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Mathematical theory for topological photonic materials in one dimension

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Abstract

This work presents a rigorous theory for topological photonic materials in one dimension. The main focus is on the existence of interface modes that are induced by topological properties of the bulk structure. For a general 1D photonic structure with time-reversal symmetry, we investigate the existence of an interface mode that is induced by a Dirac point upon perturbation. Specifically, we establish conditions on the perturbation which guarantee the opening of a band gap around the Dirac point and the existence of an interface mode. For a periodic photonic structure with both time-reversal and inversion symmetry, the Zak phase is quantized, taking only two values $0, \pi$. We show that the Zak phase is determined by the parity (even or odd) of the Bloch modes at the band edges. For a photonic structure consisting of two semi-infinite systems on the two sides of an interface with distinct topological indices, we show the existence of an interface mode inside the common gap. The stability of the mode under perturbations is also investigated. Finally, we study resonances for finite topological structures. Our results are based on the transfer matrix method and the oscillation theory for Sturm–Liouville operators. The methods and results can be extended to general topological Sturm–Liouville systems in one dimension.

Keywords: topological photonics, interface mode, Dirac point, Zak phase, bulk-edge correspondence

(Some figures may appear in colour only in the online journal)

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1. Introduction

Topological insulators are electronic materials that conduct electrons on their edges or surfaces without backscattering. The underlying protected edge/interface mode is robust in the presence of large impurities, which prevents the degradation of device performance due to fabrication imperfections. Tremendous progress has been made in the past several decades in the studies of topological insulators and quantum topological materials in general in electron systems [8, 23]. In recent years, there has been increasing interest in exploring the analogue of the quantum topological materials for periodic photonic/phononic band gap materials [21, 26, 30–33].

From the mathematical point of view, there are several important questions in the studies of topological materials.

(a) The first one concerns with the Dirac points of the band structure for the topological material. Dirac points are special vertices located at the Brillouin zone corners when two bands in the spectrum touch in a linear conical fashion and degeneracies occur for the corresponding Bloch modes [21]. We refer to [19] for the rigorous mathematical studies of Dirac points in 1D periodic Schrödinger operator with double-well potential and [17] for the investigation of Dirac points for Schrödinger operator with the Honeycomb lattice potentials in 2D. In general, a topological phase transition takes place near the Dirac point and interesting physics phenomena occurs as a result. This is exemplified in photonic graphene and subwavelength resonators in [1, 4, 18, 28] and references therein.
(b) The second question is the existence of interface modes (also called edge modes or edge states) that are supported at the interface of two structures with distinct topological invariants. This is typically formulated as the so-called bulk-edge correspondence, which formally states that the bulk index is equal to the edge index. The former is a topological quantity that can be computed from the bulk media, while the latter is related to the number of edge modes supported by the structure. A variety of tools have been developed for the study of the bulk-edge correspondence in different settings, including K-theory, functional analysis, microlocal analysis, etc [6, 7, 11, 13, 15, 16, 22, 25, 34, 35].
(c) The third one is the stability of the interface modes supported by the topological materials. Such modes are 'topologically protected' in the sense that they are stable against the system perturbations that are not necessarily small; see, for instance, [19] and [5] for the mathematical investigation of stability for edge mode in 1D Schrödinger system and subwavelength resonators respectively.

In this paper, we study one-dimensional photonic structures with time-reversal symmetry. The corresponding periodic differential operator is given by:

$$\mathcal{L}\psi = -\frac{1}{\varepsilon(x)}\frac{d}{dx}\left(\frac{1}{\mu(x)}\frac{d\psi}{dx}\right) \quad \text{for } x \in \mathbf{R}, \tag{1.1}$$

and the coefficients satisfy the following:

Assumption 1. The permittivity $\varepsilon(x)$ and the permeability $\mu(x)$ are positively valued and are piece-wisely Lipschitz continuous with period one:

$$\varepsilon(x) = \varepsilon(x+1), \quad \mu(x) = \mu(x+1). \tag{1.2}$$

We aim to provide a rigorous mathematical theory for the one-dimensional topological structures, especially on the existence and stability of the interface modes. Based on the transfer matrix method, we characterize the Dirac points of the structure precisely. We present explicit conditions for the perturbation of the structure so that a band gap can be opened near the Dirac point and an interface mode can be generated (cf theorem 4.7). For structures with additional inversion symmetry, we provide explicit formulas for the Berry phase, which is also called the Zak phase for one-dimensional structures. In this scenario, the Zak phase is closely related to the parity of the Bloch modes at the band edges. It is quantized by taking the value of 0 or π only and hence becomes a natural bulk topological index. We establish the existence and investigate the stability of interface modes when two semi-infinite periodic structures attain distinct topological indices. These results are formulated in theorems 5.10 and 5.11. Furthermore, we study the resonances for finite topological structures, for which the eigenvalues are complex-valued and they converge exponentially fast to the eigenvalues of the infinite topological structure as the size of the structure increases (cf theorem 6.3).

We mention several closely related works [12, 14, 19, 20], where one-dimensional Schrödinger equations with periodic potentials are studied using multiscale analysis or scattering theory for highly oscillatory operators, etc. It is shown in [19] that for a class of background periodic Schrödinger operators with Dirac points, localized edge states can be induced via small and adiabatic modulation of the periodic potentials with a domain wall, and the bifurcation of these states are associated with the discrete eigenmodes of an effective Dirac operator. In [14], the authors study a topological system where the background periodic Schrödinger operator is perturbed by a small and adiabatic dislocation. It is shown that all the edge states of the dislocated system are associated with the eigenmodes of an effective Dirac operator. Moreover, the full asymptotic expansions of the eigenpairs are derived. In [12, 20], the bulk-edge correspondence is rigorously established for a family of operators, wherein each operator corresponds to a dislocation of the background periodic Schrödinger operator. In this work, we apply the transfer matrix method and the mathematical theories for the Sturm–Liouville operators to derive the existence of interface modes for the underlying differential operator in the photonic system. We also investigate the stability of interface mode and study the resonant finite topological structure. Compared to [12, 14, 19, 20], the approach developed and the results obtained in this paper have the following new features:

- The analytical approach is based on the transfer matrix method and can be applied to differential operators with discontinuous coefficients. In addition, the Prüfer transform is used to study the stability of interface modes.
- The concept of **impedance function** for semi-infinity periodic structures is introduced and their basic properties are established. They are used to derive conditions on the existence of interface modes at the interface of two different periodic operators. This new approach can address the case of a sharp interface that separates two different materials.
- A characterization of Dirac points for the periodic operator (1.1) is given in terms of the discriminant of the transfer matrix.
- A general perturbation theory for opening a band gap near a Dirac point of the periodic operator (1.1) is established for the first time. So is the theory for the bifurcation of interface eigenvalue from a Dirac point under a general perturbation.
- The existence of the interface modes is proved for topological materials with inversion symmetry and a bulk-interface correspondence type result is established.

Finally, we also refer to [3, 5] for the studies on topologically protected edge states in a one-dimensional chain of subwavelength resonators in three dimensions and [10] for using a combination of the transfer matrix method and the homogenization approach to study Su–Schrieffer–Heeger model.

The rest of the paper is organized as follows. In section 2, we recall the band structure theory for the 1D periodic differential operators, and introduce the Dirac point and Zak phase for the structure. The concepts of impedance function and interface mode for a joint photonic structure are introduced in section 3. Section 4 studies the perturbation of a time-reversal symmetric photonic structure with a Dirac point and the existence of an interface mode for the perturbed system. Section 5 focuses on time-reversal symmetric structures that attain inversion symmetry. The existence of an interface mode that is predicted by the bulk topological indices and its stability under perturbations that are not necessarily small are established. Finally, the studies of resonances for finite topological structures is provided in section 6.

2. Band structure, Dirac point and Zak phase for the periodic structure

In this section, we present the band structure theory for the periodic differential operator $\mathcal{L}$ defined in (1.1). We also introduce Dirac points at the corners of the reduced Brillouin zone and the Zak phase for the periodic structure.

2.1. The transfer matrix and the spectrum of the operator $\mathcal{L}$

The spectrum of the operator $\mathcal{L}$ can be characterized using the Floquet–Bloch theory and the transfer matrix. For completeness we collect several key results in this section. The readers are referred to [27] for more details about the Floquet–Bloch theory for periodic differential operators.

Throughout, we let $L^2(\mathbf{R})$ be the Hilbert space equipped with the inner product:

$$(\psi, \phi) = \int_{\mathbf{R}} \varepsilon(x)\psi(x)\bar{\phi}(x)dx.$$

Here and throughout the notation $\bar{\phi}(x)$ means the complex conjugate of the number $\phi(x)$. We use the notation X for the Hilbert space $L^2[0,1]$ equipped with the inner product:

$$(u, v) = \int_0^1 \varepsilon(x)u(x)\bar{v}(x)dx. \tag{2.1}$$

Let $\mathcal{B} = [-\pi, \pi]$ be the Brillouin zone and $[0, \pi]$ be the reduced Brillouin zone. For each Bloch wavenumber $k \in \mathcal{B}$, we consider the following one-parameter family of Floquet–Bloch eigenvalue problems:

$$\mathcal{L}\psi(x) = E\psi(x) \quad x \in \mathbf{R}, \tag{2.2}$$

in the function space:

$$L^2_k = \{u \in L^2_{loc} : u(x+1) = e^{ik}u(x)\},$$

where $L^2_{loc} := \{u : \int_K \varepsilon(x)|u(x)|^2\, dx < \infty,\ \forall \text{ compact set } K \subset \mathbf{R}\}$. For each $k \in \mathcal{B}$, the eigenvalue problem (2.2) is self-adjoint with the inner product (2.1) and attains a discrete set of real eigenvalues:

$$E_1(k) \leqslant E_2(k) \leqslant \cdots \leqslant E_j(k) \leqslant \cdots.$$

The function $E_j(k)$ is called the dispersion relation of the jth band and the eigenfunction associated with $E_j(k)$ is called the jth Bloch mode with quasi-momentum k. In electromagnetism, the eigenvalue E and the frequency of the photonic mode ω is related by $E = \omega^2$ [24]. For convenience of notations, we present the band theory using the eigenvalue E. The corresponding band theory using the frequency ω is similar.

For each integer j, let,

$$E_j^- = \min\{E_j(k) : k \in \mathcal{B}\}, \quad E_j^+ = \max\{E_j(k) : k \in \mathcal{B}\}.$$

Then the entire spectrum of the operator $\mathcal{L}$ on $L^2(\mathbf{R})$ is given by:

$$\sigma(\mathcal{L}) = \bigcup_{j\geqslant 1} [E_j^-, E_j^+],$$

which corresponds to the essential spectrum of the operator. If $E_j^+ < E_{j+1}^-$, the spectrum forms a band gap between the two bands $E_j(k)$ and $E_{j+1}(k)$.

The band structure of the spectrum can be characterized using the transfer matrix. To this end, for each $E \in \mathbf{R}$, we let $\psi_{E,1}$ and $\psi_{E,2}$ to be the solutions to the following initial value problems in $\mathbf{R}$ respectively:

$$(\mathcal{L} - E)\psi_{E,1} = 0, \quad \psi_{E,1}(0) = 1, \ \frac{1}{\mu(0)}\psi'_{E,1}(0) = 0, \tag{2.3}$$

$$(\mathcal{L} - E)\psi_{E,2} = 0, \quad \psi_{E,2}(0) = 0, \ \frac{1}{\mu(0)}\psi'_{E,2}(0) = 1. \tag{2.4}$$

For $x \in \mathbf{R}$, let

$$\Psi_E(x) := (\Psi_{E,1}(x), \Psi_{E,2}(x)) = \begin{pmatrix} \psi_{E,1}(x) & \psi_{E,2}(x) \\ \frac{1}{\mu(x)}\psi'_{E,1}(x) & \frac{1}{\mu(x)}\psi'_{E,2}(x) \end{pmatrix}. \tag{2.5}$$

It is clear that the second-order initial value problems (2.3) and (2.4) are equivalent to the following first-order initial value problem in $\mathbf{R}$:

$$\frac{d}{dx}\Psi_E(x) = J(B + EW)\Psi_E(x), \quad \Psi_E(0) = Id, \tag{2.6}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = B(x) = \begin{pmatrix} 0 & 0 \\ 0 & \mu(x) \end{pmatrix}, \quad W = W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.7}$$

Recall that the coefficients $\varepsilon(x)$ and $\mu(x)$ are real-valued and are piece-wisely Lipschitz continuous. It follows from the theory of first-order linear ordinary differential equations that the solution $\Psi_E(x)$ is uniquely defined for all x in $\mathbf{R}$. Moreover, it is real-valued and absolutely continuous in $\mathbf{R}$, and it depends on E smoothly. In particular, the functions $\psi'_{E,j}(x)$ and $\frac{1}{\mu(x)}\psi'_{E,j}(x)$ $(j = 1, 2)$ are real-valued and continuous for $x \in \mathbf{R}$.

Remark 1. At a point of discontinuity of μ, say $x = x_0$, we should interpret the value $\frac{1}{\mu(x_0)}\psi'_{E,1}(x_0)$ as either the left-sided limit $\frac{1}{\mu(x_0^-)}\psi'_{E,1}(x_0^-) =: \lim_{x\to x_0^-} \frac{1}{\mu(x)}\psi'_{E,1}(x)$ or the right-sided limit $\frac{1}{\mu(x_0^+)}\psi'_{E,1}(x_0^+) =: \lim_{x\to x_0^+} \frac{1}{\mu(x)}\psi'_{E,1}(x)$. The two-sided limits are equal by the continuity of the solution $\Psi_E(x)$. For ease of notation, we use the notation $\frac{1}{\mu(x_0)}\psi'_{E,1}(x_0)$ for either of the two-sided limit in subsequent analysis.

The solution matrix $\Psi_E(x)$ is referred to as the **transfer matrix**, which can be used to characterize the solutions to the initial value problem for the differential operator $\mathcal{L} - E$. Next we shall use it to characterize the spectrum of the operator $\mathcal{L}$.

Let $M(E) = \Psi_E(1)$, which is called the **monodromy matrix**. The following lemma states the relation between the eigenvalues of $\mathcal{L}$ in L_k^2 and the eigenvalues of the 2×2 matrix-valued function $M(E)$.

Lemma 2.1. *(a) For each $k \in \mathcal{B}$, if (E, ψ) is an eigenpair of the operator $\mathcal{L}$ in L_k^2, then $(\psi(0), \frac{1}{\mu(0)}\psi'(0))^T$ is an eigenvector of the monodromy matrix M(E) with the corresponding eigenvalue e^{ik}.*

(b) *If the matrix M(E) attains an eigenpair* $(e^{ik}, (c_1, c_2)^T)$ *for some* $k \in \mathcal{B}$*, then* $\psi(x) = c_1\psi_{E,1}(x) + c_2\psi_{E,2}(x)$ *is a Bloch mode of the differential operator* $\mathcal{L}$ *in* L^2_k *with the eigenvalue E.*

Proof. Given $k \in \mathcal{B}$, assume that (E, ψ) is an eigenpair of $\mathcal{L}$ in L^2_k. We can write:

$$\psi(x) = c_1\psi_{E,1}(x) + c_2\psi_{E,2}(x), \quad x \in [0,1], \tag{2.8}$$

where $c_1 = \psi(0), c_2 = \frac{1}{\mu(0)}\psi'(0)$. Together with the quasi-periodicity of the function in L^2_k, we have:

$$\psi(1) = c_1\psi_{E,1}(1) + c_2\psi_{E,2}(1) = e^{ik}\psi(0) = e^{ik}c_1, \tag{2.9}$$

$$\frac{1}{\mu(1)}\psi'(1) = \frac{1}{\mu(1)}\left(c_1\psi'_{E,1}(1) + c_2\psi'_{E,2}(1)\right) = e^{ik}\frac{1}{\mu(0)}\psi'(0) = e^{ik}c_2, \tag{2.10}$$

or equivalently,

$$M(E)c = e^{ik}c, \quad \text{where } c = (c_1, c_2)^T.$$

Thus the monodromy matrix $M(E)$ attains an eigenvalue e^{ik} and the associated eigenvector c. On the other hand, assuming that $M(E)$ has an eigenpair (e^{ik}, c). We construct ψ by (2.8). It is straightforward to show that ψ is a Bloch mode of $\mathcal{L}$ in L^2_k. □

Lemma 2.2. *det M(E) = 1.*

Proof. For each fixed E, consider $f(x) = \det \Psi_E(x)$. A direct calculation shows that $f'(x) = 0$. As a result, $\det \Psi_E(x)$ is independent of x. The proof is complete by noting that $\Psi_E(0) = Id$. □

Define

$$D(E) := TrM(E) = \psi_{E,1}(1) + \frac{1}{\mu(1)}\psi'_{E,2}(1).$$

$D(E)$ is called the discriminant of the $M(E)$. It is clear that D is real valued. The two eigenvalues of the matrix $M(E)$ are given by:

$$\lambda_{E,1} = \frac{D(E) - \sqrt{D(E)^2 - 4}}{2}, \quad \lambda_{E,2} = \frac{D(E) + \sqrt{D(E)^2 - 4}}{2}. \tag{2.11}$$

If $|D(E)| \leqslant 2$, then $\lambda_{E,1}$ and $\lambda_{E,2}$ are conjugate pairs with $|\lambda_{E,1}| = |\lambda_{E,2}| = 1$. It follows from lemma 2.1 that $E \in \sigma(\mathcal{L})$. On the other hand, if $|D(E)| > 2$, then both $\lambda_{E,1}$ and $\lambda_{E,2}$ are real numbers satisfying $|\lambda_{E,1}| < 1 < |\lambda_{E,2}|$ or $|\lambda_{E,2}| < 1 < |\lambda_{E,1}|$. In this case, $E \notin \sigma(\mathcal{L})$ and it lies in the band gap. In summary, we have the following lemma characterizing the spectrum of $\mathcal{L}$ using the discriminant $D(E)$.

Lemma 2.3. *The real number* $E \in \sigma(\mathcal{L})$ *if and only if* $|D(E)| \leqslant 2$.

Let

$$\mathcal{S} = \{E \in \mathbf{R} : |D(E)| < 2\} \quad \text{and} \quad \mathcal{I} = \{E \in \mathbf{R} : |D(E)| > 2\}.$$

Then the following lemma holds. We refer to theorem 1.6.1 in [9] for its proof.

Lemma 2.4. *(a) The function D(E) is strictly monotonic on each subinterval of* $\mathcal{S}$.
(b) *For* $E \in \mathbf{R}$*, it holds that* $D(E) = 2$ *and* $D'(E) = 0$ *if and only if* $M(E) = Id$*. In this case,* $D''(E) < 0$.
(c) *For* $E \in \mathbf{R}$*, it holds that* $D(E) = -2$ *and* $D'(E) = 0$ *if and only if* $M(E) = -Id$*. In this case,* $D''(E) > 0$.

Using the above lemmas, a quantitative characterization of the spectrum for the operator $\mathcal{L}$ is given in the theorem below.

Theorem 2.5. *Let $\mathcal{L}$ be a periodic operator in the form of (1.1) that attains periodic coefficients in the sense of assumption 1.*

(a) The following inequalities hold for the spectrum $\sigma(\mathcal{L}) = \bigcup_{j\geqslant 1} [E_j^-, E_j^+]$:

$$0 = E_1^- < E_1^+ \leqslant E_2^- < E_2^+ \leqslant E_3^- < E_3^+ \cdots$$

(b) The dispersion relation $E_j = E_j(k)$ can be obtained by solving the equation:

$$2\cos k = D(E) \tag{2.12}$$

for $k \in \mathcal{B}$ and $E \in [E_j^-, E_j^+]$.

(c) $E_j(k)$ are strictly monotonic on each of the half Brillouin zone $(-\pi, 0)$ and $(0, \pi)$.

(d) For each $j \geqslant 1$, we have either:

$$E_j^+ = \max\{E_j(k) : k \in \mathcal{B}\} = E_j(0),\ E_{j+1}^- = \min\{E_{j+1}(k) : k \in \mathcal{B}\} = E_{j+1}(0),$$

or

$$E_j^+ = \max\{E_j(k) : k \in \mathcal{B}\} = E_j(\pi),\ E_{j+1}^- = \min\{E_{j+1}(k) : k \in \mathcal{B}\} = E_{j+1}(\pi).$$

(e) It holds that $D(E^*) = \pm 2$ and $D'(E^*) = 0$ if and only if $E^* = E_j^+ = E_{j+1}^-$ for some $j \geqslant 1$. If this is the case, then $E^* = E_j(0) = E_{j+1}(0)$ if $D(E^*) = 2$ and $E^* = E_j(\pi) = E_{j+1}(\pi)$ if $D(E^*) = -2$.

