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Convergence to a diffusive contact wave for solutions to a system of hyperbolic balance laws

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Abstract. We consider a 2 × 2 system of hyperbolic balance laws that is the converted form under inverse Hopf–Cole transformation of a Keller–Segel type chemotaxis model. We study Cauchy problem when Cauchy data connect two different end-states as $x \to \pm\infty$. The background wave is a diffusive contact wave of the reduced system.

We establish global existence of solution and study the time asymptotic behavior. In the special case where the cellular population initially approaches its stable equilibrium value as $x \to \pm\infty$, we obtain nonlinear stability of the diffusive contact wave under smallness assumption. In the general case where the population initially does not approach to its stable equilibrium value at least at one of the far fields, we use a correction function in the time asymptotic ansatz, and show that the population approaches logistically to its stable equilibrium value. Our result shows two significant differences when comparing to Euler equations with damping. The first one is the existence of a secondary wave in the time asymptotic ansatz. This implies that our solutions converge to the diffusive contact wave slower than those of Euler equations with damping. The second one is that the correction function logistically grows rather than exponentially decays.

Keywords: Nonlinear stability; asymptotic behavior; logarithmic chemotactic singularity.

1. Introduction

We consider the Cauchy problem of a two by two hyperbolic system,

$$\begin{cases} v_t + u_x = 0, \\ u_t + (uv)_x = ru(1-u), \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

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$$(v,u)(x,0) = (v_0,u_0)(x), \tag{1.2}$$

where the parameter $r > 0$ is a constant, and the initial data satisfy

$$\lim_{x\to\pm\infty}(v_0,u_0)(x) = (v_\pm,u_\pm) \tag{1.3}$$

with $v_- \neq v_+$ and $u_\pm > 0$. The goal is to establish the existence of global-intime solution and to study the time asymptotic behavior of the solution under appropriate assumptions.

1.1. Background

Consider a Keller–Segel type chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate for a non-diffusive chemical signal and a non-diffusive cellular population,

$$\begin{cases} s_t = -\mu u s - \sigma s, \\ u_t = -\chi[u(\ln s)_x]_x + au\left(1 - \dfrac{u}{K}\right), \end{cases} \qquad x \in \mathbb{R}, \quad t > 0 . \tag{1.4}$$

Here the unknown functions are $s = s(x,t)$ and $u = u(x,t)$ for the concentration of the chemical signal and the density of the cellular population, respectively. The system parameters are constants and have the following meaning: $\mu \neq 0$ is the coefficient of density-dependent production/consumption rate of the chemical signal; $\sigma \geq 0$ the natural degradation rate of the signal; $\chi \neq 0$ the coefficient of chemotactic sensitivity; $a > 0$ the natural growth rate of the cellular population and $K > 0$ the typical carrying capacity of the environment for the population. For a more detailed discussion on the model see [23–25] and references therein.

The logarithmic function in (1.4) can be removed via the inverse Hopf–Cole transformation [5]:

$$v = (\ln s)_x = \frac{s_x}{s}. \tag{1.5}$$

Under the new variables v and u, the reaction-diffusion system (1.4) is converted into a system of balance laws,

$$\begin{cases} v_t + \mu u_x = 0, \\ u_t + \chi(uv)_x = au\left(1 - \dfrac{u}{K}\right) \end{cases} . \tag{1.6}$$

We assume $\chi\mu > 0$, which implies $\chi,\mu > 0$, or $\chi,\mu < 0$. The former is interpreted as cells are attracted to and consume the chemical, while the latter indicates that cells deposit the signal to modify the local environment for succeeding passages [15]. Mathematically, (1.6) is hyperbolic in biologically relevant regimes when $\chi\mu > 0$, while it may change type when $\chi\mu < 0$ [23].

Under the assumption $\chi\mu > 0$, we introduce rescaled and dimensionless variables,

$$\check{t} = \chi\mu K t, \quad \check{x} = \sqrt{\chi\mu K}x, \quad \check{v} = \mathrm{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}\,v, \quad \check{u} = \frac{u}{K}. \tag{1.7}$$

Dropping the check accent after the change of variables, we obtain (1.1) with

$$r = \frac{aD}{\chi\mu K} > 0. \qquad (1.8)$$

Therefore, (1.1) is the converted form of the chemotaxis model (1.4) under the transformation (1.5) followed by the rescaling (1.7).

Correspondingly, (1.4) is considered with Cauchy data

$$(s,u)(x,0) = (s_0,u_0)(x), \qquad x \in \mathbb{R}. \qquad (1.9)$$

Here u_0 in (1.9) is the unscaled version of u_0 in (1.2). Therefore, based on the physical relevance, we set $u_\pm > 0$ in (1.3). The other initial function s_0 is related to v_0 in (1.2) by the transformation (1.5) and the rescaling (1.7). Assuming $\chi > 0$, for simplicity and without loss of generality, we bypass the scaling to discuss the connection between v_0 and s_0. That is,

$$v_0(x) = v(x,0) = (\ln s_0)'(x) = \frac{s_0'(x)}{s_0(x)},$$

which implies

$$s_0(x) = s_0(0)e^{\int_0^x v_0(y)dy}, \quad s_0(0) > 0. \qquad (1.10)$$

Note that the Cauchy data (1.2) satisfy (1.3). This includes (but is not limited to) the following special cases.

