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Embeddings Among Quantum Affine $\mathfrak{sl}_n$

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Abstract We establish an explicit embedding of a quantum affine $\mathfrak{sl}_n$ into a quantum affine $\mathfrak{sl}_{n+1}$. This embedding serves as a common generalization of two natural, but seemingly unrelated, embeddings, one on the quantum affine Schur algebra level and the other on the non-quantum level. The embedding on the quantum affine Schur algebras is used extensively in the analysis of canonical bases of quantum affine $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$. The embedding on the non-quantum level is used crucially in a work of Riche and Williamson on the study of modular representation theory of general linear groups over a finite field. The same embedding is also used in a work of Maksimau on the categorical representations of affine general linear algebras. We further provide a more natural compatibility statement of the embedding on the idempotent version with that on the quantum affine Schur algebra level. A $\widehat{\mathfrak{gl}}_n$-variant of the embedding is also established.

Keywords Quantum affine $\mathfrak{sl}_n$, embeddings

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1 Introduction

Consider the following rule

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & d \end{bmatrix}. \tag{1.1}$$

It defines an embedding of $\mathfrak{sl}_2(R)$ into $\mathfrak{sl}_3(R)$ over any commutative ring R where a, b, c and d belong to. Further, by regarding a, b, c and d as block matrices, the embedding generalizes to a natural embedding of $\mathfrak{sl}_n(R)$ into $\mathfrak{sl}_{n+1}(R)$. When R is the local field $\mathbb{C}((t))$, such an embedding plays a key role in the study of categorical representations of affine general Lie algebras in Maksimau's work [10, 11]. An affine $\mathfrak{gl}$ variant of the embedding is further used in the study of modular representations of general linear groups over finite fields in a recent work [12] of Riche and Williamson.

Now consider the n-step affine flag variety $\mathfrak{F}_{n,d}$ associated with $\mathrm{GL}_d(\mathbb{C})$. It is well known that quantum affine Schur algebras $\mathbf{S}_{n,d}$ admit a geometric realization as the convolution algebra on $\mathfrak{F}_{n,d} \times \mathfrak{F}_{n,d}$. The ind-variety $\mathfrak{F}_{n,d}$ parametrizes lattice chains in a d-dimensional vector space

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over $\mathbb{C}((t))$. The operation of adding an extra copy of the lattice in a prescribed step in lattice chains defines a natural embedding $\mathbf{S}_{n,d} \subset \mathbf{S}_{n+1,d}$. Such a natural embedding of quantum affine Schur algebras is used crucially in the study of canonical bases and multiplication formulas of quantum affine $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$ in [4, 7]. We refer to [2] for a Hecke-algebra approach to, and further applications of, this embedding.

So we have a natural embedding of affine $\mathfrak{sl}_n$ into affine $\mathfrak{sl}_{n+1}$ on the one hand and a natural embedding $\mathbf{S}_{n,d} \subset \mathbf{S}_{n+1,d}$ on the other hand. Note that $\mathbf{S}_{n,d}$ receives an algebra homomorphism from quantum affine $\mathfrak{sl}_n$. To this end, it is desirable to understand the relationship between the two kinds of seemingly unrelated embeddings. In this paper, we establish an explicit embedding of quantum affine $\mathfrak{sl}_n$ into its higher rank. It is a natural quantization of the embedding on affine $\mathfrak{sl}_n$ defined by (1.1) and used in the works [10–12]. We further provide a compatibility of the embedding with that on the quantum affine Schur algebra level. It is somewhat unnatural. However when consider the embedding on Lusztig's modified quantum affine $\mathfrak{sl}_n$, it becomes natural again. So the embeddings established in this paper can be regarded as a common generalization of the previous two kinds of embeddings. It is worthwhile to point out that the quantization is not unique, depending on a parameter $\varepsilon \in \{\pm 1\}$. In an upcoming paper [6], the author will extend the embeddings established in this paper to a much broader setting.

Note that if one replaces the zero in the center of the right matrix by $-\mathrm{trace}(a) - \mathrm{trace}(d)$ in the assignment (1.1), it induces an embedding $\mathfrak{gl}_n(R) \subseteq \mathfrak{sl}_{n+1}(R)$. As a quantization of this embedding, we further provide an embedding of quantum affine $\mathfrak{gl}_n$ in the sense of Green [5] into quantum affine $\mathfrak{sl}_{n+1}$ of level 1. There is a second version of quantum affine $\mathfrak{gl}_n$ studied by Du and Fu [3]. It is very interesting to see if there is a similar embedding as a common generalization of the embeddings on the non-quantum level and on the quantum affine Schur algebra level.

Let θ_n be an involution of $\mathfrak{sl}_n(R)$ by sending a matrix to the matrix whose (i, j) entry is the $(n + 1 - i, n + 1 - j)$ entry of the original matrix. The pair $(\mathfrak{sl}_n(R), \mathfrak{sl}_n(R)^{\theta_n})$ is a quasi-split symmetric pair. The embedding defined in (1.1) with appropriate block sizes of a and d is compatible with the involutions θ_n and θ_{n+1} on $\mathfrak{sl}_n(R)$ and $\mathfrak{sl}_{n+1}(R)$ respectively. Thus it induces an embedding on the fixed-point set: $\mathfrak{sl}_n(R)^{\theta_n} \subset \mathfrak{sl}_{n+1}(R)^{\theta_{n+1}}$. On the geometry side, there is a quantum affine Schur algebras $\mathbf{S}^{\imath}_{n,d}$ defined as the convolution algebras of n-step affine isotropic flag varieties in the works [4] (see also [1]). These algebras are homomorphic images of a quantum version of $\mathfrak{sl}_n^{\theta_n}$, i.e., the coideal subalgebras in an affine quantum symmetric pair of quasi-split type A. There are natural embeddings $\mathbf{S}^{\imath}_{n,d} \subseteq \mathbf{S}^{\imath}_{n+1,d}$, which play a key role in *loc. cit.* It is a natural question to see if there exists an embedding for these types of coideal subalgebras as a common generalization of the embedding $\mathfrak{sl}_n(R)^{\theta_n} \subset \mathfrak{sl}_{n+1}(R)^{\theta_{n+1}}$ and $\mathfrak{sl}_n(R)^{\theta_n} \subset \mathfrak{sl}_{n+1}(R)^{\theta_{n+1}}$.

The embedding established in this paper certainly calls for a further investigation of a more intrinsic connection between the above lines of research.

2 Preliminaries

This section recalls the basic definitions of quantum affine $\mathfrak{sl}_n$ and its Schur quotients.

2.1 Definition

Let $\mathbb{Q}(v)$ be the field of rational functions with the variable v. For each integer $a \in \mathbb{N}$, we define

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}}, \quad [a]^! = [a][a-1]\cdots[1].$$

If x is an element in an associative algebra over $\mathbb{Q}(v)$, we write

$$x^{(a)} = x^a/[a]^!$$

Let $I_n = \mathbb{Z}/n\mathbb{Z}$. If there is no danger of confusion, we write elements in I_n by $0, 1, \ldots, n-1$. Recall that the Cartan matrix of affine type $A^{(1)}_{n-1}$ is the n by n matrix $C = (c_{ij})_{i,j\in I_n}$ such that $c_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ be the quantum affine $\widehat{\mathfrak{sl}}_n$ associated with C. It is an associative algebra over $\mathbb{Q}(v)$ and it admits a generator-and-relation presentation. Precisely, the generators are standard Chevalley generators E_i, F_i, and $K_i^{\pm 1}$ for all $i \in I_n$ and the defining relations are given as follows.

