TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 1–47 (January 17, 2023)
https://doi.org/10.1090/btran/117

ORTHOGONAL RATIONAL FUNCTIONS WITH REAL POLES, ROOT ASYMPTOTICS, AND GMP MATRICES

BENJAMIN EICHINGER, MILIVOJE LUKIĆ, AND GIORGIO YOUNG

Abstract. There is a vast theory of the asymptotic behavior of orthogonal polynomials with respect to a measure on $\mathbb{R}$ and its applications to Jacobi matrices. That theory has an obvious affine invariance and a very special role for ∞. We extend aspects of this theory in the setting of rational functions with poles on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, obtaining a formulation which allows multiple poles and proving an invariance with respect to $\overline{\mathbb{R}}$-preserving Möbius transformations. We obtain a characterization of Stahl–Totik regularity of a GMP matrix in terms of its matrix elements; as an application, we give a proof of a conjecture of Simon – a Cesàro–Nevai property of regular Jacobi matrices on finite gap sets.

1. Introduction

There is a vast theory of orthogonal polynomials with respect to measures on $\mathbb{C}$ and their root asymptotics, exemplified by the Ullman–Stahl–Totik theory of regularity. Let μ be a compactly supported probability measure and $\{p_n\}_{n=0}^{\infty}$ the corresponding orthonormal polynomials, obtained by the Gram–Schmidt process from $\{z^n\}_{n=0}^{\infty}$ in $L^2(d\mu)$. Then

$$\liminf_{n\to\infty} |p_n(z)|^{1/n} \geq e^{G_{\mathsf{E}}(z,\infty)} \tag{1.1}$$

for z outside the convex hull of $\operatorname{supp}\mu$, where E is the essential support of μ and G_{E} denotes the potential theoretic Green function for the domain $\overline{\mathbb{C}} \setminus \mathsf{E}$; if that domain is not Greenian, one takes $G_{\mathsf{E}} = +\infty$ instead. For measures compactly supported in $\mathbb{R}$, this theory can be interpreted in terms of self-adjoint operators. In particular, for any bounded half-line Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

with $a_\ell > 0$, $b_\ell \in \mathbb{R}$,

$$\limsup_{n\to\infty} \left(\prod_{\ell=1}^{n} a_\ell \right)^{1/n} \leq \operatorname{Cap} \sigma_{\mathrm{ess}}(J), \tag{1.2}$$

Received by the editors October 30, 2020, and, in revised form, December 1, 2020, and February 16, 2022.

2020 *Mathematics Subject Classification.* Primary 47B36, 42C05.

The first author was supported by Austrian Science Fund FWF, project no: J 4138-N32. The second author was supported in part by NSF grant DMS–1700179. The third author was supported in part by NSF grant DMS–1745670.

where Cap denotes logarithmic capacity. For both of these universal inequalities, the case of equality (and existence of limit) is called Stahl–Totik regularity [27]; the theory originated with the case $\mathsf{E} = [-2, 2]$, first studied by Ullman [30].

We extend aspects of this theory to the setting of rational functions with poles in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. One motivation for this is the search for a more conformally invariant theory. Statements such as (1.1), (1.2) rescale in obvious ways with respect to affine transformations (automorphisms of $\mathbb{C}$) which preserve $\mathbb{R}$, so it is obvious that an affine pushforward of a Stahl–Totik regular measure is Stahl–Totik regular. However, the point ∞ has a very special role throughout the theory: for a Möbius transformation f which does not preserve ∞, $p_n \circ f$ are rational functions with a pole at $f^{-1}(\infty)$, and $f(J)$ as defined by the functional calculus is not a finite band matrix. Thus, it is a nontrivial question whether a Möbius pushforward of a Stahl–Totik regular measure is Stahl–Totik regular.

The set of Möbius transformations which preserve $\overline{\mathbb{R}}$ is the semidirect group product $\mathrm{PSL}(2, \mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}$, whose normal subgroup $\mathrm{PSL}(2, \mathbb{R})$ corresponds to the orientation preserving case. Denote by $f_*\mu$ the pushforward of μ, defined by $(f_*\mu)(A) = \mu(f^{-1}(A))$ for Borel sets A. As an example of our techniques, we obtain the following:

Theorem 1.1. *Let $f \in \mathrm{PSL}(2, \mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}$. If μ is a Stahl–Totik regular measure on $\mathbb{R}$ and $\infty \notin \mathrm{supp}(f_*\mu)$, then the pushforward measure $f_*\mu$ is also Stahl–Totik regular.*

However, we will mostly work in the more general setting when multiple poles on $\overline{\mathbb{R}}$ are allowed, which arises naturally in the spectral theory of self-adjoint operators. Denote by $T_{f,d\mu}$ the multiplication operator by f in $L^2(d\mu)$. The matrix representation for $T_{x,d\mu(x)}$ in the basis of orthogonal polynomials is a Jacobi matrix, and through this classical connection, the theory of orthogonal polynomials is inextricably linked to the spectral theory of Jacobi matrices. In this matrix representation, resolvents $T_{(\mathbf{c}-x)^{-1},d\mu(x)}$ are not finite-diagonal matrices. However, in a basis of orthogonal rational functions with poles at $\mathbf{c}_1, \dots, \mathbf{c}_g, \infty$, the multiplication operators $T_{(\mathbf{c}_1-x)^{-1},d\mu(x)}, \dots, T_{(\mathbf{c}_g-x)^{-1},d\mu(x)}, T_{x,d\mu(x)}$ all have precisely $2g+1$ nontrivial diagonals. The corresponding matrix representations are called GMP matrices; they were introduced by Yuditskii [32].

Self-adjoint operators and their matrix representations are an important part of this work, so we choose to present the theory in a more self-contained way, using self-adjoint operators from the ground up; this has similarities with [22]. Some proofs could be shortened by using orthogonal polynomials with respect to varying weights [27, Chapter 3], but some facts rely on the precise structure obtained by the periodically repeating sequence of poles.

We should also compare this to the case of CMV matrices: for a measure supported on the unit circle, Stahl–Totik regularity is still defined in terms of orthogonal polynomials, but the CMV basis [4,22] is given in terms of positive and negative powers of z, i.e., orthonormal rational functions with poles at ∞ and 0. The symmetries in that setting lead to explicit formulas for the CMV basis in terms of the orthogonal polynomials; it is then a matter of calculation to relate the exponential growth rate of the CMV basis to that of the orthogonal polynomials, and to interpret regularity in terms of the CMV basis. In our setting, there is no such symmetry and no formula for orthonormal rational functions in terms of orthonormal polynomials.

In order to state our results in a conformally invariant way, we will use the following notations and conventions throughout the paper. The measure μ will be a probability measure on $\overline{\mathbb{R}}$. We denote by $\operatorname{supp}\mu$ its support in $\overline{\mathbb{R}}$, and we consider its essential support (the support with isolated points removed), denoted

$$\mathsf{E} = \operatorname{ess\,supp}\mu.$$

We will always assume that μ is nontrivial; equivalently, $\mathsf{E} \neq \emptyset$.

Fix a finite sequence with no repetitions, $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{g+1})$ with $\mathbf{c}_k \in \overline{\mathbb{R}} \setminus \operatorname{supp}\mu$ for all k. Consider the sequence $\{r_n\}_{n=0}^{\infty}$ where $r_0 = 1$ and for $n = j(g+1)+k$, $1 \le k \le g+1$,

$$r_n(z) = \begin{cases} \frac{1}{(\mathbf{c}_k - z)^{j+1}} & \mathbf{c}_k \in \mathbb{R}, \\ z^{j+1} & \mathbf{c}_k = \infty. \end{cases} \tag{1.3}$$

Applying the Gram–Schmidt process to this sequence in $L^2(d\mu)$ gives the sequence of orthonormal rational functions $\{\tau_n\}_{n=0}^{\infty}$ whose behavior we will study. We note that the special case $\operatorname{supp}\mu \subset \mathbb{R}$, $g = 0$, $\mathbf{C} = (\infty)$ corresponds to the standard construction of orthonormal polynomials associated to the measure μ (note that, since we denote by $\operatorname{supp}\mu$ the support in $\overline{\mathbb{R}}$, the statement $\operatorname{supp}\mu \subset \mathbb{R}$ implies that μ is compactly supported in $\mathbb{R}$), and our first results are an extension of the same techniques.

The first result is a universal lower bound on the growth of $\{\tau_n\}_{n=0}^{\infty}$ in terms of a potential theoretic quantity. If E is not a polar set, we use the (potential theoretic) Green function for the domain $\overline{\mathbb{C}} \setminus \mathsf{E}$, denoted G_{E}, and we define

$$\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) = \begin{cases} \frac{1}{g+1} \sum_{k=1}^{g+1} G_{\mathsf{E}}(z, \mathbf{c}_k) & \mathsf{E} \text{ is not polar}, \\ +\infty & \mathsf{E} \text{ is polar}. \end{cases} \tag{1.4}$$

Theorem 1.2. *For all $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{R}}$,*

$$\liminf_{n\to\infty} |\tau_n(z)|^{1/n} \ge e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}.$$

This is a good place to point out that our current setup is not related to the recent paper [13], in which the behavior was compared to a Martin function at a boundary point of the domain. Here, the behavior is compared to a combination of Green functions (1.4), all the poles are in the interior of the domain $\overline{\mathbb{C}} \setminus \mathsf{E}$, and the difficulty comes instead from the multiple poles.

Another universal inequality for orthonormal polynomials comes from comparing their leading coefficients to the capacity of E. In our setting, the analog of the leading coefficient must be considered in a pole-dependent way. Denote

$$\mathcal{L}_n = \operatorname{span}\{r_\ell \mid 0 \le \ell \le n\}.$$

By the nature of the Gram–Schmidt process, there is a $\kappa_n > 0$ such that

$$\tau_n - \kappa_n r_n \in \mathcal{L}_{n-1}.$$

The Gram–Schmidt process can be reformulated as the $L^2(d\mu)$-extremal problem

$$\kappa_n = \max\left\{\operatorname{Re}\kappa : f = \kappa r_n + h, h \in \mathcal{L}_{n-1}, \|f\|_{L^2(d\mu)} \le 1\right\}. \tag{1.5}$$

By strict convexity of the L^2-norm, these L^2-extremal problems have unique extremizers given by $f = \tau_n$, and κ_n is explicitly characterized as a kind of leading coefficient for τ_n with respect to the pole at $\mathbf{c}_k$ where $n = j(g+1)+k$, $1 \le k \le g+1$.

Below, we will also relate the constants κ_n to off-diagonal coefficients of certain matrix representations.

The growth of the leading coefficients κ_n will be studied along sequences $n = j(g+1)+k$ for a fixed k, and bounded by quantities related to the pole $\mathbf{c}_k$. If E is not a polar set, it is a basic property of the Green function that the limits

$$\gamma_{\mathsf{E}}^k = \begin{cases} \lim_{z\to\mathbf{c}_k}(G_{\mathsf{E}}(z,\mathbf{c}_k) + \log|z-\mathbf{c}_k|), & \mathbf{c}_k \neq \infty, \\ \lim_{z\to\mathbf{c}_k}(G_{\mathsf{E}}(z,\mathbf{c}_k) - \log|z|), & \mathbf{c}_k = \infty \end{cases}$$

exist. Note that if $\mathbf{c}_k = \infty$, γ_{E}^k is precisely the Robin constant for the set E. We further define constants λ_k by

$$\log \lambda_k = \begin{cases} \gamma_{\mathsf{E}}^k + \sum_{\substack{1\le\ell\le g+1 \\ \ell\neq k}} G_{\mathsf{E}}(\mathbf{c}_k,\mathbf{c}_\ell) & \mathsf{E} \text{ is not polar}, \\ +\infty & \mathsf{E} \text{ is polar}. \end{cases} \tag{1.6}$$

Theorem 1.3. *For all $1 \le k \le g+1$, for the subsequence $n(j) = j(g+1)+k$,*

$$\liminf_{j\to\infty} \kappa_{n(j)}^{1/n(j)} \ge \lambda_k^{1/(g+1)}. \tag{1.7}$$

Theorem 1.4. *The following are equivalent:*

(i) *For some $1 \le k \le g+1$, for the subsequence $n(j) = j(g+1)+k$,*
$$\lim_{j\to\infty} \kappa_{n(j)}^{1/n(j)} = \lambda_k^{1/(g+1)};$$

(ii) *For all $1 \le k \le g+1$, for the subsequence $n(j) = j(g+1)+k$,*
$$\lim_{j\to\infty} \kappa_{n(j)}^{1/n(j)} = \lambda_k^{1/(g+1)};$$

(iii)
$$\lim_{n\to\infty} \left(\prod_{\ell=1}^{g+1} \kappa_{n+\ell}\right)^{1/n} = \left(\prod_{k=1}^{g+1} \lambda_k\right)^{1/(g+1)};$$

(iv) *For q.e. $z \in \mathsf{E}$, we have $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \le 1$;*

(v) *For some $z \in \mathbb{C}_+$, $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \le e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;*

(vi) *For all $z \in \mathbb{C}$, $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \le e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;*

(vii) *Uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$, $\lim_{n\to\infty} |\tau_n(z)|^{1/n} = e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$.*

Definition 1.5. The measure μ is $\mathbf{C}$-regular if it obeys one (and therefore all) of the assumptions of Theorem 1.4.

In this terminology, Stahl–Totik regularity is precisely (∞)-regularity, i.e., $\mathbf{C}$-regularity for the special case $\operatorname{supp}\mu \subset \mathbb{R}$, $g = 0$, $\mathbf{C} = (\infty)$. Theorems 1.2, 1.3, 1.4 are closely motivated by foundational results for Stahl–Totik regularity. A new phenomenon appears through the periodicity with which poles are taken in (1.3) and the resulting subsequences $n(j) = j(g+1)+k$: since κ_n is a normalization constant for τ_n, it is notable that control of κ_n along a single subsequence $n(j) = j(g+1)+k$ in Theorem 1.4(i) provides control over the entire sequence. This phenomenon doesn't have an exact analog for orthogonal polynomials, where $g = 0$. We will also see below that this is essential in order to characterize the regularity of a GMP matrix using only the entries of the matrix itself and not its resolvents.

Moreover, we show that the regular behavior described by Theorem 1.4 is independent of the set of poles $\mathbf{C}$:

Theorem 1.6. *Let $\mathbf{C}_1, \mathbf{C}_2$ be two finite sequences of elements from $\overline{\mathbb{R}} \setminus \operatorname{supp}\mu$, not necessarily of the same length. Then μ is $\mathbf{C}_1$-regular if and only if it is $\mathbf{C}_2$-regular.*

Corollary 1.7. *Let $\operatorname{supp}\mu \subset \mathbb{R}$. Let $\mathbf{C}$ be a finite sequence of elements from $\overline{\mathbb{R}} \setminus \operatorname{supp}\mu$. Then μ is $\mathbf{C}$-regular if and only if it is Stahl–Totik regular.*

Thus, Theorem 1.4 should not be seen as describing equivalent conditions for a new class of measures, but rather a new set of regular behaviors for the familiar class of Stahl–Totik regular measures.

We consistently work with poles on $\overline{\mathbb{R}}$ since our main interest is tied to self-adjoint problems. Some of our results are in a sense complementary to the setting of [27, Section 6.1], where poles are allowed in the complement of the convex hull of $\operatorname{supp}\mu$, and the behavior of orthogonal rational functions is considered with respect to a Stahl–Totik regular measure. Due to this, it is natural to expect that these results hold more generally, for measures on $\mathbb{C}$ and general collections of poles and Möbius transformations. Moreover, in our setup the poles are repeated exactly periodically, but we expect this can be generalized to a sequence of poles which has a limiting average distribution. Related questions for orthogonal rational functions were also studied by [3, 10].

As noted in [27, Section 6.1], poles in the gaps of $\operatorname{supp}\mu$ can cause interpolation defects in the problem of interpolation by rational functions. In our work, these interpolation defects show up as possible reductions in the order of the poles. For example, consider $\mathbf{C} = (\infty, 0)$. Then, by construction, τ_{2j+1} is allowed a pole at 0 of order at most j. However, if μ is symmetric with respect to $z \mapsto -z$, the functions τ_n will have an even/odd symmetry. Since τ_{2j+1} contains a nontrivial multiple of z^{j+1}, it follows that $\tau_{2j+1}(z) = (-1)^{j+1}\tau_{2j+1}(-z)$. By this symmetry, the actual order of the pole at 0 is $j+1-k$ for some even k, so it cannot be equal to j (it will follow from our results that in this case, the order of the pole is $j-1$). The same effect can be seen for the pole at ∞ for $\mathbf{C} = (0, \infty)$. In the polynomial case, this does not occur: p_n always has a pole at ∞ of order exactly n.

We will consider at once the distribution of zeros of τ_n and the possible reductions in the order of the poles. We will prove that all zeros of τ_n are real and simple, and that $n - g \le \deg\tau_n \le n$. We define the normalized zero counting measure

$$\nu_n = \frac{1}{n} \sum_{w:\tau_n(w)=0} \delta_w.$$

Although we normalize by n, ν_n may not be a probability measure: however $1 - g/n \le \nu_n(\overline{\mathbb{R}}) \le 1$. Therefore, normalizing by $\deg\tau_n$ instead of by n would not affect the limits as $n \to \infty$.

We will now describe the weak limit behavior of the measures ν_n as $n \to \infty$. To avoid pathological cases, we assume that E is not polar; in that case, denoting by $\omega_{\mathsf{E}}(dx, w)$ the harmonic measure for the domain $\overline{\mathbb{C}} \setminus \mathsf{E}$ at the point w, we define the probability measure on E,

$$\rho_{\mathsf{E},\mathbf{C}} = \frac{1}{g+1} \sum_{j=1}^{g+1} \omega_{\mathsf{E}}(\mathrm{d}x, \mathbf{c}_j).$$

The results below describe weak limits of measures in the topology dual to $C(\overline{\mathbb{R}})$.

Theorem 1.8. *Let μ be a probability measure on $\overline{\mathbb{R}}$. Assume that E is not a polar set.*

(a) *If* μ *is* $\mathbf{C}$ *regular, then* $\text{w-lim}_{n\to\infty} \nu_n = \rho_{\mathsf{E},\mathbf{C}}$.
(b) *If* $\text{w-lim}_{n\to\infty} \nu_n = \rho_{\mathsf{E},\mathbf{C}}$, *then* μ *is* $\mathbf{C}$ *regular or there exists a polar set* $X \subset \mathsf{E}$ *such that* $\mu(\overline{\mathbb{R} \setminus X}) = 0$.

We now turn to matrix representations of self-adjoint operators. Fix a sequence $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{g+1})$ such that $\mathbf{c}_{k_\infty} = \infty$ for some $1 \le k_\infty \le g+1$. A half-line GMP matrix [32] is the matrix representation for multiplication by x in the basis $\{\tau_n\}_{n=0}^\infty$ for this sequence $\mathbf{C}$; its matrix elements are

$$A_{mn} = \int \overline{\tau_m(x)} x \tau_n(x)\, d\mu(x).$$

The condition that $\mathbf{c}_{k_\infty} = \infty$ for some k_∞ guarantees that $A_{mn} = 0$ for $|m-n| > g+1$, so these matrix elements generate a bounded operator A on $\ell^2(\mathbb{N}_0)$ such that $A_{mn} = \langle e_m, Ae_n\rangle$, where $(e_n)_{n=0}^\infty$ denotes the standard basis of $\ell^2(\mathbb{N}_0)$. We say that $A \in \mathbb{A}(\mathbf{C})$.

GMP matrices have the property that some of their resolvents are also GMP matrices; namely, for any $k \ne k_\infty$, $(\mathbf{c}_k - A)^{-1} \in \mathbb{A}(f(\mathbf{C}))$ where f is the Möbius transform $f : z \mapsto (\mathbf{c}_k - z)^{-1}$ and $f(\mathbf{C}) = (f(\mathbf{c}_1), \dots, f(\mathbf{c}_{g+1}))$.

Note that the special case $g = 0$, $\mathbf{C} = (\infty)$ gives precisely a Jacobi matrix. A Jacobi matrix is said to be regular if it is obtained by this construction from a regular measure; analogously, we will call a GMP matrix regular if it is obtained from a regular measure. Just as regularity of a Jacobi matrix can be characterized in terms of its off-diagonal entries, we will show that regularity of a GMP matrix can be characterized in terms of its entries in the outermost nontrivial diagonal. We will also obtain a GMP matrix analog of the inequality (1.2).

The GMP matrix has an additional block matrix structure; in particular, for a GMP matrix with $\mathbf{c}_{k_\infty} = \infty$, on the outermost nonzero diagonal $m = n - g - 1$, the only nonzero terms appear for $n = j(g+1) + k_\infty$, and those are strictly positive. Thus, we denote

$$\beta_j = \langle e_{j(g+1)+k_\infty}, Ae_{(j+1)(g+1)+k_\infty}\rangle. \tag{1.8}$$

Theorem 1.9. *Fix a probability measure* μ *with* $\operatorname{supp}\mu \subset \mathbb{R}$ *and a sequence* $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{g+1})$ *with* $\mathbf{c}_k = \infty$. *Then*

$$\limsup_{j\to\infty} \left(\prod_{\ell=1}^{j} \beta_\ell\right)^{1/j} \le \lambda_{k_\infty}^{-1}. \tag{1.9}$$

Moreover, the measure μ *is Stahl–Totik regular if and only if*

$$\lim_{j\to\infty} \left(\prod_{\ell=1}^{j} \beta_\ell\right)^{1/j} = \lambda_{k_\infty}^{-1}. \tag{1.10}$$

The proof will use a relation between the sequence $\{\beta_j\}_{j=1}^\infty$ and the constants $\{\kappa_{j(g+1)+k_\infty}\}_{j=1}^\infty$. In particular, the characterization of regularity in Theorem 1.9 is made possible by the characterization of regularity in terms of the subsequence $\{\kappa_{j(g+1)+k}\}_{j=1}^\infty$ for any single k. Theorem 1.9 also corroborates the perspective that regularity of the measure is the fundamental notion which manifests itself equally well in many different matrix representations.

Since the resolvents $(\mathbf{c}_k - A)^{-1}$ are also GMP matrices and their measures are pushforwards of the original measure, they are also regular GMP matrices; in this

sense, Theorem 1.9 provides $g+1$ criteria for regularity, one corresponding to each subsequence $n(j) = j(g+1)+k$, $1 \le k \le g+1$.