(1) $0 < v_- < \infty$ and $\int_0^\infty |v_0(y)|\, dy < \infty$. In this case, (1.3) and (1.10) imply

$$\lim_{x\to\infty} s_0(x) = 0, \qquad \lim_{x\to-\infty} s_0(x) = s_+ < \infty.$$

(2) $-\infty < v_+ < 0$ and $\int_{-\infty}^0 |v_0(y)|\, dy < \infty$. Similarly, (1.3) and (1.10) imply

$$\lim_{x\to-\infty} s_0(x) = s_- < \infty, \qquad \lim_{x\to\infty} s_0(x) = 0.$$

(3) $0 < v_- < \infty$ and $-\infty < v_+ < 0$. In this case,

$$\lim_{x\to\pm\infty} s_0(x) = 0.$$

We observe that in those special cases, s_0 is not bounded away from zero while $v_- > v_+$. Similarly, if we assume $\chi < 0$, there are also special cases where s_0 is not bounded away from zero while $v_- < v_+$. In other words, the singularity of the logarithmic function in (1.4) is intrinsic, which is further translated into technical difficulties associated with $v_- \neq v_+$ in (1.1)–(1.3).

1.2. Connection with existing literature

There is an abundant literature on chemotaxis models similar to (1.4). For a short discussion and a list of related references see [23, 24]. In fact, the model studied in [23, 24] differs from (1.4) in that it is for a diffusive cellular population. Since the focus of this paper is on the mathematical theories for the hyperbolic system (1.1) (assuming $u_\pm > 0$ in (1.3)), especially those related to nonlinear stability, we give a brief discussion in that regard.

A general system of hyperbolic balance laws takes the form

$$w_t + f(w)_x = g(w), \tag{1.11}$$

where $w,f,g \in \mathbb{R}^n$. Here w is the unknown density function, f the flux function and g the reaction term. In physical applications, the Jacobian matrix f has real, distinct eigenvalues or can be symmetrized with an entropy function, and the Jacobian matrix g is rank deficient.

We observe that (1.1) is an example of (1.11), where

$$w = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(w) = \begin{pmatrix} u \\ uv \end{pmatrix}, \quad g(w) = \begin{pmatrix} 0 \\ ru(1-u) \end{pmatrix}. \tag{1.12}$$

It is clear that

$$f'(w) = \begin{pmatrix} 0 & 1 \\ u & v \end{pmatrix}$$

has two real, distinct eigenvalues $\lambda_\pm = \frac{1}{2}(v \pm \sqrt{v^2 + 4u})$ in the biologically relevant regime $u > 0$. It is also clear that g is rank deficient.

There is an extensive literature on the general system (1.11) when Cauchy data are around constant equilibrium states, see [19] and references therein. In particular, pointwise time asymptotic behavior of solutions is studied in [22]. The results apply to (1.1), (1.2) with

$$\lim_{x\to\pm\infty} (v_0,u_0)(x) = (0,1),$$

e.g. see [20, 17]. Here the constant equilibrium state (0,1) is the stable one in the physically interesting scenario. That is, it corresponds to

$$\lim_{x\to\pm\infty} s_0(x) = s_\pm, \qquad 0 < s_\pm < \infty, \tag{1.13}$$

see [23] for details. In view of Sec. 1.1, $s_0(x)$ and hence $s(x,t)$ in the original model is bounded away from zero, or the logarithmic singularity in (1.4) does not play an intrinsic role in such a scenario. This paper is to advance the research to the much more complicated situation when $v_- \neq v_+$ for arbitrary $v_\pm$ and arbitrary $u_\pm > 0$ in (1.3), provided $|v_- - v_+| + |u_- - u_+|$ is small.

In the case of Cauchy data connecting two different end-states as $x \to \pm\infty$, a system in the form (1.11) is usually studied around a permanent wave, which is a diffusive version of an elementary wave of the corresponding equilibrium system/equation. Therefore, the type of the permanent wave depends on the system under consideration. For (1.1), the equilibrium equation is a linear equation as to be seen below. The only relevant elementary waves are contact discontinuities. Thus, we consider the Cauchy problem (1.1)–(1.3) around a diffusive contact wave in this paper. For a 2×2 hyperbolic system that admits diffusive rarefaction waves and traveling waves (shock waves) readers are referred to [8].

In the existing literature several models in the form (1.11) have been studied around diffusive contact waves, also known as diffusion waves. Among them are Euler equations with linear or nonlinear damping, bipolar hydrodynamical models for semiconductors, etc. Comparing to those models, (1.1)–(1.3) exhibits significant new features that may shed light on the future study of the general system (1.11) around a diffusive contact wave. Since Euler equations with damping are perhaps most extensively studied, we give a more detailed discussion and make a comparison with (1.1)–(1.3) below.

1.3. Comparison to Euler equations with damping

We consider Euler equations with damping for isentropic flows,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -ru, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \tag{1.14}$$

where $r > 0$ is a constant. The unknown functions v and u stand for the specific volume and velocity, respectively, while p is the pressure, a given smooth function of v satisfying $p'(v) < 0$. The model describes a compressible flow through a porous medium.

We observe two differences between (1.1) and (1.14). The first is that the nonlinear flux function in $(1.14)_2$ is $p(v)$, depending only on v, while the counterpart in $(1.1)_2$ is uv, depending on both unknown functions. The second one is the lower order term in the second equation, which governs the evolution of u as $x \to \pm\infty$.

In (1.14), it is exponential decay while in (1.1) it is logistic growth. Those two differences alter the components in the time asymptotic solution beyond the primary wave.

It is shown in [4] that the solution of (1.14), (1.2), (1.3) time asymptotically behaves like one of the following systems:

$$\begin{cases} v_t = -\frac{1}{r} p(v)_{xx}, \\ r- \end{cases} \tag{1.15}$$

$$\left| \left\lfloor p(v)_x = \quad ru, \right.\right.$$

where the first equation is the porous medium equation and the second one is Darcy's law. Also see [13, 14, 16] and references therein. The intuitive observation is as follows. The equilibrium manifold of (1.14) is $u = 0$. Substituting it into $(1.14)_1$ gives us the equilibrium equation

$$v_t = 0, \tag{1.16}$$

which is also known as the reduced equation. A better (the next order) approximation is obtained by first dropping u_t, the higher order term in time decay in $(1.14)_2$. This gives us $(1.15)_2$. Then we substitute it into $(1.14)_1$ to have $(1.15)_1$. The idea employed here is Chapman–Enskog expansion.