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1 = K_i^{-1}K_i, \tag{R1}$$

$$K_iE_j = v^{c_{ij}}E_jK_i, \quad K_iF_j = v^{-c_{ij}}F_jK_i, \tag{R2}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{v - v^{-1}}, \tag{R3}$$

$$\sum_{p=0}^{1-c_{ij}} (-1)^p E_i^{(p)} E_j E_i^{(1-c_{ij}-p)} = 0, \tag{R4}$$

$$\sum_{p=0}^{1-c_{ij}} (-1)^p F_i^{(p)} F_j F_i^{(1-c_{ij}-p)} = 0, \quad \forall i, j \in I_n. \tag{R5}$$

The algebra $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ can be endowed with a Hopf algebra structure with the comultiplication Δ defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad \forall i \in I_n.$$

2.2 Idempotent Form

Let n be a positive integer greater than 1. Let $\mathfrak{S}_n$ be the set of all sequences $(a_i)_{i\in\mathbb{Z}}$ such that $a_i \in \mathbb{Z}$ and $a_i = a_{i+n}$ for all i. Note that any sequence $(a_i)_{i\in\mathbb{Z}}$ in $\mathfrak{S}_n$ is completely determined by the entries $(a_1, \ldots, a_n)$, which we shall use to denote the sequence itself. The set $\mathfrak{S}_n$ carries a natural abelian group structure with the component-wise addition. It is naturally isomorphic to $\mathbb{Z}^n$. Let Y_n be the subset of $\mathfrak{S}_n$ consisting of all sequences $(a_i)_{i\in\mathbb{Z}}$ such that $\sum_{1\le i\le n} a_i = 0$. Let $\mathbf{1}$ be the sequence in $\mathfrak{S}_n$ such that each entry is 1. Let $X_n = \mathfrak{S}_n/\langle \mathbf{1}\rangle$ be the quotient of $\mathfrak{S}_n$ by the subgroup generated by $\mathbf{1}$. Let $\langle -, -\rangle : Y_n \times X_n \to \mathbb{Z}$ be the perfect pairing defined by $\langle \mathbf{b}, \bar{\mathbf{a}}\rangle = \sum_{1\le i\le n} b_i a_i$ for all $\mathbf{b} \in Y_n$ and $\bar{\mathbf{a}} \in X_n$. Let $I_n = \mathbb{Z}/n\mathbb{Z}$. There is an embedding $I_n \to Y_n$ defined by $i \mapsto \alpha_i^\vee$ where the i (resp. $i+1$) entry of $\alpha_i^\vee$ is 1 (resp. -1) mod n and 0 otherwise. Similarly there is an embedding $I_n \to X_n$ defined by $i \mapsto \alpha_i$ where α_i is the coset of $\alpha_i^\vee$ in X_n. Let $i \cdot j = \langle \alpha_i^\vee, \alpha_j\rangle$ for all $i, j \in I_n$. Then $(I_n, \cdot)$ is the Cartan datum of affine type A_n. Further, $(Y_n, X_n, \langle -, -\rangle)$ is a root datum of $(I_n, \cdot)$. The root datum is neither X-regular nor Y-regular.

Let $\dot{\mathbf{U}}_n$ be the modified quantum group associated with the root datum (Y_n, X_n) ([8]). The algebra $\dot{\mathbf{U}}_n$ is an associative algebra over $\mathbb{Q}(v)$ without unit. Instead it has a collection of

idempotents 1_λ for $\lambda \in X_n$. It carries a natural $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$-bimodule structure generated by the 1_λ's.

2.3 Quantum Affine Schur Algebra

Let $\mathbf{V}_n$ be the vector space over $\mathbb{Q}(v)$ spanned by the symbols u_k for all $k \in \mathbb{Z}$. To avoid confusion, we write $\bar{k}$ for the congruence class of k in $I_n = \mathbb{Z}/n\mathbb{Z}$. We define a $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$-module structure on $\mathbf{V}_n$ by the following rules: for all $\bar{i} \in I_n$, $k \in \mathbb{Z}$,

$$E_{\bar{i}}.u_k = \delta_{\bar{k},\overline{i+1}} u_{k-1}, \quad F_{\bar{i}}.u_k = \delta_{\bar{k},\bar{i}} u_{k+1}, \quad K_{\bar{i}}.u_k = v^{\delta_{\bar{k},\bar{i}} - \delta_{\bar{k},\overline{i+1}}} u_k.$$

Via the comultiplication Δ, there is a $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$-module structure on the tensor product $\mathbf{V}_n^{\otimes d} = \mathbf{V}_n \otimes \cdots \otimes \mathbf{V}_n$ of d copies of $\mathbf{V}_n$. There exists a natural action of the affine Hecke algebra H_d of type A on $\mathbf{V}_n^{\otimes d}$. The quantum affine Schur algebra is defined to be the centralizer algebra $\mathbf{S}_{n,d} = \mathrm{End}_{H_d}(\mathbf{V}^{\otimes d})$. We shall not recall the precise definition of the H_d-action as it plays no role in later analyses, instead we shall recall a geometric definition of $\mathbf{S}_{n,d}$ in the following. Since the two actions on $\mathbf{V}^{\otimes d}$ are commutative, there exists an algebra homomorphism $\mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \mathbf{S}_{n,d}$. The morphism is not surjective in general. Let $\mathbf{S}'_{n,d}$ be its image, a subalgebra in $\mathbf{S}_{n,d}$. The algebra $\mathbf{S}'_{n,d}$ was first studied by Lusztig. In particular, we have a surjective algebra homomorphism

$$\pi_{n,d} : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \mathbf{S}'_{n,d}. \tag{2.1}$$

It is well known from [9] that there exists an algebra homomorphism $\phi_{d+n,d} : \mathbf{S}'_{n,d+n} \to \mathbf{S}'_{n,d}$ such that $\pi_{n,d} = \phi_{d+n,d}\pi_{d+n,d}$. Moreover $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ is in the inverse limit of the inverse system $\{\mathbf{S}_{n,d}, \phi_{d+n,d}\}_{d\in\mathbb{N}}$.

Let $\mathbf{k}$ be a finite field of q elements. Let $\mathbf{k}((t))$ be the field of formal Laurent polynomials. Let $\mathbf{k}[[t]]$ be the subring of $\mathbf{k}((t))$ consisting of all formal power series. Consider a $\mathbf{k}((t))$-vector space V of dimension d. A free $\mathbf{k}[[t]]$-module L of rank d such that $\mathbf{k}((t)) \otimes_{\mathbf{k}[[t]]} L = V$ is called a lattice of V. The collection of chains of lattices $L_\bullet = (L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subset t^{-1}L_1)$ will be denoted by $\mathfrak{F}_{n,d}$, the n-step affine flag variety. When specialized at $v = q^{1/2}$, the algebra $\mathbf{S}_{n,d}$ is isomorphic to the convolution algebra on $\mathfrak{F}_{n,d} \times \mathfrak{F}_{n,d}$.

Let $\Lambda_{n,d}$ be the subset in $\mathfrak{S}_n$ of all sequence (a_i) such that $\sum_{1\le i\le n} a_i = d$. For each $(a_i) \in \Lambda_{n,d}$, define $\mathfrak{F}_{(a_i)}$ to be the subset of $\mathfrak{F}_{n,d}$ consisting of all lattice chains $L_\bullet$ such that $\dim_{\mathbf{k}} L_i/L_{i-1} = a_i$ for all $1 \le i \le n$. For each $(a_i) \in \Lambda_{n,d}$, there is an idempotent $1_{(a_i)}$ in $\mathbf{S}_{n,d}$ such that, when specialized to $v = q^{1/2}$, it is the characteristic function of the diagonal of $\mathfrak{F}_{(a_i)} \times \mathfrak{F}_{(a_i)}$.

3 The Embedding $\Phi_{r,\varepsilon}$

In this section, we present an explicit embedding of the quantum affine $\mathfrak{sl}_n$ to its higher rank and a compatibility with the natural embedding on the Schur algebra level.

3.1 The Statement

Let us fix forever an integer $r \in [0, n-1]$ and $\varepsilon \in \{\pm 1\}$. Consider $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$. To avoid confusion, we denote the Chevalley generators in $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ by $\breve{E}_i, \breve{F}_i, \breve{K}_i^{\pm 1}$ for $i \in I_{n+1}$. Consider the following elements in $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$. For all $i \in I_n$, we define

$$\begin{cases} e_i = \begin{cases} \breve{E}_i & \text{if } i = 0, \dots, r-1, \\ \breve{E}_i \breve{E}_{i+1} - v^{\varepsilon} \breve{E}_{i+1} \breve{E}_i & \text{if } i = r, \\ \breve{E}_{i+1} & \text{if } i = r+1, \dots, n-1. \end{cases} \\ f_i = \begin{cases} \breve{F}_i & \text{if } i = 0, \dots, r-1, \\ \breve{F}_{i+1} \breve{F}_i - v^{-\varepsilon} \breve{F}_i \breve{F}_{i+1} & \text{if } i = r, \\ \breve{F}_{i+1} & \text{if } i = r+1, \dots, n-1. \end{cases} \\ k_i = \begin{cases} \breve{K}_i & \text{if } i = 0, \dots, r-1, \\ \breve{K}_i \breve{K}_{i+1} & \text{if } i = r, \\ \breve{K}_{i+1} & \text{if } i = r+1, \dots, n-1. \end{cases} \end{cases} \tag{3.1}$$

Recall that for any $x \in \mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ we write $x^{(a)} = x^a/[a]^!$. For any $p \in \mathbb{N}$, we have

$$e_r^{(p)} = \sum_{j=0}^{p} (-1)^j v^{\varepsilon j} \breve{E}_{r+1}^{(j)} \breve{E}_r^{(p)} \breve{E}_{r+1}^{(p-j)}, \tag{3.2}$$

$$f_r^{(p)} = \sum_{j=0}^{p} (-1)^j v^{-\varepsilon j} \breve{F}_r^{(j)} \breve{F}_{r+1}^{(p)} \breve{F}_r^{(p-j)}. \tag{3.3}$$

The equality (3.2) is [8, Lemma 42.1.2 (c)]. We have the following.