As an application of this theory, we show that it provides a proof of a theorem for Jacobi matrices originally conjectured by Simon [23]. Let $\mathsf{E} \subset \mathbb{R}$ be a compact finite gap set,

$$\mathsf{E} = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{k=1}^{g} (\mathbf{a}_k, \mathbf{b}_k), \tag{1.11}$$

and denote by $\mathcal{T}_{\mathsf{E}}^+$ the set of almost periodic half-line Jacobi matrices with $\sigma_{\mathrm{ess}}(J) = \sigma_{\mathrm{ac}}(J) = \mathsf{E}$ [5, 14]. Through algebro-geometric techniques and the reflectionless property, this class of Jacobi matrices has been widely studied for their spectral properties and quasiperiodicity (see also [26, 31] for more general spectral sets). They also provide natural reference points for perturbations, which is our current interest. On bounded half-line Jacobi matrices J, we consider the metric

$$d(J, \tilde{J}) = \sum_{k=1}^{\infty} e^{-k}(|a_k - \tilde{a}_k| + |b_k - \tilde{b}_k|). \tag{1.12}$$

On norm-bounded sets of Jacobi matrices, convergence in this metric corresponds to strong operator convergence. However, instead of distance to a fixed Jacobi matrix $\tilde{J}$, we will consider the distance to $\mathcal{T}_{\mathsf{E}}^+$,

$$d(J, \mathcal{T}_{\mathsf{E}}^+) = \inf_{\tilde{J} \in \mathcal{T}_{\mathsf{E}}^+} d(J, \tilde{J}) = \min_{\tilde{J} \in \mathcal{T}_{\mathsf{E}}^+} d(J, \tilde{J}).$$

Denote by S_+ the right shift operator on $\ell^2(\mathbb{N}_0)$, $S_+ e_n = e_{n+1}$. The condition $d((S_+^*)^m J S_+^m, \mathcal{T}_{\mathsf{E}}^+) \to 0$ as $m \to \infty$ is called the Nevai condition. For $\mathsf{E} = [-2, 2]$, this corresponds simply to the commonly considered condition $a_n \to 1$, $b_n \to 0$ as $n \to \infty$ [18]. In general, as a consequence of [21], the Nevai condition implies regularity. The converse is false; however:

Theorem 1.10. *If $\mathsf{E} \subset \mathbb{R}$ is a compact finite gap set and J is a regular Jacobi matrix with $\sigma_{\mathrm{ess}}(J) = \mathsf{E}$, then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} d((S_+^*)^m J S_+^m, \mathcal{T}_{\mathsf{E}}^+) = 0. \tag{1.13}$$

The condition (1.13) is described as the Cesàro–Nevai condition; it was first studied by Golinskii–Khrushchev [15] in the OPUC setting with essential spectrum equal to $\partial\mathbb{D}$. Theorem 1.10 was conjectured by Simon [23] and proved in the special case when E is the spectrum of a periodic Jacobi matrix with all gaps open by using the periodic discriminant and techniques from Damanik–Killip–Simon [7] to reduce to a block Jacobi setting. It was then proved by Krüger [17] by very different methods under the additional assumption $\inf_n a_n > 0$. While this is a common assumption in the ergodic literature, regular Jacobi matrices do not always satisfy it: [22, Example 1.4] can easily be modified to give a regular Jacobi matrix with spectrum $[-2, 2]$ and $\inf a_n = 0$. We prove Theorem 1.10 in full generality by applying Simon's strategy and, instead of the periodic discriminant and techniques from [7], using the Ahlfors function, GMP matrices, and techniques of Yuditskii [32].

For the compact finite gap set $\mathsf{E} \subset \mathbb{R}$, among all analytic functions $\overline{\mathbb{C}} \setminus \mathsf{E} \to \mathbb{D}$ which vanish at ∞, the Ahlfors function Ψ takes the largest value of $\mathrm{Re}(z\Psi(z))|_{z=\infty}$.

The Ahlfors function has precisely one zero in each gap, denoted $\mathbf{c}_k \in (\mathbf{a}_k, \mathbf{b}_k)$ for $1 \le k \le g$, a zero at $\mathbf{c}_{g+1} = \infty$, and no other zeros; see also [25, Chapter 8]. In particular, for the finite gap set E, this generates a particularly natural sequence of poles $\mathbf{C}_\mathsf{E} = (\mathbf{c}_1, \dots, \mathbf{c}_g, \infty)$.

The Ahlfors function was used by Yuditskii [32] to define a discriminant for finite gap sets,

$$\Delta_\mathsf{E}(z) = \Psi(z) + \frac{1}{\Psi(z)}. \tag{1.14}$$

This function is not equal to the periodic discriminant, but it has some similar properties and it is available more generally (even when E is not a periodic spectrum). Namely, Δ_E extends to a meromorphic function on $\overline{\mathbb{C}}$ and $(\Delta_\mathsf{E})^{-1}([-2,2]) = \mathsf{E}$. It was introduced by Yuditskii to solve the Killip–Simon problem for finite gap essential spectra. In fact, the discriminant is a rational function of the form

$$\Delta_\mathsf{E}(z) = \lambda_{g+1} z + d + \sum_{k=1}^{g} \frac{\lambda_k}{\mathbf{c}_k - z} \tag{1.15}$$

for some $d \in \mathbb{R}$; in particular, we will explain that the constants $\lambda_j > 0$ in (1.15) match the general definition (1.6).

As a first glimpse of our proof of Theorem 1.10, we note that it uses the following chain of implications. Starting with a regular Jacobi matrix with essential spectrum E, by a change of one Jacobi coefficient, which does not affect regularity, we can assume that $\mathbf{c}_k \notin \operatorname{supp} \mu$ (Lemma 7.1). Under this assumption, regularity of the Jacobi matrix implies regularity of the corresponding GMP matrix A and the resolvents $(\mathbf{c}_k - A)^{-1}$, $k = 1, \dots, g$, which can be characterized in terms of their coefficients by Theorem 1.9. By properties of the Yuditskii discriminant, this further implies regularity of the block Jacobi matrix $\Delta_\mathsf{E}(A)$. Let us briefly recall that a block Jacobi matrix is of the form

$$\mathbf{J} = \begin{bmatrix} \mathfrak{w}_0 & \mathfrak{v}_0 & & & \\ \mathfrak{v}_0^* & \mathfrak{w}_1 & \mathfrak{v}_1 & & \\ & \mathfrak{v}_1^* & \mathfrak{w}_2 & \mathfrak{v}_2 & \\ & & \mathfrak{v}_2^* & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}, \tag{1.16}$$

where $\mathfrak{v}_j$ and $\mathfrak{w}_j$ are $d \times d$ matrices, $\mathfrak{w}_j = \mathfrak{w}_j^*$, and $\det \mathfrak{v}_j \neq 0$ for each j. Type 3 block Jacobi matrices have each $\mathfrak{v}_j$ lower triangular and positive on the diagonal. An extension of regularity to block Jacobi matrices was developed by Damanik–Pushnitski–Simon [8]; in particular, $\mathbf{J}$ is regular for the set $[-2,2]$ if $\sigma_{\mathrm{ess}}(\mathbf{J}) = [-2,2]$ and

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} |\det \mathfrak{v}_j| \right)^{1/n} = 1. \tag{1.17}$$

This chain of arguments will result in Lemma 1.11:

Lemma 1.11. *Let J be a regular Jacobi matrix, $\mathsf{E} = \sigma_{\mathrm{ess}}(J)$ a finite gap set, and $\mathbf{C}_\mathsf{E}$ the corresponding sequence of zeros of the Ahlfors function. Assuming $\mathbf{c}_k \notin \sigma(J)$ for $1 \le k \le g$, denote by A the GMP matrix corresponding to J with respect to the*

sequence $\mathbf{C}_\mathsf{E}$. *Then* $\Delta_\mathsf{E}(A)$ *is a regular type 3 block Jacobi matrix with essential spectrum* $[-2,2]$.

With Lemma 1.11, it will follow that $\mathbf{J} = \Delta_\mathsf{E}(A)$ obeys a Cesàro–Nevai condition. That Cesàro–Nevai condition will imply (1.13) by a modification of arguments of [32]. The strategy is clear: just as [32] uses a certain square-summability in terms of $\mathfrak{v}_j, \mathfrak{w}_j$ to prove finiteness of ℓ^2-norm of $\{d((S_+^*)^m J S_+^m, \mathcal{T}_\mathsf{E}^+)\}_{m=0}^\infty$, we will use Cesàro decay in terms of $\mathfrak{v}_j, \mathfrak{w}_j$ to conclude the Cesàro decay (1.13). This can be expected due to a certain locality in the dependence between the terms of the series considered; this idea first appeared in [23] in the setting of periodic spectra with all gaps open. However, some care is needed, since the locality is only approximate in some steps; this is already visible in (1.12). Also, substantial modifications are needed throughout the proof due to the possibility of $\liminf \|\mathfrak{v}_j\| = 0$ (this cannot happen in the Killip–Simon class), which locally breaks some of the estimates. The fix is that this can only happen along a sparse subsequence, but the combination of a bad sparse subsequence and approximate locality means that we cannot simply ignore a bad subsequence once from the start; we must maintain it throughout the proof. A related issue arises with the Cesàro version of a Killip–Simon type functional. We will describe the necessary modifications to the detailed analysis in [32].

The rest of the paper will not exactly follow the order given in this section. In Section 2, we describe the behavior of our problem with respect to Möbius transformations, and we describe the distribution of zeros of the rational function τ_n. In Section 3, we recall the structure of GMP matrices and relate their matrix coefficients to the quantities κ_n, and use this to provide a first statement about exponential growth of orthonormal rational functions on $\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}$. In Section 4, we combine this with potential theoretic techniques to characterize limits of $\frac{1}{n}\log|\tau_n|$ as $n \to \infty$ and prove the universal lower bounds. In Section 5, we prove the results for $\mathbf{C}$-regularity and Stahl–Totik regularity. In Section 6 we describe a proof of Theorem 1.10.

2. Orthonormal rational functions and Möbius transformations

In Section 1, starting from the measure μ and sequence of poles $\mathbf{C}$, we defined a sequence $\{r_n\}_{n=0}^\infty$ and the orthonormal rational functions $\{\tau_n\}_{n=0}^\infty$. In the next statement, we will denote these by $r_n(z;\mathbf{C})$ and $\tau_n(z;\mu,\mathbf{C})$, in order to state precisely the invariance of the setup with respect to Möbius transformations.

Lemma 2.1. *If* f *is a Möbius transformation which preserves* $\overline{\mathbb{R}}$, *then*

$$\tau_n(z;\mu,\mathbf{C}) = \rho^n \tau_n(f(z); f_*\mu, f(\mathbf{C})), \tag{2.1}$$

where $f(\mathbf{C}) = (f(\mathbf{c}_1), \dots, f(\mathbf{c}_{g+1}))$ *and*

$$\rho = \begin{cases} +1 & f \in \mathrm{PSL}(2,\mathbb{R}), \\ -1 & f \in (\mathrm{PSL}(2,\mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}) \setminus \mathrm{PSL}(2,\mathbb{R}). \end{cases}$$

Proof. Note that the sequence $\{r_n\}_{n=0}^\infty$ does not have this property: $r_n(z;\mathbf{C})$ is not equal to $\rho^n r_n(f(z); f(\mathbf{C}))$. However, if we denote

$$\mathcal{L}_n(\mathbf{C}) = \mathrm{span}\{r_\ell(\cdot;\mathbf{C}) \mid 0 \le \ell \le n\},$$

then it suffices to have

$$r_n(f(z); f(\mathbf{C})) - c_n \rho^n r_n(z; \mathbf{C}) \in \mathcal{L}_{n-1}(\mathbf{C}) \tag{2.2}$$

for some constants $c_n > 0$. If (2.2) holds, then applying the Gram–Schmidt process to the sequences $\{r_n(f(z); f(\mathbf{C}))\}_{n=0}^{\infty}$ and $\{r_n(z; \mathbf{C})\}_{n=0}^{\infty}$ will give the same sequence of orthonormal functions, up to the sign change ρ^n, which is precisely (2.1).

Note that if (2.1) holds for f_1, f_2, it holds for their composition, so it suffices to verify (2.2) for a set of generators of $\mathrm{PSL}(2, \mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}$. In particular, (2.2) is checked by straightforward calculations for affine transformations and for the inversion $f(z) = -1/z$, which implies the general statement since affine maps and inversion generate $\mathrm{PSL}(2, \mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}$. □

Let us emphasize what Lemma 2.1 does and what it doesn't do. Since the Möbius transformation acts on both the measure and the sequence of poles, Lemma 2.1 does not by itself prove Theorem 1.1. Lemma 2.1 can only say that if μ is Stahl–Totik regular, then $f_*\mu$ is $(f(\infty))$-regular, which is not sufficient unless f is affine. The proof of Theorem 1.1 will be more involved.

However, Lemma 2.1 provides a very useful conformal invariance for many of our proofs. This can be compared to choosing a convenient reference frame. Since potential theoretic notions such as Green functions are conformally invariant, our results will be invariant with respect to Möbius transformations. We will often use this invariance in the proofs to fix a convenient point at ∞.

Note that this will be possible even though some objects are not conformally invariant. Some of our results compare the sequences κ_n with the λ_k, and although those objects are not preserved under conformal transformations, both sequences are affected in a compatible way so that the inequalities and equalities are preserved. Explicitly, fix k and $n = j(g+1) + k$ and a Möbius transformation $f \in \mathrm{PSL}(2, \mathbb{R})$ (a reflection can be considered separately). Let us denote a local dilation factor $f'(\mathbf{c}_k) = \lim_{z \to \mathbf{c}_k} \frac{r_k(z, \mathbf{C})}{r_k(f(z), f(\mathbf{C}))}$. Then, we use Lemma 2.1 to compute

$$\kappa_{n(j)} = \lim_{z \to \mathbf{c}_k} \frac{\tau_n(z, \mu, \mathbf{C})}{r_n(z, \mathbf{C})} = \lim_{z \to \mathbf{c}_k} \frac{\tau_n(f(z), f_*\mu, f(\mathbf{C}))}{r_n(z, \mathbf{C})} = \frac{\tilde{\kappa}_{n(j)}}{f'(\mathbf{c}_k)^{j+1}},$$

where $\tilde{\kappa}_{n(j)}$ is the leading coefficient $\tau_n(z, f_*\mu, f(\mathbf{C})) - \tilde{\kappa}_n r_n(z, f(\mathbf{c}_k)) \in \mathcal{L}_{n-1}(f(\mathbf{C}))$. If E is nonpolar, the Green function is conformally invariant so we find by another computation

$$\begin{aligned}
\log \tilde{\lambda}_k &:= \lim_{w \to f(\mathbf{c}_k)} \left(G_{f(\mathsf{E})}(w, f(\mathbf{c}_k)) - \log|r_k(w, f(\mathbf{C}))| \right) + \sum_{\substack{1 \le \ell \le k \\ \ell \ne k}} G_{f(\mathsf{E})}(f(\mathbf{c}_k), f(\mathbf{c}_\ell)) \\
&= \lim_{z \to \mathbf{c}_k} \left(G_{\mathsf{E}}(z, \mathbf{c}_k) - \log|r_k(z, \mathbf{C})| + \log \left| \frac{r_k(z, \mathbf{C})}{r_k(f(z), f(\mathbf{C}))} \right| \right) + \sum_{\substack{1 \le \ell \le k \\ \ell \ne k}} G_{\mathsf{E}}(\mathbf{c}_k, \mathbf{c}_\ell) \\
&= \log \lambda_k + \log(f'(\mathbf{c}_k)),
\end{aligned}$$

where we have used that $f \in \mathrm{PSL}(2, \mathbb{R}) \implies f' > 0$ on $\overline{\mathbb{R}}$. Thus, $\tilde{\lambda}_k = f'(\mathbf{c}_k)\lambda_k$. If E is polar, then $f(\mathsf{E})$ is as well. From these calculations, it becomes elementary to verify that statements such as those in Theorems 1.3, 1.4 are conformally invariant.

Note that technical ingredients of the proof, such as polynomial factorizations, give a preferred role to ∞ so they break symmetry. For instance, we will often use

the observation that the subspace $\mathcal{L}_n$ can be represented as

$$\mathcal{L}_n = \left\{ \frac{P}{R_n} \mid P \in \mathcal{P}_n \right\} \tag{2.3}$$

for some suitable polynomial R_n with factors which account for finite poles $\mathbf{c}_k \neq \infty$. We will use the representation (2.3) after placing a convenient point at ∞. This idea is already seen in the next proof.

Lemma 2.2. *All zeros of the rational function τ_n are simple and lie in $\overline{\mathbb{R}}$. Moreover, $n - g \leq \deg \tau_n \leq n$.*

Let $n = j(g+1) + k$, $1 \leq k \leq g+1$, and denote by I the connected component of $\mathbf{c}_k$ in $\overline{\mathbb{R}} \setminus \operatorname{supp} \mu$. Then τ_n has no zeros in I and at most one zero in any other connected component of $\overline{\mathbb{R}} \setminus \operatorname{supp} \mu$.

Proof. Fix $1 \leq k \leq g+1$ and without loss of generality, assume $\mathbf{c}_k = \infty$. Then, in the representations (2.3), we can notice that $R_{n-1} = R_n$. In particular, then $\tau_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ implies the representation $\tau_{n(j)} = \frac{P_n}{R_n}$ for some polynomial P_n of degree n.

Recall that τ_n, $n = k + (j-1)(g+1)$ is the unique minimizer for the extremal problem (1.5). By complex conjugation symmetry, the minimizer is real. To proceed further, we study zeros of P_n by using Markov correction terms.

We say that a rational function M is an admissible Markov correction term if $M > 0$ a.e. on E and $M(z)P_n(z) \in \mathcal{P}_{n-1}$. In this case, using $\langle M\tau_n, \tau_n \rangle > 0$, we see that the function $g(\epsilon) = \|\tau_n - \epsilon M \tau_n\|^2$ obeys

$$g'(0) = -2\langle M\tau_n, \tau_n \rangle < 0.$$

Thus, for small enough $\epsilon > 0$, the function

$$\tilde{\tau}_n = \tau_n - \epsilon M \tau_n$$

obeys $\|\tilde{\tau}_n\|_{L^2(d\mu)} < \|\tau_n\|_{L^2(d\mu)}$. Since $\tilde{\tau}_n$ is of the form $\tilde{\tau}_n = \kappa_n z^{j+1} + h(z)$ for some $h(z) \in \mathcal{L}_{n-1}$ and in particular has the same leading coefficient as τ_n, the function $\tilde{\tau}_n / \|\tilde{\tau}_n\|_{L^2(d\mu)} \in \mathcal{L}_n$ contradicts extremality of τ_n. In other words, for the extremizer τ_n, there cannot be any admissible Markov correction terms.

Assume that P_n has a nonreal zero $w \in \overline{\mathbb{C}} \setminus \overline{\mathbb{R}}$. Then, since τ_n is real, $P_n(\overline{w}) = 0$, so the Markov correction term $M(z; w) = \frac{1}{(z-w)(z-\bar{w})}$ would be admissible, leading to contradiction.

Assume that P_n has two zeros x_1, x_2 in the same connected component of $\overline{\mathbb{R}} \setminus \operatorname{supp} \mu$; then, the Markov correction term

$$M(z; x_1, x_2) = \frac{1}{(z - x_1)(z - x_2)}$$

would be admissible, leading to contradiction.

There are no zeros of P_n in I. Otherwise, if $x \in I$ was a zero, the Markov term

$$M(z, x) = \begin{cases} \frac{1}{z-x}, & x < \inf \mathsf{E}, \\ \frac{1}{x-z}, & x > \sup \mathsf{E} \end{cases}$$

would be admissible.

Finally, all zeros of P_n are simple: otherwise, if $x_0 \in \mathbb{R}$ was a double zero, the Markov term

$$M(z, x_0) = \frac{1}{(z - x_0)^2}$$

would be admissible.

The properties of zeros of τ_n follow from those of P_n. There may be cancellations in the representation $\tau_n = \frac{P_n}{R_n}$, but since P_n has at most a simple zero at $\mathbf{c}_\ell$, the only possible cancellations are simple factors $(z-\mathbf{c}_\ell)$, $\ell \neq k$. Thus, $n-g \leq \deg \tau_n \leq n$. □

The use of Markov correction factors is standard in the Chebyshev polynomial literature and is applied here with a modification for the L^2-extremal problem (in the L^∞-setting, singularities in M are treated with a separate argument near the singularity, which would not work here).

Corollary 2.3. *The measures ν_n are a precompact family with respect to weak convergence on $C(\overline{\mathbb{R}})$. Any accumulation point $\nu = \lim_{\ell\to\infty} \nu_{n_\ell}$ is a probability measure and* $\operatorname{supp}\nu \subset \mathsf{E}$.

Proof. By Lemma 2.2, $\nu_n(\overline{\mathbb{R}}) \leq 1$, so precompactness follows by the Banach-Alaoglu theorem. If $\nu = \lim_{\ell\to\infty} \nu_{n_\ell}$, then since $1 - \frac{g}{n_\ell} \leq \nu_{n_\ell}(\overline{\mathbb{R}}) \leq 1$, $\nu(\overline{\mathbb{R}}) = 1$.

Let $(\mathbf{a}, \mathbf{b})$ be a connected component of $\overline{\mathbb{R}} \setminus \mathsf{E}$. Let us prove that $\nu((\mathbf{a}, \mathbf{b})) = 0$. By Möbius invariance, it suffices to assume that (a, b) is a bounded subset of $\mathbb{R}$.

Fix $r \in \mathbb{N}$. As $\operatorname{supp}\mu \setminus \mathsf{E}$ is a discrete set, we have

$$\#\{x \in \operatorname{supp}\mu : \mathbf{a} + 1/r < x < \mathbf{b} - 1/r\} = M < \infty.$$

So, by Lemma 2.2, $\nu_{n_\ell}((\mathbf{a}+1/r, \mathbf{b}-1/r)) \leq \frac{2M+1}{n_\ell}$ and by the Portmanteau theorem and sending $r \to \infty$, $\nu((\mathbf{a}, \mathbf{b})) = 0$ and $\operatorname{supp}\nu \subset \mathsf{E}$. □

3. GMP matrices and exponential growth of orthonormal rational functions

In this section, we consider orthonormal rational functions through the framework of GMP matrices. We begin by recalling the structure of GMP matrices [32]. The GMP matrix has a tridiagonal block matrix structure, with the beginnings of new blocks corresponding to occurrences of $\mathbf{c}_{k_\infty} = \infty$. Explicitly,

$$A = \begin{bmatrix} \tilde{B}_0 & \tilde{A}_0 & & & \\ \tilde{A}_0^* & B_1 & A_1 & & \\ & A_1^* & B_2 & A_2 & \\ & & A_2^* & \ddots & \ddots \\ & & & \ddots & \end{bmatrix},$$

where $\tilde{B}_0$ is a $k_\infty \times k_\infty$ matrix, $\tilde{A}_0$ is a $k_\infty \times (g+1)$ matrix. For $j \geq 0$, A_j, B_j are $(g+1)\times(g+1)$ matrices; while for $j \geq 1$ these appear in A unmodified in the above, $\tilde{A}_0$ and $\tilde{B}_0$ are projections of A_0 and B_0 respectively. More precisely, let X^- denote the upper triangular part of a matrix X (excluding the diagonal) and X^+ the lower triangular part (including the diagonal). Then, indexing the entries of A_j, B_j, $j \geq 0$ from 0 to g, we see they are of the form

$$A_j = \vec{p}_j \vec{\delta}_0^{\mathsf{T}}, \quad B_j = \hat{\mathbf{C}} + (\vec{q}_j \vec{p}_j^{\mathsf{T}})^+ + (\vec{p}_j \vec{q}_j^{\mathsf{T}})^-, \tag{3.1}$$

where $\vec{p}_j, \vec{q}_j \in \mathbb{R}^{g+1}$, with $(\vec{p}_j)_0 > 0$ and $\hat{\mathbf{C}} = \operatorname{diag}\{0, \mathbf{c}_{k_\infty+1}, \ldots, \mathbf{c}_{g+1}, \mathbf{c}_1, \ldots, \mathbf{c}_{k_\infty-1}\}$ (with the obvious modification if $k_\infty = 1$ or $k_\infty = g+1$) and $\vec{\delta}_0$ denotes the standard

first basis vector of $\mathbb{R}^{g+1}$. $\tilde{A}_0$ and $\tilde{B}_0$ are projections of A_0 and B_0,

$$\tilde{A}_0 = \Pi A_0 \quad \tilde{B}_0 = \Pi B_0 \Pi^*$$

with Π the block matrix $\Pi := \left[0_{k_\infty \times (g+1-k_\infty)} | I_{k_\infty \times k_\infty}\right]$. We will refer to $\{\vec{p}_j, \vec{q}_j\}_{j\geq 1}$ as the GMP coefficients of A. While the precise structure will not be essential throughout the paper, we point out two things. First on the outermost diagonal of A in each block there is only one nonvanishing entry, given by $(\vec{p}_j)_0$, which is positive and which is at a different position depending on the position of ∞ in the sequence $\mathbf{C}$. And secondly, in general as a self-adjoint matrix B_j could depend on $(g+1)(g+2)/2$ parameters, but we see that in fact they only depend on $2(g+1)$. This is not that surprising due to their close relation to three-diagonal Jacobi matrices. A similar phenomenon also appears for their unitary analogs [6].

