2022 IEEE 61st Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

Finite-time integral control of nonlinear planar systems subject to mismatched disturbances

Xiandong Chen, Chunjiang Qian, *Fellow, IEEE* and Xianfu Zhang

2022 IEEE 61st Conference on Decision and Control (CDC) | 978-1-6654-6761-2/22/$31.00 ©2022 IEEE | DOI: 10.1109/CDC51059.2022.9992620

Abstract— **This paper investigates the finite-time bounded control of nonlinear planar systems subject to mismatched disturbances, where the perturbed uncertainties are relaxed to nonlinear functions with low-order terms. By revamping the technique of adding a power integrator and introducing new coordinates, a systematic Lyapunov method is proposed. This is achieved by three major mechanisms: (i) for the purpose of finite-time convergence, a lower-order integral dynamic is first constructed; (ii) a new structural controller is constructed to handle the mismatched disturbances; and (iii) a new structural Lyapunov function is established to provide an effective estimating tool for analyzing the finite-time boundness of the considered systems. A simulation example is given to verify the effectiveness of the proposed controller.**

I. Introduction

It is well-known that planar systems are widely applied to model various practical systems in circuit analysis, mechanical and thermal processes, image processing, and digital filtering, etc. [1]. As one of the key topics in the control field, global feedback stabilization of the control systems is a very practical and some remarkable results have been obtained for planar systems [2], [3].

In practice, planar systems are often affected by perturbed uncertainties and have received fully consideration[4]. When the perturbed uncertainties satisfy linear condition with known growth rate, the global stabilization problem has been studied by the backstepping approach [4] and the static high-gain method [5]. When the perturbed uncertainties satisfy linear condition with unknown growth rate, a domination approach was proposed in [6] to construct linear controllers making system states converge to zero. Time-varying high-gain method has been introduced in [7] to study the regulation problem. Furthermore, finite-time stability [8] achieved for control systems has many features such as faster convergence rates, higher accurateness, as well as better disturbance rejection properties [9], [10], [11]. When the perturbed uncertainties meet Hölder growth condition, a design methodology called adding a power integrator was proposed in [12] to achieve the global finite-time stabilization. The finite-time stabilization problem was then considered in [13] via the dynamic gain method, where systems nonlinearities are dominated by a lower-triangular model with time-varying gains.

Mismatched disturbances widely exist in various control engineering systems [14] such as the buck converter system [15]. Due to the presence of mismatched disturbances, the aforementioned methods bring great difficulties to control this kind of systems. In order to tackle the mismatched disturbances, sliding mode control [14], integral sliding mode control [16], disturbance observer technique [17] and extended state observer technique [18] have been proposed. For example, an integral sliding-mode control approach was introduced in [16] to tackle system's mismatched uncertainties. [19] proposed an extended state observer-based sliding mode control for pulse-width modulation-based DC-DC buck converter systems subject to mismatched disturbances. Recently, [20] studied the finite-time bounded control problem of nonlinear planar systems with mismatched disturbances, but the method of how to design the proposed controllers and choose these kinds of Lyapunov functions is not given clearly. Then, the following interesting question arises naturally: **is there a systematic method one can adopt to construct the desired finite-time controllers and choose the Lyapunov functions correspondingly for nonlinear planar systems subject to mismatched disturbances?**

This work attempts to answer this question. In particular, inspired by [21], [12], [20], this paper investigates the global finite-time bounded problem of perturbed planar systems subject to mismatched disturbances. In comparison with those relevant existing literatures, this work has the following two distinctive features. First, in contrast to those relevant existing works [21], [12], a revamped adding a power integrator technique is proposed and based on this method, a controller with a new structure is designed. Moreover, a robustness analysis of the proposed control strategy is also given. Second, significantly different from [20], a systematic Lyapunov method is proposed, and it is shown that a novel integral controller consisting of a nonlinear integral dynamic is constructed and the required Lyapunov functions are given correspondingly. Moreover, the proposed control strategy can solve the considered systems with the actuator being undergoing partial loss of effectiveness.

Notations: Define the parameters $r_1 = 1$, $r_2 =$

The work was supported by the U.S. National Science Foundation (Grant No. 1826086), the National Natural Science Foundation of China (Grant No. 62073190, 62203243, 61973189 and 61873334), the Research Fund for the Taishan Scholar Project of Shandong Province of China (Grant No. ts20190905), and the Natural Science Foundation of Shandong Province of China (Grant No. ZR2020ZD25).

Xiandong Chen is with School of Control Science and Engineering, Shandong University, Jinan, 250061, P.R. China and Department of Electrical and Computer Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, USA chenxiandong@hotmail.com

Chunjiang Qian is with Department of Electrical and Computer Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, USA chunjiang.qian@utsa.edu

Xianfu Zhang is with School of Control Science and Engineering, Shandong University, Jinan, 250061, P.R. China zhangxianfu@sdu.edu.cn

$\frac{r_1+\tau}{p}$, $r_3 = r_2 + \tau = 1$ and $\sigma = r_2$, where $\tau = \frac{q}{d}$ with q an even integer and d an odd integer, $p \in \mathbb{R}^+_{\rm odd}$ satisfying $p < 1$, and the set $\mathbb{R}^+_{\rm odd} := \{m/n|\ m$ and n are positive odd intergers$\}$. $t \in \mathbb{R}_{\geq 0}$ denotes $t \geq 0$. Lemmas A.1, A.2 and A.3 can be found in our previous work [21].