Theorem 3.1 *There is an injective homomorphism of associative algebras*

$$\Phi_{r,\varepsilon} : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \mathbf{U}(\widehat{\mathfrak{sl}}_{n+1}) \text{ defined by } E_i \mapsto e_i, F_i \mapsto f_i, K_i^{\pm 1} \mapsto k_i^{\pm 1}, \quad \forall i \in I_n. \tag{3.4}$$

Proof We shall show the existence of $\Phi_{r,\varepsilon}$ in Section 4: for $n > 2$, it is given in Section 4.1 and for $n = 2$ it is given in Section 4.2.

Assume now that the algebra homomorphism $\Phi_{r,\varepsilon}$ is well-defined. We shall show that the map $\Phi_{r,\varepsilon}$ is injective. Let $\mathbf{U}_{r,\varepsilon}$ be the subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ generated by $e_i, f_i, k_i^{\pm 1}$ for all $i \in I_n$. We consider the module $\mathbf{V}_{n+1}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$. It restricts automatically to a $\mathbf{U}_{r,\varepsilon}$-module. To avoid confusion, we write its basis element by $\breve{u}_k$ for all $k \in \mathbb{Z}$. We define an embedding $\mathbf{V}_n \to \mathbf{V}_{n+1}$ by

$$u_k \mapsto \begin{cases} \breve{u}_{k+\lfloor k/n \rfloor}, & \text{if } \bar{k} \in \{\overline{0}, \overline{1}, \dots, \overline{r-1}\} \subseteq I_n, \\ \breve{u}_{k+\lceil k/n \rceil}, & \text{if } \bar{k} \in \{\overline{r}, \dots, \overline{n-1}\} \subseteq I_n. \end{cases}$$

Let $\mathbf{W}_n$ be the image of the embedding $\mathbf{V}_n \to \mathbf{V}_{n+1}$. One can check that $\mathbf{W}_n$ is a $\mathbf{U}_{r,\varepsilon}$-submodule of $\mathbf{V}_n$, and moreover the actions of the generators in $\mathbf{U}_{r,\varepsilon}$ on $\mathbf{W}_n$ is compatible with the action of the generators in $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ under the isomorphism $\mathbf{V}_n \to \mathbf{W}_n$. Observe that

$$\Delta(e_r) = e_r \otimes 1 + k_r \otimes e_r + (v^{-1} - v^{\varepsilon}) \breve{E}_{r+1} \breve{K}_r \otimes \breve{E}_r + (v - v^{\varepsilon}) \breve{K}_{r+1} \breve{E}_r \otimes \breve{E}_{r+1}, \tag{3.5}$$

$$\Delta(f_r) = f_r \otimes k_r^{-1} + 1 \otimes f_r + (v - v^{-\varepsilon}) \breve{F}_r \otimes \breve{K}_r^{-1} \breve{F}_{r+1} + (v^{-1} - v^{-\varepsilon}) \breve{F}_{r+1} \otimes \breve{F}_r \breve{K}_{r+1}^{-1}. \tag{3.6}$$

This indicates that when restricting to $\mathbf{W}_n^{\otimes d}$, as a summand of $\mathbf{V}_{n+1}^{\otimes d}$, the last two operators in $\Delta(e_r)$ and $\Delta(f_r)$ are zero. Thus $\mathbf{W}_n^{\otimes d}$ is a $\mathbf{U}_{r,\varepsilon}$-module and thanks to (3.5) the actions of e_r and f_r on $\mathbf{W}_n^{\otimes d}$ are compatible with the actions of E_r, F_r on $\mathbf{V}_n^{\otimes d}$ via the isomorphism $\mathbf{V}_n \to \mathbf{W}_n$. Identifying $\mathbf{V}_n$ with $\mathbf{W}_n$, we see that there exists an algebra homomorphism

$\pi'_{n,d} : \mathbf{U}_{r,\varepsilon} \to \mathbf{S}'_{n,d}$ such that $\pi'_{n,d}(e_i) = \pi_{n,d}(E_i)$, $\pi'_{n,d}(f_i) = \pi_{n,d}(F_i)$ and $\pi'_{n,d}(k_i) = \pi_{n,d}(K_i)$ for all $i \in I_n$. Since $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ is in the inverse limit of an inverse system $\{\mathbf{S}'_{n,d}, \mathbf{S}'_{n,d+n} \to \mathbf{S}'_{n,d}\}_{d\in\mathbb{N}}$, by the universality there exists a unique algebra homomorphism $\Psi_{r,\varepsilon} : \mathbf{U}_{r,\varepsilon} \to \mathbf{U}(\widehat{\mathfrak{sl}}_n)$ such that $\Psi_{r,\varepsilon}(e_i) = E_i$, $\Psi_{r,\varepsilon}(f_i) = F_i$ and $\Psi_{r,\varepsilon}(k_i^{\pm 1}) = K_i^{\pm 1}$ for all $i \in I_n$. This is the inverse of the map $\Phi'_{r,\varepsilon} : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \mathbf{U}_{r,\varepsilon}$ induced by $\Phi_{r,\varepsilon}$. So $\Phi'_{r,\varepsilon}$ is an isomorphism, and in other words, $\Phi_{r,\varepsilon}$ is injective. The theorem is thus proved. □

We end this section with a remark.

Remark 3.2 (1) The assumption that $r \in [0, n-1]$ is not essential. One can define the morphism $\Phi_{r,\varepsilon}$ when $r = n$ as well in a similar manner.

(2) The $v = 1$ version of the embedding $\mathbf{V}_n \to \mathbf{V}_{n+1}$ and (3.1) first appeared in [11].

(3) The analysis in Proof of Theorem 3.1 yields the following commutative diagram of linear maps.

$$\begin{array}{ccc} \mathbf{U}(\widehat{\mathfrak{sl}}_n) & \xrightarrow{\Phi_{r,\varepsilon}} & \mathbf{U}_{r,\varepsilon} \\ {\scriptstyle \pi_{n,d}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{n+1,d}} \\ \mathbf{S}'_{n,d} & \xleftarrow{\pi} & \mathbf{S}'_{n+1,d} \end{array}$$

where $\tilde{\pi}_{n+1,d}$ is the restriction of the map $\pi_{n+1,d}$ in (2.1) to $\mathbf{U}_{r,\varepsilon}$ and π is the projection of a linear endomorphism of $\mathbf{V}_{n+1}^{\otimes d}$ to a linear endomorphism of $\mathbf{V}_n^{\otimes d}$ under the embedding $\mathbf{V}_n \to \mathbf{V}_{n+1}$. (Note that π is not an algebra homomorphism.) A more natural compatibility will be given in the following section. The injectivity of $\Phi_{r,\varepsilon}$ can be proved by exploring the above commutative diagram as follows. Suppose that $x \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)$ is in the kernel of $\Phi_{r,\varepsilon}$. Then by the above commutative diagram, we have $\pi_{n,d}(x) = \pi\pi'_{n+1,d}\Phi_{r,\varepsilon}(x) = 0$ for all d. It is well-known that if $\pi_{n,d}(x) = 0$ for all d, then $x = 0$. Therefore $\Phi_{r,\varepsilon}$ is injective.

3.2 A Variant

In [5], Green defines a version of quantum affine $\mathfrak{gl}_n$. This is a variant of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$. More precisely, it is defined as an associative algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ over $\mathbb{Q}(v)$, which has generators $E_i, F_i, L_i^{\pm 1}$ for $i \in I_n$ and which subject to the following defining relations.

$$\begin{cases} L_iL_j = L_jL_i, L_iL_i^{-1} = 1, \\ L_iE_j = v^{\delta_{i,j}-\delta_{i-1,j}}E_jL_i, \\ L_iF_j = v^{-\delta_{i,j}+\delta_{i-1,j}}F_jL_i, \\ E_iF_j - F_jE_i = \delta_{i,j}\frac{L_iL_{i+1}^{-1} - L_i^{-1}L_{i+1}}{v-v^{-1}}, \\ \sum_{p=0}^{1-c_{ij}} (-1)^p E_i^{(p)} E_j E_i^{(1-c_{ij}-p)} = 0, \\ \sum_{p=0}^{1-c_{ij}} (-1)^p F_i^{(p)} F_j F_i^{(1-c_{ij}-p)} = 0, \quad \forall i, j \in I_n. \end{cases} \tag{3.7}$$

Note that in [5], there is an assumption that $n \geq 3$. We do not need this assumption.