To illustrate the differences between (1.1) and (1.14) one by one, for the moment we assume $u_- = u_+ = \bar{u}$ while $v_- \neq v_+$ in (1.3). Here $\bar{u}$ needs to be an equilibrium state. Thus, $\bar{u} = 0$ for (1.14) while $\bar{u} = 1$ for (1.1).

We now follow [4] to consider the Cauchy problem (1.14), (1.2), where

$$\lim_{x\to\pm\infty}(v_0, u_0)(x) = (v_\pm, 0), \quad v_- \neq v_+. \tag{1.17}$$

Let $\bar{v}(x,t) = \varphi(x/\sqrt{t+1})$ be the unique self-similar solution of the porous medium equation

satisfying the boundary condition

$$\lim_{x\to\pm\infty} \bar{v}(x,t) = v_\pm. \tag{1.19}$$

$$\bar{v}_t = -\frac{1}{r}p(\bar{v})_{xx}, \tag{1.18}$$

Here the uniqueness is up to a translation. Noting the equation for v in (1.14) and (1.18) for $\bar{v}$ are conservation laws, we define the translation x_0 uniquely by setting

$$\int_{-\infty}^{\infty}[v(x,t) - \bar{v}(x+x_0,t)]dx = \int_{-\infty}^{\infty}[v_0(x) - \bar{v}(x+x_0,0)]dx = 0. \tag{1.20}$$

With $(1.15)_2$ we define the primary wave in the time asymptotic ansatz as

$$(\bar{v},\bar{u})(x+x_0,t), \quad \bar{u} = -\frac{1}{r}p(\bar{v})_x. \tag{1.21}$$

The end-states $v_\pm$ form a contact discontinuity with speed zero of the reduced equation (1.16),

$$\hat{v}(x,t) = \begin{cases} v_- & \text{if } x < 0, \\ v_+ & \text{if } x > 0, \end{cases}$$

see [18]. The primary wave $\bar{v}$ defined by (1.18), (1.19) can be regarded as a diffusive version of $\hat{v}$. While it is called a diffusion wave in [4], we call $\bar{v}$ a diffusive contact wave to emphasize its relation with $\hat{v}$.

Based on (1.20) one introduces new variables,

$$V(x,t) = \int_{-\infty}^{x}[v(y,t) - \bar{v}(y+x_0,t)]dy,$$

$$U(x,t) = u(x,t) - \bar{u}(x+x_0,t).$$

Thus,

$$V_x(x,t) = v(x,t) - \bar{v}(x+x_0,t).$$

$$\|V_x(t)\|_{L^2(\mathbb{R})} \sim (t+1)^{-\frac{3}{4}}, \ \|U(t)\|_{L^2(\mathbb{R})} \sim (t+1)^{-\frac{5}{4}}$$

Nonlinear stability of a weak diffusive contact wave is studied in [4]. That is, if $|v_+ - v_-|$ is small and $V(x,0)$ and $U(x,0) = V_t(x,0)$ are small in $H^3(\mathbb{R})$ and $H^2(\mathbb{R})$, respectively, then there exists a global-in-time solution of (1.14), (1.2), (1.17). The solution converges in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $(\bar{v},\bar{u})(x + x_0,t)$ time asymptotically, with $\|(V_x,U)(t)\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})}$ decaying at the rate $(t+1)^{-\frac{1}{2}}$. The decay rate is improved to optimal ones,,

$$\|V_x(t)\|_{L^\infty(\mathbb{R})} \sim (t+1)^{-1} \qquad \|U(t)\|_{L^\infty(\mathbb{R})} \sim (t+1)^{-\frac{3}{2}}$$

and, under a variety of assumptions on the initial data [13, 14].

For (1.1), similarly, we derive an approximate system that is the counterpart of the porous medium equation and Darcy's law (1.15), see Sec. 2 for details. As a consequence, the primary wave in the time asymptotic ansatz of the solution to (1.1)–(1.3) is a diffusive contact wave. Our analysis, however, reveals a significant difference in that there exists a secondary wave in the asymptotic ansatz. The v-component of the wave has zero mass and decays with the same rates as a heat kernel. (It has positive and negative peaks as observed from numerical simulations.) The u-component, on the other hand, decays at the same rates as the first derivative of a heat kernel. See Sec. 2 for details.

Our results show that the remainder of the solution to (1.1)–(1.3) after taking out the primary and secondary waves is higher order in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. This implies that the secondary wave is the leading term in the time-asymptotic error when approximating the solution by the diffusive contact wave. Therefore, the contact wave is stable, and the L^2-convergence rates to it are $(t+1)^{-\frac{1}{4}}$ for the v-component and $(t+1)^{-\frac{3}{4}}$ for the u-component. The L^∞-convergence rates are $(t+1)^{-\frac{1}{2}}$ and $(t+1)^{-1}$, respectively. This is to compare with $(t+1)^{-\frac{3}{4}}$ and $(t+1)^{-\frac{5}{4}}$ for L^2 and $(t+1)^{-1}$ and $(t+1)^{-\frac{3}{2}}$ for L^∞, respectively, for Euler equations with damping (1.14).