Remark 1. For later reference, we provide an alternative point of view on the block structure of A. The structure provided above is chosen so that c_{k_∞} is at the first diagonal position of the B-blocks. Recall also that to these blocks we attached a column $\vec{p}$ (with positive first entry $(\vec{p})_0 > 0$) to the right and a row $\vec{p}_j^{\intercal}$ at the bottom. If, instead of viewing this as a block matrix structure with blocks of size $(g+1)\times(g+1)$, we view this structure as overlapping blocks of size $(g+2)\times(g+2)$ which overlap at the positions of c_{k_∞}, then those would contain all nonvanishing entries of the GMP matrix (i.e., it would also include the vector $\vec{p}_j$). Moreover, the positive entries are exactly at the upper right and the lower left corner of the bigger block. Now placing the window of size $(g+1)\times(g+1)$ on the top of the bigger block corresponds to the structure presented above. We will encounter in Section 6 that in other settings it may be more natural to place the block at the lower corner, and in this case the B blocks will have structure similar to (3.1).

Now the various notations for the off-diagonal blocks A_j, the vectors $\vec{p}_j$ which determine them, and the coefficients β_j defined in (1.8) are related as

$$\beta_j = \langle e_{j(g+1)+k_\infty}, A e_{(j+1)(g+1)+k_\infty}\rangle = (A_j)_{00} = (\vec{p}_j)_0.$$

The coefficients β_j are a special case of the coefficients Λ_n defined for $n = j(g+1)+k$, $1 \leq k \leq g+1$ as

$$\Lambda_n = \begin{cases} \langle e_{j(g+1)+k}, (\mathbf{c}_k - A)^{-1} e_{(j+1)(g+1)+k}\rangle & k \neq k_\infty, \\ \langle e_{j(g+1)+k}, A e_{(j+1)(g+1)+k}\rangle & k = k_\infty. \end{cases} \tag{3.2}$$

Namely $\beta_j = \Lambda_{j(g+1)+k_\infty}$, and the coefficients $\Lambda_{j(g+1)+\ell}$ for $k \neq k_\infty$ instead occur as outermost diagonal coefficients for the GMP matrix $(\mathbf{c}_k - A)^{-1}$. In our later applications to the discriminant of A, both the coefficients of A and of its resolvents will appear, so we will work with Λ_n throughout.

Next, we connect the coefficients (3.2) to the solutions of the L^2-extremal problem (1.5).

Lemma 3.1. *For all n,*

$$\frac{\kappa_n}{\kappa_{n+g+1}} = \Lambda_n. \tag{3.3}$$

Proof. Let $n = j(g+1)+k$. By self-adjointness,

$$\Lambda_n = \langle e_n, r_k(A) e_{n+g+1}\rangle = \langle r_k \tau_n, \tau_{n+g+1}\rangle = \langle \kappa_n r_{n+g+1} + h, \tau_{n+g+1}\rangle$$

for some $h \in \mathcal{L}_{n+g}$. By orthogonality, $\langle \tau_{n+g+1}, h\rangle = 0$, so $\langle \tau_{n+g+1}, r_{n+g+1}\rangle = \frac{1}{\kappa_{n+g+1}}$ implies that

$$\Lambda_n = \langle \tau_{n+g+1}, \kappa_n r_{n+g+1} + h\rangle = \frac{\kappa_n}{\kappa_{n+g+1}}. \qquad \square$$

We now adapt to GMP matrices ideas from the theory of regularity for Jacobi matrices [22].

Lemma 3.2. *Let $A \in \mathbb{A}(\mathbf{C})$. For all $j \geq 1$, $\|\vec{p}_j\| \leq \|A\|$.*

Proof. Fix $j \geq 1$ and denote $n = j(g+1) + k_\infty$. For any $\ell = 0, \dots, g$,

$$(p_j)_\ell = \langle e_{n-g-1+\ell}, Ae_n\rangle = \int \tau_{n-g-1+\ell}(x) x \tau_n(x) d\mu(x).$$

Since the vectors $\tau_{n-g-1+\ell}$ are orthonormal, by the Bessel inequality,

$$\|\vec{p}_j\|^2 \leq \int |x\tau_n(x)|^2 d\mu(x) \leq \|A\|^2 \int |\tau_n(x)|^2 d\mu(x) = \|A\|^2$$

since $\|A\| = \sup_{x \in \operatorname{supp}\mu} |x|$. $\square$

Lemma 3.3. *For $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log |\tau_n(z)| > 0. \tag{3.4}$$

Proof. We adapt the proof of [22, Proposition 2.2]. It suffices to prove (3.4) along the subsequences $n(j) = j(g+1) + k$, $j \to \infty$, for $1 \leq k \leq g+1$. Moreover, due to $\overline{\mathbb{R}}$-preserving conformal invariance, it suffices to fix k and prove

$$\liminf_{j\to\infty} \frac{1}{n(j)} \log |\tau_{n(j)}(z)| > 0 \tag{3.5}$$

under the assumption that $\mathbf{c}_k = \infty$. This allows us to use the associated GMP matrix $A \in \mathbb{A}(\mathbf{C})$.

Note that for any m, since $\{\tau_\ell\}_{\ell=0}^\infty$ is an orthonormal basis of $L^2(d\mu)$,

$$\sum_\ell A_{m\ell} \tau_\ell(z) = \sum_\ell \langle z\tau_m(z), \tau_\ell(z)\rangle \tau_\ell(z) = z\tau_m(z).$$

This equality holds in $L^2(d\mu)$, but since all functions are rational, it also holds pointwise. Thus, if we fix $z \in \mathbb{C} \setminus \mathbb{R}$, the sequence $\vec{\varphi} = \{\tau_\ell(z)\}_{\ell=0}^\infty$ is a formal eigensolution for A at energy z, i.e. $(A - z)\vec{\varphi} = 0$ componentwise. Since A is represented as a block tridiagonal matrix, let us also write $\vec{\varphi}$ in a matching block form, as $\vec{\varphi}^\top = \begin{bmatrix}\vec{u}_0^\top & \vec{u}_1^\top & \vec{u}_2^\top & \dots\end{bmatrix}$ where

$$\vec{u}_0^\top = \begin{bmatrix}\tau_0(z) & \dots & \tau_{k-1}(z)\end{bmatrix}, \quad \vec{u}_j^\top = \begin{bmatrix}\tau_{n(j-1)-1}(z) & \dots & \tau_{n(j)-1}(z)\end{bmatrix}, \quad j \geq 1.$$

We also consider the projection of $\vec{\varphi}$ onto the first $j+1$ blocks,

$$\vec{\varphi}_j^\top = \begin{bmatrix}\vec{u}_0^\top & \dots & \vec{u}_j^\top & 0 & \dots\end{bmatrix},$$

and compute $(A - z)\vec{\varphi}_j$. By the block tridiagonal structure of A, for $m < n(j-1)$ we have $\langle e_m, (A-z)\vec{\varphi}_j\rangle = 0$. For $0 \leq \ell \leq g$, we have

$$\langle e_{n(j-1)+\ell}, (A-z)\vec{\varphi}\rangle - \langle e_{n(j-1)+\ell}, (A-z)\vec{\varphi}_j\rangle = (p_j)_\ell \tau_{n(j)}(z)$$

so that $\langle e_{n(j-1)+\ell}, (A-z)\vec{\varphi}_j\rangle = -(p_j)_\ell \tau_{n(j)}(z)$. Moreover,

$$\langle e_{n(j)}, (A-z)\vec{\varphi}_j\rangle = \langle e_{n(j)}, A\vec{\varphi}_j\rangle = (\vec{p}_j)^* u_j(z).$$

For $m > n(j)$, we again have $\langle e_m, (A-z)\vec{\varphi}_j\rangle = 0$. In conclusion, $(A-z)\vec{\varphi}_j$ has only two nontrivial blocks,

$$((A-z)\vec{\varphi}_j)^\top = \begin{bmatrix} 0 & \dots & 0 & -(\vec{p}_j \tau_{n(j)}(z))^\top & ((\vec{p}_j)^* u_j)^\top & 0 & \dots \end{bmatrix}.$$

In particular, we can compute

$$\langle \vec{\varphi}_j, (A-z)\vec{\varphi}_j\rangle = -\vec{u}_j^* \tau_{n(j)}(z)\vec{p}_j. \tag{3.6}$$

Since A is self-adjoint and $\vec{\varphi}_j \in \ell^2(\mathbb{N}_0)$, by a standard consequence of the spectral theorem [29, Lemma 2.7.],

$$|\operatorname{Im} z| \|\vec{\varphi}_j\|^2 \le |\langle \vec{\varphi}_j, (A-z)\vec{\varphi}_j\rangle|.$$

Using (3.6) and the Cauchy–Schwarz inequality gives

$$|\operatorname{Im} z| \sum_{m=0}^{j} \|\vec{u}_m\|^2 \le |\tau_{n(j)}(z)| \|\vec{p}_j\| \|\vec{u}_j\|.$$

By Lemma 3.2, with $C = |\operatorname{Im} z|/\|A\|$,

$$C \sum_{m=0}^{j} \|\vec{u}_m\|^2 \le |\tau_{n(j)}(z)| \|\vec{u}_j\|. \tag{3.7}$$

Applying the AM-GM inequality to the right-hand side of (3.7) gives

$$|\tau_{n(j)}(z)| \|\vec{u}_j(z)\| \le \frac{1}{2}\left(C\|\vec{u}_j(z)\|^2 + C^{-1}|\tau_{n(j)}(z)|^2\right)$$

which together with (3.7) implies

$$|\tau_{n(j)}(z)|^2 \ge C^2 \sum_{m=0}^{j} \|\vec{u}_m\|^2. \tag{3.8}$$

Since $|\tau_{n(j)}(z)|^2 \le \|\vec{u}_{j+1}\|^2$, this implies that

$$\sum_{m=0}^{j+1} \|\vec{u}_m\|^2 \ge \left(1 + C^2\right) \sum_{m=0}^{j} \|\vec{u}_m\|^2.$$

Since $\|\vec{u}_0\| \ge |\tau_0(z)| = 1$, this implies by induction that

$$\sum_{m=0}^{j} \|\vec{u}_m\|^2 \ge \left(1 + C^2\right)^j.$$

Combining this with (3.8) gives a lower bound on $|\tau_{n(j)}(z)|$ which implies (3.5). □

The estimates in the previous proof also lead to the following:

Corollary 3.4. *For any $z \in \mathbb{C} \setminus \mathbb{R}$, the quantities*

$$\liminf_{j\to\infty} \frac{1}{j(g+1)+k} \log|\tau_{j(g+1)+k}(z)|, \qquad \limsup_{j\to\infty} \frac{1}{j(g+1)+k} \log|\tau_{j(g+1)+k}(z)|$$

are independent of $k \in \{1, \dots, g+1\}$.

Proof. Assume $j \geq 1$. For $k-g-1 \leq \ell \leq k-1$, the estimate (3.8) gives

$$|\tau_{j(g+1)+k}(z)|^2 \geq C^2 \|\vec{u}_j\|^2 \geq C^2 |\tau_{j(g+1)+\ell}(z)|^2$$

which implies

$$\liminf_{j\to\infty} \frac{1}{j(g+1)+k} \log|\tau_{j(g+1)+k}(z)| \geq \liminf_{j\to\infty} \frac{1}{j(g+1)+\ell} \log|\tau_{j(g+1)+\ell}(z)| \tag{3.9}$$

and

$$\limsup_{j\to\infty} \frac{1}{j(g+1)+k} \log|\tau_{j(g+1)+k}(z)| \geq \limsup_{j\to\infty} \frac{1}{j(g+1)+\ell} \log|\tau_{j(g+1)+\ell}(z)|. \tag{3.10}$$

Clearly, the right-hand sides don't change if ℓ is shifted by $g+1$, so (3.9), (3.10) hold for all $k, \ell \in \{1, \dots, g+1\}$ with $k \neq \ell$. By symmetry, since the roles of k, ℓ can be switched, we conclude that equality holds in (3.9), (3.10). □

4. Growth rates of orthonormal rational functions

In this section, we will combine the positivity (3.4) with potential theory techniques in order to study exponential growth rates of orthonormal rational functions. Our main conclusions will be conformally invariant, but our proofs will use potential theory arguments and objects such as the logarithmic potential of a finite measure ν,

$$\Phi_\nu(z) = \int \log|z-x| d\nu(x),$$

which is well defined when $\operatorname{supp}\nu$ does not contain ∞.

Theorem 4.1. *Fix $1 \leq k \leq g+1$ and denote by I the connected component of $\overline{\mathbb{R}} \setminus \operatorname{supp}\mu$ containing $\mathbf{c}_k$. Suppose there is a subsequence $n_\ell = j_\ell(g+1)+k$ such that $\text{w-lim}_{\ell\to\infty} \nu_{n_\ell} = \nu$ and $\frac{1}{n_\ell}\log\kappa_{n_\ell} \to \alpha \in \mathbb{R}\cup\{-\infty,+\infty\}$ as $\ell\to\infty$. Then uniformly on compact subsets of $(\overline{\mathbb{C}}\setminus\overline{\mathbb{R}})\cup(I\setminus\{\mathbf{c}_k\})$, we have*

$$h(z) := \lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)|.$$

The function h is determined by ν and α; in particular, if $\mathbf{c}_k = \infty$,

$$h(z) = \alpha + \Phi_\nu(z) - \frac{1}{g+1} \sum_{\substack{m=1 \\ m\neq k}}^{g+1} \log|\mathbf{c}_m - z|. \tag{4.1}$$

Moreover,

(a) *$\alpha = -\infty$ is impossible;*
(b) *If $\alpha = +\infty$, the limit is $h = +\infty$;*
(c) *If $\alpha \in \mathbb{R}$, the limit h extends to a positive harmonic function on $\overline{\mathbb{C}} \setminus (\mathsf{E} \cup \{\mathbf{c}_1, \dots, \mathbf{c}_{g+1}\})$ such that*

$$h(z) = -\frac{1}{g+1}\log|\mathbf{c}_m - z| + O(1), \quad z \to \mathbf{c}_m \neq \infty, \tag{4.2}$$

$$h(z) = \frac{1}{g+1}\log|z| + O(1), \quad z \to \mathbf{c}_m = \infty. \tag{4.3}$$

Proof. By using $\overline{\mathbb{R}}$-preserving conformal invariance, we can assume without loss of generality that $\mathbf{c}_k = \infty$. We will use the representation (2.3) of the subspace $\mathcal{L}_n$. For $n = j(g+1)+k$, counting degrees of the poles leads to

$$\tau_n = \frac{P_n}{R_n}, \qquad R_n(z) = \prod_{m=1}^{k-1} (\mathbf{c}_m - z) \prod_{\substack{m=1 \\ m\neq k}}^{g+1} (\mathbf{c}_m - z)^j,$$

with $\deg P_n = n$. This may not be the minimal representation of τ_n, but by the proof of Lemma 2.2, the only possible cancellations are simple factors $(\mathbf{c}_m - z)$ for each $m \neq k$, so we get the minimal representation $\tau_n(z) = P(z)/Q(z)$ with

$$P(z) = \kappa_n \prod_{w:\tau_n(w)=0} (z-w), \quad Q(z) = \prod_{\substack{m=1 \\ m\neq k}}^{g+1} (\mathbf{c}_m - z)^{j+\delta_{m,j}},$$

where $|\delta_{m,j}| \leq 1$ for each j. All that matters is that $\delta_{m,j}/j \to 0$ as $j \to \infty$. It will be useful to turn this rational function representation into a kind of Riesz representation,

$$\log|\tau_n(z)| = \log \kappa_n + n \int \log|x-z| d\nu_n(x) - \sum_{\substack{1\leq m\leq g+1 \\ m\neq k}} (j+\delta_{m,j}) \log|\mathbf{c}_m - z|. \tag{4.4}$$

Since $\mathbf{c}_k = \infty$, note that $K = \overline{\mathbb{R}} \setminus I$ is a compact subset of $\mathbb{R}$. Denote $\Omega = \mathbb{C} \setminus K$. For any $z \in \Omega$, the map $x \mapsto \log|x-z|$ is continuous on K, so $\Phi_{\nu_{n_\ell}}(z) \to \Phi_\nu(z)$ as $\ell \to \infty$. In fact, convergence is uniform on compact subsets of Ω: since $\operatorname{supp}(\nu_{n_\ell}) \subset K$ and $\nu_{n_\ell}(K) \leq 1$ for all ℓ, the estimate

$$\log\left|\frac{x-z_1}{x-z_2}\right| \leq \log\left(1 + \frac{|z_1-z_2|}{\operatorname{dist}(z_2,K)}\right) \leq \frac{|z_1-z_2|}{\operatorname{dist}(z_2,K)}, \qquad z_1, z_2 \in \Omega,$$

implies uniform equicontinuity of the potentials Φ_{n_ℓ} on compact subsets of Ω, and the Arzelà–Ascoli theorem implies uniform convergence on compacts.

Note that (b) follows from (4.1). By Corollary 2.3, $\operatorname{supp}\nu \subset \mathsf{E}$ and $\Phi_\nu(z)$ is harmonic on $\mathbb{C} \setminus \mathsf{E}$, so the right hand side extends to a harmonic function on $\mathbb{C} \setminus (\mathsf{E} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_{g+1}\})$ and we denote this extension also by h. By Lemma 3.3, h is positive on $\mathbb{C}_+ \cup \mathbb{C}_-$, so $\alpha \neq -\infty$; moreover, by the mean value property, h is positive on $\mathbb{C} \setminus (\mathsf{E} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_{g+1}\})$.

The remaining asymptotic properties follow from (4.1). Under the assumption $\mathbf{c}_k = \infty$, $\operatorname{supp}\nu$ is a compact subset of $\mathbb{R}$, and $\Phi_\nu(z) = \log|z| + O(1)$, $z \to \infty$. It then follows that $h(z) = \frac{1}{g+1}\log|z| + O(1)$ as $z \to \infty$. Of course, $h(z) = -\frac{1}{g+1}\log|z-\mathbf{c}_m| + O(1)$ near each $\mathbf{c}_m \neq \mathbf{c}_k$. □

Theorem 4.1 motivates interest in positive harmonic functions on $\overline{\mathbb{C}} \setminus (\mathsf{E} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_{g+1}\})$. If E is polar, by Myrberg's theorem [2, Theorem 5.3.8], any such function is constant. If E is not polar, knowing the asymptotic behavior of h at the poles, positivity of h improves to the following lower bound on h. Lemma 4.2 reflects a standard minimality property of the Green function [11, Section VII.10].

Lemma 4.2. *Assume that E is a nonpolar closed subset of $\overline{\mathbb{R}}$. Let h be a positive superharmonic function on $\overline{\mathbb{C}} \setminus (\mathsf{E} \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_{g+1}\})$. Suppose $h(z) + \frac{1}{g+1}\log|z-\mathbf{c}_k|$*

has an existent limit at $\mathbf{c}_k$ *for each finite* $\mathbf{c}_k$*, and* $h(z) - \frac{1}{g+1}\log|z|$ *has an existent limit at* ∞ *if one of the* $\mathbf{c}_k = \infty$*. Then*

$$h(z) \geq \mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) \tag{4.5}$$

for $z \in \overline{\mathbb{C}} \setminus \mathsf{E}$*. For* $1 \leq k \leq g+1$*, define*

$$\alpha_k = \begin{cases} \lim_{z\to\mathbf{c}_k}(h(z) + \frac{1}{g+1}\log|z-\mathbf{c}_k|), & \mathbf{c}_k \neq \infty, \\ \lim_{z\to\infty}(h(z) - \frac{1}{g+1}\log|z|), & \mathbf{c}_k = \infty. \end{cases}$$

Then

$$\alpha_k \geq \frac{\log\lambda_k}{g+1}. \tag{4.6}$$

Proof. We will use a stronger, q.e. version of the maximum principle [20, Thm 3.6.9]. Define

$$\tilde{h}(z) := \mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) - h(z),$$

which is bounded at $\mathbf{c}_k$ for $1 \leq k \leq g+1$ and so extends to a subharmonic function on $\overline{\mathbb{C}} \setminus \mathsf{E}$. Since $\mathcal{G}_{\mathsf{E}}$ vanishes q.e. on E, we have for q.e. $t \in \mathsf{E}$,

$$\limsup_{z\to t} \tilde{h}(z) = -\liminf_{z\to t} h(z) \leq 0.$$

Now we show $\tilde{h}$ is bounded above on $\overline{\mathbb{C}}\setminus\mathsf{E}$. Let $\mathcal{U}$ be a union of small neighborhoods containing the points $\mathbf{c}_k$ in $\overline{\mathbb{C}}\setminus\mathsf{E}$. By the definition of the Green function, $\mathcal{G}_{\mathsf{E}}(z, \mathbf{C})$ defines a harmonic and bounded function on $\overline{\mathbb{C}} \setminus (\mathsf{E}\cup\mathcal{U})$. That is, there exists M such that for all $z \in \overline{\mathbb{C}} \setminus (\mathcal{U}\cup\mathsf{E})$ we have

$$\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) \leq M.$$

Since $h \geq 0$, it follows on $\overline{\mathbb{C}} \setminus (\mathcal{U}\cup\mathsf{E})$ that

$$\tilde{h}(z) = \mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) - h(z) \leq \mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) \leq M.$$

On the other hand, by properties of the Green functions we have

$$\frac{\log\lambda_k}{g+1} = \begin{cases} \lim_{z\to\mathbf{c}_k}(\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) + \frac{1}{g+1}\log|z-\mathbf{c}_k|), & \mathbf{c}_k \neq \infty, \\ \lim_{z\to\infty}(\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) - \frac{1}{g+1}\log|z|), & \mathbf{c}_k = \infty. \end{cases}$$

Then, by assumption, for $1 \leq k \leq g+1$, $\tilde{h}(z) = \frac{\log\lambda_k}{g+1} - \alpha_k + o(1)$ as $z \to \mathbf{c}_k$ and, in particular, the difference is bounded in a small neighborhood of $\mathbf{c}_k$. Thus, $\tilde{h}$ is bounded above on $\overline{\mathbb{C}} \setminus \mathsf{E}$.