Recall that, to avoid confusion, a superscript $\breve{}$ is put on the Chevalley generators of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$. Recall that we fix an integer $r \in [0, n-1]$. Consider the following element in

$\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$.

$$l_i = \begin{cases} \breve{K}_i \breve{K}_{i+1} \cdots \breve{K}_r, & 1 \le i \le r, \\ \breve{K}_{r+1}^{-1} \cdots \breve{K}_i^{-1}, & r+1 \le i \le n. \end{cases} \tag{3.8}$$

Let $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})_1 \equiv \mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})/\langle \prod_{i=1}^{n+1} \breve{K}_i - 1 \rangle$ be the quotient algebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ by the two-sided ideal generated by $\prod_{i=1}^{n+1} \breve{K}_i - 1$. This is the quantum affine $\widehat{\mathfrak{sl}}_{n+1}$ at level 1. We still use the same notations for the images of the Chevalley generators of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ under the canonical projection map. We have the following $\widehat{\mathfrak{gl}}_n$-variant of Theorem 3.1

Proposition 3.3 *There is an embedding* $\Phi_{r,\varepsilon;1} : \mathbf{U}(\widehat{\mathfrak{gl}}_n) \to \mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})_1$ *defined by* $E_i \mapsto e_i, F_i \mapsto f_i, L_i \mapsto l_i$ *for all* $i \in I_n$*, where* e_i, f_i *are in* (3.1).

The proof is given in Section 4.3.

3.3 Compatibility with Quantum Affine Schur Algebras

Recall the root datum (Y_n, X_n) from Section 2.2. Define a morphism

$$\phi = (f, g) : (Y_n, X_n) \to (Y_{n+1}, X_{n+1})$$

where $f : Y_n \to Y_{n+1}$ sends a sequence $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_r, 0, a_{r+1}, \ldots, a_n)$, and $g : X_{n+1} \to X_n$ sends a coset of $(a_1, \ldots, a_{n+1})$ to the coset of $(a_1 - a_{r+1}, \ldots, a_{i_r} - a_{r+1}, a_{r+2} - a_{r+1}, \ldots, a_{n+1} - a_{r+1})$. The root datum (Y_{n+1}, X_{n+1}) can be regarded as a root datum of affine type A_n, instead of A_{n+1}, in an appropriate way such that ϕ is a morphism of root data of affine type A_n. Note that f extends naturally to a map $f : \mathfrak{S}_n \to \mathfrak{S}_{n+1}$.

There is an embedding $\Lambda_{n,d} \to X_n$ defined by $\lambda \mapsto \bar{\lambda}$ as the composition of the embedding $\Lambda_{n,d} \to \mathfrak{S}_n$ and the quotient map $\mathfrak{S}_n \to X_n$. For $\bar{\lambda} \in X_n - \Lambda_{n,d}$, then there is a $d' \neq d$ such that $\lambda \in \Lambda_{n,d'}$. Clearly different representatives of $\bar{\lambda}$ yield different d'. We fix once and for all a representative for each class in X_n so that if $\bar{\lambda} \cap \Lambda_{n,d} \neq \emptyset$ then $\lambda \in \Lambda_{n,d}$. Under this assumption, we can define a map $f_d : X_n \to X_{n+1}$ by sending $\bar{\lambda} \to \overline{f(\lambda)}$. Clearly $g f_d(\bar{\lambda}) = \bar{\lambda}$. In light of Theorem 3.1, we thus have an algebra embedding

$$\phi_{d,\varepsilon} : \dot{\mathbf{U}}_n \to \dot{\mathbf{U}}_{n+1} \tag{3.9}$$

defined by $1_{\bar{\lambda}} \mapsto 1_{\overline{f(\lambda)}}$, $E_i 1_{\bar{\lambda}} \mapsto e_i 1_{\overline{f(\lambda)}}$ and $F_i 1_{\bar{\lambda}} \mapsto f_i 1_{\overline{f(\lambda)}}$ for all $i \in I_n$ and $\lambda \in \mathfrak{S}_n$.

Recall from Section 2.3 the quantum affine Schur algebra $\mathbf{S}_{n,d}$. It has idempotents 1_λ for $\lambda \in \Lambda_{n,d}$. It is known that there is an algebra homomorphism $\dot{\pi}_{n,d} : \dot{\mathbf{U}}_n \to \mathbf{S}_{n,d}$ such that $\dot{\pi}_{n,d}(1_{\bar{\lambda}}) = 1_\lambda$ if $\lambda \in \Lambda_{n,d}$, and $\dot{\pi}_{n,d}(1_{\bar{\lambda}}) = 0$ otherwise. On the other hand, the subset $\mathfrak{F}_{n+1,d}|_r$ of $\mathfrak{F}_{n+1,d}$ consisting of all lattice chains $L_\bullet$ such that $L_r = L_{r+1}$ is naturally in bijection with $\mathfrak{F}_{n,d}$. Via this bijection, there is an algebra homomorphism $\sigma_d : \mathbf{S}_{n,d} \to \mathbf{S}_{n+1,d}$ such that $\sigma_d(1_\lambda) = 1_{f(\lambda)}$ for all $\lambda \in \Lambda_{n,d}$ (see e.g., [7]). We are ready to state the compatibility of the embeddings on $\dot{\mathbf{U}}_n$ and $\mathbf{S}_{n,d}$.

Proposition 3.4 *Let* $\varepsilon \in \{\pm 1\}$*. The following diagram is commutative.*

$$\begin{array}{ccc} \dot{\mathbf{U}}_n & \xrightarrow{\dot{\phi}_{d,\varepsilon}} & \dot{\mathbf{U}}_{n+1} \\ \downarrow{\scriptstyle \pi_{n,d}} & & \downarrow{\scriptstyle \dot{\pi}_{n+1,d}} \\ \mathbf{S}_{n,d} & \xrightarrow[\sigma_d]{} & \mathbf{S}_{n+1,d} \end{array} \tag{3.10}$$

where $\phi_{d,\varepsilon}$ is from (3.9).

Proof By definition, the two compositions $\sigma_d \circ \dot{\pi}_{n,d}$ and $\dot{\pi}_{n+1,d} \circ \phi_{d,\varepsilon}$ coincide when evaluating at the idempotents 1_λ for all $\lambda \in X_n$. Now the maps are all $\mathbf{U}_n$-module homomorphisms and hence the two compositions coincide when evaluating at any element in $\dot{\mathbf{U}}_n$. Hence the diagram must be commutative. The proposition follows. □

4 Existence of $\Phi_{r,\varepsilon}$

This section is devoted to the proof of the existence of $\Phi_{r,\varepsilon}$. In Section 4.1, we present the proof when $n > 2$ and in Section 4.2, we present the proof for the $n = 2$ case.

4.1 Existence of $\Phi_{r,\varepsilon}$ in the $n > 2$ Case

In this section we assume that $n > 2$. We shall show that $\Phi_{r,\varepsilon}$ is an algebra homomorphism, i.e., the set $\{e_i, f_i, k_i^{\pm 1} \mid i \in I_n\}$ satisfies the defining relations of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$.

It is easy to see that the elements $k_i^{\pm 1}$ satisfy the relation (R1).

