HOMFLY-PT HOMOLOGY OF COXETER LINKS

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Dedicated to the memory of Jim Humphreys

Abstract. A Coxeter link is a closure of a product of two braids, one being a quasi-Coxeter element and the other being a product of partial full twists. This class of links includes torus knots $T_{n,k}$ and torus links $T_{n,nk}$. We identify the knot homology of a Coxeter link with the space of sections of a particular line bundle on a natural generalization of the punctual locus inside the flag Hilbert scheme of points in $\mathbb{C}^2$.

1. Introduction

In the seminal paper [Jon87] Jones introduced what would be later called the HOMFLY-PT polynomial invariant $P(L)$ of a link L in $\mathbb{R}^3$. Besides the definition, the paper has many amazing results and computations. In particular, Section 9 of [Jon87] contains a proof of a formula for the HOMFLY-PT invariant of torus knots $T_{m,n}$. Later, the HOMFLY-PT invariant was upgraded to the homology theory [KR08a], [KR08b]. In this paper, we demonstrate that the Jones formula has a natural generalization to the homology theory for a special class of torus links.

Consider the plane $\mathbb{C}^2$ with the action of the group $\mathbb{C}^*$ denoted as $\mathbb{C}^*_q$: $\lambda\cdot(x,y) = (\lambda^2 x, \lambda^{-2} y)$. This action extends to the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ which is a variety of ideals $I \subset \mathbb{C}[x,y]$ of codimension n. The tautological vector bundle $\mathcal{B}$ whose fiber over I is the vector space dual to $\mathbb{C}[x,y]/I$ is naturally $\mathbb{C}^*_q$-equivariant. Combining the localization formula of Atiyah and Bott [AB83] with the result of Haiman [Hai02] we get an algebro-geometric version of the Jones formula:

$$P(T_{1+kn,n}) = \sum_{i=0}^{n} \dim_q \left(\mathrm{H}^0\big(\mathrm{Hilb}_n(\mathbb{C}^2), \mathcal{O}_Z \otimes L^k \otimes \textstyle\bigwedge^i \mathcal{B}\big)\right) a^i,$$

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where $Z \subset \mathrm{Hilb}_n(\mathbb{C}^2)$ is the punctial Hilbert scheme consisting of ideals I with support at $(0,0) \in \mathbb{C}^2$, and $\dim_q$ is the dimension graded by $\mathbb{C}^*_q$-weights.

Many authors [AS12], [GN15], [GORS14], [ORS18] suggested that the Poincare polynomial $\mathscr{P}(T_{1+kn,n})$ of the triply graded HOMFLY-PT homology [KR08a], [KR08b] has a similar interpretation if one augments the action of $\mathbb{C}^*_q$ to that of $T_{\mathrm{sc}} = \mathbb{C}^*_q \times \mathbb{C}^*_t$, where $\mathbb{C}^*_t$: $\mu \cdot (x,y) = (x, \mu^2 y)$, and uses the $\mathbb{C}^*_{q,t}$-weighted dimension:

$$\mathscr{P}(T_{1+kn,n}) = \sum_{i=0}^{n} \dim_{q,t} \left(\mathrm{H}^0\left(\mathrm{Hilb}_n, \mathcal{O}_Z \otimes L^k \otimes \wedge^i \mathcal{B}\right)\right) a^i. \tag{1.1}$$

While we were finishing the preprint of this paper, M. Hogancamp published a proof of the conjecture [Hog17], [Mel22]. He used the construction of the HOMFLYPT homology via Soergel bimodules and matched combinatorics of the complexes of bimodules that appear in knot homology of torus knots with the combinatorics of the generalized Catalan numbers, the latter related to the sections of L^k by a combination of the results [CM18], [Mel21], [Hai02].

The paper [Hog17] is a real tour de force in combinatorics and homological algebra, however it does not provide a natural explanation for the appearance of $\mathrm{Hilb}(\mathbb{C}^2)$ in knot homology. When the conjecture (1.1) appeared, the available constructions for triply graded homology had no obvious connections with coherent sheaves on this variety.

A direct relation between the triply graded knot homology and $\mathbb{C}^*_{q,t}$-equivariant coherent sheaves on $\mathrm{Hilb}(\mathbb{C}^2)$ was established by the authors [OR18b] (see also the paper [GNR20] where a K-theoretic version of this relation is suggested). Recently, it was also shown [OR20] by the authors that the link homology from [OR18b] coincides with the Khovanov–Rozansky link homology [KR08a].

From the papers [OR18b], as well as [GN15], [GNR20], it is clear that the natural home for the algebro-geometric version of the HOMFLY-PT homology is the category of the quasi-coherent sheaves on the *nested* Hilbert scheme $\mathrm{Hilb}_{1,n}$ parameterizing chains of ideals $I_1 \supset I_2 \supset \cdots \supset I_n$ with support of I_i/I_{i+1} being a point on the line $y = 0$. There is a natural analog $Z_{1,n}$ of the punctual Hilbert scheme Z in the nested case which consists of the chains of ideals with the support of I_i/I_{i+1} at $(x,y) = (0,0)$. However, the natural analogue of $\mathcal{O}_Z$ turns to be the Koszul complex of the defining equations for $Z_{1,n}$ which we denote by $[\mathcal{O}_{Z_{1,n}}]^{\mathrm{vir}}$ and define in Section 4. Finally, the weights of $\mathbb{C}^*_t$-action are combined with homological degree which means that all differentials have $\mathbb{C}^*_t$-weight one and the variable y has homological degree two.

The main result of this paper is the following.

Theorem 1.0.1. *For any positive n, k we have*

$$\mathscr{P}(T_{1+kn,n}) = \sum_{i=0}^{n} \dim_{q,t} \left(\mathrm{H}^*\left(\mathrm{Hilb}_{1,n}, [\mathcal{O}_{Z_{1,n}}]^{\mathrm{vir}} \otimes L^k \otimes \wedge^i \mathcal{B}\right)\right) a^i,$$

where $\mathscr{P}$ is the Poincare polynomial for the triply graded homology[1]*.*

[1] In this paper we use the term "the triply graded homology" for the homology theory from [OR18b]; it is shown in [OR20] that the homology from [OR18b] are isomorphic to the triply-graded homology of [KR08a].

This paper is a natural continuation of our previous papers [OR17], [OR18b]. In the second paper we prove the relation between the homology of the closure $L(\beta)$ of $\beta \in Br_n$ and of closure of $\beta \cdot \delta^{\vec{k}}$ where $\delta^{\vec{k}} := \prod_{i=1}^n \delta_i^{k_i}$ is the product of the JM elements

$$\delta_i := \sigma_i \sigma_{i+1} \cdots \sigma_{n-1}^2 \cdots \sigma_{i+1}\sigma_i, \quad i = 1, \dots, n-1,$$

where σ_i are the standard generators for the braid group $\mathfrak{Br}_n$. The above mentioned formula for the homology of $T_{1+kn,n}$ is obtained by applying result of [OR17] for $\beta = \sigma_1 \cdots \sigma_{n-1}$ and $k_1 = \ldots = k_n = k$. To apply the result of [OR17] we need to analyze the sheaf-theoretic object that the theory from [OR18b] assigns to the braid β which we call the Coxeter braid.

More generally, we study the sheaf-theoretic object that is attached by the theory from [OR18b] to the general quasi-Coxeter braid:

$$\mathrm{cox}_S := \overrightarrow{\prod_{i \notin S}} \sigma_i,$$

where $S \subset \{1, \dots, n-1\}$ is a subset and the product is taken in the descending order of the indices. In particular, we identify the homology of the closure of element $\mathrm{cox}_S \cdot \delta^k$ for any S and k. We call these closures Coxeter links. This is a wide class of links which includes the torus links $T_{m,n}$, $(m,n) = 1$. The class also contains the torus link $T_{n,kn}$.

The Khovanov–Rozansky homology of the links $T_{n,nk}$ and knots $T_{n,k}$ were studied in [EH19] and in [Hog17], [Mel22], and it would be interesting to make a connection between our results and technique of these papers.

The nested Hilbert scheme $\mathrm{Hilb}_{1,n}$ carries a natural line bundle $\mathcal{L}_i$ whose fiber over $I_\bullet$ is the quotient I_i/I_{i+1}. For any subset $S \subset \{1, \dots, n-1\}$, we define $Z_{1,n}^S \subset \mathrm{Hilb}_{1,n}$ to be a subscheme defined by the condition $\mathrm{supp}(I_{i-1}/I_i) = \mathrm{supp}(I_i/I_{i+1}))$ for all $i \notin S$. We prove the following.

Theorem 1.0.2. *For any $S \subset \{1, \dots, n-1\}$ and $\vec{k} \in \mathbb{Z}^{n-1}$,*

$$\mathscr{P}\big(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})\big) = \sum_{i=0}^n \dim_{q,t} \Big(\mathrm{H}^*\big(\mathrm{Hilb}_{1,n}, [\mathcal{O}_{Z_{1,n}^S}]^{\mathrm{vir}} \otimes \mathcal{L}^{\vec{k}} \otimes \wedge^i \mathcal{B}\big)\Big) a^i.$$

We prove this theorem in Sections 3 and 4. We also provide some short overview of the methods of [OR18b] in Section 2. If the vector $\vec{k}$ is sufficiently positive, we can use Atiyah–Bott localization [AB83] to compute the graded dimensions in this formula similar to the one from [GNR20], see Theorem 1.0.3 below.

It turns out that the localization approach only works under some vanishing conditions on the sheaf homology like in [Hai02]. In Section 6 we show that the easiest version of the vanishing condition [OR18a] implies the following.

Theorem 1.0.3. *For $\vec{k} \in \mathbb{Z}_{>0}$ such that $k_1 > k_2 > \cdots > k_{n-1}$, there is M such that we have the following explicit formula for the knot invariant:*

$$\mathscr{P}\big(L(1 \cdot \delta^{\vec{b}})\big) = \sum_{p \in \mathrm{Hilb}_{1,n}^T} \Omega_p(Q,T,a)\, Q^{\vec{b}\cdot w_x(p)}\, T^{\vec{b}\cdot w_y(p)},$$

where $Q = q^2/t^2, T = t^2$, $\vec{b} = \vec{k} + r\vec{1}$, $\vec{1} = (1, \dots, 1)$, $r > M$, *and the weight* Ω *and vectors* $w_\bullet(p)$ *can be explicitly computed and depend only on* p.

The formulas for $\Omega_p(Q, T, a)$ and $w_\bullet(p)$ are given in Section 6 and theorem is proved at the end of Section 6.4. We conjecture that the theorem can be strengthened in two direction: we can replace the Coxeter braid $1 = \mathrm{cox}_{\{1,\dots,n-1\}}$ by any Coxeter braid cox_S and we give a precise criterion for the vector $\vec{k}$ to be sufficiently positive. We formulate these conjectures in Section 6.5 and provide evidence in their support. Note that the weight Ω_p appears to be equal to the localization weight from the main formula of [GN15].

We do not expect a simple localization formula for the Poincare polynomials of nonpositive links. The examples of such links are discussed in Sections 5 and 6.

The first author was lucky to have an advice of Jim Humphreys on multiple subjects of mathematics. In particular, quasi-Coxeter braids were suggested by Jim as a class of braids that tends to lead to more computable objects. This suggestion was a starting point of this paper.

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2. Matrix factorizations and knot invariants

The construction of link invariants in [OR18b] is based on a homomorphism from the braid group to a special monoidal category of matrix factorizations. The main result of this paper follows from the explicit computation of the images of Coxeter braids.

2.1. Matrix factorizations

Matrix factorizations were introduced by Eisenbud [Eis80] and further developed by Orlov [Orl04], see [Dyc11] for a review. Here we present only the basic definitions and omit proofs.

The category of matrix factorizations $\mathrm{MF}(\mathcal{Z}, W)$ is a triangulated category based on an affine variety $\mathcal{Z}$ and a function $W \in \mathbb{C}[\mathcal{Z}]$. An object of this category is a $\mathbb{Z}_2$-graded free $\mathbb{C}[\mathcal{Z}]$-module $M = M_0 \oplus M_1$ of finite rank equipped with a degree one endomorphism D called a curved differential:

$$\mathcal{F} = (M_0 \oplus M_1, D), \quad D : M_i \to M_{i+1}, \quad D^2 = W \,\mathrm{id}_M.$$

Given $\mathcal{F} = (M, D)$ and $\mathcal{G} = (N, D')$, the linear space of morphisms $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ consists of the homomorphisms of $\mathbb{C}[Z]$-modules $\phi = \phi_0 \oplus \phi_1$, $\phi_i \in \mathrm{Hom}(M_i, N_i)$ such that $\phi \circ D = D' \circ \phi$. Two morphisms $\phi, \rho \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ are homotopic if there is homomorphism of $\mathbb{C}[\mathcal{Z}]$-modules $h = h_0 \oplus h_1$, $h_i \in \mathrm{Hom}(M_i, N_{i+1})$ such that $\phi - \rho = D' \circ h - h \circ D$.

In the paper [OR18b], we introduced a notion of the equivariant matrix factorizations which we explain below. First, recall the construction of the Chevalley–Eilenberg complex.

2.2. Chevalley–Eilenberg complex

Suppose that $\mathfrak{h}$ is a Lie algebra. Chevalley–Eilenberg complex $\mathrm{CE}_{\mathfrak{h}}$ is the complex $(V_\bullet(\mathfrak{h}), d)$ with $V_p(\mathfrak{h}) = U(\mathfrak{h}) \otimes_{\mathbb{C}} \wedge^p \mathfrak{h}$ and differential $d_{\mathrm{ce}} = d_1 + d_2$ where

$$d_1(u \otimes x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_p,$$
$$d_2(u \otimes x_1 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_p.$$

Let us denote by Δ the standard map $\mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ defined by $x \mapsto x \otimes 1 + 1 \otimes x$. Suppose V and W are modules over the Lie algebra $\mathfrak{h}$. Then we use notation $V \overset{\Delta}{\otimes} W$ for the $\mathfrak{h}$-module which is isomorphic to $V \otimes W$ as vector space, the $\mathfrak{h}$-module structure being defined by Δ. Respectively, for a given $\mathfrak{h}$-equivariant matrix factorization $\mathcal{F} = (M, D)$, we denote by $\mathrm{CE}_{\mathfrak{h}} \overset{\Delta}{\otimes} \mathcal{F}$ the $\mathfrak{h}$-equivariant matrix factorization $(\mathrm{CE}_{\mathfrak{h}} \overset{\Delta}{\otimes} \mathcal{F}, D + d_{\mathrm{ce}})$. The $\mathfrak{h}$-equivariant structure on $\mathrm{CE}_{\mathfrak{h}} \overset{\Delta}{\otimes} \mathcal{F}$ originates from the left action of $U(\mathfrak{h})$ that commutes with the right action of $U(\mathfrak{h})$ used in the construction of $\mathrm{CE}_{\mathfrak{h}}$.

