects  $Z^\bullet(\vec{I})$  determines categories assigned to closed curves: a curve  $C$  is presented as a gluing of two intervals, then its category  $Z^\bullet(C)$  must be the category of morphisms between their objects. Two curves are of special importance for our braid-related constructions.

The first type is a curve that does not intersect any defects. So the curve is a circle that lies inside of the connected component with the marking  $n$ . We denote such circle  $S_n^1$ . To a point  $p \in S_n^1$  we assign 2-category  $\mathbb{H}\mathrm{om}(\mathbf{n}^\bullet, \mathbf{0}^\bullet)$ . For brevity, we start using notation

$$[\mathbf{n}^\bullet, \mathbf{m}^\bullet] := \mathbb{H}\mathrm{om}(\mathbf{n}^\bullet, \mathbf{m}^\bullet),$$

for the corresponding 2-category.

The interval  $I$  connecting  $\mathbf{p}_n$  to itself get assigned the identity  $Z^\bullet(I) = \mathbb{L}_{\mathrm{id}}^n \in [n, n]$ . Since  $S_n^1$  is a result of gluing two such intervals, its category is the category of endomorphisms of the interval object:

$$Z^\bullet(S_n^1) = \mathrm{Hom}(\mathbb{L}_{\mathrm{id}}^n, \mathbb{L}_{\mathrm{id}}^n).$$

In the introduction explained the matrix factorization realization of the braid group (1.5) and the Drinfeld center construction of the corresponding monoidal category (1.7) from [37]. To relate these constructions to Corollary 5.1 and motivate the Conjecture 5.1, we need to study the Drinfeld category from [37]:

$$\begin{aligned} \mathbf{MF}_{\mathrm{Dr}}^{\bullet} &= \mathbf{MF}_{G_n}((\mathfrak{g}^{\bullet} \times \mathfrak{g}^{\bullet} \times G)^{\mathrm{st}}, W_{\mathrm{Dr}}^{\bullet}), \\ W_{\mathrm{Dr}}(X, Y, g) &= \mathrm{Tr} X(\mathrm{Ad}_g(Y) - Y), \\ W_{\mathrm{Dr}}^f(X, v, Y, u, g) &= W_{\mathrm{Dr}}(X, Y, g) + u^* \cdot v - u^* \cdot g v. \end{aligned} \quad (5.4)$$

**Proposition 5.2** *For  $\bullet = \emptyset, f$ , the categories  $\mathbf{MF}_{\mathrm{Dr}}^{\bullet}$  are equivalent to the categories of endomorphisms of the identity objects  $\mathbb{I}_{\mathrm{id}}^n$ :*

$$\mathbf{MF}_{\mathrm{Dr}}^{\bullet} = \mathrm{Hom}(\mathbb{I}_{\mathrm{id}}^n, \mathbb{I}_{\mathrm{id}}^n)$$

**Proof** We consider only the case  $\bullet = \emptyset$  the other case is analogous. The statement follows from the Knorrer periodicity [19]. Indeed, by our definition we have

$$\begin{aligned} \mathrm{Hom}(\mathbb{I}_{\mathrm{id}}^n, \mathbb{I}_{\mathrm{id}}^n) &= \mathbf{MF}_{\mathrm{GL}_n}(\mathrm{T}^*\mathrm{GL}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathrm{T}^*\mathrm{GL}_n / \mathrm{GL}_n, W), \\ W &= \mathrm{Tr} \phi(X - \mathrm{Ad}_g(X')) - \mathrm{Tr} \phi(X' - \mathrm{Ad}_{g'}(X)) \end{aligned}$$

where  $(\phi, g)$  and  $(\phi', g')$  are the coordinates on two copies of  $\mathrm{T}^*\mathrm{GL}_n$  and  $X, X'$  are the coordinates on two copies of  $\mathfrak{gl}_n$ . By setting  $g' = 1$ , we take slice to the  $\mathrm{GL}_n$ -orbits. On the slice, the second term in the formula for  $W$  becomes  $\mathrm{Tr} \phi'(X - X')$ . Thus, the Knorrer periodicity implies that the restriction on the locus  $X = X', \phi' = 0$  is the equivalence of the corresponding categories of matrix factorizations.  $\square$

We can linearize the potential  $W_{\mathrm{Dr}}^{\bullet}$  by introducing a new variable  $U = Yg^{-1}$ :

$$\begin{aligned} W_{\mathrm{lin}}(X, U, g) &= \mathrm{Tr}(X[g, U]), \\ W_{\mathrm{lin}}^f(X, v, U, u, g) &= W_{\mathrm{lin}}(X, U, g) + u^* \cdot v - u^* \cdot g v. \end{aligned}$$