II. Preliminaries

We consider the uncertain nonlinear planar system described as follows

$$\begin{cases}\dot{\xi}_1 = \xi_2 + f_1(\xi_1) + \theta,\\ \dot{\xi}_2 = b(t,\xi)u + f_2(\xi_1,\xi_2),\end{cases} \tag{1}$$

where $\xi = (\xi_1,\xi_2)^T \in \mathbb{R}^2$ and $u \in \mathbb{R}$ are system state and control input respectively, θ is the mismatched and non-vanishing constant disturbance, $f_1(\xi_1)$ and $f_2(\xi_1,\xi_2)$ are perturbed uncertain functions vanishing at the origin, and $b(t,\xi)$ is an unknown function satisfying the following assumptions.

Assumption 1: There are known constants c_1 and c_2 such that

$$|f_1(\xi_1)| \leq c_1|\xi_1|^{\frac{p+1}{2}},\quad |f_2(\xi_1,\xi_2)| \leq c_2|\xi_1|^p. \tag{2}$$

Assumption 2: There is a continuous function $b_0(t,\xi)$ such that $b(t,\xi) \geq b_0(t,\xi) > 0$ for all $(t,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$.

Remark 1: It should be noted that the main results obtained in [20] cannot be applied to finite-time bounded control system (1) since the existence of $b(t,\xi)$. Moreover, it follows from Assumption 2 that system (1) can be viewed as an extension of a control system with the actuator being undergoing partial loss of effectiveness when $b(t,\xi)$ is an unknown constant belonging to $(0,1)$ [22].

With the above preliminaries, our control objective is to propose a systematic Lyapunov method to design integral controllers for system (1) under Assumptions 1 and 2 to make all system states finite-time bounded, that is, there is a finite time t^* such that $\xi_1 = 0$ and $\xi_2 = -\theta$ for $t \geq t^*$. Therefore, we introduce the following three parts to elaborate the main results of this paper.

III. A second-order integrator system

In this section, we motivate our design with the following second-order integrator system

$$\begin{cases}\dot{\xi}_1 = \xi_2 + \theta,\\ \dot{\xi}_2 = u,\end{cases} \tag{3}$$

which can be used to model the rigid body plant driven by a force on a smooth surface, where ξ_1 and ξ_2 are the position and velocity respectively, the input u represents the acceleration, and θ is an unknown, mismatched and non-vanishing constant disturbance. In order to achieve the control objective, a desired controller will be proposed by revamping the adding a power integrator technique [21]. To this end, the following proposition is first introduced.

Proposition 1: For the control system

$$\dot{x}_1 = x_2^p,\quad \dot{x}_2 = x_3,\quad \dot{x}_3 = v \tag{4}$$

with $x_i \in \mathbb{R}$, $i = 1,2,3$ and $v \in \mathbb{R}$ being system states and control input respectively, there are positive constants a_i, $i = 1,2,3$ such that the controller

$$v = -a_3\big(a_1x_1 + a_2x_2^{\frac{p+1}{2}} + x_3\big)^{\frac{2p}{p+1}} \tag{5}$$

globally finite-time stabilizes system (4).

Proof. A systematic Lyapunov method is proposed to design the desired controller and choose the corresponding Lyapunov functions. To this end, the proof of Proposition 1 is divided into the following four steps.

Step 1. Choose the Lyapunov function

$$V_1 = \frac{r_1}{2\sigma - \tau}x_1^{\frac{2\sigma-\tau}{r_1}}. \tag{6}$$

Taking the time derivative of V_1 along system (4) and constructing the virtual controller

$$x_2^{*p} = -\beta_1 x_1^{\frac{r_2p}{r_1}} = -3x_1^{\frac{r_2p}{r_1}}, \tag{7}$$

we have $\dot{V}_1|_{(4)} = -3x_1^{2\sigma} + x_1^{\frac{2\sigma-r_2p}{r_1}}(x_2^p - x_2^{*p})$.

Step 2. First, choose the Lyapunov function

$$V_2 = V_1 + W_2,\ W_2 = \int_{x_2^*}^{x_2}\left(s^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_2-\tau}{\sigma}}ds. \tag{8}$$

The time derivative of V_2 along system (4) is

$$\begin{aligned}\dot{V}_2|_{(4)} \leq& -3x_1^{2\sigma} + x_1^{\frac{2\sigma-r_2p}{r_1}}(x_2^p - x_2^{*p})\\ &+ \left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}x_3^*\\ &+ \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}(x_3 - x_3^*).\end{aligned} \tag{9}$$

By the fact $p < 1$ and Lemma A.1, we have

$$x_2^p - x_2^{*p} \leq 2^{-\tau}\left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|^{r_2p}, \tag{10}$$

which indicates

$$x_1^{\frac{2\sigma-r_2p}{r_1}}\left(x_2^p - x_2^{*p}\right) \leq \frac{1}{2}x_1^{2\sigma} + c_{21}\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} \tag{11}$$

with c_{21} being a positive constant.

