Geometry & Topology

Volume 27 (2023)

High-energy harmonic maps and degeneration of minimal surfaces

CHARLES OUYANG

msp

msp

Geometry & Topology **27**:5 (2023) 1691–1746
DOI: 10.2140/gt.2023.27.1691
Published: 27 July 2023

High-energy harmonic maps and degeneration of minimal surfaces

CHARLES OUYANG

Let S be a closed surface of genus $g \geq 2$ and ρ a maximal $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ surface group representation. By a result of Schoen, there is a unique ρ–equivariant minimal surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{H}^2$. We study the induced metrics on these minimal surfaces and prove the limits are precisely mixed structures. We prove a similar result for maximal surfaces in AdS^3. In the second half of the paper, we provide a geometric interpretation: the minimal surfaces $\widetilde{\Sigma}$ degenerate to the core of a product of two $\mathbb{R}$–trees. As a consequence, we obtain a compactification of the space of maximal representations of $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

49Q05, 53C43

1 Introduction and main results

Let S be a closed, orientable, smooth surface of genus $g > 1$. For any reductive Lie group G, one can form the character variety $\mathcal{R}(\pi_1(S), G) = \mathrm{Hom}^+(\pi_1(S), G) /\!\!/ G$, consisting of conjugacy classes of reductive surface group representations into G. In the classical setting, where $G = \mathrm{PSL}(2, \mathbb{R})$, one recovers a copy of Teichmüller space. A goal in the higher Teichmüller theory is to understand geometric aspects of surface group representations into higher-rank Lie groups.

Following the work of Labourie [23], given a reductive surface group representation ρ into a semisimple Lie group G, to each complex structure J on the surface S, one can record the energy of the unique ρ–equivariant harmonic map from $(\widetilde{S}, J)$ to the Riemannian symmetric space G/K. This defines an energy functional on Teichmüller space, and Labourie proves that if the original representation ρ is *Anosov*, then the energy functional admits a critical point. Hence, to each such representation ρ, there is an associated branched immersed minimal surface in the symmetric space G/K.

The existence and uniqueness of the minimal surface in the associated symmetric space has been resolved by Labourie [24] for the rank-two real split simple Lie groups: namely $\mathrm{SL}(3, \mathbb{R})$, $\mathrm{PSp}(4, \mathbb{R})$ and G_2. Interestingly enough, the result still holds when G is merely semisimple, as the case of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ was proven by Schoen in [39].

There is also the aim in the program of the higher Teichmüller theory to understand representations as geometric objects. This is a natural goal, given that in the case of classical Teichmüller theory, where the group is $G = \mathrm{PSL}(2, \mathbb{R})$ and the representation is discrete and faithful, the associated geometric objects are given by marked hyperbolic surfaces. Moreover, it is of interest to obtain a description of boundary points associated to higher Teichmüller spaces in terms of degenerations of geometric objects. It would be interesting to have these geometric objects at the boundary be a generalization of measured laminations (see [43, Section 11]), which are the limiting geometric objects in the Thurston compactification of Teichmüller space.

In the setting $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, this paper does exactly that: we provide a parametrization of maximal surface group representations into $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, by studying the induced metrics on the ρ–equivariant minimal surfaces in the symmetric space $\mathbb{H}^2 \times \mathbb{H}^2$. If $\rho = (\rho_1, \rho_2)$, and $\widetilde{\Sigma}$ is the unique ρ–equivariant minimal surface in $\mathbb{H}^2 \times \mathbb{H}^2$, then its quotient by the action of the fundamental group via the representation

is the graph of the unique minimal lagrangian isotopic to the identity between $(S, g_1) = \mathbb{H}^2/\rho_1$ and $(S, g_2) = \mathbb{H}^2/\rho_2$.

Let $\mathrm{Ind}(S)$ be the equivalence class of induced metrics on the graph minimal surface in the product of two hyperbolic surfaces. Two such metrics are identified if one is the pullback metric of the other by a diffeomorphism homotopic to the identity map.

We study the length spectrum of these induced metrics on the minimal surface and show that we can degenerate the metrics to obtain singular flat metrics, measured laminations and mixed structures. A *mixed structure* $\eta = (S_\alpha, q_\alpha, \lambda)$ is the data of a collection of incompressible subsurfaces $\bigsqcup S_\alpha$, with a prescribed meromorphic (integrable) quadratic differential on each subsurface (collapsing the boundary components and viewing them as punctures), with a measured lamination λ supported on the complement $S \setminus \bigsqcup S_\alpha$. Observe that a holomorphic quadratic differential on S and a measured lamination on S are trivial examples of mixed structures, where $S_\alpha = S$ and $S_\alpha = \varnothing$, respectively. Define then $\mathrm{PMix}(S)$ to be the space of projectivized mixed structures. Our first main result is the following.

Theorem A *The space* $\mathrm{Ind}(S)$ *of induced metrics embeds into the space* $\mathrm{PCurr}(S)$ *of projectivized currents. Its closure is* $\mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$.

If we keep track of the ambient space, namely $\mathbb{H}^2 \times \mathbb{H}^2$, we show that by scaling the ambient space by a suitable sequence of constants (which generally will be the total energy of a harmonic map), we can obtain as limits of minimal lagrangians the core of a pair of $\mathbb{R}$–trees coming from measured foliations. In fact, we show there is an isometric embedding from a metric space obtained from the data of a mixed structure to the core of trees.

As a consequence, we have an answer to our original goal of ascribing something geometric to maximal surface group representations into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. By studying degenerations of the minimal lagrangians, we obtain natural boundary objects which are both geometric and are natural extensions of measured laminations.

Theorem B *The space of maximal representations of* $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *embeds into the space of* $\pi_1 S$*–equivariant minimal lagrangians in* $\mathbb{H}^2 \times \mathbb{H}^2$. *The scaled Gromov–Hausdorff limits of the minimal lagrangians are given by the core of a product* $T_1 \times T_2$ *of trees, where* T_1 *and* T_2 *are a pair of* $\mathbb{R}$*–trees coming from a projective pair of measured foliations.*

Minimal lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$ arise as the image of the Gauss map of the unique embedded spacelike maximal surface in a Globally Hyperbolic Maximal Compact (GHMC) AdS^3–manifold; see [22]. Mess [28] showed that the maximal representations into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ are precisely the holonomy representations of GHMC AdS^3–structures. One could have studied maximal representations into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ by looking at the induced metric on the maximal surface instead. We show an analogue of Theorem A, that the limits one obtains are also mixed structures. If $\mathrm{Max}(S)$ denotes the space of induced metrics on the maximal surface, then our final result is the following.

Theorem C *The space* $\mathrm{Max}(S)$ *of induced metrics on the maximal surfaces embeds into the space* $\mathrm{PCurr}(S)$ *of projectivized currents. Its closure is* $\mathrm{Max}(S) \sqcup \mathrm{PMix}(S)$.

There has been some recent interest in studying surface group representations to the Lie group $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ by way of geodesic currents. Work of Glorieux [15] shows that the average of two Liouville currents, $\frac{1}{2}(L_{X_1} + L_{X_2})$, yields the length spectrum of the Globally Hyperbolic Maximal Compact AdS^3 manifold with holonomy (ρ_1, ρ_2), where $X_i = \mathbb{H}^2 \setminus \rho_i$. In another recent paper of Glorieux [16], it is shown that this map which sends unordered pairs of elements in Teichmüller space to the space of projectivized currents, given by $(X_1, X_2) = (X_2, X_1) \to \frac{1}{2}(L_{X_1} + L_{X_2})$, is injective. Recent work of Burger, Iozzi, Parreau and Pozzetti [6] show the limits of this embedding are given by the projectivization of a pair of measured laminations. The limiting current η thus satisfies

$$i(\eta, \cdot) = i(\lambda_1, \cdot) + i(\lambda_2, \cdot), \tag{1-1}$$

where λ_1 and λ_2 are specific representatives of the projectivize classes $[\lambda_1]$ and $[\lambda_2]$, respectively, representing limits on the Thurston boundary.

We remark that our compactification via geodesic currents is distinct. If the limiting laminations λ_1 and λ_2 *fill*, that is, the sum of their intersection numbers with any third measured lamination is never zero, then the corresponding limiting object η' under our compactification is a singular flat metric coming from a unit-norm holomorphic quadratic differential Φ whose horizontal and vertical laminations are λ_1 and λ_2. The corresponding current is thus given by

$$l^2_{|\Phi|}(\alpha) = i^2(\eta', \alpha) = i^2(\lambda_1, \alpha) + i^2(\lambda_2, \alpha) \tag{1-2}$$

for a suitably short arc α away from the zeros of $|\Phi|$. In general, this is different from the sum of λ_1 and λ_2. Notice that for γ an arc of the horizontal lamination of Φ,

the two intersection numbers $i(\eta, \alpha)$ and $i(\eta', \alpha)$ coincide, so that the two currents η and η' are distinct even as projectivized currents. However, using their limiting currents, Burger, Iozzi, Parreau and Pozzetti are able to construct and interpret their boundary objects as subbuildings in the product of trees, endowed with the L^1–metric.

Finally, since the first version of this paper appeared, related work has been done on some of the other rank-two Lie groups; see Ouyang and Tamburelli [34; 35; 36]. More recently, together with Martone and Tamburelli [26], we have described our compactification as a closed ball, upon which the mapping class group acts.

Acknowledgements It is the author's privilege to thank his advisor Mike Wolf for first suggesting the problem and for his continued patience, support and guidance. The author would like to express his gratitude to Zeno Huang and Andrea Tamburelli for many fruitful conversations throughout this project. He would also like to thank Chris Leininger and Ivo Slegers for helpful comments on an earlier draft. Finally, he would like to thank the referee for the many careful comments to improve the paper. The author acknowledges support from the National Science Foundation through grant DMS-1564374.

2 Geometric preliminaries

2.1 Harmonic maps between surfaces

Let $(M, \sigma\,|dz|^2)$ and $(N, \rho\,|dw|^2)$ be two closed Riemannian surfaces and

$$w\colon (M, \sigma\,|dz|^2) \to (N, \rho\,|dw|^2)$$

a Lipschitz map. Then the energy of the map w is given by the integral

$$\mathcal{E}(w) := \frac{1}{2}\int_M \|dw\|^2\, d\mathrm{vol}_\sigma.$$

A critical point of the energy functional is a *harmonic map*. The energy density of the map w, defined almost everywhere, is given by

$$e(w) = \frac{\rho(w(z))}{\sigma(z)}(|w_z|^2 + |w_{\overline{z}}|^2),$$

and so the total energy is also given by the formula

$$\mathcal{E}(w) = \int_M e(w)\sigma\; dz \wedge d\overline{z} = \int_M \rho(w(z))(|w_z|^2 + |w_{\overline{z}}|^2)\, dz \wedge d\overline{z},$$

which shows the total energy depends only upon the conformal structure of the domain surface but on the metric of the target. Alternatively, a harmonic map w solves the Euler–Lagrange equation for the energy functional, a second-order nonlinear PDE,

$$w_{z\overline{z}} + (\log \rho)_w w_z w_{\overline{z}} = 0.$$

To any harmonic map $w\colon (M, \sigma\,|dz|^2) \to (N, \rho\,|dw|^2)$, the pullback of the metric tensor decomposes by type according to

$$w^*(\rho\,|dw|^2) = \Phi\,dz^2 + \sigma e\,dz\,d\overline{z} + \overline{\Phi}\,d\overline{z}^2,$$

where $\Phi\,dz^2$ is a holomorphic quadratic differential with respect to the complex structure coming from the conformal class of $(M, \sigma\,|dz|^2)$, called the *Hopf differential* of w. Much of the formulas arising from harmonic maps make use of the auxiliary functions

$$\mathcal{H} = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2 \quad \text{and} \quad \mathcal{L} = \frac{\rho(w(z))}{\sigma(z)}|w_{\overline{z}}|^2.$$

We list some of these formulas and make liberal use of them without always explicitly citing the precise one:

- The energy density $e = \mathcal{H} + \mathcal{L}$.
- The Jacobian $\mathcal{J} = \mathcal{H} - \mathcal{L}$.
- The norm of the quadratic differential
$$\frac{|\Phi|^2}{\sigma^2} = \mathcal{H}\mathcal{L}.$$
- The Laplace–Beltrami operator
$$\Delta \equiv \frac{4}{\sigma}\frac{\partial^2}{\partial z \partial \overline{z}}.$$
- Gaussian curvature of the source
$$K(\sigma) = -\frac{2}{\sigma}\frac{\partial^2 \log \sigma}{\partial z \partial \overline{z}}.$$
- Gaussian curvature of the target
$$K(\rho) = -\frac{2}{\rho}\frac{\partial^2 \log \rho}{\partial w \partial \overline{w}}.$$
- The Beltrami differential
$$\nu = \frac{w_{\overline{z}}}{w_z} = \frac{\overline{\Phi}}{\sigma\mathcal{H}}, \quad \text{with } |\nu|^2 = \frac{\mathcal{L}}{\mathcal{H}}.$$

The Bochner formula is given by

$$\Delta \log \mathcal{H} = -2K(\rho)\mathcal{H} + 2K(\rho)\mathcal{L} + 2K(\sigma) \quad \text{when } \mathcal{H}(p) \neq 0,$$
$$\Delta \log \mathcal{L} = -2K(\rho)\mathcal{L} + 2K(\rho)\mathcal{H} + 2K(\sigma) \quad \text{when } \mathcal{L}(p) \neq 0.$$

We shall often be in the setting where both the source and target are hyperbolic surfaces, that is, $K(\sigma) = K(\rho) \equiv -1$, and so some of the formulas listed above can be simplified. In the more general setting where the target has negative curvature, the existence of a harmonic map in the homotopy class is due to Eells and Sampson [11], its uniqueness is due to Hartman [19] and Al'ber [1], and the fact that if the homotopy class contains a diffeomorphism, then the harmonic map itself is a diffeomorphism and $\mathcal{H} > 0$, is due to Schoen and Yau [40] and Sampson [38].

2.2 Teichmüller space

Recall that Teichmüller space $\mathcal{T}(S)$ is the space of all hyperbolic metrics on S with the identification $g \sim h$ if there exists a diffeomorphism ϕ of the surface, homotopic to the identity map, for which $\phi^* g = h$. The topology is given by its marked length spectrum.

Alternatively, one may regard Teichmüller space as the space of marked Riemann surfaces. For a fixed surface S, two complex structures (S, J_1) and (S, J_2) are identified if there exists a biholomorphism $f : (S, J_1) \to (S, J_2)$ which is homotopic to the identity. The topology is given by the metric which, for two points of Teichmüller space, assigns the logarithm of the quasiconformal dilatation of the unique Teichmüller mapping between the marked Riemann surfaces.

Teichmüller space is topologically trivial, being homeomorphic to an open ball of dimension $6g - 6$.

2.3 Measured foliations and measured laminations

For a closed surface S, a measured foliation $(S, \mathcal{F})$ is a singular foliation (finitely many k–pronged singularities, with $k \in \{3, 4, \dots\}$) with a transverse measure, that is, a measure μ defined on each arc transverse to the foliation, such that the measure is invariant under isotopy between two arcs through transverse arcs.

To any isotopy class of measured foliations, there is an associated measured lamination. A measured geodesic lamination on a hyperbolic surface is a closed disjoint set of

geodesics with a transverse measure. Likewise, to any measured lamination, there is an associated measured foliation, so that there is a canonical way to pass from one to the other; see [7] and [37]. Hence, the space of measured laminations does not depend upon the choice of hyperbolic metric. Thurston showed that both spaces are homeomorphic to Euclidean balls of dimension $6g-6$; see [12] and [42].

2.4 Holomorphic quadratic differentials

The space of holomorphic quadratic differentials Q_g is a holomorphically trivializable vector bundle over Teichmüller space, whose fiber over the Riemann surface X is the vector space of holomorphic quadratic differentials on X. It is the vector space of holomorphic sections of the square of the canonical bundle K_X, and so may be written $H^0(X, K_X^2)$. By the Riemann–Roch theorem, the complex dimension of this vector space is $3g-3$. More concretely, if X is a Riemann surface and q is a holomorphic quadratic differential on X, then locally $q = f(z)\,dz^2$, where f is a holomorphic function and z is a chart for X.

Holomorphicity of the differential and compactness of the Riemann surface ensures the quadratic differential has precisely $4g-4$ zeros counted with multiplicity. Hence, in a neighborhood avoiding a zero of q, one may choose natural coordinates ζ so that $q = d\zeta^2$. The metric $|q|$ is well-defined on the complement of the zeros and is locally Euclidean. At the zeros, the metric has conic singularities of angle $(n+2)\pi$, where n is the order of the zero of the quadratic differential at that point.

For any point on the complement of the zeros of the quadratic differential, there is a unique direction for which $q(v,v) \in \mathbb{R}^+$. Integrating the resulting line field, one obtains a foliation, called the *horizontal foliation* of the quadratic differential q. Likewise, one can define the *vertical foliation* of q, by integrating the line field of directions for which $q(v,v) \in \mathbb{R}^-$. The foliations come equipped with a transverse measure. For any arc γ transverse to the horizontal foliation, the measure for the horizontal foliation is given by

$$\tau_h = \int_\gamma |\mathrm{Im}(\sqrt{f})(z)|\,|dz|,$$

and likewise, the transverse measure for the vertical foliation is given by integrating the real part $|\mathrm{Re}(\sqrt{q})|$ over an arc γ.

If $S_{g,n}$ is a compact surface of genus g with n punctures such that $3g-3+n>0$, then $Q_{g,n}$ will denote the space of integrable holomorphic quadratic differentials on $S_{g,n}$. At each of the punctures, the differential has a pole of order one.

2.5 Geodesic currents and marked length spectra

Let (S,σ) be a fixed closed hyperbolic surface of genus $g \geq 2$. Then its universal cover $\widetilde{S}$ may be identified isometrically with $\mathbb{H}^2$. Let $G(\widetilde{S})$ denote the space of geodesics of $\widetilde{S}$. Then a geodesic current on S is a $\pi_1(S)$–equivariant Radon measure on $G(\widetilde{S})$. The space of geodesic currents, denoted by $\mathrm{Curr}(S)$, is given by the weak* topology.

Remark A priori, the definition of a geodesic current may appear to depend upon the choice of hyperbolic metric, but it turns out $G(\widetilde{S})$ depends only upon $\pi_1(S)$; see [4]. Hence the space of geodesic currents is independent of the hyperbolic metric initially chosen for S.