Proof. (a) It is clear that $\sigma(\mathcal{L}) \subset [0, \infty)$. By lemma 2.3, we see that $|D(E)| > 2$ for $E < 0$, thus $\mathcal{S} \subset (0, \infty)$. We write

$$\mathcal{S} = \bigcup_{j\geqslant 1} S_j,$$

where S_j are the subintervals of $\mathcal{S}$ ordered in an increasing manner. By lemma 2.4, we have $S_j = (E_j^-, E_j^+)$, and it follows that $0 = E_1^- < E_1^+ \leqslant E_2^- < E_2^+ \leqslant E_3^- < E_3^+ \cdots$.

(b) follows from lemmas 2.1 and 2.2. The eigenvalues of $M(E)$ are given by $\lambda_{E,1} = e^{ik}$ and $\lambda_{E,2} = e^{-ik}$.

(c) follows from (a) in lemma 2.4.

(d) From (c), either $E_j^+ = E_j(0)$ or $E_j^+ = E_j(\pi)$ holds. We consider the former case and show that $E_{j+1}^- = E_{j+1}(0)$, and the proof for the latter case is similar. Indeed, note that $|D(E)| > 2$ on the interval (E_j^+, E_{j+1}^-). Since $D(E_j^+) = 2$, it follows that $D(E_{j+1}^-) = 2$ by lemma 2.3. That is when $k = 0$ in the equation (2.12) and we obtain $E_{j+1}^- = E_{j+1}(0)$.

(e) If $D(E^*) = \pm 2$ and $D'(E^*) = 0$, By (b) and (c) in lemma 2.4, we see that $|D(E)| < 2$ for E sufficiently close to but not equal to E^*. Therefore, E^* separates two subintervals in $\mathcal{S}$ and consequently $E^* = E_j^+ = E_{j+1}^-$ for some $j \geqslant 1$. On the other hand, if $E^* = E_j^+ = E_{j+1}^-$ for some $j \geqslant 1$, then (d) implies that either $E^* = E_j(0) = E_{j+1}(0)$ or $E^* = E_j(\pi) = E_{j+1}(\pi)$. Using (b), we further deduce that $D(E^*) = 2$ in the former case and $D(E^*) = -2$ in the latter case. Note that using lemma 2.3, $|D(E)| \leqslant 2$ for E near E^*. This is only possible if $D'(E^*) = 0$. Therefore, we have $D(E^*) = \pm 2$ and $D'(E^*) = 0$. The last assertion also follows.

□

Following the terminology in the physics literature [38], in what follows, we shall call the points $(0, E_j(0))$ and $(\pi, E_j(\pi))\big(\text{or}(-\pi, E_j(-\pi))\big)$ **edge points** of the jth spectral band. Then the above theorem implies that both the maximum and minimum of the dispersion relation for a given band $E_j(k)$ are achieved at the band edge points.

2.2. Dirac point

A pair $(k^*, E^*) \in \mathcal{B} \times \mathbf{R}$ on the dispersion curves is called a **Dirac point** for the spectrum of the differential operator $\mathcal{L}$ in (1.1) if:

(a) There exits integer $j \geqslant 1$ such that $E_j(k^*) = E_{j+1}(k^*) = E^*$. In addition, there exit constants $\alpha > 0$ and $\delta > 0$ such that the following expansions:

$$\begin{aligned} E_j(k) \quad &= E^* - \alpha|k - k^*| + O((k - k^*)^2), \\ E_{j+1}(k) &= E^* + \alpha|k - k^*| + O((k - k^*)^2), \end{aligned}$$

hold for $|k - k^*| < \delta$.

(b) The multiplicity of the Bloch modes in $L^2_{k^*}$ for the eigenvalue E^* is two.

The following proposition characterizes the Dirac points using the discriminant $D(E)$.

Proposition 2.6. *Let $\mathcal{L}$ be the differential operator in the form of (1.1) that attains periodic coefficients in the sense of assumption 1. Then the Dirac points for its spectrum can only occur at $k^* = 0$ or $k^* = \pi$ with $D(E^*) = \pm 2$ and $D'(E^*) = 0$. Furthermore, if $(k^* = 0, E^*)$ is a pair on the dispersion curves with $D'(E^*) = 0$, $D(E^*) = 2$, then $(k^* = 0, E^*)$ is a Dirac point. Similarly, the pair $(k^* = \pi, E^*)$ is a Dirac point if $D'(E^*) = 0$ and $D(E^*) = -2$.*

Proof. By virtue of (c) in theorem 2.5, Dirac points can only occur when $k^* = 0$ or $k^* = \pi$. Using (e) in theorem 2.5, we have $D(E^*) = \pm 2$ and $D'(E^*) = 0$.

We now assume that $(k^* = 0, E^*)$ is a pair on the dispersion curves such that $D'(E^*) = 0$, $D(E^*) = 2$ and aim to show that $(k^* = 0, E^*)$ is a Dirac point. First by (e) in theorem 2.5, it follows that $E^* = E_j^+ = E_{j+1}^-$ for some $j \geqslant 1$. Note that $M(E^*) = Id$ by lemma 2.4. The multiplicity of the eigenvector is 2. We deduce from lemma 2.1 that the multiplicity of the Bloch modes is 2. Furthermore, in the neighborhood of $k^* = 0$, we have $\cos k = 1 - \frac{1}{2}k^2 + O(k^4)$. Solving the equation:

$$2\cos k = D(E) = 2 + \frac{1}{2}D''(E^*)(E - E^*)^2 + O(E - E^*)^3,$$

we obtain the following expansions for dispersion curves $E_j(k)$ and $E_{j+1}(k)$:

$$E_j(k) = E^* - |k - k^*|\sqrt{\frac{2}{|D''(E^*)|}} + O((k - k^*)^2), \tag{2.13}$$

$$E_{j+1}(k) = E^* + |k - k^*|\sqrt{\frac{2}{|D''(E^*)|}} + O((k - k^*)^2). \tag{2.14}$$

This completes the proof that $(k^* = 0, E^*)$ is a Dirac point. □

From the above discussions, a Dirac point appears when two neighboring bands touch each other and the multiplicity of the Bloch modes at the Dirac point is two. In fact, the multiplicity of the Bloch modes for non-Dirac point is always one, as stated in following proposition.

Proposition 2.7. *Let $\mathcal{L}$ be a differential operator in the form of (1.1) that attains periodic coefficients in the sense of assumption 1. For each pair of $(k, E_j(k))$ ($j \geqslant 1$) that is not a Dirac point, the multiplicity of the associated Bloch modes is one.*

Proof. In light of lemma 2.1, we only need to show that the multiplicity of the eigenvector for $M(E)$ is 1. The claim is automatically true when $M(E)$ attains two different eigenvalues, thus it is sufficient to consider $M(E)$ at $k=0$ and $k=\pi$ only when $\lambda_{E,1}=\lambda_{E,2}$. Without loss of generality, we consider the former case. Let $E=E_j(0)$, then $\lambda_{E,1}=\lambda_{E,2}=1$. If $dimKer(M(E)-Id)=2$, then $M(E)=Id$, which implies that $E_j^+=E_{j+1}^-$ by lemma 2.4 and theorem 2.5, and consequently $(k=0, E=E_j(0))$ would be a Dirac point. Therefore, $dimKer(M(E)-Id)=1$. $\square$

2.3. Bloch modes

Assuming that the jth spectral band of the operator $\mathcal{L}$ does not intersect with other bands, we construct its Bloch modes using the transfer matrix. For each $E\in[E_j^-, E_j^+]$, let k be a real number in $[0,\pi]$ such that:

$$e^{ik}=\lambda_{E,1}=\frac{D(E)+i\sqrt{4-D(E)^2}}{2}, \tag{2.15}$$

the first eigenvalue of the matrix $M(E)$. We choose the corresponding eigenvector:

$$\left(\psi_{E,2}(1), e^{ik}-\psi_{E,1}(1)\right)^T.$$

It follows from lemma 2.1 that:

$$\phi_{j,k}(x) := \psi_{E,2}(1)\psi_{E,1}(x)+(e^{ik}-\psi_{E,1}(1))\psi_{E,2}(x), \quad x\in[0,1], \tag{2.16}$$

is a Bloch mode and it forms a basis of the one dimensional eigenspace. We define the normalized Bloch mode by letting:

$$\varphi_{j,k} := \frac{\phi_{j,k}}{\|\phi_{j,k}\|_X}.$$

It is clear that the above Bloch mode is well-defined as long as the eigenvector above is nonzero and the function $\phi_{j,k}\not\equiv 0$. The case $\phi_{j,k}\equiv 0$ is a degenerate case which only occurs at the band edge where $k=0$ or π. We next show that the Bloch mode $\varphi_{j,k}$ constructed above can be extended continuously in $[0,\pi]$ when such degeneracy is present.

Lemma 2.8. *Assume that the jth spectral band of the operator $\mathcal{L}$ does not intersect with other bands. If $\phi_{j,0}\equiv 0$ or $\phi_{j,\pi}\equiv 0$ at the jth band edge, then there holds:*

$$\lim_{k\to 0^+}\varphi_{j,k}=\frac{i\psi_{E^*,2}}{\|\psi_{E^*,2}\|_X}, \quad E^*=E_j(0),$$

or

$$\lim_{k\to \pi^-}\varphi_{j,k}=\frac{i\psi_{E^*,2}}{\|\psi_{E^*,2}\|_X}, \quad E^*=E_j(\pi),$$

respectively. Moreover, $\psi_{E^*,2}$ is a Bloch mode for $k=0$ and $k=\pi$ respectively.

Proof. If $\phi_{j,0} \equiv 0$, then $\psi_{E^*,2}(1) = 1 - \psi_{E^*,1}(1) = 0$ with $E^* = E_j(0)$. From (2.16) we have:

$$\frac{\partial \phi_{j,k}(x)}{\partial k} = \frac{\partial}{\partial E}\big(\psi_{E,2}(1)\psi_{E,1}(x) - \psi_{E,1}(1)\psi_{E,2}(x) + e^{ik}\psi_{E,2}(x)\big)E_j'(k) + ie^{ik}\psi_{E,2}(x).$$

Note that at $E = E^* = E_j(0)$, we have $E_j'(0) = 0$. Thus

$$\frac{\partial \phi_{j,k}(x)}{\partial k}|_{k=0} = i\psi_{E,2}(x).$$

It follows that:

$$\varphi_{j,0^+} = \lim_{k\to 0^+} \varphi_{j,k} = \frac{i\psi_{E,2}}{\|\psi_{E,2}\|_X}.$$

We now show that $\psi_{E^*,2}$ is a Bloch mode. Indeed, since $\psi_{E^*,2}(1) = 0$ and $\psi_{E^*,1}(1) = 1$, from (2.15) we have $\frac{1}{\mu(0)}\psi_{E^*,2}'(1) = 1 = \frac{1}{\mu(0)}\psi_{E^*,2}'(0)$. Therefore, it follows from lemma 2.1 that $\psi_{E^*,2}$ is a Bloch mode for $k = 0$. The proof for the other case is similar. □

From the above discussions, we see that the Bloch modes:

$$\varphi_{j,k} = \begin{cases} \frac{\phi_{j,k}}{\|\phi_{j,k}\|_X}, \text{if } \ \phi_{j,k} \not\equiv 0, \\ \frac{i\psi_{E,2}}{\|\psi_{E,2}\|_X}, \text{if } \ \phi_{j,k} \equiv 0, \end{cases} \tag{2.17}$$

are continuous in the reduced Brillouin zone $[0, \pi]$. For $\mathcal{L}$ with periodic coefficients, the above Bloch modes can be extended for $k \in (-\pi, 0)$ by letting:

$$\varphi_{j,k} = \bar{\varphi}_{j,-k} = \frac{\bar{\phi}_{j,-k}}{\|\phi_{j,-k}\|_X}.$$

We end this subsection by discussing the Bloch modes at the Dirac points.

Lemma 2.9. *Assume that the jth and $(j+1)$th bands touch at the Dirac point $(0, E^*)$, then:*

$$\varphi_{j+1,0}(x) = -\overline{\varphi_{j,0}(x)}. \tag{2.18}$$

The same relation holds at the Dirac point (π, E^*).

Proof. Let $\phi_{j,k}$ be the Bloch modes given by (2.16), where $E = E_j(k)$ adopts the expansion (2.13). We have the following expansions for $k > 0$ and $E < E^*$:

$$e^{ik} = 1 + ik + O(k^2) = 1 - i(E - E^*)\sqrt{-\frac{1}{2}D''(E^*)} + O(E - E^*)^2.$$

On the other hand,

$$\psi_{E,2}(1) = \frac{\partial \psi_{E,2}(1)}{\partial E}(E^*)(E - E^*) + O(E - E^*)^2,$$
$$\psi_{E,1}(1) = 1 + \frac{\partial \psi_{E,1}(1)}{\partial E}(E^*)(E - E^*) + O(E - E^*)^2.$$

Denote

$$b_1(E) := \frac{\partial \psi_{E,1}(1)}{\partial E}, \quad b_2(E) := \frac{\partial \psi_{E,2}(1)}{\partial E}.$$

We see that $\phi_{j,k}$ admits the expansion:

$$\phi_{j,k}(x) = (E-E^*)\left(b_2(E^*)\psi_{E,1}(x) + \left(-i\sqrt{-\frac{1}{2}D''(E^*)} - b_1(E^*)\right)\psi_{E,2}(x)\right) + O(E-E^*)^2.$$

It follows that:

$$\varphi_{j,0}(x) = -\frac{b_2(E^*)\psi_{E^*,1}(x) + \left(-i\sqrt{-\frac{1}{2}D''(E^*)} - b_1(E^*)\right)\psi_{E^*,2}(x)}{\|b_2(E^*)\psi_{E^*,1} + \left(-i\sqrt{-\frac{1}{2}D''(E^*)} - b_1(E^*)\right)\psi_{E^*,2}\|_X}.$$

Similarly, we can show that:

$$\varphi_{j+1,0}(x) = \frac{b_2(E^*)\psi_{E^*,1}(x) + \left(i\sqrt{-\frac{1}{2}D''(E^*)} - b_1(E^*)\right)\psi_{E^*,2}(x)}{\|b_2(E^*)\psi_{E^*,1} + \left(i\sqrt{-\frac{1}{2}D''(E^*)} - b_1(E^*)\right)\psi_{E^*,2}\|_X},$$

whence (2.18) follows. □

2.4. Zak phase

Given a normalized Bloch mode $\varphi_{j,k}$ for $(k, E_j(k)) \in \mathcal{B} \times \mathbf{R}$ over the jth band, one can express $\varphi_{j,k}$ in the form of:

$$\varphi_{j,k}(x) = e^{ikx}u_{j,k}(x),$$

where $u_{j,k}(x)$ is a periodic function satisfying $u_{j,k}(x) = u_{j,k}(x+1)$. $u_{j,k}$ is called the periodic part of the Bloch mode $\varphi_{j,k}$. For the jth band that is isolated, it is clear that the Bloch modes $\varphi_{j,k}$ form a closed loop as k runs over the Brillouin zone from $-\pi$ to π since $\varphi_{j,\pi}$ and $\varphi_{j,-\pi}$ only differs by a global phase constant. However, this is no longer the case for the periodic part $u_{j,k}$ since $u_{j,\pi}(x) = e^{-i2\pi x}u_{j,-\pi}(x)$ even if $\varphi_{j,\pi} = \varphi_{j,-\pi}$. To take this into account, we define the following discrete Zak phase over the jth band (cf section 3.4 in [36]):

$$\theta_j^{(N)} = \sum_{n=0}^{N-2} -\mathrm{Im}\ln\left(u_{j,k_{n+1}}, u_{j,k_n}\right)_X - \mathrm{Im}\ln\left(e^{-i2\pi x}u_{j,k_0}, u_{j,k_{N-1}}\right)_X \ mod\ 2\pi, \tag{2.19}$$

where $k_n = -\pi + \dfrac{2\pi n}{N}, n = 0, 1, \cdots N-1$.

If the Bloch mode $\varphi_{j,k}$ is smooth with respect to k over the Brillouin zone with $\varphi_{j,-\pi} = \varphi_{j,\pi}$, by taking the continuum limit of (2.19) as $N \to \infty$, we recover the well-known Zak phase formula (cf [36, 39]):

$$\theta_j = i\int_{-\pi}^{\pi}\left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk\ mod\ 2\pi. \tag{2.20}$$

The Zak phase θ_j is invariant under the gauge transformation with $u_{j,k}$ being replaced by $e^{i\beta(k)}u_{j,k}$ for certain phase function $\beta(k)$. This can be observed from the discrete formulation (2.19).

For $u_{j,k}$ that is piecewisely smooth in $(-\pi,0)\cup(0,\pi)$ with respect to k, the continuous formula (2.20) cannot be used directly. We modify (2.20) by taking into account of the possible phase jump at $k=0,\pi$, and define the Zak phase accordingly as:

$$\theta_j = i\int_{-\pi}^{0}\left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk + i\int_{0}^{\pi}\left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk + \hat{\theta}_j \quad mod\, 2\pi, \tag{2.21}$$

where $\hat{\theta}_j$ is given by:

$$\hat{\theta}_j = -\mathrm{Im}\ln(u_{j,0}, u_{j,0^-})_X - \mathrm{Im}\ln(e^{-i2\pi x}u_{j,(-\pi)^+}, u_{j,\pi})_X.$$

In the above, $u_{j,0^-} = \lim_{k\to 0^-} u_{j,k}$ and $u_{j,(-\pi)^+} = \lim_{k\to(-\pi)^+} u_{j,k}$ denote the one-side limit.

3. Impedance function and interface mode

3.1. Mode decomposition in the band gap and impedance function

In this section, we consider the decomposition of the solution to the wave equation $(\mathcal{L}-E)\psi = 0$ when $E\notin\sigma(\mathcal{L})$ lies in a band gap of the operator $\mathcal{L}$. We assume that $E_j^+ < E_{j+1}^-$ and $E\in(E_j^+, E_{j+1}^-)$. Here and henceforth, without loss of generality we assume that the trace $D(E)>0$ so that the eigenvalues defined in (2.11) satisfy $|\lambda_{E,1}|<1$ and $|\lambda_{E,2}|>1$ in the band gap. If $D(E)<0$, all the arguments follow by replacing $\lambda_{E,1}$ and $\lambda_{E,2}$ with each other.

Let ψ be a solution of $(\mathcal{L}-E)\psi = 0$ for $E\in(E_j^+, E_{j+1}^-)$. Define the vector-valued function,

$$\Phi(x) := (\psi(x), \frac{1}{\mu(x)}\psi'(x))^T.$$

Using the transfer matrix, there holds:

$$\Phi(x+n) = \Psi_E(x+n)\Phi(0) = \Psi_E(x)\Psi_E(n)\Phi(0) = \Psi_E(x)M(E)^n\Phi(0), \quad x\in[0,1].$$

Recall that,

$$M(E) = \begin{pmatrix} \psi_{E,1}(1) & \psi_{E,2}(1) \\ \frac{1}{\mu(1)}\psi'_{E,1}(1) & \frac{1}{\mu(1)}\psi'_{E,2}(1)\end{pmatrix}.$$

We denote by $V_{E,j}$ $(j=1,2)$ the eigenvector of the matrix $M(E)$ associated with the eigenvalue $\lambda_{E,j}$. By decomposing $\Phi(0)$ as $\Phi(0) = t_1V_{E,1} + t_2V_{E,2}$, it follows that:

$$M(E)^n\Phi(0) = t_1\lambda_{E,1}^n V_{E,1} + t_2\lambda_{E,2}^n V_{E,2}.$$

Therefore,

$$\Phi(x+n) = t_1\lambda_{E,1}^n\Psi_E(x)V_{E,1} + t_2\lambda_{E,2}^n\Psi_E(x)V_{E,2},$$

and

$$\psi(x+n) = t_1\lambda_{E,1}^n(\psi_{E,1}(x), \psi_{E,2}(x))V_{E,1} + t_2\lambda_{E,2}^n(\psi_{E,1}(x), \psi_{E,2}(x))V_{E,2}.$$

Note that the functions $\psi_{E,1}(x)$ and $\psi_{E,2}(x)$ are continuous in x, hence there exist constants $C_1, C_2>0$ such that:

$$C_1 < \|(\psi_{E,1}(x), \psi_{E,2}(x))V_{E,j}\|_{L^2[0,1]} < C_2, \quad j=1,2.$$

Using the relations:

$$\|\psi(x)\|^2_{L^2[0,\infty)} = \sum_{n\geqslant 0}\|\psi(x)\|^2_{L^2[n,n+1]}, \quad \|\psi(x)\|^2_{L^2[-\infty,0)} = \sum_{n\leqslant 0}\|\psi(x)\|^2_{L^2[n-1,n]},$$

and the condition that $|\lambda_{E,1}| < 1$ and $|\lambda_{E,2}| > 1$, we can conclude that:

$$\|\psi(x)\|_{L^2[0,\infty)} < \infty,$$

if and only if $t_2 = 0$ and $\Phi(0) = t_1 V_{E,1}$, and that

$$\|\psi(x)\|_{L^2(-\infty,0]} < \infty,$$

if and only if $t_1 = 0$ and $\psi(0) = t_2 V_{E,2}$.