Then, it follows from (7) and Lemmas A.1-A.3 that we have

$$|x_2|^p \leq 2^{-\tau}\left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|^{r_2p} + \beta_1|x_1|^{r_2p} \tag{12}$$

and

$$|x_2 - x_2^*| \leq c\left(\left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|^{r_2} + \left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|x_2^{*\left(1-\frac{1}{r_2}\right)}\right) \tag{13}$$

with c being a known positive constant, and it follows from $\int_{x_2^*}^{x_2}\left(s^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{\sigma-r_2-\tau}{\sigma}}ds \leq 2^{\frac{-\tau}{\sigma}}c^{\frac{\sigma-\tau}{\sigma}}\left(\left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|^{\sigma-\tau} + \beta_1^{\frac{1}{1+\tau}(\sigma-1)\frac{\sigma-\tau}{\sigma}}\left|x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right|^{\frac{\sigma-\tau}{\sigma}}|x_1|^{(\sigma-1)\frac{\sigma-\tau}{\sigma}}\right)$ that we obtain

$$\frac{\partial W_2}{\partial x_1}\dot{x}_1 \leq \frac{1}{2}x_1^{2\sigma} + c_{22}\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma}, \tag{14}$$

where c_{22} is a positive constant. Now we construct the virtual controller

$$x_3^* = -\beta_2\left(x_2^{\frac{r_3}{r_2}} - x_2^{*\frac{r_3}{r_2}}\right) \tag{15}$$

with $\beta_2 = 2^{(\sigma-1)(2\sigma-1)/\sigma}(c_{21}+c_{22}+2)$ being a positive constant and substitute (11), (14) and (15) into (9) yielding

$$\begin{aligned}\dot{V}_2|_{(4)} \leq & -2x_1^{2\sigma} - \beta_2\left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}\left(x_2^{\frac{r_3}{r_2}} - x_2^{*\frac{r_3}{r_2}}\right) \\ & + (c_{21}+c_{22})\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} \\ & + \left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}(x_3 - x_3^*).\end{aligned} \tag{16}$$

It then follows from Lemma A.1 and $r_3 = 1$ that we can easily obtain $-\beta_2\left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}\left(x_2^{\frac{r_3}{r_2}} - x_2^{*\frac{r_3}{r_2}}\right) \leq -2^{(1-\sigma)\frac{2\sigma-1}{\sigma}}\beta_2^{2\sigma}\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma}$, which makes (16) be rewritten as

$$\begin{aligned}\dot{V}_2|_{(4)} \leq & -2x_1^{2\sigma} - 2\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} \\ & + \left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}(x_3 - x_3^*).\end{aligned}$$

Step 3. Choose the following function

$$W_3(x_1,x_2,x_3) = \int_{x_3^*}^{x_3} l(x_1,x_2,s)ds, \tag{17}$$

where $l(x_1,x_2,s) = \left(s - \beta_2 x_2^{*\frac{r_3}{r_2}}\right)^{3\sigma-2} + \left(\beta_2 x_2^{\frac{r_3}{r_2}}\right)^{3\sigma-2}$ is a continuous function with respect to x_1, x_2 and s.

We first introduce the following proposition whose proof can be found in the Section VIII.

Proposition 2: $W_3(x_1,x_2,x_3)$ is a positive C^1 function. It follows from Proposition 2 that the time derivative of $V_3 = V_2 + W_3$ along system (4) is

$$\begin{aligned}\dot{V}_3|_{(4)} \leq & -2x_1^{2\sigma} - 2\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} \\ & + \left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}(x_3 - x_3^*) \\ & + l(x_1,x_2,x_3)v + \sum_{i=1}^{2}\frac{\partial W_3}{\partial x_i}\dot{x}_i.\end{aligned} \tag{18}$$

It follows from (13), (15) and Lemmas A.1-A.3 that we have

$$\begin{aligned}&\left(x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}(x_3 - x_3^*) \\ &\leq \frac{1}{3}\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} + c_{31}(x_3 - x_3^*)^{2\sigma},\end{aligned} \tag{19}$$

where c_{31} is a positive constant.

Then, it follows from (7), (12), (13) and (15), and Lemmas A.1-A.3 that we have

$$\sum_{i=1}^{2}\frac{\partial W_3}{\partial x_i}\dot{x}_i \leq \frac{2}{3}x_1^{2\sigma} + \frac{2}{3}\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} + c_{32}(x_3 - x_3^*)^{2\sigma}, \tag{20}$$

where c_{32} is a positive constant. Now we choose the controller

$$v = -\beta_3(x_3 - x_3^*)^{2-\sigma} = -\beta_3\left(x_3 + \beta_2 x_2^{\frac{1}{r_2}} + \beta_1^{\frac{1}{r_2}}\beta_2 x_1\right)^{2-\sigma} \tag{21}$$

with $\beta_3 = 2^{3\sigma-3}(c_{31}+c_{32}+1)$ and substitute (19), (20) and (21) into (18), we have

$$\begin{aligned}\dot{V}_3|_{(4)} \leq & -x_1^{2\sigma} - \left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} \\ & -\beta_3 l(x_1,x_2,x_3)(x_3 - x_3^*)^{2-\sigma} \\ & + (c_{31}+c_{32})(x_3 - x_3^*)^{2\sigma}.\end{aligned} \tag{22}$$

Similarly, it follows from Lemma A.1 and $r_3 = 1$ that we have $-\beta_3 l(x_1,x_2,x_3)(x_3 - x_3^*)^{2-\sigma} \leq -2^{3-3\sigma}\beta_3(x_3 - x_3^*)^{2\sigma}$, which indicates

$$\dot{V}_3|_{(4)} \leq -x_1^{2\sigma} - \left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} - (x_3 - x_3^*)^{2\sigma}. \tag{23}$$

Thus, we have proven that system (4) can be globally asymptotically stabilized by controller (21).