The ur-example of a geodesic current is given by a single closed geodesic γ on S. Lift γ to a discrete set of geodesics $\widetilde{\gamma}$ on $\widetilde{S}$. These lifted geodesics may be given a Dirac measure, which is $\pi_1(S)$–invariant as the lifts themselves are $\pi_1(S)$–invariant. Hence for any closed curve, by looking at its geodesic representative, one obtains a geodesic current on S. In fact, Bonahon [4] shows the space of weighted closed curves is dense in $\mathrm{Curr}(S)$ and the geometric intersection number between curves has a continuous bilinear extension to $i : \mathrm{Curr}(S)\times\mathrm{Curr}(S) \to \mathbb{R}_{\geq 0}$. Moreover, a geodesic current on S is determined by its intersection number with all closed curves [33]. The topology then on the space of geodesic currents is given by its marked length spectrum. For the fixed surface S, denote by $\mathcal{C}$ the set of isotopy classes of closed curves. The marked length spectrum of a geodesic current μ is given by the collection $\{i(\mu,\gamma)\}_{\gamma\in\mathcal{C}}$. A sequence of geodesic currents μ_n is said to converge to μ if their marked length spectra converge; that is, for each $\gamma \in \mathcal{C}$ and $\epsilon > 0$, there is an $N(\epsilon,\gamma)$ such that for $n > N(\epsilon,\gamma)$, one has $|i(\mu,\gamma) - i(\mu_n,\gamma)| < \epsilon$. It is important to note that N is allowed to depend on the curve class chosen. No requirement on uniform convergence is required.

If a current arises from a metric, the following rather useful formula applies.

Proposition 2.1 (Bonahon [4]; Otal [33]) *Let μ be a current arising from a metric σ. Then*

$$i(\mu,\mu) = \frac{\pi}{2}\,\mathrm{Area}(\sigma).$$

In the case where μ is a geodesic current arising from a measured lamination, it is not hard to see that $i(\mu,\mu) = 0$, but in fact, this turns out to be a characterization of measured laminations.

Proposition 2.2 (Bonahon [4]) *Let μ be a geodesic current such that $i(\mu,\mu)=0$. Then μ is a measured lamination.*

It is clear that if μ is a geodesic current, then so is $c\mu$ for $c\in\mathbb{R}_+$. The set of projectivized currents, denoted by $\mathrm{PCurr}(S)$, is given by $\mathrm{Curr}(S)/\sim$, where $\mu\sim\nu$ if there exists a positive constant c for which $\mu=c\nu$, and so consists of projective classes of geodesic currents. The space $\mathrm{PCurr}(S)$ is then given the quotient topology. We highlight an important property of this space.

Proposition 2.3 (Bonahon [4]) *The space $\mathrm{PCurr}(S)$ is compact.*

Several geometric structures have been shown to be embedded into $\mathrm{Curr}(S)$. The first such example was due to Bonahon [4], who showed Teichmüller space could be embedded inside $\mathrm{Curr}(S)$ via its Liouville current, namely $\sigma\mapsto L_\sigma$ with the property that for any closed curve γ, one has $l_\sigma([\gamma])=i(L_\sigma,\gamma)$, so that the length of the geodesic representative of γ with respect to the hyperbolic metric σ coincides with the intersection number between the currents L_σ and γ. As the space of measured laminations can be realized as geodesic currents, Bonahon recovers the Thurston compactification by way of projectivized geodesic currents.

Otal [33] has shown the space of negatively curved Riemannian metrics on surfaces can be realized by geodesic currents. For any simple curve class $[\gamma]$, the length of the unique geodesic representative coincides with the intersection number of the corresponding geodesic current and the curve class $[\gamma]$, extending the work of Bonahon.

Duchin, Leininger and Rafi [9] have embedded the space of singular flat metrics arising from integrable holomorphic quadratic differentials into the space of geodesic currents. We summarize a few results here, as we shall use them in what follows. Recall that to any holomorphic quadratic differential q, one can associate a singular flat metric $|q|$ via canonical coordinates.

The unit sphere $Q_g^1\subset Q_g$ consists of the holomorphic quadratic differentials with L^1–norm one. Then the space $\mathrm{Flat}(S)$ of unit-norm singular flat metrics may be identified by

$$\mathrm{Flat}(S)=Q_g^1/\mathbb{S}^1,$$

where the action of $\mathbb{S}^1$ is given by multiplication by $e^{i\theta}$, for $0\le\theta\le 2\pi$. We require this quotient because if q is a holomorphic quadratic differential, then q and $e^{i\theta}q$ will have the same singular flat metric $|q|$. For $q\in Q_g^1$, consider the transverse measure

for the vertical foliation of q, that is, $v_q = |\mathrm{Re}(\sqrt{q})|$. Denote by $v_q^\theta = |\mathrm{Re}(e^{i\theta}\sqrt{q})|$ the vertical foliation of $e^{i\theta}q$. Form the integral

$$L_q := \frac{1}{2}\int_0^\pi v_q^\theta \, d\theta.$$

Theorem 2.4 (Duchin, Leininger and Rafi [9]) *The integral L_q is a geodesic current such that to any simple closed curve γ,*

$$l_{|q|}(\gamma) = i(L_q, \gamma),$$

where $|q|$ is the singular flat metric arising from the holomorphic quadratic differential q. Furthermore, the map which sends $|q| \in \mathrm{Flat}(S)$ to $L_q \in \mathrm{PCurr}(S)$ is an embedding.

As a geodesic current is determined by its marked length spectrum, the construction of the geodesic current L_q depends only upon the $U(1)$–orbit of q. Hence we will use the notation $L_{|q|}$ to denote the geodesic current whose marked length spectrum coincides with that of the singular flat metric $|q|$.

As the space of projectivized currents is compact, one may take the closure of the space $\mathrm{Flat}(S)$, and it is shown in [9] that the limiting structures consist precisely of projectivized mixed structures. A *mixed structure* may be defined as follows. Let S' be an incompressible subsurface of S equipped with a Riemann surface structure. Then consider $Q_{S'}$, the space of integrable meromorphic quadratic differentials on S' such that with respect to the underlying complex structure on S', neighborhoods of boundary components of $\partial S'$ are conformally punctured disks. To any such quadratic differential q, the corresponding singular flat metric on S' thus assigns length zero to any peripheral curve. Let λ be a measured lamination supported on the complement $S \setminus S'$. The triple (S', q, λ) is called a mixed structure on S. For any $\eta = (S', q, \lambda)$, one obtains a geodesic current L_η given by the property

$$i(L_\eta, \gamma) = i(\lambda, \gamma) + \frac{1}{2}\int_0^{\pi/2} i(v_q^\theta, \gamma)\, d\theta,$$

where λ is a closed curve on S. We remark that in the case $S' = \varnothing$, then η is a measured lamination on S, so that the space $\mathrm{Mix}(S)$ properly contains $\mathrm{ML}(S)$. The compactification of the singular flat metrics arising from unit-norm quadratic differentials is then given by the following theorem.

Theorem 2.5 (Duchin, Leininger and Rafi [9]) *The closure of* $\mathrm{Flat}(S)$ *in* $\mathrm{PCurr}(S)$ *is given by* $\mathrm{PMix}(S)$.

2.6 Anti-de Sitter space

We are primarily concerned with the anti-de Sitter space of signature $(2,1)$, which is given by the quasisphere $x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1$ inside $\mathbb{R}^{(2,2)}$ with the metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$. More precisely,

$$\widehat{\mathrm{AdS}}^3 = \{x \in \mathbb{R}^{(2,2)} : \langle x, x\rangle = -1\}.$$

As the manifold is pseudo-Riemannian, tangent vectors $v \in T\widehat{\mathrm{AdS}}^3$ come in one of the following three types:

- timelike if $\langle v, v\rangle < 0$,
- lightlike if $\langle v, v\rangle = 0$,
- spacelike if $\langle v, v\rangle > 0$.

The anti-de Sitter space AdS^3 is given by the projectivization of $\widehat{\mathrm{AdS}}^3$, its double cover. The isometry group of AdS^3 is $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

A smooth surface $S \hookrightarrow \mathrm{AdS}^3$ is said to be *spacelike* if the restriction to S of the metric on AdS^3 is a Riemannian metric. This is equivalent to the condition that every tangent vector $v \in TS$ is spacelike.

Consider the Levi-Civita connections on S and AdS^3 given by ∇^S and ∇, respectively. For a unit normal field N on S, the second fundamental form is given by

$$\nabla_{\widetilde{v}}\widetilde{w} = \nabla^S_v w + \mathrm{II}(v,w)N,$$

where v and w are vector fields on S, and $\widetilde{v}$ and $\widetilde{w}$ are vector fields extending v and w. The shape operator is the $(1,1)$ tensor given by $B(v) = \nabla_v N$. It satisfies the property $\mathrm{II}(v,w) = \langle B(v), w\rangle$. The maximal surfaces then are governed by the condition that $\operatorname{tr} B = 0$.

An AdS^3 manifold is a Lorentzian manifold locally isometric to AdS^3. Among these manifolds, we restrict our attention to those which are *globally hyperbolic maximal compact*, henceforth written as GHMC. These manifolds are defined by those satisfying the following three properties:

(1) They contain a closed orientable spacelike surface S.

(2) Each complete timelike geodesic intersects S precisely once.

(3) They are maximal with respect to isometric embeddings.

It follows that GHMC AdS^3 manifolds must be homeomorphic to $S \times \mathbb{R}$. Mess [28] showed that the genus of S must be at least two, and that GHMC structures are parametrized by two copies of Teichmüller space. Barbot, Béguin and Zeghib [2] showed that for each such GHMC manifold, there exists a unique embedded spacelike maximal surface Σ. In fact, there is a parametrization of all such GHMC manifolds by the unique embedded maximal surface it contains, along with its second fundamental form.

Theorem 2.6 (Krasnov and Schlenker [22]) *Let M be a GHMC AdS^3–manifold and let Σ be its unique embedded spacelike maximal surface. The second fundamental form of Σ is given by the real part of a holomorphic quadratic differential on the underlying complex structure of the maximal surface. Furthermore, there is a homeomorphism between the space of all GHMC AdS^3–structures and the cotangent bundle of Teichmüller space, which assigns to a GHMC AdS^3–structure the conformal class of its unique maximal surface and the holomorphic quadratic differential for which its real part is the second fundamental form.*

The induced metric of the maximal surface is given by $e^{2u}\sigma$, where σ is the hyperbolic metric and u satisfies the PDE

$$\Delta_\sigma u = e^{2u} - e^{-2u}|\Phi|^2 - 1.$$

But the solution to this PDE is $u = \frac{1}{2}\log \mathcal{H}$, for which the PDE becomes the usual Bochner equation. Here $\mathcal{H}$ is the holomorphic energy density arising from harmonic maps between closed hyperbolic surfaces. Hence, the induced metric of the maximal surface is given by $\mathcal{H}\sigma$. As a corollary of our main result, we will describe the limiting length spectrum of any sequence of induced metrics of the maximal surface.

3 Minimal lagrangians

A diffeomorphism $\phi\colon (S, g_1) \to (S, g_2)$ is *minimal* if its graph $\Sigma \subset (S \times S, g_1 \oplus g_2)$ with the induced metric is a minimal surface. Recall that Σ is a minimal surface if the inclusion $i\colon \Sigma \to (S \times S, g_1 \oplus g_2)$ is critical point of the area functional. Observe that if ϕ is minimal then so is ϕ^{-1}. If ω_1 and ω_2 denote the area forms of g_1 and g_2, respectively, and if in addition $\Sigma \subset (S \times S, \omega_1 - \omega_2)$ is a lagrangian submanifold, then we say that ϕ is a minimal lagrangian.

Theorem 3.1 (Schoen [39]) *If g_1 and g_2 are hyperbolic metrics on S, then there is a unique minimal lagrangian map $\phi\colon (S, g_1) \to (S, g_2)$ in the homotopy class of the identity.*

Let Σ denote the graph minimal surface with the induced metric. Then its inclusion into the product $i : \Sigma \to (S \times S, g_1 \oplus g_2)$ is a conformal harmonic map. A conformal map to a product space is a product of harmonic maps whose Hopf differentials sum to zero. Hence, for any pair of points in Teichmüller space, one may record the data of both the conformal structure of the minimal surface along with one of the Hopf differentials. The harmonic-maps parametrization of Teichmüller space which we record below ensures the map is bijective. Sampson proved injectivity and continuity of the map, and Wolf showed the map was surjective and admits a continuous inverse.

Theorem 3.2 (Sampson [38], Wolf [44]) *Let (S, σ) be a fixed hyperbolic surface. For any point in Teichmüller space $[(S, \rho)]$, select the representative (S, ρ) so that the identity map $\mathrm{id}\colon (S, \sigma) \to (S, \rho)$ is the unique harmonic map in its homotopy class, and denote its Hopf differential by $\Phi(\rho)$. Then this map*

$$\Phi\colon \mathcal{T}(S) \to H^0(X, K_X^2)$$

is a homeomorphism, where X is the complex structure associated to (S, σ).

Theorem 3.3 *There is a homeomorphism*

$$\Psi\colon \mathcal{T}(S) \times \mathcal{T}(S) \to \ Q_g, \quad (X_1, X_2) \mapsto ([\Sigma], \mathrm{Hopf}(u_1)),$$

which assigns to any pair of points X_1, X_2 in Teichmüller space the conformal structure of the unique graph minimal surface $\Sigma \subset X_1 \times X_2$ along with the Hopf differential $\mathrm{Hopf}(u_1)$ of the projection $u_1\colon \Sigma \to X_1$.

Proof The discussion above ensures the map Ψ is well-defined. As the construction of the minimal surface varies continuously with the choice of X_1 and X_2, it is clear the map is continuous. To see injectivity of Ψ, suppose that $\Psi(X_1, X_2) = \Psi(Y_1, Y_2) = (\Sigma, \Phi)$. Then the harmonic maps $u_1 : \Sigma \to X_1$ and $v_1 : \Sigma \to Y_1$ have the same Hopf differentials, so by the harmonic-maps parametrization, $X_1 = Y_1$. The same argument forces $X_2 = Y_2$. Surjectivity follows similarly, as to any choice of Riemann surface $\Sigma = (S, J)$ and holomorphic quadratic differential Φ, there exists a unique hyperbolic metric $X_1 = (S, g_1)$ such that the identity map $\mathrm{id}\colon \Sigma \to X_1$ is a harmonic map with Hopf differential Φ. Similarly, one can find an X_2 arising from the Hopf differential $-\Phi$.

Hence $\Psi(X_1, X_2) = (\Sigma, \Phi)$, which gives surjectivity. The inverse is clearly continuous as given the data of a Riemann surface and a holomorphic quadratic differential, the pair of hyperbolic metrics may be written explicitly and vary continuously, which suffices for the proof. □

4 Embedding of the induced metrics

In this section we study the induced metric on the graph minimal surfaces. Recall that given a pair (X_1, X_2) of hyperbolic surfaces, Theorem 3.1 produces a graph minimal surface Σ in the 4–manifold $(S \times S, g_1 \oplus g_2)$, where $X_i = (S, g_i)$. If $m\colon (S, g_1) \to (S, g_2)$ is the unique minimal map isotopic to the identity, then $\mathrm{id}\colon (S, g_1) \to (S, m^* g_2)$ is the unique minimal map isotopic to the identity, which in this case is the identity map. The graph Σ, then, is the diagonal in $S \times S$, and there is a canonical diffeomorphism from S to the diagonal in $S \times S$. The induced metric on Σ thus furnishes a metric g on S by the pullback of this diffeomorphism. Henceforth, when we say *induced metric*, we refer to this metric g on S, and will use Σ to denote (S, g). We consider these metrics up to pullback by a diffeomorphism isotopic to the identity, and call this subspace of metrics $\mathrm{Ind}(S)$ and endowing it with the compact–open topology. The remainder of the section is devoted towards studying geometric properties of the minimal surfaces and showing that $\mathrm{Ind}(S)$ can be embedded into $\mathrm{PCurr}(S)$.

Proposition 4.1 *Let $X_1 = (S, g_1)$, $X_2 = (S, g_2)$ and $\Psi(X_1, X_2) = (\Sigma, \Phi)$. Then the induced metric on the minimal surface Σ is given by $g_1 + m^* g_2$. Consequently, the induced metric is given by twice the $(1,1)$ part of a hyperbolic metric when expressed in conformal coordinates.*

Proof As in the discussion above, we may choose a suitable hyperbolic metric $X_2 = (S, g_2)$ in the equivalence class of $[X_2]$ to ensure that the unique minimal map isotopic to the identity is the identity map. Hence, the graph of the minimal map is the diagonal in $S \times S$, so that (after identifying the diagonal with S) the harmonic map from the minimal surface Σ to X_i is given by the identity map. The first result then follows by definition of the product metric. Notice that the hyperbolic metric g_1 may be written in conformal coordinates on Σ as $\Phi\, dz^2 + \sigma e_1\, dz\, d\overline{z} + \overline{\Phi}\, d\overline{z}^2$. As the minimal surface Σ is mapped conformally into the product $X_1 \times X_2$ of hyperbolic surfaces, then one obtains a pair $u_i\colon \Sigma \to X_i$ of harmonic maps, whose Hopf differentials, $\mathrm{Hopf}(u_1)$ and $\mathrm{Hopf}(u_2)$, sum to zero. Hence g_2 may be written in conformal coordinates

on Σ as $-\Phi\,dz^2+\sigma e_2\,dz\,d\bar{z}-\bar{\Phi}\,d\bar{z}^2$, with $|\Phi|=|-\Phi|$, so by a result of Sampson (Proposition 4.4) the energy densities e_1 and e_2 will coincide. As the induced metric is given by the sum, the induced metric has local expression $2\sigma e\,dz\,d\bar{z}$. □

Proposition 4.2 *The induced minimal surfaces have sectional curvature that is strictly negative.*

Proof For any point $p\in\Sigma$, it is clear that $K_p\le 0$, since Σ is a minimal surface in an NPC space, so we wish to show that $K_p\neq 0$. The proof is by contradiction. Let $\{e_1,e_2\}$ be an orthonormal basis of $N_p\Sigma$. Now consider the 2–plane spanned by eigenvectors X and Y of the second fundamental form II. One has $\mathrm{II}(X,Y)=\sum_{j=1}^{2}\mathrm{II}_j(X,Y)e_j$. The mean curvatures of the immersion are given by

$$\begin{aligned} H_1&=\mathrm{II}_1(X,X)+\mathrm{II}_1(Y,Y)=0,\\ H_2&=\mathrm{II}_2(X,X)+\mathrm{II}_2(Y,Y)=0. \end{aligned} \tag{4-1}$$

The Gauss equation tells us that at p,

$$\begin{aligned} 0&=\mathrm{Rm}(X,Y,Y,X)\\ &=\widetilde{\mathrm{Rm}}(X,Y,Y,X)-\langle\mathrm{II}(X,X),\mathrm{II}(Y,Y)\rangle+\langle\mathrm{II}(X,Y),\mathrm{II}(X,Y)\rangle\\ &=\widetilde{\mathrm{Rm}}(X,Y,Y,X)+\sum_{j=1}^{2}\mathrm{II}_j(X,X)\mathrm{II}_j(Y,Y)-\sum_{j=1}^{2}\mathrm{II}_j(X,Y)^2, \end{aligned} \tag{4-2}$$

and as $\mathbb{H}^2\times\mathbb{H}^2$ is NPC, from (4-1) and (4-2) it follows that $\mathrm{II}\equiv 0$ at p and that $\widetilde{\mathrm{Rm}}(X,Y,Y,X)=0$ at p. As $T(\mathbb{H}^2\times\mathbb{H}^2)\cong T\mathbb{H}^2\oplus T\mathbb{H}^2$, we may write $X=X_1\oplus X_2$ and $Y=Y_1\oplus Y_2$. A simple calculation shows

$$\begin{aligned} 0&=\widetilde{\mathrm{Rm}}(X,Y,Y,X)\\ &=\mathrm{Rm}_1(X_1,Y_1,Y_1,X_1)+\mathrm{Rm}_2(X_2,Y_2,Y_2,X_2)\\ &=\kappa(X_1,Y_1)(|X_1|^2|Y_1|^2-\langle X_1,Y_1\rangle^2)+\kappa(X_2,Y_2)(|X_2|^2|Y_2|^2-\langle X_2,Y_2\rangle^2)\\ &=-1\cdot(|X_1|^2|Y_1|^2-\langle X_1,Y_1\rangle^2)-1\cdot(|X_2|^2|Y_2|^2-\langle X_2,Y_2\rangle^2), \end{aligned}$$

which by Cauchy–Schwarz implies that X_1 and Y_1 (and also X_2 and Y_2) are linearly dependent. So the map u_{1_*} drops rank, a contradiction, as our surface was a graph. □

For a choice of complex coordinates $z=x+iy$ on the minimal surface Σ, then $\partial/\partial x$ and $\partial/\partial y$ form an orthogonal frame. Denote then

$$E_1=\Big|\frac{\partial}{\partial x}\Big|_\Sigma^{-1}\frac{\partial}{\partial x}\quad\text{and}\quad E_2=\Big|\frac{\partial}{\partial y}\Big|_\Sigma^{-1}\frac{\partial}{\partial y}.$$

Let J be the almost complex structure on the 4–manifold $X_1 \times X_2$. Then $J = J_1 \oplus J_2$, where J_i is the almost complex structure arising from $X_i = (S, g_i)$.