The existence of the secondary wave and hence the slower convergence rates to the diffusive contact wave comes from the fact that the nonlinear flux in the equation for u in (1.1) contains both v and u. More precisely, in the notations of (1.11) and (1.12), $\partial_{w_2} f_2 = \partial_u(uv) = v$ does not decay in time due to the background permanent wave. In the case of Euler equations with damping, the corresponding $\partial_{w_2} f_2 = \partial_u(p(v)) = 0$ for (1.14). There are other models in the literature studied around diffusive contact waves, such as the model for heat wave in rigid solids at low temperature [7] and the bipolar hydrodynamical model for semiconductors [2]. Also see [3, 6] and references therein. In those models, the counterparts of $\partial_{w_2} f_2$ are either zero or decaying in time sufficiently fast, and hence there are no secondary waves. To the best of the author's knowledge, the existence of a secondary wave is a novelty of (1.1) that may shed light on the future study of more general systems in the form of (1.11).

The other main difference between (1.1) and (1.14) is the diffusion mechanism in the second equation. It is exponential decay in (1.14) but logistic growth in (1.1). To illustrate the difference we now let $v_- \neq v_+$ and $u_- \neq u_+$ in (1.3), and consider the Cauchy problem (1.14), (1.2). In this case, the asymptotic ansatz (1.21) does not match the end-states for the u-component. The remedy is to construct an auxiliary function. In [4], it is defined as

$$\begin{cases} \tilde{u}(x,t) = e^{-rt}\left[u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy\right] & , \\ \tilde{v}(x,t) = \dfrac{u_- - u_+}{r} e^{-rt} m_0(x), \end{cases} \tag{1.22}$$

where $m_0(x)$ is a smooth function with compact support, satisfying

$$\int_{\mathbb{R}} m_0(x)dx = 1 . \tag{1.23}$$

It is clear that $(\tilde{v},\tilde{u})$ is the solution of

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0 \\ \tilde{u}_t = -r\tilde{u} \end{cases} , \tag{1.24}$$

with $\tilde{u}(x,t) \to e^{-rt}u_\pm$ as $x \to \pm\infty$. Here the rate equation in (1.24) is from the rate equation in (1.14) when $x \to \pm\infty$, governing the evolution of u as $x \to \pm\infty$. The initial value of $\tilde{u}$ is

$$\tilde{u}(x,0) = u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy, \tag{1.25}$$

which connects $\tilde{u} = u_-$ to $\tilde{u} = u_+$ smoothly in the compact support of $m_0(x)$.

The asymptotic ansatz (1.21) is now updated as

$$(\bar{v}(x + x_0,t) + \tilde{v}(x,t), \bar{u}(x + x_0,t) + \tilde{u}(x,t)),$$

while the shift is updated accordingly so that

$$\int_{\mathbb{R}} [v_0(x) - \bar{v}(x + x_0, 0) - \tilde{v}(x, 0)]dx = 0 .$$

The new variables now become

$$V(x,t) = \int_{-\infty}^{x} [v(y,t) - \bar{v}(y + x_0,t) - \tilde{v}(y,t)]dy,$$

$$U(x,t) = u(x,t) - \bar{u}(x + x_0,t) - \tilde{u}(x,t).$$

With the additional assumption $|u_- - u_+| \ll 1$, convergence of the solution of (1.14), (1.2), (1.3) to a weak diffusive contact wave is studied. The same results on global existence and convergence rates are obtained in [4, 13, 14].

In this paper, the diffusion mechanism is the logistic growth in (1.1). Therefore, our construction of an auxiliary function is via the logistic function, see Sec. 2. There are studies on nonlinear damping in the literature, see [7, 3, 26, 11] and references therein. Those studies are either for the case $u_- = u_+ = 0$ and hence no auxiliary functions, or for a perturbation of the linear damping. In particular, in [26, 11] Euler equations with nonlinear damping in the form $-ru - \beta|u|^{q-1}u$ are considered, where $r > 0$, $\beta \neq 0$ and $q > 1$ are constants. The convergence to the diffusive contact wave is obtained in [26] with $u_- = u_+ = 0$ and $q \geq 3$, and in [11] with small $u_- \neq u_+$ (near the equilibrium state $u = 0$) and $q > 5/2$. Roughly speaking, the logistic growth in this paper is equivalent to $q = 2$ while $u_\pm$ do not need to be near the equilibrium state $u = 1$.

As a final remark of this part, we point out that the two new features associated with (1.1)–(1.3) not only apply to more general hyperbolic balance laws in the form of (1.11), but also to hyperbolic-parabolic balance laws,

$$w_t + f(w)_x = [B(w)w_x]_x + g(w). \tag{1.26}$$

Here $B \in \mathbb{R}^{n\times n}$ is the viscosity matrix (usually rank deficient). In fact, in a recent paper [21], we consider the chemotaxis model for a diffusive cellular population. The converted system is

$$\begin{cases} v_t + u_x = 0, \\ u_t + (uv)_x = u_{xx} + ru(1-u) \end{cases} \tag{1.27}$$

,

which is a prototype of (1.26). We consider the case $\lim_{x\to\pm\infty}(v_0,u_0)(x) = (v_\pm,1)$ with $v_- \neq v_+$. Therefore, there is no auxiliary function needed but we do observe a secondary wave atop of the diffusive contact wave.

1.4. Goals and plan

In this paper, we establish the global existence of solution to (1.1)–(1.3) when Cauchy data are perturbations of a weak diffusive contact wave under appropriate assumptions. We identify and justify the leading term, a secondary wave, in the time-asymptotic error and an auxiliary function for the u-component to approach the equilibrium state $u = 1$ if $\{u_-, u_+\} \neq \{1\}$. These give us the nonlinear stability of the contact wave and large time behavior of solution to (1.1)–(1.3). We expect that our results can provide an innovative insight into asymptotic solutions of systems in the forms of (1.11) and (1.26), beyond what has been understood through the Euler equations with damping. Our results are obtained via energy and weighted energy methods. Using more sophisticated methods, it is possible to obtain results in L^p-spaces, $1 \le p \le \infty$, and fine-tune the higher order terms for their better accuracy. These are left to future works.