Next, we show that the pair (k_i, e_j) satisfies the defining condition in (R2). All are a consequence of the definition, except the case (k_i, e_i), $(k_{i\pm 1}, e_i)$ for $i = r$. Observe that

$$\breve{K}_{i-1}e_i = v^{-1}e_i\breve{K}_{i-1}, \quad \breve{K}_i e_i = ve_i\breve{K}_i, \quad \breve{K}_{i+1}e_i = ve_i\breve{K}_{i+1}, \quad \breve{K}_{i+2}e_i = v^{-1}e_i\breve{K}_{i+2} \quad \text{if } i = r.$$

So we have

$$k_i e_i = \breve{K}_i\breve{K}_{i+1}e_i = v^2 e_i\breve{K}_i\breve{K}_{i+1} = v^2 e_i k_i, \quad k_{i\pm 1}e_i = v^{-1}e_i k_{i\pm 1}.$$

Thus the pair (k_i, e_j) satisfies the relation in (R2). Similarly the pair (k_i, f_j) satisfies the relation in (R2) because we have

$$\breve{K}_{i-1}f_i = vf_i\breve{K}_{i-1}, \quad \breve{K}_i f_i = v^{-1}f_i\breve{K}_i, \quad \breve{K}_{i+1}f_i = v^{-1}f_i\breve{K}_{i+1}, \quad \breve{K}_{i+2}f_i = vf_i\breve{K}_{i+2} \quad \text{if } i = r.$$

Next, we show that the triple $(e_i, f_j, k_i^{\pm 1})$ satisfies the commutator relation in (R3). The relation is satisfied automatically if $i, j \neq r$. Assume that $i = r$, $j \neq r$, and $j < r$. Then we have

$$\begin{aligned} e_i f_j - f_j e_i &= (\breve{E}_i\breve{E}_{i+1} - v^\varepsilon \breve{E}_{i+1}\breve{E}_i)\breve{F}_j - \breve{F}_j(\breve{E}_i\breve{E}_{i+1} - v^\varepsilon \breve{E}_{i+1}\breve{E}_i) \\ &= \breve{E}_i\breve{E}_{i+1}\breve{F}_j - \breve{F}_j\breve{E}_i\breve{E}_{i+1} - v^\varepsilon(\breve{E}_{i+1}\breve{E}_i\breve{F}_j - \breve{F}_j\breve{E}_{i+1}\breve{E}_i) \\ &= (\breve{E}_i\breve{F}_j - \breve{F}_j\breve{E}_i)\breve{E}_{i+1} - v^\varepsilon \breve{E}_{i+1}(\breve{E}_i\breve{F}_j - \breve{F}_j\breve{E}_i) = 0. \end{aligned}$$

So the commutator relation holds for the case $i = r$, $j \neq r$ and $j < r$. Similarly one can show that the commutator relation holds for the case $i = r$, $j \neq r$ and $j > r$ and the case $i \neq r$, $j = r$. We now show the remaining case $i = j = r$. In this case, a direct simplification yields

$$\begin{aligned} e_i f_i - f_i e_i = (\breve{E}_i\breve{E}_{i+1}\breve{F}_{i+1}\breve{F}_i - \breve{F}_{i+1}\breve{F}_i\breve{E}_i\breve{E}_{i+1}) - v^\varepsilon(\breve{E}_{i+1}\breve{F}_{i+1}\breve{E}_i\breve{F}_i - \breve{F}_{i+1}\breve{F}_i\breve{E}_{i+1}\breve{E}_i) \\ + v^\varepsilon(\breve{E}_i\breve{F}_i\breve{E}_{i+1}\breve{F}_{i+1} - \breve{F}_i\breve{F}_{i+1}\breve{E}_i\breve{E}_{i+1}) + (\breve{E}_{i+1}\breve{E}_i\breve{F}_i\breve{E}_{i+1} - \breve{F}_i\breve{F}_{i+1}\breve{E}_{i+1}\breve{E}_i). \quad (4.1) \end{aligned}$$

By using the commutator relations in $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$ on $\breve{E}_{i+1}\breve{F}_{i+1}$ and $\breve{F}_i\breve{E}_i$, we see that the term in the first parenthesis in (4.1) is equal to

$$\breve{E}_i\frac{\breve{K}_{i+1} - \breve{K}_{i+1}^{-1}}{v - v^{-1}}\breve{F}_i + \breve{F}_{i+1}\frac{\breve{K}_i - \breve{K}_i^{-1}}{v - v^{-1}}\breve{E}_{i+1}. \quad (4.2)$$

Note that $\breve{F}_{i+1}\breve{F}_i\breve{E}_{i+1}\breve{E}_i = \breve{F}_{i+1}\breve{E}_{\imath+1}\breve{F}_i\breve{E}_i$, and by applying the commutator relation on both $\breve{F}_{i+1}\breve{E}_{\imath+1}$ and $\breve{F}_i\breve{E}_i$, the term in the second parenthesis in (4.1) is simplified to

$$-v^{\varepsilon}\Big(\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}\breve{E}_i\breve{F}_i+\breve{E}_{i+1}\breve{F}_{i+1}\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}-\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}\Big). \tag{4.3}$$

By doing the same operation on the second monomial in the term in the third parenthesis in (4.1), the term is simplied to

$$-v^{-\varepsilon}\Big(\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}\breve{E}_{i+1}\breve{F}_{i+1}+\breve{E}_i\breve{F}_i\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}-\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}\Big). \tag{4.4}$$

Just like the term in the first parenthesis, the term in the last parenthesis in (4.1) is simplified to

$$\breve{E}_{i+1}\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}\breve{F}_{i+1}+\breve{F}_i\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}\breve{E}_i. \tag{4.5}$$

The sum of the terms in (4.2)–(4.5) having $\breve{E}_i\breve{F}_i$ or $\breve{F}_i\breve{E}_i$ can be simplified to

$$\frac{-v^{-1}\breve{K}_{i+1}+v\breve{K}_{i+1}^{-1}}{v-v^{-1}}\cdot\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}. \tag{4.6}$$

The sum of the terms in (4.2)–(4.5) having $\breve{E}_{i+1}\breve{F}_{i+1}$ or $\breve{F}_{i+1}\breve{E}_{i+1}$ can be simplified to

$$\frac{-v^{-1}\breve{K}_i+v\breve{K}_i^{-1}}{v-v^{-1}}\cdot\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}. \tag{4.7}$$

The term involving only $\breve{K}_i^{\pm1}$ and $\breve{K}_{i+1}^{\pm1}$ in (4.3) plus the term in (4.5) is equal to

$$\breve{K}_{i+1}^{-1}\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}. \tag{4.8}$$

The term involving only $\breve{K}_i^{\pm1}$ and $\breve{K}_{i+1}^{\pm1}$ in (4.4) plus the term in (4.6) is equal to

$$\breve{K}_i\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}. \tag{4.9}$$

So the commutator $e_if_i-f_ie_i$ for $i=r$ is equal to the sum of (4.8) and (4.9), which is

$$\breve{K}_{i+1}^{-1}\frac{\breve{K}_i-\breve{K}_i^{-1}}{v-v^{-1}}+\breve{K}_i\frac{\breve{K}_{i+1}-\breve{K}_{i+1}^{-1}}{v-v^{-1}}=\frac{\breve{K}_i\breve{K}_{i+1}-\breve{K}_i^{-1}\breve{K}_{i+1}^{-1}}{v-v^{-1}}=\frac{k_i-k_i^{-1}}{v-v^{-1}}.$$

The commutator relation (R3) for the case $i=j=r$ is proved. This finishes the proof that the triple $(e_i,f_i,k_i^{\pm1})$ satisfies the relation (R3).

Next, we show that the pair (e_i,e_j) satisfies the quantum Serre relation (R4). First we observe that all cases are a consequence of the corresponding defining relations of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$, except the cases $(r\pm1,r)$ and $(r,r\pm1)$. For the case $(r-1,r)$, we can simplify the quantum Serre relation, say $S_{r-1,r}$, as follows.

$$\begin{aligned}
&\breve{E}_{r-1}^{(2)}(\breve{E}_r\breve{E}_{r+1}-v^{\varepsilon}\breve{E}_{r+1}\breve{E}_r)-\breve{E}_{r-1}(\breve{E}_r\breve{E}_{r+1}-v^{\varepsilon}\breve{E}_{r+1}\breve{E}_r)\breve{E}_{r-1}+(\breve{E}_r\breve{E}_{r+1}-v^{\varepsilon}\breve{E}_{r+1}\breve{E}_r)\breve{E}_{r-1}^{(2)}\\
&\quad=\breve{E}_{r-1}^{(2)}\breve{E}_r\breve{E}_{r+1}-\breve{E}_{r-1}\breve{E}_r\breve{E}_{r+1}\breve{E}_{r-1}+\breve{E}_r\breve{E}_{r+1}\breve{E}_{r-1}^{(2)}\\
&\qquad-v^{\varepsilon}(\breve{E}_{r-1}^{(2)}\breve{E}_{r+1}\breve{E}_r-\breve{E}_{r-1}\breve{E}_{r+1}\breve{E}_r\breve{E}_{r-1}+\breve{E}_{r+1}\breve{E}_r\breve{E}_{r-1}^{(2)})\\
&\quad=(\breve{E}_{r-1}^{(2)}\breve{E}_r-\breve{E}_{r-1}\breve{E}_r\breve{E}_{r-1}+\breve{E}_r\breve{E}_{r-1}^{(2)})\breve{E}_{r+1}-v^{\varepsilon}\breve{E}_{r+1}(\breve{E}_{r-1}^{(2)}\breve{E}_r-\breve{E}_{r-1}\breve{E}_r\breve{E}_{r-1}+\breve{E}_r\breve{E}_{r-1}^{(2)})
\end{aligned}$$