The group  $G$  naturally embeds inside its lie algebra  $\mathfrak{g}$ ,  $j_G: G \rightarrow \mathfrak{g}$ . Induced by this embedding we have the localization functor:

$$\mathrm{loc}^{\bullet}: \mathbf{MF}_{\mathrm{Dr}}^{\bullet} \rightarrow \underline{\mathbf{MF}}_{\mathrm{Dr}}^{\bullet} = \mathbf{MF}_G((\mathfrak{g}^{\bullet} \times \mathfrak{g}^{\bullet} \times \mathfrak{g})^{\mathrm{st}}, W_{\mathrm{lin}}^{\bullet}).$$

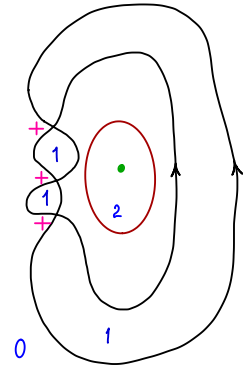
It turns out that in the framed case, the localization functor is an isomorphism:

**Proposition 5.3** [37, Proposition 5.5.2] *The localization functor  $\mathrm{loc}^f$  is an isomorphism.*

Since the potential  $W^{\mathrm{lin}}$  is linear along the last copy of  $\mathfrak{g}$ , the Koszul duality (see for example [1] or [37]) provides an equivalence:

$$\mathrm{KSZ}: \underline{\mathbf{MF}}_{\mathrm{Dr}}^f \longrightarrow \mathrm{D}^{\mathrm{per}}(\mathrm{Hilb}_n(\mathbb{C}^2)). \quad (5.5)$$

**Fig. 4** Decomposition of  $S^2_\beta$  on two discs



Thus, we have completed proof of Theorem 1.3.

The second type of a closed curve is the line that intersects our braid transversally. The line goes through the regions with the marks  $0, 1, \dots, n-1, n, n-1, \dots, 0$ . Figure 2 gives an example.

We denote such a line compactified by a point at infinity as  $S^1_{0 \uparrow 1 \uparrow \dots \uparrow n \downarrow n-1 \downarrow \dots \downarrow 0}$ . The value  $Z^\bullet$  follows immediately from the Proposition 5.1:

$$Z^f(S^1_{0 \uparrow 1 \uparrow \dots \uparrow n \downarrow n-1 \downarrow \dots \downarrow 0}) = \text{MF}_{G_n}^{\text{st}}(\mathfrak{gl}_n^\bullet \times T^* \text{Fl}_n \times T^* \text{Fl}_n, w_1 - w_2), \quad (5.6)$$

where “st” indicate that we restrict to the GIT stable locus of the corresponding space.

The categories (5.6) are closely related to the main categories of [38]. Recall that in [38], we worked with  $G_n \times B \times B$ -equivariant categories of matrix factorizations on the space

$$\mathcal{X}^\bullet := \mathfrak{gl}_n^\bullet \times (G_n \times \mathfrak{n}_n) \times (G_n \times \mathfrak{n}_n)$$

with the equivariant structure preserving the potential

$$W(X, g_1, Y_1, g_2, Y_2) = \text{Tr } X(\text{Ad}_{g_1} X_1 - \text{Ad}_{g_2} X_2),$$

where  $X$  is the coordinate in  $\mathfrak{gl}_n$  and  $g_i, Y_i$  are the coordinates in  $G$  and  $\mathfrak{n}$ .

## 5.4 Values on discs

As a final step of our construction, we need to discuss the values of TQFT on discs. The first type of disc is the disc  $D_\emptyset$  that bounds  $S^1_n$  and does not contain tautological defect point. The category  $Z^\bullet(S^2) = \text{Hom}(L_{\text{id}}^n, L_{\text{id}}^n)$  is monoidal and  $Z^\bullet(D_\emptyset)$  represents the identity object in it. Hence, we set

$$Z^f(D_\emptyset) := \mathcal{O} \in D^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2)).$$

If the disc contains the point of tautological defect then we set

$$Z^f(D_{\text{taut}}) := \Lambda^\bullet \mathcal{B} \in \text{Coh}(\text{Hilb}_n(\mathbb{C}^2)),$$

where  $\mathcal{B}$  is the tautological vector bundle.

The other important type of a disc is a half-plane  $H$  bordered by the line  $S_{0\uparrow 1\uparrow \dots \uparrow n\downarrow n-1\downarrow \dots \downarrow 0}^1$ . Its object  $Z^\bullet(H)$  lies in the monoidal category

$$Z^\bullet(S_{0\uparrow 1\uparrow \dots \uparrow n\downarrow n-1\downarrow \dots \downarrow 0}^1) = \text{End}(\vec{I}_{0\uparrow 1\uparrow \dots \uparrow n-1\uparrow n}). \quad (5.7)$$

and it depends on the configuration of defects inside  $H$ . Denote by  $H_1$  the simplest configuration which is the collection of non-intersecting curves connecting the points of the same type as in the right half-plane in Fig. 3. In this situation,  $Z^\bullet$  is the identity object:

$$Z^\bullet(H_1) = \mathcal{C}_1 \in \text{MF}_n^\bullet.$$

More generally, denote by  $H_\beta$  the half-plane containing a braid  $\beta$  as in the left of Fig. 3. The value of  $Z(H_\beta)$  for more complicated configurations of defects can be computed by using the monoidal structure of the category (5.7) through cutting  $H_\beta$  into the union:

$$H_\beta = \bigcup_k S_{\sigma_{j_k}^{\epsilon_k}},$$

where  $\beta = \sigma_{j_1}^{\epsilon_1} \dots \sigma_{j_l}^{\epsilon_l}$  and  $S_{\sigma_{j_k}^{\epsilon_k}}$  is the disc with the boundary  $S_{0\uparrow 1\uparrow \dots \uparrow n\downarrow n-1\downarrow \dots \downarrow 0}^1$  and defects inside the strip form an elementary braid on the  $j_k$ th and  $(j_k + 1)$ st stands, see the figure below for the case  $\beta = \sigma_1^3$ . Since

$$Z^\bullet(H_\beta) = Z^\bullet(S_{\sigma_{j_1}^{\epsilon_1}}) \star \dots \star Z^\bullet(S_{\sigma_{j_l}^{\epsilon_l}}),$$

it is enough to define  $Z^\bullet(S_{\sigma_k^{\pm 1}}) \in \text{MF}_n^\bullet$ , as in [38]:

$$Z^\bullet(S_{\sigma_k^{\pm 1}}) := \mathcal{C}_\pm^{(k)} \in \text{MF}^\bullet,$$

It is shown in [38] that the element  $Z(H_\beta)$  only depends on the braid  $\beta$  but not on the braid presentation, thus our disc assignment indeed is a well-defined partition function of TQFT.

Finally, we define the value of  $Z^f$  on the half-plane  $H_1^{\text{taut}}$  containing the unit braid and the tautological point defect as

$$Z^f(H_1^{\text{taut}}) := \mathcal{C}_1 \otimes \Lambda^\bullet \mathcal{B}.$$

We leave the following statement as conjecture and will provide a proof in the forthcoming publication.

**Conjecture 5.2** *The above assignments of the values of  $Z$  are the part of the data of a well-defined 3D TQFT.*

### 5.5 Value on $\mathbb{R}_\beta^2$

There are two ways to cut the plane with a closed braid defect  $\mathbb{R}_\beta^2$  into two pieces. As a result, the TQFT formalism implies two presentations of the corresponding vector space  $Z^f(\mathbb{R}_\beta^2)$  as the space of morphisms between two objects in the category of the cutting line.

The first cut splits  $\mathbb{R}_\beta^2$  in two half-planes  $H_1^{\text{taut}}$  and  $H_\beta$ , and the corresponding presentation is

$$Z^f(\mathbb{R}_\beta^2) = \text{Hom}(Z^f(H_1^{\text{taut}}), Z^f(H_\beta)) = (\text{Hom}(\mathcal{C}_\beta, \mathcal{C}_1) \otimes \Lambda^\bullet \mathcal{B})^{B^2 \times G}.$$

The vector space  $Z^f(\mathbb{R}_\beta^2)$  is triply-graded and the main result of [38] could be restated as

**Theorem 5.1** [38, Theorem 13.3] *The triply-graded vector space  $Z^f(\mathbb{R}_\beta^2)$  is an isotopy invariant of the closure of the braid  $\beta$  after a special shift of the grading.*

The second cut (see Fig. 4) splits  $\mathbb{R}_\beta^2$  into the inner disc  $D_{\text{taut}}$  (containing the tautological bundle defect) and its complement  $D_\beta^\infty$  which contains the closed braid defect. The cut goes over a circle that lies in the region marked by  $n$  and does not intersect defect lines, hence its category is

$$Z^f(S_n^1) = \text{MF}_{\text{Dr}}^f \cong \text{D}^{\text{per}}(\text{Hilb}_n(\mathbb{C}^2)) \quad (5.8)$$

of (5.4) and (5.5). The object of  $D_{\text{taut}}$  is just the defect bundle:  $Z^f(D_{\text{taut}}) = \Lambda^\bullet \mathcal{B}$ . The object of  $D_\beta^\infty$  is determined by the categorical Chern character functor

$$\text{CH}_{\text{loc}}^{\text{fs}} : \text{MF}_n^f \rightarrow Z^f(S_n^1),$$

that is,  $Z^f(D_\beta^\infty) = \text{CH}_{\text{loc}}^{\text{fs}}(Z^f(H_\beta))$ . Thus, we get the second presentation of the vector space  $Z^f(\mathbb{R}_\beta^2)$  as the Ext space between two complexes of sheaves within the derived category of 2-periodic sheaves on the Hilbert scheme  $\text{Hilb}_n(\mathbb{C}^2)$ :

$$Z^f(\mathbb{R}_\beta^2) = \mathcal{H}om(\text{CH}_{\text{loc}}^{\text{fs}}(Z^f(H_\beta)), \Lambda^\bullet \mathcal{B}) \quad (5.9)$$