A slight modification of the standard fact that $\mathrm{CE}_{\mathfrak{h}}$ is the resolution of the trivial module implies that $\mathrm{CE}_{\mathfrak{h}} \overset{\Delta}{\otimes} M$ is a free resolution of the $\mathfrak{h}$-module M.

2.3. Equivariant matrix factorizations

Let us assume that the Lie algebra $\mathfrak{h}$ acts by derivations on the ring of regular functions on $\mathcal{Z}$ and F is a function anihilated by $\mathfrak{h}$. Then we can construct the following triangulated category $\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z}, W)$.

Definition 2.3.1. The objects of the category $\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z}, W)$ are the triples:

$$\mathcal{F} = (M, D, \partial), \quad (M, D) \in \mathrm{MF}(\mathcal{Z}, W)$$

where

$$M = M^0 \oplus M^1,\ M^i = \mathbb{C}[\mathcal{Z}] \otimes V^i,\ V^i \in \mathrm{Mod}_H,\ \partial \in \bigoplus_{i>j} \mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]}(\wedge^i \mathfrak{h} \otimes M, \wedge^j \mathfrak{h} \otimes M),$$

and D is an odd endomorphism $D \in \mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]}(M, M)$ such that

$$D^2 = W \,\mathrm{id}_M, \quad D_{\mathrm{tot}}^2 = W \,\mathrm{id}_M, \quad D_{\mathrm{tot}} = D + d_{\mathrm{ce}} + \partial,$$

where the total differential D_{tot} is an endomorphism of $\mathrm{CE}_{\mathfrak{h}} \overset{\Delta}{\otimes} M$, that commutes with the $U(\mathfrak{h})$-action. The morphism ∂ is called *correction differential.*

Note that we do not impose the equivariance condition on the differential D in our definition of matrix factorizations. On the other hand, if $\mathcal{F} = (M, D) \in$

$\mathrm{MF}(\mathcal{Z},W)$ is a matrix factorization with D that commutes with $\mathfrak{h}$-action on M, then $(M,D,0)\in\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},W)$. We call such matrix factorization *strictly equivariant.*

Given two $\mathfrak{h}$-equivariant matrix factorizations $\mathcal{F}=(M,D,\partial)$ and $\widetilde{\mathcal{F}}=(\widetilde{M},\widetilde{D},\widetilde{\partial})$, the space of morphisms $\mathrm{Hom}(\mathcal{F},\widetilde{\mathcal{F}})$ consists of homotopy equivalence classes of elements $\Psi\in\mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]^{\mathfrak{h}}}(\Lambda^\bullet\mathfrak{h}\otimes M,\Lambda^\bullet\mathfrak{h}\otimes\widetilde{M})$ such that $\Psi\circ D_{\mathrm{tot}}=\widetilde{D}_{\mathrm{tot}}\circ\Psi$ and Ψ commutes with $U(\mathfrak{h})$-action on $\mathrm{CE}_{\mathfrak{h}}\overset{\Delta}{\otimes}M$. Two map $\Psi,\Psi'\in\mathrm{Hom}(\mathcal{F},\widetilde{\mathcal{F}})$ are homotopy equivalent if there is

$$h\in\mathrm{Hom}_{\mathbb{C}[\mathcal{Z}]}(\mathrm{CE}_{\mathfrak{h}}\overset{\Delta}{\otimes}M,\mathrm{CE}\mathfrak{h}\overset{\Delta}{\otimes}\widetilde{M})$$

such that $\Psi-\Psi'=\widetilde{D}_{\mathrm{tot}}\circ h+h\circ D_{\mathrm{tot}}$ and h commutes with $U(h)$-action on $\mathrm{CE}_{\mathfrak{h}}\overset{\Delta}{\otimes}M$.

Given two $\mathfrak{h}$-equivariant matrix factorizations $\mathcal{F}=(M,D,\partial)\in\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},W)$ and $\widetilde{\mathcal{F}}=(\widetilde{M},\widetilde{D},\widetilde{\partial})\in\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},\widetilde{W})$, we have $\mathcal{F}\otimes\widetilde{\mathcal{F}}\in\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},W+\widetilde{W})$ as the equivariant matrix factorization $(M\otimes\widetilde{M},D+\widetilde{D},\partial+\widetilde{\partial})$.

2.4. Push forwards, quotient by the group action

The technical part of [OR18b] is the construction of push-forwards of equivariant matrix factorizations. Here we state the main results, the details may be found in Section 3 of [OR18b]. We need push forwards along projections and embeddings. We also use the functor of taking quotient by group action for our definition of the convolution algebra.

The projection case is more elementary. Suppose $\mathcal{Z}=\mathcal{X}\times\mathcal{Y}$ and $\mathfrak{h}$ acts by derivations on $\mathbb{C}[\mathcal{X}]$ and $\mathbb{C}[\mathcal{Y}]$ and the projections $\pi:\mathcal{Z}\to\mathcal{X}$ respects these actions. Then for any $\mathfrak{h}$-invariant element $w\in\mathbb{C}[\mathcal{X}]^{\mathfrak{h}}$ there is a functor $\pi_*\colon\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},\pi^*(w))\to\mathrm{MF}_{\mathfrak{h}}(\mathcal{X},w)$ which simply forgets the action of $\mathbb{C}[\mathcal{Y}]$.

We define an embedding-related push-forward in the case when the subvariety $\mathcal{Z}_0\overset{j}{\hookrightarrow}\mathcal{Z}$ is the common zero of an ideal $I=(f_1,\dots,f_n)$ such that the functions $f_i\in\mathbb{C}[\mathcal{Z}]$ form a regular sequence. We assume that the Lie algebra $\mathfrak{h}$ acts on $\mathbb{C}[\mathcal{Z}]$ and I is $\mathfrak{h}$-invariant. Then there exists an $\mathfrak{h}$-equivariant Koszul complex $K(I)=(\Lambda^\bullet\mathbb{C}^n\otimes\mathbb{C}[\mathcal{Z}],d_K)$ over $\mathbb{C}[\mathcal{Z}]$ which has nontrivial homology only in degree zero. Then in Section 3 of [OR18b] we define the push-forward functor

$$j_*\colon\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z}_0,W|_{\mathcal{Z}_0})\longrightarrow\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},W),$$

for any $\mathfrak{h}$-invariant element $W\in\mathbb{C}[\mathcal{Z}]^{\mathfrak{h}}$.

Finally, let us discuss the quotient map. Let H be a group acting on $\mathcal{Z}$ such that $\mathrm{Lie}(H)=\mathfrak{h}$ and $\mathbb{C}[\mathcal{Z}]^{\mathfrak{h}}$ is finitely generated. The complex $\mathrm{CE}_{\mathfrak{h}}$ is a resolution of the trivial $\mathfrak{h}$-module by free modules. Thus the correct derived version of taking $\mathfrak{h}$-invariant part of the matrix factorization $\mathcal{F}=(M,D,\partial)\in\mathrm{MF}_{\mathfrak{h}}(\mathcal{Z},W)$, $W\in\mathbb{C}[\mathcal{Z}]^{\mathfrak{h}}$ is

$$\mathrm{CE}_{\mathfrak{h}}(\mathcal{F}):=(\mathrm{CE}_{\mathfrak{h}}(M),D+d_{\mathrm{ce}}+\partial)\in\mathrm{MF}(\mathcal{Z}/H,W),$$

where $\mathcal{Z}/H:=\mathrm{Spec}(\mathbb{C}[\mathcal{Z}]^{\mathfrak{h}})$ and use the general definition of $\mathfrak{h}$-module V:

$$\mathrm{CE}_{\mathfrak{h}}(V):=\mathrm{Hom}_{\mathfrak{h}}(\mathrm{CE}_{\mathfrak{h}},\mathrm{CE}_{\mathfrak{h}}\overset{\Delta}{\otimes}V).$$

2.5. Convolutions on reduced spaces

Let us use notation B for the standard upper-triangular Borel subgroup of $G_n = \mathrm{GL}_n$ and $T \subset B$ for the diagonal maximal torus. We treat B-modules as T-equivariant $\mathfrak{n} = \mathrm{Lie}([B,B])$-modules. For a space $\mathcal{Z}$ with B-action and for $W \in \mathbb{C}[\mathcal{Z}]^B$, we define $\mathrm{MF}_B(\mathcal{Z}, W)$ as a full subcategory of $\mathrm{MF}_{\mathfrak{n}}(\mathcal{Z}, W)$ whose objects are matrix factorizations (M, D, ∂), where M is a B-module and the differentials D and ∂ are T-invariant. The category $\mathrm{MF}_{B^\ell}(\mathcal{Z}, W)$ has a similar definition.

Let $H_i \subset G_n$, $i = 1, \dots, \ell$ be the subsets preserved by the left and right B-action. The backbone of the constructions of the knot invariant from [OR17] is the study of the category of matrix factorizations on the spaces

$$\overline{\mathcal{X}}_\ell(H_1, \dots, H_{\ell-1}) := \mathfrak{b} \times H_1 \times \cdots \times H_{\ell-1} \times \mathfrak{n}$$

with the following B^ℓ-action and potential $\overline{W}_\ell$:

$$(b_1, \dots, b_\ell) \cdot (X, g_1, \dots, g_{\ell-1}, Y) = (\mathrm{Ad}_{b_1}(X), b_1 g_1 b_2^{-1}, b_2 g_2 b_3^{-1}, \dots, \mathrm{Ad}_{b_\ell}(Y)),$$
$$\overline{W}_\ell(X, g_1, \dots, g_\ell, Y) = \mathrm{Tr}(X \mathrm{Ad}_g Y), \quad g = g_1 \dots g_\ell.$$

If $\ell = 1$ then $\overline{W}_\ell = 0$. Also for brevity we use notation

$$\overline{\mathcal{X}}_\ell = \overline{\mathcal{X}}_\ell(G_n, G_n, \dots, G_n), \quad \overline{W} = \overline{W}_2.$$

The space $\overline{\mathcal{X}}_\ell(H_1, \dots, H_{\ell-1})$ carries a natural $\mathbb{C}^*_q \times \mathbb{C}^*_t = T_{\mathrm{sc}}$-action:

$$(\lambda, \mu) \cdot (X, g_1, \dots, g_\ell, Y) = (\lambda^2 X, g_1, \dots, \lambda^{-2}\mu^2 Y).$$

The categories that we use in [OR18b] are the subcategories $\mathrm{MF}^{T_{\mathrm{sc}}}_{B^\ell}(\overline{\mathcal{X}}_\ell, \overline{W}_\ell) \subset \mathrm{MF}_{B^\ell}(\overline{\mathcal{X}}_\ell, \overline{W}_\ell)$ that consist of the matrix factorizations which are equivariant with respect to the action of T_{sc} and G-invariant. In particular, the space $\overline{\mathcal{X}}_2$ has the B^2-invariant potential: $\overline{W}(X, g, Y) = \mathrm{Tr}(X \mathrm{Ad}_g(Y))$, and the category $\mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$ has a structure of the convolution algebra [OR18b] that we outline below.

There are the following maps $\overline{\pi}_{ij} : \overline{\mathcal{X}}_3 \to \overline{\mathcal{X}}_2$:

$$\overline{\pi}_{12}(X, g_{12}, g_{13}, Y) = (X, g_{12}, \mathrm{Ad}_{g_{23}}(Y)_{++}), \tag{2.1}$$
$$\overline{\pi}_{13}(X, g_{12}, g_{13}, Y) = (X, g_{12}g_{23}, Y),$$
$$\overline{\pi}_{23}(X, g_{12}, g_{13}, Y) = (\mathrm{Ad}^{-1}_{g_{12}}(X)_+, g_{23}, Y). \tag{2.2}$$

Here and everywhere below X_+ and X_{++} stand for the upper and strictly-upper triangular parts of X. The map $\overline{\pi}_{12} \times \overline{\pi}_{23}$ is B^2-equivariant but not B^3-equivariant. However, in Section 5.4 of [OR18b] we show that for any $\mathcal{F}, \mathcal{G} \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$, there is a natural element

$$(\overline{\pi}_{12} \otimes_B \overline{\pi}_{23})^*(\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B^3}(\overline{\mathcal{X}}_3, \overline{\pi}^*_{13}(\overline{W})), \tag{2.3}$$

such that we can define the following binary operation on $\mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$:

$$\mathcal{F} \mathbin{\overline{\star}} \mathcal{G} := \overline{\pi}_{13*}(\mathrm{CE}_{\mathfrak{n}^{(2)}}((\overline{\pi}_{12} \otimes_B \overline{\pi}_{23})^*(\mathcal{F} \boxtimes \mathcal{G}))^{T^{(2)}}).$$

Instead of going into details of the construction of the convolution algebra let us explain the induction functors [OR18b] that provide us with an effective method of computing of the convolution product.

2.6. Induction functors

The standard parabolic subgroup P_k has the Lie algebra generated by $\mathfrak{b}$ is and $E_{i+1,i}$, $i \neq k$. Let us define the space $\overline{\mathcal{X}}_2(P_k) := \mathfrak{b} \times P_k \times \mathfrak{n}$ and let us also use the notation $\overline{\mathcal{X}}_2$ for $\overline{\mathcal{X}}_2(G_n)$. There are a natural embedding $\bar{i}_k : \overline{\mathcal{X}}_2(P_k) \to \overline{\mathcal{X}}_2$ and a natural projection $\overline{p}_k : \overline{\mathcal{X}}_2(P_k) \to \overline{\mathcal{X}}_2(G_k) \times \overline{\mathcal{X}}_2(G_{n-k})$. The embedding $\bar{i}_k$ satisfies the conditions for existence of the push-forward and we can define the induction functor

$$\overline{\mathrm{ind}}_k := \bar{i}_{k*} \circ \overline{p}_k^* \colon \mathrm{MF}^{T_{\mathrm{sc}}}_{B_k^2}(\overline{\mathcal{X}}_2(G_k), \overline{W}) \times \mathrm{MF}^{T_{\mathrm{sc}}}_{B_{n-k}^2}(\overline{\mathcal{X}}_2(G_{n-k}), \overline{W}) \\ \to \mathrm{MF}^{T_{\mathrm{sc}}}_{B_n^2}(\overline{\mathcal{X}}_2(G_n), \overline{W}).$$

Similarly we define the space $\overline{\mathcal{X}}_{2,\mathrm{fr}}(P_k) \subset \mathfrak{b} \times P_k \times \mathfrak{n} \times V$ as an open subset defined by the stability condition

$$\mathbb{C}\langle X, \mathrm{Ad}_g^{-1}(Y)\rangle u = V, \quad (X, g, Y, u) \in \mathfrak{b} \times P_k \times \mathfrak{n} \times V. \tag{2.4}$$