$= 0.$

So the quantum Serre relation holds for the case $(i,j) = (r-1, r)$. The case for $(i,j) = (r+1, r)$ can be shown in exactly the same way. Now we show the quantum Serre relation for the case $(r, r-1)$. First of all, we need the formula (3.2) for $p = 2$, which reads

$$e_r^{(2)} = \breve{E}_r^{(2)}\breve{E}_{r+1}^{(2)} - v^{\varepsilon}\breve{E}_{r+1}\breve{E}_r^{(2)}\breve{E}_{r+1} + v^{2\varepsilon}\breve{E}_{r+1}^{(2)}\breve{E}_r^{(2)}.$$

By using this formula, we see immediately that the left-hand side of the quantum Serre relation for $(r, r-1)$, i.e., $e_r^{(2)}e_{r-1} - e_{r-1}e_re_{r-1} + e_{r-1}e_r^{(2)}$, is equal to

$$\begin{aligned}
&(\breve{E}_r^{(2)}\breve{E}_{r+1}^{(2)}\breve{E}_{r-1} - \breve{E}_r\breve{E}_{r+1}\breve{E}_{r-1}\breve{E}_r\breve{E}_{r+1} + \breve{E}_{r-1}\breve{E}_r^{(2)}\breve{E}_{r+1}^{(2)})\\
&\quad - v^{\varepsilon}(\breve{E}_{r+1}\breve{E}_r^{(2)}\breve{E}_{r+1}\breve{E}_{r-1} - \breve{E}_{r+1}\breve{E}_r\breve{E}_{r-1}\breve{E}_{r+1} - \breve{E}_r\breve{E}_{r+1}\breve{E}_{r-1}\breve{E}_{r+1}\breve{E}_r + \breve{E}_{r-1}\breve{E}_{r+1}\breve{E}_r^{(2)}\breve{E}_{r+1})\\
&\quad + v^{2\varepsilon}(E_{r+1}^{(2)}\breve{E}_r^{(2)}\breve{E}_{r-1} - \breve{E}_{r+1}\breve{E}_r\breve{E}_{r-1}\breve{E}_{r+1}\breve{E}_r + \breve{E}_{r-1}\breve{E}_{r+1}^{(2)}\breve{E}_r^{(2)}). \qquad (4.10)
\end{aligned}$$

Replace the piece $\breve{E}_{r+1}\breve{E}_r\breve{E}_{r+1}$ by $\breve{E}_r\breve{E}_{r+1}^{(2)} + \breve{E}_{r+1}^{(2)}\breve{E}_r$ in the second monomial, we see that the terms in the first parenthesis in (4.10) are equal to

$$-\breve{E}_r\breve{E}_{r-1}\breve{E}_{r+1}^{(2)}\breve{E}_r. \qquad (4.11)$$

Similarly, the third term with the coefficient $v^{2\varepsilon}$ in (4.10) is equal to

$$-v^{2\varepsilon}\breve{E}_r\breve{E}_{r+1}^{(2)}\breve{E}_{r-1}\breve{E}_r. \qquad (4.12)$$

The monomials in (4.11) and (4.12) add up to

$$-v^{\varepsilon}[2]\breve{E}_r\breve{E}_{r-1}\breve{E}_{r+1}^{(2)}\breve{E}_r. \qquad (4.13)$$

The first, second and fourth monomials in the term with coefficient v^{ε} add up to zero. The remaining monomial in the term with v^{ε} is $v^{\varepsilon}\breve{E}_r\breve{E}_{r+1}\breve{E}_{r-1}\breve{E}_{r+1}\breve{E}_r = v^{\varepsilon}[2]\breve{E}_r\breve{E}_{r-1}\breve{E}_{r+1}^{(2)}\breve{E}_r$, which cancels with (4.13). So the quantum Serre relation for $(r, r-1)$ holds. This finishes the proof of the quantum Serre relation (R4) for the pair (e_i, e_j).

Similarly one can show that the quantum Serre relation (R5) for the pair (f_i, f_j) holds. One can also apply the involution on $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$, defined by $\breve{E}_i \mapsto \breve{F}_i, \breve{F}_i \mapsto \breve{E}_i, \breve{K}_i \mapsto \breve{K}_i^{-1}$ for all $i \in I_{n+1}$, to obtain the proof.

The proof of $\Phi_{r,\varepsilon}$ being an algebra homomorphism for $n > 2$ is thus finished.

4.2 Existence of $\Phi_{r,\varepsilon}$ in the $n = 2$ Case

In this section, we assume that $n = 2$, and we shall show that the set $\{e_i, f_i, k_i^{\pm 1} \mid i \in I_2\}$ satisfies the defining relations of $\mathbf{U}(\widehat{\mathfrak{sl}}_2)$. The relations (R1) and (R2) can be verified directly, just like the $n > 2$ case, with very minor modifications. The commutator relation (R3) can be proved in exactly the same way as the $n > 2$ case. So we only need to verify the quantum Serre relations (R4) and (R5).

Let us verify the relation (R4). For simplicity, we use (i, j, ℓ) for $(r, r+1, r-1)$. The first relation to show is

$$e_\ell^{(3)}e_i - e_\ell^{(2)}e_ie_\ell + e_\ell e_i e_\ell^{(2)} - e_ie_\ell^{(3)} = 0. \qquad (4.14)$$

A simplification yields that the left-hand side of (4.14) is equal to

$$(\breve{E}_\ell^{(3)}\breve{E}_i\breve{E}_j - \breve{E}_\ell^{(2)}\breve{E}_i\breve{E}_j\breve{E}_\ell + \breve{E}_\ell\breve{E}_i\breve{E}_j\breve{E}_\ell^{(2)} - \breve{E}_i\breve{E}_j\breve{E}_\ell^{(3)})$$

$$- v^{\varepsilon}(\breve{E}_{\ell}^{(3)}\breve{E}_j\breve{E}_i - \breve{E}_{\ell}^{(2)}\breve{E}_j\breve{E}_i\breve{E}_{\ell} + \breve{E}_l\breve{E}_j\breve{E}_i\breve{E}_{\ell}^{(2)} - \breve{E}_j\breve{E}_i\breve{E}_{\ell}^{(2)}). \tag{4.15}$$

Observe that if the term in the first parenthesis is zero, so is the term in the second one by switching the roles of i and j. Recall that we have

$$\breve{E}_{\ell}^{(3)}\breve{E}_i = \breve{E}_{\ell}\breve{E}_i\breve{E}_{\ell}^{(2)} - [2]\breve{E}_i\breve{E}_{\ell}^{(3)}, \breve{E}_i\breve{E}_{\ell}^{(3)} = \breve{E}_{\ell}^{(2)}\breve{E}_i\breve{E}_{\ell} - [2]\breve{E}_{\ell}^{(3)}\breve{E}_i.$$

Applying the above identities to the first and fourth monomials in the term in the first parenthesis in (4.15), we have

$$\begin{aligned}
&\breve{E}_{\ell}^{(3)}\breve{E}_i\breve{E}_j - \breve{E}_{\ell}^{(2)}\breve{E}_i\breve{E}_j\breve{E}_{\ell} + \breve{E}_{\ell}\breve{E}_i\breve{E}_j\breve{E}_{\ell}^{(2)} - \breve{E}_i\breve{E}_j\breve{E}_{\ell}^{(3)} \\
&\quad = \breve{E}_{\ell}\breve{E}_i(\breve{E}_{\ell}^{(2)}\breve{E}_j + E_j\breve{E}_{\ell}^{(2)}) - (\breve{E}_{\ell}^{(2)}\breve{E}_i + \breve{E}_i\breve{E}_{\ell}^{(2)})\breve{E}_j\breve{E}_{\ell} \\
&\quad = \breve{E}_{\ell}\breve{E}_i\breve{E}_{\ell}\breve{E}_j\breve{E}_{\ell} - \breve{E}_{\ell}\breve{E}_i\breve{E}_{\ell}\breve{E}_j\breve{E}_{\ell} = 0.
\end{aligned} \tag{4.16}$$

Therefore, the relation (4.14) holds.