The plan of the paper is as follows. In Sec. 2, we give the needed preliminaries and state main results. In Sec. 3, we prove Theorem 2.4 to establish global existence of solution. In Sec. 4, we prove Theorem 2.5 to obtain convergence rates of the solution to the asymptotic solution and hence justify the asymptotic solution. In Appendix A, we prove Lemma 2.3.

2. Preliminaries and Main Results

We first derive equations that define the primary wave for (1.1)–(1.3). Considering u as a perturbation of the equilibrium state $u = 1$ we write $(1.1)_2$ as

$$(u-1)_t + v_x + [(u-1)v]_x = -r(u-1) - r(u-1)^2.$$

Identifying the leading terms in time decay rates, we have

$$v_x \approx -r(u-1).$$

Therefore, with $(1.1)_1$ we define the leading term in the time asymptotic solution for (v,u) as $(\bar{v},1+\bar{u})$, where $(\bar{v},\bar{u})$ satisfies

$$\begin{cases} \bar{v}_t + \bar{u}_x = 0 \\ \bar{v}_x = -r\bar{u}. \end{cases} \tag{2.1}$$

Substituting $(2.1)_2$ into $(2.1)_1$ gives us

$$\begin{cases} \bar{v}_t = \frac{1}{r}\bar{v}_{xx}, \\ -r\bar{u} = \bar{v}_x. \end{cases} \tag{2.2}$$

We observe that (2.2) is the counterpart of (1.15), the porous medium equation and Darcy's law.

Now we define $\bar{v}$ as the self-similar solution of $(2.2)_1$ with

$$\lim_{x\to\pm\infty} \bar{v}(x,t) = v_\pm. \tag{2.3}$$

Consequently, $\bar{u}$ is obtained by $(2.2)_2$. Explicitly,

$$\begin{aligned} \bar{v}(x,t) &= \frac{v_-}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4(t+1)/r}}}^{\infty} e^{-y^2}\,dy + \frac{v_+}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4(t+1)/r}}} e^{-y^2}\,dy \\ &= \frac{v_-+v_+}{2} - \frac{v_--v_+}{2}\,\mathrm{erf}\,(\frac{x}{\sqrt{4(t+1)/r}}), \\ \bar{u}(x,t) &= \frac{v_--v_+}{\sqrt{4\pi r(t+1)}} e^{-\frac{rx^2}{4(t+1)}}. \end{aligned} \tag{2.4}$$

We note that the solution to $(2.2)_1$, (2.3) is unique up to a translation. We will determine the translation x_0 later and use $(\bar{v},\bar{u})(x+x_0,t)$ instead. We also note that $\bar{v}(x,t)$ in (2.4) is a diffusive contact wave of the heat equation $(2.2)_1$, with Riemann data at $t=-1$.

The leading term $(\bar{v},1+\bar{u})$ so constructed is not sufficiently accurate. To construct a secondary wave we substitute (v,u) in (1.1) by $(\bar{v}+v^*,1+\bar{u}+u^*)$, apply (2.1) and keep the leading terms in time decay only. We arrive at

$$\begin{cases} v_t^* + u_x^* = 0, \\ v_x^* + ru^* = -R(x,t) \end{cases} \quad (2.5)$$

,

$$\begin{cases} v_t^* = \frac{1}{r} v_{xx}^* + \frac{1}{r} R_x(x,t) \\ u^* = -\frac{1}{r} v_x^* - \frac{1}{r} R(x,t). \end{cases}$$

where

$$R(x,t) = (\bar{u}_x \bar{v})(x,t). \quad (2.6)$$

Substituting $(2.5)_2$ into $(2.5)_1$ gives us

,

(2.7)

Here $(2.7)_1$ is to be solved with

$$v^*(x,0) = 0. \tag{2.8}$$

Thus by Duhamel's principle, we have an explicit expression for v^*,

$$v^*(x,t) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi r(t-\tau)}} e^{-\frac{r(x-y)^2}{4(t-\tau)}} R_y(y,\tau)\,dy\,d\tau \tag{2.9}$$

and $u^*(x,t)$ is given accordingly by $(2.7)_2$.

The above derivation of $(\bar{v},\bar{u})$ and (v^*,u^*) is based on expansions with respect to time decay, the idea of Chapman–Enskog expansion. The asymptotic ansatz constructed so far satisfies the following boundary condition:

$$\lim_{x\to\pm\infty} (\bar{v} + v^*, 1 + \bar{u} + u^*)(x,t) = (v_\pm, 1), \tag{2.10}$$

see (2.4) and (2.25) below. Comparing with (1.3), (2.10) is satisfactory if $u_- = u_+ = 1$. Otherwise, an auxiliary function is needed to match the boundary data for the u-component as $x \to \pm\infty$.