The latter space has a natural projection map $\overline{p}_k : \overline{\mathcal{X}}_{2,\mathrm{fr}}(P_k) \to \overline{\mathcal{X}}_2(G_k) \times \overline{\mathcal{X}}_{2,\mathrm{fr}}(G_{n-k})$ and the embedding $\bar{i}_k : \overline{\mathcal{X}}_{2,\mathrm{fr}}(P_k) \to \overline{\mathcal{X}}_{2,\mathrm{fr}}(G_n)$, and we can define the induction functor

$$\overline{\mathrm{ind}}_k := \bar{i}_{k*} \circ \overline{p}_k^* : \mathrm{MF}^{T_{\mathrm{sc}}}_{B_k^2}(\overline{\mathcal{X}}_2(G_k), \overline{W}) \times \mathrm{MF}^{T_{\mathrm{sc}}}_{B_{n-k}^2}(\overline{\mathcal{X}}_{2,\mathrm{fr}}(G_{n-k}), \overline{W}) \\ \to \mathrm{MF}^{T_{\mathrm{sc}}}_{B_n^2}(\overline{\mathcal{X}}_{2,\mathrm{fr}}(G_n), \overline{W}).$$

It is shown in Section 6 (Proposition 6.2) of [OR18b] that the functor $\overline{\mathrm{ind}}_k$ is the homomorphism of the convolution algebras

$$\overline{\mathrm{ind}}_k(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \bar{\star} \overline{\mathrm{ind}}_k(\mathcal{G}_1 \boxtimes \mathcal{G}_2) = \overline{\mathrm{ind}}_k(\mathcal{F}_1 \bar{\star} \mathcal{G}_2 \boxtimes \mathcal{F}_2 \bar{\star} \mathcal{G}_2).$$

Let us define B^2-equivariant embedding

$$i : \overline{\mathcal{X}}_2(B_n) \to \overline{\mathcal{X}}_2,\ \overline{\mathcal{X}}_2(B) := \mathfrak{b} \times B \times \mathfrak{n}.$$

The pull-back of $\overline{W}$ along the map i vanishes and the embedding i satisfies the conditions for existence of the push-forward

$$i_* : \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2(B_n), 0) \to \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2(G_n), \overline{W}).$$

We denote by $\underline{\mathbb{C}[\overline{\mathcal{X}}_2(B_n)]} \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2(B_n), 0)$ the matrix factorization with zero differential with the support only in even homological degree. As it is shown in [OR18b, Prop. 7.1] the push-forward

$$\overline{\mathbb{1}}_n := i_*(\underline{\mathbb{C}[\overline{\mathcal{X}}_2(B_n)]})$$

is the unit in the convolution algebra.

Using the induction functor and the unit in the convolution algebra we define the insertion functor that inserts matrix factorization of smaller rank inside the higher rank one:

$$\overline{\mathrm{Ind}}_{k,k+1} : \mathrm{MF}^{T_{\mathrm{sc}}}_{B_2^2}(\overline{\mathcal{X}}_2(G_2), \overline{W}) \to \mathrm{MF}^{T_{\mathrm{sc}}}_{B_n^2}(\overline{\mathcal{X}}_2(G_n), \overline{W}),$$
$$\overline{\mathrm{Ind}}_{k,k+1}(\mathcal{F}) := \overline{\mathrm{ind}}_{k+1}(\overline{\mathrm{ind}}_{k-1}(\overline{\mathbb{1}}_{k-1} \times \mathcal{F}) \times \overline{\mathbb{1}}_{n-k-1}).$$

2.7. Generators of the braid group

Let us first discuss the case of the braids on two strands. The key to construction of the braid group action in [OR18b] is the following factorization in the case $n = 2$:

$$\overline{W}(X, g, Y) = y_{12}\big(g_{11}(x_{11} - x_{22}) + g_{21}x_{12}\big)g_{21}/\det,$$

where $\det = \det(g)$ and

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix}.$$

Thus we can define the following strongly equivariant Koszul matrix factorization:

$$\overline{\mathcal{C}}_+ := (\mathbb{C}[\overline{\mathcal{X}}_2] \otimes \wedge\langle\theta\rangle, D, 0, 0) \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B_2^2}(\overline{\mathcal{X}}_2, \overline{W}),$$
$$D = \frac{g_{12}y_{12}}{\det}\theta + (g_{11}(x_{11} - x_{22}) + g_{21}x_{12})\frac{\partial}{\partial\theta},$$

where $\wedge\langle\theta\rangle$ is the exterior algebra with one generator.

This matrix factorization corresponds to the positive elementary braid on two strands.

Using the insertion functor we can extend the previous definition on the case of the arbitrary number of strands:

$$\overline{\mathcal{C}}^{(k)}_+ := \overline{\mathrm{Ind}}_{k,k+1}(\overline{\mathcal{C}}_+).$$

Section 11 of [OR18b] is devoted to the proof of the following braid relations between these elements:

$$\overline{\mathcal{C}}^{(k+1)}_+ \bar{\star} \overline{\mathcal{C}}^{(k)}_+ \bar{\star} \overline{\mathcal{C}}^{(k+1)}_+ = \overline{\mathcal{C}}^{(k)}_+ \bar{\star} \overline{\mathcal{C}}^{(k+1)}_+ \bar{\star} \overline{\mathcal{C}}^{(k)}_+,$$

Let us now discuss the inversion of the elementary braid. In view of the inductive definition of the braid group action, it is sufficient to understand the inversion in the case $n = 2$.

Let us fix the notation for the characters of B_2. For $h \in \mathrm{Lie}(B_2)$, we define $\epsilon_i(h) = h_{ii}$ where h_{ii} it the i-th diagonal of the upper-triangular matrix h. Let us choose $\chi_l = a\epsilon_1 + b\epsilon_2$ and $\chi_r = c\epsilon_1 + d\epsilon_2$, then given $B_2 \times B_2$-equivariant module M, we define $M\langle\chi_l, \chi_r\rangle$ to be the module with the same underlying space and the action of the first copy of B_2 twisted by χ_l and the action of the second copy of B_2 twisted by χ_r. We use same convention for the matrix factorization: $\mathcal{F}\langle\chi_l, \chi\rangle$ is the matrix factorization $\mathcal{F}$ with the $\langle\chi_l, \chi_r\rangle$-twisted B_2^2-equivariant structure of the even part of the underlying free module.

Thus we define

$$\overline{\mathcal{C}}_- := \overline{\mathcal{C}}_+\langle-\epsilon_1, \epsilon_2\rangle \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B_2^2}(\overline{\mathcal{X}}_2(G_2), \overline{W});$$

respectively, we define $\overline{\mathcal{C}}^{(k)}_- := \overline{\mathrm{Ind}}_{k,k+1}(\overline{\mathcal{C}}_-)$. It is shown in [OR18b, Sect. 9] that $\overline{\mathcal{C}}^{(k)}_-$ is inverse to $\overline{\mathcal{C}}^{(k)}_+$.

2.8. Koszul matrix factorizations

The generators of the braid group from the previous subsection are examples of the Koszul matrix factorizations. Let us remind a general definition of Koszul matrix factorizations and elementary transformation of the Koszul matrix factorizations. More details on Koszul matrix factorizations in the form relevant to the current paper could be found in [OR18b].

Suppose $\mathcal{Z}$ is a variety with the action of a group G and F is a G-invariant potential. An object of the category $\mathrm{MF}^{\mathrm{str}}_{B^2}(\mathcal{Z}, F)$ is a free B^2-equivariant $\mathbb{Z}_2$-graded $\mathbb{C}[\mathcal{Z}]$-module M with the odd G-invariant differential D such that $D^2 = F\mathbb{1}_M$. In particular, a free G-equivariant $\mathbb{C}[\mathcal{Z}]$-module V with two elements $d_{\mathrm{l}} \in V$, $d_{\mathrm{r}} \in V^*$ such that $(v, w) = F$ determines a Koszul matrix factorization $\mathrm{K}(V; d_{\mathrm{l}}, d_{\mathrm{r}}) = \bigwedge^\bullet V$ with the differential $Dv = d_{\mathrm{l}} \wedge v + d_{\mathrm{r}} \cdot v$ for $v \in \bigwedge^\bullet V$. We use a more detailed notation by choosing a basis $\theta_1, \dots, \theta_n \in V$ and presenting d_{l} and d_{r} in terms of components: $d_{\mathrm{l}} = a_1\theta_1 + \dots + a_n\theta_n$, $d_{\mathrm{r}} = b_1\theta_1^* + \dots + b_n\theta_n^*$,

$$\mathrm{K}(V; d_{\mathrm{l}}, d_{\mathrm{r}}) = \begin{bmatrix} a_1 & b_1 & \theta_1 \\ \vdots & \vdots & \vdots \\ a_n & b_n & \theta_n \end{bmatrix} \tag{2.5}$$

The structure of G-module is described by specifying the action of G on the basis $\theta_1, \dots, \theta_n$. In some cases when G-equivariant structure of the module M is clear from the context we omit the last columns from the notation. We call a matrix presenting Koszul matrix factorization Koszul matrix. For example, if we change the basis $\theta_1, \dots, \theta_n$ to the basis $\theta_1, \dots, \theta_i + c\theta_j, \dots, \theta_j, \dots, \theta_n$ the i-th and j-th rows of the Koszul matrix will change as follows:

$$\begin{bmatrix} a_i & b_i & \theta_i \\ a_j & b_j & \theta_j \end{bmatrix} \mapsto \begin{bmatrix} a_i + ca_j & b_i & \theta_i + c\theta_j \\ a_j & b_j - cb_i & \theta_j \end{bmatrix}.$$

Suppose $a_1, \dots, a_n \in \mathbb{C}[\mathcal{Z}]$ is a regular sequence and $F \in (a_1, \dots, a_n)$. We can choose b_i such that $F = \sum_i a_i b_i$ and d_{l} and d_{r} are as above: $d_{\mathrm{l}} = a_1\theta_1 + \cdots a_n\theta_n$, $d_{\mathrm{r}} = b_1\theta_1^* + \dots + b_n\theta_n^*$. In general, there is no unique choice for b_i but all choices lead to homotopy equivalent Koszul matrix factorizations (in the nonequivariant case they would be simply isomorphic). In other words, if b_i' is a another collection of elements such that $F = \sum a_i b_i'$ and $d_{\mathrm{r}}' = b_1'\theta_1^* + \dots + b_n'\theta_n^*$, then [OR18b, Lem. 2.2] implies that the complexes $\mathrm{K}(V; d_{\mathrm{l}}, d_{\mathrm{r}})$ and $\mathrm{K}(V; d_{\mathrm{l}}, d_{\mathrm{r}}')$ are homotopy equivalent. Thus from now on we use notation $\mathrm{K}^F(a_1, \dots, a_n)$ for such matrix factorization.

3. Coxeter matrix factorization

In the previous section we outlined the definition of the convolution algebra on the category of matrix factorizations. In particular we explained that for any element $\beta \in \mathfrak{Br}_n$ we can associate a matrix factorization

$$\overline{\mathcal{C}}_\beta := \overline{\mathcal{C}}^{(k_1)}_{\epsilon_1} \,\bar{\star}\, \cdots \,\bar{\star}\, \overline{\mathcal{C}}^{(k_l)}_{\epsilon_l},$$

where $\beta = \sigma_{k_1}^{\epsilon_1} \cdot \dots \cdot \sigma_{k_l}^{\epsilon_l}$ is an expression for β is terms of elementary braids.

We could not expect a simple formula for $\overline{\mathcal{C}}_\beta$ for a general element $\beta \in \mathfrak{Br}_n$. In particular, as one can see from the computations in [OR18b, Sect. 11] the matrix factorizations $\overline{\mathcal{C}}_\beta$ is not always Koszul. Thus it is a bit surprising, at least for us, that for the Coxeter braid matrix factorization, $\overline{\mathcal{C}}_\beta$ is Koszul and quite simple. To describe the answer we need the following coordinates on the space $\overline{\mathcal{X}}_2 = \mathfrak{b}_n \times G_n \times \mathfrak{n}_n$:

$$X = (x_{ij})_{i\le j}, \quad g = (g_{ij}), \quad Y = (y_{ij})_{i\le j}.$$

Let us introduce $i \times i$ matrix $M_i := [g^{(i)}_{\bullet,1}, \dots, g^{(i)}_{\bullet,i-1}, \widehat{X}^{(i)}_{\bullet,i}]$ where $v^{(i)}$ is an abbreviation for the vector consisting of first i entries of $v \in \mathbb{C}^n$ and $\widehat{X} = X - x_{11}\mathrm{Id}_n$. Respectively, we define functions $F_i := \det(M_{i+1}) \in \mathbb{C}[\mathfrak{b} \times G]$ and the ideal $I_F \subset \mathbb{C}[\overline{\mathcal{X}}_2]$ generated by these functions F_i, $i = 1, \dots, n-1$.

Proposition 3.0.1. *Consider the ideal $I_g = (\{g_{ij}\}_{i-j>1})$ in $\mathbb{C}[\overline{\mathcal{X}}_2]$. Then the ideal $I_{\mathrm{cox}} := I_g + I_F$ contains $\overline{W}$.*

Proof. We show below that the above equations imply that

$$\mathrm{Ad}_g^{-1}(X) \in \mathfrak{b}.$$

Indeed, the ideal I_g defines the sublocus $Hess$ of Hessenberg matrices of G_n. On the other hand if $g \in Hess$ then the condition $F_i = 0$ implies that the column $\widehat{X}_{\bullet,i+1}$ is a linear combination of the columns $g_{\bullet,1}, \dots, g_{\bullet,i}$. Let us denote by K the matrix of these coefficients. Then we have K is strictly upper-triangular and $\widehat{X} = g \cdot K$.

Hence, $\mathrm{Ad}_g^{-1}(\widehat{X}) = K \cdot g$ but the product of the Hessenberg matrix and strictly upper-triangular matrix is upper-triangular. □

Proposition 3.0.2. *The functions $\{g_{ij}\}_{i-j>1}, F_1, \dots, F_{n-1}$ form a regular sequence in $\mathbb{C}[\mathfrak{b} \times G]$*

Proof. We proceed by induction. We assume that $\{g_{ij}\}_{i-j>1}, F_1, \dots, F_{n-2}$ form a regular sequence. Then we observe that G_n is covered by the open sets U_i defined by $g_i = \det(\Delta_{in}) \neq 0$ where Δ_{in} is the minor of M_n obtained by removal of the in-th entry. It is enough to show regularity at every open chart. But in the chart U_i we have $F_n/g_i = \widehat{X}_{in} + \dots$ and F_i, $i < n$ do not depend on $\widehat{X}_{\bullet,n}$. Hence the regularity follows. □

Thus we can apply [OR18b, Lem. 2.2] to imply that there is a unique up to homotopy Koszul matrix factorization $\mathrm{K}^{\overline{W}}(\{g_{ij}\}_{i-j>1}, F_1, \dots, F_{n-1})$ and we show the following.

Theorem 3.0.3. *There is a strictly equivariant Koszul matrix factorization that realizes $\overline{\mathcal{C}}_{\mathrm{cox}}$:*

$$\overline{\mathcal{C}}_{\mathrm{cox}} = \mathrm{K}^{\overline{W}}(\{g_{ij}\}_{i-j>1}, F_1, \dots, F_{n-1}).$$

The construction of the induction functors implies the following.