So, by the maximum principle $\tilde{h} \leq 0 \implies \mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) \leq h(z)$ on $\overline{\mathbb{C}} \setminus \mathsf{E}$. Since $0 \geq \lim_{z\to\mathbf{c}_k} \tilde{h}(z) = \frac{\log\lambda_k}{g+1} - \alpha_k$, we have (4.6). □

Lemma 4.3. *Under the same assumptions as Lemma 4.2, the following are equivalent:*

(i) *Equality in (4.6) for all* k *with* $1 \leq k \leq g+1$
(ii) *Equality in (4.6) for a single* k *with* $1 \leq k \leq g+1$
(iii) *Equality holds in (4.5)*

Proof. (i) $\implies$ (ii) is trivial. Suppose then (ii); with the notation of Lemma 4.2, by assumption, $\tilde{h}(\mathbf{c}_k) = 0$ and $\tilde{h}$ achieves a global maximum. By the maximum principle for subharmonic functions [20, Theorem 2.3.1], $\tilde{h} \equiv 0$ on $\overline{\mathbb{C}} \setminus \mathsf{E}$. Finally, if (iii) holds, then evaluating $\tilde{h}(\mathbf{c}_k)$ for each $1 \leq k \leq g+1$ yields (i). □

We will now prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Using conformal invariance, we take $\mathbf{c}_k = \infty$. Fix $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{R}}$ and select a sequence $(n_\ell)_{\ell=1}^{\infty}$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log|\tau_n(z)| = \lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)|.$$

By precompactness of the (ν_n), we may pass to a further subsequence, which we denote again by $(n_\ell)_{\ell=1}^{\infty}$, so that w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$ and $\frac{1}{n_\ell} \log \kappa_{n_\ell} \to \alpha$ for some ν and α. Then for h as in Theorem 4.1,

$$\lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)| = h(z),$$

on $\mathbb{C} \setminus \mathbb{R}$. If $\alpha = +\infty$, then there is nothing to show. Suppose $\alpha < \infty$. If E is not polar we apply (a) of Theorem 4.1 to find $\alpha \in \mathbb{R}$, and we may use (c) of the same theorem and Lemma 4.2 to conclude.

If instead E is polar, by Myrberg's theorem, h is constant on $\mathbb{C}\setminus(\mathsf{E}\cup\{\mathbf{c}_1, \ldots, \mathbf{c}_{g+1}\})$. Computing the limit at $\mathbf{c}_k$ we see $h \equiv +\infty$. In particular, $\liminf_{n\to\infty} \frac{1}{n} \log|\tau_n(z)| = +\infty$ for $z \in \mathbb{C} \setminus \mathbb{R}$. □

Proof of Theorem 1.3. Fix $1 \le k \le g+1$ and assume again by conformal invariance that $\mathbf{c}_k = \infty$. Using precompactness of the measures (ν_n), we find a subsequence $n_\ell = j_\ell(g+1) + k$ with

$$\lim_{\ell\to\infty} \frac{1}{n_\ell} \log \kappa_{n_\ell} = \liminf_{j\to\infty} \frac{1}{n(j)} \log \kappa_{n(j)} =: \alpha$$

and w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$. If $\alpha = +\infty$, we are done. Suppose then $\alpha < \infty$, then we have by Theorem 4.1(a), $\alpha \in \mathbb{R}$. Furthermore, if E is nonpolar, by (c) and Lemma 4.2, $h(z) \ge \mathcal{G}_{\mathsf{E}}(z, \mathbf{C})$ on $\overline{\mathbb{C}} \setminus \mathsf{E}$. In particular, by the representation (4.1) we see that $\alpha = \lim_{z\to\infty}(h(z) - \frac{1}{g+1} \log|z|)$, and so (4.6) yields the desired inequality.

If instead E is polar, by Theorem 1.2, for each $z \in \mathbb{C} \setminus \mathbb{R}$,

$$h(z) = \lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}| \ge \liminf_{n\to\infty} \frac{1}{n} \log|\tau_n(z)| = +\infty,$$

and so by Theorem 4.1(b), $\alpha = +\infty$. □

5. Regularity

We will begin by proving a version of Theorem 1.4 for a fixed k.

Lemma 5.1. *Fix $k \in \{1, \ldots, g+1\}$. Along the subsequence $n(j) = j(g+1) + k$, the following are equivalent:*

(i) $\lim_{j\to\infty} \kappa_{n(j)}^{1/n(j)} = \lambda_k^{1/(g+1)}$;
(ii) *For q.e. $z \in \mathsf{E}$, we have* $\limsup_{j\to\infty} |\tau_{n(j)}(z)|^{1/n(j)} \le 1$;
(iii) *For some $z \in \mathbb{C}_+$,* $\limsup_{j\to\infty} |\tau_{n(j)}(z)|^{1/n(j)} \le e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;
(iv) *For all $z \in \mathbb{C}$,* $\limsup_{j\to\infty} |\tau_{n(j)}(z)|^{1/n(j)} \le e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;
(v) *Uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$,* $\lim_{j\to\infty} |\tau_{n(j)}(z)|^{1/n(j)} = e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$.

Proof. Using conformal invariance, we will assume throughout the proof that $\mathbf{c}_k = \infty$. First, suppose that E is polar. In this case (ii) is vacuous, and since $\mathcal{G}_{\mathsf{E}} \equiv +\infty$, (iii) and (iv) are trivially true. Since $\lambda_k = +\infty$, (i) follows from Theorem 1.3. As in the proof of Theorem 4.1, weak convergence of measures implies uniform

on compacts convergence of their potentials. Thus, since ν_n are a precompact family, so are Φ_{ν_n}. Thus, the convergence $\lim_{j\to\infty} \frac{1}{n(j)} \log \kappa_{n(j)} = +\infty$ implies that $\lim_{j\to\infty} \frac{1}{n(j)} \log|\tau_{n(j)}(z)| = +\infty$ uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$, so (v) holds.

For the remainder of the proof, we will assume E is not polar. Moreover, we will repeatedly use the fact that if any subsequence of a sequence in a topological space has a further subsequence which converges to a limit, then the sequence itself converges to this limit. In particular, when concluding (v), we apply this fact in the Fréchet space of harmonic functions on $\mathbb{C} \setminus \mathbb{R}$ with the topology of uniform convergence on compact sets.

(iii) $\implies$ (v): Given a subsequence of $n(j) = j(g+1)+k$, using precompactness of the measures ν_n, we pass to a further subsequence $n_\ell = j_\ell(g+1)+k$ with w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$ and $\lim_{\ell\to\infty} \frac{1}{n_\ell} \log \kappa_{n_\ell} =: \alpha$, with α real or infinite. By Theorem 4.1, uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$,

$$h(z) = \lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)|$$

with h given by (4.1). Using the assumption, for some $z_0 \in \mathbb{C}_+$, we have

$$h(z_0) \le \limsup_{j\to\infty} \frac{1}{n(j)} \log|\tau_{n(j)}(z_0)| < \infty.$$

So, by Theorem 4.1, $\alpha \in \mathbb{R}$ and h has a harmonic extension to $\overline{\mathbb{C}}\setminus(\mathsf{E}\cup\{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\})$. Furthermore, by Lemma 4.2, $h \ge \mathcal{G}_{\mathsf{E}}$. By assumption, we have the opposite inequality at $z_0 \in \mathbb{C}_+$, and so, by the maximum principle for harmonic functions, $h = \mathcal{G}_{\mathsf{E}}$ on $\mathbb{C} \setminus (\mathsf{E} \cup \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\})$, and in particular on $\mathbb{C} \setminus \mathbb{R}$. Thus, we have (v).

(v) $\implies$ (iv): For $z \in \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\}$, $\mathcal{G}_{\mathsf{E}}(z,\mathbf{C}) = +\infty$ and there is nothing to show. Fix $z \in \mathbb{C} \setminus \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\}$ and let $n_\ell = j_\ell(g+1)+k$ be a subsequence with $\lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)| = \limsup_{j\to\infty} \frac{1}{n(j)} \log|\tau_{n(j)}(z)|$. By passing to a further subsequence, we may assume w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$, and $\lim_{\ell\to\infty} \frac{1}{n_\ell} \log \kappa_{n_\ell} =: \alpha$ where α is real or infinite. By the assumption, we have $h = \lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}| = \mathcal{G}_{\mathsf{E}}$ on $\mathbb{C} \setminus \mathbb{R}$. So, by (a) and (b), $\alpha \in \mathbb{R}$ and h extends to a harmonic function on $\overline{\mathbb{C}} \setminus (\mathsf{E} \cup \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\})$. By the representation (4.1), we may extend h subharmonically to $\overline{\mathbb{C}} \setminus \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\}$. On this set, $\mathcal{G}_{\mathsf{E}}$ is also subharmonic, so, by the weak identity principle [20, Theorem 2.7.5], $h = \mathcal{G}_{\mathsf{E}}$ on $\overline{\mathbb{C}} \setminus \{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\}$. Thus, by the principle of descent [27, A.III], we have

$$\lim_{\ell\to\infty} \frac{1}{n_\ell} \log|\tau_{n_\ell}(z)| \le h(z) = \mathcal{G}_{\mathsf{E}}(z,\mathbf{C}) \tag{5.1}$$

and (iv) follows.

(v) $\implies$ (i): Given a subsequence of $n(j) = j(g+1)+k$, we use precompactness of the ν_n to pass to a further subsequence $n_\ell = j_\ell(g+1)+k$ with $\lim_{\ell\to\infty} \frac{1}{n_\ell} \log \kappa_{n_\ell} =: \alpha \in \mathbb{R} \cup \{-\infty,+\infty\}$ and w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$. Then in the notation of Theorem 4.1 and by assumption, for a $z \in \mathbb{C} \setminus \mathbb{R}$

$$\lim_{\ell\to\infty} \log|\tau_{n_\ell}(z)| = h(z) = \mathcal{G}_{\mathsf{E}}(z,\mathbf{C}).$$

So by Lemma 4.3, $\alpha = \frac{\log \lambda_k}{g+1}$. Thus, $\lambda_k^{1/(g+1)}$ is the only accumulation point of $\kappa_{n(j)}^{1/n(j)}$ in $\mathbb{R} \cup \{-\infty,+\infty\}$ and we have (i).

(i) $\implies$ (v): As before, we fix a subsequence of $n(j) = j(g+1)+k$ and use precompactness to pass to a further subsequence $n_\ell = j_\ell(g+1)+k$ with w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$.

Then, by Theorem 4.1 and in the notation introduced there, uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$,

$$\lim_{\ell\to\infty}\frac{1}{n_\ell}\log|\tau_{n_\ell}(z)| = h(z),$$

where h is given by (4.1) with $\alpha = \frac{\log\lambda_k}{g+1}$. Thus, by Lemma 4.3(ii), $h(z) = \mathcal{G}_{\mathsf{E}}(z,\mathbf{C})$ on $\mathbb{C}\setminus\mathbb{R}$. Since the initial subsequence was arbitrary, we have (v).

(iv) $\Longrightarrow$ (ii): Recalling that the Green function vanishes q.e. on E, the claim follows.

(ii) $\Longrightarrow$ (v): Fixing a subsequence of $n(j)$, we again use precompactness to select a further subsequence $n_\ell = j_\ell(g+1)+k$ such that w-$\lim_{\ell\to\infty}\nu_{n_\ell} = \nu$ and $\lim_{\ell\to\infty}\frac{1}{n_\ell}\log\kappa_{n_\ell} =: \alpha$, $\alpha\in\mathbb{R}\cup\{-\infty,+\infty\}$. By the upper envelope theorem, there is a polar set $X_1\subset\mathbb{C}$ such that on $\mathbb{C}\setminus X_1$, $\limsup_{\ell\to\infty}\Phi_{\nu_{n_\ell}} = \Phi_\nu$. Now, we let $X_2 := \{t\in\mathsf{E} : \limsup_{n\to\infty}\frac{1}{n}\log|\tau_n(t)|>0\}$, which is polar by assumption, and $X_3 := \{z\in\mathbb{C} : \Phi_\infty(z) = -\infty\}$, which is polar by [20, Theorem 3.5.1]. Then, for a $t\in\mathsf{E}\setminus(X_1\cup X_2\cup X_3)$, we have

$$\alpha \le \limsup_{n\to\infty}\frac{1}{n}\log|\tau_n(t)| - \Phi_\nu(t) + \frac{1}{g+1}\sum_{\substack{m=1\\ m\neq k}}^{g+1}\log|\mathbf{c}_m - t| < \infty.$$

So $\alpha\in\mathbb{R}$ by Theorem 4.1(a). Thus, by (c) of the same theorem, uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$

$$h(z) = \lim_{\ell\to\infty}\frac{1}{n_\ell}\log|\tau_{n_\ell}(z)|$$

and h extends to a positive harmonic function on $\overline{\mathbb{C}}\setminus(\mathsf{E}\cup\{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\})$ with logarithmic poles at each of the $\mathbf{c}_m$. So, $h-\mathcal{G}_{\mathsf{E}}$ extends to a harmonic function on $\overline{\mathbb{C}}\setminus\mathsf{E}$, and $h-\mathcal{G}_{\mathsf{E}}\ge 0$ there by Lemma 4.2. We now show that in fact $h=\mathcal{G}_{\mathsf{E}}$ using the stronger, q.e. maximum principle.

We use the equality in (4.1) to extend h to a subharmonic function on $\mathbb{C}\setminus\{\mathbf{c}_1,\dots,\mathbf{c}_{g+1}\}$. By the upper envelope theorem and the assumption again, for $t\in\mathsf{E}\setminus(X_1\cup X_2)$

$$h(t) = \limsup_{\ell\to\infty}\frac{1}{n_\ell}\log|\tau_{n_\ell}(t)|\le 0.$$

Then, for these t, since $\mathcal{G}_{\mathsf{E}}$ is positive, we have

$$\limsup_{\substack{z\to t\\ z\in\mathbb{C}\setminus\mathsf{E}}}(h(z)-\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})) \le \limsup_{\substack{z\to t\\ z\in\mathbb{C}\setminus\mathsf{E}}} h(z)\le h(t)\le 0$$

by upper semicontinuity. So, $\limsup_{\substack{z\to t\\ z\in\mathbb{C}\setminus\mathsf{E}}}(h(z)-\mathcal{G}_{\mathsf{E}}(z,\mathbf{C}))\le 0$ for q.e. $t\in\mathsf{E}$.

Since h is upper semicontinuous on the compact set E, there is an M so that $\sup_{t\in\mathsf{E}}h(t)\le M$. As in the above, now for any $t\in\mathsf{E}$, we have

$$\limsup_{\substack{z\to t\\ z\in\mathbb{C}\setminus\mathsf{E}}}(h(z)-\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})) \le \limsup_{\substack{z\to t\\ z\in\mathbb{C}\setminus\mathsf{E}}} h(z)\le h(t)\le M.$$

So, there is a neighborhood $\mathcal{U}$ of E with $\sup_{z\in\mathcal{U}\cap(\overline{\mathbb{C}}\setminus\mathsf{E})}(h-\mathcal{G}_{\mathsf{E}})\le M+1$. Since the difference is harmonic on $\overline{\mathbb{C}}\setminus\mathcal{U}$, we conclude that $\sup_{z\in\overline{\mathbb{C}}\setminus\mathsf{E}}(h(z)-\mathcal{G}_{\mathsf{E}}(z,\mathbf{C}))<\infty$. Thus, by the maximum principle and the reverse inequality, $h=\mathcal{G}_{\mathsf{E}}$ on $\overline{\mathbb{C}}\setminus\mathsf{E}$. Since the first sequence was arbitrary, we have (v).

Since the implication (iv) $\Longrightarrow$ (iii) is clear, we may conclude. □

We now put the subsequences together and use Corollary 3.4 to show that regular behavior occurs for one k if and only if it happens for all.

Proof of Theorem 1.4. Applying Lemma 5.1 for all k implies equivalence of conditions (ii), (iv), (v), (vi), (vii) from Theorem 1.4. By Corollary 3.4, for some $z \in \mathbb{C}_+$, the condition

$$\limsup_{j\to\infty} \frac{1}{j(g+1)+k} \log|\tau_{j(g+1)+k}(z)| \le \mathcal{G}_{\mathsf{E}}(z, \mathbf{C})$$

holds for one value of k if and only if it holds for all. Due to Lemma 5.1, this immediately implies equivalence of conditions (i) and (iii) from Theorem 1.4. It remains to prove equivalence of (ii), (iii).

(ii) $\implies$ (iii): For $n \in \mathbb{N}$ and $1 \le k \le g+1$, denote by $N(n,k)$ the integer such that $n+1 \le N(n,k) \le n+g+1$ and $N(n,k)-k$ is divisible by $g+1$. Then $N(n,k)/n \to 1$ as $n \to \infty$ so (ii) implies $\lim_{n\to\infty} \kappa_{N(n,k)}^{1/n} = \lambda_k^{1/(g+1)}$. Taking the product over $k = 1, \dots, g+1$ gives (iii).

(iii) $\implies$ (ii): Similarly to the above, Theorem 1.3 shows that for all k,

$$\liminf_{n\to\infty} \kappa_{N(n,k)}^{1/n} \ge \lambda_k^{1/(g+1)}. \tag{5.2}$$

Thus, if (ii) was false, this would mean that for some $k = m$, $\limsup_{n\to\infty} \kappa_{N(n,m)}^{1/n} > \lambda_m^{1/(g+1)}$. Taking products over k, we would have

$$\begin{aligned}\limsup_{n\to\infty} \left(\prod_{k=1}^{g+1} \kappa_{N(n,k)}\right)^{1/n} &\ge \limsup_{n\to\infty} \kappa_{N(n,m)}^{1/n} \liminf_{n\to\infty} \left(\prod_{\substack{1\le k\le g+1\\ k\neq m}} \kappa_{N(n,k)}\right)^{1/n} \\ &> \left(\prod_{k=1}^{g+1} \lambda_k\right)^{1/(g+1)}\end{aligned}$$

(the last step again uses (5.2) for all $k \neq m$). This would contradict (iii), so the proof is complete. □

We now prove a seemingly special case of Corollary 1.7.

Proposition 5.2. *Assume that the sequence* $\mathbf{C}$ *contains* ∞*. Then* μ *is Stahl–Totik regular if and only if it is* $\mathbf{C}$*-regular.*

Proof. Let us assume that μ is $\mathbf{C}$-regular and let p_n denote the orthonormal polynomial with respect to μ. Fix $z \in \mathbb{C}$. Since ∞ is in $\mathbf{C}$, $p_n \in \mathcal{L}_{n(g+1)}$, so the orthonormal polynomials can be expressed on the basis of orthonormal rational functions as

$$p_n(z) = \sum_{m=0}^{n(g+1)} c_m \tau_m(z), \qquad \sum_{m=0}^{n(g+1)} |c_m|^2 = 1.$$

Thus, in particular, $|c_\ell| \le 1$ and we get

$$|p_n(z)| \le (1 + n(g+1)) \sup_{0\le m\le n(g+1)} |\tau_m(z)|. \tag{5.3}$$

By Theorem 1.4, for q.e. $z \in \mathsf{E}$,

$$\limsup_{\ell\to\infty} \frac{1}{\ell} \log|\tau_\ell(z)| \le 0. \tag{5.4}$$

Thus, for q.e. $z \in \mathsf{E}$, (5.3) implies

$$\limsup_{n\to\infty} \frac{1}{n} \log |p_n(z)| \le 0. \tag{5.5}$$

Thus, μ is Stahl–Totik regular.

Conversely, assume that μ is Stahl–Totik regular. For $n = j(g+1)+k$, the polynomial R_n is a divisor of R_{g+1}^{j+1}, so we can write $\tau_n = \frac{P_n}{R_{g+1}^{j+1}}$ where $\deg P_n \le n+g$. For any $\epsilon > 0$ there exists a polynomial Q_ϵ such that $1-\epsilon \le Q_\epsilon R_{g+1} \le 1+\epsilon$ on E. Thus,

$$|\tau_n(z)| \le (1-\epsilon)^{-j-1} |P_n(z) Q_\epsilon^{j+1}(z)| \tag{5.6}$$

and $\|P_n Q_\epsilon^{j+1}\| \le (1+\epsilon)^{j+1}$ since τ_n is normalized. Since $P_n Q_\epsilon^{j+1}$ is a polynomial of degree at most $n+g+(j+1)\deg Q_\epsilon$, similarly to the above, representing it in the basis of polynomials shows

$$|P_n(z) Q_\epsilon^{j+1}(z)| \le (1+\epsilon)^{j+1} (n+g+1+(j+1)\deg Q_\epsilon) \sup_{0\le m\le n+g+(j+1)\deg Q_\epsilon} |p_n(z)|. \tag{5.7}$$

Since $n+g+(j+1)\deg Q_\epsilon = O(n)$ as $n\to\infty$, the supremum in (5.7) grows subexponentially whenever (5.5) holds. By (5.6), this implies

$$\limsup_{n\to\infty} \frac{1}{n} \log|\tau_n(z)| \le \frac{1}{g+1} \log\left(\frac{1+\epsilon}{1-\epsilon}\right).$$

Since $\epsilon > 0$ is arbitrary, we conclude that (5.5) implies (5.4), so (5.4) holds q.e. on E. □

From this seemingly special case, Theorem 1.6 and Corollary 1.7 follow easily:

Proof of Theorem 1.6. By applying a conformal transformation, the special case shows that μ is $\mathbf{C}_1$-regular if and only if it is $(\mathbf{c}_k)$-regular for any single $\mathbf{c}_k$ in $\mathbf{C}_1$. By applying this twice, we conclude that if $\mathbf{C}_1$, $\mathbf{C}_2$ have a common element, then μ is $\mathbf{C}_1$-regular if and only if it is $\mathbf{C}_2$-regular.