We now consider two scenarios when $\psi_{E,2}(1) = 0$ or not. If $\psi_{E,2}(1) \neq 0$, we set:

$$V_{E,1} = \begin{pmatrix} \psi_{E,2}(1) \\ \lambda_{E,1} - \psi_{E,1}(1) \end{pmatrix} \quad \text{and} \quad V_{E,2} = \begin{pmatrix} \psi_{E,2}(1) \\ \lambda_{E,2} - \psi_{E,1}(1) \end{pmatrix}, \tag{3.1}$$

and the following results hold.

Lemma 3.1. *Let $E \in (E_j^+, E_{j+1}^-)$ and $\psi_{E,2}(1) \neq 0$. Let ψ be a nontrivial solution to $\mathcal{L}\psi = E\psi$.*

(a) $\|\psi(x)\|_{L^2[0,\infty)} < \infty$ if and only if, up to certain constant,

$$\psi(x) = \psi_{E,2}(1)\psi_{E,1}(x) + (\lambda_{E,1} - \psi_{E,1}(1))\psi_{E,2}(x).$$

(b) $\|\psi(x)\|_{L^2(-\infty,0]} < \infty$ if and only if, up to certain constant,

$$\psi(x) = \psi_{E,2}(1)\psi_{E,1}(x) + (\lambda_{E,2} - \psi_{E,1}(1))\psi_{E,2}(x).$$

If $\psi_{E,2}(1) = 0$ at $E = E^*$, the monodromy matrix:

$$M(E^*) = \begin{pmatrix} \psi_{E^*,1}(1) & 0 \\ \frac{1}{\mu(1)}\psi'_{E^*,1}(1) & \frac{1}{\mu(1)}\psi'_{E^*,2}(1) \end{pmatrix}.$$

We see that either $\psi_{E^*,1}(1) = \lambda_{E^*,1}$ or $\psi_{E^*,1}(1) = \lambda_{E^*,2}$ depending on whether $|\psi_{E^*,1}(1)| < 1$ or $|\psi_{E^*,1}(1)| > 1$. If $\psi_{E^*,1}(1) = \lambda_{E^*,1}$, $V_{E^*,1}$ defined in (3.1) is a zero vector and hence cannot be used as an eigenvector for the eigenvalue $\lambda_{E^*,1}$. In this degenerate case, noting that the matrix $M(E)$ is smooth in E, so the family of eigenspace associated with eigenvalue $\lambda_{E,1}$ depends on E continuously. Therefore, for E near E^*, we choose,

$$V_{E,1} = \frac{1}{E - E^*}(\psi_{E,2}(1), \lambda_{E,1} - \psi_{E,1}(1))^T,$$

for $E \neq E^*$ and,

$$V_{E^*,1} = \lim_{E \to E^*} \frac{1}{E - E^*}(\psi_{E,2}(1), \lambda_{E,1} - \psi_{E,1}(1))^T,$$

if the limit exists. We choose $V_{E,2}$ in a similar way if $\psi_{E^*,1}(1) = \lambda_{E^*,2}$.

We now introduce the concept of impedance function, which is used in the proof of the existence of interface modes in the subsequent sections. From the above discussions, it is known that for each E in the band gap, all the solutions to the equation $(\mathcal{L} - E)\psi = 0$ with finite energy over the left half-line $(-\infty, 0]$ spans a one-dimensional space. Let $\psi_{L,E}$ be one of such solutions, and define:

$$\xi_L(E) := \frac{\psi_{L,E}(0)}{\frac{1}{\mu(0)}\psi'_{L,E}(0)}, \quad \text{if } \psi'_{L,E}(0) \neq 0.$$

Here L denotes that the solution $\psi_{L,E}$ attains finite energy on the left of the real axis. In the case when $\psi'_{L,E}(0) = 0$, we set $\xi_L(E) = \infty$. Note that $\psi'_{L,E}(0)$ and $\psi_{L,E}(0)$ will not vanish simultaneously, otherwise $\psi_{L,E} \equiv 0$.

It is clear $\xi_L(E)$ defined above is independent of the choice of the solution $\psi_{L,E}$. We call $\xi_L(E)$ the **impedance function** for the operator $\mathcal{L}$ defined over the half-line $(-\infty,0]$. Using lemma 3.1, we have:

$$\xi_L(E) = \frac{\psi_{E,2}(1)}{\lambda_{E,2} - \psi_{E,1}(1)}.$$

We remark that the above definition can be extended to those E^* in the band gap satisfying $\psi_{E^*,2}(1) = \lambda_{E^*,2} - \psi_{E^*,1}(1) = 0$.

In such scenario, we interpret:

$$\xi_L(E^*) = \lim_{E\to E^*} \xi_L(E),$$

provided that the limit exists. Using the fact that the functions $\lambda_{E,1}, \psi_{E,2}(1)$ and $\psi_{E,1}(1)$ are continuous in E (since the matrix $\Psi_E(1)$ is smooth in E), we see that $\xi_L(E)$ defined above is continuous in E in the band gap (E_j^+, E_{j+1}^-) except at those points where $\xi_L(E) = \infty$.

The impedance function $\xi_L(E)$ can also be used to define the Robin boundary condition for the operator $\mathcal{L} - E$ defined in $(-\infty,0]$. Indeed, all the solutions ψ to $(\mathcal{L}-E)\psi = 0$ in $(-\infty,0]$ that decays at $-\infty$ satisfies the Robin boundary condition:

$$\psi(0) - \xi_L(E)\frac{1}{\mu(0)}\psi'(0) = 0.$$

In the case when $\xi_L(E) = \infty$, the above boundary condition is interpreted as the Neumann boundary condition.

In a similar way, we define the impedance function for the periodic operator $\mathcal{L}$ defined in the right half-line $[0,\infty)$ by:

$$\xi_R(E) := \frac{\psi_{R,E}(0)}{\frac{1}{\mu(0)}\psi'_{R,E}(0)} = \frac{\psi_{E,2}(1)}{\lambda_{E,1} - \psi_{E,1}(1)},$$

where $\psi_{R,E}$ attains finite energy over the right half-line $[0,\infty)$.

3.2. Interface mode

We now introduce the definition of interface modes:

Definition 1. Let $\mathcal{L}_-$ and $\mathcal{L}_+$ be two periodic operators with the coefficients (ε_-, μ_-) and (ε_+, μ_+) satisfying assumption 1. Let $\tilde{\mathcal{L}}$ be the 'glued' operator over the whole real line such that as $\tilde{\mathcal{L}} = \mathcal{L}_-$ for $x<0$ and $\tilde{\mathcal{L}} = \mathcal{L}_+$ for $x>0$. A function ψ is called an interface mode of the structure associated with the operator $\tilde{\mathcal{L}}$ if:

$$\psi \in L^2(\mathbf{R}) \quad \text{and} \quad (\tilde{\mathcal{L}} - E)\psi(x) = 0 \quad \text{for } x\in\mathbf{R},$$

for some real number E. E is called the energy level of the interface mode ψ.

The existence of interface modes can be formulated in terms of impedance functions.

Lemma 3.2. *Let $\mathcal{L}_-$, $\mathcal{L}_+$ and $\tilde{\mathcal{L}}$ be given as in definition 1. Assume that E lies in a common spectral band gap of $\mathcal{L}_-$ and $\mathcal{L}_+$, and $\xi_{L,-}(E)$ and $\xi_{R,+}(E)$ are the corresponding impedance functions at the interface $x=0$. Then there exists an interface mode at energy level E for the operator $\tilde{\mathcal{L}}$ if and only if:*

$$\xi_{L,-}(E) = \xi_{R,+}(E).$$

Proof. If ψ is an interface mode for $\tilde{\mathcal{L}}$, then $\|\psi(x)\|_{L^2(\mathbf{R})} < \infty$. By restricting ψ over $(-\infty,0]$ and $[0,\infty)$ respectively, we obtain:

$$\frac{\psi(0^-)}{\frac{1}{\mu_-(0^-)}\psi'(0^-)} = \xi_{L,-}(E) \quad \text{and} \quad \frac{\psi(0^+)}{\frac{1}{\mu_+(0^+)}\psi'(0^+)} = \xi_{R,+}(E),$$

where $0^{\pm}$ denotes the one-sided limit at the interface. The continuity of the solution implies that $\xi_{L,-}(E) = \xi_{R,+}(E)$. On the other hand, given $\xi_{L,-}(E) = \xi_{R,+}(E)$, we can construct ψ_L and ψ_R with $\|\psi_L\|_{L^2(-\infty,0]} < \infty$ and $\|\psi_R\|_{L^2[0,\infty)} < \infty$ respectively, and they satisfy:

$$(\mathcal{L}_L - E)\psi_L = 0, \quad (\mathcal{L}_R - E)\psi_R = 0$$

with

$$\psi_L(0^-) = \psi_R(0^+), \quad \frac{1}{\mu_-(0^-)}\psi_L'(0^-) = \frac{1}{\mu_+(0^+)}\psi_R'(0^+).$$

Then $\psi(x)$ defined by $\psi_L(x)$ for $x<0$ and $\psi_R(x)$ for $x>0$ is a solution to $(\tilde{\mathcal{L}} - E)\psi = 0$. This completes the proof of the lemma. □

4. Interface modes induced by Dirac points for time-reversal symmetric structures

In this section, we study the perturbation to a general time-reversal symmetric photonic structure with a Dirac point. Throughout this section, we assume that the following holds.

Assumption 2. The operator $\mathcal{L}$ for the unperturbed photonic structure as defined in (1.1) attains periodic coefficients in the sense of assumption 1 and attains a Dirac point $(k = 0, E^*)$ at the intersection of the jth and the $j+1$th band, i.e.:

$$E^* = E_j^+ = E_{j+1}^- = E_j(0) = E_{j+1}(0).$$

The configuration when the Dirac point is $(k = \pi, E^*)$ can be treated similarly, and we omit here for conciseness of the presentation. We shall derive conditions on the perturbation of the parameters such that a band gap opens near the Dirac point and an interface mode exists for the perturbed system. The main result of this section is given in theorem 4.7.

4.1. Perturbation of periodic system with a Dirac point

We perturb the periodic operator $\mathcal{L}$ in the following way:

$$\begin{cases} \mu(x) \to \mu(x) + \delta\tilde{\mu}(x), \\ \varepsilon(x) \to \varepsilon(x) + \delta\tilde{\varepsilon}(x), \end{cases}$$

where $|\delta| \ll 1$, $\tilde{\mu}(x)$ and $\tilde{\varepsilon}(x)$ are two piecewisely continuous periodic functions (with period one) satisfying $\|\tilde{\mu}\|_{L^\infty} + \|\tilde{\varepsilon}\|_{L^\infty} = 1$. The perturbed operator is denoted by:

$$\mathcal{L}_\delta\psi(x) := -\frac{1}{\varepsilon(x) + \delta\tilde{\varepsilon}(x)}\left(\frac{1}{\mu(x) + \delta\tilde{\mu}(x)}\psi'(x)\right)'.$$

We have $\mathcal{L} = \mathcal{L}_0$. For each $E \in \mathbf{R}$, let $\psi_{E,1,\delta}$ and $\psi_{E,2,\delta}$ be the unique solution to the following equations respectively:

$$(\mathcal{L}_\delta - E)\psi_{E,1,\delta} = 0, \quad \psi_{E,1,\delta}(0) = 1, \ \frac{1}{\mu(0) + \delta\tilde{\mu}(0)}\psi'_{E,1,\delta}(0) = 0,$$
$$(\mathcal{L}_\delta - E)\psi_{E,2,\delta} = 0, \quad \psi_{E,2,\delta}(0) = 0, \ \frac{1}{\mu(0) + \delta\tilde{\mu}(0)}\psi'_{E,2,\delta}(0) = 1.$$

Let $\Psi_{E,\delta}$ denote the perturbed transfer matrix which solves the Ordinary Differential Equation (ODE) system:

$$\frac{d}{dx}\Psi_{E,\delta}(x) = J(B + EW + \delta\tilde{F})\Psi_{E,\delta}(x), \quad \Psi_{E,\delta}(0) = Id, \tag{4.1}$$

where

$$\tilde{F} = \begin{pmatrix} E\tilde{\varepsilon}(x) & 0 \\ 0 & \tilde{\mu}(x) \end{pmatrix}. \tag{4.2}$$

It follows from the theory of first-order linear ordinary differential equations that the solution matrix $\Psi_{E,\delta}(x)$ depends on the two parameters E and δ smoothly. Let $M(E,\delta) = \Psi_{E,\delta}(1)$ be the transfer matrix for one period with two eigenvalues be $\lambda_{E,1,\delta}$ and $\lambda_{E,2,\delta}$. The trace of $M(E,\delta)$ is denoted as $D(E,\delta)$. Under assumption 2, we see from proposition 2.6 that $\Psi_{E^*,0}(1) = Id$, $D(E^*,0) = 2$. For ease of notation, we write:

$$U := \begin{pmatrix} \psi_{E^*,1,0}(x) \\ \frac{1}{\mu(x)}\psi'_{E^*,1,0}(x) \end{pmatrix}, \quad V := \begin{pmatrix} \psi_{E^*,2,0}(x) \\ \frac{1}{\mu(x)}\psi'_{E^*,2,0}(x) \end{pmatrix}. \tag{4.3}$$

Lemma 4.1. *The following identities hold:*

$$\frac{\partial D}{\partial E}(E^*,0) = 0, \ \frac{\partial D}{\partial \delta}(E^*,0) = 0.$$
$$\frac{1}{2}\frac{\partial^2 D}{\partial E^2}(E^*,0) = \left(\int_0^1 U^T WV dx\right)^2 - \left(\int_0^1 V^T WV dx\right)\cdot\left(\int_0^1 U^T WU dx\right)$$
$$\frac{1}{2}\frac{\partial^2 D}{\partial \delta^2}(E^*,0) = \left(\int_0^1 U^T \tilde{F}V dx\right)^2 - \left(\int_0^1 V^T \tilde{F}V dx\right)\cdot\left(\int_0^1 U^T \tilde{F}U dx\right)$$
$$\frac{1}{2}\frac{\partial^2 D}{\partial E\partial \delta}(E^*,0) = \left(\int_0^1 U^T WV dx\right)\cdot\left(\int_0^1 U^T \tilde{F}V dx\right) - \frac{1}{2}\left(\int_0^1 V^T WV dx\right)\cdot\left(\int_0^1 U^T WU dx\right)$$
$$- \frac{1}{2}\left(\int_0^1 V^T \tilde{F}V dx\right)\cdot\left(\int_0^1 U^T \tilde{F}U dx\right).$$

The proof the lemma is given in the appendix.

4.2. Band gap opening for the perturbed system

Let us denote,

$$a_1 := \frac{\partial^2 D}{\partial E^2}(E^*,0), \quad a_2 := \frac{\partial^2 D}{\partial E\partial\delta}(E^*,0), \quad a_3 := \frac{\partial^2 D}{\partial \delta^2}(E^*,0).$$

Lemma 4.2. *There holds* $a_1 = \frac{\partial^2 D}{\partial E^2}(E^*, 0) < 0$.

Proof. Note that

$$\int_0^1 V^T W V = \int_0^1 \psi_{E,2}^2(x)\varepsilon(x)dx > 0, \int_0^1 u^T W U = \int_0^1 \psi_{E,1}^2(x)\varepsilon(x)dx > 0,$$

and that

$$\int_0^1 U^T W V = \int_0^1 \psi_{E,2}(x)\psi_{E,1}(x)\varepsilon(x)dx.$$

Since $\psi_{E,2}$ and $\psi_{E,1}$ are linearly independent, using Cauchy-Schwarz type inequality, we can derive that:

$$\left|\int_0^1 U^T W V dx\right|^2 < \int_0^1 V^T W V dx \cdot \int_0^1 U^T W U dx,$$

whence $a_1 < 0$ follows. □

Theorem 4.3. *Let the unperturbed operator* $\mathcal{L}$ *defined as in (1.1) satisfy assumption 2. Let* $\delta > 0$ *be a sufficiently small number. Assume that:*

$$a_2^2 - a_1 a_3 > 0. \tag{4.4}$$

Then there exists a band gap $(E_{j,\delta}^+, E_{j+1,\delta}^-)$ *between the jth and the* $(j+1)$*th band for the perturbed operator* $\mathcal{L}_\delta$. *Moreover,*

$$\begin{aligned} E_{j,\delta}^+ &= E_{j,\delta}(0) = E^* + \eta^- \delta + O(\delta^2), \\ E_{j+1,\delta}^- &= E_{j+1,\delta}(0) = E^* + \eta^+ \delta + O(\delta^2), \end{aligned}$$

where

$$\eta^- = \frac{-a_2 + \sqrt{a_2^2 - a_1 a_3}}{a_1}, \quad \eta^+ = \frac{-a_2 - \sqrt{a_2^2 - a_1 a_3}}{a_1}. \tag{4.5}$$

Proof. By Taylor expansion, we have:

$$D(E,\delta) = 2 + a_1(E - E^*)^2 + 2a_2(E - E^*)\delta + a_3\delta^2 + r_1(E - E^*, \delta),$$

where $r_1(E,\delta) = O(E - E^*)^3 + O(\delta^3)$ is a smooth function of E and δ for $|\delta| \ll 1$. Solving $D(E,\delta) = 2$ yields:

$$\left(E - E^* + \frac{a_2}{a_1}\delta\right)^2 = \frac{a_2^2 - a_1 a_3}{a_1^2}\delta^2 - \frac{1}{a_1} r_1(E,\delta).$$

By substituting $t = \frac{E - E^*}{\delta}$ and $r_2(t,\delta) = -\frac{1}{a_1\delta^2} r_1(t\delta, \delta)$, we further obtain:

$$\left(t + \frac{a_2}{a_1}\right)^2 = \frac{a_2^2 - a_1 a_3}{a_1^2} + r_2(t,\delta).$$

Since $a_2^2 - a_1a_3 > 0$ and $r_2(t,\delta) = O(\delta)$ for $t = O(1)$, we see that $\frac{a_2^2-a_1a_3}{a_1^2} + r_2(t,\delta) > 0$ for $t = O(1)$ and $|\delta| \ll 1$. Therefore the solution to $D(E,\delta) = 2$ can be solved from the following two equations:

$$f_1(t,\delta) : = t + \frac{a_2}{a_1} - \sqrt{\frac{a_2^2 - a_1a_3}{a_1^2} + r_2(t,\delta)} = 0,$$

$$f_2(t,\delta) : = t + \frac{a_2}{a_1} + \sqrt{\frac{a_2^2 - a_1a_3}{a_1^2} + r_2(t,\delta)} = 0.$$

Using implicit function theorem, we can deduce that there exists a unique solution that depend on δ smoothly for $\delta \ll 1$ to each of the two equations above. Moreover, the two solutions can be expressed as respectively as:

$$t_1 = -\frac{a_2}{a_1} - \sqrt{\frac{a_2^2 - a_1a_3}{a_1^2}} + O(\delta), \quad t_2 = -\frac{a_2}{a_1} + \sqrt{\frac{a_2^2 - a_1a_3}{a_1^2}} + O(\delta).$$

It follows that the solution to $D(E,\delta) = 0$ has the following form:

$$E = E^* + \frac{-a_2 \pm \sqrt{a_2^2 - a_1a_3}}{a_1}\delta + O(\delta^2).$$

On the other hand, it is clear that $D(E,\delta) > 2$ holds for $E \in (E^* + \frac{-a_2+\sqrt{a_2^2-a_1a_3}}{a_1}\tau\delta, E^* + \frac{-a_2-\sqrt{a_2^2-a_1a_3}}{a_1}\tau\delta)$ for some constant $0 < \tau < 1$. The opening of the band gap follows by lemma 2.3.