The isomorphism between the vector spaces (5.8) and (5.9) is one of the key properties of the functor  $\text{CH}_{\text{loc}}^{\text{fs}}$  (one may call it a simple case of the categorical Riemann-Roch formula):

**Theorem 5.2** [37, Theorem 6.0.4] *For any  $\mathcal{C} \in \text{MF}_n$ , we have:*

$$\mathcal{H}om(\Lambda^\bullet \mathcal{B}, \text{CH}_{\text{loc}}^{\text{fs}}(\mathcal{C}))^G = \mathcal{H}om(\mathcal{C}_1 \otimes \Lambda^\bullet \mathcal{B}, \mathcal{C})^{B^2 \times G}.$$



Thus, we constructed a complex of sheaves  $S_\beta := \mathrm{CH}_{\mathrm{loc}}^{\mathrm{fs}}(\mathcal{C}_\beta)$  such that its global sections are the knot homology. In the language of TQFT the categorical Riemann–Roch and our main theorem is just a gluing property of TQFT: the Figs. 3 and 4 present two different ways of computing the same partition sum  $Z^{\mathrm{f}}(\mathbb{R}_\beta^2)$ .

## 6 Categorification of traces

In this section, we expand on the table from the introduction. The first column of the table is a summary of the previous sections. The second column is a physical interpretation of the results of [40] where a fully-faithful functor from the category of Soergel bimodules to the category of stable matrix factorizations was constructed. The last column is closely related to the work [2, 32] and forthcoming preprint of the authors [41].

### 6.1 Soergel bimodules

A slight modification of the computation in the proof of Proposition 5.1 shows that:

$$\tilde{\mathcal{F}}_n = \widehat{\blacksquare}_1 \underbrace{\widehat{\square}_1 \cdots \widehat{\square}_1}_{n-1} \mathcal{O}, \quad \tilde{\mathcal{F}}_n = ((\tilde{\mathfrak{gl}}_n \times V_n)^{\mathrm{st}}, \tilde{W}_n),$$

where  $\tilde{\mathfrak{gl}}_n = \mathrm{GL}_n \times \mathfrak{b}/B$  is the Grothendieck–Springer alteration with the moment map  $\mu : \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$  and  $\tilde{W}_n(X, z) = \mathrm{Tr}(X\mu(z))$ . The stability condition is given by requiring  $\mathbb{C}[\mu(z)]v = V_n$ .

Indeed, the computation in the Proposition 5.1 shows that the composition results into the object the  $B$ -equivariant object  $(\mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathrm{GL}_n \times \mathfrak{n} \times V_n, W'')$  where  $W''(Y', g, X, Y, v) = \mathrm{Tr}(Y'X) + \mathrm{Tr}(X\mathrm{Ad}_g Y)$ . The last potential has a quadratic term:

$$W''(Y', g, X, Y, v) = \mathrm{Tr}((\mathrm{Ad}_g^{-1}X)_{--}(Y + \mathrm{Ad}_g^{-1}Y')) + \mathrm{Tr}((\mathrm{Ad}_g^{-1}X)_{+}\mathrm{Ad}_g^{-1}Y).$$

where the sub-indices  $--$  and  $+$  stand for the strictly lower-triangular and the upper-triangular parts of the matrices. Thus, the Knörrer reduced object has potential  $\mathrm{Tr}(X'\mathrm{Ad}_g^{-1}Y)$ ,  $X' = (\mathrm{Ad}_g^{-1}X)_{+}$  and it is isomorphic to the object  $\tilde{\mathcal{F}}_n$ .

The algebra of the MOY graphs [27] has an algebraic incarnation in terms of the algebra  $\mathfrak{Br}_n^{\flat}$  of flat braids. The last algebra can be realized by means of endomorphisms of  $\tilde{\mathcal{F}}_n$

**Proposition 6.1** [40, Theorem 1.0.1] *For any  $n$ , there is a monoidal functor*

$$\Phi_n^{\flat} : \mathfrak{Br}_n \rightarrow \mathrm{Hom}(\tilde{\mathcal{F}}_n, \tilde{\mathcal{F}}_n).$$

To connect the previous discussion with the second column of the table from the introduction, we examine the picture (1.11) for case when the region  $\mathbb{B}$  contains the

circular defect of type  $NS5^{(n)}$ . In this case, the ray  $I_{rad}$  that starts at the center of the circle from the picture (1.11) intersects the domains with labels  $0, n, n-1, \dots, 1, 0$  and the corresponding category of morphisms  $Z^f(I_{rad}) = \widehat{\blacksquare}_{-n}(\tilde{\mathcal{F}})$  is computed in

**Proposition 6.2** *For any  $n$ , we have*

$$\widehat{\blacksquare}_{-n}(\tilde{\mathcal{F}}_n) = \text{Mod}(\mathbb{C}[y_1, \dots, y_n]).$$