Therefore, we consider the case $\{u_-, u_+\} \neq \{1\}$. From $(1.1)_2$ we observe that $u(\pm\infty,t)$ evolves according to the logistic equation

$$\phi_t = r\phi(1 - \phi). \tag{2.11}$$

The solution to (2.11) is the logistic function,

$$\varphi(x,t) = \frac{\varphi(x,0)}{\varphi(x,0) + [1-\varphi(x,0)]e^{-rt}}. \tag{2.12}$$

We want $\phi(x,0)$ to connect u_- to u_+ smoothly. For this we use the same function as in [4] for (1.14), which is the initial function on the right-hand side of $(1.22)_1$. That is,

$$\varphi_0(x) \equiv \varphi(x,0) = u_- + (u_+ - u_-) \int_{-\infty}^x m_0(y)\,dy, \tag{2.13}$$

where $m_0(y)$ is a smooth, nonnegative function with compact support, satisfying

$$\int_{\mathbb{R}} m_0(y)\,dy = 1. \tag{2.14}$$

With the addition of an auxiliary function $(\tilde{v},\tilde{u})$, we expect that the new ansatz $(\bar{v} + v^* + \tilde{v}, 1 + \bar{u} + u^* + \tilde{u})$ matches $(\bar{v},\phi)$ as $x \to \pm\infty$. For this we set $1 + \tilde{u} = \phi$. Thus, with (2.12) and (2.13) we define

$$\tilde{u}(x,t) = \varphi(x,t) - 1 = \frac{[\varphi_0(x) - 1]e^{-rt}}{\varphi_0(x) + [1 - \varphi_0(x)]e^{-rt}}. \tag{2.15}$$

Correspondingly, we set

$$\tilde{v}(x,t) = \frac{\varphi_0'(x)e^{-rt}}{r\varphi_0(x)\{\varphi_0(x) + [1 - \varphi_0(x)]e^{-rt}\}}. \tag{2.16}$$

We observe that $(\tilde{v},\tilde{u})$ satisfies

$$\begin{cases} \tilde{v}_t + \tilde{u}_x = 0, \\ \tilde{u}_t = -r\tilde{u}(1+\tilde{u}) \end{cases} \tag{2.17}$$

Recall that $\bar{v}$ as a solution to $(2.2)_1$, (2.3) is unique up to a translation. As the final step in the construction of an asymptotic ansatz, we now determine the translation x_0 by setting

$$\int_{\mathbb{R}} [v_0(x) - \bar{v}(x+x_0,0) - \tilde{v}(x,0)]dx = 0 \tag{2.18}$$

From (2.16) and (2.13), this is equivalent to

$$\int_{\mathbb{R}} [v_0(x) - \bar{v}(x,0)]dx - (v_+ - v_-)x_0 - \frac{1}{r}\ln\frac{u_+}{u_-} = 0.$$

Thus, the constant x_0 is uniquely determined by

$$x_0 = \frac{1}{v_+ - v_-}\left\{\int_{\mathbb{R}} [v_0(x) - \bar{v}(x,0)]dx - \frac{1}{r}\ln\frac{u_+}{u_-}\right\}. \tag{2.19}$$

Combining $(1.1)_1$, $(2.1)_1$, $(2.5)_1$ and $(2.17)_1$, we have

$$\frac{d}{dt}\int_{\mathbb{R}} [v(x,t) - \bar{v}(x+x_0,t) - v^*(x+x_0,t) - \tilde{v}(x,t)]dx = 0.$$

With (2.8) and (2.18) we further have

$$\int_{\mathbb{R}} [v(x,t) - \bar{v}(x+x_0,t) - v^*(x+x_0,t) - \tilde{v}(x,t)]dx = 0. \tag{2.20}$$

This allows us to define a new variable

$$V(x,t) = \int_{-\infty}^{x} [v(y,t) - \bar{v}(y+x_0,t) - v^*(y+x_0,t) - \tilde{v}(y,t)]dy. \tag{2.21}$$

That is,

$$V_x(x,t) = v(x,t) - \bar{v}(x+x_0,t) - v^*(x+x_0,t) - \tilde{v}(x,t).$$

We also introduce a new variable,

$$U(x,t) = u(x,t) - 1 - \bar{u}(x+x_0,t) - u^*(x+x_0,t) - \tilde{u}(x,t). \tag{2.22}$$

These give us the following decomposition:

$$\begin{cases} v(x,t) = \bar{v}(x+x_0,t) + v^*(x+x_0,t) + \tilde{v}(x,t) + V_x(x,t), \\ u(x,t) = 1 + \bar{u}(x+x_0,t) + u^*(x+x_0,t) + \tilde{u}(x,t) + U(x,t) \end{cases} \tag{2.23}$$

In (2.23), the time asymptotic ansatz for $(v,u)(x,t)$ is $(\bar{v}, 1+\bar{u})(x+x_0,t) + (v^*,u^*)(x+x_0,t)$ with the correction $(\tilde{v},\tilde{u})(x,t)$, while $(V_x,U)(x,t)$ is the remainder. Here all the components of the ansatz including the shift are uniquely defined by (2.4), (2.9), $(2.7)_2$, (2.15), (2.16), (2.13) and (2.19). In particular, if $u_- = u_+ = 1$, we have $\tilde{u} = \tilde{v} = 0$ and (2.23) is simplified. The exact, explicit formulation of the components in the ansatz, however, does not provide a clear, convenient picture when comparing them with the remainder. Thus, we give their precise, pointwise estimates before we state the main results concerning the remainder.

Lemma 2.1. *For a fixed* $t \ge 0$, $\bar{v}(x,t)$ *monotonically increases or decreases from* v_- *to* v_+ *on* $\mathbb{R}$. *Let* $0 < r' < r/4$ *be an arbitrarily fixed constant. For* $x \in \mathbb{R}$, $t \ge 0$, *we have*

$$\partial_x^l \bar{v}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l}{2}} e^{-\frac{r' x^2}{t+1}}, \quad l \ge 1,$$
$$\partial_x^l \bar{u}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+1}{2}} e^{-\frac{r' x^2}{t+1}}, \quad l \ge 0 \tag{2.24}$$

.