Finally, for $\delta > 0$, we have:

$$E_{j,\delta}^{+} = E_{j,\delta}(0) = E^* + \frac{-a_2 + \sqrt{a_2^2 - a_1a_3}}{a_1}\delta + O(\delta^2) = E^* + \eta^{-}\delta + O(\delta^2)$$

$$E_{j+1,\delta}^{-} = E_{j+1,\delta}(0) = E^* + \frac{-a_2 - \sqrt{a_2^2 - a_1a_3}}{a_1}\delta + O(\delta^2) = E^* + \eta^{+}\delta + O(\delta^2).$$

This completes the proof. □

Before we end this section, we present scenarios for which the assumption (4.4) holds.

Proposition 4.4. *Let $\tilde{\mu}, \tilde{\varepsilon}$ be such that $\|\tilde{\mu}\|_{L^\infty} + \|\tilde{\varepsilon}\|_{L^\infty} = 1$. If $\tilde{\mu} \geqslant 0, \tilde{\varepsilon} \geqslant 0$, then there holds* $a_2^2 - a_1a_3 \geqslant 0$.

Proof. See the appendix for the detailed proof. □

4.3. *Impedance function in the band gap for the perturbed system*

We assume that (4.4) holds and a band gap between the jth and the $(j+1)$th band is opened for the perturbed operator $\mathcal{L}_\delta$. Following section 3.1, for each E in the band gap, we define the following impedance functions at the boundary point $x = 0$ for the perturbed operator $\mathcal{L}_\delta$ defined in the left half-line $(-\infty, 0]$ and right half-line $[0, \infty)$ respectively:

$$\xi_{L,\delta}(E) := \frac{\psi_{E,2,\delta}(1)}{\lambda_{E,2,\delta} - \psi_{E,1,\delta}(1)}, \quad \xi_{R,\delta}(E) := \frac{\psi_{E,2,\delta}(1)}{\lambda_{E,1,\delta} - \psi_{E,1,\delta}(1)}.$$

We have the following asymptotic expansions for $\xi_{L,\delta}$ and $\xi_{R,\delta}$.

Lemma 4.5. *let E be in the band gap $(E^+_{j,\delta}, E^-_{j+1,\delta})$, say $\tau\eta^-\delta < E - E^* < \tau\eta^+\delta$ for some $0 < \tau < 1$. Here $\eta^\pm$ are defined in (4.5).*

Then, we have:

$$\xi_{L,\delta}(E) = \frac{\beta_2(E-E^*) + \tilde{\beta}_2\delta + O(\delta^2)}{\sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} - \beta_1(E-E^*) - \tilde{\beta}_1\delta + O(\delta^2)};$$

$$\xi_{R,\delta}(E) = \frac{\beta_2(E-E^*) + \tilde{\beta}_2\delta + O(\delta^2)}{-\sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} - \beta_1(E-E^*) - \tilde{\beta}_1\delta + O(\delta^2)},$$

where

$$\beta_1 = \int_0^1 V^T W U dx, \quad \tilde{\beta}_1 = \int_0^1 V^T \tilde{F} U dx,$$

$$\beta_2 = \int_0^1 V^T W V dx, \quad \tilde{\beta}_2 = \int_0^1 V^T \tilde{F} V dx.$$

Proof. First note that $a_1 < 0$ (by lemma 4.2) and that,

$$D(E,\delta) = 2 + a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2 + O(E-E^*)^3 + O(\delta^3),$$

we have,

$$a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2 > 0.$$

Then the two eigenvalues for the matrix $M(E,\delta)$ are given by:

$$\lambda_{E,1,\delta} = \frac{D(E,\delta) - \sqrt{D(E,\delta)^2 - 4}}{2} = 1 - \sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} + O(\delta^2),$$

$$\lambda_{E,2,\delta} = \frac{D(E,\delta) + \sqrt{D(E,\delta)^2 - 4}}{2} = 1 + \sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} + O(\delta^2).$$

Using Taylor expansion and the formulas (A.1), (A.2) and (A.6), we can derive that:

$$\lambda_{E,1,\delta} = 1 - \sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} + O(\delta^2),$$

$$\lambda_{E,2,\delta} = 1 + \sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} + O(\delta^2),$$

and that,

$$\psi_{E,1,\delta}(1) = 1 + \beta_1(E-E^*) + \tilde{\beta}_1\delta + O(E-E^*)^2 + O(\delta^2);$$

$$\psi_{E,2,\delta}(1) = \beta_2(E-E^*) + \tilde{\beta}_2\delta + O(E-E^*)^2 + O(\delta^2).$$

Therefore the asymptotic of $\xi_{L,\delta}(E)$ and $\xi_{R,\delta}(E)$ follows. □

4.4. Existence of an interface mode for the perturbed system

In this section, we establish the existence of the interface mode that is generated by perturbing a periodic system with a Dirac point. We fix $\delta > 0$ and denote:

$$\varepsilon_{\delta,\pm}(x) := \varepsilon(x) \pm \delta\tilde{\varepsilon}(x), \quad \mu_{\delta,\pm}(x) := \mu(x) \pm \delta\tilde{\mu}(x),$$

$$\mathcal{L}_{\delta,\pm}\psi(x) := -\frac{1}{\varepsilon_{\delta,\pm}(x)}\frac{d}{dx}\left(\frac{1}{\mu_{\delta,\pm}(x)}\frac{d\psi}{dx}\right).$$

We also define:

$$\varepsilon_\delta(x) := \begin{cases} \varepsilon(x) - \delta\tilde{\varepsilon}(x), & x < 0, \\ \varepsilon(x) + \delta\tilde{\varepsilon}(x), & x > 0, \end{cases} \qquad \mu_\delta(x) := \begin{cases} \mu(x) - \delta\tilde{\mu}(x), & x < 0, \\ \mu(x) + \delta\tilde{\mu}(x), & x > 0, \end{cases}$$

and the associated differential operator:

$$\tilde{\mathcal{L}}_\delta\psi(x) := -\frac{1}{\varepsilon_\delta(x)}\frac{d}{dx}\left(\frac{1}{\mu_\delta(x)}\frac{d\psi}{dx}\right), \text{ or equivalently, } \tilde{\mathcal{L}}_\delta := \begin{cases} \mathcal{L}_{\delta,-}, & x < 0; \\ \mathcal{L}_{\delta,+}, & x > 0. \end{cases} \tag{4.6}$$

We shall make the following assumption on the perturbation which ensures the existence of a common band gap for the left and right semi-infinite perturbed periodic systems.

Assumption 3

$$\frac{a_3}{2} = \frac{1}{2}\frac{\partial^2 D(E,\delta)}{\partial\delta^2}(E^*,0) = \left(\int_0^1 U^T\tilde{F}Vdx\right)^2 - \left(\int_0^1 V^T\tilde{F}Vdx\right)\cdot\left(\int_0^1 U^T\tilde{F}Udx\right) > 0,$$

where $\tilde{F}$ is the matrix defined in (4.2) and U and V are defined in (4.3).

Lemma 4.6. *Under assumptions 2 and 3, the intersection of two band gaps* $(E^+_{j,\delta,+}, E^-_{j+1,\delta,+}) \cap (E^+_{j,\delta,-}, E^-_{j+1,\delta,-})$ *for the two operators* $\mathcal{L}_{\delta,+}$ *and* $\mathcal{L}_{\delta,-}$ *is not empty.*

Proof. Recall that $a_1 < 0$ (by lemma 4.2). We see that the inequality (4.4) holds. In light of theorem 4.3, the perturbation $\tilde{\varepsilon}, \tilde{\mu}$ will create a band gap $(E^+_{j,\delta,\pm}, E^-_{j+1,\delta,\pm})$ at E^* for the operators $\mathcal{L}_{\delta,\pm}$. In addition,

$$E^+_{j,\delta,\pm} = E^* \pm \eta^\mp\delta + O(\delta^{3/2}),$$
$$E^-_{j+1,\delta,\pm} = E^* \pm \eta^\pm\delta + O(\delta^{3/2}),$$

where $\eta^\pm$ are defined in (4.5). Since $a_1 < 0, a_3 > 0$, we have $\eta^- < 0 < \eta^+$. If $a_2 < 0$, then $|\eta^-| < |\eta^+|$, and consequently:

$$(E^+_{j,\delta,+}, E^-_{j+1,\delta,+}) \cap (E^+_{j,\delta,-}, E^-_{j+1,\delta,-}) = (E^* - |\eta^-|\delta + O(\delta^{3/2}), E^* + |\eta^-|\delta + O(\delta^{3/2})).$$

On the other hand, if $a_2 > 0$, then $|\eta^-| > |\eta^+|$ and there holds,

$$(E^+_{j,\delta,+}, E^-_{j+1,\delta,+}) \cap (E^+_{j,\delta,-}, E^-_{j+1,\delta,-}) = (E^* - |\eta^+|\delta + O(\delta^{3/2}), E^* + |\eta^+|\delta + O(\delta^{3/2})).$$

□

We are ready to investigate the existence of an interface mode in the band gap for the operator $\tilde{\mathcal{L}}_\delta$.

Theorem 4.7. *Let* $\mathcal{L}$ *be the operator defined in (1.1) satisfying assumption 2 and* $\mathcal{L}_\delta$ *be its perturbation defined in (4.6) satisfying assumption 3, then there exists an interface mode for* $\tilde{\mathcal{L}}_\delta$ *if* δ *is sufficiently small. The same holds when the Dirac point occurs at* $k^* = \pi$.

Proof. Without loss of generality, we only consider the case $k^* = 0$ and $a_2 < 0$. The other cases can be proved similarly. For $a_2 < 0$, the common band gap is given by:

$$(E^+_{j,\delta,+}, E^-_{j+1,\delta,+}) \cap (E^+_{j,\delta,-}, E^-_{j+1,\delta,-}) = \left(E^* - \eta^+\delta + O(\delta^{3/2}), E^* + \eta^+\delta + O(\delta^{3/2})\right).$$

We further consider two cases depending on whether $\tilde{\beta}_2 = \int_0^1 V^T \tilde{F} V dx = 0$ or not. We first consider the case $\tilde{\beta}_2 = 0$. For the operator $\mathcal{L}_{\delta,+}$, we consider the impedance function for the right half-line $[0,\infty)$ at the boundary $x = 0$:

$$\xi_{R,\delta}(E) = \frac{\psi_{E,2,\delta}(1)}{\lambda_{E,1,\delta} - \psi_{E,1,\delta}(1)}.$$

By lemma 4.5, we have:

$$\xi_{R,\delta}(E) = \frac{\beta_2(E-E^*) + \tilde{\beta}_2\delta + O(\delta^2)}{-\sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} - \beta_1(E-E^*) - \tilde{\beta}_1\delta + O(\delta^2)}.$$

Recall that,

$$\frac{a_3}{2} = \left(\int_0^1 U^T \tilde{F} V dx\right)^2 - \left(\int_0^1 V^T \tilde{F} V dx\right) \cdot \left(\int_0^1 U^T \tilde{F} U dx\right).$$

In the case $\tilde{\beta}_2 = 0$ we have:

$$a_3 = 2\tilde{\beta}_1^2.$$

Hence we can find $0 < \tau_1 < \eta^+$ such that for all E satisfying $|E - E^*| \leqslant \tau_1\delta$,

$$|\sqrt{a_1(E-E^*)^2 + 2a_2(E-E^*)\delta + a_3\delta^2} + \beta_1(E-E^*) + \tilde{\beta}_1\delta| \geqslant c_1\delta$$

for some constant $c_1 > 0$. Therefore, $\xi_{R,\delta}(E)$ is well-defined for E satisfying $|E - E^*| \leqslant \tau_1\delta$ and for δ sufficiently small.

For the operator $\mathcal{L}_{\delta,-}$, similarly we consider the impedance function for the left half-line $(-\infty, 0]$ at the boundary $x = 0$:

$$\xi_{L,-\delta}(E) = \frac{\psi_{E,2,-\delta}(1)}{\lambda_{E,2,-\delta} - \psi_{E,1,-\delta}(1)}.$$

Using lemma 4.5 again, we have,

$$\xi_{L,-\delta}(E) = \frac{\beta_2(E-E^*) - \tilde{\beta}_2\delta + O(\delta^2)}{\sqrt{a_1(E-E^*)^2 - 2a_2(E-E^*)\delta + a_3\delta^2} - \beta_1(E-E^*) + \tilde{\beta}_1\delta + O(\delta^2)}.$$

Again $\xi_{L,-\delta}(E)$ is well-defined for E satisfying $|E - E^*| \leqslant \tau_1\delta$.

By lemma 3.2, there exists an interface mode for $\tilde{\mathcal{L}}_\delta$ at energy level E if and only if:

$$\xi_{R,\delta}(E) = \xi_{L,-\delta}(E).$$

Let $t = (E - E^*)/\delta$. For $-\tau_1 \leqslant t \leqslant \tau_1$, we define:

$$\begin{aligned}\xi(t) = \xi_{R,\delta}(E) - \xi_{L,-\delta}(E) = &\frac{\beta_2 t + \tilde{\beta}_2 + O(\delta)}{-\sqrt{a_1t^2 + 2a_2t + a_3} - \beta_1 t - \tilde{\beta}_1 + O(\delta)} \\ &- \frac{\beta_2 t - \tilde{\beta}_2 + O(\delta)}{\sqrt{a_1t^2 - 2a_2t + a_3} - \beta_1 t + \tilde{\beta}_1 + O(\delta)}.\end{aligned}$$

It is clear that:

$$\xi(t) = \tilde{\xi}(t) + O(\delta),$$

where

$$\tilde{\xi}(t) = \frac{\beta_2 t + \tilde{\beta}_2}{-\sqrt{a_1 t^2 + 2a_2 t + a_3} - \beta_1 t - \tilde{\beta}_1} + \frac{-\beta_2 t + \tilde{\beta}_2}{\sqrt{a_1 t^2 - 2a_2 t + a_3} - \beta_1 t + \tilde{\beta}_1}.$$

One can check directly that $\tilde{\xi}(t)$ is odd, i.e. $\tilde{\xi}(t) + \tilde{\xi}(-t) = 0$.

We now choose $0 < \tau_2 < 1$ such that $|\tau_2 \eta^+| < \tau_1$ and $\tilde{\xi}(\tau_2 \eta^+) \neq 0$. It then follows that:

$$\tilde{\xi}(\tau_2 \eta^+) \cdot \tilde{\xi}(-\tau_2 \eta^+) < 0.$$

Therefore for δ small enough, we have,

$$\xi(\tau_2 \eta^+) \cdot \xi(-\tau_2 \eta^+) < 0.$$

Hence there exists a root to $\xi(t) = 0$ in the interval $(-\tau_2 \eta^+, \tau_2 \eta^+)$. This root gives the existence of an interface mode.

Finally, we consider the case when $\tilde{\beta}_2 \neq 0$. The above argument may not get through since the leading order term in the denominators of the two impedance functions $\xi_{R,\delta}(E)$ and $\xi_{L,-\delta}(E)$ may vanish for E in the common band gap. To address this issue, we consider the reciprocal of the impedance functions instead. Then a similar argument leads to the conclusion that:

$$\frac{1}{\xi_{R,\delta}(E)} = \frac{1}{\xi_{L,-\delta}(E)},$$

for some E in the common band gap. This implies the existence of an interface mode. □

Remark 2. The Bulk-interface correspondence for the interface mode is not formulated in the above theorem. Another question is the stability of the interface mode under perturbations that are not necessarily small. One formulation of the stability question is to show the persistence of the interface mode under a continuous family of perturbations to the operator $\tilde{\mathcal{L}}_\delta$ such that the band-gap structure below the gap for the two periodic operators defined over the two semi-infinite intervals remains unchanged during the perturbation. We leave these as the future work.

Finally, we present the scenarios for which assumption 3 holds, and consequently the theorem above can be applied. We leave the proof to the appendix.

Lemma 4.8. *Let the periodic operator $\mathcal{L}$ in the form of (1.1) that attains periodic coefficients in the sense of assumption 1. Assume that μ and ε are even functions and that $(k = 0, E = E^*)$ is a Dirac point. Let U, V be defined by (4.3). Furthermore, we assume that $\tilde{\mu}$ and $\tilde{\varepsilon}$ are odd, then:*

$$\left(\int_0^1 V^T \tilde{F} V dx \right) \cdot \left(\int_0^1 U^T \tilde{F} U dx \right) = 0, \quad \frac{a_3}{2} = \left(\int_0^1 U^T \tilde{F} V dx \right)^2.$$

Moreover, we can choose $\tilde{\mu}$ and $\tilde{\varepsilon}$ such that:

$$\int_0^1 U^T \tilde{F} V dx \neq 0.$$

5. Photonic structures with inversion symmetry

In this section, we investigate time-reversal symmetric photonic structure with additional inversion symmetry. That is, the following assumption holds for the periodic operator $\mathcal{L}$ in (1.1).

Assumption 4. The operator $\mathcal{L}$ attains periodic coefficients in the sense of assumption 1. Moreover,

$$\mathcal{P}\mathcal{L} = \mathcal{L}\mathcal{P},$$

where $\mathcal{P}$ is the **parity operator** defined by:

$$\mathcal{P}\psi(x) = \psi(-x),$$

for any function ψ defined over $\boldsymbol{R}$.

Under the above assumption, we see that $\varepsilon(x) = \varepsilon(-x)$, $\mu(x) = \mu(-x)$, or equivalently, $\varepsilon(x) = \varepsilon(1-x)$, $\mu(x) = \mu(1-x)$. Such topological structures were investigated in [38] and it was shown that localized mode exists at the interface of the two semi-infinite periodic structures with different bulk topological indices. Inspired by this work, we would like to provide a rigorous theory for the existence of an interface mode for such a structure and its connection to the bulk topological index, which is defined via the quantized Zak phase. In addition, we investigate the stability of the interface mode under perturbations that are not necessarily small.

5.1. Bloch modes and parity

We begin with some properties of the Bloch modes when the structure attains inversion symmetry. The following result is obvious.

Lemma 5.1. *Under assumption 4, if $\varphi_{j,k}(x)$ is a Bloch mode of the operator $\mathcal{L}$ for the jth band with the Bloch wavenumber k, then $\mathcal{P}\varphi_{j,k}$ is a Bloch mode with the Bloch wavenumber $-k$.*

Lemma 5.2. *Under assumption 4, the Bloch modes $\varphi_{j,k}$ are even or odd when $k = 0$ or π over an isolated band $E_j(k)$. In addition, when $k = 0$ or π, there holds $\varphi_{j,k} = c\,\psi_{E,1}$ or $\varphi_{j,k} = c\,\psi_{E,2}$ over the periodic cell $[0,1]$ for certain constant c depending on whether $\varphi_{j,k}$ is even or odd, and $\psi_{E,1}$ and $\psi_{E,2}$ is the solution of the initial value problems (2.3) and (2.4) respectively.*

Proof. Consider the Bloch mode $\varphi_{j,0}$ for $k = 0$ which solves the following boundary problem:

$$(\mathcal{L} - E_j(0))\varphi_{j,0} = 0, \quad \varphi_{j,0}(x+1) = \varphi_{j,0}(x).$$

One can check that $\mathcal{P}\varphi_{j,0}$ is also a Bloch mode for $k = 0$. Since the multiplicity of the Bloch mode for $k = 0$ is one (propositon 2.7) and that $\varphi_{j,0}$ is real-valued, it follows that $\mathcal{P}\varphi_{j,0} = \pm\varphi_{j,0}(x)$, i.e.:

$$\varphi_{j,0}(-x) = \pm\varphi_{j,0}(x).$$

Note that if $\varphi_{j,0}$ is even, then $\varphi'_{j,0}(0) = 0$ and $\varphi_{j,0} = c\psi_{E,1}$ for some constant c. Similarly, if $\varphi_{j,0}$ is odd, then $\varphi_{j,0} = c\psi_{E,2}$ for some constant c. A parallel argument leads to the conclusion for the Bloch mode $\varphi_{j,\pi}$.

□

Definition 2. We say that the Bloch mode $\varphi_{j,k}$ attains an even-parity (odd-parity) if $\varphi_{j,k}$ is an even (odd) function.