Proposition 4.3 *Let E_1, E_2 and J be as above. The second fundamental form is given by*

$$\begin{aligned}
\mathrm{II}(E_1, E_1) &= \frac{-\mathrm{Re}\,\Phi(\sigma e)_y - \sigma e(\mathrm{Im}\,\Phi)_x + \mathrm{Im}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\qquad + \frac{\mathrm{Im}\,\Phi(\sigma e)_y - \sigma e(\mathrm{Re}\,\Phi)_x + \mathrm{Re}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2, \\
\mathrm{II}(E_2, E_2) &= \frac{\mathrm{Re}\,\Phi(\sigma e)_y + \sigma e(\mathrm{Im}\,\Phi)_x - \mathrm{Im}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\qquad + \frac{-\mathrm{Im}\,\Phi(\sigma e)_y + \sigma e(\mathrm{Re}\,\Phi)_x - \mathrm{Re}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2, \\
\mathrm{II}(E_1, E_2) &= \frac{\mathrm{Im}\,\Phi(\sigma e)_y - \sigma e(\mathrm{Re}\,\Phi)_x + \mathrm{Re}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_1 \\
&\qquad + \frac{-\sigma e(\mathrm{Re}\,\Phi)_y + \mathrm{Re}\,\Phi(\sigma e)_y - \mathrm{Im}\,\Phi(\sigma e)_x}{\sigma e\sqrt{2\sigma e(\sigma^2 e^2 - 4|\Phi|^2)}} JE_2.
\end{aligned}$$

Proof As $\Sigma \subset X_1 \times X_2$ is a lagrangian submanifold, $\{E_1, E_2, JE_1, JE_2\}$ forms an orthonormal basis of $T(X_1 \times X_2) \cong TX_1 \oplus TX_2$ in this neighborhood. The second fundamental form then is given by

$$\mathrm{II}(X, Y) = \sum_{j=1}^{2} \widetilde{g}(\widetilde{\nabla}_X Y, JE_j) JE_j,$$

where $\widetilde{g} = g_1 \oplus g_2$ and $\widetilde{\nabla} = \nabla_1 \oplus \nabla_2$. We first calculate $\mathrm{II}(E_1, E_1)$. As the minimal surface metric is given by $2\sigma e\,|dz|^2 = 2\sigma e\,(dx^2 + dy^2)$, one has

$$2\sigma e(dx^2 + dy^2)\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = 2\sigma e = \Big\|\frac{\partial}{\partial x}\Big\|_{\Sigma}^2,$$

so that

$$E_1 = \frac{1}{\sqrt{2\sigma e}}\,\frac{\partial}{\partial x}.$$

Similarly, E_2 is given by

$$E_2 = \frac{1}{\sqrt{2\sigma e}}\,\frac{\partial}{\partial y}.$$

To calculate JE_1, we project E_1 to each of its factors and apply the almost complex structure on each of its factors; namely, we find the vector which has the same length and forms angle $\frac{\pi}{2}$ with the projected factor using the hyperbolic metric. This is the complex structure arising from the conformal class of the metric. To find $J_1E_1 = a(\partial/\partial x) + b(\partial/\partial y)$, for instance, we observe first that the hyperbolic metric on X_1 is given by

$$\begin{aligned}\rho_1 &= \Phi\, dz^2 + \sigma e\, dz\, d\overline{z} + \overline{\Phi}\, d\overline{z}^2 \\ &= (2\operatorname{Re}\Phi + \sigma e)\, dx^2 - 4\operatorname{Im}\Phi\, dx\, dy + (-2\operatorname{Re}\Phi + \sigma e)\, dy^2.\end{aligned}$$

Hence we want to solve for $a \neq 0$ and $b > 0$ which satisfy

$$g_1\Big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, E_1\Big) = 0, \tag{4-3}$$

$$g_1\Big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\Big) = g_1(E_1, E_1) = \frac{2\operatorname{Re}\Phi + \sigma e}{2\sigma e}. \tag{4-4}$$

Some basic algebra yields that

$$a = \frac{2\operatorname{Im}\Phi}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}} \quad \text{and} \quad b = \frac{2\operatorname{Re}\Phi + \sigma e}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}},$$

so that

$$J_1E_1 = \frac{2\operatorname{Im}\Phi}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial x} + \frac{2\operatorname{Re}\Phi + \sigma e}{\sqrt{(2\sigma e)((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial y}.$$

Now J_2E_1 is found similarly, and is given by

$$J_2E_1 = \frac{-2\operatorname{Im}\Phi}{\sqrt{2\sigma e((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial x} + \frac{-2\operatorname{Re}\Phi + \sigma e}{\sqrt{2\sigma e((\sigma e)^2 - 4|\Phi|^2)}}\frac{\partial}{\partial y}.$$

The tangent vector given by $\widetilde{\nabla}_{E_1}E_1$ splits as $\nabla^1_{E_1}E_1 \oplus \nabla^2_{E_1}E_1$. The Christoffel symbols for g_1 and g_2 can be readily calculated:

$$\begin{aligned}\nabla^1_{E1}E_1 &= \nabla^1_{(1/\sqrt{2\sigma e})\partial/\partial x}\frac{1}{\sqrt{2\sigma e}}\frac{\partial}{\partial x} \\ &= \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\nabla^1_{\partial/\partial x}\frac{\partial}{\partial x} + \left(\frac{1}{\sqrt{2\sigma e}}\right)_x\frac{\partial}{\partial x}\right) \\ &= \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\left({}^1\Gamma^1_{11}\frac{\partial}{\partial x} + {}^1\Gamma^2_{11}\frac{\partial}{\partial y}\right) + \left(\frac{1}{\sqrt{2\sigma e}}\right)_x\frac{\partial}{\partial x}\right) \\ &= \left(\frac{1}{2\sigma e}{}^1\Gamma^1_{11} + \frac{1}{\sqrt{2\sigma e}}\left(\frac{1}{\sqrt{2\sigma e}}\right)_x\right)\frac{\partial}{\partial x} + \frac{1}{2\sigma e}{}^2\Gamma^2_{11}\frac{\partial}{\partial y},\end{aligned}$$

where ${}^1\Gamma^1_{11}$ and ${}^1\Gamma^2_{11}$ are the usual Christoffel symbols, with the extra superscript denoting that these are the ones for the metric g_1. They are given explicitly by

$$
\begin{aligned}
{}^1\Gamma^1_{11} = \frac{1}{2}\Bigg(& \frac{-2\operatorname{Re}\Phi + \sigma e}{\sigma^2 e^2 - 4|\Phi|^2}(2\operatorname{Re}\Phi + \sigma e)_x \\
& + \frac{2\operatorname{Im}\Phi}{\sigma^2 e^2 - 4|\Phi|^2}\big((-4\operatorname{Im}\Phi_x) - (2\operatorname{Re}\Phi + \sigma e)_y\big)\Bigg), \\
{}^2\Gamma^1_{11} = \frac{1}{2}\Bigg(& \frac{2\operatorname{Im}\Phi}{\sigma^2 e^2 - 4|\Phi|^2}(2\operatorname{Re}\Phi + \sigma e)_x \\
& + \frac{2\operatorname{Re}\Phi + \sigma e}{\sigma^2 e^2 - 4|\Phi|^2}\big((-4\operatorname{Im}\Phi_x) - (2\operatorname{Re}\Phi + \sigma e)_y\big)\Bigg).
\end{aligned}
$$

Similarly, the same can be done for the metric g_2 and using the formula for II, one gets $\mathrm{II}(E_1, E_1)$. The same can be done for the rest. □

It would be curious to see under what conditions different points in Q_g would yield the same induced metric. One might hope that the space of induced metrics would be homeomorphic to Q_g, but the following result of Sampson shows this is not possible.

Proposition 4.4 (Sampson) *For a fixed closed hyperbolic surface $X = (S, \sigma)$, if Φ_1 and Φ_2 are two Hopf differentials on X arising from harmonic maps from X to closed hyperbolic surfaces of the same genus such that the norms $|\Phi_1|$ and $|\Phi_2|$ coincide, then the energy densities coincide, that is, $e_1 = e_2$.*

Hence, if we select two elements of Q_g, say (X, Φ_1) and (X, Φ_2), where $|\Phi_1| = |\Phi_2|$ but $\Phi_1 \neq \Phi_2$, then the corresponding energy densities are the same and hence the corresponding induced metrics are the same.

The following proposition is a converse to the result of Sampson and shows this is the only situation for which the corresponding induced metrics coincide.

Lemma 4.5 *On a fixed closed hyperbolic surface, we have $e_1 = e_2$ if and only if $|\Phi_1| = |\Phi_2|$.*

Proof That $|\Phi_1| = |\Phi_2|$ implies $e_1 = e_2$ is due to Sampson. Now suppose $e_1 = e_2$. Then $\mathcal{H}_1 + \mathcal{L}_1 = \mathcal{H}_2 + \mathcal{L}_2$, so the Bochner formula $\triangle \log \mathcal{H}_i = 2\mathcal{H}_i - 2\mathcal{L}_i - 2$ may be rewritten as $\triangle \log \mathcal{H}_i = 4\mathcal{H}_i - 2e_i - 2$. Subtracting the two equations for $i = 1, 2$ yields

$$\triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 4(\mathcal{H}_1 - \mathcal{H}_2).$$

Now $\mathcal{H}_i > 0$, so that the quotient $\mathcal{H}_1/\mathcal{H}_2$ attains its maximum on the surface; we claim this is 1, for if the maximum of $\mathcal{H}_1/\mathcal{H}_2$ were greater than 1, then at the maximum (which is also the maximum of $\log(\mathcal{H}_1/\mathcal{H}_2)$) we would have

$$0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 4(\mathcal{H}_1 - \mathcal{H}_2) = 4\mathcal{H}_2\left(\frac{\mathcal{H}_1}{\mathcal{H}_2} - 1\right) > 0,$$

a contradiction, so that $\mathcal{H}_1/\mathcal{H}_2 \leq 1$ and symmetrically $\mathcal{H}_2/\mathcal{H}_1 \leq 1$. Hence $\mathcal{H}_1 = \mathcal{H}_2$ and so $\mathcal{L}_1 = \mathcal{L}_2$, by the assumption on the energy densities. From the formula $|\Phi|^2/\sigma^2 = \mathcal{H}\mathcal{L}$, the conclusion follows. □

Corollary 4.6 *The space of induced metrics* $\mathrm{Ind}(S)$ *may be identified with* $Q_g/\sim$, *where* $(X, \Phi_1) \sim (Y, \Phi_2)$ *if* $X = Y$ *and* $|\Phi_1| = |\Phi_2|$.

We conclude this section by proving that the space $\mathrm{Ind}(S)$ can be embedded into the space of currents, and that the embedding remains injective after projectivization, thereby obtaining an embedding into projectivized currents.

Proposition 4.7 *The space* $\mathrm{Ind}(S)$ *can be realized as geodesic currents.*

Proof From Proposition 4.2, the induced metrics have strictly negative curvature, so by Otal [33], there is a well-defined embedding $\mathcal{C}\colon \mathrm{Ind}(S) \to \mathrm{Curr}(S)$ from the space of induced metrics on S to the space of geodesic currents, which sends $2\sigma e \mapsto L_{2\sigma e}$, so that if γ is a closed curve, then $l_{2\sigma e}([\gamma]) = i(L_{2\sigma e}, \gamma)$. □

The following lemma is a statement concerning energy densities and their failure to scale linearly.

Lemma 4.8 *On a fixed closed hyperbolic surface, if* $e_1 = ce_2$, *then* $c = 1$, *and hence* $|\Phi_1| = |\Phi_2|$.

Proof Without loss of generality, suppose $c \geq 1$, else we may re-index so that $c \geq 1$. Then $\mathcal{H}_1/\mathcal{H}_2 \leq c$, for if $\mathcal{H}_1/\mathcal{H}_2 > c$, we locate the maximum of $\mathcal{H}_1/\mathcal{H}_2$, and the Bochner formula at that point yields

$$\begin{aligned}
0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} &= 4(\mathcal{H}_1 - \mathcal{H}_2) - 2(e_1 - e_2) = 4(\mathcal{H}_1 - \mathcal{H}_2) - 2(ce_2 - e_2) \\
&= 4\mathcal{H}_2\left(\frac{\mathcal{H}_1}{\mathcal{H}_2} - 1\right) - 2e_2\,(c-1) \\
&> 4\mathcal{H}_2\,(c-1) - 2e_2\,(c-1) \\
&= (c-1)(4\mathcal{H}_2 - 2e_2) = (c-1)(2\mathcal{H}_2 - 2\mathcal{L}_2) = 2(c-1)\,\mathcal{J}_2 > 0,
\end{aligned}$$

a contradiction. Notice the upper bound is actually attained, for at a zero of $|\Phi_1|$, we have that $\mathcal{L}_1$ vanishes and so at such a zero we have the equation

$$\mathcal{H}_1 = c\mathcal{H}_2 + c\mathcal{L}_2,$$

and as we have $\mathcal{H}_1/\mathcal{H}_2 \leq c$, it follows that $\mathcal{L}_2$ must also vanish whenever $\mathcal{L}_1$ does. In fact, we can say more about the zeros of $\mathcal{L}_i$. The condition on the energy densities yields the equality

$$0 = (c\mathcal{H}_2 - \mathcal{H}_1) + (c\mathcal{L}_2 - \mathcal{L}_1),$$

and the bound on the quotient $\mathcal{H}_1/\mathcal{H}_2$ implies that the first term is nonnegative so the second term is nonpositive, that is, $c\mathcal{L}_2 - \mathcal{L}_1 \leq 0$ or $c \leq \mathcal{L}_1/\mathcal{L}_2$ or $\mathcal{L}_2/\mathcal{L}_1 \leq 1/c$, so that the order of the zeros of $\mathcal{L}_2$ is greater than or equal to the order of zeros of $\mathcal{L}_1$. As $|\Phi|^2/\sigma^2 = \mathcal{H}\mathcal{L}$ and $\mathcal{H} > 0$, both $\mathcal{L}_1$ and $\mathcal{L}_2$ have exactly $8g-8$ zeros counted with multiplicity, so the order of vanishing of $\mathcal{L}_1$ is the same as that of $\mathcal{L}_2$ at every point of the surface. Hence the quadratic differentials Φ_1 and Φ_2 differ by a multiplicative constant $k \in \mathbb{C}$, that is, $\Phi_1 = k\Phi_2$. At the zero of $|\Phi_2|$ (and so also a zero of $|\Phi_1|$), which is a maximum of the quotient $\mathcal{H}_1/\mathcal{H}_2$, the Bochner equation now reads

$$0 \geq \triangle \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 2\mathcal{H}_1 - \frac{2|\Phi_1|^2}{\sigma^2\mathcal{H}_1} - 2\mathcal{H}_2 + \frac{2|\Phi_2|^2}{\sigma^2\mathcal{H}_2} = 2(\mathcal{H}_1 - \mathcal{H}_2) = 2\mathcal{H}_2(c-1) \geq 0,$$

which implies $c = 1$, and by the previous lemma $|k| = 1$. □

Theorem 4.9 *The space of induced metrics* $\mathrm{Ind}(S)$ *embeds into* $\mathrm{PCurr}(S)$.

Proof Let $\pi : \mathrm{Curr}(S) \to \mathrm{PCurr}(S)$ be the natural projection map. It suffices to show that the map $\pi \circ \mathcal{C} : \mathrm{Ind}(S) \to \mathrm{PCurr}(S)$ is injective. If the images of two induced metrics under the map $\pi \circ \mathcal{C}$ coincide, then by Otal's theorem [33] on the marked length spectrum rigidity of negatively curved Riemannian metrics, we have

$$\sigma\, dz\, d\overline{z} = c\sigma' e'\, dz\, d\overline{z},$$

where $c \in \mathbb{R}_{>0}$. Then they will be in the same conformal class, so that $\sigma = \sigma'$. Then $e = ce'$, and by Lemma 4.8, $c = 1$. □

Remark As the induced metrics are not scalar multiples of each other, we make a slight modification by dividing the induced metrics by 2 to ensure these metrics are now precisely the $(1,1)$–part of a hyperbolic metric when written in conformal coordinates, rather than twice that.

5 Compactification of the induced metrics

In this section we identify the elements in the closure $\overline{\mathrm{Ind}(S)} \subset \mathrm{PCurr}(S)$. As the space of projectivized currents is compact, we obtain a compactification $\mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$ of the induced metrics from the embedding obtained in the previous section.

5.1 Flat metrics as limits

In a simple scenario where the conformal structure of the minimal surface remains fixed, we can describe the asymptotic behavior of the induced metric. We consider the simplest case, where $X_{1,n}$ (and consequently $X_{2,n}$) lie along a harmonic-maps ray, that is, the sequence of Hopf differentials of the projection map onto the first factor is given by $t_n\Phi$, where $\Phi \neq 0$ and $t_n \to \infty$.