Step 4. We will prove that all states of the closed-loop system (4)-(21) converge to the origin in a finite time [8].

It follows from (6), (8), (17) and Lemmas A.1-A.3 that we can easily obtain

$$\begin{cases} V_1^{\frac{2\sigma}{2\sigma-\tau}} \leq \delta_1|x_1|^{2\sigma}, \\ W_2^{\frac{2\sigma}{2\sigma-\tau}} \leq \delta_2|x_1|^{2\sigma} + \delta_3\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma}, \\ W_3^{\frac{2\sigma}{2\sigma-\tau}} \leq \delta_4|x_1|^{2\sigma} + \delta_5\left(x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}\right)^{2\sigma} + \delta_6(x_3 - x_3^*)^{2\sigma}, \end{cases}$$

where $\frac{2\sigma}{2\sigma-\tau} < 1$ and $\delta_i,\ i = 1,\ldots,6$ are positive constants. Thus, we finally arrive at

$$\dot{V}_4|_{(4)} \leq -\delta\left(V_1^{\frac{2\sigma}{2\sigma-\tau}} + W_2^{\frac{2\sigma}{2\sigma-\tau}} + W_3^{\frac{2\sigma}{2\sigma-\tau}}\right) \leq -\delta V_4^{\frac{2\sigma}{2\sigma-\tau}},$$

where δ is a positive constant. Then, we can see that

$$0 \leq V_4(t)^{1-\frac{2\sigma}{2\sigma-\tau}} \leq V_4(0)^{1-\frac{2\sigma}{2\sigma-\tau}} - \delta\left(1 - \frac{2\sigma}{2\sigma-\tau}\right)t$$

for $t \leq t^* = \frac{V_4(0)^{1-\frac{2\sigma}{2\sigma-\tau}}}{\delta\left(1-\frac{2\sigma}{2\sigma-\tau}\right)}$ and $V_4(t) = 0$, for $t \geq t^*$. Therefore, all states of the closed-loop system will converge to the origin in a finite time.

Based on Proposition 1, we state the following theorem.

Theorem 1: There are positive constants $a_i,\ i = 1,2,3$ such that the following integral controller

$$\begin{cases} u = -a_3\left(a_1\xi_0 + a_2\xi_1^{\frac{p+1}{2}} + \xi_2\right)^{\frac{2p}{p+1}}, \\ \dot{\xi}_0 = \xi_1^p \end{cases} \tag{24}$$

makes system (3) finite-time bounded.

Proof. Define

$$\begin{cases} x_1 = \xi_0 - \dfrac{\theta}{a_1}, \\ x_2 = \xi_1, \\ x_3 = \xi_2 + \theta, \end{cases} \tag{25}$$

and it follows from the new coordinates (25) that the closed-loop system (3) and (24) can be converted into (4) and (5). Then, by Proposition 1, we can find constants $a_i,\ i=1,2,3$ such that system (3) is globally finite-time bounded under controller (24), that is, there is a finite time t^* determined in Proposition 1 such that $\xi_1=0$ and $\xi_2=-\theta$ for $t\geq t^*$.

IV. The perturbed uncertain planar systems

In this section, we consider the case when system (3) is affected by the perturbed uncertainties, that is,

$$\begin{cases}\dot{\xi}_1=\xi_2+f_1(\xi_1)+\theta,\\ \dot{\xi}_2=u+f_2(\xi_1,\xi_2),\end{cases}\tag{26}$$

where $f_1(\xi_1)$ and $f_2(\xi_1,\xi_2)$ satisfy Assumption 1. By Proposition 1 and Theorem 1 that we can state the following theorem.

Theorem 2: Under Assumption 1, there are constants $a_i,\ i=1,2,3$ such that the integral controller (24) makes system (3) finite-time bounded.