**Proof** The functor  $\widehat{\blacksquare}_{-n}$  corresponds to the object  $\tilde{\mathbb{I}}_{0|n} = (\mathfrak{gl}_n, \text{Tr}(ZX)) \in \mathbb{H}\text{om}(\mathbf{n}^f, \mathbf{0}^f)$ . Respectively, the composition with the object  $\tilde{\mathcal{F}}_n = ((\tilde{\mathfrak{gl}}_n \times V_n)^{st}, \tilde{W}_n) \in \mathbb{H}\text{om}(\mathbf{0}^f, \mathbf{n}^f)$  is by the  $G \times B$ -equivariant object:

$$\widehat{\blacksquare}_{-n}(\tilde{\mathcal{F}}_n) = \tilde{\mathbb{I}}_{0|n} \circ \tilde{\mathcal{F}}_n = (\mathfrak{gl}_n \times \tilde{\mathfrak{gl}}_n \times V_n, \text{Tr}(ZX) + \text{Tr}(X\text{Ad}_g Y)). \quad (6.1)$$

The potential of the last object is quadratic in  $Z$  and  $X$  thus after the Knörrer reduction we obtain the  $G \times B$  equivariant object  $G \times B$ -equivariant object  $(G \times \mathfrak{n} \times V_n)^{st}$ . The action of  $G \times B$  on this object is free and a slice to the orbits of the action is defined by the triple  $(1, K(y), v^0)$ , where  $K(y)_{ij} = \sum_{i=1}^n y_i \delta_{i,i} + \sum_{i=1}^{n-1} \delta_{i,j-1}$ ,  $v_i^0 = \delta_{i,n}$ .  $\square$

The category of endomorphism of  $\text{Mod}(R_n)$ ,  $R_n = \mathbb{C}[y_1, \dots, y_n]$  is the category of two-periodic complexes of  $R_n$ -bimodules  $\mathbb{B}\text{im}_n$ . Respectively, the formula (6.1) implies that there is a monoidal functor:

$$\mathbb{B} : \text{Hom}(\tilde{\mathcal{F}}_n, \tilde{\mathcal{F}}_n) : \rightarrow \mathbb{B}\text{im}_n.$$

Combining the last functor with  $\Phi^b$ , we obtain a realization of the algebra  $\mathfrak{B}\mathfrak{r}_n^b$  inside  $\mathbb{B}\text{im}_n$ . Thus if we have two radial segments  $I_{rad}, I'_{rad}$  with defects between the rays forming  $\beta \in \mathfrak{B}\mathfrak{r}_n^b$ , the value of the partition function  $Z^f$  on the segment is  $\mathbb{B} \circ \Phi^b(\beta)$ .

The bimodule realization of  $\mathfrak{B}\mathfrak{r}_n^b$  inside  $\mathbb{B}\text{im}$  used for  $\text{HHH}_{alg}$  is due to Soergel [43]:

$$\Phi^S : \mathfrak{B}\mathfrak{r}_n^b \rightarrow \mathbb{S}\text{bim}_n \subset \mathbb{B}\text{im}_n, \quad \text{HHH}_{alg}(\beta) := \text{HH}_*(\Phi^S(\beta)),$$

where  $\text{HH}_* = \text{Tor}_R$  is functor of Hochschild homology. The Hochschild homology functor is an trace functor  $\text{HHH}_{alg}(\alpha \cdot \beta) = \text{HHH}_{alg}(\beta \cdot \alpha)$

We show in [40] that two realizations of  $\mathfrak{B}\mathfrak{r}_n^b$  actually match and there is a natural isomorphism between the corresponding trace functors:

**Theorem 6.1** [40, Theorem 1.3.1] *For any  $n$  we have a commuting diagram of monoidal functors:*

$$\begin{array}{ccc} \mathfrak{B}\mathfrak{r}_n^b & \xrightarrow{\Phi^S} & \mathbb{S}\text{bim}_n \\ \downarrow \Phi^b & & \downarrow \\ \text{Hom}(\tilde{\mathcal{F}}_n, \tilde{\mathcal{F}}_n) & \xrightarrow{\mathbb{B}} & \mathbb{B}\text{im}_n \end{array}$$

Moreover, the functors intertwine the trace functors:

$$HH^*(\Phi^S(\beta)) = \mathcal{H}om(\Phi^b(\beta), \Phi^b(1) \otimes \Lambda^* V_n).$$

## 6.2 Annular homology

In this section, we explain the relations between the objects in the last column of the table from the introduction. The column has a natural physical explanation in terms of NS5 and D5 branes as we explained before, see Sects. 1.2 and 2.5.3. The mathematical side of the story is mostly contained in the work of Nakajima–Takayama [31, 32] and Anno–Nandakumar [2].

The work of Nakajima–Takayama [32] explains realization of the  $GL_n$ -Slodowy slices and their resolutions in terms of Cherkis quivers. These quiver varieties are related to the objects we study by the linear Koszul duality. We provide more details below.