Proposition 5.1 *Let $\sigma_n e_n$ be the induced metric where $\sigma_n = \sigma$ for all n, and the Hopf differentials of the harmonic maps $u_{1,n}\colon (S,\sigma) \to X_{1,n}$ are given by $t_n\Phi_0$, where Φ_0 is a unit-norm quadratic differential on (S,σ). Suppose $\mathcal{E}_n \to \infty$. Then everywhere away from the zeros of $|\Phi_0|$, one has*

$$\lim_{n\to\infty} \frac{\sigma_n e_n}{\mathcal{E}_n} = |\Phi_0|.$$

Proof By construction, the Hopf differential of the harmonic map from (S,σ) to $X_{1,n}$ is given by $t_n\Phi_0$, where Φ_0 is a unit-norm quadratic differential. In a neighborhood away from any zero of Φ_0, consider then the horizontal foliation of $\Phi_n = t_n\Phi_0$. By the estimates on the geodesic curvature of its image [45], a horizontal arc of the foliation in this neighborhood will be mapped close to a geodesic in $X_{1,n}$; we do not reproduce the techniques here, as we will do so later in a slightly modified setting. Using normal coordinates (x, y) for the target adapted to this geodesic and estimates on stretching [44], we have that

$$(x, y) \mapsto (2t_n^{1/2}x, 0) + o(e^{-ct}),$$

where the constant c only depends upon the domain Riemann surface and the distance from the zero of the quadratic differential. For the harmonic map from (S,σ) to $X_{2,n}$, its Hopf differential is given by $-t_n\Phi_0$, so that an arc of its horizontal foliation, which is an arc of the vertical foliation of $t_n\Phi_0$, gets mapped close to a geodesic, yielding

$$(x, y) \mapsto (0, 2t_n^{1/2}y) + o(e^{-ct}).$$

Hence, as a map from Σ to the 4–manifold $X_{1,n} \times X_{2,n}$ with the product metric, we have that the induced metric $\sigma_n e_n$ in this neighborhood has the form

$$(4t_n + o(e^{-ct}))\,dx^2 + 2o(e^{-ct})\,dx\,dy + (4t_n + o(e^{-ct}))\,dy^2.$$

Dividing by $4t_n$ and observing that for a high-energy harmonic map, the total energy is comparable to twice the L^1–norm of the quadratic differential (Proposition 5.8), and taking the limit, yields the conclusion. □

Proposition 5.2 *Suppose $\sigma_n e_n$ is a sequence of induced metrics such that $\sigma_n \to \sigma$ in $\mathcal{T}(S)$ and $\mathcal{E}_n \to \infty$. Then, after passing to a subsequence, there exists a sequence t_n and a unit-norm quadratic differential Φ_0 on $[\sigma]$ such that*

$$\lim_{n\to\infty} \frac{\sigma_n e_n}{t_n} \to |\Phi_0|.$$

Proof Let $t_n = \mathcal{E}_n$. Then the result follows from the compactness of unit-norm holomorphic quadratic differentials over a compact set in Teich(S), and the argument in the previous proposition. □

As the previous results only show C^0 convergence in any neighborhood away from a zero of the quadratic differential, it is not quite so obvious we have convergence in the sense of length spectrum. The following technical proposition shows we actually do have convergence when the metrics are regarded as projectivized geodesic currents. With the length spectrum embedding (as given in Theorem 4.9), we now have sequences of points whose limits are the flat structures in the space of geodesic currents.

Proposition 5.3 *Let $\sigma_n e_n$ and $\mathcal{E}_n$ be in the same setting as above. Then, as currents,*

$$\frac{L_{\sigma_n e_n}}{\mathcal{E}_n^{1/2}} \to L_{|\Phi_0|}.$$

Proof As the topology of geodesic currents is determined by the intersection number against closed curves, it suffices to show that given any closed, nonnull homotopic curve class $[\gamma]$ and $\epsilon > 0$, there is an $N([\gamma], \epsilon)$ such that for $n > N$, one has that $|i(L_{\sigma_n e_n/\mathcal{E}_n}, \gamma) - i(L_{|\Phi_0|}, \gamma)| < \epsilon$. We choose a representative γ of $[\gamma]$ to be a $|\Phi_0|$–geodesic with length $L = i(L_{|\Phi_0|}, \gamma)$ with some fixed orientation. As the estimate in Proposition 5.2 does not hold near a zero z_i of $|\Phi_0|$, the first step is to construct open balls V_i of radius ϵ in the $|\Phi_0|$–metric about each zero z_i of Φ_0 (choosing ϵ sufficiently

small) so that

(i) balls centered about distinct zeros do not intersect,

(ii) if the curve γ enters one of the neighborhoods V_i, then the curve γ must intersect the zero z_i before γ exits V_i,

(iii) $(1-\epsilon)\,C-(4g-2)\,\pi\,\epsilon>0$, where C is the systolic length of the surface $(S,|\Phi_0|)$.

As Φ_0 is holomorphic, the zeros are isolated, so we can easily ensure that (i) is satisfied. If the curve γ does not intersect z_i, then as γ is a closed curve, the distance from z_i to the curve γ in the $|\Phi_0|$–metric is bounded away from zero, guaranteeing condition (ii). Finally, condition (iii) follows as the systolic length C of $(S,|\Phi_0|)$ and the genus of surface are fixed.

As the complement of the union of the V_i forms a compact set, by Proposition 5.2 we can find an N so that for $n>N$ the metrics $\sigma_n e_n/\mathcal{E}_n$ and $|\Phi_0|$ differ by at most ϵ. Now each time γ enters V_i, say at p, then hits the zero z_i and exits V_i for the first time thereafter, say at q, we may replace that segment of γ with a segment running along the boundary of V_i connecting p and q. Notice that this does not change the homotopy class of γ. We make this alteration for each instance γ enters a V_i, and denote the new curve by γ'. Observe that each time we make such an alteration, the length of the curve (in the $|\Phi_0|$–metric) increases by at most $K_i\epsilon$, where K_i is a constant depending only upon the $|\Phi_0|$ and the order of the zero z_i. In fact $K_i\le(4g-2)\pi$. Hence the $|\Phi_0|$–length of γ' is bounded above by $L+\sum_{i=1}^{j}n_iK_i\epsilon$, where n_i is the number of times γ enters V_i. But as γ' now lies in the complement of the union of the V_i, by Proposition 5.2, the length of γ' in the $\sigma_n e_n/\mathcal{E}_n$–metric is at most $(1+\epsilon)\big(L+\sum_{i=1}^{j}n_iK_i\epsilon\big)$. But the length of γ' in the $\sigma_n e_n/\mathcal{E}_n$–metric must be at least the length of the geodesic in its homotopy class, which has length $L_n'=i\,(L_{\sigma_n e_n/\mathcal{E}_n},\gamma)$; hence

$$(1+\epsilon)\left(L+\sum_{i=1}^{j}n_iK_i\epsilon\right)\ge L_n'.$$

Distributing on the left-hand side and subtracting both sides by L yields

$$\sum_{i=1}^{j}n_iK_i\epsilon+\epsilon\left(L+\sum_{i=1}^{j}n_iK_i\epsilon\right)\ge L_n'-L:=i\,(L_{\sigma_n e_n/\mathcal{E}_n},\gamma)-i\,(L_{|\Phi_0|},\gamma).$$

Now if $L_n'-L\ge0$, we are done, for K_i is independent of ϵ and n_i is constant in ϵ.

So consider the case where $L'_n - L < 0$, that is, $L > L'_n$. Consider the $\sigma_n e_n/\mathcal{E}_n$–geodesic $\widetilde{\gamma}_n$ in the homotopy class of γ, and again we give $\widetilde{\gamma}_n$ an orientation. Naturally $\widetilde{\gamma}_n$ can enter and exit the V_i neighborhoods multiple times, but we remark that as the distance function on a NPC space from a convex set is itself convex, then each time the curve leaves V_i, it must pick up some topology before returning, that is, the part of the curve rel endpoints lying on the boundary of V_i is not homotopic to a segment along the boundary of V_i.

However, now if $\widetilde{\gamma}$ enters and exits V_i say a total of r times, we consider the pairs of entry and exit points ordered accordingly as $p_1, q_1, \dots, p_r, q_r$ using the chosen orientation. Now look at the segment of $\widetilde{\gamma}_n$ between p_s and p_{s+1}. If this is homotopic rel endpoints to a segment of the boundary of V_i, then we look at the segment of $\widetilde{\gamma}_n$ between p_s and p_{s+2} (using a cyclic ordering, so $r+1$ is identified with 1) and see if that segment is homotopic relative endpoints to a segment along the boundary of V_i. We repeat this until the segment of $\widetilde{\gamma}_n$ between p_s and $p_{s'}$ is not homotopic rel endpoints to the boundary of V_i. Then we repeat this process for p_s and p_{s-1} (again using a cyclic ordering) until we find the segment of $\widetilde{\gamma}_n$ between p_s and $p_{s''}$ which is not homotopic rel endpoints to the boundary of V_i. Then we replace the segment of $\widetilde{\gamma}$ between $p_{s''+1}$ and $p_{s'-1}$ with a segment along the boundary of V_i connecting these two points. We repeat this for each i, so that when the curve leaves V_i, it picks up some topology before reentering V_i. Altering $\widetilde{\gamma}_n$ in this fashion yields a curve $\widetilde{\gamma}'_n$ lying outside of all the V_i. Switching over to the $|\Phi_0|$–metric yields the inequality

$$(1+\epsilon)L'_n + \sum_{i=1}^{4g-4} m_i K_i \epsilon \geq L,$$

where m_i is the number of segments of the altered curve $\widetilde{\gamma}'_n$ lying on the boundary of V_i, and once again K_i is a constant depending solely on the order of the zero z_i. By the assumption that $L > L'_n$, we have actually that

$$L'_n + \epsilon L + \sum_{i=1}^{4g-4} m_i K_i \epsilon \geq L, \quad \text{so} \quad \epsilon L + \sum_{i=1}^{4g-4} m_i K_i \epsilon \geq L - L'_n.$$

It suffices to show that m_i can be bounded independently of n. This follows from an estimate on the systolic length of the metric $\sigma e_n/\mathcal{E}_n$. Let C' denote the systolic length among all homotopically nontrivial curves which avoid the V_i for the metric $|\Phi|$. Then $C' \geq C$. Then by Proposition 5.1, the systolic length among all homotopically nontrivial curves which avoid all the V_i for the metric $\sigma e_n/\mathcal{E}_n$ is at least $(1-\epsilon)C'$.

If K denotes the largest constant among the K_i, then one has that

$$\sum_{i=1}^{4g-4} m_i \leq \frac{L}{(1-\epsilon)C - K\epsilon},$$

for by construction we had m_i segments of $\widetilde{\gamma}_n'$ each of which is not homotopic rel endpoints to the boundary of V_i, so that if we connect the endpoints of the segment with a segment along the boundary of V_i, we add at most $K\epsilon$ to the length of the segment. But we now have a closed curve not homotopic to the boundary of any of the V_i, so the length of this closed curve is bigger than C'. This suffices for the proof. □

The resulting flat metrics arising from unit-norm holomorphic quadratic differentials are distinct as Riemannian metrics from the induced metrics as the quadratic differential metrics have zero curvature away from the zeros, whereas the induced metrics have negative curvature everywhere (Proposition 4.2). In fact, the flat metrics are distinct as geodesic currents, as work of Frazier [13] shows that the marked length spectrum distinguishes nonpositively curved Euclidean metrics from the negatively curved Riemannian metrics.

5.2 Measured laminations as limits

However, not all limits of induced metrics are given by flat metrics. One can also obtain measured laminations. This is most readily seen in the setting where one takes a hyperbolic metric and looks at the minimal lagrangian to itself. The induced metric of the minimal surface is then twice the hyperbolic metric. We thus have a copy of Teichmüller space inside the space of induced metrics inside the space of projectivized currents. From Bonahon [4], we know we must have projectivized measured laminations in our compactification of the induced metrics. However, there are more ways to obtain measured laminations than by degenerating only the induced metrics which are scalar multiples of hyperbolic metrics, as the following proposition shows.

Proposition 5.4 *Suppose that $L_{\sigma_n e_n}$ leaves all compact sets, but that the sequence $\mathcal{E}_n$ of total energies is bounded. Then, in* $\mathrm{PCurr}(S)$, *we have* $[L_{\sigma_n e_n}] \to [\lambda] \in \mathrm{PMF}(S)$. *Furthermore, if* $[L_{\sigma_n}] \to [\lambda']$ *in the Thurston compactification, then* $i(\lambda, \lambda') = 0$, *where* $\lambda \in [\lambda]$ *and* $\lambda' \in [\lambda']$.

Proof By the compactness of $\mathrm{PCurr}(S)$, any sequence $[L_{\sigma_n e_n}]$ subconverges to $[\lambda] \in \mathrm{PCurr}(S)$. Hence, there is a sequence of positive real numbers so that $t_n L_{\sigma_n e_n} \to \lambda \in \mathrm{Curr}(S)$. We claim $t_n \to 0$.

Consider a finite set of curves $\gamma_1, \gamma_2, \dots, \gamma_k$ which fill the surface S. Then the current $\gamma_1 + \gamma_2 + \cdots + \gamma_k$ is a *binding current*, that is to say, it has positive intersection number with any nonzero geodesic current.

As $L_{\sigma_n e_n}$ leaves all compact sets in $\mathrm{Curr}(S)$,

$$\lim_{n\to\infty} i(L_{\sigma_n e_n}, \gamma_1 + \cdots + \gamma_k) \to \infty,$$

so by continuity of the intersection form, one has

$$\lim_{n\to\infty} t_n i(L_{\sigma_n e_n}, \gamma_1 + \cdots + \gamma_k) = i(\lambda, \gamma_1 + \cdots + \gamma_k).$$

But the intersection number on the right-hand side is finite, hence $t_n \to 0$. From Proposition 2.1, one has $i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) = \frac{\pi}{2}\,\mathrm{Area}(S, \sigma_n e_n)$, which in this case is $\frac{\pi}{2}\mathcal{E}_n$. Then

$$i(\lambda, \lambda) = \lim_{n\to\infty} i(t_n L_{\sigma_n e_n}, t_n L_{\sigma_n e_n}) = \lim_{n\to\infty} t_n^2 \frac{\pi}{2}\mathcal{E}_n = 0,$$

where the last equality follows from the boundedness of total energy, hence $\lambda \in \mathrm{MF}(S)$. Now if $[L_{\sigma_n}] \to [\lambda']$, then there is a sequence $t_n' \to 0$ such that $t_n' L_{\sigma_n} \to \lambda'$. Then

$$i(\lambda, \lambda') = \lim_{n\to\infty} i(t_n L_{\sigma_n e_n}, t_n' L_{\sigma_n}) \le \lim_{n\to\infty} t_n t_n' i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) = \lim_{n\to\infty} t_n t_n' \mathcal{E}_n = 0,$$

where the inequality follows from $\sigma_n \le \sigma_n e_n$ as metrics, and the last equality by the boundedness of the sequence of total energy $\mathcal{E}_n$ along with the sequences t_n, t_n' tending towards zero. □

5.3 Mixed structures as limits

As some of the possible limits are the singular flat metrics arising from a holomorphic quadratic differential, the closure of the space of induced metrics on the minimal surface must include mixed structures, as these arise as limits of singular flat metrics. The main theorem asserts these are precisely all the possible limits of the degenerating minimal surfaces.

Theorem 5.5 *Let $\sigma_n e_n$ be a sequence of induced metrics such that σ_n leaves all compact sets in $\mathcal{T}(S)$ or $\mathcal{E}_n \to \infty$. Then there exists a sequence $t_n \to 0$ such that, up to a subsequence, we have $t_n L_{\sigma_n e_n} \to \eta = (S', q, \lambda) \in \mathrm{Mix}(S) \subset \mathrm{Curr}(S)$. Furthermore, given any $\eta \in \mathrm{Mix}(S)$, there exists a sequence of induced metrics $\sigma_n e_n$, and a sequence of constants $t_n \to 0$, such that $t_n L_{\sigma_n e_n} \to \eta$. Hence, the closure of the space of induced metrics in the space of projectivized currents is $\overline{\mathrm{Ind}(S)} = \mathrm{Ind}(S) \sqcup \mathrm{PMix}(S)$.*

The proof of the main theorem will follow from a series of intermediate results, and will be at the end of the section. The strategy is to show that if the sequence of currents coming from the induced metrics is not converging projectively to a measured lamination, then scaling the induced metrics to have total area 1 is enough to ensure convergence in length spectrum. To each normalized induced metric, we produce a quadratic differential metric in the same conformal class as the induced metric, which will serve as a lower bound. Convergence of the quadratic differential metric to a mixed structure will yield a decomposition of the surface into a flat part and a laminar part. On each flat part, we will prove the conformal factor between the normalized induced metric and the quadratic differential converges to 1 uniformly (away from finitely many points). An area argument will show the complement is laminar.

The following proposition allows us to analyze sequences of induced metrics which are not converging to projectivized measured laminations. If the sequence of induced metrics is not converging to a projectivized measured lamination, we may scale the current associated to the induced metric by the square root of its area (which is also the total energy). We remark that in the case where the limiting geodesic current is not a measured lamination, scaling the induced metrics by total energy of the associated harmonic map is strong enough to ensure length-spectrum convergence, yet delicate enough to ensure the limiting length spectrum is not identically zero. This should be compared to the situation in [45; 8], where one always scales the metric by the total energy.

Proposition 5.6 *Suppose the conformal class of the minimal surface leaves all compact sets in $\mathcal{T}(S)$, and the sequence of total energy is unbounded, that is, $\mathcal{E}_n \to \infty$. Then, up to a subsequence, there exists a sequence $c_n \to 0$ and a geodesic current μ such that $c_n L_{\sigma_n e_n} \to \mu$. If μ is a measured lamination, then $c_n = o(\mathcal{E}_n^{-1/2})$. If μ is not a measured lamination, then $c_n \asymp \mathcal{E}_n^{-1/2}$.*

Proof By Theorem 4.9, one has an embedding of the space of induced metrics into the space of projectivized geodesic currents, which is compact. Taking the closure implies the first result. If $[\mu]$ is the limiting projective geodesic current, then one can choose a fixed representative; call it μ.

If μ is a measured lamination, then dividing the current $L_{\sigma_n e_n}$ by $\mathcal{E}_n^{1/2}$ normalizes the current to have self-intersection number 1. Then, as the measured laminations have self-intersection 0, the second result follows.

Suppose then that μ is not a measured lamination. Then its self-intersection number is positive and finite. But

$$\begin{aligned} i(\mu,\mu) &= \lim_{n\to\infty} i(c_n L_{\sigma_n e_n}, c_n L_{\sigma_n e_n}) = \lim_{n\to\infty} c_n^2\, i(L_{\sigma_n e_n}, L_{\sigma_n e_n}) \\ &= \lim_{n\to\infty} c_n^2 \frac{\pi}{2} \operatorname{Area}(S, \sigma_n e_n) = \lim_{n\to\infty} c_n^2 \frac{\pi}{2} \int_S \sigma_n e_n \, dz_n \wedge d\overline{z}_n \\ &= \lim_{n\to\infty} c_n^2 \frac{\pi}{2} \mathcal{E}_n, \end{aligned}$$

so that $0 < \lim_{n\to\infty} c_n^2 \mathcal{E}_n < \infty$; that is, $c_n \asymp \mathcal{E}_n^{-1/2}$, as desired. □

With this normalization, the self-intersection of the current will be $\frac{\pi}{2}$; that is to say we have scaled the induced metric to have total area 1.