By applying that conclusion twice, we will finish the proof. Namely, for arbitrary $\mathbf{C}_1$, $\mathbf{C}_2$, choose a sequence $\mathbf{C}_3$ which has common elements with both $\mathbf{C}_1$ and $\mathbf{C}_2$. Then μ is $\mathbf{C}_1$-regular if and only if it is $\mathbf{C}_3$-regular if and only if it is $\mathbf{C}_2$-regular. □

Proof of Corollary 1.7. The result follows by taking $\mathbf{C}_2 = (\infty)$ in Theorem 1.6. □

Proof of Theorem 1.1. By Lemma 2.1, $f_*\mu$ is Stahl–Totik regular if and only if μ is $(f^{-1}(\infty))$-regular, and by Corollary 1.7, this is equivalent to Stahl–Totik regularity of μ. □

Proof of Theorem 1.8. (a) We note that by Corollary 1.7 we may use Theorem 1.4. Fix $1 \le k \le g+1$, and use conformal invariance to assume $\mathbf{c}_k = \infty$. Given a subsequence of $n(j) = j(g+1)+k$, we use precompactness to pass to a further subsequence $n_\ell = j_\ell(g+1)+k$ with w-$\lim_{\ell\to\infty} \nu_{n_\ell} = \nu$. We write

$$\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) = \Phi_{\rho_{\mathsf{E},\mathbf{C}}}(z) + \frac{1}{g+1} \log \lambda_k - \frac{1}{g+1} \sum_{\substack{m=1 \\ m\ne k}}^{g+1} \log|z - \mathbf{c}_m| \tag{5.8}$$

which we will use to show $\Phi_\nu = \Phi_{\rho_{\mathsf{E},\mathbf{C}}}$. By (ii), we may apply Theorem 4.1 with $\alpha = \frac{1}{g+1} \log \lambda_k$. Then, (vii) yields $h = \mathcal{G}_{\mathsf{E}}$ off the real line, and thus the equality

between the representations (4.1) and (5.8) gives $\Phi_\nu(z) = \Phi_{\rho_{\mathsf{E},\mathbf{C}}}(z)$ on $\mathbb{C}\setminus\mathbb{R}$. By the weak identity principle, this equality extends to $\mathbb{C}$. Applying the distributional Laplacian to both sides gives $\nu = \rho_{\mathsf{E},\mathbf{C}}$. Thus, w-$\lim_{n\to\infty}\nu_n = \rho_{\mathsf{E},\mathbf{C}}$.

(b) The main ingredient is a variant of Schnol's theorem; for any n, $\int|\tau_n|^2\,d\mu = 1$, so

$$\sum_{n=1}^{\infty} n^{-2}\int|\tau_n|^2\,d\mu < \infty.$$

By Tonelli's theorem, it follows that $\sum_{n=1}^{\infty} n^{-2}|\tau_n|^2 < \infty$ μ-a.e., so there exists a Borel set $B\subset\mathbb{C}$ with $\mu(\mathbb{C}\setminus B) = 0$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log|\tau_n(z)| \le 0, \qquad \forall z\in B. \tag{5.9}$$

Suppose μ is not regular. Then, by Theorem 1.4(ii), there is a $1\le k\le g+1$ with

$$\limsup_{j\to\infty}\frac{1}{n(j)}\log\kappa_{n(j)} > \frac{1}{g+1}\log\lambda_k.$$

Using conformal invariance, we may assume $\mathbf{c}_k = \infty$, and we can pass to a subsequence $n_\ell = j_\ell(g+1)+k$ such that $\alpha := \lim_{\ell\to\infty}\frac{1}{n_\ell}\log\kappa_{n_\ell} > \frac{1}{g+1}\log\lambda_k$, where $\alpha\in\mathbb{R}\cup\{+\infty\}$ by Theorem 4.1(a). Since w-$\lim_{n\to\infty}\nu_n = \rho_{\mathsf{E},\mathbf{C}}$, by comparing (4.1) and (5.8), we have for $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\lim_{\ell\to\infty}\frac{1}{n_\ell}\log|\tau_{n_\ell}(z)| = \mathcal{G}_{\mathsf{E}}(z,\mathbf{C}) + d, \tag{5.10}$$

where $d = \alpha - \frac{\log\lambda_k}{g+1} > 0$. By the upper envelope theorem applied to the sequence $\{\nu_{n_\ell}\}_{\ell\in\mathbb{N}}$, there exists a polar set X such that (5.10) also holds for all $z\in\mathbb{C}\setminus X$. Moreover, since $\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})\ge 0$ for all $z\in\mathbb{C}$, we conclude that

$$\limsup_{n\to\infty}\frac{1}{n}\log|\tau_n(z)| \ge \lim_{\ell\to\infty}\frac{1}{n_\ell}\log|\tau_{n_\ell}(z)| \ge d, \qquad \forall z\in\mathbb{C}\setminus X.$$

Comparing with (5.9) shows that $B\subset X$, so μ is supported on the polar set X. □

Proof of Theorem 1.9. Defining $n(j) = j(g+1)+k_\infty$ and using Lemma 3.1 to compute a telescoping product,

$$\left(\prod_{\ell=1}^{j}\beta_\ell\right)^{1/j} = \left(\prod_{\ell=1}^{j}\frac{\kappa_{n(\ell)}}{\kappa_{n(\ell+1)}}\right)^{1/j} = \kappa_{n(1)}^{1/j}\kappa_{n(j+1)}^{-1/j}. \tag{5.11}$$

The first term on the right-hand side is independent of j, so $\kappa_{n(1)}^{1/j}\to 1$ as $j\to\infty$. For the second factor, using Theorem 1.3 we compute

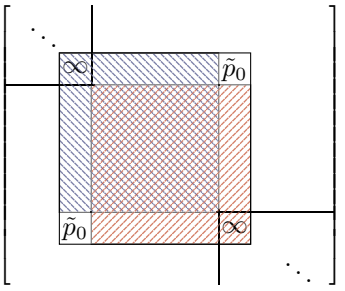
$$\liminf_{j\to\infty}\kappa_{n(j+1)}^{1/j} \ge \lambda_{k_\infty}$$

and we have the upper bound (1.9) for the lim sup of (5.11). Similarly, using the criterion Theorem 1.4(ii), it follows from (5.11) that μ is $\mathbf{C}$-regular if and only if (1.10) holds. □

6. GMP matrices 2

The proof of Theorem 1.10 will rely heavily on the results of [32]. In this section we will recall some properties of GMP matrices from [32] which we will use in the proof of Theorem 1.10. However, in order to justify the use of those constructions, we need to add some explanation of the structure of GMP matrices. This technical explanation is necessary in order to understand the action on Jacobi matrices caused by a single coefficient stripping step on GMP matrices; since such a step changes the location of ∞ in the sequence of poles, it links our GMP matrices which naturally arise from ORF expansions, and those in [32], which naturally arise from functional models of reflectionless operators. This link will allow us to use parts of the analysis of [32].

As noted in the beginning of Section 3, GMP matrices split up into blocks due to the appearance of some $c_{k_\infty} = \infty$. However, there is a choice whether to place the "window" of block size $(g+1) \times (g+1)$ so that c_{k_∞} is the last element of the previous block, or the first element of the next block. In this paper, the latter choice has been more natural (i.e., to split before ∞), because it corresponds to the choice $\tau_0 = 1$ in the rational function construction. From now on, we will call this the RF structure. On the other hand, in [32] the first choice was more natural (i.e., to split after ∞) for the functional model construction, and we will call this the FM structure. Alternatively, recall that we discussed in Remark 1 that one could view the GMP structure also as overlapping $(g+2) \times (g+2)$ blocks. The RF-structure then corresponds to placing the $(g+1) \times (g+1)$ B block at the upper left corner of the bigger block, whereas the FM structure corresponds to placing the B blocks at the lower right corner. This is shown in the figure below, where the blue lines indicate a B block corresponding to the RF structure and the red lines a B block corresponding to the FM structure. Moreover, $\tilde{p}_0$ denotes the positive entry on the outermost diagonal:



The two structures can be translated into each other, by means of the formulas (6.1). Moreover, we will show below that they are also linked by a coefficient stripping formula.

For the reader's convenience, we recall the FM structure of GMP matrices as introduced in [32]. Although the RF and FM structures are just a different interpretation of the same object, namely a GMP matrix, it will be convenient to have a separate notation. For a GMP matrix written in the FM structure we will use A, respectively for its blocks A_k, B_k, and we will use $\tilde{A}, \tilde{A}_k, \tilde{B}_k$, for GMP matrices written in the RF structure. Note that this is a change from the notation used in previous sections.

Fix a finite sequence $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_g)$ and recall that X^- denotes the upper triangular part of a matrix X (excluding the diagonal), and X^+ the lower triangular part (including the diagonal). Then we say that A acting on $\ell^2(\mathbb{Z})$ is GMP structured, and denote it by $A \in \mathbb{A}$, if it is a $(g+1)$-block Jacobi matrix

$$A = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & A_{-1}^* & B_{-1} & A_0 & & \\ & & A_0^* & B_0 & A_1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

such that

$$A_j = \delta_g \vec{p}_j^{\,*}, \quad B_j = (\vec{q}_j \vec{p}_j^{\,*})^- + (\vec{p}_j \vec{q}_j^{\,*})^+ + \hat{\mathbf{C}}, \quad \vec{p}_j, \vec{q}_j \in \mathbb{R}^{g+1},$$

and

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{c}_1 & & & \\ & \ddots & & \\ & & \mathbf{c}_g & \\ & & & 0 \end{bmatrix}, \ \vec{p}_j = \begin{bmatrix} p_0^{(j)} \\ \vdots \\ p_g^{(j)} \end{bmatrix}, \ \vec{q}_j = \begin{bmatrix} q_0^{(j)} \\ \vdots \\ q_g^{(j)} \end{bmatrix}, \quad p_g^{(j)} > 0.$$

We then say an operator $A \in \mathbb{A}$ is a two-sided GMP matrix if the resolvents $(\mathbf{c}_\ell - A)^{-1}$ exist for all $1 \le \ell \le g$ and $S^{-\ell}(\mathbf{c}_\ell - A)^{-1} S^\ell \in \mathbb{A}$. In this case we write $A \in \mathrm{GMP}(\mathbf{C})$. Again we call the generating coefficients $\{\vec{p}_j, \vec{q}_j\}_{j \in \mathbb{Z}}$ the GMP coefficients of A.

We encounter several differences compared to the RF structure presented in Section 3. First of all the 0 in $\hat{\mathbf{C}}$ is now in the last, rather than in the first, position. Moreover, in the definition of A_j, the vector $\vec{p}_j$ is now a row vector in the last row, rather than a column vector in the first column. This is exactly due to shifting the position of ∞ as described above. Extending the structure of GMP matrices to two-sided operators on $\ell^2(\mathbb{Z})$, it is not hard to see that the RF and the FM structures can be translated into each other, simply by shifting the window of size $(g+1) \times (g+1)$ by one. In this process the role of p_j and q_j changes, that is, for $1 \le k \le g$ we have

$$\tilde{p}_k^{(j)} = q_{k-1}^{(j)} p_g^{(j)}, \quad \tilde{q}_k^{(j)} = \frac{p_{k-1}^{(j)}}{p_g^{(j)}}. \tag{6.1}$$

More importantly for us is that the positive entries are the same, i.e.,

$$p_g^{(j)} = \tilde{p}_0^{(j)}.$$

Following [32], we explain how to associate to a given GMP matrix a Jacobi matrix. Let e_j denote the standard basis vectors in $\ell^2(\mathbb{Z})$; recall that $\{e_{-1}, e_0\}$ forms a spectral basis for two-sided Jacobi matrices in the sense that $\{J^n e_j \mid n \in \mathbb{N}_0, j = -1, 0\}$ is dense in $\ell^2(\mathbb{Z})$. Define the matrix resolvent function by

$$R^J(z) = \begin{bmatrix} \langle (J-z)^{-1} e_{-1}, e_{-1} \rangle & \langle (J-z)^{-1} e_0, e_{-1} \rangle \\ \langle (J-z)^{-1} e_{-1}, e_0 \rangle & \langle (J-z)^{-1} e_0, e_0 \rangle \end{bmatrix}.$$

Let $\ell^2_+ = \ell^2(\mathbb{N}_0)$ and $\ell^2_- = \ell^2(\mathbb{Z}) \ominus \ell^2_+$ and $\Pi_\pm$ denote the projection onto $\ell^2_\pm$. Define $J_\pm = \Pi_\pm J \Pi_\pm$ and define the half-line resolvent functions by

$$m_+^J(z) = \langle (J_+ - z)^{-1} e_0, e_0 \rangle, \quad m_-^J(z) = \langle (J_- - z)^{-1} e_{-1}, e_{-1} \rangle.$$

Then, essentially due to the structure

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + a_0(\langle \cdot, e_{-1}\rangle e_0 + \langle \cdot, e_0\rangle e_{-1}),$$

one can see that

$$R^J(z) = \begin{bmatrix} m_-^J(z)^{-1} & a_0 \\ a_0 & m_+^J(z)^{-1} \end{bmatrix}^{-1}; \tag{6.2}$$

cf. [9, pg 758].

For GMP matrices, we need to modify the spectral basis. Define

$$\tilde{e}_0 = \frac{1}{a_0}\Pi_+ A e_{-1}, \quad a_0 = \|\Pi_+ A e_{-1}\|, \tag{6.3}$$

with the natural embedding into $\ell^2(\mathbb{Z})$. Note that

$$a_0 \tilde{e}_0^\intercal = \left[\ldots \quad 0 \quad | \quad p_0^{(0)} \quad p_1^{(0)} \quad \ldots \quad p_g^{(0)} \quad 0 \quad \ldots\right].$$

Then $\{e_{-1}, \tilde{e}_0\}$ form a spectral basis for A and similarly as for Jacobi matrices we have

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} + a_0(\langle \cdot, e_{-1}\rangle \tilde{e}_0 + \langle \cdot, \tilde{e}_0\rangle e_{-1}).$$

This allows us to define

$$R^A(z) = \begin{bmatrix} \langle (A-z)^{-1} e_{-1}, e_{-1}\rangle & \langle (A-z)^{-1}\tilde{e}_0, e_{-1}\rangle \\ \langle (A-z)^{-1} e_{-1}, \tilde{e}_0\rangle & \langle (A-z)^{-1}\tilde{e}_0, \tilde{e}_0\rangle \end{bmatrix},$$

and

$$m_-^A(z) = \langle (A_- - z)^{-1} e_{-1}, e_{-1}\rangle, \quad m_+^A(z) = \langle (A_+ - z)^{-1}\tilde{e}_0, \tilde{e}_0\rangle \tag{6.4}$$

and find similar to the Jacobi case that

$$R^A(z) = \begin{bmatrix} m_-^A(z)^{-1} & a_0 \\ a_0 & m_+^A(z)^{-1} \end{bmatrix}^{-1}.$$

For a given GMP matrix A, the associated Jacobi matrix is simply defined by setting the resolvent functions to be equal, i.e.,

$$R^J(z) = R^A(z). \tag{6.5}$$

Note that this defines J uniquely. Due to the common vector e_{-1}, it follows that

$$\begin{aligned} b_{-1} &= \langle J e_{-1}, e_{-1}\rangle = \langle A e_{-1}, e_{-1}\rangle = p_g^{(-1)} q_g^{(-1)}, \\ a_0 &= \|\Pi_+ J e_{-1}\| = \|\Pi_+ A e_{-1}\| = \|\vec{p}_0\|, \end{aligned} \tag{6.6}$$

which explains in hindsight the definition of a_0 in (6.3).

6.1. **Shifts on GMP matrices.** For a vector $x \in \ell^2(\mathbb{Z})$, let $|$ denote the splitting of ℓ^2_- and ℓ^2_+, i.e., we write $x^\intercal = \begin{bmatrix} \dots & x_{-1} | x_0 & \dots \end{bmatrix}$. We chose the vector of poles in the following way $\begin{bmatrix} \dots & \infty | \mathbf{c}_1 & \dots & \mathbf{c}_g & \infty & \mathbf{c}_1 & \dots \end{bmatrix}$. That is for $A_+ = \Pi_+ A \Pi_+$ the first pole is $\mathbf{c}_1 \in \mathbb{R}$. However, if we consider $\tilde{A}_+ = \Pi_+ SAS^{-1}\Pi_+$, where $Se_k = e_{k+1}$ denotes the right shift, then the first pole of $\tilde{A}_+$ is ∞.

$$\tilde{A}_+ = \left[\begin{array}{c|cc} \infty & & \\ \hline & \mathbf{c}_1 & \\ & & A_+ \end{array}\right].$$

The resolvent functions of A_+ and $\tilde{A}_+$ are related by a coefficient stripping formula:

Lemma 6.1. *Let $A \in \mathrm{GMP}(\mathbf{C})$, $A_+ = \Pi_+ A \Pi_+$, $\tilde{e}_0, a_0, b_{-1}$ as in (6.3), (6.6) and define $\tilde{A}_+ = \Pi_+ SAS^{-1}\Pi_+$. Then the resolvent functions*

$$m_+(z) = \langle (A_+ - z)^{-1}\tilde{e}_0, \tilde{e}_0 \rangle, \quad \tilde{m}_+(z) = \langle (\tilde{A}_+ - z)^{-1} e_0, e_0 \rangle$$

are related by the coefficient stripping formula

$$\tilde{m}_+(z) = \frac{1}{b_{-1} - z - a_0^2 m_+(z)}. \tag{6.7}$$

Proof. Recall that S_+ denotes the right shift on ℓ^2_+ and define

$$f_0 = S_+ \tilde{e}_0 = \frac{1}{a_0}\begin{bmatrix} 0 & p_0^{(0)} & p_1^{(0)} & \dots & p_g^{(0)} & 0 & \dots \end{bmatrix}.$$

Then we have

$$\tilde{A}_+ = \begin{bmatrix} b_{-1} & 0 \\ 0 & A_+ \end{bmatrix} + a_0(\langle \cdot, e_0\rangle f_0 + \langle \cdot, f_0\rangle e_0).$$

Then as for Jacobi matrices this implies (6.7); cf. [24, Theorem 3.2.4]. □

Lemma 6.1 has a very natural interpretation. As we discussed above, GMP matrices split into blocks where $c_{k_\infty} = \infty$ and then there is some choice if we place ∞ as the last or the first element of a block. However, this discussion is irrelevant for Jacobi matrices, where all $c_k \equiv \infty$. Thus, if we associate to A a Jacobi matrix J by (6.5) and define $J_+ = \Pi_+ J \Pi_+$ and $\tilde{J}_+ = \Pi_+ SJS^{-1}\Pi_+$ and the associated m_+, $\tilde{m}_+$, then (6.7) becomes the standard coefficient stripping for Jacobi matrices.

There is another natural shift on GMP matrices. Namely, since the shift $A^{(1)} = S^{-(g+1)}AS^{(g+1)}$ preserves the GMP structure, one can describe how the resolvent functions of A and $A^{(1)}$ are related. This will be done by so-called elementary Blaschke-Potapov factors of the third kind with poles at $\mathbf{c}_1, \dots, \mathbf{c}_g, \infty$; cf. [1, 19]. In the following it will be convenient to use the notation

$$\mathbf{p} = (p, q), \quad \vec{\mathbf{p}} = (\vec{p}, \vec{q}).$$

Definition 6.2. For $p, q, \mathbf{c} \in \mathbb{R}$

$$\mathfrak{a}(z, \mathbf{c}; \mathbf{p}) = I - \frac{1}{\mathbf{c} - z}\begin{bmatrix} p \\ q \end{bmatrix}\begin{bmatrix} p & q \end{bmatrix}\mathrm{j} = \exp\left(-\frac{1}{\mathbf{c} - z}\begin{bmatrix} p \\ q \end{bmatrix}\begin{bmatrix} p & q \end{bmatrix}\mathrm{j}\right), \quad \mathrm{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{6.8}$$

represents the so-called Blaschke-Potapov factor of the third kind with a real pole $\mathbf{c}$. If $\mathbf{c} = \infty$ it is of the form

$$\mathfrak{a}(z;\mathbf{p}) = \mathfrak{a}(z,\infty;\mathbf{p}) = \begin{bmatrix} 0 & -p \\ \frac{1}{p} & \frac{z-pq}{p} \end{bmatrix}. \tag{6.9}$$

Define the matrix function

$$\mathfrak{A}(z,\vec{\mathbf{p}}) = \begin{bmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix}(z,\vec{\mathbf{p}}) = \mathfrak{a}(z,\mathbf{c}_1;\mathbf{p}_0)\dots\mathfrak{a}(z,\mathbf{c}_g;\mathbf{p}_{g-1})\mathfrak{a}(z;\mathbf{p}_g). \tag{6.10}$$

The important role of the function $\mathfrak{A}(z,\vec{\mathbf{p}})$ will become clear by Theorem 6.3.

Theorem 6.3 ([32, Theorem 2.13 and Theorem 2.15])**.** *Let $A \in \mathrm{GMP}(\mathbf{C})$, $A^{(1)} = S^{-(g+1)}AS^{(g+1)}$ and A_+ and $A_+^{(1)}$ the projections onto ℓ^2_+. Let m_+^A and $m_+^{A^{(1)}}$ be the resolvent functions defined by (6.4). Let $a_0^2 = \|\vec{p}_0\|^2$, $(a_0^{(1)})^2 = \|\vec{p}_0^{(1)}\|^2$. Then*

$$a_0^2 m_+^A(z) = \frac{\mathfrak{A}_{11}(z,\vec{\mathbf{p}}_0)((a_0^{(1)})^2 m_+^{A^{(1)}}(z)) + \mathfrak{A}_{12}(z,\vec{\mathbf{p}}_0)}{\mathfrak{A}_{21}(z,\vec{\mathbf{p}}_0)((a_0^{(1)})^2 m_+^{A^{(1)}}(z)) + \mathfrak{A}_{22}(z,\vec{\mathbf{p}}_0)}. \tag{6.11}$$

6.2. **Periodic GMP matrices.** We call a two-sided GMP matrix 1-periodic or simply periodic if $S^{g+1}AS^{-(g+1)} = A$. In this case $m_+^{A^{(1)}} = m_+^A$ and (6.11) is a quadratic equation for m_+^A. This allows to describe the spectrum of A in terms of the function $\mathfrak{A}(z)$.

Theorem 6.4 ([12, Theorem 1.8])**.** *Let $A = A(\vec{\mathbf{p}}) \in \mathrm{GMP}(\mathbf{C})$ be a periodic GMP matrix and $\mathfrak{A}(z,\vec{\mathbf{p}})$ be as in (6.10) and define the discriminant by*

$$\Delta(z) = \operatorname{tr}\mathfrak{A}(z,\vec{\mathbf{p}}).$$

Then the spectrum of A is a finite union of intervals, it is purely absolutely continuous and of multiplicity 2 and it is given by

$$\sigma(A) = \Delta^{-1}([-2,2]) = \{z \in \mathbb{C} |\ \Delta(z) \in [-2,2]\}.$$

The inverse problem can also be answered explicitly. Namely, given a finite union of intervals E, are there periodic GMP matrices with the given spectrum and if so can one describe the set of all such matrices? Crucially, the answer to both questions is positive for the special choice $\mathbf{C} = \mathbf{C}_\mathsf{E}$, where $\mathbf{C}_\mathsf{E}$ denotes the zeros of the Ahlfors function associated to E. We define the isospectral torus of periodic two-sided GMP matrices by

$$\mathcal{T}_\mathsf{E}(\mathbf{C}_\mathsf{E}) = \{\mathring{A} \in \mathrm{GMP}(\mathbf{C}_\mathsf{E}), \mathring{A} \text{ is periodic and } \sigma(\mathring{A}) = \mathsf{E}\}.$$

Henceforth, we will use $\mathring{A}$ for elements from $\mathcal{T}_\mathsf{E}(\mathbf{C}_\mathsf{E})$. We point out that for arbitrary finite gap sets, the isospectral torus of Jacobi matrices usually consists of almost periodic operators, whereas for GMP matrices we can always work with periodic operators. This also makes it possible to characterize the isospectral torus by a magic formula for GMP matrices. Moreover, this then can be used to describe $\mathcal{T}_\mathsf{E}(\mathbf{C}_\mathsf{E})$ also as an algebraic manifold. Recall that Λ_n denotes the outermost positive entry of the resolvents $(\mathbf{c}_\ell - A)^{-1}$. That is if $A \in \mathrm{GMP}(\mathbf{C})$ is a periodic GMP matrix let

$$\begin{aligned} \Lambda_\ell(A) &= \langle e_\ell, (\mathbf{c}_{\ell+1} - A)^{-1} e_{\ell+g+1}\rangle \quad \text{for } 0 \le \ell \le g-1, \\ \Lambda_g(A) &= \langle e_g, Ae_{2g+1}\rangle. \end{aligned}$$

The resolvent entries can again be given explicitly in terms of $\Delta(z)$.