Let us also point out that the relation between the Slodowy slices has a long history. Indeed, a quiver realization of the  $GL_n$ -Slodowy slices was conjectured by Nakajima [13] and proven by Maffei [26]. On other hand, the relation between the Slodowy slices, quiver varieties and slices in the Beilinson–Drinfeld Grassmannian were studied by Mirkovic and Vybornov [28, 29].

The quiver realization of the Slodowy slices from the above-mentioned work is related to Nakajima–Takayama realization [32] by the Hannany–Witten transitions [16]. The mathematical theory of Hannany–Witten transitions was developed by Nakajima–Takayama [32].

Let us denote by  $Fl_\mu$ ,  $\mu_n \leq \mu_{n-1} \leq \dots$ ,  $\sum_{i=1}^m \mu_i = n$  the partial flag variety consisting of nested subspaces of dimension  $M_1 = \mu_1$ ,  $M_2 = \mu_1 + \mu_2$ ,  $M_3 = \mu_1 + \mu_2 + \mu_3$ ,  $\dots$ . Similarly, we denote by  $G_\mu$  the product of the groups  $GL_{M_1} \times GL_{M_2} \times \dots \times GL_{M_{m-1}}$ . Respectively,  $\mathcal{N}(\mu) \subset \mathfrak{gl}_n$  is the nilpotent orbit with the Jordan blocks of size  $\mu_i$  and  $\mathcal{S}(\mu)$  is a transversal slice to the orbit  $\mathcal{N}(\mu)$ .

**Proposition 6.3** [32, Theorem 7.11] *The  $G_\lambda \times GL_n \times G_\mu$ -quotient of the quiver  $Q^{bow}(\lambda, \mu)$  pictured below*

$$\begin{array}{ccccccc} \mathbb{C} & & \mathbb{C} & & \dots & & \mathbb{C} \\ \uparrow & \searrow & \uparrow & \searrow & & \searrow & \uparrow \\ \mathbb{C}^{L_1} & \rightarrow & \mathbb{C}^{L_3} & \rightarrow & \dots & \rightarrow & \mathbb{C}^{L_{l-1}} \rightarrow \mathbb{C}^n \rightleftharpoons \mathbb{C}^{M_{m-1}} \rightleftharpoons \dots \rightleftharpoons \mathbb{C}^{M_2} \rightleftharpoons \mathbb{C}^{M_1} \end{array}$$

is isomorphic to  $\overline{\mathcal{N}}(\mu) \cap \mathcal{S}(\lambda)$  and the quotient of the stable locus is the simultaneous resolution  $\mathcal{R}_{\lambda, \mu} = p^{-1}(\overline{\mathcal{N}}(\mu) \cap \mathcal{S}(\lambda)) \subset T^*Fl_\mu$  with  $p : T^*Fl(\mu) \rightarrow \mathfrak{g}$  being the Springer resolution map.

The relevant object in the TQFT setting discussed in our paper is a segment  $I^{\lambda, \mu}$  with ends in the regions marked with  $\mathbf{0}$  and such that the list of defects it traverses is  $NS5^{\lambda_1}, NS5^{\lambda_2}, \dots, NS5^{\lambda_l}, D5^{\mu_m}, \dots, D5^{\mu_1}$ . As we explained in Sect. 2.5.3, the

value  $Z(I^{\lambda, \mu})$  of partition function is given by composition of morphisms

$$Z(I^{\lambda, \mu}) = \check{\square}_{\lambda_1} \circ \check{\square}_{\lambda_2} \circ \dots \circ \check{\square}_{\lambda_l} \circ \widehat{\square}_{\mu_m} \circ \dots \circ \widehat{\square}_{\mu_2} \circ \widehat{\square}_{\mu_1}.$$

Thus, the previous proposition gives a geometric description of the morphism

$$Z(I^{\lambda, \mu}) = (Q^{bow} \times \mathrm{Lie}(G_\lambda \times \mathrm{GL}_n \times G_\mu), W^{\lambda, \mu}(z, x) = \mathrm{Tr}(\mu(z)x)).$$

To specialize to the case treated in the table in the introduction, we need to set  $\mu = (1^n)$  and  $\lambda = (k, n-k)$ . Since the braid group  $\mathfrak{B}\mathfrak{r}_n$  acts on  $\widehat{\square}_1^n$ , there is monoidal functor

$$\Phi^\lambda : \mathfrak{B}\mathfrak{r}_n \rightarrow \mathrm{Hom}(Z(I^{\lambda, 1^n}), Z(I^{\lambda, 1^n})).$$

In the setting of the last column of the table from the introduction, the radial rays in the picture (1.11) are the homotopic to the interval  $I^{(k, n-k), (1^n)}$ . Thus, the last column of the table is equivalent to the following conjecture that we plan to address in the forthcoming paper [41]:

**Conjecture 6.1** *For any  $n$ , a trace functor  $\mathrm{Tr} : \mathfrak{B}\mathfrak{r}_n \rightarrow 2 - \mathrm{grVect}$*

$$\mathrm{Tr}_{k, n-k}(\beta) = \mathcal{H}om(\Phi_{(k, n-k)}(\beta), \Phi_{(k, n-k)}(1_n))$$

*categorifies the trace  $\mathrm{Tr}(\beta)[H_{1^n, \lambda+k}^\lambda]$ .*

The space  $H_{1^n, \lambda+k}^\lambda$  is the weight  $k$  part of  $L_1^{\otimes n}$ , it is of dimension  $\binom{n}{k}$ . To provide an evidence to the conjecture, we compute the Euler characteristic of the trace for the identity.