Next we investigate the change of parity for the Bloch modes at $k = 0$ or π when the energy crosses a band gap. A crucial tool we used is the oscillation theory for Sturm–Liouville operators, see for instance [37]. To be more precise, let us denote $E_j^P, E_j^S, E_j^D, E_j^N$ the jth eigenvalues of the operator $\mathcal{L}$ restricted to the unit cell $[0,1]$ with the following boundary conditions respectively:

(a) Periodic boundary conditions: $\psi(1) = \psi(0), \quad \psi'(1) = \psi'(0)$;
(b) Semi-periodic boundary conditions: $\psi(1) = -\psi(0), \quad \psi'(1) = -\psi'(0)$;
(c) Dirichlet boundary conditions: $\psi(1) = \psi(0) = 0$;
(d) Neumann boundary conditions: $\psi'(1) = \psi'(0) = 0$.

We have the following theorem on the eigenvalues above, see for instance theorem 13.10 in [37].

Theorem 5.3. *The eigenvalues $E_j^P, E_j^S, E_j^D, E_j^N$ $(j = 1, 2, 3, \ldots)$ defined above for the periodic operator $\mathcal{L}$ in the form of (1.1) attain the following interlacing property:*

$$
\begin{aligned}
&E_1^N \leqslant E_1^P < E_1^S \leqslant \{E_2^N, E_1^D\} \leqslant E_2^S < E_2^P \leqslant \{E_3^N, E_2^D\} \leqslant \cdots \\
&\quad \leqslant E_{2n-1}^P < E_{2n-1}^S \leqslant \{E_{2n}^N, E_{2n-1}^D\} \leqslant E_{2n}^S < E_{2n}^P \leqslant \{E_{2n+1}^N, E_{2n}^D\} \leqslant E_{2n+1}^P < \cdots,
\end{aligned}
$$

here the expression $E_1^S \leqslant \{E_2^N, E_1^D\} \leqslant E_2^S$ means that $E_1^S \leqslant \min\{E_2^N, E_1^D\} \leqslant \max\{E_2^N, E_1^D\} \leqslant E_2^S$, and the same meaning applies to others.

Based on the above theorem, we are able to show the change of parity for Bloch modes across band gaps, which is stated in the theorem below:

Theorem 5.4. *Let $\mathcal{L}$ be a periodic operator in the form of (1.1) that satisfies assumption 4. Assume that there is a band gap between the jth and $(j+1)$th bands, then the Bloch modes at (k, E_j^+) and (k, E_{j+1}^-) attain different parity, where $k = 0$ or π.*

Proof. Without loss of generality, we assume that $k = 0$ so that $E_j^+ = E_j(0)$, $E_{j+1}^- = E_{j+1}(0)$, and the Bloch mode $\varphi_{j,0}$ at $(0, E_j^+)$ is even. Then $\varphi_{j,0}$ satisfies the following boundary value problem:

$$
\begin{cases}
(\mathcal{L} - E_j^+)\varphi_{j,0} = 0, \\
\varphi_{j,0}(0) = \varphi_{j,0}(1), \\
\varphi'_{j,0}(0) = \varphi'_{j,0}(1) = 0.
\end{cases}
$$

Hence E_j^+ is a common eigenvalue to the operator $\mathcal{L}$ for both the periodic boundary condition and the Neumann boundary condition. We prove by contradiction that $\varphi_{j+1,0}$ is odd. Otherwise, if $\varphi_{j+1,0}$ is even, then E_{j+1}^- is also a common eigenvalue to the operator $\mathcal{L}$ for both the periodic boundary condition and the Neumann boundary condition. Note that $E_j^+ < E_{j+1}^-$ are two neighboring eigenvalues to $\mathcal{L}$ with the periodic boundary condition. We either have $E_j^+ = E_{2n-1}^P, E_{j+1}^- = E_{2n}^P$ or $E_j^+ = E_{2n}^P, E_{j+1}^- = E_{2n+1}^P$ for some integer n. By theorem 5.3, the former is impossible since there is no eigenvalue to $\mathcal{L}$ with the semi-periodic boundary condition inside the band gap. The latter is also impossible since both E_j^+, E_{j+1}^- are eigenvalues to $\mathcal{L}$ with the Neumann boundary condition. This contradiction proves that $\varphi_{j+1,0}$ should be an odd-parity mode and this completes the proof for the case $k = 0$. The case $k = \pi$ can be proved in a similar manner. $\square$

5.2. Zak phase under inversion symmetry

Consider the periodic operator $\mathcal{L}$ that satisfies assumption 4. Following section 2.3 and in view of lemma 5.1, we construct the Bloch modes for the jth band of $\mathcal{L}$ as follows:

$$\varphi_{j,k}(x) = \begin{cases} \frac{\phi_{j,k}(x)}{\|\phi_{j,k}\|_X}, 0 \leqslant k \leqslant \pi, \quad \phi_{j,k} \not\equiv 0, \\ \frac{i\psi_{E,2}(x)}{\|\psi_{E,2}\|_X}, \ k \in \{0, \pi\} \text{ and } \phi_{j,k} \equiv 0, \\ \varphi_{j,-k}(-x), \quad -\pi < k < 0. \end{cases} \tag{5.1}$$

The periodic part of $\varphi_{j,k}$ is given by $u_{j,k}(x) = \varphi_{j,k}(x)e^{-ikx}$, and it satisfies:

$$u_{j,k}(x) = u_{j,-k}(-x), \quad -\pi < k < 0.$$

We calculate the Zak phase using the formula (2.21). First, there holds,

$$\begin{aligned}
\int_{-\pi}^{0} \left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk &= -\int_0^{\pi} \left(\frac{\partial u_{j,-k}}{\partial k}, u_{j,-k}\right)_X dk \\
&= -\int_0^{\pi}\int_0^1 \frac{\partial u_{j,-k}(x)}{\partial k}\bar{u}_{j,-k}(x)\varepsilon(x)dxdk \\
&= -\int_0^{\pi}\int_0^1 \frac{\partial u_{j,k}(-x)}{\partial k}\bar{u}_{j,k}(-x)\varepsilon(-x)dxdk \\
&= -\int_0^{\pi}\int_0^1 \frac{\partial u_{j,k}(1-x)}{\partial k}\bar{u}_{j,k}(1-x)\varepsilon(1-x)dxdk \\
&= -\int_0^{\pi}\int_0^1 \frac{\partial u_{j,k}(x)}{\partial k}\bar{u}_{j,k}(x)\varepsilon(x)dxdk \\
&= -\int_0^{\pi} \left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk.
\end{aligned}$$

On the other hand, note that:

$$\begin{aligned}
u_{j,0^-}(x) &= \lim_{k\to 0^-} u_{j,k}(x) = \lim_{k\to 0^+} u_{j,-k}(x) = \lim_{k\to 0^+} u_{j,k}(-x) = u_{j,0}(-x); \\
u_{j,(-\pi)^+}(x) &= \lim_{k\to(-\pi)^+} u_{j,k}(x) = \lim_{k\to\pi^-} u_{j,-k}(x) = \lim_{k\to\pi^-} u_{j,k}(-x) = u_{j,\pi}(-x).
\end{aligned}$$

It follows that:

$$\begin{aligned}
(u_{j,0}, u_{j,0^-})_X &= \int_0^{1/2} u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx + \int_{1/2}^1 u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx \\
&= \int_0^{1/2} u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx + \int_{-1/2}^0 u_{j,0}(x+1)\bar{u}_{j,0}(-x-1)\varepsilon(x+1)dx \\
&= \int_0^{1/2} u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx + \int_{-1/2}^0 u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx \\
&= \int_{-1/2}^{1/2} u_{j,0}(x)\bar{u}_{j,0}(-x)\varepsilon(x)dx = \int_{-1/2}^{1/2} \varphi_{j,0}(x)\bar{\varphi}_{j,0}(-x)\varepsilon(x)dx \\
&= \begin{cases} 1, & \text{if } \varphi_{j,0}(x) = \varphi_{j,0}(-x), \\ -1, & \text{if } \varphi_{j,0}(x) = -\varphi_{j,0}(-x). \end{cases}
\end{aligned}$$

Similarly, we have:

$$(e^{-i2\pi x}u_{j,(-\pi)^+}, u_{j,\pi})_X = \begin{cases} 1, & \text{if } \varphi_{j,\pi}(x) = \varphi_{j,\pi}(-x), \\ -1, & \text{if } \varphi_{j,\pi}(x) = -\varphi_{j,\pi}(-x). \end{cases} \tag{5.2}$$

Therefore, by substituting the above into the formula (2.21), we obtain the following theorem.

Theorem 5.5. *Let $\mathcal{L}$ be a periodic operator in the form of (1.1). Assume that assumption 4 holds. Then the Zak phase for the jth band is given by:*

$$\theta_j = \begin{cases} 0, & \text{if } \varphi_{j,0}(x) \text{ and } \varphi_{j,\pi}(x) \text{ attain the same parity}, \\ \pi, & \text{if } \varphi_{j,0}(x) \text{ and } \varphi_{j,\pi}(x) \text{ attain different parity}. \end{cases}$$

In the presence of a Dirac point, we have the following result on the Zak phase, which is a direct consequence of the theorem above and lemma 2.9.

Theorem 5.6. *Let $\mathcal{L}$ be a periodic operator in the form of (1.1). Assume that assumption 4 holds and that the jth and $(j+1)$th bands touch at the Dirac point $(k=0, E_j^+)$, then*

$$\theta_j + \theta_{j+1} = \begin{cases} 0, & \text{if } \varphi_{j,\pi}(x), \varphi_{j+1,\pi}(x) \text{ attain the same parity}; \\ \pi, & \text{if } \varphi_{j,\pi}(x), \varphi_{j+1,\pi}(x) \text{ attain different parities}. \end{cases} \tag{5.3}$$

Similar result holds if the jth band and the $(j+1)$th band cross at the Dirac point $(k=\pi, E_j^+)$.

5.3. Bulk topological indices

Assume that there is a gap between the jth and $(j+1)$th bands of the periodic operator $\mathcal{L}$ with the inversion symmetry. We define an index for the jth band of the spectrum as:

$$\gamma_j := (-1)^{j+\ell-1} e^{i\sum_{m=1}^{j}\theta_m}, \tag{5.4}$$

in which θ_m is the Zak phase for the mth band, and ℓ is the number of Dirac points below the jth band. The relation between the parity of the Bloch mode $\psi_{k,j}$ at band edge (k, E_j^+) and the bulk index γ_j is given in the following theorem.

Theorem 5.7. *Let $\mathcal{L}$ be an operator in the form of (1.1). Assume that assumption 4 holds and that there is a band gap between the jth and $(j+1)$th bands. Then the bulk topological index for the jth band γ_j only takes the values ± 1. In addition, γ_j is 1 and -1 when the Bloch mode at band edge (k, E_j^+) is even and odd respectively.*

Proof. Note that for the first band, we have $E_1^- = E_1(0) = 0$ and the associated Bloch mode is a constant function. Since $E_1^+ = E_1(\pi)$, by virtue of theorem 5.5, $\gamma_1 = 1$ and -1 when the Bloch mode $\psi_{\pi,1}$ at (π, E_1^+) is even and odd respectively. Now we prove by induction and assume that the statement holds for the band $E_n(k)$ with $n < j$. If $E_{j-1}(k)$ does not cross with $E_j(k)$, then an application of theorems 5.4 and 5.5 yields $\gamma_j = -e^{i\theta_j}\gamma_{j-1}$, where θ_j is 0 or π. Otherwise, if $E_{j-1}(k)$ and $E_j(k)$ cross at the Dirac point (k, E_j^-), applying theorem 5.6 gives $\gamma_j = -e^{i(\theta_{j-1}+\theta_j)}\gamma_{j-2}$, where $\theta_{j-1} + \theta_j = 0$ or π. The proof is complete. □

5.4. Impedance functions in the band gap

In this section, we derive several properties for the following two impedance functions:

$$\xi_L(E) = \frac{\psi_{E,2}(1)}{\lambda_{E,2} - \psi_{E,1}(1)}, \quad \xi_R(E) = \frac{\psi_{E,2}(1)}{\lambda_{E,1} - \psi_{E,1}(1)},$$

defined in section 3.1 for E in the band gap and for the system with inversion symmetry. We first present a preliminary lemma.

Lemma 5.8. *Assume that the operator $\mathcal{L}$ satisfies assumption 4. Further assume that the jth and $j+1$th bands attain a gap (E_j^+, E_{j+1}^-).*

(a) For all $E \in (E_j^+, E_{j+1}^-)$, we have $\psi_{E,2}(1) \neq 0$, $\lambda_{E,1} - \psi_{E,1}(1) \neq 0$, and $\lambda_{E,2} - \psi_{E,1}(1) \neq 0$.
(b) If $\psi_{E,2}(1) = 0$ for some $E \notin (E_j^+, E_{j+1}^-)$, then

$$\psi'_{E,2}(1) \cdot \frac{\partial \psi_{E,2}(1)}{\partial E} > 0.$$

Proof. We first show that $\psi_{E,2}(1) \neq 0$. Assume otherwise that $\psi_{E,2}(1) = 0$. Then both functions $\psi_{E,2}(1-x)$ and $\psi_{E,2}(x)$ are solutions to the following boundary value problem:

$$(\mathcal{L} - E)\psi = 0, \quad \psi(0) = \psi(1) = 0.$$

Thus $\psi_{E,2}(1-x)$ and $\psi_{E,2}(x)$ must be linearly dependent. Since both are real-valued, we see that $\psi_{E,2}(1-x) = \pm\psi_{E,2}(x)$. It follows that $\psi'_{E,2}(1^-) = \mp\psi'_{E,2}(0^+) = \mp\mu(0^+)$. Since μ has period one and is inversion symmetry, we have $\mu(0^+) = \mu(0^-) = \mu(1^-)$. Therefore

$$M(E) = \begin{pmatrix} \psi_{E,1}(1^-) & \psi_{E,2}(1^-) \\ \frac{1}{\mu(1^-)}\psi'_{E,1}(1^-) & \frac{1}{\mu(1^-)}\psi'_{E,2}(1^-) \end{pmatrix} = \begin{pmatrix} \psi_{E,1}(1^-) & 0 \\ \frac{1}{\mu(1^-)}\psi'_{E,1}(1^-) & \pm 1 \end{pmatrix},$$

We see that $\pm 1 \in \{\lambda_{E,1}, \lambda_{E,2}\}$, which is a contradiction to the fact that $|\lambda_{E,1}| < 1$ and $|\lambda_{E,2}| > 1$ in the band gap.

We next prove that $\lambda_{E,1} - \psi_{E,1}(1) \neq 0$. If $\lambda_{E,1} - \psi_{E,1}(1) = 0$, then by using $det(M(E) - \lambda_{E,1}) = 0$, we have:

$$\psi_{E,2}(1) \cdot \frac{1}{\mu(1)}\psi'_{E,1}(1) = 0,$$

which yields $\psi'_{E,1}(1) = 0$. Then both functions $\psi_{E,1}(1-x)$ and $\psi_{E,1}$ are solutions to the following boundary value problem:

$$(\mathcal{L} - E)\psi = 0, \quad \psi'(0) = \psi'(1) = 0.$$

We can thus derive that $\psi = \pm\psi_{E,1}$. Then,

$$\psi_{E,1}(0) = \pm\psi_{E,1}(1) = \pm 1.$$

Hence

$$M(E) = \begin{pmatrix} \psi_{E,1}(1) & \psi_{E,2}(1) \\ \frac{1}{\mu(1)}\psi'_{E,1}(1) & \frac{1}{\mu(1)}\psi'_{E,2}(1) \end{pmatrix} = \begin{pmatrix} \pm 1 & \psi_{E,2}(1) \\ 0 & \pm\frac{1}{\mu(1)}\psi'_{E,2}(1) \end{pmatrix}.$$

Again, this leads to $\pm 1 \in \{\lambda_{E,1}, \lambda_{E,2}\}$, which contradicts to the fact that the eigenvalues are in the band gap. The inequality $\lambda_{E,2} - \psi_{E,1}(1) \neq 0$ can be proved in a similar manner. This completes the proof of (a).

We now prove (b). Note that,

$$\begin{cases}(\mathcal{L}-E)\psi_{E,2}=0,\\ \psi_{E,2}(0)=0,\frac{1}{\mu(0)}\psi'_{E,2}(0)=1.\end{cases}$$

Let $\psi(x)=\frac{\partial\psi_{E,2}(x)}{\partial E}$, then

$$\begin{cases}(\mathcal{L}-E)\psi=\psi_{E,2},\\ \psi(0)=0,\psi'(0)=0.\end{cases}$$

Multiplying both sides of the equation $(\mathcal{L}-E)\psi=\psi_{E,2}$ by $\varepsilon(x)\psi_{E,2}(x)$, we obtain:

$$\int_0^1\left((\frac{1}{\mu}\psi')'+E\varepsilon(x)\psi\right)\psi_{E,2}(x)dx=-\int_0^1\psi_{E,2}^2(x)\varepsilon(x)dx.$$

Using Green's theorem and the boundary conditions that $\psi(0)=0,\psi'(0)=0,\psi_{E,2}(0)=0$, we further derive that:

$$\frac{1}{\mu(1)}\psi'_{E,2}(1)\psi(1)=\int_0^1\psi_{E,2}^2(x)\varepsilon(x)dx>0,$$

This gives the desired inequality and completes the proof. □

Lemma 5.9. *Assume that the operator $\mathcal{L}$ satisfies assumption 4. Further assume that there is a band gap between the jth and the $(j+1)$th bands. Then the following holds for $E\in(E_j^+,E_{j+1}^-)$:*

(a) If the Bloch mode at the band edge (k,E_j^+) attains the odd-parity for $k=0$ or π, then $\xi_R(E)<0$, and $\xi_R(E)\to0$ as $E\to E_j^+$ and $\xi_R(E)\to-\infty$ as $E\to E_{j+1}^-$ respectively; On the other hand, $\xi_L(E)>0$, and $\xi_L\to0$ as $E\to E_j^+$ and $\xi_L\to+\infty$ as $E\to E_{j+1}^-$ respectively.
(b) If the Bloch edge mode at (k,E_j^+) attains the even-parity, then $\xi_R(E)>0$, and $\xi_R(E)\to+\infty$ as $E\to E_j^+$ and $\xi_R(E)\to0$ as $E\to E_{j+1}^-$ respectively; On the other hand, $\xi_L(E)<0$ and $\xi_L(E)\to-\infty$ as $E\to E_j^+$ and $\xi_L(E)\to0$ as $E\to E_{j+1}^-$ respectively.

Proof. Without loss of generality, we consider only the case $k=0$ and the Bloch mode $\varphi_{j,0}$ at $(0,E_j^+)$ is odd. The proof for other cases is similar. It also suffices to prove for the function $\xi_R(E)$ since the function $\xi_L(E)$ can be treated similarly. First, by lemma 5.2, $\varphi_{j,0}(x)=c\psi_{E_j^+,2}(x)$ for some constant c. Therefore $\psi_{E_j^+,2}(1)=\psi_{E_j^+,2}(0)=0$ since $\varphi_{j,0}$ is periodic of period one. By lemma 5.8, we have:

$$\frac{\partial\psi_{E,2}(1)}{\partial E}(E_j^+)\cdot\psi'_{E_j^+,2}(1)>0.$$

Since $\psi'_{E_j^+,2}(1)=\psi'_{E_j^+,2}(0)>0$, $\frac{\partial\psi_{E,2}(1)}{\partial E}(E_j^+)>0$, and consequently, $\psi_{E,2}(1)>0$ for $E\in(E_j^+,E_{j+1}^-)$.

We next define

$$g(E):=\lambda_{E,1}-\psi_{E,1}(1).$$

By theorem 5.4, $\varphi_{j+1,0}$ is even. Thus $\varphi_{j+1,0}=c\psi_{E_{j+1}^-,1}$ for some constant c and we have $\psi_{E_{j+1}^-,1}(1)=\psi_{E_{j+1}^-,1}(0)=1$ using the periodicity of $\varphi_{j+1,0}$. It follows that:

$$g(E_{j+1}^-)=\lambda_{E_{j+1}^-,1}-\psi_{E_{j+1}^-,1}(1)=1-1=0.$$

On the other hand, since

$$\lambda_{E,1} = \frac{D(E) - \sqrt{D(E)^2 - 4}}{2},$$

we have

$$g'(E) = \frac{1}{2} D'(E) \left(1 - \frac{D(E)}{\sqrt{D(E)^2 - 4}}\right) - \frac{\partial \psi_{E,1}(1)}{\partial E}.$$

It is clear $D(E) \to 2$ and $D'(E) < 0$ as $E \to E_{j+1}^-$. Therefore

$$\lim_{E \to E_{j+1}^-} g'(E) = \infty, \tag{5.5}$$

whence $g(E) < 0$ near E_{j+1}^- and hence over the whole interval (E_j^+, E_{j+1}^-). This proves that $\xi_R(E) = \frac{\psi_{E,2}(1)}{g(E)} < 0$ over (E_j^+, E_{j+1}^-).