Proof. It follows from the new coordinates (25) that system (26) satisfying Assumption 1 can be converted into

$$\begin{cases}\dot{x}_1=x_2^p,\\ \dot{x}_2=x_3+f_1(\xi_1),\\ \dot{x}_3=u+f_2(\xi_1,\xi_2).\end{cases}\tag{27}$$

Next, we will prove Theorem 2 based on Proposition 1. By Proposition 1, we have $v=u$. Due to the existence of $f_1(\xi_1)$ in system (27), there exists a term $\left(x_2^{\frac{\sigma}{r_2}}-x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}f_1(\xi_1)$ in (9), and the terms $l(x_1,x_2,x_3)f_2(\xi_1,\xi_2)$ and $\frac{\partial W_3}{\partial x_2}f_1(\xi_1)$ in (18), and it then follows from Assumption 1 that the estimate of these terms is given as follows

$$\left(x_2^{\frac{\sigma}{r_2}}-x_2^{*\frac{\sigma}{r_2}}\right)^{\frac{2\sigma-r_3}{\sigma}}f_1(\cdot)\leq x_1^{2\sigma}+c_{23}\left(x_2^{\frac{1}{r_2}}-x_2^{*\frac{1}{r_2}}\right)^{2\sigma},\tag{28}$$

$$l(\cdot)f_2(\cdot)\leq c_{33}\left(x_1^{2\sigma}+\left(x_2^{\frac{1}{r_2}}-x_2^{*\frac{1}{r_2}}\right)^{2\sigma}+(x_3-x_3)^{2\sigma}\right)\tag{29}$$

and

$$\frac{\partial W_3}{\partial x_2}f_1(\cdot)\leq c_{34}\left(x_1^{2\sigma}+\left(x_2^{\frac{1}{r_2}}-x_2^{*\frac{1}{r_2}}\right)^{2\sigma}+(x_3-x_3)^{2\sigma}\right).\tag{30}$$

Therefore, by Proposition 1, we can rechoose constants $a_i,\ i=1,2,3$ such that system (26) satisfying Assumption 1 is globally finite-time bounded under controller (24), that is, there is a finite time t^* such that $\xi_1=0$ and $\xi_2=-\theta$ for $t\geq t^*$.

Remark 2: Based on the homogeneous system theory [23], [24], [25], it should be noted that a system needs a negative homogeneous degree to achieve finite-time convergence. Thus, a nonlinear integral auxiliary equation $\dot{\xi}_0=\xi_1^p$ with $p<1$ is proposed in this paper. However, when $p<1$, the proposed controller using traditional adding a power integrator approach [21] cannot be used to handle the unknown, mismatched and non-vanishing constant disturbance θ. Therefore, the novel controller based on new chosen Lyapunov functions is constructed in Theorem 1 to tackle the mismatched disturbance.

Furthermore, in the case when $p=1$ for the introducing integral term, the following corollary can be easily obtained.

Corollary 1: Under Assumption 1, there are constants $a_i,\ i=1,2,3$ such that the following integral controller

$$\begin{cases}u=-a_3(a_1\xi_0+a_2\xi_1+\xi_2),\\ \dot{\xi}_0=\xi_1\end{cases}\tag{31}$$

makes system (3) asymptotically bounded.

V. Main results

We recall the system (1) satisfying Assumptions 1 and 2 of interest. It follows from Proposition 1, and Theorems 1 and 2 that we can now state our main result.

Theorem 3: Under Assumptions 1 and 2, there are constants $a_i,\ i=1,2,3$ such that the following integral controller

$$\begin{cases}u=-\dfrac{a_3}{b_0(t,\xi)}\left(a_1\xi_0+a_2\xi_1^{\frac{p+1}{2}}+\xi_2\right)^{\frac{2p}{p+1}},\\ \dot{\xi}_0=\xi_1^p\end{cases}\tag{32}$$

makes system (1) finite-time bounded.

Proof. In order to achieve the proof of Theorem 3, we aim to prove that Proposition 1 holds for system (4) when the input is infected by $b(t,\xi)$. It follows from the new coordinates (25) and the proof of Proposition 1 and (18) that we can easily have

$$\begin{aligned}\dot{V}_3\leq&-x_1^{2\sigma}-\left(x_2^{\frac{1}{r_2}}-x_2^{*\frac{1}{r_2}}\right)^{2\sigma}+b(t,\xi)l(x_1,x_2,x_3)v\\&+(c_{31}+c_{32})(x_3-x_3^*)^{2\sigma}.\end{aligned}\tag{33}$$

It follows from Assumption 2 that we can design the controller

$$\begin{aligned}v=&-\frac{\beta_3}{b_0(t,\xi)}(x_3-x_3^*)^{2-\sigma}\\=&-\frac{\beta_3}{b_0(t,\xi)}\left(x_3+\beta_2x_2^{\frac{1}{r_2}}+\beta_1^{\frac{1}{r_2}}\beta_2x_1\right)^{2-\sigma}\end{aligned}\tag{34}$$

with $\beta_3=2^{3\sigma-3}(c_{31}+c_{32}+1)$. Then, substituting controller (34) into (33), (22) is satisfied, which implies that Proposition 1 holds. Finally, it follows from Assumption 1 and the proof of Theorem 2 that we can achieve the proof of Theorem 3.

Now, we proceed to discuss the robustness of the proposed control strategy for the uncertain nonlinear planar systems

$$\begin{cases}\dot{\xi}_1=\xi_2+f_1(\xi_1)+\theta,\\ \dot{\xi}_2=b(t,\xi)u+f_2(\xi_1,\xi_2)+\triangle(t),\end{cases}\tag{35}$$

where the perturbation $\triangle(t),\ t\in\mathbb{R}_{\geq0}$ is uniformly bounded with a known constant $\bar{\triangle}$, i.e., $|\triangle(t)|\leq\bar{\triangle}$.