The following proposition shows the relation of the induced metric to the corresponding Hopf differential metric.

Proposition 5.7 *Away from the zeros of* Φ, *one has the identity*

$$\sigma e = |\Phi|\left(\frac{1}{|\nu|} + |\nu|\right).$$

Consequently,

$$\sigma_n e_n \geq 2|\Phi_n|.$$

Proof This result follows immediately by manipulation of the formulae involving $\mathcal{H}$ and $\mathcal{L}$. One has

$$\sigma^2 e^2 = \sigma^2(\mathcal{H}^2 + 2\mathcal{H}\mathcal{L} + \mathcal{L}^2) = \sigma^2 \mathcal{H}\mathcal{L}\left(\frac{\mathcal{H}}{\mathcal{L}} + 2 + \frac{\mathcal{L}}{\mathcal{H}}\right) = |\Phi|^2\left(\frac{1}{|\nu|^2} + 2 + |\nu|^2\right).$$

Taking a square root on both sides yields the result. □

For a given sequence $\sigma_n e_n$ we consider the associated smooth (away from the zeros of the quadratic differential) function $1/|\nu_n| + |\nu_n|$. This function is well-defined for each n by Lemma 4.5.

The following proposition due to Wolf allows us to pass freely between the L^1–norm of a Hopf differential and the total energy of the corresponding harmonic map. The original proof was for a fixed Riemann surface as the domain, but the argument holds when the domain is allowed to change. For the ease of the reader, we have included the adapted proof.

Proposition 5.8 [44, Lemma 3.2] *For any Riemann surface (S, J) and hyperbolic surface (S, σ), if* $\mathrm{id}\colon (S, J) \to (S, \sigma)$ *is a harmonic map with Hopf differential Φ and total energy $\mathcal{E}$, then*

$$\mathcal{E} + 2\pi\chi(s) \leq 2\|\Phi\| \leq \mathcal{E} - 2\pi\chi(S).$$

Proof As $\mathcal{H} - \mathcal{L} = \mathcal{J}$ and $\int \mathcal{J}\sigma \, dz \, d\overline{z} = -2\pi\chi$, we have

$$\int \mathcal{H}\sigma \, dz \, d\overline{z} + 2\pi\chi = \int \mathcal{L}\sigma \, dz \, d\overline{z} = \int \Phi\nu \, dz \, d\overline{z},$$

as the integrands agree. But, recalling that $|\nu| < 1$, we have

$$\int \Phi\nu \, dz \, d\overline{z} \leq \int |\Phi| \, dz \, d\overline{z} = \int \mathcal{H}|\nu|\sigma \, dz \, d\overline{z} \leq \int \mathcal{H}\sigma \, dz \, d\overline{z} = \int \mathcal{L}\sigma \, dz \, d\overline{z} - 2\pi\chi.$$

Summing up the first two and last two integrals respectively yields

$$\int e\sigma \, dz \, d\overline{z} + 2\pi\chi \leq 2\int |\Phi| \, dz \, d\overline{z} \leq \int e\,\sigma \, dz \, d\overline{z} - 2\pi\chi,$$

proving the proposition. □

Corollary 5.9 *If the sequence $\Phi_{0,n}$ of unit-norm quadratic differential metrics converges projectively to a measured lamination, then so does the associated sequence $L_{\sigma_n e_n}/\mathcal{E}_n^{1/2}$ of geodesic currents.*

Proof Suppose that $L_{|\Phi_{0,n}|} \to [\lambda]$ in the space of projectivized currents. Since $i(L_{|\Phi_{0,n}|}, L_{|\Phi_{0,n}|}) = \frac{\pi}{2}$, while $i(\lambda, \lambda) = 0$, there exists a sequence $t_n \to 0$ such that the length spectrum of $t_n|\Phi_{0,n}|$ converges to that of some $\lambda \in [\lambda]$. This is to say, there is a curve class $[\gamma]$ for which the length of its geodesic representative against the metric $|\Phi_{0,n}|$ is unbounded, so by Propositions 5.7 and 5.8, the sequence of lengths of the $[\gamma]$–geodesic against the metrics $\sigma_n e_n/\mathcal{E}_n$ is unbounded. Hence there is a sequence $s_n \to 0$ such that $s_n L_{\sigma_n e_n}/\mathcal{E}_n^{1/2}$ converges to a current μ. But as the self-intersection of $L_{\sigma_n e_n}/\mathcal{E}_n$ is exactly $\frac{\pi}{2}$, the intersection of μ with itself is zero, from which the result follows. □

The previous corollary allows us to exclude the case where the sequence of flat metrics tends towards a projectivized measured lamination, for in that case, we have that the sequence of induced metrics also tends towards a projectivized measured lamination. Hence, we need only consider the case where the sequence of flat metrics converges to a nontrivial mixed structure, say η. The data of η gives us a subsurface S' for which the

restriction of η is a flat metric arising from a quadratic differential. Here we consider S' up to isotopy.

The remainder of the section is devoted towards showing that if the sequence of unit-norm quadratic differential metrics converges to a mixed structure that is not entirely laminar, then so does the sequence of unit-area induced metrics. This will then complete the proof of Theorem 5.5.

We begin by recording the following useful bound due to Minsky, for the function $\mathcal{G} = \log(1/|\nu|)$.

Proposition 5.10 [29, Lemma 3.2] *Let $p \in S$ be a point with a neighborhood U such that U contains no zeros of Φ and in the $|\Phi|$–metric is a round disk of radius r centered at p. Then there is a bound*

$$\mathcal{G}(p) \leq \sinh^{-1}\left(\frac{|\chi(S)|}{r^2}\right).$$

Proof The PDE $\Delta\mathcal{G} = 2\mathcal{J} > 0$ shows that $\mathcal{G}$ is subharmonic in U. It suffices therefore to bound the average of $\mathcal{G}$ on U in the $|\Phi|$–metric. Some algebra yields

$$\sinh \mathcal{G} = \frac{1}{2}\frac{\sigma}{|\Phi|}\mathcal{J}.$$

Using the concavity of $\sinh^{-1}$ on the positive real axis, we obtain

$$\begin{aligned}
\mathcal{G}(p) &\leq |\Phi|\text{–}\mathrm{Avg}_U(\mathcal{G}) && \text{(by subharmonicity of } \mathcal{G}) \\
&= |\Phi|\text{–}\mathrm{Avg}_U\left(\sinh^{-1}\frac{1}{2}\frac{\sigma}{|\Phi|}\right) \\
&\leq \sinh^{-1}\left(|\Phi|\text{–}\mathrm{Avg}_U\left(\frac{1}{2}\frac{\sigma}{|\Phi|}\right)\right) && \text{(by concavity of } \sinh^{-1}) \\
&= \sinh^{-1}\left(\frac{1}{2\pi r^2}\int_U \frac{\sigma}{|\Phi|}\,\mathcal{J}\,dA(|\Phi|)\right) \\
&\leq \sinh^{-1}\left(\frac{|\chi(S)|}{r^2}\right) && \text{(by Gauss–Bonnet).}
\end{aligned}$$

□

As we are in the setting where the sequence $L_{\Phi_{0,n}}$ of currents coming from unit-area holomorphic quadratic differential metrics converges to a nontrivial mixed structure $\eta = (S', \Phi_\infty, \lambda)$, we have that the restriction of the metric $|\Phi_{0,n}|$ to S' converges to the metric $|\Phi_\infty|$. On this systole positive collection S' of subsurfaces, we have the following proposition.

Proposition 5.11 *Given $\epsilon, \epsilon' > 0$, there exists $N = N(\epsilon, \epsilon')$ such that for $n > N$,*

$$m_{|\Phi_{0,n}|}\left(\left\{p \in S : \left(\frac{1}{|\nu_n|} + |\nu_n|\right)(p) \geq 2 + \epsilon'\right\}\right) < \epsilon.$$

Consequently the limiting function $1/|\nu| + |\nu|$ is equal to 2 almost everywhere with respect to the $|\Phi_\infty|$–metric.

Proof By Proposition 5.7, one has the equality

$$\frac{\sigma_n e_n}{\mathcal{E}_n} = \frac{|\Phi_n|}{\mathcal{E}_n}\left(\frac{1}{|\nu_n|} + |\nu_n|\right) = \frac{\|\Phi_n\|}{\mathcal{E}_n}\frac{|\Phi_n|}{\|\Phi_n\|}\left(\frac{1}{|\nu_n|} + |\nu_n|\right).$$

Defining

$$\frac{1 - c_n}{2} := \frac{\|\Phi_n\|}{\mathcal{E}_n},$$

one has $c_n \to 0$ by virtue of Proposition 5.8. Observe that $c_n \geq 0$, as the function $1/|\nu_n| + |\nu_n| \geq 2$, the area of $|\Phi_{0,n}| = |\Phi_n|/\|\Phi_n\|$ is 1 and the area of the scaled metric $\sigma_n e_n/\mathcal{E}_n$ is also 1. If m_n then denotes the $|\Phi_{0,n}|$–measure of the set of points for which the function $1/|\nu_n| + |\nu_n|$ is at least $2 + \epsilon'$, then one has

$$\begin{aligned}\int_{\{p:(1/|\nu_n|+|\nu_n|)(p)\geq 2+\epsilon'\}} &\left(\frac{1}{|\nu_n|} + |\nu_n|\right)\left(\frac{1-c_n}{2}\right) dA(|\Phi_{0,n}|) \\ &+ \int_{\{p:(1/|\nu_n|+|\nu_n|)(p)<2+\epsilon'\}} \left(\frac{1}{|\nu_n|} + |\nu_n|\right)\left(\frac{1-c_n}{2}\right) dA(|\Phi_{0,n}|) \\ &= \int dA\left(\frac{\sigma_n e_n}{\mathcal{E}_n}\right) = 1.\end{aligned}$$

The integrand in the first integral is at least $(2 + \epsilon')(1 - c_n)/2$, whereas the second integrand is at least $2(1 - c_n)/2$. Multiplying these lower bounds with the measures of their respective sets yields

$$(2 + \epsilon')\left(\frac{1 - c_n}{2}\right)m_n + 2\left(\frac{1 - c_n}{2}\right)(1 - m_n) \leq 1.$$

Some basic algebraic manipulation leads from

$$m_n\left((2 + \epsilon')\left(\frac{1 - c_n}{2}\right) - 2\left(\frac{1 - c_n}{2}\right)\right) \leq c_n \quad \text{to} \quad m_n \leq \frac{2c_n}{(1 - c_n)(\epsilon')},$$

and as ϵ' is now fixed, one may find a sufficiently large N to guarantee $m_n < \epsilon$. As the metric $|\Phi_\infty|$ has finite total area, convergence in measure of the sequence of functions $1/|\nu_n| + |\nu_n|$ to the constant function 2 implies that up to a subsequence, one has convergence to the constant function 2 almost everywhere. □

Sets of measure zero can be rather problematic if we wish to say something about length of curves. The following proposition shows that we actually have convergence off the zeros and poles of $|\Phi_\infty|$.

Proposition 5.12 *Suppose $\mathcal{E}_n \to \infty$. Then, up to a subsequence,*

$$\frac{1}{|\nu_n|} + |\nu_n| \to 2$$

everywhere on S' except at the zeros and poles of $|\Phi_\infty|$.

Proof Observe that the function $1/|\nu_n| + |\nu_n|$ is not defined at the zeros of $|\Phi_n|$, but is well-defined everywhere else. Moreover, the auxiliary function $\mathcal{G} = \log(1/|\nu|)$ satisfies the partial differential equation

$$\Delta \log \frac{1}{|\nu_n|} = 2\mathcal{J}_n > 0,$$

so that the function $\mathcal{G}$ and hence $1/|\nu_n| + |\nu_n|$ never attains an interior maximum on the complement of the zeros. It follows that $1/|\nu_n| + |\nu_n|$ is only unbounded in a neighborhood of a zero of a corresponding quadratic differential Φ_n. The sequence of flat metrics $|\Phi_{0,n}|$ on S' converges geometrically to $|\Phi_\infty|$, and so the zeros of $|\Phi_n|$ on S' will converge to the zeros of $|\Phi_\infty|$. For any $\epsilon > 0$, consider balls of radius 3ϵ about each zero of $|\Phi_\infty|$, choosing ϵ sufficiently small that balls about distinct zeros do not intersect. Call this collection B. Then for large n, balls of radius ϵ in the $|\Phi_{0,n}|$–metric about the zeros of $|\Phi_n|$ will be contained in B. For each boundary component of S', which in the geometric limit is collapsed to a puncture, choose a geodesic curve with respect to the $|\Phi_\infty|$–metric, homotopic to the puncture and enclosing the puncture, of length $l_\epsilon > 3\epsilon$, so that the $|\Phi_\infty|$–distance of each point of the curve to the puncture is at least 3ϵ, possibly choosing a smaller ϵ until such a configuration is possible. This gives an annulus for each boundary component of S'. Call the collection of these annuli A.

For any point in the complement of both A and B, for large n, the injectivity radius with respect to the $|\Phi_{0,n}|$–metric is at least ϵ and the distance to any of the zeros is at least ϵ. Moreover, each point p in the region satisfies the property that any $q \in B_{\epsilon/2}(p)$ has injectivity radius at least $\frac{\epsilon}{2}$ and distance at least $\frac{\epsilon}{2}$ to any zero or the boundary of the cylindrical region. Hence, by Proposition 5.10, the value of $\log(1/|\nu_n|)$ is at most $M_{\epsilon/2}$, where the constant no longer depends on n, once n is chosen sufficiently large.

As the function $\log(1/|\nu_n|)$ is subharmonic, by the mean-value property, one has at any point p in this set that

$$\log \frac{1}{|\nu_n|}(p) \leq \int_{B_{\epsilon/2}(p)} \log \frac{1}{|\nu_n|}\, dA_{|\Phi_{0,n}|} \leq \big(|\Phi_{0,n}|\text{–Area}(B_{\epsilon/2}(p))\big)\epsilon' + M_{\epsilon/2}\epsilon''$$

for n large enough, so that $\log(1/|\nu_n|) < \epsilon'$ outside a set of measure at most ϵ'' by Proposition 5.11. As the choice of ϵ was arbitrary, the conclusion follows. □

This collection of propositions proves the following result:

Theorem 5.13 *Suppose $L_{|\Phi_{0,n}|}$ converges to a nontrivial mixed structure μ. Then the corresponding metrics $\sigma_n e_n/\mathcal{E}_n$ as $\mathcal{E}_n \to \infty$, restricted to S', converge in length spectrum to $|\Phi_\infty|$.*

Proof Defining A and B as in the previous proof, on the region $S' \setminus (A \cup B)$ Proposition 5.12 guarantees that we have uniform bounds on the sequence of functions $1/|\nu_n| + |\nu_n|$ whose limit was the constant function 2. Hence, by Arzelá–Ascoli, up to a subsequence, we have uniform convergence on this region. Hence, by the same argument as that of Proposition 5.3, the length spectrum of the scaled induced metric on this domain converges to the limiting length spectrum of the sequence $|\Phi_{0,n}|$, which is $|\Phi_\infty|$. □

Proof of Theorem 5.5 Recall that for any flat metric arising from a holomorphic quadratic differential, one can find a sequence of induced metrics so that the chosen flat metric is the limit in the space of geodesic currents (Proposition 5.3). Hence by Theorem 2.5, any mixed structure η can be obtained by a sequence $L_{\sigma_n e_n}$ of currents coming from the induced metrics. On the other hand, to any sequence of induced metrics leaving all compact sets, then either it converges projectively to a measured lamination or it does not. If it does not converge to a measured lamination, then the energy is unbounded and the corresponding sequence of normalized Hopf differential metrics must converge to a mixed structure μ which is not purely laminar. The previous theorem thus ensures there is a nonempty collection of incompressible subsurfaces, S', on which the limiting current η is a flat metric. But on the complement of S', the current η restricts to a measured lamination (as on this complement the areas of the metric tend to zero), so the proof of Theorem 5.5 is complete. □

5.4 Dimension of the boundary

We end this section with a remark about the compactification of the induced metrics. Recall the dimension of the space of induced metrics (being homeomorphic to $\mathcal{Q}_g/\mathbb{S}^1$) was $12g-13$. The dimension of the singular flat metrics can be readily seen to be $12g-14$. The actual mixed structures are stratified by the subsurfaces for which the mixed structure is a flat metric. A subsurface of lower complexity yields fewer free parameters in the choice of a flat structure, and the extra choices one gains for a measured lamination on the complementary subsurface is strictly less in our loss of choice for the flat structure. Hence the boundary of the compactification of the induced metrics via projectivized geodesic currents is of codimension one.

6 Analysis of the limits

In this section, we wish to relate the mixed structures with cores of $\mathbb{R}$–trees arising from measured laminations. To this end, we elucidate the relation between the mixed structure and the pair of projective measured laminations obtained from the pair of degenerating hyperbolic surfaces.

6.1 $\mathbb{R}$–trees

Here we recall some basic facts about $\mathbb{R}$–trees. An $\mathbb{R}$–tree T is a metric space for which any two points are connected by a unique topological arc, and such that the arc is a geodesic. Equivalently, if (X,d) is a metric space, for any pair of points $x, y \in X$, define the segment $[x,y]=\{z\in X : d(x,y)=d(x,z)+d(z,y)\}$. Then an $\mathbb{R}$–tree is a real nonempty metric space (T,d) satisfying:

(i) For all $x, y \in T$, the segment $[x,y]$ is isometric to a segment in $\mathbb{R}$.

(ii) The intersection of two segments with an endpoint in common is a segment.

(iii) The union of two segments of T whose intersection is a single point which is an endpoint of each is itself a segment.

A group Γ acts on T *by isometry* if there is a group homomorphism $\theta\colon \Gamma \to \mathrm{Isom}(T)$. The action is from the left. An action is said to be *small* if the stabilizer of each arc does not contain a free group of rank two. An action is said to be *minimal* if no proper subtree is invariant under Γ.