Proof. The monotonicity $\bar{v}(x,t)$ comes from $(2.1)_2$ and $(2.4)_2$, which imply $\bar{v}_x \lesseqgtr 0$ if $v_- \gtreqless v_+$. The estimates in (2.24), on the other hand, are direct consequence of $\bar{u}(x,t)$ in $(2.4)_2$ and $\bar{v}_x = -r\bar{u}$, see $(2.1)_2$. □

Lemma 2.2 ([21]). *Let* $0 < r' < r/4$ *be an arbitrarily fixed constant and* $l \ge 0$ *be an integer. For* $x \in \mathbb{R}$, $t \ge 0$, *we have*

$$\partial_x^l v^*(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+1}{2}} e^{-\frac{r' x^2}{t+1}},$$
$$\partial_x^l u^*(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+2}{2}} e^{-\frac{r' x^2}{t+1}}. \tag{2.25}$$

Lemma 2.2 is proved in [21], applying a result from [9]. For the auxiliary function we give L^p-estimates instead.

Lemma 2.3. *For a fixed* $t \ge 0$, $\tilde{u}(x,t)$ *monotonically connects* $(u_- - 1)e^{-rt}/[u_- + (1-u_-)e^{-rt}]$ *to* $(u_+ - 1)e^{-rt}/[u_+ + (1-u_+)e^{-rt}]$ *on* $\mathbb{R}$. *Let* $1 \le p \le \infty$ *and* l *be an integer. We have the following estimates for* $t \ge 0$:

$$\|\tilde{u}(t)\|_{L^\infty(\mathbb{R})} \le Ce^{-rt},$$

$$\|\partial_x^l \tilde{u}(t)\|_{L^p(\mathbb{R})} \leq C|u_- - u_+|e^{-rt}, \quad l \geq 1$$
$$\|\partial_x^l \tilde{v}(t)\|_{L^p(\mathbb{R})} \leq C|u_- - u_+|e^{-rt}, \quad l \geq 0, \tag{2.26}$$

where $C > 0$ *is a constant.*

We postpone the proof of Lemma 2.3 to Appendix A. Here we introduce some notations. We use the following abbreviations for Sobolev norms with respect to x:

$$\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R})}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}.$$

We set

$$V_0(x) \equiv V(x,0) = \int_{-\infty}^{x} [v_0(y) - \bar{v}(y + x_0, 0) - \tilde{v}(y,0)]\, dy,$$
$$U_0(x) \equiv U(x,0) = u_0(x) - 1 - \bar{u}(x + x_0, 0) - u^*(x + x_0, 0) - \tilde{u}(x,0)$$
$$= u_0(x) - \bar{u}(x + x_0, 0) - u^*(x + x_0, 0) - \varphi_0(x). \tag{2.27}$$

Our first result is on global existence when the two end-states are sufficiently close and V_0 and U_0 are small.

$$v_- \neq v_+$$

Theorem 2.4.$\in^{m+1}$(R) *andLet*$U_0 \in H$ *and* $u_\pm > 0$ *be constants. Let* $m \geq 2$ *be an integer,*

V_0 H $_m$(R)*. Then there exists a constant* $\varepsilon_0 > 0$*, such that if*

$$|v_- - v_+| + |u_- - u_+| + \|V_0\|_{m+1} + \|U_0\|_m \leq \varepsilon_0, \tag{2.28}$$

the Cauchy problem (1.1)–(1.3) *has a unique, global-in-time solution* (v,u)*. With the decomposition* (2.23)*, the solution satisfies* $V \in C^i(0,\infty;H^{m+1-i}(\mathbb{R}))$, $0 \leq i \leq 2$*,* $U \in C^j(0,\infty;H^{m-j}(\mathbb{R}))$, $0 \leq j \leq 1$*, and the following energy estimate*:

$$\sup_{t\geq 0}\{\|V(t)\|_{m+1}^2 + \|U(t)\|_m^2\} + \int_0^\infty [\|V_x(t)\|_m^2 + \|U(t)\|_m^2]dt$$
$$\leq C(\|V_0\|_{m+1}^2 + \|U_0\|_m^2 + |v_- - v_+|^2 + |u_- - u_+|^2), \tag{2.29}$$

where $C > 0$ *is a constant.*

Our next theorem is on L^2 decay rates of the remainder.

Theorem 2.5. *Under the same assumptions as in Theorem* 2.4*, and with sufficiently small* $\varepsilon_0 > 0$*, the global solution* (v,u) *of* (1.1)–(1.3) *has the following estimates for* $t \geq 0$:

$$\sum_{k=0}^{2}(t+1)^{k+1}[\|\partial_x^k V_x(t)\|_{m-k}^2 + \|\partial_x^k U(t)\|_{m-k}^2]$$
$$+\sum_{k=0}^{2}\int_0^t (\tau+1)^{k+1}\|\partial_x^k U(\tau)\|_{m-k}^2\,d\tau$$
$$+\sum_{k=0}^{1}\int_0^t (\tau+1)^{k+1}\|\partial_x^{k+1} V_x(\tau)\|_{m-k-1}^2\,d\tau$$
$$\leq C(\|V_0\|_{m+1}^2 + \|U_0\|_m^2 + |v_- - v_+|^2 + |u_- - u_+|^2), \tag{2.30}$$

$$\|\partial_x^k U(t)\|_{m-k}$$
$$\leq C(\|V_0\|_{m+1} + \|U_0\|_m + |v_- - v_+| + |u_- - u_+|)(t+1)^{-1-\frac{k}{2}}, \quad k = 0, 1, \tag{2.31}$$

where $C > 0$ *is a constant.*

With the Sobolev inequality, see (3.14), one obtains L^∞ rates for V and U as follows.