The second relation we need to verify is

$$S_{i\ell} \equiv e_i^{(3)}e_{\ell} - e_i^{(2)}e_{\ell}e_i + e_ie_{\ell}e_i^{(2)} - e_{\ell}e_i^{(3)} = 0. \tag{4.17}$$

Instead of proving it directly, which involves monomials of degree 7, we shall make use of Lusztig's twisted derivation r_k on the positive half of a quantum group in [8, 1.2.13]. Let $\mathbf{U}^+(\widehat{\mathfrak{sl}}_3)$ be the positive half of $\mathbf{U}(\widehat{\mathfrak{sl}}_3)$ generated by $\breve{E}_k$ for all $k \in I_3$. To each k, there is a linear map $r_k : \mathbf{U}^+(\widehat{\mathfrak{sl}}_3) \to \mathbf{U}^+(\widehat{\mathfrak{sl}}_3)$ such that

$$r_k(1) = 0, \quad r_k(\breve{E}_{k'}) = \delta_{k,k'}, \quad r_k(xy) = v^{k\cdot|y|}r_k(x)y + xr_k(y),$$

for all homogeneous element y. Here $|y| = (y_0, y_1, y_2) \in \mathbb{Z}^{I_3}$ is the degree of y and $k \cdot |y| = 2y_k - y_{k+1} - y_{k-1}$. Recall $(r, r+1, r-1) = (i, j, \ell)$. For now we assume that $\varepsilon = -1$. We will freely use the following formulas in later analysis.

Lemma 4.1 *For all $p \geq 0$, we have*

$$r_{\ell}(e_i^{(p)}) = 0, \tag{s1}$$

$$r_i(e_i^{(p)}) = 0, \tag{s2}$$

$$r_j(e_i^{(p)}) = (v - v^{-1})v^{p-2}\sum_{a=0}^{p-1}(-1)^a v^{-2a}\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-1-a)}, \tag{s3}$$

$$r_ir_j(e_i^{(p)}) = (v - v^{-1})v^{p-2}e_i^{(p-1)}, \tag{s4}$$

$$r_jr_j(e_i^{(p)}) = (v - v^{-1})(v^2 - v^{-2})v^{2(p-3)}\sum_{a=0}^{p-2}v^{-3a}\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-2-a)}. \tag{s5}$$

Proof The equality (s1) is because the ℓ-degree of e_i is zero. For (s2), we only need to show that $r_i(e_i) = 0$, which can be done as follows.

$$r_i(e_i) = r_i(\breve{E}_i\breve{E}_j - v^{-1}\breve{E}_j\breve{E}_i) = v^{i\cdot j}r_i(\breve{E}_i)\breve{E}_j - v^{-1}\breve{E}_jr_i(\breve{E}_i) = v^{-1}\breve{E}_j - v^{-1}\breve{E}_j = 0.$$

We show (s3) by induction. When $p = 1$, we have

$$r_j(e_i) = r_j(\breve{E}_i\breve{E}_j - v^{-1}\breve{E}_j\breve{E}_i) = \breve{E}_i - v^{-1}v^{i\cdot j}r_j(\breve{E}_j)\breve{E}_i = (v - v^{-1})v^{-1}\breve{E}_i.$$

Assume that (s3) holds for p, we want to show that it holds for $p+1$. We have

$$r_j(e_i^{(p+1)}) = \frac{1}{[p+1]}r_j(e_i^{(p)}.e_i) = \frac{1}{[p+1]}(vr_j(e_i^{(p)})e_i + e_i^{(p)}r_j(e_i)). \tag{4.18}$$

We simplify the first term as follows.

$$\begin{aligned}\frac{1}{[p+1]}vr_j(e_i^{(p)})e_i &= \frac{(v-v^{-1})v^{p-1}}{[p+1]}\sum_{a=0}^{p-1}(-1)^a v^{-2a}\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-1-a)}(\breve{E}_i\breve{E}_j - v^{-1}\breve{E}_j\breve{E}_i)\\ &= (v-v^{-1})v^{p-1}\sum_{a=0}^{p-1}(-1)^a v^{-2a}\breve{E}_j^{(a)}\breve{E}_i^{(p+1)}\breve{E}_j^{(p-a)}\\ &\quad + \frac{(v-v^{-1})v^{p-1}}{[p+1]}\sum_{a=0}^{p-1}(-1)^a(v^{-2a}[p-1-a]-v^{-1-2a}[p-a])\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-1}\breve{E}_i, \qquad (4.19)\end{aligned}$$

where we use the fact

$$\breve{E}_j^{(p-1-a)}\breve{E}_i\breve{E}_j = \breve{E}_i\breve{E}_j^{(p-a)} + [p-1-a]\breve{E}_j^{(p-a)}\breve{E}_i.$$

The second term can be simplified as follows:

$$\begin{aligned}\frac{1}{[p+1]}e_i^{(p)}r_j(e_i) &= (v-v^{-1})v^{p-1}(-1)^p v^{-2p}\breve{E}_j^{(p)}\breve{E}_i^{(p+1)}\\ &\quad + \frac{(v-v^{-1})v^{-1}}{[p+1]}\sum_{a=0}^{p-1}(-1)^a v^{-a}\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_i^{(p-a)}\breve{E}_i. \qquad (4.20)\end{aligned}$$

The second terms in (4.19) and (4.20) cancel once we observe that

$$v^{p-2a}[p-1-a] - v^{p-1-2a}[p-a] + v^{-a} = 0.$$

The equality (s3) is then followed by adding the first terms in (4.19) and (4.20). Now we show (s4). By (s3), we have

$$\begin{aligned}r_i r_j(e_i^{(p)}) &= (v-v^{-1})v^{p-2}\sum_{a=0}^{p-1}(-1)^a v^{-2a}\breve{E}_j^{(a)}v^{-(p-1-a)}r_i(\breve{E}_i^{(p)})\breve{E}_j^{(p-1-a)}\\ &= (v-v^{-1})v^{p-2}\sum_{a=0}^{p-1}(-1)^a v^{-a}\breve{E}_j^{(a)}\breve{E}_i^{(p-1)}\breve{E}_j^{(p-1-a)}\\ &= (v-v^{-1})v^{p-2}e_i^{(p-1)}. \qquad (4.21)\end{aligned}$$

So (s4) holds. Finally we show (s4). By (s3), we have

$$\begin{aligned}&r_j r_j(e_i^{(p)})\\ &= (v-v^{-1})v^{p-2}\sum_{a=0}^{p-1}(-1)^a v^{-2a}(v^{p-2a-2}r_j(\breve{E}_j^{(a)})\breve{E}_i^{(p)}\breve{E}_j^{(p-1-a)} + \breve{E}_j^{(a)})\breve{E}_i^{(p)}r_j(\breve{E}_j^{(p-1-a)})\\ &= (v-v^{-1})v^{p-2}\sum_{a=0}^{p-1}(-1)^a v^{-2a}(v^{p-a-3}\breve{E}_j^{(a-1)}\breve{E}_i^{(p)}\breve{E}_j^{(p-1-a)} + v^{p-2-a}\breve{E}_j^{(a)})\breve{E}_i^{(p)}\breve{E}_j^{(p-2-a)}\\ &= (v-v^{-1})v^{p-2}\sum_{a=0}^{p-2}(-1)^a(v^{p-3a-6} - v^{p-3a-2})\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-a-2)}\\ &= (v-v^{-1})(v^2-v^{-2})v^{2(p-3)}\sum_{a=0}^{p-2}(-1)^a v^{-3a}\breve{E}_j^{(a)}\breve{E}_i^{(p)}\breve{E}_j^{(p-a-2)}. \qquad (4.22)\end{aligned}$$

So (s5) holds. The lemma is therefore proved. □

In light of [8, Lemma 1.2.15], to show that $S_{i\ell}=0$, it is sufficient to show that

$$r_i(S_{i\ell})=0,\quad r_j(S_{i\ell})=0,\quad r_\ell(S_{i,\ell})=0. \tag{4.23}$$

Due to (s2) and $r_i(\breve{E}_\ell)=0$, we have immediately that $r_i(S_{i\ell})=0$. Moreover, we have

$$\begin{aligned} r_\ell(S_{i\ell}) &= e_i^{(3)} - e_i^{(2)} r_\ell(e_\ell e_i) + e_i r_\ell(e_\ell e_i^{(2)}) - v^{\ell\cdot 3(i+j)} e_i^{(3)} \\ &= e_i^{(3)} - v^{-2} e^{(2)} e_i + v^{-4} e_i e_i^{(2)} - v^{-6} e_i^{(3)} \\ &= (1 - v^{-2}[3] + v^{-4}[3] - v^{-6}) e_i^{(3)} = 0. \end{aligned}$$