A particularly important class of $\mathbb{R}$–trees comes from the leaf space of a lift of a measured foliation on a closed surface to its universal cover. Any measured foliation $\mathcal{F}$ on a closed surface of genus $g \geq 2$ may be lifted to a $\pi_1 S$–equivariant measured foliation on its universal cover. The leaf space can be made into a metric space, by letting the distance be induced from the transverse measure. This is an $\mathbb{R}$–tree with a $\Gamma = \pi_1 S$ action by isometries. Naturally, not all $\mathbb{R}$–trees with a $\pi_1 S$ action arise from this construction. A theorem of Skora [41] shows that an $\mathbb{R}$–tree with a $\pi_1 S$ action comes from a measured foliation if and only if the action is small and minimal. Alternatively, one may start with a measured lamination (λ, μ) on S and lift it to a measured lamination $(\widetilde{\lambda}, \mu)$ on the universal cover. Then an $\mathbb{R}$–tree may be formed by taking the connected components of $\widetilde{S} \setminus \widetilde{\lambda}$ with edges between two vertices if the two components were adjacent (separated by a geodesic), and then metrically completing the distance induced by the transverse measure. The $\mathbb{R}$–tree comes equipped with a $\pi_1 S$ action, and is $\pi_1 S$–equivariantly isometric to the $\mathbb{R}$–tree constructed from the corresponding measured foliation. In what follows, we will deal exclusively with $\mathbb{R}$–trees with a $\pi_1 S$ action coming from the leaf space of the lift of a measured foliation. There is a rich theory of convergence of hyperbolic space to $\mathbb{R}$–trees in the literature from a number of different perspectives; see [3; 30; 31; 41; 46].

6.2 Convergence of metric spaces

In this section, we construct noncompact metric spaces admitting a $\pi_1 S$ action by isometries.

Definition 6.1 Let X and X' be two metric spaces and let $\epsilon > 0$. An *ϵ–approximation* between X and X' is a relation R in $X \times X'$ that is onto, and such that for every $x, y \in X$ and every $x', y' \in X'$, the conditions xRx' and yRy' imply $|d_X(x, y) - d_{X'}(x', y')| < \epsilon$.

Definition 6.2 Let X_n be a sequence of metric spaces, each admitting an isometric action by a group Γ, and let X_∞ be a supposed limiting metric space, also admitting an isometric action by the same group Γ. We say X_n converges to X_∞ in the sense of Gromov–Hausdorff if for every $\epsilon > 0$ and every finite set $A \subset \Gamma$, and for every compact subset $K \subset X_\infty$, then, for n sufficiently large, there is a compact set $K_n \subset X_n$ and an ϵ–approximation R_n which is A–equivariant between K_n and K in the following sense: for every $x \in K$, every $x_n, y_n \in K_n$, and every $\alpha \in A$, the conditions $\alpha x \in K$ and $x_n R_n x$ and $y_n R_n \alpha x$ imply $d(\alpha x_n, y_n) < \epsilon$.

We construct a sequence of noncompact metric spaces X_n with an isometric action by $\Gamma = \pi_1 S$, as follows. Take the induced metric $(S, \sigma_n e_n)$ and lift the metric to the universal cover $(\widetilde{S}, \widetilde{\sigma_n e_n})$. We will deal with the case where the induced metric converges in length spectrum to a mixed structure that is not entirely laminar — this is to ensure so that we can scale our metric spaces by total energy; for the case of a mixed structure that is entirely laminar, the same discussion holds after amending the sequence of constants. The sequence of noncompact metric spaces thus will be $X_n = (\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$. The following proposition is clear.

Proposition 6.1 *The manifold $X_n = (\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$ is a noncompact metric space admitting an isometric action by the group $\Gamma = \pi_1 S$.*

Proof As X_n itself is a noncompact Riemannian manifold with $\Gamma = \pi_1 S$ acting on it by isometries, the result follows immediately. □

Up to a subsequence, the metrics $(S, \sigma_n e_n/\mathcal{E}_n)$ will converge in length spectrum to a nontrivial mixed structure $\eta = (S', q, \lambda)$. We construct a noncompact metric space $X_\infty = X_\eta$ from the mixed structure η. Regard η as a geodesic current on $(\widetilde{S}, g)$. To any two distinct points $x, y \in \widetilde{S}$, one can form the geodesic arc α connecting the two points. Let c be the set of bi-infinite geodesics which intersect α transversely. Then the intersection number $i(\eta, \alpha)$ is given by the η–measure of c. This yields a pseudometric space coming from the geodesic current η. Notice that it is possible for the intersection number to be zero, for instance if the geodesic arc is disjoint from the support of the current, or if it forms no nontransverse intersection with the support of η. Taking the quotient by identifying points which are distance 0 from each other, and then taking the metric completion, yields a noncompact metric space X_∞. As $\Gamma = \pi_1 S$ acted on η equivariantly, Γ acts by isometries on X_∞. For a more detailed discussion about the construction of a metric space from the data of a geodesic current, see [5].

Remark In the setting where η is a measured foliation, the metric space X_η is a familiar one. It is an $\mathbb{R}$–tree dual to the foliation. The space is constructed by collapsing the leaves of the foliation with the distance on the tree inherited by intersection number and then completing; see [32]. The case where η is a nontrivial mixed structure follows the same spirit of this construction. The laminar part is treelike, formed on the universal cover by collapsing leaves of the supported lamination and then completing. On the flat part, the metric space is formed by the product of the trees dual to the vertical and horizontal lamination of a quadratic differential whose metric is the given flat metric.

The preceding discussion is summarized by the following proposition.

Proposition 6.2 *To any mixed structure η, the construction above gives a noncompact metric space X_η admitting an isometric action by $\Gamma = \pi_1 S$.*

Using the Gromov–Hausdorff topology, one has the following.

Theorem 6.3 *A subsequence of the metric spaces $(\widetilde{S}, \widetilde{\sigma_n e_n}/\mathcal{E}_n)$ converges in the sense of Gromov–Hausdorff to a noncompact metric space X_η coming from a mixed structure η acted upon by $\Gamma = \pi_1 S$.*

Before presenting the proof, we record one useful fact regarding convergence of maps. This follows from work of Korevaar and Schoen.

Theorem 6.4 (Korevaar and Schoen [21]; see also [8]) *Let $\widetilde{M}$ be the universal cover of a compact Riemannian manifold. Let $u_k : \widetilde{M} \to X_k$ be a sequence of maps such that*

(a) *each X_k is an NPC space, and*

(b) *the u_k have uniform modulus of continuity: for each x, there is a monotone function $\omega(x, \cdot)$ such that*

$$\lim_{R \to 0} \omega(x, R) = 0 \quad \text{and} \quad \max_{B(x,R)} d(u_k(x), u_k(y)) \leq \omega(x, R).$$

Then the pullback metrics d_{u_k} converge (possibly after passing to a subsequence) pointwise, locally uniformly to a pseudometric d_∞.

Proof of Theorem 6.3 Recall from Theorem 5.13 that on S' we have uniform convergence of the induced metric to the flat metric. For the complementary subsurface, recall that metric spaces were obtained as the induced metric on the minimal surface, so that the metric came from a pullback of a harmonic map. By Proposition 5.6, the scaled metric is the pullback metric of a harmonic map with energy at most 1. Hence, by Theorem 6.4 (see [21, Proposition 3.7] or [8, Theorem 2.2]), the metrics converge uniformly. As the lifts of the induced metrics admitted an $\pi_1 S$ action by isometries, so does the limit. □

6.3 Convergence of harmonic maps

Not only do the metric spaces converge in a suitable topology, the harmonic maps do as well. As we have shown in the preceding section that the domains converge in the sense

of Gromov–Hausdorff to a metric space arising from a mixed structure, and as shown in work of Wolf [46], one has that the lifts of a sequence of degenerating hyperbolic metrics, when properly scaled, subconverge in the sense of Gromov–Hausdorff to $\mathbb{R}$–trees dual to a particular measured lamination in the projective class of the associated point on the Thurston boundary. Hence we have both domain and target converging in the same topology to noncompact metric spaces with isometric actions by $\Gamma = \pi_1 S$. It is natural to expect some sort of convergence in the harmonic maps. In Wolf [46], the domain is a fixed Riemann surface, and the target is changing. Here, we have both domain and target changing (and converging). We begin by reviewing the necessary definitions.

Definition 6.3 Let X_n and X_∞ be metric spaces admitting an action of a group Γ and let (Y_n, d_n) and (Y_∞, d_∞) be metric spaces admitting an isometric action of Γ. Suppose $f_n\colon X_n \to Y_n$ and $f_\infty\colon X_\infty \to Y_\infty$ are equivariant maps. Then we say that f_n converges (uniformly) to f if

(i) both X_n and Y_n converge (uniformly) to X and Y respectively in the sense of Gromov, and

(ii) for every $\epsilon > 0$, there is an $N(\epsilon)$ such that for $n > N(\epsilon)$, the ϵ–approximations R_n, R_n' satisfy the condition that for every $x_n R_n x$, one has $f_n(x_n) R_n' f(x)$.

We will require a notion of harmonic map for maps between singular spaces. The following can be found in more detail in [10]. While the general theory of harmonic maps between Riemannian polyhedra is covered there, in what follows, we only deal with singular flat metrics and metric graphs.

Definition 6.4 Let X be an admissible Riemannian polyhedron and Y a metric space. Let $\phi \in L^2_{\mathrm{loc}}(X, Y)$. The approximate energy density is defined for $\epsilon > 0$ by

$$e_\epsilon(\phi)(x) = \int_{B_X(x,\epsilon)} \frac{d_Y^2(\phi(x), \phi(x'))}{\epsilon^{m+2}}\, d\mu_g(x').$$

Definition 6.5 The energy $E(\phi)$ of a map ϕ of class $L^2_{\mathrm{loc}}(X, Y)$ is

$$E(\phi) = \sup_{f \in C_c(X,[0,1])} \left(\limsup_{\epsilon \to 0} \int_X f e_\epsilon(\phi)\, d\mu_g \right).$$

Definition 6.6 A harmonic map $\phi\colon X \to Y$ is a continuous map of class $W^{1,2}_{\mathrm{loc}}(X,Y)$ which is bi-locally E–minimizing in the sense that X can be covered by relatively compact subdomains U for each of which there is an open set $V \supset \phi(U)$ in Y such that

$$E(\phi|_U) \leq E(\psi|_U)$$

for every continuous map $\psi \in W^{1,2}_{\mathrm{loc}}(X,Y)$ with $\psi(U) \subset V$ and $\psi = \phi$ in $X \setminus U$.

We obtain a classification of the flat parts of the mixed structure arising from the data of the limits of the sequences $X_{1,n}$ and $X_{2,n}$. Let S' be a connected subsurface for which the limiting mixed structure η is a flat metric. For each n, denote by S'_n the subsurface isotopic to S' such that the boundary components are geodesics with respect to the induced metric $\sigma_n e_n/\mathcal{E}_n$. Let $X'_{1,n}$ denote the restriction of the hyperbolic metric $X_{1,n}$ to the subsurface of S, in the same isotopy class of S', but which has geodesic boundary with respect to the hyperbolic metric. Then let $u'_{i,n}$ denote the restriction to S'_n of the harmonic map $u_{i,n}\colon (S, \sigma_n e_n) \to X_{i,n}$.

Theorem 6.5 *Consider a connected component of S'. The sequence of harmonic maps $u'_{1,n}\colon (S'_n, \sigma_n e_n/\mathcal{E}_n) \to X_{1,n}/2\mathcal{E}_n$ converges to a $\pi_1(S')$–equivariant harmonic map $u'\colon (S', |\Phi_\infty|) \to T_1$, where T_1 is the $\mathbb{R}$–tree dual to $\lambda_1 = \lim_{n\to\infty} X_{1,n}/2\mathcal{E}_n$. The Hopf differential is given by Φ_∞. Likewise, the same holds for λ_2 and $-\Phi_\infty$. Hence, the laminations are the vertical and horizontal foliations of Φ_∞.*

Proof We begin by showing that λ_1 is a well-defined measured lamination in the projective class of $[\lambda_1]$, which is the limit on the Thurston boundary of the sequence $X_{1,n}$. This will follow from standard estimates on stretching and geodesic curvature of an arc of the horizontal foliation which avoids the zeros. This will be an adaptation of the argument employed in [45], for the case where the domain conformal structure is fixed and the Hopf differentials lie along a ray.

We first show boundedness of the Jacobian. For any neighborhood U of the surface which avoids a zero of $\Phi_{0,n}$, one has the usual PDE

$$\Delta_{\sigma_n} \log \frac{1}{|\nu_n|^2} = 4\mathcal{J}_n > 0, \tag{6-1}$$

and, consequently,

$$\Delta_{\sigma_n} \|\Phi_n\| \log \frac{1}{|\nu_n|^2} = 4\|\Phi_n\|\mathcal{J}_n > 0. \tag{6-2}$$

Using the conformal invariance of harmonic maps, we replace the metric σ_n on the neighborhood U with a metric σ_n' in the same conformal class as σ_n, but one which is flat on U. Subharmonicity of the function $\|\Phi_n\| \log(1/|\nu_n|^2)$ yields

$$\|\Phi_n\| \log \frac{1}{|\nu_n|^2}(p) \leq \frac{1}{\pi R^2} \int_{B_R(p)} \|\Phi_n\| \log \frac{1}{|\nu_n|^2}\, dA(\sigma_n') \tag{6-3}$$

on a ball of σ_n' radius R contained in U. Some algebra yields

$$\mathcal{J}_n(p) \frac{\|\Phi_n\| \log |\nu_n|^{-2}(p)}{\mathcal{J}_n(p)} \leq \frac{1}{\pi R^2} \int_{B_R(p)} \frac{\mathcal{J}_n}{\mathcal{J}_n} \|\Phi_n\| \log \frac{1}{|\nu_n|^2}\, dA(\sigma_n'), \tag{6-4}$$

and hence

$$\mathcal{J}_n(p) \leq \frac{\mathcal{J}_n(p)}{\|\Phi_n\| \log |\nu_n|^{-2}(p)} \left(\sup_{q \in B_R(p)} \frac{\|\Phi_n\| \log |\nu_n|^{-2}(q)}{\mathcal{J}_n(q)} \right) \frac{1}{\pi R^2} \int_{B_R(p)} \mathcal{J}_n\, dA(\sigma_n'). \tag{6-5}$$

But one has that

$$\frac{\mathcal{J}_n}{\|\Phi_n\| \log |\nu_n|^{-2}} = \frac{|\Phi_{0,n}|}{\sigma_n |\nu_n|} \frac{(1-|\nu_n|^2)}{\log |\nu_n|^{-2}}, \tag{6-6}$$

so that in applying Proposition 5.12 to the expression (6-6), one obtains that (6-5) may be rewritten as

$$\mathcal{J}_n(p) \leq c_n \int_{B_R(p)} \mathcal{J}_n\, dA(\sigma_n'), \tag{6-7}$$

where c_n will depend on the metric $|\Phi_{0,n}|$, $|\nu_n|$, R and σ_n. But, on the neighborhood U, we know for sufficiently large n that $|\Phi_n| \to |\Phi_\infty|$, and $|\nu_n| \to 1$ and $\sigma_n \to \sigma_\infty$, where σ_∞ is the uniformizing metric of Φ_∞. Hence c_n remains bounded on U. But, finally,

$$\begin{aligned} \int_{B_R(p)} \mathcal{J}_n\, dA(\sigma_n') = \int_{B_R(p)} \frac{\sigma_n'}{\sigma_n} \mathcal{J}_n\, dA(\sigma_n) &\leq \sup_U \frac{\sigma_n'}{\sigma_n} \int_M \mathcal{J}_n\, dA(\sigma_n) \\ &\leq -2\pi \chi(S) c_n', \end{aligned} \tag{6-8}$$

where here c_n' will only depend upon the injectivity radius of the metric σ_n on the neighborhood U, which for large n will be close to the injectivity radius of σ_∞.

From (6-7), (6-8) and the PDE in (6-1), by elliptic regularity (see [14, Problem 4.8a]) one obtains that $|\nu_n| \to 1$ in $C^{1,\alpha}(U)$, where U does not contain a zero or pole of Φ_∞.

In the natural coordinates of the quadratic differential, the hyperbolic metric $g_{1,n}$ is given by $(\sigma_n e_n + 2\|\Phi_n\|)\, d\zeta_n^2 + (\sigma_n e_n - 2\|\Phi_n\|)\, d\eta_n^2$.

Recall that the geodesic curvature of an arc of the horizontal foliation of $\Phi_{0,n}$ in the natural coordinates for $\Phi_{0,n} = d\zeta_n^2 = d\xi_n^2 + d\eta_n^2$ is given by the equation

$$\kappa(\gamma)_{\eta=\text{constant}} = -\frac{1}{2g_{11,n}\sqrt{g_{22,n}}}\frac{\partial g_{11,n}}{\partial \eta_n}, \tag{6-9}$$

so that for γ an arc of the horizontal foliation of $\Phi_{0,n}$ avoiding the zeros, one has

$$\begin{aligned}\kappa(\gamma)_{\eta=\text{constant}} &= -\frac{1}{2(\sigma_n e_n + 2\|\Phi_n\|)(\sigma_n e_n - 2\|\Phi_n\|)^{1/2}}\frac{\partial}{\partial \eta_n}(\sigma_n e_n + 2\|\Phi_n\|) \\ &= -\frac{1}{2\mathcal{J}_n(\sigma_n e_n + 2\|\Phi_n\|)^{1/2}}\frac{\partial}{\partial \eta_n}\sigma_n e_n.\end{aligned} \tag{6-10}$$

But simple algebra yields that $\sigma_n e_n = \|\Phi_n\||\Phi_{0,n}|(|\nu_n|^{-1} + |\nu_n|)$, so that in the natural coordinates as $|\Phi_{0,n}| \equiv 1$, one actually has $\sigma_n e_n = \|\Phi_n\|(|\nu_n|^{-1} + |\nu_n|)$. Hence

$$\kappa(\gamma) = \tfrac{1}{2}\|\Phi_n\|(1-|\nu_n|^2)\mathcal{J}_n^{-1}|\nu_n|^{-2}(\sigma_n e_n + 2\|\Phi_n\|)^{-1/2}\frac{\partial}{\partial \eta_n}|\nu_n| \tag{6-11}$$

$$= \tfrac{1}{2}\|\Phi_n\|\mathcal{H}_n^{-1}|\nu_n|^{-2}(\sigma_n e_n + 2\|\Phi_n\|)^{-1/2}\frac{\partial}{\partial \eta_n}|\nu_n|, \tag{6-12}$$

as $\mathcal{J}_n = \mathcal{H}_n(1-|\nu_n|^2)$. As $\|\Phi_n\|\mathcal{H}_n^{-1} = |\nu_n|/|\Phi_{0,n}|$, rewriting (6-12) gives

$$\kappa(\gamma) = \frac{1}{2}\frac{1}{(|\Phi_{0,n}|\cdot|\nu_n|)(\sigma_n e_n + 2\|\Phi_n\|)^{1/2}}\cdot\frac{\partial}{\partial \eta_n}|\nu_n|, \tag{6-13}$$

and as $|\nu_n| \to 1$ in $C^{1,\alpha}(U)$, one obtains $\kappa_{g_{1,n}}(\gamma) = o(\|\Phi_n\|^{-1/2}) = o(\mathcal{E}_n^{-1/2})$.