**Proposition 6.4** *For any  $n, k$ , we have:*

$$\chi(\mathrm{Tr}_{k, n-k}(1_n)) = \binom{n}{k}.$$

**Proof** First, let us observe that by the linear Koszul duality for the matrix factorizations and by the Proposition 6.3, the category of morphism  $\mathrm{Hom}(Z(I^{\lambda, \mu}), Z(I^{\lambda, \mu}))$  is equivalent to the dg category two-periodic complexes  $D_{\mathbb{C}^* \times \mathbb{C}^*}^{per}(\mathcal{R}_{\lambda, \mu} \times \mathcal{R}_{\lambda, \mu})$ . Respectively,  $\Phi_\lambda(1_n)$  is the structure sheaf of the diagonal  $\Delta \subset \mathcal{R}_{\lambda, 1^n} \times \mathcal{R}_{\lambda, 1^n}$ .

Thus,  $\mathrm{Tr}(1_n)$  is the homology self-intersection of the diagonal and since  $\mathcal{R}_{\lambda, 1^n}$  is smooth we get  $\mathrm{Tr}(1_n) = H^*(\oplus_i \Omega_{\mathcal{R}_{\lambda, 1^n}}^i)$ . Since the  $\mathbb{C}^* \times \mathbb{C}^*$ -fixed locus in  $\mathcal{R}_{\lambda, 1^n}$  is zero-dimensional, the localization formula implies that Euler characteristics  $\chi(\mathrm{Tr}(1_n))$  are equal to the number of torus fixed points.

The 3D mirror partner of  $\mathcal{R}_{(k, n-k), 1^n}$  is the Grassmannian  $T^*Gr(k, n)$  [32]. The mirror symmetric partners have the same number of the torus fixed points and  $T^*Gr(k, n)$  has  $\binom{n}{k}$  fixed points.  $\square$

Finally, let us remark that in the work [2], the category of coherent sheaves on  $\mathcal{R}_{(k,k),(1^{2k})}$  was used to construct the invariants of the affine tangles and relate this invariant to the annular Khovanov homology [3]. The under/over-crossing the generators of the affine tangles in [2] match with the braid generators used in this paper. The cap/cup of the affine tangle monoid from [2] have a natural interpretation in terms of NS5 defects. Thus, in the forthcoming paper [41], we also address the

**Conjecture 6.2** *The construction of the trace functor  $\mathrm{Tr}_{(k,k)}$  can be extended to affine tangles and the corresponding invariant provides a realization of the  $sl_2$  annular Khovanov homology.*

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## Declarations

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

## References

1. Arkhipov, S., Kanstrup, T.: Equivariant matrix factorizations and Hamiltonian reduction. *Bull. Korean Math. Soc.* **54**(5), 1803–1825 (2015). [arXiv:1510.07472v1](#)
2. Anno, R., Nandakumar, V.: Exotic t-structures for two-block Springer fibers (2016). [arXiv:1602.00768v1](#)
3. Asaeda, M., Przytycki, J., Sikora, A.: Categorification of the Kauffman bracket skein module of I-bundles over surfaces. *Algebra Geom. Topol.* **4**(2), 1177–1210 (2004)
4. Arnold, V.: *Mathematical Methods of Classical Mechanics*. Springer, New York (1989)
5. Baez, J., Neuchl, M.: Higher dimensional algebra. *Adv. Math.* **121**(2), 196–244 (1996)
6. Caldararu, A., Willerton, S.: The Mukai pairing, I: a categorical approach (2007). [arXiv:0707.2052v1](#)
7. Dimofte, T., Garner, N., Hilburn, J., Oblomkov, A., Rozansky, L.: (**in preparation**) (2023)
8. Dunfield, N., Gukov, S., Rasmussen, J.: The superpolynomial for knot homologies. *Exp. Math.* **15**(2), 129–159 (2006)
9. Etingof, P., Gelaki, S., Nikshych, D., Ostrik, V.: *Tensor categories*. Mathematical Surveys and Monographs (2015)
10. Garner, R., Gurski, N.: The low-dimensional structures formed by tricategories. *Math. Proc. Camb. Philos. Soc.* **146**(03), 551 (2009)
11. Gaitsgory, D., Rozenblyum, N.: *A study in derived algebraic geometry*. Mathematical Surveys and Monographs (2017)
12. Gaiotto, D., Witten, E.: Supersymmetric boundary conditions in  $\mathcal{N} = 4$  super yang-mills theory. *J. Stat. Phys.* **135**(5–6), 789–855 (2009)
13. Nakajima, H.: Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras. *Duke Math. J.* **76**(2), 365–416 (1994)
14. Hiraku, N.: Introduction to quiver varieties-for ring and representation theorists. In: *Proceedings of the 49th Symposium on Ring Theory and Representation Theory* (2017)
15. Hanany, A., Witten, E.: Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nucl. Phys. B* **492**(1–2), 152–190 (1997)
16. Hanany, Amihay, Witten, Edward: Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nucl. Phys. B* **492**(1–2), 152–190 (1997)