We now prove that $\xi_R(E) \to 0$ as $E \to E_j^+$ and $\xi_R(E) \to -\infty$ as $E \to E_{j+1}^-$. Recall that $\psi_{E_j^+,2}(1) = 0$. There are two cases: $g(E_j^+) \neq 0$ or $g(E_j^+) = 0$. In the former case it is clear that $\xi_R(E) \to 0$ as $E \to E_j^+$. In the latter case,

$$\lim_{E \to E_j^+} \xi_R(E) = \lim_{E \to E_j^+} \frac{\psi_{E,2}(1)}{g(E)} = \lim_{E \to E_j^+} \frac{\frac{\partial \psi_{E,2}(1)}{\partial E}}{g'(E)} = 0,$$

where we used the fact that $\lim_{E \to E_j^+} g'(E) = -\infty$ if $g(E_j^+) = 0$ (the proof is similar to (5.5)). Therefore, in both cases we have $\xi_R(E) \to 0$ as $E \to E_j^+$.

Finally, we show that $\xi_R(E) \to -\infty$ as $E \to E_{j+1}^-$. Since $g(E_{j+1}^-) = 0$, we need only to show that $\psi_{E_{j+1}^-,2}(1) \neq 0$. Indeed, assume otherwise $\psi_{E_{j+1}^-,2}(1) = 0 = \psi_{E_{j+1}^-,2}(0)$. Recall that $\psi_{E_{j+1}^-,1}(1) = 1$ and that:

$$\det M(E_{j+1}^-) = \det \begin{pmatrix} \psi_{E_{j+1}^-,1}(1) & \psi_{E_{j+1}^-,2}(1) \\ \frac{1}{\mu(1)} \psi'_{E_{j+1}^-,1}(1) & \frac{1}{\mu(1)} \psi'_{E_{j+1}^-,2}(1) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ \frac{1}{\mu(1)} \psi'_{E_{j+1}^-,1}(1) & \frac{1}{\mu(1)} \psi'_{E_{j+1}^-,2}(1) \end{pmatrix}.$$

We have,

$$\frac{1}{\mu(1)} \psi'_{E_{j+1}^-,2}(1) = 1 = \frac{1}{\mu(0)} \psi'_{E_{j+1}^-,2}(0).$$

Since $\mu(0) = \mu(1)$, $\psi'_{E_{j+1}^-,2}(1) = \psi'_{E_{j+1}^-,2}(0)$. Therefore we can derive that $\psi_{E_{j+1}^-,2}$ is a periodic function with period one and hence $\varphi_{j+1,0} = c\psi_{E_{j+1}^-,2}$ for some constant c. This contradicts to the established fact that $\varphi_{j+1,0} = c\psi_{E_{j+1}^-,1}$. This completes the proof of the lemma. □

5.5. Interface modes induced by bulk topological indices

We consider a photonic system which consists of two semi-infinite periodic structures for $x < 0$ and $x > 0$ respectively.

The corresponding periodic differential operator is:

$$\mathcal{L}_j \psi = -\frac{1}{\varepsilon_j(x)} \frac{d}{dx} \left(\frac{1}{\mu_j(x)} \frac{d\psi}{dx} \right), \quad j = 1, 2.$$

We assume that both operators satisfy assumption 4. The differential operator for the joint structure is given by:

$$\tilde{\mathcal{L}}\psi(x) := \begin{cases} \mathcal{L}_1\psi(x), & x < 0, \\ \mathcal{L}_2\psi(x), & x > 0. \end{cases} \tag{5.6}$$

We investigate the existence of interface modes for the operator $\tilde{\mathcal{L}}$. In what follows, we denote the quantities associated with the operator $\mathcal{L}_j$ using the superscript j $(j = 1, 2)$, such as the energy level $E_m^{(j)}$, the Bloch mode $\psi_{m,k}^{(j)}$, etc.

Theorem 5.10. *Assume that the following holds:*

(a) The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy assumption 4 and attain a common band gap:

$$I := (E_{m_1}^{(1),+}, E_{m_1+1}^{(1),-}) \cap (E_{m_2}^{(2),+}, E_{m_2+1}^{2,-}) \neq \emptyset$$

for certain positive integers m_1 and m_2.

(b) The bulk topological indices $\gamma_{m_1}^{(1)} \neq \gamma_{m_2}^{(2)}$ for the operator $\mathcal{L}_1$ and $\mathcal{L}_2$.

Then there exists an interface mode for the operator $\tilde{\mathcal{L}}$ defined in (5.6). In addition, the number of interface modes are given by the number of roots to the equation:

$$\xi(E) := \xi_L^{(1)}(E) - \xi_R^{(2)}(E) = 0 \quad \textit{for } E \in I. \tag{5.7}$$

Proof. By lemma 3.2, there is an interface mode of $\tilde{\mathcal{L}}$ at energy level E if and only if:

$$\xi(E) := \xi_L^{(1)}(E) - \xi_R^{(2)}(E) = 0.$$

Without loss of generality, we consider the case when the common band gap of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ is given by $I = (E_{m_1}^{(1),+}, E_{m_1+1}^{(1),-})$. Moreover, $\gamma_{m_1}^{(1)} = 1$ and $\gamma_{m_2}^{(2)} = -1$ for the two operators. Then the Bloch mode $\psi_{m_1,k}^{(1)}$ at the band edge $(k, E_{m_1}^{(1),+})$ for the operator $\mathcal{L}_1$ is even while the Bloch mode $\psi_{m_2,k}^{(2)}$ at the band edge $(k, E_{m_2}^{(2),+})$ for the operator $\mathcal{L}_2$ is odd. By lemma 5.9, $\xi_L^{(1)}(E) < 0$ and $\xi_L^{(1)}(E) \to -\infty$ as $E \to E_{m_1}^{(1),+}$ and $\xi_L^{(1)}(E) \to 0$ as $E \to E_{m_1+1}^{(1),-}$ respectively. On the other hand, $\xi_R^{(2)}(E) < 0$ and $\xi_R^{(2)}(E) \to 0$ as $E \to E_{m_2}^{(1),+}$ and $\xi_R^{(2)}(E) \to -\infty$ as $E \to E_{m_2+1}^{(1),-}$ respectively. Therefore, for E in the common gap I, we see that $\xi(E) < 0$ for E near $E_{m_1}^{(1),+}$ and $\xi(E) > 0$ for E near $E_{m_1+1}^{(1),-}$. It follows that there exists a root over the interval I for $\xi(E) = 0$.

□

To illustrate the interface mode of the operator $\tilde{\mathcal{L}}$, we consider a joint structure, where the periodic medium on the left consists of two layers in each period, with a thickness of $\ell_a^{(1)} = 0.42$ and $\ell_b^{(1)} = 0.58$ respectively. The permittivity values of the two layers are $\varepsilon_a^{(1)} = 3.8$ and $\varepsilon_b^{(1)} = 1$, and the permeability values are $\mu_a^{(1)} = \mu_b^{(1)} = 1$. The periodic medium on the right also consists of two layers in each period, with the physical parameters in each period given by $\ell_a^{(2)} = 0.38$, $\ell_b^{(2)} = 0.62$, $\varepsilon_a^{(2)} = 4.2$, $\varepsilon_b^{(2)} = 1$, and $\mu_a^{(2)} = \mu_b^{(2)} = 1$. The band structures of the two periodic media and the Zak phase for each band are shown in figure 1. The bulk indices in the band gap between the seventh and eighth band are $\gamma_7^{(1)} = 1$ and $\gamma_7^{(2)} = -1$ respectively. The operator $\tilde{\mathcal{L}}$ attains an interface mode at a frequency in the band gap, which is shown in figure 2.

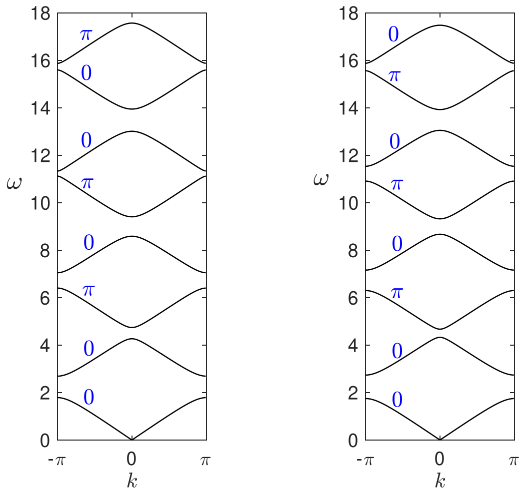

Figure 1. The band structures of the two periodic media and the Zak phase for each band.

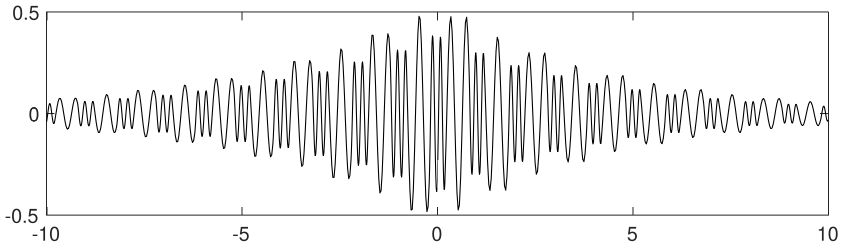

Figure 2. The interface mode of the operator $\tilde{\mathcal{L}}$ in the band gap between the seventh and eighth band.

5.6. Stability of interface modes

Consider a photonic system $\tilde{\mathcal{L}}$ of the form (5.6) that attains an interface mode over a common spectral band gap I of two operators $\mathcal{L}_1$ and $\mathcal{L}_2$. Assume that the structure is perturbed locally with a defect region (d_1, d_2), in which $d_1 < 0 < d_2$, and the relative permittivity and permeability of the structure attain the following values:

$$\varepsilon(x) = \begin{cases} \varepsilon_1(x-d_1), & x < d_1, \\ \varepsilon_d(x), & d_1 < x < d_2, \\ \varepsilon_2(x-d_2), & x > d_2. \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} \mu_1(x-d_1), & x < d_1, \\ \mu_d(x), & d_1 < x < d_2, \\ \mu_2(x-d_2), & x > d_2. \end{cases}$$

We denote the differential operator for the perturbed system by $\tilde{\mathcal{L}}_d$, and denote:

$$\Phi(x;E) = \left(\psi(x), \frac{1}{\mu}\psi'(x)\right)^T,$$

where ψ solves the differential equation $(\tilde{\mathcal{L}}_d - E)\psi = 0$.

For $j = 1, 2$, let

$$V_{E,1}^{(j)} = \begin{pmatrix} \psi_{E,2}^{(j)}(1) \\ \lambda_{E,1}^{(j)} - \psi_{E,1}^{(j)}(1) \end{pmatrix} \quad \text{and} \quad V_{E,2}^{(j)} = \begin{pmatrix} \psi_{E,2}^{(j)}(1) \\ \lambda_{E,2}^{(j)} - \psi_{E,1}^{(j)}(1), \end{pmatrix}$$

be the eigenvectors of the transfer matrix $M^{(j)}(E)$ as defined in (3.1). For each $E \in I$, we normalize the eigenvectors $V_{E,1}^{(j)}$ and $V_{E,2}^{(j)}$ by letting $\tilde{V}_{E,1}^{(j)} = V_{E,1}^{(j)}/\|V_{E,1}^{(j)}\|$ and $\tilde{V}_{E,2}^{(j)} = V_{E,2}^{(j)}/\|V_{E,2}^{(j)}\|$ and extend them continuously over the closure of the interval I. Here the norm $\|\cdot\|$ is the standard Euclidean norm in $\mathbf{R}^2$. Let $M_d(E)$ be the transfer matrix over the defect region (d_1, d_2) such that $\Phi(d_2; E) = M_d(E)\Phi(d_1; E)$. We see that the localized state is retained for the perturbed system if and only if:

$$M_d(E)\tilde{V}_{E,2}^{(1)} = c\tilde{V}_{E,1}^{(2)}, \tag{5.8}$$

holds for certain $E \in I$ and some nonzero real number c. A natural question is how large perturbation is allowed for the defect medium parameters so that the condition (5.8) holds and the interface mode persists for the operator $\tilde{\mathcal{L}}_d$.

Theorem 5.11. *Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ attain the same band gap $I := (E_{m_1}^{(1),+}, E_{m_1+1}^{(1),-}) = (E_{m_2}^{(2),+}, E_{m_2+1}^{(2),-})$ and the bulk topological indices $\gamma_{m_1}^{(1)}$ and $\gamma_{m_2}^{(2)}$ are different for the two operators. If:*

$$\max\left\{\|\mu\|_{L^\infty(d_1,d_2)}, E\|\varepsilon\|_{L^\infty(d_1,d_2)}\right\} \cdot (d_2 - d_1) < \frac{\pi}{2}, \tag{5.9}$$

holds for any $E \in I$, then the operator $\tilde{\mathcal{L}}_d$ attains an interface mode.

To prove the theorem, we express the solution vector Φ as:

$$\Phi(x; E) = \rho\,[\sin\theta, \cos\theta]^T = \rho\,[\cos\tilde{\theta}, \sin\tilde{\theta}]^T,$$

in which the polar angle $\tilde{\theta} := \frac{\pi}{2} - \theta$ represents the angle between the x-axis and the vector Φ on the plane. The radius ρ and the angle θ are called Prüfer radius and angle, respectively [9]. Both θ and $\tilde{\theta}$ are unique up to an additive constant integer multiple of 2π. By a direct calculation, ρ, θ and $\tilde{\theta}$ satisfy the following equations:

$$(\log\rho)' = \frac{1}{2}(\mu - E\varepsilon)\sin(2\theta), \tag{5.10}$$

$$\theta' = \mu\cos^2\theta + E\varepsilon\sin^2\theta, \tag{5.11}$$

$$\tilde{\theta}' = -\mu\sin^2\tilde{\theta} - E\varepsilon\cos^2\tilde{\theta}. \tag{5.12}$$

In what follows, we view ρ, θ and $\tilde{\theta}$ as functions of x and E.

Lemma 5.12. *Let $\theta(x_0; E) = \theta_0$, then for any fixed $E > 0$, the Prüfer angle $\theta(x; \cdot)$ is an increasing function and the polar angle $\tilde{\theta}(x; \cdot)$ is a decreasing function.*

This is obvious by noting that $\theta' \geqslant 0$ and $\tilde{\theta}' \leqslant 0$. Hence the solution vector Φ rotates clockwisely as x increases for fixed E.

Lemma 5.13. *Let $\theta_1(x; E_1)$ and $\theta_2(x; E_2)$ be the Prüfer angle of the solution vector with the energy $E_1 \leqslant E_2$ respectively. If $\theta_1(x_0; E_1) \leqslant \theta_2(x_0; E_2)$, then $\theta_1(x; E_1) \leqslant \theta_2(x; E_2)$ for all $x > x_0$.*

Lemma 5.14. *Let $\theta(x_0; E) = \theta_0$, then for any $x > x_0$, $\theta(x; E)$ is an increasing function of E.*

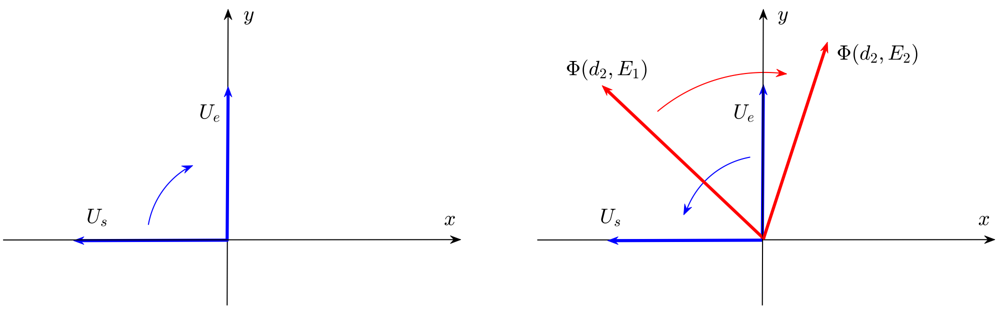

Figure 3. Left: Rotation of the vector $\tilde{V}_{E,2}^{(1)}$ from $U_s := (-1,0)^T$ to $U_e := (0,1)^T$ as E increases from E_1 to E_2; Right: Rotation of the vector $\tilde{V}_{E,1}^{(2)}$ from $U_e := (0,1)^T$ to $U_s := (-1,0)^T$ (blue) and rotation of the vector $\Phi(d_2;E)$ (red) as E increases from E_1 to E_2.

The proofs of lemmas 5.13 and 5.14 can be found in corollary 2.3.2 and theorem 2.3.3 of [9].

Proof of theorem 5.11. Let $I := (E_1, E_2)$ be the common band gap of the two operators $\mathcal{L}_1$ and $\mathcal{L}_2$. Without loss of generality, we assume that $\gamma_{m_1}^{(1)} = 1$ and $\gamma_{m_2}^{(2)} = -1$ so that the Bloch modes $\psi_{m,k}^{(1)}$ and $\psi_{m,k}^{(2)}$ at the band edge (k, E_m^+) for the operator $\mathcal{L}_1$ and $\mathcal{L}_2$ are even and odd respectively. In view of lemma 5.9 and observing that the impedance function $\xi_L^{(1)}$ is equal to the ratio between the first and the second component of $\tilde{V}_{E,2}^{(1)}$, as E increases from E_1 to E_2, either $\tilde{V}_{E,2}^{(1)}$ or $-\tilde{V}_{E,2}^{(1)}$ rotates from $U_s := (-1,0)^T$ to $U_e := (0,1)^T$ in the second quadrant (cf figure 3, left). On the other hand, either $\tilde{V}_{E,1}^{(2)}$ or $-\tilde{V}_{E,1}^{(2)}$ rotates from U_e to U_s in the second quadrant.

We only consider the case when $\tilde{V}_{E,2}^{(1)}$ rotates from U_s to U_e. The other scenarios can be proved in a similar fashion. If one sets $\Phi(d_1;E) = \tilde{V}_{E,2}^{(1)}$, then by lemmas 5.12 and 5.13, the vector $\Phi(d_2;E) := M_d(E)\tilde{V}_{E,2}^{(1)}$ rotates clockwisely as E increases from E_1 to E_2. The corresponding Prüfer angle $\theta(d_2;E)$ increases continuously. If (5.9) holds, by (5.11), $\theta' \leqslant \frac{\pi}{2(d_2-d_1)}$ for all $x \in (d_1, d_2)$ and $E \in I$. We obtain:

$$\Delta\theta := \theta(d_2;E) - \theta(d_1;E) < \frac{\pi}{2} \qquad \forall E \in I. \tag{5.13}$$

As such $\Phi(d_2;E_1)$ is located in the second quadrant while $\Phi(d_2;E_2)$ is located in the first quadrant. Therefore, the continuity of the Prüfer angle $\theta(d_2;E)$ implies that (5.8) holds for certain E in the band gap as $\tilde{V}_{E,1}^{(2)}$ or $-\tilde{V}_{E,1}^{(2)}$ rotates from U_e to U_s in the second quadrant with increasing E (cf figure 3, right).

□

If the condition (5.9) is violated, the stability question is more subtle. Here we provide an answer for a special scenario when the defect only consists of one layer.

Theorem 5.15. *Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ attain the same band gap and the bulk topological indices are different for the two operators. If $\varepsilon_d(x) \equiv \varepsilon_0$ and $\mu_d(x) \equiv \mu_0$ for certain constants ε_0 and μ_0, then the operator $\tilde{\mathcal{L}}_d$ attains a localized state for any $\varepsilon_0 \geqslant 1$, $\mu_0 \geqslant 1$, and $d := d_2 - d_1 \geqslant 0$.*

Proof. Similar to theorem 5.11, we assume that the two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ attain a common band gap $I := (E_1, E_2)$, and the toplogical indices for the two operators are 1 and -1. We denote the trajectory of the end point for the solution vector $\Phi(d_2; E)$ by γ as E increases from E_1 to E_2 in the band gap. Since $\tilde{V}^{(2)}_{E,1}$ or $-\tilde{V}^{(2)}_{E,1}$ rotates from $U_e := (0,1)^T$ to $U_s := (-1,0)^T$ in the second quadrant, while the vector $\Phi(d_2, E)$ rotates clockwisely as E increases, we deduce that (5.8) holds as long as γ crosses the x or y axis on the plane. Next we show that this is true for any $\varepsilon_0 \geqslant 1$, $\mu_0 \geqslant 1$, and d in the defect layer.