So it remains to show that $r_j(S_{i\ell})=0$. This is further reduced to show that

$$r_i r_j(S_{i\ell})=0,\quad r_j r_j(S_{i\ell})=0,\quad r_\ell r_j(S_{i\ell})=0. \tag{4.24}$$

By a direct computation, we get

$$\begin{aligned} r_j(S_{i\ell}) = {} & v^{-1} r_j(e_i^{(3)})\breve{E}_\ell - r_j(e_i^{(2)})\breve{E}_\ell e_i - v^{-1}(v-v^{-1}) e_i^{(2)}\breve{E}_\ell\breve{E}_i \\ & + (v-v^{-1})\breve{E}_i\breve{E}_\ell e_i^{(2)} + e_i\breve{E}_\ell r_j(e_i^{(2)}) - \breve{E}_\ell r_j(e_i^{(3)}). \end{aligned} \tag{4.25}$$

By applying r_i to the formula (4.25) and a direction computation, we have

$$\begin{aligned} r_i r_j(S_{i\ell}) = {} & v^{-2} r_i r_j(e_i^{(3)})\breve{E}_\ell - (v-v^{-1}) e_i\breve{E}_\ell e_i \\ & - v^{-1}(v-v^{-1}) e_i^{(2)}\breve{E}_\ell + v(v-v^{-1})\breve{E}_\ell e_i^{(2)} + (v-v^{-1}) e_i\breve{E}_\ell e_i - \breve{E}_\ell r_i r_j(e_i^{(3)}) \\ = {} & v^{-1}(v-v^{-1}) e_i^{(2)}\breve{E}_\ell - v^{-1}(v-v^{-1}) e_i^{(2)}\breve{E}_\ell + v(v-v^{-1})\breve{E}_\ell e_i^{(2)} - v(v-v^{-1})\breve{E}_\ell e_i^{(2)} = 0. \end{aligned}$$

So to show that $r_j(S_{i\ell})=0$, it remains to show that $r_\ell r_j(S_{i\ell})=0$ and $r_j r_j(S_{i\ell})=0$. By a direct computation, we get

$$\begin{aligned} r_\ell r_j(S_{i\ell}) = {} & v^{-3}(v^2-v^{-2}) r_j(e_i^{(3)}) - v^{-2} r_j(e_i^{(2)}) e_i - v^{-2}(v-v^{-1}) e_i^{(2)}\breve{E}_i \\ & + v^{-4}(v-v^{-1})\breve{E}_i e_i^{(2)} + v^{-3} e_i r_j(e_i^{(2)}). \end{aligned}$$

Since there is no $\breve{E}_\ell$ in $r_\ell r_j(S_{i\ell})$, so $r_\ell r_\ell r_j(S_{i\ell})=0$. Next, we compute $r_i r_\ell r_j(S_{i\ell})$ as follows.

$$r_i r_\ell r_j(S_{i\ell}) = [v^{-2}(v^2-v^{-2})(v-v^{-1}) - v^{-1}(v-v^{-1})[2] + v^{-3}(v-v^{-1})[2]] e_i^{(2)} = 0.$$

Further we compute $r_j r_\ell r_j(S_{i\ell})$. We have

$$r_j r_\ell r_j(S_{i\ell}) = v^{-3}(v-v^{-1})^3(\breve{E}_i^{(3)}\breve{E}_j - v^{-1}\breve{E}_i^{(2)}\breve{E}_j\breve{E}_i + v^{-2}\breve{E}_i\breve{E}_j\breve{E}_i^{(2)} - v^{-3}\breve{E}_j\breve{E}_i^{(3)}) = 0.$$

The above analysis shows that $r_\ell r_j(S_{i\ell})=0$. So to show that $r_j(S_{i\ell})=0$ it remains to show that $r_j r_j(S_{i\ell})=0$. By a direct computation, we have

$$\begin{aligned} \frac{r_j^2(S_{i\ell})}{v^{-2}(v-v^{-1})(v^2-v^{-2})} = {} & (\breve{E}_i^{(3)}\breve{E}_j - v^{-3}\breve{E}_j\breve{E}_i^{(3)})\breve{E}_\ell - \breve{E}_i^{(2)}\breve{E}_\ell e_i \\ & - v^{-1}(v\breve{E}_i^{(2)}\breve{E}_j - v^{-1}\breve{E}_j\breve{E}_i^{(2)})\breve{E}_\ell\breve{E}_i + \breve{E}_i\breve{E}_\ell(v\breve{E}_i^{(2)}\breve{E}_j - v^{-1}\breve{E}_j\breve{E}_i^{(2)}) \\ & + e_i\breve{E}_\ell\breve{E}_i^{(2)} - v^2\breve{E}_\ell(\breve{E}_i^{(3)}\breve{E}_j - v^{-3}\breve{E}_j\breve{E}_i^{(3)}). \end{aligned} \tag{4.26}$$

Now substitute e_i by $\breve{E}_i\breve{E}_{i+1} - v^{-1}\breve{E}_{i+1}\breve{E}_i$, we see that (4.26) is equal to

$$\begin{aligned} & \breve{E}_i^{(3)}\breve{E}_j\breve{E}_\ell - \breve{E}_i^{(2)}\breve{E}_j\breve{E}_\ell\breve{E}_i + \breve{E}_i\breve{E}_j\breve{E}_\ell\breve{E}_i^{(2)} \\ & - v^{-3}\breve{E}_j\breve{E}^{(3)}\breve{E}_\ell + v^{-2}\breve{E}_\ell\breve{E}_i^{(2)}\breve{E}_\ell\breve{E}_i - v^{-1}\breve{E}_j\breve{E}_i\breve{E}_\ell\breve{E}_i^{(2)} \end{aligned}$$

$$- \breve{E}_i^{(2)} \breve{E}_\ell \breve{E}_i \breve{E}_j + v \breve{E}_i \breve{E}_\ell \breve{E}_i^{(2)} \breve{E}_j - v^2 \breve{E}_\ell \breve{E}_i^{(3)} \breve{E}_j$$
$$+ \breve{E}_i^{(2)} \breve{E}_\ell \breve{E}_j \breve{E}_i - v^{-1} \breve{E}_i \breve{E}_\ell \breve{E}_j \breve{E}_i^{(2)} + v^{-1} \breve{E}_\ell \breve{E}_j \breve{E}_i^{(3)}. \tag{4.27}$$

By (4.16), the first row in (4.27) is equal to $\breve{E}_j \breve{E}_\ell \breve{E}_i^{(3)}$, which cancels with the second row. The third row in (4.27) is equal to $-v^{-1} \breve{E}_i^{(3)} \breve{E}_\ell \breve{E}_j$, which cancels with the fourth row. So we get $r_j^2(S_{i\ell}) = 0$. This finishes the proof of (4.24), and therefore $r_j(S_{i\ell}) = 0$. In turn, this shows that (4.23) holds, and thus $S_{i\ell} = 0$, i.e., (4.17), as desired. The above proof assumes that $\varepsilon = -1$. The case for $\varepsilon = 1$ can be proved by rewriting e_i as $v^\varepsilon(\breve{E}_j \breve{E}_i - v^{-\varepsilon} \breve{E}_i \breve{E}_j)$ and the proof for $\varepsilon = 1$ case applies by switching the roles of i and j.

The relation (R5) is a consequence of the relation (R4) by applying the involution on $\mathbf{U}(\widehat{\mathfrak{sl}}_3)$ sending $\breve{E}_k$ to $\breve{F}_k$ $\forall k \in I_3$. This finishes the proof of $\Phi_{r,\varepsilon}$ being an algebra homomorphism for $n = 2$.

4.3 Proof of Proposition 3.3

In this section, we provide a proof of Proposition 3.3. The last two relations in (3.7) have been verified in the proof of Theorem 3.1. The relations in the first three rows of (3.7) can be checked directly. By using the relation $\prod_{i=1}^{n+1} \breve{K}_i = 1$, we see that $k_i = l_i l_{i+1}^{-1}$ for all $i \in I_n$. Indeed, the equality holds for all $i \in I_n - \{n\}$ obviously. If $i = n$, then

$$l_n l_{n+1}^{-1} = l_n l_1^{-1} = \breve{K}_{r+1}^{-1} \cdots \breve{K}_n^{-1} (\breve{K}_1 \cdots \breve{K}_r)^{-1} = \breve{K}_{n+1} = k_n.$$

So the commutator relation in (3.7) is a consequence of the commutator relation of $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})$. This implies that the map $\Phi_{r,\varepsilon:1}$ is an algebra homomorphism. The injective property follows from the triangular decompositions of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}(\widehat{\mathfrak{sl}}_{n+1})_1$. This finishes the proof.

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