Then to any arc γ of the horizontal foliation of Φ_n, one has that it is mapped close to its geodesic in the target hyperbolic surface. The following standard calculation on the stretching shows that by normalizing the target hyperbolic manifold by the total energy, the resulting length is given by the intersection number with the measured lamination λ_1. One has

$$\begin{aligned}l_{g_{1,n}}(\gamma) &= \int_\gamma \mathcal{H}_n^{1/2} + \mathcal{L}_n^{1/2}\, ds_{\sigma_n} = \int_\gamma \mathcal{H}_n^{1/2}(1+|\nu_n|)\, ds_{\sigma_n} \\ &= \int_\gamma \frac{\|\Phi_n\|^{1/2}|\Phi_0|^{1/2}}{|\nu_n|^{1/2}}(1+|\nu_n|)\,\frac{ds_{\sigma_n}}{\sigma_n^{1/2}} \\ &= \|\Phi_n\|^{1/2}\int_\gamma\left(1+\left(\frac{1}{|\nu_n|^{1/2}}-1\right)\right)(2-(1-|\nu_n|))\, ds_{|\Phi_0|} \\ &= 2\|\Phi_n\|^{1/2} l_{|\Phi_{0,n}|}(\gamma) + O\big(\|\Phi_n\|^{1/2}(1-|\nu_n|)\big),\end{aligned}$$

recalling that in order to obtain the metric σe, one has to divide both hyperbolic surfaces by twice the energy, which is approximately four times the L^1–norm of the Hopf differential for sufficiently large energy, independent of the Riemann surface structure; see Proposition 5.8. Meanwhile, a similar calculation shows that an arc of the vertical foliation of Φ_n, say α, has length in the target hyperbolic surface given by

$$\begin{aligned} l_{g_{1,n}}(\alpha) &= \int_\alpha \mathcal{H}_n^{1/2} - \mathcal{L}_n^{1/2}\, d_{s_{\sigma_n}} = \int_\alpha \mathcal{H}_n^{1/2}(1-|\nu_n|)\, ds_{\sigma_n} \\ &= \int_\alpha \frac{\|\Phi_n\|^{1/2}|\Phi_{0,n}|^{1/2}}{\sigma_n^{1/2}|\nu_n|^{1/2}}(1-|\nu_n|)\, ds_{\sigma_n} \\ &= \|\Phi_n\|^{1/2}\int_\alpha \frac{1-|\nu_n|}{|\nu_n|^{1/2}}\, ds_{|\Phi_{0,n}|} \\ &= o(\mathcal{E}_n^{1/2}). \end{aligned}$$

Noting that a horizontal arc of Φ_n is a vertical arc of $-\Phi_n$, one sees the λ_1 and λ_2 are the horizontal and vertical foliations of Φ_∞ (the geometric limit of Φ_n; see [27]), respectively.

To get our desired harmonic map from the flat subsurface to the two trees, notice that the above estimates show that a horizontal arc of $\Phi_{0,n}$ gets mapped close to a geodesic in the target space which is a hyperbolic surface scaled by the reciprocal of total energy. As the scaled induced metric limits to the flat metric $|\Phi_\infty|$, a horizontal arc of Φ_∞ will thus be mapped by an isometry to the tree T_1 and any vertical arc collapsed, so that the limiting map in the universal cover is given by a projection onto the leaf space of the vertical foliation of Φ_∞. The same argument holds for T_2. □

Proposition 6.6 *For any closed curve γ on the surface S, one has the inequalities*

$$l_{g_{1,n}}(\gamma) \le l_{\sigma_n e_n}(\gamma) \quad \text{and} \quad l_{g_{2,n}}(\gamma) \le l_{\sigma_n e_n}(\gamma).$$

Consequently, if $t_n L_{\sigma_n e_n} \to \eta$ as currents, then the length spectra of $\lim_{n\to\infty} t_n L_{g_{i,n}}$ are well-defined. If the limiting currents are denoted by λ_j, then

$$i(\lambda_j, \cdot) \le i(\eta, \cdot).$$

Proof As the minimal surface has induced metric of the form $g_{1,n} + g_{2,n}$, where the $g_{i,n}$ is a hyperbolic metric, both inequalities follow immediately. The final comment follows from choosing a closed curve $\gamma = \gamma_n$ to be a $\sigma_n e_n$–geodesic and using the inequality $l_{t_n^2 g_{i,n}}([\gamma]) \le l_{t_n^2 g_{i,n}}(\gamma)$. □

Combining Proposition 6.6 and Theorem 6.5, we obtain a necessary and sufficient condition on the pair of measured laminations λ_1 and λ_2 to determine a corresponding flat part on the mixed structure. Recall that a pair of measured laminations λ_1, λ_2 *fill* if for any measured lamination λ_3, one has $i(\lambda_1, \lambda_3) + i(\lambda_2, \lambda_3) > 0$.

Corollary 6.7 *Let $\lambda_i' = \lim_{n\to\infty} X_{i,n}'/2\mathcal{E}_n$ be a pair of nonzero measured laminations on a subsurface S'. Then the pair of laminations fill if and only if the restriction of the mixed structure η to S' is flat.*

Proof If η is flat on S', the preceding theorem shows the pair of laminations are dual and hence fill. If the pair of laminations do fill, then for any third lamination λ' one has by Proposition 6.6 that $i(\eta, \lambda') > 0$, so that it cannot be a lamination, and hence must be flat by definition of a mixed structure. □

Proposition 6.8 *On the subsurface $S'' = S \setminus S'$, the laminations λ_1 and λ_2 restrict to a pair of measured laminations which have no transverse intersection. If λ denotes the measured lamination part of the mixed structure, then $i(\lambda, \lambda_1) = i(\lambda, \lambda_2) = 0$.*

Proof By Proposition 6.6, since $i(\lambda, \lambda) = 0$, one has that $i(\lambda_1, \lambda) = i(\lambda_2, \lambda) = 0$. Using the inequality again yields $i(\lambda_1, \lambda_2) \le i(\lambda, \lambda_2) = 0$, from which the result follows. □

In the setting where both singular spaces are finite metric graphs, the resulting harmonic maps are *affine maps*. Each edge of the domain graph is mapped via the constant map, or mapped linearly to the target graph. The following result of Lebeau characterizes all such harmonic maps.

Theorem 6.9 (Lebeau [25]) *Given two finite metric graphs G and G', every continuous map between G and G' is homotopic to an affine map which minimizes the energy within its homotopy class. Furthermore, the map is unique up to parallel transport.*

Proposition 6.10 *Suppose $L_{\sigma_n e_n/C_n}$ converges to λ, where λ is a Jenkins–Strebel lamination (measured lamination supported on finitely many closed curves). Then the sequence of metric spaces $(S, \sigma_n e_n/C_n)$ converges geometrically to a finite metric graph.*

Proof This follows immediately from Theorem 6.4 (see also [21, Proposition 3.7]), as the induced metrics are the pullback metrics of a harmonic map from $\mathbb{H}^2$ to $\mathbb{H}^2 \times \mathbb{H}^2$,

which is NPC. The assumption on the modulus of continuity follows from the bound on the total energy of the maps u_n to the rescaled target, so that total energy is at most 1. Hence, the limiting metric space is the dual graph of λ, which is a finite metric graph. □

Theorem 6.11 *Let $C_n \to \infty$, so that $L_{\sigma_n e_n/C_n} \to \eta$, where η is a mixed structure with laminar part supported on a finite collection of simple closed curves. Suppose $L_{X_{i,n}/C_n} \to \lambda_i$, where λ_i are measured laminations also supported on a finite collection of simple closed curves. Then the sequence of harmonic maps $u_{i,n}\colon (S, \sigma_n e_n/C_n) \to X_{i,n}/C_n$ converges to a map $u_i\colon X_\eta \to T_i$, which is a union of harmonic maps.*

Proof Recall that X_η is the metric completion of the metric space obtained from the geodesic current η by creating a pseudometric space from the intersection number with η, and then identifying points with 0 distance.

As the case where η is flat has been previously handled in Theorem 6.5, we first construct a $\pi_1 S$–equivariant map between the laminar part of X_η and T_1 (here we will consider only the case where η is a Jenkins–Strebel lamination). The same construction will produce a similar map to T_2. Let D be a connected fundamental domain of the laminar region of X_η. Then D is a finite metric graph. We embed the graph D into the laminar region S'' of the minimal surface as follows: we map each vertex of D to its corresponding thick region on S''. The geometric convergence of the minimal surfaces to D from Proposition 6.10 allows us to determine which region of the minimal surface will converge to a given vertex. Once we have made our choice of where to send each vertex of D, if there is an edge e connecting two vertices of D, then we send the edge e to the geodesic arc connecting the two points on the minimal surface where we have mapped our two vertices. (The limiting map we will obtain later will not depend on this choice, as distances will converge uniformly.)

As we have convergence in length spectrum and as there are only finitely many edges, we can ensure that for large $n > N(\epsilon)$, the length of the image of each edge has changed by at most ϵ. We require that the embedding is proportional to arclength. Then there is a collection of continuous maps $\phi_n\colon D \to X_n$ with the property that given $\epsilon > 0$, there is an $N = N(\epsilon)$ so that ϕ_n is a $(1+\epsilon)$–quasi-isometry.

Likewise, as $\widetilde{X}_{1,n}/C_n$ converges geometrically to an $\mathbb{R}$–tree, a fundamental domain of $\widetilde{X}_{1,n}/C_n$ will converge geometrically to a finite graph G_1; see for instance [46]. Hence,

there is a collection of continuous maps $\psi_n\colon X_{1,n}/C_n \to G_1$ with the same property as ϕ_n.

Form the composition $g_n = \psi_n \circ u_{1,n} \circ \phi_n\colon D \to G_1$, where $u_{1,n}\colon (S, \sigma_n e_n/C_n) \to X_{1,n}/C_n$ is a harmonic map with total energy at most 1. We claim this sequence of maps g_n is uniformly bounded and equicontinuous. Uniform boundedness is clear as the target graph G_1 is a finite graph. To see it is equicontinuous, we note that, as ϕ_n and ψ_n were $(1+\epsilon)$–quasi-isometries, and since there is a uniform Lipschitz constant of the maps $u_{1,n}$, as the total energy of the maps are bounded by 1 (see [20, Theorem 2.4.6]), equicontinuity follows. Hence, by the Arzelà–Ascoli theorem, we have a subsequence g_k converging uniformly to a map $g\colon D \to G_1$.

We have that g is harmonic as a map between singular spaces, for we have uniform convergence of distances (see [21]) between the approximate metric spaces coming from our scaled induced metrics and the limiting $\mathbb{R}$–tree. Hence all the quantities in the definitions of the approximate energy density, and the energy, converge. As there is a unique energy minimizer (up to parallel transport, by Theorem 6.9) between the limiting spaces (which are finite graphs), the map g must be this unique energy minimizer. (If g were not the energy minimizer, it would have larger energy than the unique energy minimizer, by say δ. One could then construct a map between the approximate Riemannian manifolds, which would have energy lower than the harmonic maps $u_{1,n}$, contradicting the harmonicity of $u_{1,n}$.)

From Theorem 6.5, we obtained a limiting harmonic map u' on the flat part of X_η to the tree T_1, and now we have a limiting harmonic map g from the laminar part of X_η to the tree T_1. Taking the union yields the desired $u\colon X_\eta \to T_1$. The same argument holds for T_2. □

6.4 Cores of trees

Here we review some basics of cores of $\mathbb{R}$–trees. A more detailed overview of this material may be found in [18; 46].

For any $\mathbb{R}$–tree, a *direction* at a point $x \in T$ is a connected component of $T \setminus x$. A *quadrant* in $T_1 \times T_2$ is the product $\delta_1 \times \delta_2$ of two directions $\delta_1 \subset T_1$ and $\delta_2 \subset T_2$. We will say that the quadrant is based at $(x_1, x_2) \in T_1 \times T_2$, where x_i is the basepoint for the direction δ_i.

Let T_1, T_2 be a pair of trees with a common group action by Γ. Let $x = (x_1, x_2) \in T_1 \times T_2$ be a basepoint.

Definition 6.7 Consider a quadrant $Q = \delta_1 \times \delta_2 \subset T_1 \times T_2$. Then Q is said to be *heavy* if there exists a sequence $\gamma_k \in \Gamma$ such that

(i) $\gamma_k \cdot x \in Q$,

(ii) $d_i(\gamma_k \cdot x_i, x_i) \to \infty$ as $k \to \infty$ for $i = 1, 2$.

Otherwise we say Q is *light*.

We define the *core* of a product of trees to be the product $T_1 \times T_2$ with all light quadrants removed.

Definition 6.8 (Guirardel [18]) The *core* $\mathcal{C}$ of $T_1 \times T_2$ is the subset

$$\mathcal{C} = T_1 \times T_2 \setminus \Big[\bigcup_{Q \text{ light quadrant}} Q \Big].$$

Take a pair γ_1, γ_2 of simple closed geodesics on a hyperbolic surface and let T_1 and T_2 be the trees dual to the laminations. On the surface S, foliate by parallel curves a small open tubular neighborhood $A_i = \gamma_i \times (-\epsilon, \epsilon)$ of each of the curves. Define the map $p_i \colon \widetilde{S} \to T_i$ which maps the connected components of $\widetilde{S} \setminus \widetilde{A}_i$ to the corresponding vertex of T_i and each $\widetilde{A}_i$ to the corresponding edge of T_i. This construction extends to measured laminations, as the simple closed curves are dense in the space of measured laminations. The following proposition characterizes the core in terms of the map $p = (p_1, p_2)$.

Proposition 6.12 [18] *Let T_1 and T_2 be dual to a pair of measured laminations λ_1 and λ_2, respectively. Consider the map $p = (p_1, p_2) \colon \widetilde{S} \to T_1 \times T_2$, as defined above. Then $\mathcal{C}(T_1 \times T_2) = p(\widetilde{S})$.*

Proof The result will follow from the claim that any quadrant $Q = \delta_1 \times \delta_2$ in $T_1 \times T_2$ is light if and only if $p_1^{-1}(\delta_1) \cap p_2^{-1}(\delta_2) = \varnothing$. It is clear that if $p_1^{-1}(\delta_1) \cap p_2^{-1}(\delta_2) = \varnothing$, then Q is light, as for each point $x \in \widetilde{S}$, the orbit of $(p_1(x), p_2(x))$ does not intersect Q. Conversely, if $p_1^{-1}(\delta_1)$ intersects $p_2^{-1}(\delta_2)$, then take U_{δ_i} to be an open half-plane in $\widetilde{S}$ with bounded Hausdorff distance from $p_i^{-1}(\delta_i)$, where U_{δ_i} is bounded by a geodesic in $\widetilde{\lambda}_i$. As $p_1^{-1}(\delta_1)$ has nonempty intersection with $p_2^{-1}(\delta_2)$, so do U_{δ_1} and U_{δ_2}. Moreover, there exists an $h \in \pi_1 S$ whose axis γ intersects the pair of geodesics bounding U_{δ_1} and U_{δ_2}. Then h is hyperbolic in both T_1 and T_2, and h makes Q heavy. □

Remark This characterization of the core of two trees is particularly useful in our setting. When the trees come from a pair of dual measured laminations, the map $p = p_1 \times p_2$ has the same image as the map which sends $\mathbb{H}^2$ to the corresponding leaf space of each of the measured foliations. However, the map defined above is not quite projection to the leaf space when the two laminations are not dual. One has to refine the pair of laminations to λ_1^r, λ_2^r, so that each now has the same support as $\operatorname{supp}(\lambda_1) \cup \operatorname{supp}(\lambda_2)$. We describe the measure on λ_1^r by describing the case where λ_1 and λ_2 are a collection of simple closed curves. To λ_1 we add the weighted curves in λ_2 not in λ_1 and vice versa for λ_2^r. We now take the image of the projection of $\mathbb{H}^2$ onto the leaf space of the trees dual to λ_i^r, followed by projection of the tree T_i^r to T_i. This map now has image coinciding with the core. This slight modification is required to ensure the core is one-dimensional when the laminations have no transverse intersection; see [18, Theorem 6.1].

We present our next main result concerning the relation between the mixed structures we obtain as limits of the induced metrics and the limits of the corresponding graphs of the minimal lagrangians.

Theorem 6.13 *Suppose $C_n \to \infty$, so that $L_{\sigma_n e_n / C_n} \to \eta$ and $X_{1,n}/C_n \to T_1$ and $X_{2,n}/C_n \to T_2$. Then the metric space X_η is isometric to the core of the pair of trees (T_1, T_2). Consequently, the minimal lagrangians $\widetilde{\Sigma}_n/C_n \subset \mathbb{H}^2/C_n \times \mathbb{H}^2/C_n$ converge geometrically to the core $\mathcal{C}(T_1 \times T_2) \subset T_1 \times T_2$.*

Proof Define the auxiliary map $\Psi\colon \mathrm{P}(\mathrm{ML} \times \mathrm{ML}) \to \mathrm{PMix}(S)$ by

$$\Psi([\lambda_1, \lambda_2]) = \lim_{n \to \infty} [L_{\sigma_n e_n}],$$

where $\Sigma_n \subset X_{1,n} \times X_{2,n}$ is the minimal lagrangian with induced metric $2\sigma_n e_n$ and the $(X_{1,n}, X_{2,n})$ converge projectively to $[(\lambda_1, \lambda_2)]$. We claim the map is well-defined.

Choose $[(\lambda_1, \lambda_2)] \in \mathrm{P}(\mathrm{ML} \times \mathrm{ML})$ and a representative $(\lambda_1, \lambda_2) \in [(\lambda_1, \lambda_2)]$. Then if both $(X_{1,n}/k_n, X_{2,n}/k_n)$ and $(Y_{1,n}/d_n, Y_{2,n}/d_n)$ converge in length spectrum to (λ_1, λ_2), then for large enough n, we will have that $X_{1,n}/k_n$ will be close to $Y_{1,n}/d_n$ as negatively curved Riemannian surfaces (and likewise for $X_{2,n}/k_n$ and $Y_{2,n}/d_n$) by [33]. Hence the induced metrics on the respective pairs of minimal lagrangians will have close length spectra, so that Ψ is well-defined.

To see that Ψ is continuous, observe that the induced metric on the minimal surface varies continuously as a map defined on $\mathcal{T}(S) \times \mathcal{T}(S)$, and since the length spectrum of

the induced metric varies continuously as one takes a sequence of hyperbolic surfaces $(X_{1,n}, X_{2,n}) \to [(\lambda_1, \lambda_2)] \in \mathrm{P}(\mathrm{ML} \times \mathrm{ML})$, one finds the space of mixed structures varies continuously on $\mathrm{P}(\mathrm{ML} \times \mathrm{ML})$ by a diagonal argument.

But we now have a union of harmonic maps from X_η to $T_1 \times T_2$. From Theorem 6.5, the harmonic map on the flat part is given by projection to its vertical and horizontal lamination. By Theorem 6.11, the harmonic map from the laminar part is given by an affine map, when both trees come from Jenkins–Strebel differentials.

As the homotopy classes of the maps were given by the identity map, one sees that vertices on the domain graph are mapped to the vertices of the target graph — the thick regions of the minimal surface are necessarily mapped to the thick regions of the target scaled hyperbolic surface; for if a vertex were to be mapped away from vertices, the approximating thick region of the minimal surface would be mapped deep into a thin region of the target scaled hyperbolic surface, so that the thick region of the minimal surface would not have diameter going to zero, contradicting the geometric convergence of the thick region to a vertex. Hence by Theorem 6.9, the map is an affine map which maps vertices to the corresponding vertices.