Note that either $\tilde{V}^{(1)}_{E,2}$ or $-\tilde{V}^{(1)}_{E,2}$ rotates from U_s to U_e as E increases from E_1 to E_2, for brevity we only consider the former. The transfer matrix M_d is explicitly given by:

$$M_d(E) = \begin{pmatrix} \cos(\omega n_d\, d) & \dfrac{\mu}{\omega n_d}\sin(\omega n_d\, d) \\ -\dfrac{\omega n_d}{\mu}\sin(\omega n_d\, d) & \cos(\omega n_d\, d) \end{pmatrix}, \tag{5.14}$$

in which $\omega = \sqrt{E}$ and $n_d = \sqrt{\varepsilon_d \mu_d}$. Let $\tilde{V}^{(1)}_{E,2} = (-v_1(E), v_2(E))$ with $v_1(E) \geqslant 0$ and $v_2(E) \geqslant 0$, and $\omega_d = \omega n_d\, d$, then:

$$\Phi(d_2; E) = v_1(E)\begin{pmatrix} -\cos(\omega_d) \\ \dfrac{\omega n_d}{\mu}\sin(\omega_d) \end{pmatrix} + v_2(E)\begin{pmatrix} \dfrac{\mu}{\omega n_d}\sin(\omega_d) \\ \cos(\omega_d) \end{pmatrix}. \tag{5.15}$$

In particular,

$$\Phi(d_2; E_1) = \left(-\cos(\omega_{d,1}), \frac{\omega_1 n_d}{\mu}\sin(\omega_{d,1})\right)^T, \quad \Phi(d_2; E_2) = \left(\frac{\mu}{\omega_2 n_d}\sin(\omega_{d,2}), \cos(\omega_{d,2})\right)^T,$$

with $\omega_j = \sqrt{E_j}$ and $\omega_{d,j} = \omega_j n_d\, d$ $(j = 1, 2)$.

Now assume that $\Phi(d_2; E_1)$ lies in the first quadrant with $\omega_{d,1} \in 2n_1\pi + \left[\frac{\pi}{2}, \pi\right]$ for certain integer $n_1 \geqslant 0$. We only need to consider the case when $\Phi(d_2; E_2)$ also lies in the first quadrant. We observe that $\omega_{d,2} \in 2n_2\pi + \left[0, \frac{\pi}{2}\right]$ for certain integer $n_2 > n_1$. Note that $\Phi(d_2; E)$ is located in the lower half plane when $\omega \in 2n_1\pi + \left(\pi, \frac{3\pi}{2}\right)$, and in the left half plane when $\omega \in 2n_1\pi + \left(\frac{3\pi}{2}, 2\pi\right)$. Thus the trajectory γ crosses both the x and (or) y axis. One can draw the same conclusion if $\Phi(d_2, E_1)$ lies in other quadrants, and the proof is complete. □

For a generic defect, the existence of interface modes for the perturbed topological structure is not guaranteed when the condition (5.9) is violated. Here we construct counter examples when the defect consists of two layers and the interface mode disappears. The permittivity and permeability values in the defect regions are given by:

$$\varepsilon_d(x) = \begin{cases} \varepsilon_{d,1}, & d_1 < x < d_*, \\ \varepsilon_{d,2}, & d_* < x < d_2 \end{cases} \quad \text{and} \quad \mu_d(x) = \begin{cases} \mu_{d,1}, & d_1 < x < d_*, \\ \mu_{d,2}, & d_* < x < d_2, \end{cases}$$

where the constants $\varepsilon_{d,j}$ and $\mu_{d,j}$ $(j = 1, 2)$ are to be specified in the following. Similar to the previous discussions, we assume that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ attain the same band gap $I := (E_1, E_2)$ and the bulk topological indices for the two operators are 1 and -1, respectively. Furthermore, as E increases in the band gap, the eigenvector $\tilde{V}^{(1)}_{E,2}$ rotates from $U_s := (-1,0)^T$ to $U_e := (0,1)^T$ in the second quadrant.

Let $\Phi_s(x; E)$ and $\Phi_e(x; E)$ be the solution vector for the equation $(\tilde{\mathcal{L}}_d - E)\psi = 0$ with $\Phi_s(d_1; E) = U_s$ and $\Phi_e(d_1; E) = U_e$, respectively. The corresponding polar angles $\tilde{\theta}_s(x; E)$ and $\tilde{\theta}_e(x; E)$ satisfy the equation (5.12). Define $\Delta\tilde{\theta} := \tilde{\theta}_s - \tilde{\theta}_e$, then $\Delta\tilde{\theta}$ solves the equation:

$$(\Delta\tilde{\theta})' = (E\varepsilon_d - \mu_d)\sin(\tilde{\theta}_s + \tilde{\theta}_e)\sin(\Delta\tilde{\theta}) \quad \text{in } (d_1, d_2). \tag{5.16}$$

First, we choose $\varepsilon_{d,1}$ and $\mu_{d,1}$ such that $E_1\varepsilon_{d,1}-\mu_{d,1}>0$. Note that $\tilde{\theta}_s(d_1;E_1)+\tilde{\theta}_e(d_1;E_1)=\frac{3\pi}{2}$. Since both $\tilde{\theta}_s$ and $\tilde{\theta}_e$ are decreasing functions of x, one can choose d_* such that:

$$\tilde{\theta}_s(d_*;E_1)+\tilde{\theta}_e(d_*;E_1)=\pi. \tag{5.17}$$

Noting that $\Delta\tilde{\theta}(d_1;E_1)=\frac{\pi}{2}$ and using (5.16), it follows that $(\Delta\tilde{\theta})'<0$ in (d_1,d_*) and consequently:

$$0<\Delta\tilde{\theta}(d_*,E_1)<\frac{\pi}{2}. \tag{5.18}$$

A combination of (5.17) and (5.18) yields:

$$\frac{\pi}{2}<\tilde{\theta}_s(d_*;E_1)<\pi \quad \text{and} \quad 0<\tilde{\theta}_e(d_*;E_1)<\frac{\pi}{2}. \tag{5.19}$$

Next we choose $\varepsilon_{d,2}$ and $\mu_{d,2}$ such that $E_1\varepsilon_{d,2}-\mu_{d,2}<0$. Furthermore, let d_2 be a real number such that:

$$\tilde{\theta}_s(d_2;E_1)=\frac{\pi}{2}. \tag{5.20}$$

We deduce from (5.18) that $0<\tilde{\theta}_e(d_2;E_1)<\frac{\pi}{2}$, since $\frac{\pi}{2}<\tilde{\theta}_s+\tilde{\theta}_e<\pi$ and $(\Delta\tilde{\theta})'<0$ in (d_*,d_2). If E_2-E_1 is sufficiently small, one can conclude that:

$$0<\tilde{\theta}_e(d_2;E_2)<\frac{\pi}{2}. \tag{5.21}$$

From (5.20) and (5.21), it is seen that the solution vector $\Phi(d_2;E):=M_d(E)\tilde{V}_{E,2}^{(1)}$ rotates in the first quadrant for E in the band gap. On the other hand, the eigenvector $\tilde{V}_{E,1}^{(2)}$ or $-\tilde{V}_{E,1}^{(2)}$ rotates from U_e to U_s in the second quadrant. Therefore, the condition (5.8) for the existence of interface modes does not hold for any $E\in I$.

6. Resonance of the finite topological structure

In this section, we consider the topological structure of finite size that is extended over the interval (N_1,N_2), where N_1 is a negative integer and N_2 is a positive integer. The structure is periodic on the left and right of the origin respectively. More precisely, the permittivity $\varepsilon_N(x)$ and permeability $\mu_N(x)$ of the finite structure takes the following form:

$$\varepsilon_N(x):=\begin{cases}\varepsilon_1(x), & N_1<x<0,\\ \varepsilon_2(x), & 0<x<N_2,\\ 1, & \text{elsewhere.}\end{cases} \quad \text{and} \quad \mu_N(x):=\begin{cases}\mu_1(x), & N_1<x<0,\\ \mu_2(x), & 0<x<N_2,\\ 1, & \text{elsewhere.}\end{cases}$$

in which ε_j and μ_j $(j=1,2)$ are piecewisely continuous periodic functions with period one. The corresponding differential operator is:

$$\tilde{\mathcal{L}}_N\psi:=-\frac{1}{\varepsilon_N(x)}\frac{d}{dx}\left(\frac{1}{\mu_N(x)}\frac{d\psi}{dx}\right).$$

When an incident wave $\psi^{inc}=e^{i\omega x}$ impinges from the left of the structure, where ω is the frequency, the structure gives rise to the transmitted field $\psi^{tran}=t(\omega)\,e^{i\omega x}$ and the reflected field $\psi^{ref}=r(\omega)\,e^{-i\omega x}$. The total field is $\psi=\psi^{inc}+\psi^{ref}$ for $x<N_1$ and $\psi=\psi^{tran}$ for $x>N_2$, and it satisfies:

$$(\tilde{\mathcal{L}}_N-\omega^2)\psi=0, \quad N_1<x<N_2. \tag{6.1}$$

The above scattering problem attains a unique solution for real frequency ω. If the resolvent associated with the scattering problem is extended to the whole complex plane by analytic continuation, it attains complex-valued poles that are called the resonances of the scattering problem, and the associated nontrivial solutions are called quasi-normal modes. Equivalently, the pole ω and the corresponding quasi-normal mode ψ solve the following homogeneous scattering problem when $\psi^{inc} = 0$:

$$(\tilde{\mathcal{L}}_N - \omega^2)\psi = 0, \quad N_1 < x < N_2, \tag{6.2}$$

$$\frac{1}{\mu_N(N_1)} \frac{d\psi(N_1)}{dx} + i\omega\psi(N_1) = 0, \tag{6.3}$$

$$\frac{1}{\mu_N(N_2)} \frac{d\psi(N_2)}{dx} - i\omega\psi(N_2) = 0. \tag{6.4}$$

The last two conditions are outgoing waves conditions imposed on the boundary of the structure. They are obtained by the continuity of the field across the boundary and the fact that the outgoing wave takes the form $\psi = c_- e^{-i\omega x}$ and $\psi = c_+ e^{i\omega x}$ for $x < N_1$ and $x > N_2$ respectively.

Lemma 6.1. *Let $\omega \in \mathbb{C}\backslash\{0\}$ be a resonance of (6.2)–(6.4), then ω attains negative imaginary part.*

Proof. Multiply the differential equation in (6.2) by $\varepsilon\bar{\psi}$ and integrate by part, it follows that:

$$\int_{N_1}^{N_2} \frac{1}{\mu_N} \left|\frac{d\psi}{dx}\right|^2 - \omega^2 \varepsilon_N |\psi|^2 \, dx + \frac{1}{\mu_N} \frac{d\psi(N_1)}{dx} \bar{\psi}(N_1) - \frac{1}{\mu_N} \frac{d\psi(N_2)}{dx} \bar{\psi}(N_2) = 0.$$

An application of the boundary conditions yields:

$$\int_{N_1}^{N_2} \frac{1}{\mu_N} \left|\frac{d\psi}{dx}\right|^2 - \omega^2 \varepsilon |\psi|^2 \, dx - i\omega |\psi(N_1)|^2 - i\omega |\psi(N_2)|^2 = 0. \tag{6.5}$$

Let $\omega = \omega_1 + i\omega_2$, where ω_1 and ω_2 are real numbers. First let us consider the case when the real part $\omega_1 \neq 0$. Note that the imaginary part of the left hand side of (6.5) is:

$$-\omega_1 \left(2\omega_2 \int_0^a |\psi|^2 \, dx + |\psi(N_1)|^2 + |\psi(N_2)|^2 \right).$$

If $\omega_2 \geqslant 0$, then $\psi(N_1) = \psi(N_2) = 0$. This implies that $\frac{d\psi(N_1)}{dx} = \frac{d\psi(N_2)}{dx} = 0$, and consequently $\psi \equiv 0$ in (N_1, N_2). Hence, we deduce that $\omega_2 < 0$. Now if the real part $\omega_1 = 0$, the left hand side of (6.5) is:

$$\int_{N_1}^{N_2} \frac{1}{\mu_N} \left|\frac{d\psi}{dx}\right|^2 + \omega_2^2 |\psi|^2 \, dx + \omega_2 |\psi(N_1)|^2 + \omega_2 |\psi(N_2)|^2 = 0.$$

If $\omega_2 > 0$, then a similar argument shows that $\psi \equiv 0$ in (N_1, N_2). The proof is complete. □

We denote the differential operator for the infinite structure (namely when $|N_1| = N_2 = \infty$) by:

$$\tilde{\mathcal{L}}_\infty \psi(x) = \begin{cases} \mathcal{L}_1 \psi(x), & x < 0, \\ \mathcal{L}_2 \psi(x), & x > 0, \end{cases}$$

where $\mathcal{L}_j$ $(j = 1, 2)$ is the differential operator with the physical parameters ε_j and μ_j. Assume that the structure attains an interface mode ψ_∞ with the energy E_∞. From the discussions in sections 4 and 5, E_∞ is located in a common spectral band gap of two operators. We would like to investigate resonances for the finite structure that are near the eigenvalue $\omega_\infty = \sqrt{E_\infty}$.

In the sequel, we set $E = \omega^2$ and let $M^{(j)}(E)$ be the transfer matrix associated with the equation $(\mathcal{L}_j - E)\psi = 0$. $\lambda_{E,1}^{(j)}$ and $\lambda_{E,2}^{(j)}$ are the eigenvalues of $M^{(j)}(E)$ defined by (2.11), with the corresponding eigenvectors $V_{E,1}^{(j)}$ and $V_{E,2}^{(j)}$ given in (3.1). Note that E_∞ is located in the common spectral band gap of $\mathcal{L}_1$ and $\mathcal{L}_2$, there holds $|\lambda_{E,1}^{(j)}| < 1 < |\lambda_{E,2}^{(j)}|$ for E in the neighborhood of E_∞. Without loss of generality, it is assumed that $\psi_{E,2}^{(j)}(1) \neq 0$ so that the two eigenvectors $V_{E,1}^{(j)}$ and $V_{E,2}^{(j)}$ defined above are linearly independent. We have the following lemma for the eigenvectors $V_{E,1}^{(j)}$ and $V_{E,2}^{(j)}$.

Lemma 6.2. *Let $E_0 = \omega_0^2$ for $\omega_0 \in \mathbb{R}^+$, and $\psi_{E_0,1}^{(j)}(1)$ and $\psi_{E_0,2}^{(j)}(1)$ are analytic at ω_0 over the complex plane. If $|\lambda_{E_0,1}^{(j)}| < 1 < |\lambda_{E_0,2}^{(j)}|$, then $V_{E_0,1}^{(j)}$ and $V_{E_0,2}^{(j)}$ are analytic at ω_0 over the complex plane. Furthermore, there holds $\det\left(V_{E_0,1}^{(j)}, \frac{dV_{E_0,1}^{(j)}}{d\omega}\right) > 0$ and $\det\left(V_{E_0,2}^{(j)}, \frac{dV_{E_0,2}^{(j)}}{d\omega}\right) < 0$.*

The proof follow the same lines as theorem 4.4 in [29], and we omit here for conciseness.

Theorem 6.3. *Let $N = \min\{|N_1|, N_2\}$. There exists an integer N_0 such that for any $N \geqslant N_0$, there is complex-valued resonance ω of (6.2)–(6.4) in the neighborhood of ω_∞. Furthermore, there holds:*

$$|\,\omega - \omega_\infty| \leqslant Ce^{-\alpha(\omega)N},$$

in which C is a positive constant independent of N and $\alpha(\omega) > 0$ is a function defined in the neighborhood of ω.

Proof. Let $\Phi(x) = \left(\psi(x), \frac{1}{\mu(x)}\psi'(x)\right)^T$ the solution vector, where ψ is the solution of (6.2)–(6.4) with the complex-valued frequency ω. Note that $|\lambda_{E,1}^{(j)}| < 1 < |\lambda_{E,2}^{(j)}|$ for ω in the neighborhood of ω_∞, in which $E = \omega^2$, one can expand $\Phi(N_1)$ as:

$$\Phi(N_1) = c_1(\omega)V_{E,1}^{(1)} + c_2(\omega)V_{E,2}^{(1)},$$

where the coefficients $c_1(\omega)$ and $c_2(\omega)$ are:

$$c_1(\omega) = \frac{\det(\Phi(N_1), V_{E,2}^{(1)})}{\det(V_{E,1}^{(1)}, V_{E,2}^{(1)})} \quad \text{and} \quad c_2(\omega) = \frac{\det(V_{E,1}^{(1)}, \Phi(N_1))}{\det(V_{E,1}^{(1)}, V_{E,2}^{(1)})}. \tag{6.6}$$

The field at $x = 0$ can be expressed as $\Phi(0) = c_1(\omega)\left(\lambda_{E,1}^{(1)}\right)^{|N_1|} V_{E,1}^{(1)} + c_2(\omega)\left(\lambda_{E,2}^{(1)}\right)^{|N_1|} V_{E,2}^{(1)}$. By decomposing $V_{E,1}^{(1)}$ and $V_{E,2}^{(1)}$ as:

$$V_{E,1}^{(1)} = c_{11}(\omega)V_{E,1}^{(2)} + c_{12}(\omega)V_{E,2}^{(2)}, \quad V_{E,2}^{(1)} = c_{21}(\omega)V_{E,1}^{(2)} + c_{22}(\omega)V_{E,2}^{(2)},$$

where

$$c_{11}(\omega) = \frac{\det\left(V_{E,1}^{(1)}, V_{E,2}^{(2)}\right)}{\det\left(V_{E,1}^{(2)}, V_{E,2}^{(2)}\right)}, \quad c_{12}(\omega) = \frac{\det\left(V_{E,1}^{(2)}, V_{E,1}^{(1)}\right)}{\det\left(V_{E,1}^{(2)}, V_{E,2}^{(2)}\right)}, \tag{6.7}$$

$$c_{21}(\omega) = \frac{\det\left(V_{E,2}^{(1)}, V_{E,2}^{(2)}\right)}{\det\left(V_{E,1}^{(2)}, V_{E,2}^{(2)}\right)}, \quad c_{22}(\omega) = \frac{\det\left(V_{E,1}^{(2)}, V_{E,2}^{(1)}\right)}{\det\left(V_{E,1}^{(2)}, V_{E,2}^{(2)}\right)}, \tag{6.8}$$

it follows that:

$$\Phi(0)=\left(c_1(\omega)c_{11}(\omega)\left(\lambda_{E,1}^{(1)}\right)^{|N_1|}+c_2(\omega)c_{21}(\omega)\left(\lambda_{E,2}^{(1)}\right)^{|N_1|}\right)V_{E,1}^{(2)}$$
$$+\left(c_1(\omega)c_{12}(\omega)\left(\lambda_{E,1}^{(1)}\right)^{|N_1|}+c_2(\omega)c_{22}(\omega)\left(\lambda_{E,2}^{(1)}\right)^{|N_1|}\right)V_{E,2}^{(2)}.$$

We deduce that the filed at $x=N_2$ is:

$$\begin{aligned}\Phi(N_2)&=\left(c_1(\omega)c_{11}(\omega)\left(\lambda_{E,1}^{(1)}\right)^{|N_1|}+c_2(\omega)c_{21}(\omega)\left(\lambda_{E,2}^{(1)}\right)^{|N_1|}\right)\left(\lambda_{E,1}^{(2)}\right)^{|N_2|}V_{E,1}^{(2)}\\&\quad+\left(c_1(\omega)c_{12}(\omega)\left(\lambda_{E,1}^{(1)}\right)^{|N_1|}+c_2(\omega)c_{22}(\omega)\left(\lambda_{E,2}^{(1)}\right)^{|N_1|}\right)\left(\lambda_{E,2}^{(2)}\right)^{|N_2|}V_{E,2}^{(2)}\\&=\begin{pmatrix}\psi(N_2)\\ i\omega\psi(N_2)\end{pmatrix}.\end{aligned}$$

This leads to the equation:

$$\begin{aligned}&\left(c_1(\omega)c_{11}(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2}+c_2(\omega)c_{21}(\omega)\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2}\right)\\&\cdot\left(\lambda_{E,1}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right)+\left(c_1(\omega)c_{12}(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}+c_2(\omega)c_{22}(\omega)\right)\\&\cdot\left(\lambda_{E,2}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right)=0,\end{aligned}\tag{6.9}$$

which is the equation of resonance.