It follows from (25), Assumptions 1 and 2, and the proof of Proposition 1, and Theorems 2 and 3 that we can have

$$\begin{aligned}l(\cdot)\triangle(\cdot)\leq&c_{35}\left(x_1^{2\sigma}+\left(x_2^{\frac{1}{r_2}}-x_2^{*\frac{1}{r_2}}\right)^{2\sigma}+(x_3-x_3)^{2\sigma}\right)\\&+\hat{\delta}_2|\bar{\triangle}|^{\frac{2p}{p+1}}\end{aligned}$$

with c_{35} being a positive constant, which further indicates

$$\dot{V}_4 \leq -\hat{\delta}_1 V_4^{\frac{2\sigma}{2\sigma-\tau}} + \hat{\delta}_2 |\bar{\triangle}|^{\frac{2p}{p+1}} \tag{36}$$

under controller (32) with gains $a_i,\ i = 1,2,3$, where $\hat{\delta}_1$ and $\hat{\delta}_2$ are known positive constants. It then follows from (36) and [26] that the states of system (35) under controller (32) converges into a bounded region in finite-time. Thus, the proposed control strategy is robust to $\triangle(t)$.

VI. An Example

To show the feasibility of the proposed control strategy, we apply it to the following example

$$\begin{cases} \dot{\xi}_1 = \xi_2 + d(t)\sin(\xi_1) + \theta, \\ \dot{\xi}_2 = b(t,\xi)u, \end{cases} \tag{37}$$

where θ is the mismatched disturbance, $d(t)$ is an unknown satisfying $|d(t)| \leq 1$. It can be verified that $|d(t)\sin(\xi_1)| \leq |\xi_1|^{\frac{11}{13}}$ satisfying Assumption 1 with $c_1 = 1$ and $p = \frac{9}{13}$.

It follows from Theorem 3 that we can construct the integral controller as follow

$$\begin{cases} u = -\dfrac{a_3}{b_0(t,\xi)} \left(a_1\xi_0 + a_2\xi_1^{\frac{11}{13}} + \xi_2 \right)^{\frac{9}{11}}, \\ \dot{\xi}_0 = \xi_1^{\frac{9}{13}}. \end{cases} \tag{38}$$

Since the existence of $b(t,\xi)$ in system (37), the control strategy proposed in [20] cannot be used to solve the finite-time bounded control of system (37). Moreover, in order to verify the effectiveness of the proposed controller (38), we compare it with the following conventional controller

$$\begin{cases} u = -\dfrac{a_3}{b_0(t,\xi)} \left(a_1\xi_0^{\frac{13}{11}} + a_2\xi_1 + \xi_2^{\frac{13}{11}} \right)^{\frac{9}{13}}, \\ \dot{\xi}_0 = \xi_1^{\frac{9}{13}}, \end{cases} \tag{39}$$

which is designed in our previous work [12].

The simulations results are shown in Figs. 1-3, where the parameters are chosen as $b(t,\xi) = \frac{3}{2} + \sin(\xi_1)$, $d(t) = \sin(t)$, $a_1 = 1$, $a_2 = 3$ and $a_3 = 50$ with $(\xi_0, \xi_1, \xi_2) = (1, 3, -1)$. More specifically, it follows from Fig. 1 that all states of system (37) under controller (38) with the mismatched disturbance $\theta = 0$ converge to zero before $10s$, which indicates that the proposed control strategy is still valid for the systems without mismatched constant disturbances. When $\theta = 5$, it follows from Fig 2 that we can see that under controller (38), the states ξ_0 and ξ_2 are bounded and finite-time convergent to the mismatched disturbance θ and $-\theta$ respectively due to the constant $a_1 = 1$ and the state ξ_1 is finite-time convergent to zero before $10s$. However, from Fig 3, we can see that under controller (39), all states of system (37) oscillates, which indicates that the previous controller (39) is invalid for the system (37) with mismatched constant disturbances.

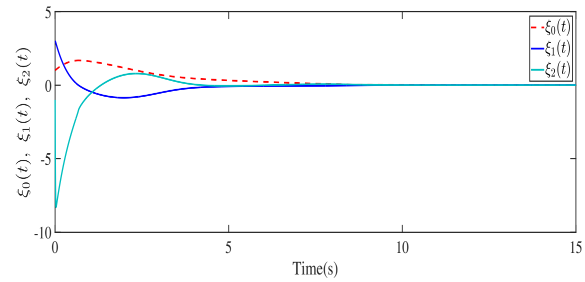

Fig. 1. Trajectories of system (37) under controller (38) with $\theta = 0$.

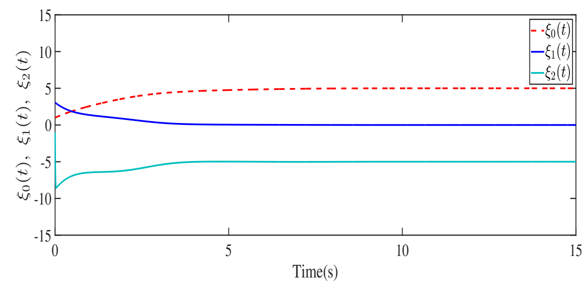

Fig. 2. Trajectories of system (37) under controller (38) with $\theta = 5$.