17. Jones, V.F.R.: Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126**(2), 335–392 (1987)
18. Khovanov, M.: Triply-graded link homology and Hochschild homology of Soergel bimodules. *Int. J. Math.* **18**(08), 869–885 (2007)
19. Knorrer, H.: Cohen–Macaulay modules on hypersurface singularities. I. *Invent. Math.* **1**, 153–164 (1987)
20. Khovanov, M., Rozansky, L.: Matrix factorizations and link homology. II. *Geom. Topol.* **12**, 1387–1425 (2008)
21. Kapustin, A., Rozansky, L.: Three-dimensional topological field theory and symplectic algebraic geometry II. *Commun. Number Theory Phys.* **4**(3), 463–549 (2010)
22. Kapustin, A., Rozansky, L., Saulina, N.: Three-dimensional topological field theory and symplectic algebraic geometry I. *Nucl. Phys. B* **816**(3), 295–355 (2009)
23. Kapranov, M.M., Voevodsky, V.A.: 2-categories and Zamolodchikov tetrahedra equations. In: *Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods*, pp. 177–259 (1994)
24. Kapustin, A., Witten, E.: Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.* **1**(1), 1–236 (2007)
25. Lurie, J.: On the classification of topological field theories. *Curr. Dev. Math.* **2008**(1), 129–280 (2008)
26. Maffei, A.: Quiver varieties of type A. *Comment. Math. Helv.* **80**, 1–27 (2005)
27. Murakami, H., Ohtsuki, T., Yamada, Sh.: HOMFLY polynomial via an invariant of colored plane graphs. *Enseign. Math.* **44**(2), 325–360 (1998)
28. Mirković, Ivan, Vybornov, Maxim: On quiver varieties and affine Grassmannians of type A. *Comptes Rendus Math.* **336**(3), 207–212 (2003)
29. Mirković, Ivan, Vybornov, Maxim, Krylov, Vasily: Comparison of quiver varieties, loop Grassmannians and nilpotent cones in type A. *Adv. Math.* **407**, 108397 (2022)
30. Nakajima, H.: *Lectures on Hilbert Schemes of Points on Surfaces*. University Lecture Series (1999)
31. Nakajima, H.: Geometric Satake correspondence for affine Kac–Moody Lie algebras of type A (2018). [arXiv:1812.11710v1](https://arxiv.org/abs/1812.11710v1)
32. Nakajima, H., Takayama, Y.: Cherkis bow varieties and coulomb branches of quiver gauge theories of affine type A. *Sel. Math. New Ser.* **23**(4), 2553–2633 (2017)
33. Oblomkov, A., Rozansky, L.: HOMFLYPT homology of Coxeter links (2017). [arXiv:1706.00124v1](https://arxiv.org/abs/1706.00124v1)
34. Oblomkov, A., Rozansky, L.: 3D TQFT and HOMFLYPT homology (2018) [arXiv:1812.06340](https://arxiv.org/abs/1812.06340)
35. Oblomkov, A., Rozansky, L.: A categorification of a cyclotomic Hecke algebra (2018). [arXiv:1801.06201v1](https://arxiv.org/abs/1801.06201v1)
36. Oblomkov, A., Rozansky, L.: Affine braid group, JM elements and knot homology. *Transformation Groups* (2018)
37. Oblomkov, A., Rozansky, L.: Categorical Chern character and braid groups (2018). [arXiv:1811.03257v1](https://arxiv.org/abs/1811.03257v1)
38. Oblomkov, A., Rozansky, L.: Knot homology and sheaves on the Hilbert scheme of points on the plane. *Sel. Math. New Ser.* **24**(3), 2351–2454 (2018)
39. Oblomkov, A., Rozasky, L.: Categorical Chern character and Hall algebras (**in preparation**) (2019)
40. Oblomkov, A., Rozansky, L.: Soergel bimodules and matrix factorizations (2020). [arXiv:2010.14546v1](https://arxiv.org/abs/2010.14546v1)
41. Oblomkov, A., Rozansky, L.: Matrix factorizations and annular homology (**in preparation**) (2022)
42. Rimanyi, R., Rozansky, L.: New quiver-like varieties and Lie superalgebras (2021). [arXiv:2105.11499v1](https://arxiv.org/abs/2105.11499v1)
43. Soergel, W.: Langlands’ philosophy and Koszul duality. In: *Algebra-Representation Theory* (Constanta, 2000), pp. 379–414 (2001)

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