Lemma 6.5 ([32, Theorem 2.17])**.** *Let $A \in \mathrm{GMP}(\mathbf{C})$ be a periodic GMP matrix. Then for $0 \leq \ell \leq g-1$*

(6.12)

$$\begin{aligned}\Lambda_\ell(A) &= -(\mathrm{Res}_{\mathbf{c}_{\ell+1}} \Delta)^{-1} \\ &= -\left(\mathrm{tr}\left(\prod_{k=0}^{\ell-1} \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{c}_{k+1}; \mathbf{p}_k) \begin{bmatrix} p_\ell \\ q_\ell \end{bmatrix} \begin{bmatrix} p_\ell & q_\ell \end{bmatrix} \mathfrak{j} \prod_{k=\ell+1}^{g-1} \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{c}_{k+1}; \mathbf{p}_k) \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{p}_g) \right)\right)^{-1}.\end{aligned}$$

This allows to describe $\mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}})$ as an algebraic manifold. Let us fix a finite union of $g+1$ intervals and let Δ_{E} denote the associated discriminant defined in terms of the Ahlfors function (1.15). Then for coefficients $\vec{\mathbf{p}}$ let $A(\vec{\mathbf{p}}) \in \mathrm{GMP}(\mathbf{C}_{\mathsf{E}})$ be a periodic GMP matrix and define

$$\begin{aligned} f_0(\vec{\mathbf{p}}) &= \lambda_{g+1}\langle \vec{p}, \vec{q} \rangle + d, \\ f_\ell(\vec{\mathbf{p}}) &= \Lambda_{\ell-1}(A(\vec{\mathbf{p}}))\lambda_\ell - 1, \qquad \text{for } 1 \leq \ell \leq g+1, \end{aligned}$$

and $\mathbf{F}_{\mathsf{E}} : U \subset \mathbb{R}^{2(g+1)} \to \mathbb{R}^{g+2}$ by

$$\mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}}) = \begin{pmatrix} f_0(\vec{\mathbf{p}}), & \dots & f_{g+1}(\vec{\mathbf{p}}) \end{pmatrix}. \tag{6.13}$$

We then define the isospectral manifold by

$$\mathcal{IS}_{\mathsf{E}} = \{\vec{\mathbf{p}} \in \mathbb{R}^{2g} : \mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}}) = 0\}.$$

The name is justified by Theorem 6.6:

Theorem 6.6 ([12, Theorem 1.6 and Theorem 1.10])**.** *Let $A \in \mathrm{GMP}(\mathbf{C}_{\mathsf{E}})$, then*

$$A \in \mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}}) \iff \Delta_{\mathsf{E}}(A) = S^{g+1} + S^{-(g+1)}. \tag{6.14}$$

Moreover, for $\vec{\mathbf{p}}$ such that $A(\vec{\mathbf{p}}) \in \mathrm{GMP}(\mathbf{C}_{\mathsf{E}})$ we have that

$$A(\vec{\mathbf{p}}) \in \mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}}) \iff \mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}}) = 0.$$

6.3. **Resolvents in the general case and the Jacobi flow.** Similar to (6.12) one can also find explicit expressions for Λ_n for general (not necessarily periodic) GMP matrices. Let $A \in \mathrm{GMP}(\mathbf{C})$ and for $n = j(g+1)+\ell$ for $j \in \mathbb{Z}$ and $0 \leq \ell \leq g$ set

$$\Lambda_n(A) = \begin{cases} \langle e_n, (\mathbf{c}_{\ell+1} - A)^{-1} e_{n+g+1} \rangle, & \ell \neq g, \\ \langle e_n, A e_{n+g+1} \rangle. \end{cases}$$

Lemma 6.7 ([32, Lemma 3.2])**.** *Let $A \in \mathrm{GMP}(\mathbf{C})$. Then for $n = j(g+1)+\ell$ and $\ell \neq g$ we have*

(6.15)

$$\begin{aligned}&\Lambda_n(A) \\ &= -\left(\mathrm{tr}\left(\prod_{k=0}^{\ell-1} \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{c}_{k+1}; \mathbf{p}_k^{(j+1)}) \begin{bmatrix} p_\ell^{(j+1)} \\ q_\ell^{(j+1)} \end{bmatrix} \begin{bmatrix} p_\ell^{(j)} & q_\ell^{(j)} \end{bmatrix} \mathfrak{j} \prod_{k=\ell+1}^{g-1} \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{c}_{k+1}; \mathbf{p}_k^{(j)}) \mathfrak{a}(\mathbf{c}_{\ell+1}, \mathbf{p}_g^{(j)}) \right)\right)^{-1}.\end{aligned}$$

The explicit representation will be crucial in the following. Moreover, let us mention that due to the finite band block structure of GMP matrices, building (formal) resolvents is a purely local computation (compare e.g. [32, eq (3.8)]). This can be seen by the formula above, where only the entries of A from the blocks j and $j+1$ are needed to compute Λ_n.

Recall that we discussed already in the beginning of this section that to any GMP matrix A we can associate a Jacobi matrix J, namely by setting the resolvent functions equal to (6.5). Let us denote this map by $\mathcal{F}$. It is a deep result from [32, Proposition 5.5.] that this map is (up to a certain identification) invertible. An important question is if we can express the Jacobi parameters of $J = \mathcal{F}A$ in terms of the coefficients of A. Let $\{\vec{\mathbf{p}}_j\}$ denote the GMP coefficients and $\{a_j, b_j\}$ the Jacobi coefficients. Then we have already seen that

$$a_0 = \|\vec{p}_0\|, \quad b_{-1} = q_g^{(-1)} p_g^{(-1)}.$$

Let

$$\mathcal{S}J = S^{-1}JS,$$

and note that

$$a_0(\mathcal{S}J) = a_1(J), \quad b_{-1}(\mathcal{S}J) = b_0(J),$$

where by $a_k(J), b_k(J)$ we mean the Jacobi parameters of the Jacobi matrix J. Thus, if one understands the transform on GMP matrices which is induced by the shift action on Jacobi matrices, one can inductively obtain the Jacobi parameters by the formulas above. This leads to the definition of the Jacobi flow on GMP matrices, which is defined by the following commutative diagram:

$$\begin{array}{ccc} \text{GMP} & \xrightarrow{\mathcal{J}} & \text{GMP} \\ {\scriptstyle\mathcal{F}}\downarrow & & {\scriptstyle\mathcal{F}}\downarrow \\ \text{Jacobi} & \xrightarrow{\mathcal{S}} & \text{Jacobi} \end{array} \tag{6.16}$$

Let us mention that this is one of the reasons why it is convenient to work with two-sided operators. If in this construction we considered the shift action on ℓ^2_+, which is not unitary, then it is possible that for some m, $\mathbf{c}_k \in \sigma((S_+^*)^m J_+ S_+^m)$, and thus the corresponding half-line GMP matrix would not be well defined.

The Jacobi flow is defined and discussed in [32, Section 4]. We provide the motivating ideas of the Jacobi flow and its precise definition below. First, note that in [32], we have the ordering of the poles

$$\mathbf{C}_A := \left[\ \dots \quad \infty \ \middle| \ \mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_g \quad \infty \ \middle\vdots \ \mathbf{c}_1 \quad \dots \ \right]$$

and recall that we anchored the blocks between ∞ (at position -1) and $\mathbf{c}_1$ (at position 0). Note that for Jacobi matrices, all poles are equal to ∞, and $\mathcal{S}J$ corresponds to shifting an ∞-pole from position 0 to position -1. Now applying the spacial shift to GMP matrices would be of a different flavor, as it shifts $\mathbf{c}_1$ from 0 to -1. Thus, one first has to shift ∞, which is now at position $g+1$, to the front, and then one may apply the spacial shift. This is done in g-steps. The $\mathcal{O}$ transform defined below corresponds changing the order from $\mathbf{C}_A$ to

$$\tilde{\mathbf{C}}_A = \left[\ \dots \quad \mathbf{c}_g \ \middle| \ \mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_{g-1} \quad \infty \quad \mathbf{c}_g \ \middle\vdots \ \mathbf{c}_1 \dots \ \right].$$

Letting

$$\mathbf{o}(\phi) = \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix},$$

we make Definition 6.8.

Definition 6.8. We define the map:

$$\mathcal{O} : \mathrm{GMP}(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_g) \to \mathrm{GMP}(\mathbf{c}_g, \mathbf{c}_1, \dots, \mathbf{c}_{g-1})$$

in the following way. Let $O = O_A$ be the block-diagonal matrix

$$O = \begin{bmatrix} \ddots & & & & \\ & O_{-1} & & & \\ & & O_0 & & \\ & & & \ddots & \end{bmatrix},$$

where O_k are the $(g+1) \times (g+1)$ orthogonal matrices

$$O_k = \begin{bmatrix} I_{g-2} & 0 \\ 0 & \mathbf{o}(\phi_k) \end{bmatrix}, \quad \begin{bmatrix} \sin\phi_k & \cos\phi_k \end{bmatrix} = \frac{\begin{bmatrix} p_{g-1}^{(k)} & p_g^{(k)} \end{bmatrix}}{\sqrt{(p_{g-1}^{(k)})^2 + (p_g^{(k)})^2}}. \tag{6.17}$$

Then

$$\mathcal{O}A := SO_A^* A O_A S^{-1}. \tag{6.18}$$

As explained above, the Jacobi flow then is defined by applying $\mathcal{O}$ g-times, in order to shift ∞ through the full block. This leads to Definition 6.9:

Definition 6.9. We define the Jacobi flow transform

$$\mathcal{J} : \mathrm{GMP}(\mathbf{C}_A) \to \mathrm{GMP}(\mathbf{C}_A)$$

by

$$\mathcal{J}A := S^{-(g+1)} \mathcal{O}^{\circ g} A S^{g+1}.$$

It is shown in [32, Equation (4.8) and Lemma 4.4] that there exists a block-diagonal unitary mapping U_A, such that

$$\mathcal{J}A = S^{-1} U_A^* A U_A S. \tag{6.19}$$

Let us also note that

$$S^{-(g+1)} \mathcal{O}(A) S^{(g+1)} = \mathcal{O}(S^{-(g+1)} A S^{(g+1)}), \tag{6.20}$$

which has the consequences

$$\mathcal{O}(\mathcal{J}^{\circ m} A) = \mathcal{J}^{\circ m}(\mathcal{O}A) \tag{6.21}$$

and

$$S^{-(g+1)} (\mathcal{J}^{\circ m} A) S^{(g+1)} = \mathcal{J}^{\circ m}(S^{-(g+1)} A S^{(g+1)}).$$

7. Proof of Theorem 1.10

Lemma 7.1 allows us to extend J_+ to a two-sided Jacobi matrix J acting on $\ell^2(\mathbb{Z})$ in a way such that $\mathbf{c}_k$ belong to the resolvent domain of J.

Lemma 7.1. *Let μ be a compactly supported probability measure such that $\mathsf{E} = \operatorname{ess\,supp} \mu$ is a union of $g+1$ intervals. Let*

$$m_+(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} d\mu(x)$$

and J_+ the associated Jacobi matrix and let $\mathbf{c}_k \in \mathbb{R} \setminus \mathsf{E}$ for $1 \le k \le g$. Then there exists a two-sided Jacobi matrix J with the following properties:

(i) $J_+ = \Pi_+ J \Pi_+$;

(ii) *there exists* $\mathring{J}\in\mathcal{T}(\mathsf{E})$ *so that* $J_-:=\Pi_- SJS^{-1}\Pi_-$ *obeys* $J_-=\Pi_-S\mathring{J}S^{-1}\Pi_-$*;*
(iii) $\mathbf{c}_k$ *belong to the resolvent domain of* J*;*
(iv) $\mathbf{c}_k$ *belong to the resolvent domain of* $\tilde{J}_+ := \Pi_+SJS^{-1}\Pi_+$.

Proof. Let J denote the extended two-sided matrix. Note that J is defined by J_+, a_0, a_{-1}, b_{-1} and J_-. We fix J_+ and a_0, a_{-1} and choose J_- and b_{-1} appropriately.

By (6.7) we have

$$\tilde{m}_+(z) = \frac{1}{b_{-1} - z - a_0^2 m_+(z)}.$$

Thus, $\mathbf{c}_k$ is a pole of $\tilde{m}_+(z)$ if and only if it is a zero of $b_{-1} - z - a_0^2 m_+(z)$. Choose b_{-1} so that

$$b_{-1} - \mathbf{c}_k - a_0^2 m_+(\mathbf{c}_k) \neq 0.$$

This already defines $\tilde{J}_+$.

Let us write (6.2) at position -1 rather than at position 0 and let m_- be the resolvent function of J_- and $\tilde{m}_+$ the resolvent function of $\tilde{J}_+$. Then we see that

$$-\frac{1}{R_{-2-2}(z)} = -\frac{1}{m_-(z)} + a_0^2\tilde{m}_+(z), \quad -\frac{1}{R_{-1,-1}(z)} = -\frac{1}{\tilde{m}_+(z)} + a_0^2 m_-(z).$$

If $\tilde{m}_+(\mathbf{c}_k) \in \{0,\infty\}$, we choose m_- so that $m_-(\mathbf{c}_k) \notin \{0,\infty\}$ and if $\tilde{m}_+(\mathbf{c}_k) \notin \{0,\infty\}$ we set $m_-(\mathbf{c}_k) = 0$. In both cases $R_{-2,-2}(\mathbf{c}_k) \neq \infty$ and $R_{-1,-1}(\mathbf{c}_k) \neq \infty$ and we obtain (iii). □

We will apply Lemma 7.1 in the following way. First we choose $\mathbf{c}_1, \dots \mathbf{c}_g$ as the zeros of the Ahlfors function of $\overline{\mathbb{C}}\setminus\mathsf{E}$. Let μ be a given Stahl-Totik regular measure and $\mathsf{E} = \operatorname{ess\,supp}\mu$. To this measure we construct J as above. Let further $\tilde{\mu}$ be the spectral measure of $\tilde{J}_+$. Clearly $\mathsf{E} = \operatorname{ess\,supp}\tilde{\mu}$ and from the characterization of regularity by existence of the limit and equality in (1.2) it follows that also $\tilde{\mu}$ is regular. Due to (iii) we can form orthogonal rational functions with respect to the periodic sequence $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_g, \infty, \mathbf{c}_1, \mathbf{c}_2, \dots)$. On the other hand (iv) allows us to associate to J a two-sided GMP matrix in the sense of [32]. In particular, $\tilde{J}_+$ satisfies the assumptions of Lemma 1.11.

It was noted in [32, Section 2.2] that

$$-\log|\Psi(z)| = \sum_{k=1}^{g+1} G_{\mathsf{E}}(z, \mathbf{c}_k) \tag{7.1}$$

and that the Yuditskii discriminant has the form (1.15) for some $\lambda_k > 0$ and $d \in \mathbb{R}$. Note that the constants λ_k can be found by computing the residue of Δ_{E} at the poles $\mathbf{c}_k$. By using (1.14) and (7.1), we find the residues to be the same constants λ_k defined in a more general setting in (1.6).

Proof of Lemma 1.11. Denote by μ the canonical spectral measure for J. Note that

$$\sigma_{\mathrm{ess}}(A) = \operatorname{ess\,supp}\mu = \mathsf{E} = \Delta_{\mathsf{E}}^{-1}([-2,2]).$$

Since Δ_{E} maps $\mathbb{R}\setminus\{\mathbf{c}_1, \dots, \mathbf{c}_g\}$ to $\mathbb{R}$ and is piecewise strictly monotone, by a spectral mapping theorem, this implies that for $\mathbf{J} = \Delta_{\mathsf{E}}(A)$, $\sigma_{\mathrm{ess}}(\mathbf{J}) = [-2,2]$.

As noted in Section 1, regularity of the Jacobi matrix J implies $\mathbf{C}_{\mathsf{E}}$-regularity by Corollary 1.7, and this can be characterized in terms of GMP matrix coefficients by Theorem 1.9. The GMP matrix structure together with (1.15) implies that

$\mathbf{J} = \Delta_{\mathsf{E}}(A)$ is a type 3 block Jacobi matrix (1.16); the diagonal entries of the off-diagonal blocks $\mathfrak{v}_j$ are given by $\lambda_k \Lambda_{j(g+1)+k}$ for $k = 0, \dots, g$, with the convention $\lambda_0 = \lambda_{g+1}$. Thus,

$$\det \mathfrak{v}_j = \prod_{k=0}^{g} \lambda_k \Lambda_{j(g+1)+k}.$$

By applying the criterion for regularity in Theorem 1.9 to the GMP matrix A and to its resolvents $(\mathbf{c}_k - A)^{-1}$, we conclude that $\mathbf{J}$ obeys (1.17). It follows that $\mathbf{J}$ is regular with $\sigma_{\mathrm{ess}}(\mathbf{J}) = [-2, 2]$. □

If $\tilde{A}_+$ is such that $\sigma_{\mathrm{ess}}(\tilde{A}_+) = \mathsf{E}$ and the corresponding measure is regular on E, then $\Delta_{\mathsf{E}}(\tilde{A}_+)$ is a block Jacobi matrix which due to Lemma 1.11 is regular for $[-2, 2]$. Therefore, if $\{\mathfrak{v}_\ell, \mathfrak{w}_\ell\}$ denote the block Jacobi coefficients of $\Delta_{\mathsf{E}}(\tilde{A}_+)$, by [23, Theorem 3.1] we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\ell=1}^{N} \|\mathfrak{v}_\ell - I\| + \|\mathfrak{w}_\ell\| = 0. \tag{7.2}$$

We note that since $C = \sup_\ell(\|\mathfrak{v}_\ell(A) - I\| + \|\mathfrak{w}_\ell\|) < \infty$, it follows from Cauchy-Schwarz and the AM-GM inequality that

$$\left(\frac{1}{N}\sum_{\ell=1}^{N} \|\mathfrak{v}_\ell - I\| + \|\mathfrak{w}_\ell\|\right)^2 \leq \frac{2}{N}\sum_{\ell=1}^{N} \|\mathfrak{v}_\ell - I\|^2 + \|\mathfrak{w}_\ell\|^2 \leq 2C\frac{1}{N}\sum_{\ell=1}^{N} \|\mathfrak{v}_\ell - I\| + \|\mathfrak{w}_\ell\|$$

and thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\ell=1}^{N} \|\mathfrak{w}_\ell\|^2 + \|\mathfrak{v}_\ell - I\|^2 = 0 \iff \lim_{N\to\infty} \frac{1}{N} \sum_{\ell=1}^{N} \|\mathfrak{w}_\ell\| + \|\mathfrak{v}_\ell - I\| = 0. \tag{7.3}$$

We will use this equivalence freely in the following.

In the setting of periodic Jacobi matrices and polynomial discriminants (i.e., Δ is a polynomial and $\{\mathfrak{v}_\ell, \mathfrak{w}_\ell\}$ are the coefficients of the block Jacobi matrix $\Delta(J_+)$) it is shown in [7] that

$$\sum_{\ell=1}^{\infty} \|\mathfrak{w}_\ell\|^2 + \|\mathfrak{v}_\ell - I\|^2 < \infty \iff \sum_{m=1}^{\infty} d((S_+^*)^m J S_+^m, \mathcal{T}_{\mathsf{E}}^+)^2 < \infty. \tag{7.4}$$

It was then stated in [23] that since all the arguments in [7] are local, in this setting (7.3) yields (1.13). Let us emphasize that finite gap sets whose isospectral torus consists of periodic Jacobi matrices are very special and the arguments in [23] only apply to this setting. Yuditskii [32] has extended the work of [7] and one has the same localness, but since the construction is quite involved, we will provide the main ideas of proof. In this case, the condition on the right-hand side of (7.4) is still the same, i.e., a condition for a Jacobi matrix J_+, but on the left-hand side $\{\mathfrak{v}_\ell, \mathfrak{w}_\ell\}$ are the coefficients of the block Jacobi matrix $\Delta_{\mathsf{E}}(A)$, where A is an associated GMP matrix and Δ_{E} is the rational function as defined in (1.15).

We will start with the main ingredients of the proof that the left-hand side in (7.4) implies the right-hand side and mention certain modifications to our setting. After this preparatory work will show how this can be applied to our setting.

We concluded from regularity that $\mathbf{J} = \Delta_{\mathsf{E}}(\tilde{A}_+)$ satisfies (7.2). As may be seen in [7], and [32], it is convenient to rewrite this condition into a "multiplicative form".

This leads to the notion of the Killip-Simon functional that we will define below. For a GMP matrix $A \in \mathrm{GMP}(\mathbf{C}_{\mathsf{E}})$, we define the functional as in [32, Section 6] by

$$H_+(A) = \sum_{\ell=0}^{\infty} h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}), \tag{7.5}$$

where

$$h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) = \frac{1}{2}\operatorname{tr}(\mathfrak{v}_\ell^*\mathfrak{v}_\ell + \mathfrak{w}_\ell^2 + \mathfrak{v}_{\ell+1}\mathfrak{v}_{\ell+1}^*) - (g+1) - \log\det \mathfrak{v}_\ell\mathfrak{v}_{\ell+1}.$$

For a square matrix X its modulus is defined by $|X| := \sqrt{X^*X}$. Moreover, define $G(|X|) = |X|^2 - I - \log|X|^2$. Then we have

$$2h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) = \operatorname{tr}\left(\mathfrak{w}_\ell^2 + G(|\mathfrak{v}_\ell|) + G(|\mathfrak{v}_{\ell+1}^*|)\right).$$

In particular, it follows from $|\mathfrak{v}_\ell|, |\mathfrak{v}_{\ell+1}| > 0$ that $h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) > 0$. In fact even more is true. There exists $\tilde{C} > 1$ so that if $\|\mathfrak{v}_\ell - I\| < \frac{1}{2}$ then by [7, Proposition 11.12]

$$\frac{1}{\tilde{C}}\|\mathfrak{v}_\ell - I\| \le \||\mathfrak{v}_\ell| - I\| \le \tilde{C}\|\mathfrak{v}_\ell - I\|.$$

Thus, if $\tilde{C}\|\mathfrak{v}_\ell - I\| < \frac{1}{2}$ we conclude that $\||\mathfrak{v}_\ell| - I\| < \frac{1}{2}$ and thus the eigenvalues of $|\mathfrak{v}_\ell|$ are greater than $\frac{1}{2}$. Under this assumption (for ℓ and $\ell+1$) it is shown in [7, Theorem 11.13] that there exists a constant C so that

$$\frac{1}{C}h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) \le \left(\|\mathfrak{v}_\ell - I\|^2 + \|\mathfrak{w}_\ell\|^2 + \|\mathfrak{v}_{\ell+1} - I\|^2\right) \le Ch(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}). \tag{7.6}$$

A key observation is that the functional $H_+(A)$ is related to the shift action of S^{g+1} on the GMP matrix A. But finally we want to conclude something about

$$\mathcal{S}J = S^*JS,$$

i.e., the shift action on J. This is another motivation of the Jacobi flow as defined above.