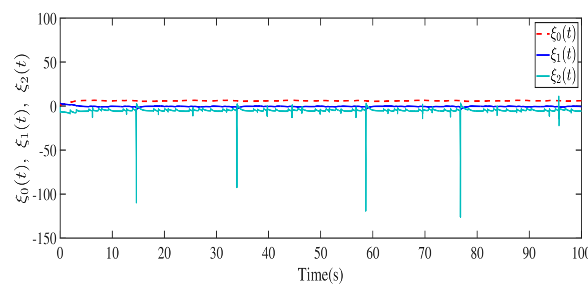

Fig. 3. Trajectories of system (37) under controller (39) with $\theta = 5$.

VII. Conclusion

This paper has proposed a new approach to design finite-time integral controllers to make all states of uncertain nonlinear planar systems with mismatched disturbances bounded. Compared to traditional control methods to tackle mismatched disturbances, our proposed controller have a simple structure and does not rely on designing an observer for the mismatched disturbances. Owing to the use of a homogeneous integral element and a new structural Lyapunov function, the system states will be bounded, and the mismatched disturbances can be recovered from the integral state in a finite time.

References

[1] C. Qian, S. He, and Y. Zou, "Compensator-based output feedback stabilizers for a class of planar systems with unknown structures and measurements," *IEEE Transactions on Automatic Control*, vol. 67, no. 4, pp. 2138–2143, 2022.

[2] C. Zhao and L. Guo, "PID controller design for second order nonlinear uncertain systems," *Science China Information Sciences*, vol. 60, no. 022201, pp. 1–13, 2017.

[3] J. Zhang and L. Guo, "Theory and design of PID controller for nonlinear uncertain systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 643–648, 2019.

[4] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, *Constructive Nonlinear Control*. Springer, New York, 1997.

[5] X. Zhang and Z. Cheng, "Global stabilization of a class of time-delay nonlinear systems," *International Journal of Systems Science*, vol. 36, no. 8, pp. 461–468, 2005.

[6] J. Tsinias, "A theorem on global stabilization of nonlinear systems by linear feedback," *Systems and Control Letters*, vol. 17, no. 5, p. 357–362, 1991.

[7] Y. Liu, "Global asymptotic regulation via time-varying output feedback for a class of uncertain nonlinear systems," *SIAM Journal on Control and Optimization*, vol. 51, no. 6, pp. 4318–4342, 2013.

[8] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[9] H. Du and S. Li, "Finite-time attitude stabilization for a spacecraft using homogeneous method," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 3, pp. 740–748, 2012.

[10] S. Li, H. Liu, and S. Ding, "A speed control for a pmsm using finite-time feedback control and disturbance compensation," *Transactions of the Institute of Measurement and Control*, vol. 32, no. 2, pp. 170–187, 2010.

[11] Y. Hong, J. Huang, and Y. Xu, "On an output feedback finite-time stabilization problem," *IEEE Transactions on Automatic Control*, vol. 46, no. 2, pp. 305–309, 2001.

[12] J. Polendo and C. Qian, "An expanded method to robustly stabilize uncertain nonlinear systems," *Communications in Information and Systems*, vol. 8, no. 1, pp. 55–70, 2008.

[13] X. Zhang, G. Feng, and Y. Sun, "Finite-time stabilization by state feedback control for a class of time-varying nonlinear systems," *Automatica*, vol. 48, no. 3, pp. 499–504, 2012.

[14] J. Yang, S. Li, and X. Yu, "Sliding-mode control for systems with mismatched uncertainties via a disturbance observer," *IEEE Transactions on Industrial Electronics*, vol. 60, no. 1, pp. 160–169, 2013.

[15] J. Mahdavi, A. Emadi, and H. Toliyat, "Application of state space averaging method to sliding mode control of PWM DC/DC converters," in *IEEE Industry Applications Conference*. IEEE, 1997, pp. 820–827.

[16] M. Rubagotti, A. Estrada, F. Castanos, A. Ferrara, and L. Fridman, "Integral sliding mode control for nonlinear systems with matched and unmatched perturbations," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2699–2704, 2011.

[17] W. Chen, D. Ballance, P. Gawthrop, and J. O'Reilly, "A nonlinear disturbance observer for robotic manipulators," *IEEE Transactions on Industrial Electronics*, vol. 47, no. 4, pp. 923–938, 2000.

[18] S. Li, J. Yang, W. Chen, and X. Chen, "Generalized extended state observer based control for systems with mismatched uncertainties," *IEEE Transactions on Industrial Electronics*, vol. 59, no. 12, pp. 4792–4802, 2011.

[19] J. Wang, S. Li, J. Yang, B. Wu, and Q. Li, "Extended state observer-based sliding mode control for pwm-based dc–dc buck power converter systems with mismatched disturbances," *IET Control Theory and Applications*, vol. 9, no. 4, pp. 579–586, 2015.

[20] K. Cao and C. Qian, "Finite-time controllers for a class of planar nonlinear systems with mismatched disturbances," *IEEE Control Systems Letters*, vol. 5, no. 6, pp. 1928–1933, 2021.

[21] J. Polendo and C. Qian, "A generalized homogeneous domination approach for global stabilization of inherently nonlinear systems via output feedback," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 17, no. 7, pp. 605–629, 2007.