Corollary 3.0.4. *For any $S\subset\{1,\dots,n-1\}$, we have a strictly equivariant matrix factorization*

$$\overline{\mathcal{C}}_{cox_S}=\mathrm{K}^{\overline{W}}(\{g_{ij}\}_{ij\in S'},\{F_i\}_{i\notin S})\in \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2(G_n),\overline{W}),$$

where $S'=\{(i,j)\}_{i-j>1}\cup\{(i,i+1)\}_{i\in S}$.

Before we proceed to the proof let us describe the most efficient method of computation of the matrix factorization corresponding to the braid $\beta=\alpha\cdot\sigma_k^\epsilon$ from already known $\mathcal{C}_\alpha$. The justification of the construction is given in [OR18b, Sect. 8].

Indeed, consider $\mathcal{F}=(M,D_1,\partial_l,\partial_r)\in\mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2(G_n),\overline{W})$ where $\partial_l,\partial_r\in\mathrm{Hom}_{\mathbb{C}[\overline{\mathcal{X}}_2(G_n)]}(\wedge^*\mathfrak{n}\otimes M,M)$ are correcting differential for the equivariant structure; ∂_l corrects equivariance for the first copy of B and ∂_r for the second copy. Respectively, $\overline{\mathcal{C}}_\epsilon=(R^2_\epsilon,D_2,0,0)\in\mathrm{MF}^{T_{\mathrm{sc}}}_{B_2^2}(\overline{\mathcal{X}}(G_2),\overline{W})$ is the strongly equivariant matrix factorization corresponding to the elementary braid σ_k^ϵ; here R_ϵ is the ring $\mathbb{C}[\overline{\mathcal{X}}_2(G_2)]$ with the appropriately twisted B_2^2-structure.

The auxiliary space $\overline{\mathcal{X}}_3(G_n,G_{k,k+1}):=\mathfrak{b}_n\times G_n\times G_2\times\mathfrak{n}_n$ is naturally embedded into the convolution space $\overline{\mathcal{X}}_3$ via the map $i_{k,k+1}:=\mathrm{Id}^2\times i'_{k,k+1}\times\mathrm{Id}$ where $i'_{k,k+1}:G_2\to G_n$ is the embedding of G_2 as 2×2-block with entries at the positions ij with $i,j\in\{k,k+1\}$. Hence we can restrict the maps $\overline{\pi}_{ij}$ on the auxiliary space and we can also endow the auxiliary space with the $B_n\times B_2\times B_n$-equivariant structure by restriction from the large space.

The maps $\overline{\pi}_{ij}$ are B_n^2-equivariant but not B_2-equivariant, thus a priori the tensor product $\overline{\pi}_{12}^*(\mathcal{F})\otimes\overline{\pi}_{23}^*(\mathcal{C}_\epsilon)$ has only B_n^2-equivariant structure. But as explained in [OR18b, Sect. 8] there is a natural $B_n\times B_2\times B_n$-equivariant matrix factorization $\mathcal{G}$ that could be imposed on $\overline{\pi}_{12}^*(\mathcal{F})\otimes\overline{\pi}_{23}^*(\mathcal{C}_\epsilon)$:

$$\mathcal{G}:=(\overline{\pi}_{12}^*(M)\otimes\overline{\pi}_{23}^*(R_\epsilon),\overline{\pi}_{12}^*(D_1)+\overline{\pi}_{23}^*(D_2);\partial_l,\partial_r'+\partial',0)$$

where $\partial_r'\in\mathrm{Hom}_{R_3}(\mathfrak{n}_2\otimes M',M')$, $M'=M\oplus M=\overline{\pi}_{12}^*(M)\otimes\overline{\pi}_{23}^*(R_\epsilon)$, $R_3=\mathbb{C}[\overline{\mathcal{X}}_3(G_n,G_{k,k+1})]$ is the restriction of the map ∂_r on the subalgebra $\mathfrak{n}_2$ and $\partial'\in\mathrm{Hom}_{R_3}(\mathfrak{n}_2\otimes M',M')$ is defined by the formula

$$\partial':=\frac{\partial\overline{\pi}_{12}^*(D_1)}{\partial\widetilde{Y}^2_{k,k+1}}(\widetilde{Y}^2_{k+1,k+1}-\widetilde{Y}^2_{kk})+\left(\frac{\partial\overline{\pi}_{23}^*(D_2)}{\partial\widetilde{X}^2_{kk}}-\frac{\partial\overline{\pi}_{23}^*(D_2)}{\partial\widetilde{X}^2_{k+1,k+1}}\right),$$

where $\widetilde{X}^2=\mathrm{Ad}_{g_{12}}(X)$, $\widetilde{Y}^2:=\mathrm{Ad}_{g_{12}}(Y)$ and X,g_{12},g_{23},Y are the coordinates on $\overline{\mathcal{X}}_3(G_n,G_{k,k+1})$.

The key observation about this matrix factorization is that up to homotopy we have (see [OR18b, Sect. 8])

$$\overline{\mathcal{C}}_\beta=\overline{\pi}_{13*}(\mathrm{CE}_{\mathfrak{n}_2}(\mathcal{G})^{T_2}).$$

Thus we reduce the complexity of the computation of matrix factorization $\overline{\mathcal{C}}_\beta$; we only need to analyze the rank one Chevalley–Eilenberg complex for $\mathfrak{n}_2$ and we use this method in our proof.

Proof of Theorem 3.0.3. Let us first notice that the case $n = 2$ of the theorem is a tautology. The case $n = 3$ was proven in [OR18b, Sect. 10]. For general n, our inductive argument is essentially identical to the computation from [OR18b, Sect. 10].

Let $\alpha = \mathrm{cox}_{n-1}$. Then by the induction and the corollary we have a presentation of $\overline{\mathcal{C}}_\alpha$ as a strongly equivariant Koszul matrix factorization. As induction step we need to analyze the equivariant matrix factorization $C_{12} := \overline{\pi}_{12}^*(\overline{\mathcal{C}}_\alpha) \otimes \overline{\pi}_{23}^*(\overline{\mathcal{C}}_+)$ (with the B_2-equivariant structure discussed before this proof) on the auxiliary space $\overline{\mathcal{X}}_3(G_n, G_{n-1,n})$.

We introduce coordinates on our auxiliary space $\overline{\mathcal{X}}_3(G_n, G_{n-1,n})$ as follows: $\overline{\mathcal{X}}_3(G_n, G_{n-1,n}) = \{(X, g_{12}, g_{23}, Y)\}$. Since $\overline{\mathcal{X}}_3(G_n, G_{n-1,n})$ is a subspace of the space $\overline{\mathcal{X}}_3(G_n, G_n)$, we can define the maps $\overline{\pi}_{12}, \overline{\pi}_{13} : \overline{\mathcal{X}}_3(G_n, G_{n-1,n}) \to \overline{\mathcal{X}}_2(G_n)$ by formulas (2.1). Respectively, we define the map $\overline{\pi}_{23} : \overline{\mathcal{X}}_3(G_n, G_{n-1,n}) \to \overline{\mathcal{X}}_2(G_2)$ by composing the map from (2.2) with the projection $\mathfrak{b}_n \times G_2 \times \mathfrak{n}_n \to \mathfrak{b}_2 \times G_2 \times \mathfrak{n}_2$.

To simplify notations we set

$$g_{12} = a, \quad g_{23} = b, \quad g_{13} = g_{12}g_{23} = c.$$

That is $g_{12} = (a_{ij})_{i,j\in[1,n]}$, $g_{23} = (b_{ij})_{i,j\in[1,n]}$, and $b_{ij} = \delta_{ij}$ if $i, j < n-1$. Respectively, we have $g_{13} := g_{12}g_{23} = (c_{ij})_{i,j\in[1,n]}$ and

$$\widetilde{X}_2 := \mathrm{Ad}_{g_{12}}^{-1}(X) = (\widetilde{x}_{ij})_{i,j\in[1,n]}, \quad \widetilde{Y}_2 = \mathrm{Ad}_{g_{23}}(Y) = (\widetilde{y}_{i,j})_{i,j\in[1,n]}.$$

We also use shorthand notations $\Delta_a = \det(a), \Delta_b = \det(b), \Delta_c = \det(c)$.

The matrix factorization C_{12} is the Koszul matrix factorization

$$C_{12} = \mathrm{K}^{\overline{\pi}_{13}^*(\overline{W})}\Big(\{a_{ij}\}_{ij\in S'}, \{F_i(X, a)\}_{i\in[1,n-2]}, \widetilde{f}\Big)$$

where $\widetilde{f} := (\widetilde{x}_{n-1,n-1} - \widetilde{x}_{nn})b_{n-1,n-1} + \widetilde{x}_{n-1,n}b_{n,n-1}$ and $S' = \{i-j > 1\} \cup \{(n, n-1)\}$. Next let us notice that $a_{\bullet,i} = c_{\bullet,i}$, $i \le n-2$ and since $F_i(X, a)$ depends on $a_{\bullet,j}$, $j \le i$ we obtain a presentation of the complex C_{12} as a tensor product

$$\mathrm{K}^{W'}\big(\{c_{ij}\}_{i-j>1}, \{F_i(X, c)_{i\in[1,n-2]}\}\big) \otimes \mathrm{K}^{W''}(a_{n,n-1}, \widetilde{f}),$$

where $W' + W'' = \overline{\pi}_{13}^*(\overline{W})$ and we can assume that W' only depends on c, X but not on b.

Let us denote the first term in the product by C_{12}' and the second term by C_{12}''. The complex C_{12}' is $\mathfrak{n}_2$-invariant thus $\mathrm{CE}_{\mathfrak{n}_2}(C_{12}) = C_{12}' \otimes \mathrm{CE}_{\mathfrak{n}_2}(C_{12}'')$ and to complete our proof we need to analyze the last complex in the product. In particular, we need to understand the $\mathfrak{n}_2$-equivariant structure of the complex C_{12}''.

Let h be an element of $B_2 \subseteq B_n$, that is $h_{ij} = 0$ if $(i, j) \notin \{(n, n), (n-1, n-1), (n-1, n)\}$. The action of h on the space $\overline{\mathcal{X}}_2(G_n, G_{n-1,n})$ is given by the formulas

$$g_{12} \mapsto g_{12}h^{-1}, \quad g_{23} \mapsto hg_{23}, \quad \widetilde{X}_2 \mapsto \mathrm{Ad}_h\widetilde{X}_2.$$

We denote by δ the element of $\mathrm{Lie}(B_2)$ corresponding to the $(n-1, n)$-th entry and below we investigate its action on the complex C_{12}''.

First, let us notice that the function $a_{n,n-1}$ is $\mathfrak{n}_2$-invariant but the function $\widetilde{f}$ is not. Thus the complex C''_{12} is not strongly $\mathfrak{n}$-equivariant and correction differentials will appear. In more detail, we have

$$C''_{12} = \begin{bmatrix} a_{n,n-1} & * & \theta_1 \\ \widetilde{f} & * & \theta_2 \end{bmatrix},$$

where the action of $\mathfrak{n}$ is given by

$$\delta(\theta_1) = k\theta_2, \quad \delta(\theta_2) = 0$$

for some function $k \in \mathbb{C}[\mathfrak{b} \times G]$ which we need to compute.

One way to approach the computation of k is to use differentials of the $\mathfrak{n}_2$-equivariant structure on C_{12} from the discussion before the proof and to derive a formula for k by the careful analysis of the effects of the elementary transformations on the differentials. However, we choose a different method: we follow the same path as in the proof of [OR18b, Lem. 10.4]. Namely, the function k is uniquely defined by the condition that $a_{n,n-1}\theta_1 + \widetilde{f}\theta_2$ is δ-invariant. Thus we only need to compute $\delta(\widetilde{f})$.

Instead of computing $\delta(\widetilde{f})$ by brute force we use the following argument. First, we present the matrix $\widetilde{X}_2$ as a sum of the upper-triangular and strictly lower-triangular parts: $\widetilde{X}_2 = \widetilde{X}_{2,+} + \widetilde{X}_{2,--}$. Next, we observe that we have $\widetilde{f}b_{n,n-1} = -\big(\mathrm{Ad}^{-1}_{g_{23}}\widetilde{X}_{2,+}\big)_{n,n-1}$ and since $\delta(b_{n,n-1}) = 0$ we obtain

$$\delta(\widetilde{f}) = -\delta\big(\mathrm{Ad}^{-1}_{g_{23}}(\widetilde{X}_{2,+})\big)_{n,n-1}/b_{n,n-1}.$$

On the other hand, $\mathrm{Ad}^{-1}_{g_{23}}(\widetilde{X}_2)$ is δ-invariant and thus we get

$$\delta(\widetilde{f}) = \delta\big(\mathrm{Ad}^{-1}(\widetilde{X}_{2,--})\big)_{n,n-1}/b_{n,n-1}.$$

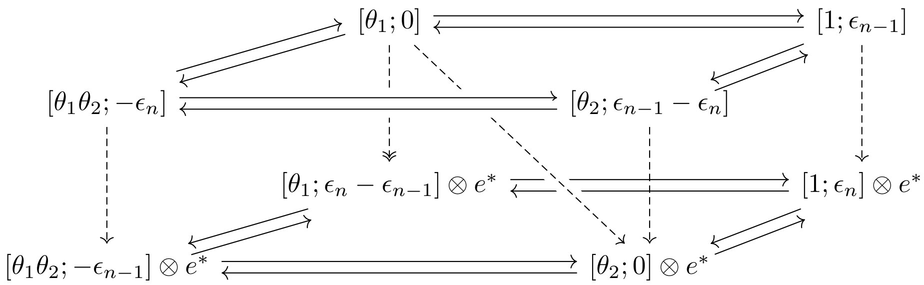
A direct computation shows that $\mathrm{Ad}^{-1}_{g_{23}}(\widetilde{X}_{2,--})_{n,n-1} = b^2_{n-1,n-1}\widetilde{x}_{n,n-1}/\Delta_b$ and since $\widetilde{x}_{n,n-1}$ is δ-invariant while $\delta b_{n-1,n-1} = b_{n,n-1}$, we obtain

$$\delta(\widetilde{f}) = 2\widetilde{x}_{n,n-1}b_{n-1,n-1}/\Delta_b.$$

Modulo relations from I_g, the matrix element $(a^{-1})_{nk}$, $k < n$ is divisible by $a_{n,n-1}$: $(a^{-1})_{nk} = (-1)^{k+n}a_{n,n-1}\det(M^{n,k}_{n,n-1}(a))/\Delta_a$ where $M^{kl}_{ij}(a)$ is the minor of a obtained by removing i,j-th columns and k,l-th rows. By putting all formulas together we finally obtain the following formula for k:

$$k = -2b_{n-1,n-1}\Delta_b^{-1}\Big((a^{-1})_{nn}x_{nn} + \sum_{k=1}^{n-1}(-1)^{k+n}\det(M^{n,k}_{n,n-1}(a))/\Delta_a\sum_{l=k}^{n} x_{kl}a_{l,n-1}\Big).$$

Let us denote by $\epsilon_{n-1}, \epsilon_n$ the generators for character group of B_2: $\epsilon_k(h) = h_{k,k}$. We use the same notation for the characters of the torus $T^{(2)} = (\mathbb{C}^*)^2 \subset B_2$. Now recall that the action of torus $T^{(2)} = (\mathbb{C}^*)^2 \subset B_2 \subset B$ on $a_{n,n-1}$ and $\widetilde{f}$ has weights ϵ_n and ϵ_{n-1} and respectively the weights of θ_1, θ_2 are $-\epsilon_n$ and $-\epsilon_{n-1}$. Thus $T^{(2)}$-invariant part of the complex $\mathrm{CE}_{\mathfrak{n}_2}(C''_{12})$ is of the shape



where the expression $[\alpha;\rho]$ stands for $\mathrm{Hom}_{T^{(2)}}(\rho, R)\alpha$, $R = \mathbb{C}[\overline{\mathcal{X}}(G_n, G_{n,n-1})]$ and ρ is a character of $T^{(2)}$; e^* is a generator of $\mathfrak{n}_2^* = \mathrm{Hom}(\mathfrak{n}_2, \mathbb{C}) = \langle e^* \rangle$. In the picture the vertical dashed arrows are the Chevalley–Eilenberg differentials. The slanted dashed arrow is the correction differential. Below we show that the left and right dashed arrows are isomorphisms and can be contracted.