The following key lemma, which follows essentially from (6.19), allows for the computation of the "derivative" in the Jacobi flow direction, and is essential in order to extract from the finiteness of $H_+(A)$ properties of the associated Jacobi matrix J.

Lemma 7.2 ([32, Lemma 6.1])**.** *Let $v_{jk}^{(\ell)}, w_{jk}^{(\ell)}$ denote the matrix entries of $\mathfrak{v}_\ell, \mathfrak{w}_\ell$ and*

$$\delta_J H_+(A) = \frac{1}{2}\langle \Delta_{\mathsf{E}}(\mathcal{J}A)e_{-1}, \Delta_{\mathsf{E}}(\mathcal{J}A)e_{-1}\rangle - 1 - \log(\mathcal{J}v)_{g,g}^{(-1)}(\mathcal{J}v)_{g,g}^{(0)}.$$

Then

$$H_+(A) = H_+(\mathcal{J}A) + \delta_J H_+(A). \tag{7.7}$$

Proof. Using (6.19) the proof is based on the realization that due to the diagonal structure of U_A, conjugating A by U_A does not affect H_+. Thus, $\delta_J H_+(A)$ corrects for the term which is omitted in $H_+(\mathcal{J}A)$ due to the shift. □

For later reference let us mention that due to (6.18) we can also relate $H_+(A)$ and $H_+(\tilde{O}A)$. Moreover, it is easy to see that we can also relate $H_+(A)$ and

$H_+(S^{-(g+1)}AS^{(g+1)})$ explicitly. Moreover, Lemma 7.2 allows to obtain ℓ^2 conditions for the coefficients of $\mathcal{J}^{\circ m}(A)$ from finiteness of $H_+(A)$. We sketch the idea in the following. Let us define

$$\tilde{H}_+(A) = \sum_{m=0}^{\infty} \delta_J H_+(A(m)), \qquad \text{where } A(m) = \mathcal{J}^{\circ m}(A).$$

Since all terms are positive, iterating (7.7) yields

$$\tilde{H}_+(A) \leq H_+(A).$$

In particular, $H_+(A) < \infty$ implies $\tilde{H}_+(A) < \infty$. The vector $\Delta_{\mathsf{E}}(A(m))e_{-1}$ has only $2g+3$ nonvanishing entries which are entries of the last columns of $\mathfrak{v}_{-1}(m), \mathfrak{w}_{-1}(m)$ and $\mathfrak{v}_0(m)$. Let us denote this $2g+3$-dimensional vector by $x(m)$ and note that the first and the last components are the positive entries $x_0(m) = (v(m))_{g,g}^{(-1)}$ and $x_{2g+2}(m) = (v(m))_{g,g}^{(0)}$. With this notation we have

$$\delta_J H_+(A(m)) = \frac{1}{2}\Big(G(x_0(m)) + G(x_{2g+2}(m)) + \sum_{j=1}^{2g+1} x_j(m)^2 \Big). \tag{7.8}$$

Thus, $\tilde{H}_+(A) < \infty$ implies already ℓ^2-conditions for the vector $x(m)$. This is used to conclude from $\tilde{H}_+(A) < \infty$ that $A(m)$ is ℓ^2-close to be periodic and that the periodic operator is ℓ^2-close to $\mathcal{IS}_{\mathsf{E}}$. That is, if $\{\vec{p}_j(m), \vec{q}_j(m)\}_{m\in\mathbb{N}_0}$ denote the GMP parameters of $A(m)$, then [32, Theorem 1.20]

$$\begin{aligned} \{\vec{\mathbf{p}}_0(m) - \vec{\mathbf{p}}_{-1}(m)\}_{m\in\mathbb{N}_0} &\in \ell^2(\mathbb{N}_0, \mathbb{R}^{2(g+1)}), \\ \{\mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}}_0(m))\}_{m\in\mathbb{N}_0} &\in \ell^2(\mathbb{N}_0, \mathbb{R}^{g+2}). \end{aligned} \tag{7.9}$$

To show how one obtains from (7.9) convergence of $(S_+^*)^m J S_+^m$ to $\mathcal{T}_{\mathsf{E}}^+$ in the sense of (7.4), we need one more ingredient: it is well known that there are continuous functions, $\mathcal{A}, \mathcal{B}$, on $\mathbb{R}^g/\mathbb{Z}^g$, which can be expressed explicitly in terms of the Riemann theta function associated to E[1] [28, Theorem 9.4.], and a fixed element $\chi \in \mathbb{R}^g/\mathbb{Z}^g$, such that

$$\mathcal{T}_{\mathsf{E}} = \{J(\alpha) : \quad \alpha \in \mathbb{R}^g/\mathbb{Z}^g\} \tag{7.10}$$

and $J(\alpha)$ is the Jacobi matrix built from the coefficients

$$a_m(\alpha) = \mathcal{A}(\alpha - m\chi), \quad b_m(\alpha) = \mathcal{B}(\alpha - m\chi). \tag{7.11}$$

Recall that by the definition of the Jacobi flow, if J is the Jacobi matrix associated to A, then $S^{-m}JS^m$ is the Jacobi matrix associated to $A(m)$. Since every point of $\mathcal{IS}_{\mathsf{E}} = \mathbf{F}_{\mathsf{E}}^{-1}(0)$ is regular for $\mathbf{F}_{\mathsf{E}}$, by [7, Lemma 11.3] there exists a constant $C > 0$, such that

$$\operatorname{dist}(\vec{\mathbf{p}}, \mathcal{IS}_{\mathsf{E}}) \leq C\|\mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}})\|, \tag{7.12}$$

where $\vec{\mathbf{p}}$ are chosen from a fixed compact neighborhood of $\mathcal{IS}_{\mathsf{E}}$, see also [32, page 755]. Taking an element $\mathring{A}_m \in \mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}})$ so that

$$\operatorname{dist}(\vec{\mathbf{p}}(n), \mathcal{IS}_{\mathsf{E}}) = \operatorname{dist}(A(\vec{\mathbf{p}}(n)), \mathring{A}_m),$$

[1] To be precise it is the Riemann theta function of the Riemann surface of the function $\sqrt{\prod_{k=0}^{g}(z-\mathbf{a}_k)(z-\mathbf{b}_k)}$, where $\mathbf{a}_k, \mathbf{b}_k$ denote the gap edges of E, cf. (1.11).

one can conclude from (7.9) and (7.12) that

$$\sum_{n\geq 0} \operatorname{dist}(A(\vec{\mathfrak{p}}(n)), \mathring{A}_m)^2 < \infty. \tag{7.13}$$

Letting $J(\alpha_m) \in \mathcal{T}_{\mathsf{E}}$ be the Jacobi matrix with $F(\mathring{A}_m) = J(\alpha_m)$, then (6.6) implies that

$$a(m)^2 - \mathcal{A}(\alpha_m) \in \ell^2_+, \quad b(m) - \mathcal{B}(\alpha_m) \in \ell^2_+.$$

Using in addition the smoothness of the Jacobi flow, one can show that

$$\alpha_n = \sum_{j=1}^{n} \varepsilon^{\alpha}_m - m\chi, \quad \varepsilon^{\alpha}_m \in \ell^2(\mathbb{N}_0, \mathbb{R}^g).$$

This is even stronger than (7.4); cf. [32, Lemma 7.2].

Before we start with our construction, we have to mention a certain technical issue. If $\{f_m\}$ is a sequence, then clearly $\{f_m - 1\} \in \ell^2$ implies

$$\liminf_{m\to\infty} f_m > 0.$$

If $|f_m - 1|$ is only Cesáro summable, then this is not necessarily the case. However, for any $\delta > 0$ the set with $f_m < \delta$ will be sparse in the following sense. Let us introduce the notation $\{f_m\} \in \mathrm{CS}$ for sequences $\{f_m\}$ satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} |f_m| = 0,$$

and we call a set $T \subset \mathbb{N}$ sparse if

$$\lim_{N\to\infty} \frac{|T \cap \{1, 2, \dots, N\}|}{N} = 0.$$

An elementary observation, which will be used repeatedly, is that for $f \in \mathrm{CS}$, the set $\{m \in \mathbb{N} \mid |f_m| \geq \delta\}$ is sparse for any $\delta > 0$. This follows immediately from Markov's inequality.

We have already concluded from regularity that one and hence both of the conditions in (7.3) hold. Due to the phenomena described above and the log in the definition of $h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1})$ it is not immediately clear that (7.3) also implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N} h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) = 0. \tag{7.14}$$

However, using in addition once again regularity, we can show (7.14).

Lemma 7.3. *Let a Jacobi matrix satisfy the conditions of Lemma 1.11 and $\{\mathfrak{v}_\ell, \mathfrak{w}_\ell\}$ denote the coefficients of the associated block Jacobi matrix $\mathbf{J} = \Delta_{\mathsf{E}}(A)$. Then (7.14) holds.*

Proof. Recall that

$$h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1}) = \frac{1}{2} \operatorname{tr}\left((|\mathfrak{v}_\ell|^2 - I) + (|\mathfrak{v}^*_{\ell+1}|^2 - I) + \mathfrak{w}^2_\ell\right) - \log\det \mathfrak{v}_\ell \mathfrak{v}_{\ell+1}.$$

Regularity allows us to consider the terms in $h(\mathfrak{v}_\ell, \mathfrak{w}_\ell, \mathfrak{v}_{\ell+1})$ separately. It follows directly from (7.3) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\ell=0}^{N} \operatorname{tr} \mathfrak{w}^2_\ell = 0.$$

Moreover, (1.17) implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\ell=0}^{N}\log\det\mathfrak{v}_\ell=0.$$

Thus it remains to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(|\mathfrak{v}_\ell|^2-I)=0,\quad \lim_{N\to\infty}\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(|\mathfrak{v}_\ell^*|^2-I)=0. \tag{7.15}$$

For a matrix $A\in\mathrm{Mat}(n,\mathbb{R})$, let $\sigma_i(A)$ denote its singular values and note that $\operatorname{tr}|A|=\sum\sigma_i(A)$. For $A,B\in\mathrm{Mat}(n,\mathbb{R})$ we will need the following inequalities

$$|\operatorname{tr}A|\le\operatorname{tr}|A|,$$

$$\sum_{i=1}^{n}\sigma_i(AB)\le\sum_{i=1}^{n}\sigma_i(A)\sigma_i(B), \tag{7.16}$$

which can be found for instance in [16, eq. (3.3.35) and Theorem 3.3.14]. Thus we have

$$\left|\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(|\mathfrak{v}_\ell|^2-I)\right|\le\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(\big||\mathfrak{v}_\ell|^2-I\big|)=\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(|(|\mathfrak{v}_\ell|-I)(|\mathfrak{v}_\ell|+I)|).$$

Using (7.16) and a uniform bound on $\sigma_i(|\mathfrak{v}_\ell|+I)$ we get

$$\operatorname{tr}(|(|\mathfrak{v}_\ell|-I)(|\mathfrak{v}_\ell|+I)|)=\sum_{j=0}^{g}\sigma_j((|\mathfrak{v}_\ell|-I)(|\mathfrak{v}_\ell|+I))\le C\sum_{j=0}^{g}\sigma_j(|\mathfrak{v}_\ell|-I),$$

where C does not depend on ℓ. The last sum is the trace norm for $|\mathfrak{v}_\ell|-I$ and thus by the equivalence of norms on $\mathrm{Mat}(n,\mathbb{R})$ we find C_2 so that

$$\left|\frac{1}{N}\sum_{\ell=0}^{N}\operatorname{tr}(|\mathfrak{v}_\ell|^2-I)\right|\le C_2\frac{1}{N}\sum_{\ell=0}^{N}\||\mathfrak{v}_\ell|-I\|.$$

Define the set

$$I_N=\left\{\ell:\ \tilde{C}\|\mathfrak{v}_\ell-I\|>\frac{1}{2}\right\}\cap[1,N]$$

and note that (7.2) implies

$$\lim_{N\to\infty}\frac{|I_N|}{N}=0. \tag{7.17}$$

It follows as in [7, Proposition 11.12] that for $\ell\notin I_N$, there exists a constant C_3 so that

$$\||\mathfrak{v}_\ell|-I\|\le C_3\|\mathfrak{v}_\ell-I\|.$$

For $\ell\in I_N$ we can estimate $\||\mathfrak{v}_\ell|-I\|$ uniformly and using (7.17) and (7.3) we obtain (7.15). The proof for $\mathfrak{v}_\ell^*$ works the same by using [24, Lemma 4.6.5.] instead of [7, Proposition 11.12]. This finishes the proof. □

We are now ready to adapt Yuditskii's construction [32] to our setting. Let μ be a regular measure with $\operatorname{ess\,supp}\mu=\mathsf{E}$ and let J_+ be the associated Jacobi matrix. As already described after Lemma 7.1, we find J and $\tilde{J}_+$ such that all $\mathbf{c}_k\in\mathbf{C}_E$ belong to the resolvent set of $\tilde{J}_+$ and J and $\tilde{J}_+$ is also regular. Let $\tilde{A}_+$ and A denote the GMP matrix associated to $\tilde{J}_+$ and J respectively and $\{\mathfrak{v}_\ell,\mathfrak{w}_\ell\}$

denote the block Jacobi coefficients of $\Delta_{\mathsf{E}}(A)$. Let us further truncate A after N positive blocks before ∞ (i.e. before the position $-1+N(g+1)$) and extend it by some element $\mathring{A} \in \mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}})$ so that $\mathbf{c}_k \notin \sigma(A_N)$. To be precise, we first truncate A and consider its resolvent function $a_0^2 r_-$, then we can extend it as in Lemma 7.1 by some reflectionless r_+ so that all $\mathbf{c}_k \in \mathbf{C}_E$ belong to the resolvent set of the associated Jacobi matrix and then we consider the associated GMP matrix by [32, Proposition 5.5]. Since elements from the isospectral torus satisfy the magic formula and computing resolvents is a purely local process, we would like to conclude from the compactness of $\mathcal{T}_{\mathsf{E}}(\mathbf{C}_{\mathsf{E}})$ that

$$H_+(A_N) = \sum_{\ell=1}^{N} h(\mathfrak{v}_{\ell-1}, \mathfrak{w}_\ell, \mathfrak{v}_\ell) + O(1),$$

where A_N denotes the truncation described above. However, due to the log-term in the definition of $h(\mathfrak{v}_{\ell-1}, \mathfrak{w}_\ell, \mathfrak{v}_\ell)$ one must be careful. At the place where we modify A by extending it by $\mathring{A}$, by formula (6.15), when computing Λ_n, in a certain range of n given precisely below, one mixes coefficients from A and $\mathring{A}$. Thus we need to argue that

$$-\log \Lambda_n$$

does not grow too fast so that we can still conclude that

$$\lim_{N\to\infty} \frac{1}{N} H_+(A_N) = 0. \tag{7.18}$$

However, looking at the formula (6.15) and the definition of the Blaschke-Potapov factors, if all the coefficients can be bounded uniformly, we see that if $p_g^{(j)} > \delta$ we find a constant C only depending on the bounds of the coefficients and of δ so that

$$\Lambda_n(A_N) \geq C. \tag{7.19}$$

Note now that

$$\Lambda_{-1+(N-1)(g+1)}(A_N) = p_g^{(N-1)},$$

which is still a coefficient of A. But

$$\Lambda_{-1+N(g+1)}(A_N) =: \mathring{p}_g$$

is already a coefficient from $\mathring{A}$. The mixing of coefficients of A and $\mathring{A}$ in computing $\Lambda_n(A_N)$ happens for $-1+(N-1)(g+1) < n < -1+N(g+1)$. But in this case the only value that can make $\Lambda_n(A_N)$ small is $\mathring{p}_g$, and for elements of the isospectral torus we know that

$$\mathring{p}_g = \frac{1}{\lambda_{g+1}}$$

and thus we can conclude (7.19) and therefore (7.18).

Together with $\tilde{H}_+(A_N) \leq H_+(A_N)$, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \tilde{H}_+(A_N) = 0. \tag{7.20}$$

Realizing that all the arguments in [32, Theorem 1.20] are local, using $2N$ blocks of A, we can obtain a local version of this theorem.

Proposition 7.4. *Let J be constructed as above and A be the associated GMP matrix. Then, there exists an N independent constant C and a sparse set I_N such that*

$$\begin{aligned} \sum_{m=1}^{N} \|\vec{\mathbf{p}}_0(m) - \vec{\mathbf{p}}_{-1}(m)\|^2 &\leq C(\tilde{H}_+(A_{2N}) + |I_N|), \\ \sum_{m=1}^{N} \|\mathbf{F}_{\mathsf{E}}(\vec{\mathbf{p}}_0(m))\|^2 &\leq C(\tilde{H}_+(A_{2N}) + |I_N|). \end{aligned} \tag{7.21}$$

We will need a more quantitative version of [32, Lemma 6.6]:

Lemma 7.5. *Let ψ_n, $\tilde{\psi}_n$, τ_n and $\tilde{\tau}_n$ be given sequences and assume that there exists $\eta > 0$ such that*

$$\cos\psi_n \geq \eta,\ \cos\tilde{\psi}_n \geq \eta,\ 0 \leq \tau_n \leq \frac{1}{\eta},\ 0 \leq \tilde{\tau}_n \leq \frac{1}{\eta}. \tag{7.22}$$

Define

$$\alpha_n := \begin{bmatrix} \tau_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\psi_n & \cos\psi_n \\ \cos\psi_n & -\sin\psi_n \end{bmatrix} - \begin{bmatrix} \sin\tilde{\psi}_n & \cos\tilde{\psi}_n \\ \cos\tilde{\psi}_n & -\sin\tilde{\psi}_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\tau}_n \end{bmatrix}. \tag{7.23}$$

Then, there exists C depending only on η so that

$$\begin{aligned} \|\{\cos\psi_n - \cos\tilde{\psi}_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})} &\leq C\|\{\alpha_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})^{2\times 2}}, \\ \|\{\sin\psi_n - \sin\tilde{\psi}_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})} &\leq C\|\{\alpha_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})^{2\times 2}}. \end{aligned}$$

Proof. If $\|\{\alpha_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})^{2\times 2}} = \infty$ the claim is trivial. If it is finite, set $S := \|\{\alpha_n\}\|_{\ell^2(\mathbb{N},\mathbb{C})^{2\times 2}}$. The constant $C > 0$ may increase throughout the proof. Directly from (7.23) we have

$$\|\{\cos\psi_n - \cos\tilde{\psi}_n\}\|_{\ell^2} \leq S \quad \text{and} \quad \|\{\tau_n\cos\psi_n - \tilde{\tau}_n\cos\tilde{\psi}_n\}\|_{\ell^2} \leq S.$$

Since

$$\tau_n\cos\psi_n - \tilde{\tau}_n\cos\tilde{\psi}_n - \tilde{\tau}_n(\cos\psi_n - \cos\tilde{\psi}_n) = (\tau_n - \tilde{\tau}_n)\cos\psi_n,$$

using $\tilde{\tau}_n \leq \frac{1}{\eta}$ and $\cos\psi_n \geq \eta$ we find $C > 0$ so that

$$\|\{\tau_n - \tilde{\tau}_n\}\|_{\ell^2} \leq CS.$$

Now, we have another two conditions

$$\|\{\tau_n\sin\psi_n - \sin\tilde{\psi}_n\}\|_{\ell^2} \leq S \quad \text{and} \quad \|\{\sin\psi_n - \tilde{\tau}_n\sin\tilde{\psi}_n\}\|_{\ell^2} \leq S.$$

Using

$$\sin\psi_n - \tilde{\tau}_n\sin\tilde{\psi}_n = \sin\psi_n - \tau_n\tilde{\tau}_n\sin\psi_n - \tilde{\tau}_n(\sin\tilde{\psi}_n - \tau_n\sin\psi_n)$$

and $\|\{\tau_n\sin\psi_n - \sin\tilde{\psi}_n\}\| \leq S$ and $\tilde{\tau}_n \leq \frac{1}{\eta}$ we conclude that

$$\|\{\sin\psi_n(1 - \tau_n\tilde{\tau}_n)\}\|_{\ell^2} \leq CS.$$

Now we have

$$1 - \tau_n^2 = 1 - \tau_n\tilde{\tau}_n + \tau_n(\tilde{\tau}_n - \tau_n)$$

and since $|\sin\psi_n| \leq 1$ and $|\tau_n| \leq \frac{1}{\eta}$, we conclude

$$\|\{(\tau_n^2 - 1)\sin\psi_n\}\| \leq CS.$$

Again by, $|\tau_n| \leq \frac{1}{\eta}$ we also get a bound for $\{(\tau_n - 1)\sin\psi_n\}$. Finally, since

$$\sin\psi_n - \sin\tilde{\psi}_n = \tau_n \sin\psi_n - \sin\tilde{\psi}_n - (\tau_n - 1)\sin\psi_n,$$

we obtain the also the estimate for $\{\sin\psi_n - \sin\tilde{\psi}_n\}$. □

Proof of Proposition 7.4. In the proof we will find constants $C > 0$ and sparse sets I_N. These quantities will change throughout the proof. Note that the union of sparse sets is clearly sparse. First we mention an important locality property of the Jacobi flow. In the following we will derive estimates for entries of $A_{2N}(m)$ in the blocks 0 and -1. Due to the locality property of the Jacobi flow, for $0 < m \leq 2N - 1$, the coefficients of $A_{2N}(m)$ and $A(m)$ coincide; this is nicely visualized in the diagram [32, eq. (4.12)]. Similarly, we have already mentioned that computing entries of the resolvents, due to the band structure, can also be done locally. Thus, our estimates will be derived for the coefficients of $A_{2N}(m)$, but by restricting it to $0 < m \leq N$ they agree with the coefficients associated to A. For this reason we will also notationally not distinguish between the coefficients of A and the ones of A_{2N}.