[22] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, "Adaptive compensation for actuator failures with event-triggered input," *Automatica*, vol. 85, pp. 129–136, 2017.

[23] H. Hermes, "Homogeneous coordinates and continuous asymptotically stabilizing feedback controls," *Differential Equations, Stability and Control*, vol. 109, no. 1, pp. 249–260, 1991.

[24] M. Kawski, "Homogeneous stabilizing feedback laws," *Control Theory and Advanced Technology*, vol. 6, no. 4, pp. 497–516, 1990.

[25] C. Qian and W. Lin, "Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 9, pp. 1457–1471, 2006.

[26] B. Tian, Z. Zuo, X. Yan, and H. Wang, "A fixed-time output feedback control scheme for double integrator systems," *Automatica*, vol. 80, pp. 17–24, 2017.

[27] C. Qian and W. Lin, "A continuous feedback approach to global strong stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1061–1079, 2001.

VIII. Appendix

Proof of Proposition 2. By definition (17) and from $\frac{3\sigma-2}{r_2} > 1$, it is clear that

$$\partial W_3(x_1,x_2,x_3)/\partial x_3 = l(x_1,x_2,x_3). \tag{40}$$

Then, similar to [27], denote

$$\triangle_1 = (x_1+\triangle, x_2),\ \triangle_2 = (x_1, x_2+\triangle),\ X_2 = (x_1,x_2)$$

and consider the limit as shown below

$$\begin{aligned}&\lim_{\triangle\to 0}\frac{W_3(x_1+\triangle,x_2,x_3)-W_3(x_1,x_2,x_3)}{\triangle}\\ &= \lim_{\triangle\to 0}\frac{\int_{x_3^*(\triangle_1)}^{x_3^*(X_2)} l(x_1+\triangle,x_2,s)ds}{\triangle}\\ &\quad + (3\sigma-2)\beta_1^{\frac{1}{r_2 p}}\beta_2\int_{x_3^*}^{x_3}\left(s-\beta_2 x_2^{*\frac{r_3}{r_2}}\right)^{3\sigma-3} ds.\end{aligned}$$

Observe that

$$\begin{aligned}&\left|\frac{\int_{x_3^*(\triangle_1)}^{x_3^*(X_2)} l(x_1+\triangle,x_2,s)ds}{\triangle}\right|\\ &\le \left|\left(x_3^*(X_2)-\beta_2 x_2^{*\frac{r_3}{r_2}}(\triangle_1)\right)^{3\sigma-2} + \left(\beta_2 x_2^{\frac{r_3}{r_2}}\right)^{3\sigma-2}\right| \\ &\quad\times\left|\frac{x_3^*(X_2)-x_3^*(\triangle_1)}{\triangle}\right|.\end{aligned} \tag{41}$$

Using (7) and (15), it is clear that $x_3^*(x_1,x_2)$ is C^1 with respect to x_1. Hence, it follows from the inequality above that $\lim_{\triangle\to 0}\int_{x_3^*(\triangle_1)}^{x_3^*(X_2)} l(x_1+\triangle,x_2,s)ds/\triangle = 0$.

On the other hand, we can easily have

$$\begin{aligned}&\lim_{\triangle\to 0}\frac{W_3(x_1,x_2+\triangle,x_3)-W_3(x_1,x_2,x_3)}{\triangle}\\ &= \lim_{\triangle\to 0}\frac{\int_{x_3^*(\triangle_2)}^{x_3^*(X_2)} l(x_1,x_2+\triangle,s)ds}{\triangle}\\ &\quad + \frac{3\sigma-2}{\sigma}\beta_2^{3\sigma-2}x_2^{\frac{2\sigma-2}{\sigma}}(x_3-x_3^*)\end{aligned}$$

and observe that $\left|\int_{x_3^*(\triangle_2)}^{x_3^*(X_2)} l(x_1,x_2+\triangle,s)ds/\triangle\right| \le \beta_2^{3\sigma-2}\left|\left((x_2+\triangle)^{\frac{3\sigma-2}{\sigma}} - x_2^{\frac{3\sigma-2}{\sigma}}\right)/\triangle\right||x_3^*(X_2)-x_3^*(\triangle_2)|$.

By means of the fact $p < 1$ and Lemma A.3, we can easily have $\lim_{\triangle\to 0}\int_{x_3^*(\triangle_2)}^{x_3^*(X_2)} l(x_1,x_2+\triangle,s)ds/\triangle = 0$ Consequently, from (40) and (41), $W_3(x_1,x_2,x_3)$ is C^1 because of continuity of $\frac{\partial W_3}{\partial x_i}, i = 1,2,3$.

Furthermore, let $v = s - \beta_2 x_2^{*\frac{r_3}{r_2}}$ and we have $W_3 = \int_{-\beta_2 x_2^{\frac{r_3}{r_2}}}^{x_3-\beta_2 x_2^{*\frac{r_3}{r_2}}}\left(v^{3\sigma-2} + \left(\beta_2 x_2^{\frac{r_3}{r_2}}\right)^{3\sigma-2}\right)dv$, which indicates $W_3 \ge 0$. Thus, we have proven that W_3 is a positive C^1 function.