In the tensor product $R = \overline{\pi}_{13}^*(\mathbb{C}[\overline{\mathcal{X}}_2]) \otimes \mathbb{C}[G_2]$ the first term is B_2-invariant. Hence the vertical arrows in the diagram above compute homology $\mathrm{H}^*(G_2/B_2, \mathcal{O}(k))$ for the corresponding value of k. The value of k could read from the bottom side of cube in the diagram: the entry $[\star; \epsilon_{n-1}^a \epsilon_n^b] e^*$ corresponds to $k = a - b$. After contracting the vertical arrows we arrive to the diagram

$$\begin{array}{ccc}
\mathrm{H}^*(\mathbb{P}^1, \mathcal{O}(-2)) \otimes \mathbb{C}[\overline{\mathcal{X}}_2] & \Longleftrightarrow & \mathrm{H}^*(\mathbb{P}^1, \mathcal{O}(-1)) \otimes \mathbb{C}[\overline{\mathcal{X}}_2] \\
\Updownarrow & \searrow & \Updownarrow \\
\mathrm{H}^*(\mathbb{P}^1, \mathcal{O}(-1)) \otimes \mathbb{C}[\overline{\mathcal{X}}_2] & \Longleftrightarrow & \mathrm{H}^*(\mathbb{P}^1, \mathcal{O}(0)) \mathbb{C}[\overline{\mathcal{X}}_2].
\end{array}$$

Since only two vertices of the last diagram are actually nonzero we only need to compute the diagonal arrow. The target of this arrow is $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes \overline{\pi}_{13}^*(\mathbb{C}[\mathcal{X}_2]) = \mathrm{H}^1_{\mathrm{Lie}}(\mathfrak{n}, R[\epsilon_{n-1} - \epsilon_n])$, hence we can replace the coefficients of the differential by the expressions that are homologous with respect to the differential δ. Below we take advantage of this observation. Indeed, note that

$$\delta b_{n-1,n-1} = b_{n,n-1}, \qquad \delta b_{n-1,n} = b_{nn},$$

so, first, $\delta(b_{n-1,n-1}^2) = 2b_{n-1,n-1}b_{n,n-1}$, hence $b_{n-1,n-1}b_{n,n-1}$ is exact and, second,

$$\delta(b_{n-1,n-1}b_{nn}) = b_{n,n-1}b_{n-1,n} + b_{n-1,n-1}b_{nn},$$

hence in view of $b_{n-1,n-1}b_{nn} - b_{n,n-1}b_{n-1,n} = \Delta_b$ we find $b_{22}b_{33} \sim \frac{1}{2}\Delta_b$. Since $b_{n-1,n-1}a = c \cdot (b_{n-1,n-1}b^{-1})$ we obtain

$$2b_{n-1,n-1}a_{i,n-1} \sim c_{i,n-1}.$$

Next, let us notice that since $a_{\bullet,i} = c_{\bullet,i}$ for $i < n-1$, by expanding along the $(n-2)$-th column of the determinant in the definition of $(a^{-1})_{nn}$, we can use the above homotopy equivalence and get

$$2b_{n-1,n-1}(a^{-1})_{nn} \sim \Delta_c \Delta_a^{-1}(c^{-1})_{nn} = \Delta_b(c^{-1})_{nn}.$$

We can combine the last formula with the observation that $M^{n,k}_{n,n-1}(a) = M^{n,k}_{n,n-1}(c)$ to obtain

$$k \sim (c^{-1})_{nn}x_{nn} + \sum_{k=1}^{n-1}(-1)^{k+n}\det(M^{n,k}_{n,n-1}(c))/\Delta_c \sum_{l=k}^{n} x_{kl}c_{l,n-1}.$$

Next, let us observe that if we collect all the terms in the last sum with $l = n$, we obtain

$$\sum_{k=1}^{n-1}(c^{-1})_{nk}x_{kn} = F_{n-1}(x,c)/\Delta_c + (c^{-1})_{nn}(-x_{nn} + x_{11}).$$

On the other hand, if we collect all the terms in the sum with $l = s$ for $s \neq n$, we get

$$\begin{aligned} c_{s,n-1}\Delta_c^{-1}&\sum_{k=1}^{n-1}(-1)^{k+n}\det(M^{n,k}_{n,n-1}(c))x_{ks} \\ &= (-1)^{s+n}\Delta^{-1}(c)c_{s,n-1}\det(M^{n,s}_{n,n-1}(c)x_{11}) \bmod (F_{s-1}). \end{aligned}$$

Thus combination of the last two observations implies that modulo the ideal $I_g + (F_1, \dots, F_{n-2})$ we have the following homotopy

$$\begin{aligned} k \sim \big((c^{-1})_{nn}x_{nn}\big) &+ \big(F_{n-1}(x,c)/\Delta_c + (c^{-1})_{nn}(-x_{nn}+x_{11})\big) \\ &+ \Big(\Delta_c^{-1}x_{11}\sum_{s=1}^{n-1}(-1)^{n+s}c_{s,n-1}\det(M^{n,s}_{n,n-1}(c))\Big) = F_{n-1}(x,c)/\Delta_c^{-1}. \end{aligned}$$

Finally, let us remark that B^2 preserves F_i and acts linearly on the generators of I_g. Thus $\mathrm{K}^{\overline{W}}(\{g_{ij}\}_{i-j>1}, F_1, \dots, F_{n-1})$ is strictly B^2-equivariant. □

4. Link homology computation

4.1. Link homology

In this subsection, we remind our construction for link invariant from [OR18b] and its connection with sheaves on the nested Hilbert scheme.

The free nested Hilbert scheme $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ is a $B \times \mathbb{C}^*$-quotient of the sublocus $\widetilde{\mathrm{Hilb}}^{\mathrm{free}}_{1,n} \subset \mathfrak{b}_n \times \mathfrak{n}_n \times V_n$ of the cyclic triples $\{(X,Y,v) \mid \mathbb{C}\langle X,Y\rangle v = V_n\}$. In other words, $\widetilde{\mathrm{Hilb}}^{\mathrm{free}}_{1,n}$ is the stable part of the product $\overline{\mathcal{X}}_1(G_n) \times V_n$. Also notice that the corresponding potential W_1 vanishes on $\mathcal{X}_1$.

The usual nested Hilbert scheme $\mathrm{Hilb}^L_{1,n}$ is the subvariety of $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$, it is defined by the commutativity of the matrices X, Y. Thus we have the pull-back morphism

$$j_e^* : \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2 \times V_n, \overline{W}) \to \mathrm{MF}^{T_{\mathrm{sc}}}_{B}(\widetilde{\mathrm{Hilb}^{\mathrm{free}}_{1,n}}, 0).$$

In more detail, the morphism j_e^* can be only applied to B-equivariant matrix factorizations. Thus to define $j_e^*(\mathcal{F})$ we need to pass from B^2-equivariant structure to B-equivariant structure. For that, we use the auxiliary B^2-equivariant map $j_B : \overline{\mathcal{X}}_2(B) \to \overline{\mathcal{X}}_2(G)$, $j_B(X, b, Y) = (X, b, \mathrm{Ad}_b(Y))$. Let us set $B^2 = B^{(1)} \times B^{(2)}$. The action of $B^{(2)}$ on $\overline{\mathcal{X}}_2(B)$ is free, hence we define $j_e^*(\mathcal{F}) = \mathrm{CE}_{\mathfrak{n}^{(2)}}(j_B^*(\mathcal{F}))^{T^{(2)}}$. For a strongly-equivariant matrix factorization $\mathcal{F}$ the defined $j_e^*(\mathcal{F})$ is isomorphic to the usual pull-back of $\mathcal{F}$ with the B^2-equivariant structure restricted to the B-equivariant structure.

The complex $\mathbb{S}_\beta := j_e^*(\overline{\mathcal{C}}_\beta)$ is naturally an element of the derived category $D^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ of two-periodic complexes of coherent sheaves on $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$. The hypercohomology functor $\mathbb{H}$ is the functor $D^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}_{1,n}) \to \mathrm{Vect}_{\mathrm{gr}}$ to the space of doubly-graded vector spaces. There is an obvious analog of the vector bundle $\mathcal{B}$ over $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ and we define

$$\mathbb{H}^k(\beta) := \mathbb{H}\Big(\mathrm{CE}_{\mathfrak{n}}\left(\mathbb{S}_\beta \otimes \wedge^k \mathcal{B}\right)^T\Big).$$

The next theorem is the main result of [OR18b].

Theorem 4.1.1 ([OR18b]). *For any $\beta \in \mathfrak{Br}_n$, we have the following.*

- *The cohomology of the complex $\mathbb{S}_\beta$ is supported on $\mathrm{Hilb}_{1,n} \subset \mathrm{Hilb}^{\mathrm{free}}_{1,n}$.*
- *The vector space $\mathbb{H}^*(\beta)$ is (up to an explicit grading shift) an isotopy invariant of the closure $L(\beta)$.*

4.2. Koszul complex for link homology

Let $S \subset \{1, \dots, n-1\}$ be given. The virtual structure sheaf $[\mathcal{O}_{Z^S_{1,n}}]^{\mathrm{vir}}$ of the subscheme $Z^S_{1,n} \subset \mathrm{Hilb}_{1,n}$ is defined as Koszul complex of the equivariant coherent sheaves on $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$:

$$[\mathcal{O}_{Z^S_{1,n}}]^{\mathrm{vir}} := \mathrm{K}(\{x_{ii} - x_{i+1,i+1}\}_{i \notin S}, \{[X,Y]_{ij}\}_{(ij) \in \widetilde{S}}),$$

where $X = (x_{ij})$, $Y = (y_{ij})$ are the coordinates on $\mathfrak{b}$ and $\mathfrak{n}$ respectively and

$$\widetilde{S} = \{(ij)\}_{i-j>1}, \{i+1, i\}_{i \in S}. \tag{4.1}$$

The zero-th homology of $[\mathcal{O}^S_{Z_{1,n}}]^{\mathrm{vir}}$ is the structure sheaf of $Z^S_{1,n}$ but the complex has higher homology too. All homology are supported on $\mathrm{Hilb}_{1,n}$ and we have the following.

Proposition 4.2.1. *If $\beta = \mathrm{cox}_S$, we have*

$$\mathbb{S}_\beta = [\mathcal{O}_{Z^S_{1,n}}]^{\mathrm{vir}}.$$

Proof. We have shown that $\overline{\mathcal{C}}_\beta$ is the Koszul matrix factorization with the differential

$$D = \sum_{ij\in\widetilde{S}} \Big(g_{ij}\theta_{ij} + k_{ij}\frac{\partial}{\partial\theta_{ij}}\Big) + \sum_{i\notin S)} \Big(F_i\theta_i + h_i\frac{\partial}{\partial\theta_u}\Big),$$

where θ_{ij} and θ_i are odd variables. The functions h_{ij} and k_i were not discussed previously since the Koszul matrix factorization $\overline{\mathcal{C}}_\beta$ is uniquely up to homotopy determined by the regular sequence $\{g_{ij}\}_{ij\in\widetilde{S}}, \{F_i\}_{i\notin\widetilde{S}}$. For concreteness, let us construct these functions.

For that, let us order the elements of the sets $\widetilde{S}$ and $\overline{S} = [1, n-1]\setminus S$. Then we define

$$k_{ij} = (\overline{W}_{i'j'} - \overline{W}_{ij})/g_{ij}, \quad \overline{W}_{ij} := \overline{W}_{i'j'}|_{g_{ij}=0},$$

where the element $i'j'$ immediately precedes the element ij, and if ij is the largest element of $\widetilde{S}$, then $\overline{W}_{i'j'} = \overline{W}$.

Providing the explicit formulas for h_i is a bit harder but later we work with our matrix factorization in the neighborhood of $g = 1$ hence we can assume that $d_i := \det([g^{(i)}_{1\bullet}, \dots, g^{(i)}_{i,\bullet}]) \neq 0$ and let us also assume that the order of $\overline{S}$ extends the natural order. Then from the first assumption we obtain that $F_i/d_i = (x_{i+1,i+1} - x_{11}) + R_i$ where R_i does not depend on variable $x_{i+1,i+1}$. We define

$$h_i = (\overline{W}_{i'} - \overline{W}_i)/F_i, \quad \overline{W}_i := \overline{W}_{i'}|_{x_{i+1,i+1}=x_{11}+R_i},$$

where i' immediately precedes i, and if i is the largest element of $\overline{S}$, then $\overline{W}_i = \overline{W}_{kl}$ where kl is the smallest element of $\widetilde{S}$.

Finally, let us observe that from our formulas immediately follows that

$$k_{ij}|_{g=1} = \frac{\partial\overline{W}}{\partial g_{ij}}\Big|_{g=1} = [X,Y]_{ij}, \quad F_i|_{g=1} = (x_{i+1,i+1} - x_{11}).$$

Moreover, since $\overline{W}$ has linear dependence on X, we also get that

$$h_i|_{g=1} = d_i^{-1}\frac{\partial\overline{W}}{\partial x_{i+1,i+1}}\Big|_{g=1} = 0. \qquad \square$$

Let us also remark that the dg-scheme from [GNR20, Prop. 3.25] seems to be closely related to the dg-scheme defined by the complex $[\mathcal{O}_{Z^S_{1,n}}]^{\mathrm{vir}}$. We hope to explore this relation in the future. For more explicit connections with [GNR20], see the last section of this paper.