Let $c(\omega)=\det\left(V_{E,1}^{(1)},V_{E,2}^{(1)}\right)\cdot\det\left(V_{E,1}^{(2)},V_{E,2}^{(2)}\right)$, and define the following complex-valued functions:

$$\begin{aligned}F(\omega)&=c(\omega)c_2(\omega)c_{22}(\omega)\cdot\left(\lambda_{E,2}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right),\\G_1(\omega)&=c(\omega)c_1(\omega)c_{11}(\omega)\cdot\left(\lambda_{E,1}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right),\\G_2(\omega)&=c(\omega)c_2(\omega)c_{21}(\omega)\cdot\left(\lambda_{E,1}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right),\\G_3(\omega)&=c(\omega)c_1(\omega)c_{12}(\omega)\cdot\left(\lambda_{E,2}^{(2)}-\psi_{E,1}^{(2)}(1)-i\omega\psi_{E,2}^{(2)}(1)\right).\end{aligned}$$

The nonlinear equation (6.9) can be written as:

$$G_1(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2}+G_2(\omega)\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2}+G_3(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}+F(\omega)=0.\tag{6.10}$$

Since the infinite structure attains an interface mode ψ_∞ with the energy $E_\infty=\omega_\infty^2$, we have $F(\omega_\infty)=c_{22}(\omega)=0$.

It can be shown that $F(\omega)$ and $G(\omega)$ are analytic in the neighborhood of the frequency ω_∞ over the complex plane. By Taylor's theorem [2], there exists an analytic function $\tilde{F}(\omega)$ such that:

$$F(\omega)=\tilde{F}(\omega)(\omega-\omega_\infty),\quad\text{where}\quad\tilde{F}(\omega_\infty)=F'(\omega_\infty).\tag{6.11}$$

Table 1. Resonances for finite structures with different number of period.

N	2	4	8	16
$\mathrm{Re}(\omega-\omega_\infty)$	-0.0077	-0.0035	-0.0008	4.30×10^{-5}
$\mathrm{Im}(\omega-\omega_\infty)$	-0.2117	-0.0625	-0.0090	-2.80×10^{-4}
$\lvert\omega-\omega_\infty\rvert$	0.2118	0.0626	0.0090	2.84×10^{-4}

Substituting into (6.10) yields:

$$\begin{aligned}\tilde{F}(\omega)(\omega-\omega_\infty) = -G_1(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2} - G_2(\omega)\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2} \\ - G_3(\omega)\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|}.\end{aligned}\tag{6.12}$$

Now a direct calculation leads to:

$$F'(\omega_\infty)=\det\left(V_{E_\infty,1}^{(1)},\Phi(N_1)\right)\cdot\frac{d}{d\omega}\left(\det\left(V_{E_\infty,1}^{(2)},V_{E_\infty,2}^{(1)}\right)\right)\cdot\left(\lambda_{E_\infty,2}^{(2)}-\psi_{E_\infty,1}^{(2)}(1)-i\omega\psi_{E_\infty,2}^{(2)}(1)\right).$$

Noting that $V_{E_\infty,1}^{(2)}=sV_{E_\infty,2}^{(1)}$ for some nonzero constant s, we obtain:

$$\begin{aligned}\frac{d}{d\omega}\left(\det\left(V_{E_\infty,1}^{(2)},V_{E_\infty,2}^{(1)}\right)\right) &= \det\left(\frac{dV_{E_\infty,1}^{(2)}}{d\omega},V_{E_\infty,2}^{(1)}\right)+\det\left(V_{E_\infty,1}^{(2)},\frac{dV_{E_\infty,2}^{(1)}}{d\omega}\right)\\ &= \frac{1}{s}\det\left(\frac{dV_{E_\infty,1}^{(2)}}{d\omega},V_{E_\infty,1}^{(2)}\right)+s\det\left(V_{E_\infty,2}^{(1)},\frac{dV_{E_\infty,2}^{(1)}}{d\omega}\right).\end{aligned}$$

From lemma 6.2 we deduce that there exists a constant $\gamma>0$ such that $\left|\tilde{F}(\omega)\right|\geqslant\gamma$ in the neighborhood of ω_∞.

Consequently, we obtain,

$$|\omega-\omega_\infty|\lesssim\max\left\{\left(\frac{\lambda_{E,1}^{(1)}}{\lambda_{E,2}^{(1)}}\right)^{|N_1|},\left(\frac{\lambda_{E,1}^{(2)}}{\lambda_{E,2}^{(2)}}\right)^{N_2}\right\}\lesssim e^{-\alpha(\omega)N},\quad N=\min\{|N_1|,N_2\},$$

where the last inequality above follows from the fact that $\frac{|\lambda_{E,1}^{(j)}|}{|\lambda_{E,2}^{(j)}|}<1$ for E in the neighborhood of E_∞.

If one rewrites the condition (6.12) as $\omega=T(\omega)$, then using the inequality $\frac{|\lambda_{E,1}^{(j)}|}{|\lambda_{E,2}^{(j)}|}<1$ again, it can be shown that T is a contraction map in the neighborhood of ω_∞. Hence the existence of the resonance follows. □

We illustrate the exponential decay of the distance $|\omega-\omega_\infty|$ by considering the layered period structure in section 5.5. Table 1 shows the value $\omega-\omega_\infty$ and $|\omega-\omega_\infty|$ when $|N_1|=N_2=2,4,8,16$. The distance $|\omega-\omega_\infty|$ decays exponentially with respect to the number of period N as illustrated in figure 4. Now considering the scattering problem (6.1) with the incident wave

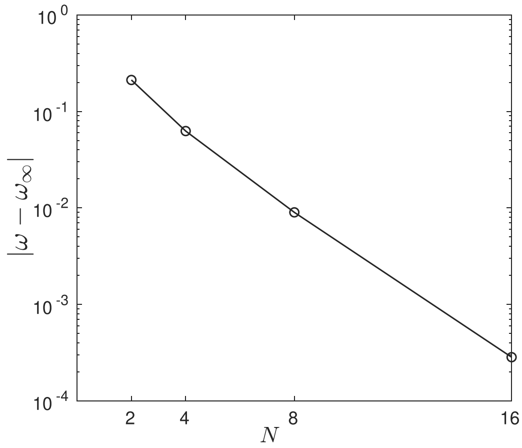

Figure 4. The distance $|\omega - \omega_\infty|$ for $N = 2, 4, 8, 16$.

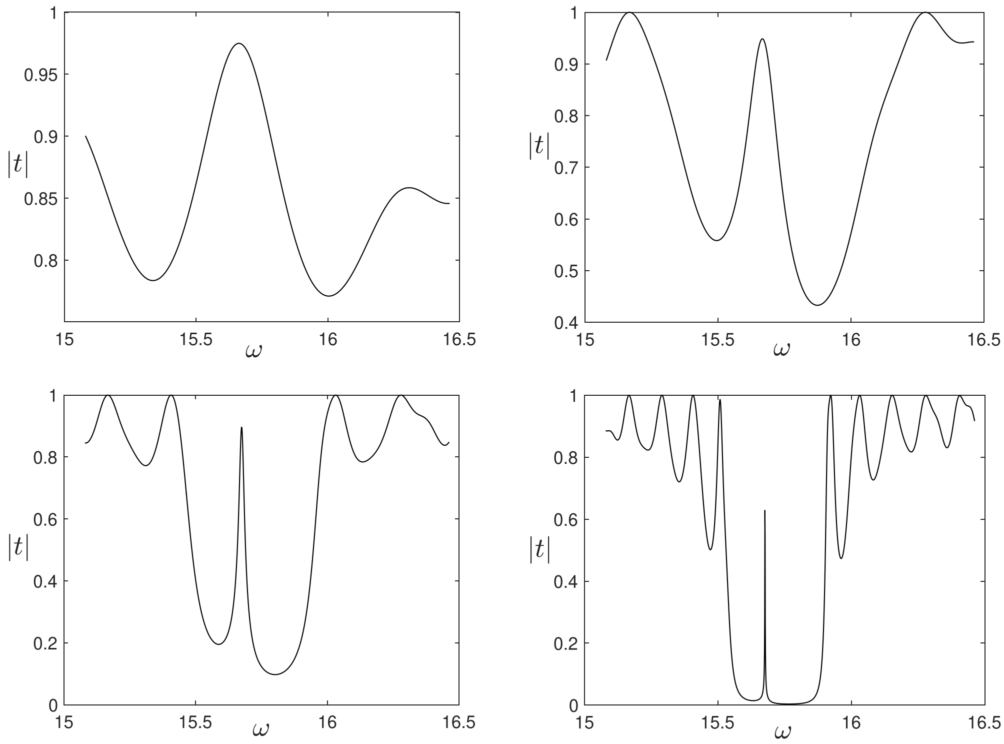

Figure 5. Transmission value $|t|$ near the resonant frequency for $|N_1| = N_2 = 2, 4, 8, 16$. The infinite structure attains an interface mode at the frequency $\omega_\infty = 15.6765$.

$\psi^{inc} = e^{i\omega x}$. The transmission $|t|$ exhibits peaks at resonant frequencies. As shown in figure 5, when N increases, the resonant peaks become sharper as the imaginary part of the resonance decreases.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix

Proof of lemma 4.1

The partial derivatives of the solution matrix $\Psi_{E,\delta}(x)$ with respect to the two parameters E and δ have the following expressions (see for instance chapter one in [9]),

$$\frac{\partial \Psi_{E,\delta}(x)}{\partial E} = \Psi_{E,\delta}(x)\int_0^x \Psi_{E,\delta}^{-1}(t) J W(t) \Psi_{E,\delta}(t) dt, \tag{A.1}$$

$$\frac{\partial \Psi_{E,\delta}(x)}{\partial \delta} = \Psi_{E,\delta}(x)\int_0^x \Psi_{E,\delta}^{-1}(t) J \tilde{F}(t) \Psi_{E,\delta}(t) dt, \tag{A.2}$$

and

$$\frac{\partial^2 \Psi_{E,\delta}(x)}{\partial E^2} = 2\Psi_{E,\delta}(x)\int_0^x \Psi_{E,\delta}^{-1}(t) J W(t) \frac{\partial \Psi_{E,\delta}(t)}{\partial E} dt, \tag{A.3}$$

$$\frac{\partial^2 \Psi_{E,\delta}(x)}{\partial \delta^2} = 2\Psi_{E,\delta}(x)\int_0^x \Psi_{E,\delta}^{-1}(t) J \tilde{F}(t) \frac{\partial \Psi_{E,\delta}(t)}{\partial \delta} dt, \tag{A.4}$$

$$\frac{\partial^2 \Psi_{E,\delta}(x)}{\partial E \partial \delta} = \Psi_{E,\delta}(x)\int_0^x \Psi_{E,\delta}^{-1}(t) J \left(W(t) \frac{\partial \Psi_{E,\delta}(t)}{\partial \delta} + \tilde{F}(t) \frac{\partial \Psi_{E,\delta}(t)}{\partial E} \right) dt. \tag{A.5}$$

We write

$$\begin{cases} Q_1(x) = \Psi_{E^*,0}^{-1}(x) J W(x) \Psi_{E^*,0}(x), \\ Q_2(x) = \Psi_{E^*,0}^{-1}(x) J \tilde{F}(x) \Psi_{E^*,0}(x), \end{cases}$$

Using the fact that $det\,\Psi_{E^*,0}(x) = 1$, we have:

$$\Psi_{E^*,0}^{-1}(x) = \begin{pmatrix} \frac{1}{\mu(x)}\psi'_{E^*,2,0}(x) & -\psi_{E^*,2,0}(x) \\ -\frac{1}{\mu(x)}\psi'_{E^*,1,0}(x) & \psi_{E^*,1,0}(x) \end{pmatrix}.$$

It follows from a direct calculation that:

$$Q_1 = \begin{pmatrix} -V^T W U & -V^T W V \\ U^T W U & U^T W V \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -V^T \tilde{F} U & -V^T \tilde{F} V \\ U^T \tilde{F} U & U^T \tilde{F} V \end{pmatrix}. \tag{A.6}$$

Now, let $x = 1$ in (A.1). By noting that $\Psi_{E^*,0}(1) = Id$, we have:

$$\frac{\partial M}{\partial E}(E^*, 0) = \int_0^1 \Psi_{E^*,0}^{-1}(t) J W(t) \Psi_{E^*,0}(t) dt.$$

Taking the trace and using the fact that $TrAB = TrBA$, and that $TrJW(t) = 0$, we obtain:

$$\frac{\partial D}{\partial E}(E^*, 0) = \int_0^1 TrJW(t)dt = 0.$$

$\frac{\partial D}{\partial \delta}(E^*, 0) = 0$ follows similarly by using (A.2).

We next show that:

$$\frac{\partial^2 D}{\partial E^2}(E^*, 0) = Tr\left[\int_0^1 Q_1(x)dx\right]^2, \tag{A.7}$$

$$\frac{\partial^2 D}{\partial \delta^2}(E^*, 0) = Tr\left[\int_0^1 Q_2(x)dx\right]^2, \tag{A.8}$$

$$\frac{\partial^2 D}{\partial E \partial \delta}(E^*, 0) = Tr\left[\int_0^1 Q_1(x)dx\right]\left[\int_0^1 Q_2(x)dx\right]. \tag{A.9}$$

In light of (A.3), we have:

$$\begin{aligned}
\frac{\partial^2 M}{\partial E^2}(E^*, 0) &= 2\int_0^1 \Psi_{E,0}^{-1}(x)JW(x)\frac{\partial \Psi_{E,0}(x)}{\partial E}dx \\
&= 2\int_0^1 \Psi_{E^*,0}^{-1}(x)JW(x)\Psi_{E^*,0}(x)\int_0^x \Psi_{E^*,0}^{-1}(t)JW(t)\Psi_{E^*,0}(t)dtdx \\
&= 2\int_0^1\int_0^x Q_1(x)Q_1(t)dtdx.
\end{aligned}$$

Taking the trace, we get:

$$\begin{aligned}
\frac{\partial^2 D}{\partial E^2}(E^*, 0) &= 2\int_0^1\int_0^x TrQ_1(x)Q_1(t)dtdx \\
&= \int_0^1\int_0^x TrQ_1(x)Q_1(t)dtdx + \int_0^1\int_0^t TrQ_1(t)Q_1(x)dxdt \\
&= \int_0^1\int_0^x TrQ_1(x)Q_1(t)dtdx + \int_0^1\int_x^1 TrQ_1(x)Q_1(t)dtdx \\
&= \int_0^1\int_0^1 TrQ_1(x)Q_1(t)dtdx \\
&= Tr\int_0^1\int_0^1 Q_1(x)Q_1(t)dtdx = Tr(\int_0^1 Q_1(x)dx)^2,
\end{aligned}$$

which proves (A.7). The equality (A.8) and (A.9) can be proved in a similar manner. Finally, the desired results follow from (A.6) and a direct calculation. □

Proof of proposition 4.4

Since $\tilde{\mu} \geqslant 0, \tilde{\varepsilon} \geqslant 0$, it follows that:

$$\int_0^1 V^T(x)\tilde{F}(x)V(x)dx > 0, \quad \int_0^1 U^T(x)\tilde{F}(x)U(x)dx > 0,$$

and we can derive the following Cauchy-Schwarz type inequality:

$$\left|\int_0^1 U^T(x)\tilde{F}(x)V(x)dx\right|^2 \leqslant \left|\int_0^1 V^T(x)\tilde{F}(x)V(x)dx \cdot \int_0^1 U^T(x)\tilde{F}(x)U(x)dx\right|.$$

Let

$$|U|_1 = \left(\int_0^1 U^T W U dx\right)^{1/2}, \quad |V|_1 = \left(\int_0^1 V^T W V dx\right)^{1/2},$$
$$|U|_2 = \left(\int_0^1 U^T \tilde{F} U dx\right)^{1/2}, \quad |V|_2 = \left(\int_0^1 V^T \tilde{F} V dx\right)^{1/2},$$

and define,

$$t_1 = \frac{\int_0^1 U^T W V dx}{(\int_0^1 V^T W V dx)^{1/2} \cdot (\int_0^1 U^T W U dx)^{1/2}} = \frac{\int_0^1 U^T W V dx}{|V|_1 \cdot |U|_1}$$
$$t_2 = \frac{\int_0^1 U^T \tilde{F} V dx}{(\int_0^1 V^T \tilde{F} V)^{1/2} \cdot (\int_0^1 U^T \tilde{F} U dx)^{1/2}} = \frac{\int_0^1 U^T \tilde{F} V dx}{|V|_2 \cdot |U|_2}.$$

Then it is clear that $|t_1| \leqslant 1, |t_2| \leqslant 1$. We obtain,

$$\begin{aligned}\frac{1}{2}a_2 = \frac{1}{2}\frac{\partial^2 D}{\partial E \partial \delta}(E^*,0) &= t_1 t_2 \cdot |U|_1 \cdot |V|_1 \cdot |U|_2 \cdot |V|_2 - \frac{1}{2}|U|_2^2 \cdot |V|_1^2 - \frac{1}{2}|U|_1^2 \cdot |V|_2^2 \\ &= |U|_1 \cdot |V|_1 \cdot |U|_2 \cdot |V|_2 \cdot \left(t_1 t_2 - \frac{|U|_2 \cdot |V|_1}{2|U|_1 \cdot |V|_2} - \frac{|U|_1 \cdot |V|_2}{2|U|_2 \cdot |V|_1}\right).\end{aligned}$$

On the other hand:

$$\frac{1}{2}a_1 = \frac{1}{2}\frac{\partial^2 D}{\partial E^2}(E^*,0) = (t_1^2 - 1) \cdot |U|_1^2 \cdot |V|_1^2,$$
$$\frac{1}{2}a_3 = \frac{1}{2}\frac{\partial^2 D}{\partial \delta^2}(E^*,0) = (t_2^2 - 1) \cdot |U|_2^2 \cdot |V|_2^2.$$

Using the inequality:

$$(1 - t_1 t_2)^2 \geqslant (1 - t_1^2)(1 - t_2^2),$$

and

$$|t_1 t_2 - \frac{|U|_2 \cdot |V|_1}{2|U|_1 \cdot |V|_2} - \frac{|U|_1 \cdot |V|_2}{2|U|_2 \cdot |V|_1}| \geqslant 1 - t_1 t_2,$$

we can conclude that $a_2^2 - a_1 a_3 \geqslant 0$. □

Proof of lemma 4.8

For ease of presentation, we set $\tilde{\mu} = 0$. Then,

$$\int_0^1 U^T \tilde{F} U dx = \int_0^1 \psi_{E^*,1,0}^2(x) E\tilde{\varepsilon}(x) dx; \quad \int_0^1 U^T \tilde{F} V dx = \int_0^1 \psi_{E^*,1}(x)\psi_{E^*,2,0}(x) E\tilde{\varepsilon}(x) dx.$$

Under the assumption that μ and ε are even functions, we can show that $\psi_{E^*,1,0}(x)$ is even and $\psi_{E^*,2,0}(x)$ is odd. In addition, since $\Psi_{E^*,0}(1) = Id$, we have $\Psi_{E^*,0}(x+1) = \Psi_{E^*,0}(x)$.

Therefore both functions $\psi_{E^*,1,0}(x)$ and $\psi_{E^*,2,0}(x)$ are periodic with period one. Hence we can derive that:

$$\int_0^1 U^T\tilde{F}Udx = \int_{-1/2}^{1/2} \psi^2_{E^*,1,0}(x)E\tilde{\varepsilon}(x)dx = 0,$$

and that

$$\int_0^1 U^T\tilde{F}Vdx = \int_{-1/2}^{1/2} \psi_{E^*,1,0}(x)\psi_{E^*,2,0}(x)E\tilde{\varepsilon}(x)dx = 2E\int_0^{1/2} \psi_{E^*,1,0}(x)\psi_{E^*,2,0}(x)\tilde{\varepsilon}(x)dx.$$

It is clear that we can choose $\tilde{\varepsilon}(x)$ to make $\int_0^1 U^T\tilde{F}Vdx \neq 0$. This completes the proof of the lemma. □

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