By the explanation following Lemma 7.2 and (7.20), we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \left(G(x_0(m)) + G(x_{2g+2}(m)) + \sum_{j=1}^{2g+1} x_j(m)^2 \right) = 0.$$

Notice that G obeys

$$c_\epsilon^{-1}(x-1)^2 \leq G(x) \leq c_\epsilon (x-1)^2, \quad \forall x \in (\epsilon, \epsilon^{-1}).$$

Thus, we find a sparse set I_N and $C > 0$ so that

$$\begin{aligned} &\sum_{m=1}^{N} \left((x_0(m)-1)^2 + (x_{2g+2}(m)-1)^2 + \sum_{j=1}^{2g+1} x_j(m)^2 \right) \\ &\qquad \leq C \Bigg(\sum_{m=1}^{N} \left(G(x_0(m)) + G(x_{2g+2}(m)) + \sum_{j=1}^{2g+1} x_j(m)^2 \right) + |I_N| \Bigg). \end{aligned} \tag{7.24}$$

Thus, for $1 \leq j \leq 2g+1$,

$$\|\{x_j(m)\}_{m=1}^N\|_2 \leq C \left(\tilde{H}_+(A_N) + |I_N| \right)$$

and

$$\|\{x_0(m) - 1\}_{m=1}^N\|_2 \leq C \left(\tilde{H}_+(A_N) + |I_N| \right),$$
$$\|\{x_{2g+2}(m) - 1\}_{m=1}^N\|_2 \leq C \left(\tilde{H}_+(A_N) + |I_N| \right).$$

We note that

$$x_{2g+2}(m) = \lambda_0 \Lambda_{-1}(m) = \lambda_0 p_g^{(0)}(m),$$

and thus

$$\frac{1}{N} \sum_{m=1}^{N} (\lambda_0 p_g^{(0)}(m) - 1)^2 = 0. \tag{7.25}$$

This is one component of F_{E}.

Let us now show the first inequality in (7.21). Denote $\hat{A} = \mathcal{O}A$, where $\mathcal{O}A$ is the transform defined in (6.18). We use the hat for all entries related to $\hat{A}$ and

$\Delta_{\mathsf{E}}(\hat{A})$, respectively. The entries of $A(m)$ are denoted by $\{p_k^{(j)}(m), q_k^{(j)}(m)\}$. Recall that m corresponds to application of the Jacobi flow, j denotes the block and k the component of the vector $\vec{p}_j(m)$. We use similar notation for $\hat{A}$, $\Delta_{\mathsf{E}}(A)$ and $\Delta_{\mathsf{E}}(\hat{A})$. Due to the definition (6.18), we find

(7.26)
$$\begin{bmatrix} v_{g-1,g-1}^{(0)}(m) & 0 \\ v_{g,g-1}^{(0)}(m) & \lambda_0 p_g^{(0)}(m) \end{bmatrix} \mathbf{o}(\phi_g^{(0)}(m)) = \mathbf{o}(\phi_g^{(-1)}(m)) \begin{bmatrix} \lambda_0 \hat{p}_g^{(0)}(m) & 0 \\ \hat{w}_{0,g}^{(0)}(m) & \hat{v}_{0,0}^{(1)}(m) \end{bmatrix}.$$

Note that $v_{g,g-1}^{(0)}(m) = x_{2g+1}(m)$. It was mentioned after Lemma 7.2 that $H_+(\hat{A})$ can be expressed in terms of $H_+(A)$. Therefore, we conclude by (6.20) that (7.25) also holds for $\hat{p}_g^{(0)}(m)$. Note that $x_{g+2}(m) = \hat{w}_{0,g}^{(-1)}(m)$. Since shifting by a full block in the very beginning only adds a fixed constant, and $\mathcal{J}$ commutes with this shift by (6.21), we can apply Lemma 7.5 to (7.26) and obtain by (7.24) that

$$\|\{\sin\phi_g^{(-1)}(m) - \sin\phi_g^{(0)}(m)\}_{m=1}^N\|_{\ell^2} \leq C(\tilde{H}_+(A_N) + |I_N|).$$

Thus, by (6.17)

$$\|\{p_{g-1}^{(-1)} - p_{g-1}^{(0)}\}_{m=1}^N\|_{\ell^2} \leq C(\tilde{H}_+(A_N) + |I_N|). \tag{7.27}$$

Since by [32, eq (4.2)] one can pass from j to $j-1$ by using $\hat{A}$, we obtain (7.27) for $0 \leq j \leq g$. Similarly, by [32, eq (4.2)], one obtains the estimates for the q_j-coefficients. This finishes the proof of the first inequality in (7.21).

It remains to prove (7.21) for the other components of F_{E}. The proof of Lemma 7.5 yields an estimate for $\|\{(v_{g-1,g-1}^{(-1)}(n) - 1)\sin\phi_g^{(-1)}(n)\}\|_2$ or, equivalently, it shows

$$\|\{(\Lambda_{-2}(m)\lambda_g - 1)p_{g-1}^{(-1)}(m)\}\| \leq C(\tilde{H}_+(A_N) + |I_N|).$$

Since $p_{g-1}^{(-1)}(m)$ may approach zero, it does not imply yet give an estimate for $\{(\Lambda_{-2}(m)\lambda_g - 1)\}$. If we can also estimate

$$\|\{(\Lambda_{-2}(m)\lambda_g - 1)q_{g-1}^{(-1)}(m)\}_{m=1}^N\|,$$

then $\inf_m \left((q_{g-1}^{(-1)}(m))^2 + (p_{g-1}^{(-1)}(m))^2\right) > 0$ yields

$$\|\{(\Lambda_{-2}(m)\lambda_g - 1)\}_{m=1}^N\| \leq C(\tilde{H}_+(A_N) + |I_N|).$$

To this end, we note that

$$\Lambda_{-2}(m+1) = \frac{\cos\phi_g^{(-1)}(m)}{\cos\phi_g^{(-2)}(m)}\Lambda_{-2}(m). \tag{7.28}$$

Indeed, by definition of the Jacobi flow

$$U(\vec{p}_{-2}(m)) \begin{bmatrix} v_{g,g}^{(-2)} & & & \\ * & v_{0,0}^{(-1)} & & \\ * & * & \ddots & \\ * & * & * & v_{g-1,g-1}^{(-1)} \end{bmatrix} (m+1) = \mathfrak{v}_{-1}(m)U(\vec{p}_{-1}(m)),$$

the second from below entry in the last column in this matrix identity means exactly (7.28). Since by the above, we can estimate $\|\{\cos\phi_g^{(-1)}(m) - \cos\phi_g^{(-2)}(m)\}_{m=1}^N\|_{\ell^2}$ we obtain

$$\|\{\Lambda_{-2}(m+1) - \Lambda_{-2}(m)\}_{m=1}^N\|_{\ell^2} \leq C(\tilde{H}_+(A_N) + |I_N|).$$

Now by [32, (4.10)] we have

$$p_{g-1}^{(-1)}(m) = -q_{g-1}^{(-1)}(m+1)f(m),$$

where $f(m)$ is an explicit function that can be small only on a sparse set. Combining this with

$$(\Lambda_{-2}(m)\lambda_g - 1)p_{g-1}^{(-1)}(m) = -(\Lambda_{-2}(m)\lambda_g - 1)q_{g-1}^{(-1)}(m+1)f(m),$$

we also get an estimate for $\|\{(\Lambda_{-2}(m)\lambda_g - 1)q_{g-1}^{(-1)}(m)\}_{m=1}^N\|_{\ell^2}$, which shows

$$\|\{(\Lambda_{-2}(m)\lambda_g - 1)\}\| \le C(\tilde{H}_+(A_N) + |I_N|).$$

The same arguments with respect to $\mathcal{O}^k A$, $k = 1, \dots, g-1$, in a combination with (6.20), yield the estimates for all other components of F_{E}. □

Lemma 7.6. *There exist* $\{\epsilon_m^\alpha\} \in \mathrm{CS}(\mathbb{N}, \mathbb{R}^g/\mathbb{Z}^g)$ *and* $\{\epsilon_m^a\} \in \mathrm{CS}(\mathbb{N}, \mathbb{R}), \{\epsilon_n^b\} \in \mathrm{CS}(\mathbb{N}, \mathbb{R})$ *so that*

$$a_m^2 = \mathcal{A}\Big(\sum_{j=1}^m \varepsilon_j^\alpha - m\chi\Big) + \varepsilon_m^a,$$

$$b_m = \mathcal{B}\Big(\sum_{j=1}^m \varepsilon_j^\alpha - m\chi\Big) + \varepsilon_m^b,$$

where $\mathcal{A}, \mathcal{B}$ *are given in* (7.11).

Proof. Let $A(\vec{\mathbf{p}}_0(m))$ be the periodic GMP matrix with coefficients $\vec{\mathbf{p}}_0(m)$ and $A(\alpha_m) \in \mathcal{T}_{\mathsf{E}}(\mathbf{C}_E)$, so that

$$\mathrm{dist}(\vec{\mathbf{p}}_0(m), \mathcal{IS}_{\mathsf{E}}) = \mathrm{dist}(\vec{\mathbf{p}}_0(m), \mathring{\vec{\mathbf{p}}}(\alpha_m)).$$

Thus, using (7.12) we obtain

$$\sum_{m=1}^N \mathrm{dist}(\vec{\mathbf{p}}_0(m), \mathring{\vec{\mathbf{p}}}(\alpha_m))^2 \le C(\tilde{H}_+(A_N) + |I_N|)$$

and by (6.6) we get

$$\sum_{m=1}^N (a_m^2 - \mathcal{A}(\alpha_m))^2 \le C(\tilde{H}_+(A_N) + |I_N|),$$

$$\sum_{m=1}^N (b_m - \mathcal{B}(\alpha_m))^2 \le C(\tilde{H}_+(A_N) + |I_N|),$$

where again $\{a_m, b_m\}_{m\in\mathbb{N}_0}$ denote the coefficients of J_+. Thus, dividing by N and sending $N \to \infty$, we obtain by (7.17) and (7.20) that

$$\{a_m^2 - \mathcal{A}(\alpha_m)\}_{m\in\mathbb{N}_0}, \{b_m - \mathcal{B}(\alpha_m)\}_{m\in\mathbb{N}_0} \in \mathrm{CS}\,. \tag{7.29}$$

The smoothness of the Jacobi flow transform, provided that $p_g^{(0)}, p_g^{(1)} > \delta$, allows for the definition of a sparse set I_N so that

$$\begin{aligned}\mathrm{dist}(\vec{\mathbf{p}}_0(m+1), \mathring{\vec{\mathbf{p}}}(\alpha_m - \chi)) &= \mathrm{dist}(\mathcal{J}(\vec{\mathbf{p}}_0(m), \vec{\mathbf{p}}_1(m)), \mathcal{J}(\vec{\mathbf{p}}(\alpha_m)) \\ &\le C(\mathsf{E}, J, \delta)\{\mathrm{dist}(\vec{\mathbf{p}}_0(m), \mathring{\vec{\mathbf{p}}}(\alpha_m)) \\ &\quad + \mathrm{dist}(\vec{\mathbf{p}}_0(m), \vec{\mathbf{p}}_1(m)) + |I_N|\}.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{dist}(\mathring{\vec{\mathbf{p}}}(\alpha_{m+1}), \mathring{\vec{\mathbf{p}}}(\alpha_m - \chi)) \leq C(\mathsf{E}, J, \delta)(&\operatorname{dist}(\vec{\mathbf{p}}_0(m), \mathring{\vec{\mathbf{p}}}(\alpha_m)) \\ &+ \operatorname{dist}(\vec{\mathbf{p}}_0(m), \vec{\mathbf{p}}_1(m)) + |I_N|) \\ &+ \operatorname{dist}(\vec{\mathbf{p}}_0(m+1), \mathring{\vec{\mathbf{p}}}(\alpha_{m+1})).\end{aligned}$$

Moreover, we have

$$\|\alpha - \beta\| \leq C_1(E) \operatorname{dist}(\mathring{\vec{\mathbf{p}}}(\alpha), \mathring{\vec{\mathbf{p}}}(\beta)).$$

Thus, defining $\epsilon_\alpha(m) = \alpha_{m+1} - (\alpha_m - \chi)$, we conclude from (7.21) that

$$\{\epsilon_\alpha\} \in \mathrm{CS}(\mathbb{N}, \mathbb{R}^g/\mathbb{Z}^g).$$

□

Lemma 7.7. *For fixed $L \in \mathbb{N}$ and $\delta > 0$, the set*

$$B_{L,\delta} = \left\{ m : \left\| \sum_{j=m+1}^{m+\ell} \epsilon_j^\alpha \right\| \leq \delta \text{ for all } \ell = 0, \dots, L-1 \right\}$$

has a sparse complement, i.e., $\frac{|B_{L,\delta} \cap \{1,\dots,N\}|}{N} \to 1$ as $N \to \infty$.

Proof. Since shifts and linear combinations of CS sequences are in CS, $\left\{ \sum_{j=m+1}^{m+\ell} \epsilon_j^\alpha \right\}_{m=0}^\infty \in \mathrm{CS}$ for any ℓ. Thus, for any ℓ, the set $\left\{ m : \left\| \sum_{j=m+1}^{m+\ell} \epsilon_j^\alpha \right\| > \delta \right\}$ is sparse; the complement of $B_{L,\delta}$ is a union of finitely many sparse sets, so it is sparse. □

Proof of Theorem 1.10. It remains to prove that, for every $\epsilon > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \operatorname{dist}(\mathcal{T}_{\mathsf{E}}^+, (S_+^*)^m J_+ S_+^m) \leq \epsilon. \tag{7.30}$$

Fix L so that $\sum_{\ell=L}^\infty e^{-\ell} \|J_+\| \leq \epsilon/16$. Choose $\delta > 0$ so that

$$|\mathcal{A}(\beta_1) - \mathcal{A}(\beta_2)| \leq \frac{\epsilon}{8L}, \qquad |\mathcal{B}(\beta_1) - \mathcal{B}(\beta_2)| \leq \frac{\epsilon}{8L} \tag{7.31}$$

whenever $|\beta_1 - \beta_2| \leq \delta$.

Since $\operatorname{dist}(\mathcal{T}_{\mathsf{E}}^+, (S_+^*)^m J_+ S_+^m)$ is uniformly bounded in m and the complement of $B_{L,\delta}$ is sparse,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\substack{1 \leq m \leq N \\ m \notin B_{L,\delta}}} \operatorname{dist}(\mathcal{T}_{\mathsf{E}}^+, (S_+^*)^m J_+ S_+^m) = 0.$$

Set $\alpha_m = \sum_{j=1}^m \varepsilon_j^\alpha$. For $m \in B_{L,\delta}$, estimating the distance to $\mathcal{T}_{\mathsf{E}}^+$ by the distance to $J(\alpha_m - m\chi)$ gives

$$\begin{aligned}&\operatorname{dist}(\mathcal{T}_{\mathsf{E}}^+, (S_+^*)^m J_+ S_+^m) \\ &\qquad \leq \sum_{\ell=0}^{\infty} e^{-\ell} (|a_{m+\ell} - \mathcal{A}(\alpha_m - (m+\ell)\chi)| + |b_{m+\ell} - \mathcal{B}(\alpha_m - (m+\ell)\chi)|).\end{aligned}$$

Using (7.31) for $\ell < L$ and using our choice of L to bound the tail of the series, we obtain

$$\begin{aligned}&\operatorname{dist}(\mathcal{T}_{\mathsf{E}}^{+}, (S_{+}^{*})^{m} J_{+} S_{+}^{m}) \\ &\leq \frac{\epsilon}{2} + \sum_{\ell=0}^{L-1} e^{-\ell}(|a_{m+\ell} - \mathcal{A}(\alpha_{m+\ell} - (m+\ell)\chi)| + |b_{m+\ell} - \mathcal{B}(\alpha_{m+\ell} - (m+\ell)\chi)|).\end{aligned}$$

Thus, to prove (7.30), it remains to prove

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\substack{1\leq m\leq N \\ m\in B_{L,\delta}}} \sum_{\ell=0}^{L-1} e^{-\ell} g_{m+\ell} \leq \frac{\epsilon}{2}, \tag{7.32}$$

where $g_p = |a_p - \mathcal{A}(\alpha_p - p\chi)| + |b_p - \mathcal{B}(\alpha_p - p\chi)|$. Note $g \in \mathrm{CS}$ by (7.29). Enlarging the range of summation, we obtain

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{\substack{1\leq m\leq N \\ m\in B_{L,\delta}}} \sum_{\ell=0}^{L-1} e^{-\ell} g_{m+\ell} \leq \limsup_{N\to\infty} \frac{1}{N} \sum_{p=1}^{N+L} \sum_{\ell=0}^{L-1} e^{-\ell} g_p.$$

Now the sum in ℓ can be separated as an explicit constant, so this lim sup is zero since $g \in \mathrm{CS}$. Then (7.32) follows, and the proof of (7.30) is complete. □

References

[1] Damir Z. Arov and Harry Dym, *J-contractive matrix valued functions and related topics*, Encyclopedia of Mathematics and its Applications, vol. 116, Cambridge University Press, Cambridge, 2008, DOI 10.1017/CBO9780511721427. MR2474532

[2] David H. Armitage and Stephen J. Gardiner, *Classical potential theory*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2001, DOI 10.1007/978-1-4471-0233-5. MR1801253

[3] Adhemar Bultheel, Pablo González-Vera, Erik Hendriksen, and Olav Njåstad, *Orthogonal rational functions*, Cambridge Monographs on Applied and Computational Mathematics, vol. 5, Cambridge University Press, Cambridge, 1999, DOI 10.1017/CBO9780511530050. MR1676258

[4] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56, DOI 10.1016/S0024-3795(02)00457-3. MR1955452

[5] Jacob S. Christiansen, Barry Simon, and Maxim Zinchenko, *Finite gap Jacobi matrices: a review*, Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday, Proc. Sympos. Pure Math., vol. 87, Amer. Math. Soc., Providence, RI, 2013, pp. 87–103, DOI 10.1090/pspum/087/01429. MR3087900

[6] Jacob S. Christiansen, Benjamin Eichinger, and Tom VandenBoom, *Finite-gap CMV matrices: periodic coordinates and a magic formula*, Int. Math. Res. Not. IMRN **18** (2021), 14016–14085, DOI 10.1093/imrn/rnz213. MR4320803

[7] David Damanik, Rowan Killip, and Barry Simon, *Perturbations of orthogonal polynomials with periodic recursion coefficients*, Ann. of Math. (2) **171** (2010), no. 3, 1931–2010, DOI 10.4007/annals.2010.171.1931. MR2680401

[8] David Damanik, Alexander Pushnitski, and Barry Simon, *The analytic theory of matrix orthogonal polynomials*, Surv. Approx. Theory **4** (2008), 1–85. MR2379691

[9] David Damanik and Peter Yuditskii, *Counterexamples to the Kotani-Last conjecture for continuum Schrödinger operators via character-automorphic Hardy spaces*, Adv. Math. **293** (2016), 738–781, DOI 10.1016/j.aim.2016.02.023. MR3474334

[10] Karl Deckers and Doron S. Lubinsky, *Christoffel functions and universality limits for orthogonal rational functions*, Anal. Appl. (Singap.) **10** (2012), no. 3, 271–294, DOI 10.1142/S0219530512500133. MR2948895

[11] Joseph L. Doob, *Classical potential theory and its probabilistic counterpart*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1984 edition, DOI 10.1007/978-3-642-56573-1. MR1814344

[12] Benjamin Eichinger, *Periodic GMP matrices*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 066, 19, DOI 10.3842/SIGMA.2016.066. MR3519566

[13] B. Eichinger and M. Lukić. 2020. *Stahl–Totik regularity for continuum Schrödinger operators*, arXiv:2001.00875.

[14] Fritz Gesztesy, Helge Holden, Johanna Michor, and Gerald Teschl, *Soliton equations and their algebro-geometric solutions. Vol. II*, Cambridge Studies in Advanced Mathematics, vol. 114, Cambridge University Press, Cambridge, 2008. (1 + 1)-dimensional discrete models, DOI 10.1017/CBO9780511543203. MR2446594

[15] Leonid Golinskii and Sergei Khrushchev, *Cesàro asymptotics for orthogonal polynomials on the unit circle and classes of measures*, J. Approx. Theory **115** (2002), no. 2, 187–237, DOI 10.1006/jath.2001.3655. MR1901215

[16] Roger A. Horn and Charles R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original. MR1288752

[17] Helge Krüger, *Probabilistic averages of Jacobi operators*, Comm. Math. Phys. **295** (2010), no. 3, 853–875, DOI 10.1007/s00220-010-1014-y. MR2600036

[18] Paul G. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **18** (1979), no. 213, v+185, DOI 10.1090/memo/0213. MR519926

[19] V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Amer. Math. Soc. Transl. (2) **15** (1960), 131–243, DOI 10.1090/trans2/015/07. MR0114915

[20] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995, DOI 10.1017/CBO9780511623776. MR1334766

[21] Christian Remling, *The absolutely continuous spectrum of Jacobi matrices*, Ann. of Math. (2) **174** (2011), no. 1, 125–171, DOI 10.4007/annals.2011.174.1.4. MR2811596

[22] Barry Simon, *Equilibrium measures and capacities in spectral theory*, Inverse Probl. Imaging **1** (2007), no. 4, 713–772, DOI 10.3934/ipi.2007.1.713. MR2350223

[23] Barry Simon, *Regularity and the Cesàro-Nevai class*, J. Approx. Theory **156** (2009), no. 2, 142–153, DOI 10.1016/j.jat.2008.04.016. MR2494549

[24] Barry Simon, *Szegő's theorem and its descendants*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011. Spectral theory for L^2 perturbations of orthogonal polynomials. MR2743058

[25] Barry Simon, *Basic complex analysis*, A Comprehensive Course in Analysis, Part 2A, American Mathematical Society, Providence, RI, 2015, DOI 10.1090/simon/002.1. MR3443339

[26] Mikhail Sodin and Peter Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), no. 3, 387–435, DOI 10.1007/BF02921627. MR1674798

[27] Herbert Stahl and Vilmos Totik, *General orthogonal polynomials*, Encyclopedia of Mathematics and its Applications, vol. 43, Cambridge University Press, Cambridge, 1992, DOI 10.1017/CBO9780511759420. MR1163828

[28] Gerald Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, vol. 72, American Mathematical Society, Providence, RI, 2000, DOI 10.1090/surv/072. MR1711536

[29] Gerald Teschl, *Mathematical methods in quantum mechanics*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators, DOI 10.1090/gsm/157. MR3243083

[30] J. L. Ullman, *On the regular behaviour of orthogonal polynomials*, Proc. London Math. Soc. (3) **24** (1972), 119–148, DOI 10.1112/plms/s3-24.1.119. MR291718

[31] A. Volberg and P. Yuditskii, *Kotani-Last problem and Hardy spaces on surfaces of Widom type*, Invent. Math. **197** (2014), no. 3, 683–740, DOI 10.1007/s00222-013-0495-7. MR3251833

[32] P. Yuditskii, *Killip-Simon problem and Jacobi flow on GMP matrices*, Adv. Math. **323** (2018), 811–865, DOI 10.1016/j.aim.2017.11.005. MR3725892

Institute of Analysis, Johannes Kepler University of Linz, 4040 Linz, Austria
Email address: benjamin.eichinger@jku.at

Department of Mathematics, Rice University MS-136, Box 1892, Houston, Texas 77251-1892
Email address: milivoje.lukic@rice.edu

Department of Mathematics, Rice University MS-136, Box 1892, Houston, Texas 77251-1892
Email address: gfy1@rice.edu