4.3. Proof of Theorem 1.0.2

Theorem 1.1.1. from [OR17] implies that

$$\mathbb{S}_{\beta\cdot\delta^k} = \mathbb{S}_\beta \otimes \mathcal{L}^{\vec{k}}.$$

If we apply this formula for $\beta = \mathrm{cox}_S$ and combine it with the previous proposition, we obtain the statement of the theorem.

5. Explicit computations

In this subsection, we explain how the above geometric computations translate into straightforward homological algebra. We discuss the subtleties of our construction of the knot homology that are related to the t-grading. We also discuss how these subtleties prevent us from using localization techniques in a naive way. All complexity of the situation could be seen in the case $n = 2$ which we discuss at the end of the section.

5.1. Details on t-grading

Since $\deg_t \overline{W} = 2$, we need to explain how we assign the t-degree shifts in our matrix factorizations. We fix convention for $\mathbf{t}^k \cdot M$, the shifted version of a module M. For example, for $1 \in \mathbf{t^k}\mathbb{C}[\overline{\mathcal{X}}_2]$, we have $\deg_t(1) = k$.

Thus let us provide a clarification for the $T_{\rm sc}$-equivariant elements of our category $\mathrm{MF}^{T_{\rm sc}}(\overline{\mathcal{X}}_2, \overline{W})$. An element of $\mathrm{MF}^{T_{\rm sc}}(\overline{\mathcal{X}}_2, \overline{W})$ is the two-periodic complex

$$\cdots \xrightarrow{d_{-1}} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots,$$

where M_i are the free modules, $M_i = M_{i+2}$, $d_i = d_{i+2}$, and the differentials d_i preserve q-degree and shift t-degree by 1. Let us call this property *degree one property.* The category $\mathrm{MF}^{T_{\rm sc}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$ is the appropriate equivariant enhancement of the previous category.

For example, the element $\overline{\mathcal{C}}_+ \in \mathrm{MF}^{T_{\rm sc}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$ is the two-periodic complex

$$\cdots \xrightarrow{d_1} R \xrightarrow{d_0} \mathbf{t}R \xrightarrow{d_1} R \xrightarrow{d_2} \mathbf{t}R \xrightarrow{d_3} \cdots,$$

where $R = \mathbb{C}[\mathcal{X}_2(G_2)]$ and $d_i = (x_{11} - x_{22})g_{11} + x_{12}g_{21}$ for odd i, and $d_i = y_{12}g_{21}$ for even i.

The elements in the ring $\mathbb{C}[\overline{\mathcal{X}}_2]$ have even t degrees thus the only source for the elements of t-degree in $\mathbb{S}_\beta$ are the shifts $\mathbf{t}^k$ in our complexes. Since the convolution needs to preserve the degree one property, we require that the degree t shifts in the Chevalley–Eilenberg complex are defined by the condition that the Chevalley–Eilenberg differentials shift t-degree by 1.

As a final step of the construction of $\mathbb{S}_\beta$ we apply the pull-back j_e^* to the complex $\overline{\mathcal{C}}_\beta$ where j_e is the embedding of $\widetilde{\mathrm{Hilb}}^{\rm free}_{1,n}$ inside $\mathfrak{n} \times \mathfrak{b}$. To construct $j_e^*(\overline{\mathcal{C}}_\beta)$ we need to choose an affine cover $\widetilde{\mathrm{Hilb}}^{\rm free}_{1,n} = \bigcup_i U_i$ by the B-equivariant charts U_i, then the pull-back $j_e^*(\overline{\mathcal{C}}_\beta)$ is the Cech complex $\check{C}_{U_\bullet}(\overline{\mathcal{C}}_\beta)$. Moreover, since we would like to preserve the degree one property, we shift t-degrees in the Cech complex so that the Cech differentials are of t-degree 1.

Since we are working with $T_{\rm sc}$-equivariant complexes of sheaves on the Hilbert scheme, it is very tempting to use localization technique to obtain explicit formulas for the super-polynomial for the links. However, the degree one property effectively prevents us from doing this in most of the cases. We expand on this issue in Section 5.3 where we discuss the two-strand case but for now let us point out that formulas obtained by localization could only produce super-polynomial that has only even powers of t because the elements $\mathbb{C}[\overline{\mathcal{X}}_2]$ have even t-degree. On the other hand, there are many examples of the links with the knot homology that are not t-even.

To end the discussion on a positive note let us point out that HOMFLY-PT polynomial is well suited for localization technique, exactly because of the degree one property. Let us denote by $\chi_q(\mathbb{S})$ the $\mathbb{C}^*$-equivariant Euler characteristics of a two-periodic complex $\mathbb{S} \in D^b_{\mathbb{C}^*}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ where $\mathbb{C}^*$ acts with opposite weights on $\mathfrak{n}$ and $\mathfrak{b}$.

Theorem 5.1.1 ([OR18b]). *For any β, we have*

$$P(L(\beta)) = \sum_i \chi_q(\mathbb{S}_\beta \otimes \wedge^i \mathcal{B}).$$

5.2. Conjectures for Coxeter links

Let $j_\triangle : \mathfrak{b} \times \mathfrak{n}$ be the B-equivariant embedding inside $\overline{\mathcal{X}}_2$ and let $\mathfrak{b}_S \subset \mathfrak{b}$ be the subspace defined by equations $x_{ii} = x_{i+1,i+1}$ for $i \notin S$. The results of the previous section imply that we have the homotopy of the two-periodic complexes:

$$j^*_\triangle(\overline{\mathcal{C}}_{\mathrm{cox}_S}) \sim K_{\mathrm{cox}_S} \otimes \mathcal{O}_{\mathfrak{b}_S \times \mathfrak{n}}, \quad K_{\mathrm{cox}_S} := \bigotimes_{ij \in \widetilde{S}} \Big[R \xrightarrow{[X,Y]_{ij}} \mathbf{t} \cdot R \Big], \tag{5.1}$$

where $\widetilde{S} = \{j - i > 1\} \cup S$ and $R = \mathbb{C}[\overline{\mathcal{X}}_2]$. The tensor product above is a restriction of the complex to the subvariety $\mathfrak{b}_S \times \mathfrak{n}$ and to simplify notations we abbreviate the restriction by K_{cox_S}.

Let us cover $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ by the affine charts U_i. Then we have the following expression for the homology:

$$\mathbb{H}^m(\mathrm{cox}_S \cdot \delta^{\vec{k}}) = \mathrm{CE}_{\mathfrak{n}} \left(\check{C}_{U_\bullet}(K_{\mathrm{cox}_S} \otimes \wedge^m \mathcal{B} \otimes \chi_{\vec{k}}) \right)^T,$$

where $\chi_{\vec{k}}$ is the notation for the character of the torus T.

We slightly simplify the above formula by eliminating the Chevalley–Eilenberg complex with the following trick. In the next section we describe the affine subspaces $\mathbb{A}_\bullet \subset \mathrm{Hilb}^{\mathrm{free}}_{1,n}$ such that the affine varieties $B\mathbb{A}_\bullet$ form an affine cover of $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ and the B-stabilizer is trivial. Hence if we choose $B\mathbb{A}_\bullet$ as our Cech cover, then because of the triviality of the stabilizers, the Chevalley–Eilenberg complex is acyclic on every chart and extracting its zero-th homology on the chart $B\mathbb{A}_S$ corresponds to the restriction on the affine subvariety $T\mathbb{A}_S$ which we denote by $\mathbb{T}_S$. Thus we have the following least geometry rich statement.

Corollary 5.2.1. *For any $\vec{k}$ and S, we have*

$$\mathbb{H}^m\big(\mathrm{cox}_S \cdot \delta^{\vec{k}}\big) = \big(\check{C}_{\mathbb{A}_\bullet}(K_{\mathrm{cox}_S}) \otimes \wedge^m \mathcal{B} \otimes \chi_{\vec{k}}\big)^T.$$

Since the line bundle $\mathcal{L}^{\vec{k}}$ is very ample for sufficiently positive $\vec{k}$, for such $\vec{k}$, the Cech complex becomes acyclic a we have the following.

Corollary 5.2.2. *For sufficiently positive $\vec{k}$, we have*

$$\mathbb{H}^i\left(\mathrm{cox}_S \cdot \delta^{\vec{k}}\right) = \left(H^0_{\check{C}}(K_{\mathrm{cox}_S} \otimes \wedge^i \mathcal{B} \otimes \chi_{\vec{k}})\right)^T.$$

In the last formula, we eliminated all possible sources of odd t-degree shifts with exception of the shifts inside the complex K_{cox_S}. Thus as it is we still can not apply localization methods to extract the explicit formulas. So let us correct the complex K_{cox_S} to make it comply with the localization formula:

$$K^{\mathrm{even}}_{\mathrm{cox}_S} = \bigotimes_{ij \in \widetilde{S}} \left[R \xrightarrow{[X,Y]_{ij}} \mathbf{t}^2 R\right], \tag{5.2}$$

and let us introduce the computationally friendly 'invariant'

$$\mathscr{P}^{\mathrm{even}}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})) = \sum_{i,j} (-1)^j \dim_{q,t} \left(\mathrm{H}^j(\check{C}_{\mathbb{A}_\bullet}(K^{\mathrm{even}}_{\mathrm{cox}_S} \otimes \wedge^i \mathcal{B} \otimes \chi_{\vec{k}}))\right) a^i.$$

This invariant is the equivariant Euler characteristic of the complex and in the next section we explain how one can obtain explicit localization formulas for this Euler characteristic with the localization technique.

Several recent preprints [Hog17], [Mel22] suggest that at least for a sufficiently positive $\vec{k}$ the sum above has nonzero terms only for $j = 0$. In other words, it is reasonable to pose the following.

Conjecture 5.2.3. *For sufficiently positive $\vec{k}$, we have*

$$\mathscr{P}^{\mathrm{even}}\left(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})\right) = \mathscr{P}\left(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})\right).$$

As we will see in the next subsection this conjecture is false without the assumption of positivity. It is false for very negative $\vec{k}$.

In the last section we discuss th following stronger and more geometric version of the conjecture for $\beta = \mathrm{cox}$.

Conjecture 5.2.4. *If the vector $\vec{k}$ is sufficiently positive, then the higher degree hyper-cohomology of the complex $\mathrm{CE}_{\mathfrak{n}}(\mathbb{S}_{\beta \cdot \delta^{\vec{k}}} \otimes \wedge^\bullet \mathcal{B})^T$ vanishes.*

5.3. Two strand case

In this subsection, we compute homology for the links obtained by closing braids on two strands. This illustrates our computational technique; also one can compare our computations with that in [GNR20, Sect. 5]. The results of computations in [GNR20] and in our paper match and that provides yet another evidence for the existence of a close relation between the theory outlined in [GNR20] and our.

First, let us describe the computation of the homology of $T_{2,2n+1} = L(\sigma_1^{2n+1})$. Since $\sigma_1 = \mathrm{cox}_S$, $S = \varnothing$, in this case $\mathfrak{b}_S = \mathfrak{n} \oplus \mathbb{C}$. Let us fix coordinates on it: $\mathfrak{b}_S = \{x_{12}E_{12} + x(E_{11} + E_{22})\}$. Respectively, we fix the notation $R = \mathbb{C}[x, x_{12}, y_{12}]$ for the coordinate ring on $\mathfrak{b}_S \times \mathfrak{n}$.

In this case, K_{cox_S} is just R. Moreover, the intersection $\widetilde{\mathrm{Hilb}}^{\mathrm{free}}_{1,n} \cap \mathfrak{b}_S \times \mathfrak{n}$ is covered by two charts $\mathbb{A}_1 = \{x_{12} \neq 0\}$, $\mathbb{A}_2 = \{y_{12} \neq 0\}$. That is, the homology $\mathbb{H}^k(T_{2,2n+1})$ is equal to the homology of the complex

$$\left((R_{x_{12}} \oplus R_{y_{12}}) \otimes \chi^{n-k} \to \mathbf{t}R_{x_{12}y_{12}} \otimes \chi^{n-k}\right)^T,$$

where $\chi : T \to \mathbb{C}^*$ is the character $(\lambda, \mu) \mapsto \lambda$.

Thus the knot homology of $T_{2,2n+1}$ is a triply graded vector space which is the tensor product of $\mathbb{C}[x]$ and the space

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n)) \oplus \mathbf{t}\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n)) \oplus \mathbf{a}\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n-1)) \oplus \mathbf{at}\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n-1))$$

shifted by $(\mathbf{a}/\mathbf{t})^n$. To compute the super-polynomial we just need the following formulas for the dimensions of the homology of the line bundles:

$$\dim_{q,t}(\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n))) = \sum_{i=0}^{n} q^{2i}(t/q)^{2n-2i},$$
$$\dim_{q,t}(\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(n))) = \sum_{i=0}^{-n-2} (q)^{2i}(t/q)^{-2n-2i-4}.$$

The case of the torus link $T_{2,2n}$ is more involved. Since $S = \{1\}$, in this case $\mathfrak{b}_S = \mathfrak{b}$. Let us denote by R the ring of functions on $\mathfrak{b} \times \mathfrak{n}$: $R = \mathbb{C}[x_+, x_-, x_{12}, y_{12}]$ where $x_+ = x_{11} + x_{22}$, $x_- = x_{11} - x_{22}$. In this notation we have

$$K_{\mathrm{cox}_S} = [R \xrightarrow{y_{12}x_-} \mathbf{t}R].$$

The Cech cover in this case is basically the same as in the previous case: $\mathbb{A}_1 = \{x_{12} \neq 0\}$ and $\mathbb{A}_2 = \{y_{12} \neq 0\}$. Thus the homology of the torus link $T_{2,2n}$ is the sum of vector spaces $\mathbb{H}^0 \oplus \mathbf{a}\mathbb{H}^1$ shifted by $(\mathbf{a}/\mathbf{t})^n$ where $\mathbb{H}^i$ is homology of the complex

$$\begin{array}{ccc} \mathbf{t}R_{x_{12}y_{12}}[n-i] & \xrightarrow{y_{12}x_-} & \mathbf{t}^2R_{x_{12}y_{12}}[n-i+1] \\ \uparrow & & \uparrow \\ R_{y_{12}}[n-i] \oplus R_{x_{12}}[n-i] & \xrightarrow{y_{12}x_-} & \mathbf{t}R_{y_{12}}[n-i-1] \oplus \mathbf{t}R_{x_{12}}[n-i-1] \end{array}, \qquad (5.3)$$

and $R[m]$ stands for the degree m part of the ring R with the following degrees of the generators:

$$\deg x_{12} = \deg y_{12} = 1, \quad \deg_{x_-} = \deg_{x_+} = 0.$$

The complex above is the tensor product of $\mathbb{C}[x_+]$ and the complex with x_+ set to zero. Thus to make our computations easier we work modulo ideal (x_+), $R' = R/(x_+)$.

Geometrically the homology of the last complex could be interpreted as homology of the line bundle $\mathcal{O}(n-i)$ on the union of a projective line and an affine line that intersect transversally at a single point. But for illustration of our methods we proceed algebraically.