But this yields the product metric for the core of the two trees; see Proposition 6.12 and the remark which follows. The equality of the metric space associated to the mixed structure and the core of the trees then holds for pairs of $\mathbb{R}$–trees dual to a pair of Jenkins–Strebel foliations, which is a dense set in $\mathrm{P}(\mathrm{ML} \times \mathrm{ML})$, and both quantities vary continuous for $\mathrm{P}(\mathrm{ML} \times \mathrm{ML})$, thus the theorem follows. □

7 Applications to maximal surfaces in AdS^3

In this section, we prove the required analogues of the minimal lagrangian setting to show a similar result for limits of maximal surfaces.

Proposition 7.1 *On a fixed hyperbolic surface (S, σ) one has $\mathcal{H}_1 = \mathcal{H}_2$ if and only if $e_1 = e_2$.*

Proof If $e_1 = e_2$, then $|\Phi_1| = |\Phi_2|$ by Lemma 4.5. From $|\Phi_1| = |\Phi_2|$, one ha,s by some basic algebra, $\mathcal{L}_2 = \mathcal{H}_1\mathcal{L}_1/\mathcal{H}_2$. From the Bochner formula, one has

$$\Delta \log \mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2,$$

$$\frac{1}{2}\Delta \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = (\mathcal{H}_1 - \mathcal{H}_2) - (\mathcal{L}_1 - \mathcal{L}_2) = (\mathcal{H}_1 - \mathcal{H}_2) - \mathcal{L}_1\left(1 - \frac{\mathcal{H}_1}{\mathcal{H}_2}\right).$$

At a point $p \in S$ for which the quotient $\mathcal{H}_1/\mathcal{H}_2$ achieves its maximum (which without loss of generality we may assume to be greater than 1, or else as before we may re-index), the left-hand side of the preceding calculation must be nonpositive, but the right-hand side is positive, hence $\mathcal{H}_1 = \mathcal{H}_2$ everywhere. □

Proposition 7.2 *On a fixed hyperbolic surface (S, σ), if $\mathcal{H}_1 = c\mathcal{H}_2$ then $c = 1$.*

Proof Without loss of generality, take $c > 1$ or we may re-index to ensure this is the case. Once again by the Bochner formula,

$$\Delta \log \frac{\mathcal{H}_1}{\mathcal{H}_2} = 2(\mathcal{H}_1 - \mathcal{H}_2) - 2(\mathcal{L}_1 - \mathcal{L}_2),$$
$$0 = \Delta \log c = 2(c\mathcal{H}_2 - \mathcal{H}_2) - 2(\mathcal{L}_1 - \mathcal{L}_2) = 2\mathcal{H}_2(c-1) - 2(\mathcal{L}_1 - \mathcal{L}_2).$$

Hence, everywhere one has

$$\mathcal{L}_1 - \mathcal{L}_2 = \mathcal{H}_2(c-1) > 0.$$

But $\mathcal{L}_1$ vanishes at the zeros of the quadratic differential Φ_1, a contradiction. Hence $c = 1$. □

Proposition 7.3 *Let $H = \int \mathcal{H}\, dA(\sigma)$. Then $\mathcal{E} = 2H + 4\pi\chi$. Consequently, if $\mathcal{E}_n \to \infty$, then $\lim_{n\to\infty} \mathcal{E}_n/H_n = 2$.*

Proof As $\mathcal{J} = \mathcal{H} - \mathcal{L}$ and $\int \mathcal{J}\sigma\, dz\, d\overline{z} = -2\pi\chi$, one has

$$\int \mathcal{H}\sigma\, dz\, d\overline{z} + 2\pi\chi = \int \mathcal{L}\sigma\, dz\, d\overline{z}.$$

Adding the terms yields

$$\mathcal{E} = \int (\mathcal{H} + \mathcal{L})\sigma\, dz\, d\overline{z} = 2\int \mathcal{H}\sigma\, dz\, d\overline{z} + 4\pi\chi = 2H + 4\pi\chi.$$ □

Recall from Section 2.6 the existence and uniqueness of a spacelike, embedded maximal surface in any GHMC AdS^3 manifold.

Proposition 7.4 [22, Lemma 3.6] *The induced metric on the maximal surface is of the form $\mathcal{H}\sigma$.*

Proposition 7.5 *The induced metric on the maximal surface has strictly negative curvature.*

Proof The formula for curvature is given by

$$K_{\mathcal{H}\sigma} = -\frac{1}{2\mathcal{H}\sigma}\Delta \log \mathcal{H}\sigma = -\frac{1}{2}\frac{1}{\mathcal{H}}\left(\frac{\Delta \log \mathcal{H}}{\sigma} + \frac{\Delta \log \sigma}{\sigma}\right) = \frac{-\mathcal{J}}{\mathcal{H}},$$

where the last step comes from the Bochner equation and the curvature of the hyperbolic metric. □

Theorem 7.6 *There exists an embedding of the space of maximal surfaces into the space of projectivized currents.*

Proof As the induced metrics on the maximal surfaces are negatively curved, they may be realized as geodesic currents. By Proposition 7.2, the projectivization remains injective. □

Theorem 7.7 *The closure of the space of induced metrics on the maximal surfaces is given by the space of flat metrics arising from unit-norm holomorphic quadratic differentials and projectivized mixed structures.*

Proof To any induced metric $\mathcal{H}\sigma$ on the maximal surface, there is a unique singular quadratic differential metric $|\Phi|$ associated to it. Some algebra shows that

$$\mathcal{H}\sigma = \frac{|\Phi|}{|\nu|} \geq |\Phi|,$$

which for high energy, Proposition 7.3 tells us H approximates the L^1–norm of the quadratic differential, so that if the sequence of unit-norm quadratic differentials converges to measured lamination, then so does the projective current associated to the induced metric on the maximal surface. (If the energy is bounded, an adaptation of the proof of Proposition 5.4 shows that the limit of induced metrics will be a measured lamination.) Hence, we assume the sequence of unit-norm quadratic differential metrics converges to a mixed structure. On the flat part of the mixed structure, from the proof of Proposition 5.12 we know that up to a subsequence the Beltrami differentials converge uniformly to 1 outside of a small region about the zeros of the differential and a cylindrical neighborhood of the boundary curves. But then we know that on this subsurface the maximal surface metric will converge to $|\Phi_\infty|$ in terms of its length spectrum. As the total area of the mixed structure is 1 and we have normalized the maximal surface metric by the total holomorphic energy, on the complement, the area of the metric tends to 0, so that the restriction of the limiting current is a measured lamination. □

We observe there is a rather interesting trichotomy at play here. For high energy, on the subsurface S', if the quadratic differentials converge to $|\Phi_\infty|$ then so do the associated sequence of minimal surface metrics and the sequence of maximal surface metrics.

8 Compactification of maximal representations to PSL(2 , ℝ) × PSL(2 , ℝ)

In this final section, we provide an application of our work to compactifying the maximal component of the character variety $\chi(\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R}))$. The theory of maximal representations is defined for general Hermitian Lie groups G and is considerably more straightforward to define in our specific setting of $G=\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$. Nevertheless, we will define a maximal representation in the general setting before providing a straightforward characterization in our setting.

Let G be a *Hermitian* Lie group, that is, a noncompact simple Lie group whose symmetric space G/K is a Kähler manifold. Equivalently, there is a G–invariant two-form ω on G/K. Let S be a closed, orientable, smooth surface of genus $g\geq 2$. Then given a representation $\rho\colon \pi_1 S\to G$, there is a ρ–equivariant map $\widetilde{f}\colon \widetilde{S}\to G/K$ defined by taking any smooth section of the flat bundle $E_\rho=\widetilde{S}\times_\rho G/K\to S$. Define the *Toledo invariant* to be

$$T(\rho):=\frac{1}{2\pi}\int_S \widetilde{f}^*\omega.$$

The Toledo invariant will be well-defined for each such representation as the number obtained will not depend on the choice of section chosen above; a different section would yield another map differing by a ρ–equivariant homotopy, giving the same number. A well-known *Milnor–Wood* type inequality holds for the Toledo invariant,

$$|T(\rho)|\leq|\chi(s)|\cdot\mathrm{rank}(G/K).$$

Representations whose Toledo invariant attains the upper bound are known as *maximal representations*. We now restrict our attention specifically to the group $G=\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$, whose associated symmetric space is $\mathbb{H}^2\times\mathbb{H}^2$.

For each representation to the group $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$, one obtains a pair of representations to the group $\mathrm{PSL}(2,\mathbb{R})$. By work of Goldman [17], the Euler number of representations to $\mathrm{PSL}(2,\mathbb{R})$ characterizes the connected components of the representation variety. The maximal representations are precisely those whose projections live in the Hitchin component of $\mathrm{PSL}(2,\mathbb{R})$ representations, that is, those

representations that are both discrete and faithful. Hence, such a representation yields a pair of points in Teichmüller space and an associated minimal surface. We may parametrize such representations by the equivariant minimal lagrangian in $\mathbb{H}^2 \times \mathbb{H}^2$ from Theorem 3.1. As a final consequence of our study of these minimal lagrangians, we obtain a compactification of the maximal component of surface group representations to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Theorem 8.1 *Let S be a closed surface of genus $g > 1$. The space of maximal representations of $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ embeds into the space of $\pi_1 S$–equivariant minimal lagrangians in $\mathbb{H}^2 \times \mathbb{H}^2$. The scaled Gromov–Hausdorff limits of the minimal lagrangians are given by cores in the product $T_1 \times T_2$ of trees, where T_1 and T_2 are a pair of $\mathbb{R}$–trees coming from a projective pair of measured foliations.*

Proof For any maximal representation $\rho = (\rho_1, \rho_2)$, we may look at the two closed hyperbolic surfaces given by $X_1 = \mathbb{H}^2 \setminus \rho_1$ and $X_2 = \mathbb{H}^2 \setminus \rho_2$. This gives a clear homeomorphism between the maximal component and two copies of Teichmüller space and thus, by Theorem 3.3, to the bundle of holomorphic quadratic differentials over Teichmüller space. By Theorem 3.1, we obtain a minimal lagrangian between X_1 and X_2 which respects the marking. Taking the lift gives a $\pi_1 S$–equivariant minimal lagrangian in $\mathbb{H}^2 \times \mathbb{H}^2$. As distinct representations have distinct minimal lagrangians (distinguished by both the metric via Corollary 4.6 and the second fundamental form via Proposition 4.3), we have our desired embedding.

If $\rho_n = (\rho_{1,n}, \rho_{2,n})$ is a sequence of representations leaving all compact sets, then there exists a sequence of constants $C_n \to \infty$ such that passing to a subsequence one has $\widetilde{X}_{1,n}/C_n \to T_1$ and $\widetilde{X}_{2,n}/C_n \to T_2$, where T_1 and T_2 are both $\mathbb{R}$–trees, and at most one of the trees is just a single vertex. By Theorem 6.13, the Gromov–Hausdorff limit of the minimal lagrangians scaled by C_n converges to the core of the product $T_1 \times T_2$, which suffices for the proof. □

References

[1] **S I Al'ber**, *Higher-dimensional problems in global variational calculus*, Dokl. Akad. Nauk SSSR 156 (1964) 727–730 MR Zbl In Russian; translated in Sov. Math. Dokl. 5 (1964) 700–704

[2] **T Barbot**, **F Béguin**, **A Zeghib**, *Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on* AdS_3, Geom. Dedicata 126 (2007) 71–129 MR Zbl

[3] **M Bestvina**, *Degenerations of the hyperbolic space*, Duke Math. J. 56 (1988) 143–161 MR Zbl

[4] **F Bonahon**, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988) 139–162 MR Zbl

[5] **M Burger**, **A Iozzi**, **A Parreau**, **M B Pozzetti**, *Currents, systoles, and compactifications of character varieties*, Proc. Lond. Math. Soc. 123 (2021) 565–596 MR Zbl

[6] **M Burger**, **A Iozzi**, **A Parreau**, **M B Pozzetti**, *Weyl chamber length compactification of the* $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *maximal character variety*, preprint (2021) arXiv 2112.13624

[7] **A J Casson**, **S A Bleiler**, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts 9, Cambridge Univ. Press (1988) MR Zbl

[8] **G Daskalopoulos**, **S Dostoglou**, **R Wentworth**, *Character varieties and harmonic maps to* $\mathbb{R}$*–trees*, Math. Res. Lett. 5 (1998) 523–533 MR Zbl

[9] **M Duchin**, **C J Leininger**, **K Rafi**, *Length spectra and degeneration of flat metrics*, Invent. Math. 182 (2010) 231–277 MR Zbl

[10] **J Eells**, **B Fuglede**, *Harmonic maps between Riemannian polyhedra*, Cambridge Tracts in Mathematics 142, Cambridge Univ. Press (2001) MR Zbl

[11] **J Eells, Jr**, **J H Sampson**, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964) 109–160 MR Zbl

[12] **A Fathi**, **F Laudenbach**, **V Poénaru**, *Thurston's work on surfaces*, Mathematical Notes 48, Princeton Univ. Press (2012) MR Zbl

[13] **J Frazier**, *Length spectral rigidity of non-positively curved surfaces*, PhD thesis, University of Maryland, College Park (2012) MR Available at `https://www.proquest.com/docview/1033322768`

[14] **D Gilbarg**, **N S Trudinger**, *Elliptic partial differential equations of second order*, Grundl. Math. Wissen. 224, Springer (1977) MR Zbl

[15] **O Glorieux**, *Counting closed geodesics in globally hyperbolic maximal compact AdS 3–manifolds*, Geom. Dedicata 188 (2017) 63–101 MR Zbl

[16] **O Glorieux**, *The embedding of the space of negatively curved surfaces in geodesic currents*, preprint (2019) arXiv 1904.02558

[17] **W M Goldman**, *Discontinuous groups and the Euler class*, PhD thesis, University of California, Berkeley (1980) MR Available at `https://www.proquest.com/docview/302998874`

[18] **V Guirardel**, *Cœur et nombre d'intersection pour les actions de groupes sur les arbres*, Ann. Sci. École Norm. Sup. 38 (2005) 847–888 MR Zbl

[19] **P Hartman**, *On homotopic harmonic maps*, Canadian J. Math. 19 (1967) 673–687 MR Zbl

[20] **N J Korevaar**, **R M Schoen**, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. 1 (1993) 561–659 MR Zbl

[21] **N J Korevaar**, **R M Schoen**, *Global existence theorems for harmonic maps to non-locally compact spaces*, Comm. Anal. Geom. 5 (1997) 333–387 MR Zbl

[22] **K Krasnov**, **J-M Schlenker**, *Minimal surfaces and particles in 3–manifolds*, Geom. Dedicata 126 (2007) 187–254 MR Zbl

[23] **F Labourie**, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, Ann. Sci. Éc. Norm. Supér. 41 (2008) 437–469 MR Zbl

[24] **F Labourie**, *Cyclic surfaces and Hitchin components in rank 2*, Ann. of Math. 185 (2017) 1–58 MR Zbl

[25] **E Lebeau**, *Applications harmoniques entre graphes finis et un théorème de superrigidité*, Ann. Inst. Fourier (Grenoble) 46 (1996) 1183–1203 MR Zbl

[26] **G Martone**, **C Ouyang**, **A Tamburelli**, *A closed ball compactification of a maximal component via cores of trees*, preprint (2021) arXiv 2110.06106

[27] **C McMullen**, *Amenability, Poincaré series and quasiconformal maps*, Invent. Math. 97 (1989) 95–127 MR Zbl

[28] **G Mess**, *Lorentz spacetimes of constant curvature*, Geom. Dedicata 126 (2007) 3–45 MR Zbl

[29] **Y N Minsky**, *Harmonic maps, length, and energy in Teichmüller space*, J. Differential Geom. 35 (1992) 151–217 MR Zbl

[30] **J W Morgan**, *Group actions on trees and the compactification of the space of classes of $\mathrm{SO}(n,1)$–representations*, Topology 25 (1986) 1–33 MR Zbl

[31] **J W Morgan**, **P B Shalen**, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. of Math. 120 (1984) 401–476 MR Zbl

[32] **J W Morgan**, **P B Shalen**, *Free actions of surface groups on $\mathbb{R}$–trees*, Topology 30 (1991) 143–154 MR Zbl

[33] **J-P Otal**, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. 131 (1990) 151–162 MR Zbl

[34] **C Ouyang**, **A Tamburelli**, *Limits of Blaschke metrics*, Duke Math. J. 170 (2021) 1683–1722 MR Zbl

[35] **C Ouyang**, **A Tamburelli**, *Length spectrum compactification of the $\mathrm{SO}_0(2,3)$–Hitchin component*, Adv. Math. 420 (2023) art. id. 108997 MR Zbl

[36] **C Ouyang**, **A Tamburelli**, *Boundary of the Gothen components*, Topology Appl. 326 (2023) art. id. 108420 MR Zbl

[37] **R C Penner**, **J L Harer**, *Combinatorics of train tracks*, Annals of Mathematics Studies 125, Princeton Univ. Press (1992) MR Zbl

[38] **J H Sampson**, *Some properties and applications of harmonic mappings*, Ann. Sci. École Norm. Sup. 11 (1978) 211–228 MR Zbl

[39] **R M Schoen**, *The role of harmonic mappings in rigidity and deformation problems*, from "Complex geometry" (G Komatsu, Y Sakane, editors), Lecture Notes in Pure and Appl. Math. 143, Dekker, New York (1993) 179–200 MR Zbl

[40] **R Schoen**, **S T Yau**, *On univalent harmonic maps between surfaces*, Invent. Math. 44 (1978) 265–278 MR Zbl

[41] **R K Skora**, *Splittings of surfaces*, J. Amer. Math. Soc. 9 (1996) 605–616 MR Zbl

[42] **W P Thurston**, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. 19 (1988) 417–431 MR Zbl

[43] **A Wienhard**, *An invitation to higher Teichmüller theory*, from "Proceedings of the International Congress of Mathematicians" (B Sirakov, P N de Souza, M Viana, editors), volume 2, World Sci., Hackensack, NJ (2018) 1013–1039 MR Zbl

[44] **M Wolf**, *The Teichmüller theory of harmonic maps*, J. Differential Geom. 29 (1989) 449–479 MR Zbl

[45] **M Wolf**, *High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space*, Topology 30 (1991) 517–540 MR Zbl

[46] **M Wolf**, *Harmonic maps from surfaces to $\mathbb{R}$–trees*, Math. Z. 218 (1995) 577–593 MR Zbl

Department of Mathematics, Washington University
St Louis, MO, United States

ouyang@math.wustl.edu

Proposed: Anna Wienhard
Seconded: Tobias H Colding, Mladen Bestvina
Received: 6 January 2020
Revised: 25 July 2021

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2023 is US $740/year for the electronic version, and $1030/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 27 Issue 5 (pages 1657–2048) 2023