First, let us observe that the horizontal differential is injective and we can contract the complex in this direction. For that, we need to describe the cokernel of the map. Since we have

$$R'_{y_{12}}[m] = \mathbb{C}[x_{12}/y_{12}, x_-]y_{12}^m,$$
$$R'_{x_{12}}[m] = \mathbb{C}[y_{12}/x_{12}, x_-]x_{12}^m.$$

the cokernel of the map on $R'_{y_{12}}[m]$ is $\mathbb{C}[x_{12}/y_{12}]y_{12}^m$, and the cokernel on $R'_{x_{12}}[m]$ is the sum

$$\mathbb{C}[y_{12}/x_{12}]x_{12}^m \oplus x_-\mathbb{C}[x_-]x_{12}^m.$$

Finally, $R'_{x_{12}y_{12}}[m] = \mathbb{C}[x_-, (x_{12}/y_{12})^{\pm 1}]x_{12}^m$ and the cokernel of the map on this space is $\mathbb{C}[(x_{12}/y_{12})^{\pm 1}]y_{12}^m$ There is the induced Cech differential d_C on the cokernels

$$\mathbb{C}[x_{12}/y_{12}]y_{12}^m \oplus \mathbb{C}[y_{12}/x_{12}]x_{12}^m \oplus x_-\mathbb{C}[x_-]x_{12}^m \xrightarrow{d_C} \mathbb{C}[(x_{12}/y_{12})^{\pm 1}]y_{12}^m.$$

If $m \geq 0$ this induced differential is surjective and the kernel spanned by

$$\langle y_{12}^m, x_{12}y_{12}^{m-1}, \dots, x_{12}^m\rangle \oplus x_-\mathbb{C}[x_-]x_{12}^m.$$

Let us denote the last vector space by V_m.

On other hand, if m is negative, then the kernel and cokernel of the induced differentials are the vector spaces

$$x_-\mathbb{C}[x_-]x_{12}^m, \quad \langle y_{12}^{-m-2}, y_{12}^{-m-1}x_{12}, \dots, x_{12}^{-m-2}\rangle.$$

Let us denote the first vector space by V'_m and the second by V''_m.

Thus for $n \geq 0$, the knot homology of $T_{2,2n}$ is triply graded vector space

$$\mathbf{a}/\mathbf{t}^n \cdot (\mathbf{t}V_n \oplus \mathbf{at}V_{n-1}) \otimes \mathbb{C}[x_+],$$

and for negative n the knot homology of $T_{2,2n}$ is the vector space

$$\mathbf{a}/\mathbf{t}^n \cdot (\mathbf{t}V'_n \oplus \mathbf{t}^2V''_n \oplus \mathbf{at}V'_{n-1} \oplus \mathbf{at^2}V''_{n-1}) \otimes \mathbb{C}[x_+].$$

To convert the last formula into super-polynomial we only need to remember that

$$\deg_{q,t} x_{12} = \deg_{q,t} x_- = \deg_{q,t} x_+ = q^2, \quad \deg_{q,t} y_{12} = t^2/q^2.$$

We would like to point out that the case of the links $T_{2,2n}$ is more complex than the case of the knots $T_{2,2n+1}$. For example, in the case of knots, elements of the knot homology of $T_{2,2n+1}$ for any n have the same parity of t-degree. It is no longer true for the links: the homology of $T_{2,2n}$ for negative n contains elements of odd and even t-degree. Thus it seems to be very unlikely that there is some localization type formula that produces the super-polynomial of $T_{2,2n}$ for (very) negative n.

6. Localization and explicit formulas for homology

In this section, we present an explicit formulas for the graded dimension of the homology of the Coxeter links under the assumption that the corresponding braid is sufficiently positive. First, we discuss the geometry of $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ since that is the space where we perform our localization computation.

6.1. Local charts

It is shown in [OR18b] that the free Hilbert scheme $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ could be covered by affine charts. In this subsection, we remind this construction. First, we describe the combinatorial data used for labeling of the charts.

Let us denote by NS_n the set of the nested pairs of sets with the following properties. An element $\mathbf{S} \in NS_n$ is a pair of nested sets

$$\mathbf{S}^1_x \supset \mathbf{S}^2_x \supset \cdots \supset \mathbf{S}^{n-1}_x \supset \mathbf{S}^n_x = \varnothing,$$
$$\mathbf{S}^1_y \supset \mathbf{S}^2_y \supset \cdots \supset \mathbf{S}^{n-1}_y \supset \mathbf{S}^n_y = \varnothing,$$

such that

$$\mathbf{S}^k_x, \mathbf{S}^k_y \subset \{k+1, \dots, n\}, \quad |\mathbf{S}^i_x| + |\mathbf{S}^i_y| = n - i.$$

Let us define the sets of pivots of $\mathbf{S}$ as the sets $P_x(\mathbf{S}), P_y(\mathbf{S})$ consisting of the pairs

$$P_x(\mathbf{S}) = \{(ij) \mid j \in \mathbf{S}^i_x \setminus \mathbf{S}^{i+1}_x\}, \quad P_y(\mathbf{S}) = \{(ij) \mid j \in \mathbf{S}^i_y \setminus \mathbf{S}^{i+1}_y\}.$$

To an element $\mathbf{S} \in NS_n$ we attach the following affine space $\mathbb{A}_{\mathbf{S}} \subset \mathfrak{n} \times \mathfrak{n}$:

$$(X, Y) \in \mathbb{A}_{\mathbf{S}} \text{ if } x_{ij} = 1, ij \in P_x(\mathbf{S}),\ y_{ij} = 1, ij \in P_y(\mathbf{S}), \text{ and}$$
$$x_{i,j} = 0 \text{ if } j \in \mathbf{S}^i_x,\ y_{i,j} = 0 \text{ if } j \in \mathbf{S}^i_y.$$

For a given $\mathbf{S}$, we denote by $N_x(\mathbf{S})$ and $N_y(\mathbf{S})$ the indices (ij) such that x_{ij}, respectively y_{ij} are not constant on $\mathbb{A}_{\mathbf{S}}$. From the construction, we see that $|N(\mathbf{S})| = n(n-1)/2$.

Let us denote by $\mathfrak{h}$ the subspace of the diagonal matrices inside $\mathfrak{b}$. The sum $\mathfrak{h} + \mathbb{A}_{\mathbf{S}}$ is an affine subspace inside $\mathfrak{b} \times \mathfrak{n}$ and we show in [OR18b] the following.

Proposition 6.1.1. *The space* $\widetilde{\mathrm{Hilb}^{\mathrm{free}}_{1,n}} \subset \mathfrak{b} \times \mathfrak{n}$ *is covered by the orbits of affine spaces* $B(\mathfrak{h} + \mathbb{A}_{\mathbf{S}})$, $\mathbf{S} \in NS_n$. *Moreover, the points in* $\mathfrak{h} + \mathbb{A}_{\mathbf{S}}$ *have trivial stabilizers.*

Thus the proposition implies that the affine subspaces $\mathfrak{h} + \mathbb{A}_{\mathbf{S}}$ provide an affine cover for the quotient $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$. As our system for labeling of the charts might look a bit artificial for the people studying Hilbert schemes, let us introduce an equivalent but somewhat more familiar system.

6.2. Combinatorics of the cover

Also it is probably a good place to enrich our notation to make it more compatible with the notation in [GNR20]. The free Hilbert scheme has a natural map $\rho : \mathrm{Hilb}^{\mathrm{free}}_{1,n} \to \mathfrak{h}$ given by the eigenvalues of the first matrix. Respectively, we define $\mathrm{Hilb}^{\mathrm{free}}_{1,n}(Z)$ to be the preimage $\rho^{-1}(Z)$.

Now recall another definition of the free Hilbert scheme as the space of the nested chains of the left ideals

$$\mathrm{Hilb}^{\mathrm{free}}_{1,n} = \{I_n \subset \cdots \subset I_1 \subset I_0 = \mathbb{C}\langle X,Y\rangle | \mathbb{C}\langle X,Y\rangle/I_i = \mathbb{C}^i\rangle\}.$$

Given a sequence of noncommutative monomials $\vec{m} = (m_1, \dots, m_n)$, we define the following sublocus of the free Hilbert scheme:

$$\mathbb{A}_{\vec{m}} = \{I_\bullet \mid \mathbb{C}\langle X,Y\rangle/I_k = \langle m_1, \dots, m_k\rangle\}.$$

Now let us explain how one could produce a vector of monomials $\vec{m}(\mathbf{S})$ from the element of $\mathbf{S} \in NS_n$. Essentially, we just retrace the definition of the free Hilbert scheme. We construct the vector inductively starting with $m_1(\mathbf{S})$ which is X if $(n-1,n) \in P_x(\mathbf{S})$ and is Y if $(n-1,n) \in P_y(\mathbf{S})$. The inductive step is the following:

$$m_k(\mathbf{S}) = \begin{cases} Xm_{n-j}(\mathbf{S}) & \text{if } (k,j) \in P_x(\mathbf{S}), \\ Ym_{n-j}(\mathbf{S}) & \text{if } (k,j) \in P_y(\mathbf{S}). \end{cases}$$

In the case of the usual nested Hilbert scheme it is convenient to label the torus fixed points by the standard Young tableaux (SYT). By analogy with the commutative case we also introduce an analog of the SYT for noncommutative case. The generalized SYT, abbreviated GYT_n, are labelings L of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by the subsets of $[1,n]$ such that every element appears once in the labeling sets. That is, an element of GYT_n is a map

$$L : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \text{subsets of } [1,n]$$

with the above mentioned properties.

It is natural to think about the labels as the labels on 1×1 squares that pave the first quadrant. We also require that the set of squares with nonempty labeling is connected, in other words, all our generalized tableaux are connected. The standard Young tableaux are examples of generalized Young tableaux but obviously there are many GYT which are not SYT.

There is a natural map $\mathrm{GYT} : NS_n \to \mathrm{GYT}_n$ that could be described by the condition $k \in L(\mathrm{GYT}(\mathbf{S}))(ij)$ if $\deg_X(m_k(\mathbf{S})) = i$ and $\deg_Y(m_k(\mathbf{S})) = j$. Since the noncommutative Hilbert scheme contains the commutative one, the image of the above map contains the set SYT_n. But we do not understand the combinatorics well. For example, we do not understand the image of this map, the answer to the following question is probably known to the experts:

Question. What is the image of the map $NS_n \to \mathrm{GYT}_n$? Is this map injective?

We checked on computer the injectivity for small n. Let us also give a few examples of GYT's that are not SYT and appear in the image:

$$\begin{matrix} 1 & 2 & \\ 3 & 4 & 5 \\ 6 & & \end{matrix}\,, \qquad \begin{matrix} 1 & 2 \\ 4 & * \end{matrix}\,, \qquad \begin{matrix} 1 & 2 & 6 \\ 3 & & 7 \\ 4 & 5 & \end{matrix}\,,$$

where $* = \{3, 5\}$.

Finally, let us observe that size of the set NS_n is $n!$. We expect that there is a natural correspondence between this set and the group $\mathfrak{S}_n$ of all permutations of $[1, n]$. On the other hand, the RS algorithm assigns to an element of $\mathfrak{S}_n$ a pair of SYT of the same shape. Thus we expect the existence of a modification of the map GYT that has as the target the set of the pairs from the RS algorithm. We leave this problem for the future publications where we plan to study the connection between the geometry of the noncommutative Hilbert scheme and the Young projectors in $\mathbb{C}[\mathfrak{S}_n]$.

6.3. Geometry of the torus fixed locus

Given a element $\mathbf{S} \in \widetilde{NS_n}$, we denote by $M_x(\mathbf{S})$ and $M_y(\mathbf{S})$ the corresponding pair of matrices from $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$. The entries x_{ij}, $ij \in N_x(\mathbf{S})$ and y_{ij}, $ij \in N_y(\mathbf{S})$ together with the coordinates along $\mathfrak{h}$ provide local coordinates in the neighborhood of the point $M_x(\mathbf{S}), M_y(\mathbf{S})$. Below we provide a formula for the weights of the T_{sc}-action on these coordinates.

First, let us define the pair of vectors of the weights $w_x(\mathbf{S})$ and $w_y(\mathbf{S})$. We define them inductively, starting with $w_x^n(\mathbf{S}) = 0$ and $w_y^n(\mathbf{S}) = 0$. The inductive step is provided by

$$w_x^j(\mathbf{S}) = \begin{cases} w_x^k(\mathbf{S}) + 1 & \text{if } (jk) \in P_x(\mathbf{S}), \\ w_x^k(\mathbf{S}) & \text{if } (jk) \in P_y(\mathbf{S}), \end{cases} \quad w_y^j(\mathbf{S}) = \begin{cases} w_y^k(\mathbf{S}) + 1 & \text{if } (jk) \in P_y(\mathbf{S}), \\ w_y^k(\mathbf{S}) & \text{if } (jk) \in P_x(\mathbf{S}). \end{cases}$$

The weights above are defines in such a way that

$$\begin{aligned} t^{-1}\mathrm{Ad}_{t_x}(X) \in \mathbb{A}_{\mathbf{S}}, \qquad & \mathrm{Ad}_{t_x}(Y) \in \mathbb{A}_{\mathbf{S}}, \\ \mathrm{Ad}_{t_y}(Y) \in \mathbb{A}_{\mathbf{S}}, \quad & t^{-1}\mathrm{Ad}_{t_y}(Y) \in \mathbb{A}_{\mathbf{S}}, \end{aligned}$$

for any $(X, Y) \in \mathbb{A}_{\mathbf{S}}$ and $t_x = \mathrm{diag}(t^{w_x^1}, \ldots, t^{w_x^n})$, $t_y = \mathrm{diag}(t^{w_y^1}, \ldots, t^{w_y^n})$.

From the discussion it is immediate that the weights of the action are given by the formula

$$\begin{aligned} d_x(ij) &= w_x^i - w_x^j + 1, \quad d_y(ij) = w_y^i - w_y^j, \quad ij \in N_x(\mathbf{S}), \\ d_x(ij) &= w_x^i - w_x^j, \qquad\quad d_y(ij) = w_y^i - w_y^j + 1, \quad ij \in N_y(\mathbf{S}). \end{aligned}$$

Now let us write the localization formula for $\chi(K^{\mathrm{even}}_{\mathrm{cox}_S} \otimes \mathcal{L}^{\vec{k}} \otimes \wedge_a(\mathcal{B}))$. For the localization formula, we need the weights of the differentials in the complex. Informally, we call these weights *weights of obstruction space*:

$$o_x(ij) = w_x^i - w_x^j + 1, \quad o_y(ij) = w_y^i - w_y^j + 1.$$

We denote by $T_{\mathbf{S}}$ the tangent space at $(M_x(\mathbf{S}), M_x(\mathbf{S}))$ and by $\mathrm{Ob}_{\mathbf{S}}$ the 'obstruction' space spanned by the vectors with the weights $o(ij)$, $i - j > 0$.

Armed with the above formulas we can write the localization formula for the sum $\sum_i \chi(K^{\mathrm{even}}_{\mathrm{cox}} \otimes \mathcal{L}^{\vec{k}} \otimes \wedge^i \mathcal{B})a^i$ as

$$\sum_{\mathbf{S}\in NS_n} Q^{\vec{k}\cdot w_x} T^{\vec{k}\cdot w_y} \Omega_{\mathbf{S}}(Q,T,a;\mathrm{cox}_S),$$

$$\Omega_{\mathbf{S}}(Q,T,a;\mathrm{cox}_S) = \frac{(1-Q)^{n-|S|}\prod_{ij\in\widetilde{S}}(1-Q^{o_x}T^{o_y})}{\prod_{ij\in N_x(S)}(1-Q^{d_x}T^{d_y})\prod_{ij\in N_y(S)}(1-Q^{d_x}T^{d_y})} \cdot \prod_{i=1}^{n-1}(1-aQ^{w_x^i}T^{w_y^i})$$

where $\widetilde{S}$ is given by (4.1), $o_x = o_x(ij), o_y = o_y(ij)$, $d_x = d_x(ij), d_y = d_y(ij)$, and Q, T variables are related to the standard variables q, t by

$$Q = q^2, \quad T = t^2/q^2.$$

Unfortunately, the sum above is not well-defined because for some $\mathbf{S}$ the vector (d_x, d_y) vanishes. It is a manifestation of the fact that the scheme $\left(\mathrm{Hilb}^{\mathrm{free}}_{1,n}\right)^{T_{\mathrm{sc}}}$ is not zero-dimensional. For example, the family of the matrices

$$X = \begin{bmatrix} 0 & u & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where u is any, lies inside $\mathbb{A}_{\mathbf{S}}$ for $\mathbf{S}$ with $\mathbf{S}_x = \{4,3\} \supset \{3\} \supset \{\varnothing\} \supset \{\varnothing\}$ and $\mathbf{S}_y = \{4\} \supset \{4\} \supset \{4\} \supset \{\varnothing\}$. It is also fixed by the torus T_{sc}.

Remark 6.3.1. As we see above the torus fixed locus is not discrete in general but we expect the locus to have virtual dimension zero. Indeed, the computer experiment suggests that for any $\mathbf{S} \in NS_n$, we have the inequality

$$\dim\left(\mathrm{Obs}_{\mathbf{S}}\right)^{T_{\mathrm{sc}}} \geq \dim\left(T_{\mathbf{S}}\right)^{T_{\mathrm{sc}}}.$$

However, on the commutative Hilbert scheme the torus fixed locus is zero-dimensional and the torus fixed points are labeled by the SYT_n. Let us identify the corresponding subset NS_n:

$$[M_x(\mathbf{S}), M_y(\mathbf{S})] = 0 \text{ iff } \mathbf{S} \in NS_n^{\mathrm{syt}},$$

where $NS_n^{syt} \subset NS_n$ consists of $\mathbf{S}$ such that $\mathrm{GYT}(\mathbf{S})$ is a standard Young tableaux.

We propose the following.

Proposition 6.3.2. *For sufficiently positive $\vec{k}$, we have the following localization formula*

$$\mathcal{P}^{\mathrm{even}}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})) = \sum_{\mathbf{S}\in NS_n^{\mathrm{syt}}} Q^{\vec{k}\cdot w_x} T^{\vec{k}\cdot w_y} \Omega_{\mathbf{S}}(Q,T,a;\mathrm{cox}_S).$$

Proof. Let $\beta = \mathrm{cox}\cdot\delta^{\vec{k}}$. Since the complex $\mathbb{S}_\beta$ is supported on the commutative Hilbert scheme, the complex $\mathbb{S}_\beta$ is contractible in the affine neighborhood of $(M_x(\mathbf{S}), M_y(\mathbf{S}))$ if $\mathbf{S} \notin SYT_n$. The union $\mathrm{Hilb}'_{1,n} := \bigcup_{\mathbf{S}\in NS_n^{\mathrm{syt}}} \mathbb{A}_{\mathbf{S}}$ is an open subset inside $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$ and by the previous remark the restriction on this open subset does not change the total homology. Since T_{sc}-fixed locus inside $\mathrm{Hilb}'_{1,n}$ is zero-dimensional, the formula in the proposition is the standard localization formula. □

The above formula is equivalent to the formula from [GNR20, Cor. 1.3]: the formula in [GNR20] is also a sum over SYT's and the corresponding terms in [GNR20] and in our formula coincide after we cancel the matching factors in the numerator and denominator.

Besides the similarity to the previous conjectures there are other observations that support the conjecture. For example, it is elementary to show that

$$[M_x(\mathbf{S}), M_y(\mathbf{S})] = 0 \text{ iff } \mathbf{S} \in \mathrm{SYT}_n.$$

Hence since the complex $\mathcal{C}_\beta$ is supported on the commutative Hilbert scheme, the complex $\mathcal{C}_\beta$ is contractible in some affine neighborhood of $(M_x(\mathbf{S}), M_y(\mathbf{S}))$ if $\mathbf{S} \notin \mathrm{SYT}_n$.

6.4. Fourth grading and localization

In this section, we provide an explanation for the even super-polynomial $\mathscr{P}^{\mathrm{even}}$ as well as some conjectures for the cases when the even super-polynomials coincide with the usual super-polynomial. As we explain below, both $\mathscr{P}^{\mathrm{even}}$ and $\mathscr{P}$ are specializations of a conjectural richer invariant.

As it is explained in [OR18b] and outlined in Section 5.2, for a braid $\beta \in \mathfrak{Br}$, one can construct an element $\overline{\mathcal{C}}_\beta \in \mathrm{MF}^{T_{\mathrm{sc}}}_{B^2}(\overline{\mathcal{X}}_2, \overline{W})$. To compute the triply-graded homology of the link closure $L(\beta)$, we need to work with $\mathbb{S}_\beta = j_e^*(\overline{\mathcal{C}}_\beta)$.

Two periodic complex $\mathbb{S}_\beta \in \mathrm{MF}^{\mathrm{str}}_B(\widetilde{\mathrm{Hilb}_{1,n}}, 0)$ has differential of degree t with respect to the T_{sc}-action. It is shown in [OR18a] that $\overline{\mathcal{C}}_\beta$ is isomorphic to a strictly B^2-equivariant matrix factorization; thus we can assume that $\mathbb{S}_\beta \in \mathrm{MF}^{\mathrm{str}}_B(\widetilde{\mathrm{Hilb}_{1,n}}, 0) = \mathrm{D}^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$.

The objects of the derived category $\mathrm{D}^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ are two periodic complexes of coherent T_{sc}-equivariant sheaves with differentials of degree t with respect to T_{sc}. It is more natural to consider the category $\mathrm{D}^b_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ of bounded complexes of T_{sc}-equivariant coherent sheaves with differentials of degree t. There is a folding functor that relates these categories:

$$\mathrm{Fold}\colon \mathrm{D}^b_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n}) \to \mathrm{D}^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n}), \quad \mathcal{C} \mapsto \bigoplus_{n\in\mathbb{Z}} \mathcal{C}[2n],$$

where $[n]$ is the notation for the homological shift and $\mathcal{C}$ is the complex of locally-free sheaves.

Clearly, not all objects in $\mathrm{D}^{\mathrm{per}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ are foldings of bounded complexes. However, as explained in the previous sections $\mathbb{S}_{\mathrm{cox}_S} = \mathrm{Fold}(K_{\mathrm{cox}_S})$, where formula (5.1) for K_{cox_S} is interpreted as a tensor product of bounded complexes. Let us call the two-periodic complexes in the image of Fold *unrollable.*

It is an interesting question for which $\beta \in \mathfrak{Br}_n$ the corresponding two-periodic complex $\mathbb{S}_\beta$ is unrollable. For example, a two-periodic complex for the half-twist on three strands $\beta = \sigma_1 \cdot \sigma_2 \cdot \sigma_1$, the two-periodic complex $\mathbb{S}_\beta$ does not appear to be unrollable, see [OR18b].

Let us define $\mathrm{D}^{\mathrm{b}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})_{\mathrm{even}}$ to be the derived category of bounded complexes of T_{sc}-equivariant complexes of coherent sheaves with T_{sc}-invariant differentials. There is a shifting functor that relates the last two categories:

$$\mathrm{Sh}_{\mathrm{even}} : \mathrm{D}^{\mathrm{b}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n}) \to \mathrm{D}^{\mathrm{b}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})_{\mathrm{even}}, \quad \bigoplus_i \mathcal{C}_i \mapsto \bigoplus_i \mathbf{t}^i \cdot \mathcal{C}_i,$$

where $\mathcal{C} = (\bigoplus \mathcal{C}_i, D)$, $D : \mathcal{C}_i \to \mathcal{C}_{i+1}$.

In the context of the paper, the relevant example is $\mathrm{Sh}_{\mathrm{even}}(K_{\mathrm{cox}_S}) = K^{\mathrm{even}}_{\mathrm{cox}_S}$ where we interpret formula (5.2) for $K^{\mathrm{even}}_{\mathrm{cox}_S}$ as tensor product of bounded complexes. That motivates us to define a super-polynomial of four-variables. Suppose $\mathbb{S}_\beta = \mathrm{Fold}(\widehat{\mathbb{S}}_\beta)$, $\widehat{\mathbb{S}}_\beta \in \mathrm{D}^{\mathrm{b}}_{T_{\mathrm{sc}}}(\mathrm{Hilb}^{\mathrm{free}}_{1,n})$ for some $\beta \in \mathfrak{Br}_n$ then we define

$$\mathfrak{P}(\beta) = \sum_{i,j} h^j \dim_{q,t} \mathrm{H}^j(\mathrm{Sh}_{\mathrm{even}}(\widehat{\mathbb{S}}_\beta \otimes \textstyle\bigwedge^i \mathcal{B})).$$

Thus $\mathfrak{P}(\beta)$ is a common generalization of $\mathscr{P}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}}))$ and of $\mathscr{P}^{\mathrm{even}}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}}))$:

$$\begin{aligned} \mathscr{P}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})) &= \mathfrak{P}(\mathrm{cox}_S \cdot \delta^{\vec{k}})|_{h=t^{-1}}, \\ \mathscr{P}^{\mathrm{even}}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}})) &= \mathfrak{P}(\mathrm{cox}_S \cdot \delta^{\vec{k}})|_{h=-1} \end{aligned} \tag{6.1}$$

Proposition 6.4.1. *The following statements are equivalent:*

(1) $\mathscr{P}^{\mathrm{even}}(L(\beta)) = \mathscr{P}(L(\beta))$,
(2) $\mathfrak{P}(\beta) = \mathscr{P}(L(\beta))$,
(3) $\mathfrak{P}(\beta) = \mathscr{P}^{\mathrm{even}}(L(\beta))$,
(4) $\mathrm{H}^j(\mathrm{Sh}_{\mathrm{even}}(\widehat{\mathbb{S}}_\beta \otimes \bigwedge^i \mathcal{B})) = 0$ *for* $j \neq 0$ *for all* i.

Proof. The last three conditions are formally equivalent. Also the last condition implies the first one. Let us show that the first condition implies the last one. Indeed, $\mathscr{P}^{\mathrm{even}}(L(\beta))|_{t=1} = \mathscr{P}(L(\beta))|_{t=1}$ implies vanishing of $\mathrm{H}^j(\mathrm{Sh}_{\mathrm{even}}(\widehat{\mathbb{S}}_\beta \otimes \bigwedge^i \mathcal{B}))$ for odd j. Thus both $\mathscr{P}^{\mathrm{even}}(L(\beta))$ and $\mathscr{P}(L(\beta))$ are the sums of monomials of q and t with positive coefficients. Hence formula (6.1) implies the statement. □

As explained above, the super-polynomial $\mathscr{P}^{\mathrm{even}}(L(\mathrm{cox}_S \cdot \delta^{\vec{k}}))$ can be computed by localization technique. Moreover, using different methods we show in [OR18a] the following.

Proposition 6.4.2 ([OR18a]). *For any $\vec{k} \in \mathbb{Z}_{>0}$ such that $k_1 > k_2 > \cdots > k_{n-1}$, there is M such that for any $m > M$, we have*

$$\mathscr{P}(\delta^{\vec{k}} \cdot FT^m) = \mathscr{P}^{\mathrm{even}}(\delta^{\vec{k}} \cdot FT^m).$$

Thus the combination of the last proposition and Proposition 6.3.2 implies Theorem 1.0.3.

6.5. Conjectures

Motivated by the previous section we state some vanishing conjectures for super-polynomial $\mathfrak{P}(\mathrm{cox}\cdot\delta^{\vec{k}})$.

The structure sheaf of $\widetilde{\mathrm{Hilb}}^{\mathrm{free}}_{1,n}$ twisted by a B-character χ descend to a line bundle on $\mathrm{Hilb}^{\mathrm{free}}_{1,n}$. Let us denote this line bundle by $\mathcal{L}^{\vec{k}}$. The line bundle $\mathcal{L}^{\vec{1}}$ corresponds to FT. Based on the discussion in [GNR20] and constructions in [OR18a] we propose the following.

Conjecture 6.5.1. *For any $\vec{k}\in\mathbb{Z}^{n-1}_{>0}$ such that $k_i\ge k_{i+1}-1$, $i=1,\dots,n-2$, there is M such that $\mathrm{H}^j(\mathrm{Sh}_{\mathrm{even}}(\mathcal{L}^{\vec{k}+r\vec{1}}\otimes\widehat{\mathbb{S}}_{\mathrm{cox}})\otimes\wedge^i\mathcal{B}))=0$ for any i, $j\neq 0$ and $r>M$.*

As we mentioned before, for any m,n, $(m,n)=1$, there is $\vec{k}$ such that $L(\mathrm{cox}\cdot\delta^{\vec{k}})=T_{m,n}$ is an m,n torus knot. Previous studies of the homology of torus knots and the related combinatorics allow us to provide an evidence for the above conjecture.

Proposition 6.5.2. *Suppose $L(\mathrm{cox}\cdot\delta^{\vec{k}})=T_{m,n}$, then Conjecture 6.5.1 holds.*

Proof. It was shown in [OR20] that for any $\beta\in\mathfrak{Br}_n$, $\mathscr{P}(L(\beta))$ is equal to the super-polynomial for the Khovanov–Rozansky homology. On the other hand, the Khovanov–Rozansky super-polynomial for $T_{m,n}$ was computed in [HM19] and it is shown in [Mel21] that this super-polynomial is equal to the super-polynomial from Proposition 6.3.2.

Finally, let us notice that $L(\mathrm{cox}\cdot\delta^{\vec{k}+r\vec{1}})=T_{m+rn,n}$ Thus we have the equality

$$\mathscr{P}(L(\mathrm{cox}\cdot\delta^{\vec{k}+r\vec{1}}))=\mathscr{P}^{\mathrm{even}}(L(\mathrm{cox}\cdot\delta^{\vec{k}+r\vec{1}}))$$

for all $r\ge 0$ and the statement follows from Proposition 6.4.1. □

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