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq C(\|V_0\|_{m+1} + \|U_0\|_m + |v_- - v_+| + |u_- - u_+|)(t+1)^{-\frac{1}{4}}$$

Corollary 2.6. *Under the same assumptions as in Theorem* 2.4, *and with sufficiently small* $\varepsilon_0 > 0$, *the global solution* (v,u) *of* (1.1)–(1.3) *has the following estimates for* $t \geq 0$:

,

$$\|V_x(t)\|_{L^\infty(\mathbb{R})} \le C(\|V_0\|_{m+1} + \|U_0\|_m + |v_- - v_+| + |u_- - u_+|)(t+1)^{-\frac{3}{4}},$$
$$\|U(t)\|_{L^\infty(\mathbb{R})} \le C(\|V_0\|_{m+1} + \|U_0\|_m + |v_- - v_+| + |u_- - u_+|)(t+1)^{-\frac{5}{4}}, \tag{2.32}$$

where $C > 0$ *is a constant.*

Consider the special case of $u_- = u_+ = 1$. In view of (2.10), the asymptotic solution $(\bar{v} + v^*, 1 + \bar{u} + u^*)(x + x_0, t)$ no longer needs a correction (i.e. $\tilde{u} = \tilde{v} = 0$ in (2.15) and (2.16)). Thus, corresponding to (2.19), the shift x_0 is uniquely determined by

$$x_0 = \frac{1}{v_+ - v_-}\int_{\mathbb{R}} [v_0(x) - \bar{v}(x,0)]dx. \tag{2.33}$$

The solution (v,u) of (1.1)–(1.3) is decomposed as

$$\begin{cases} v(x,t) = \bar{v}(x + x_0, t) + v^*(x + x_0, t) + V_x(x,t), \\ u(x,t) = 1 + \bar{u}(x + x_0, t) + u^*(x + x_0, t) + U(x,t) \end{cases}, \tag{2.34}$$

where

$$V(x,t) = \int_{-\infty}^{x} [v(y,t) - \bar{v}(y + x_0, t) - v^*(y + x_0, t)]dy. \tag{2.35}$$

These are counterparts of (2.23) and (2.21). The initial data for (V,U) is simplified to

$$\begin{aligned} V_0(x) \equiv V(x,0) &= \int_{-\infty}^{x} [v_0(y) - \bar{v}(y + x_0, 0)]dy, \\ U_0(x) \equiv U(x,0) &= u_0(x) - 1 - \bar{u}(x + x_0, 0) - u^*(x + x_0, 0) \end{aligned} \tag{2.36}$$

Applying Theorems 2.4 and 2.5 and Corollary 2.6 to this special case, we have the following results.

Corollary 2.7. *Let* $v_- \neq v_+$ *be constants and* $u_- = u_+ = 1$*. Let* $m \ge 2$ *be an integer,* $V_0 \in H^m(\mathbb{R})$ *and* $u_0 - 1 \in H^m(\mathbb{R})$*. Then there exists a constant* $\varepsilon_0 > 0$*, such that if*

$$|v_- - v_+| + \|V_0\|_{m+1} + \|u_0 - 1\|_m \le \varepsilon_0, \tag{2.37}$$

the Cauchy problem (1.1)–(1.3) *has a unique, global-in-time solution* (v,u)*. With the decomposition* (2.34)*, the solution satisfies* $V \in C^i(0,\infty; H^{m+1-i}(\mathbb{R}))$, $0 \le i \le 2$, $U \in C^j(0,\infty; H^{m-j}(\mathbb{R}))$, $0 \le j \le 1$, *and the following energy estimate:*

$$\sup_{t \ge 0}\{\|V(t)\|_{m+1}^2 + \|U(t)\|_m^2\} + \int_0^\infty [\|V_x(t)\|_m^2 + \|U(t)\|_m^2]dt$$
$$\le C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2). \tag{2.38}$$

With sufficiently small ε_0, the solution also has the following decay estimates for $t \geq 0$*:*

$$\begin{aligned}
&\sum_{k=0}^{2}(t+1)^{k+1}[\|\partial_x^k V_x(t)\|_{m-k}^2 + \|\partial_x^k U(t)\|_{m-k}^2] \\
&\qquad + \sum_{k=0}^{2}\int_0^t (\tau+1)^{k+1}\|\partial_x^k U(\tau)\|_{m-k}^2\,d\tau \\
&\qquad + \sum_{k=0}^{1}\int_0^t (\tau+1)^{k+1}\|\partial_x^{k+1} V_x(\tau)\|_{m-k-1}^2\,d\tau \\
&\quad \leq C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2),
\end{aligned} \tag{2.39}$$

$$\|\partial_x^k U(t)\|_{m-k} \leq C(\|V_0\|_{m+1} + \|u_0-1\|_m + |v_- - v_+|)(t+1)^{-1-\frac{k}{2}}, \quad k = 0, 1, \tag{2.40}$$

$$\begin{aligned}
&\|V(t)\|_{L^\infty(\mathbb{R})} \leq C(\|V_0\|_{m+1} + \|u_0-1\|_m + |v_- - v_+|)(t+1)^{-\frac{1}{4}}, \\
&\|V_x(t)\|_{L^\infty(\mathbb{R})} \leq C(\|V_0\|_{m+1} + \|u_0-1\|_m + |v_- - v_+|)(t+1)^{-\frac{3}{4}}, \\
&\|U(t)\|_{L^\infty(\mathbb{R})} \leq C(\|V_0\|_{m+1} + \|u_0-1\|_m + |v_- - v_+|)(t+1)^{-\frac{5}{4}}.
\end{aligned} \tag{2.41}$$

Here in (2.38)–(2.41)*, $C > 0$ is a constant.*

We comment that in the statement of Corollary 2.7, the assumptions on U_0 are replaced by those on $u_0 - 1$ due to (2.24) and (2.25). We also comment that Corollary 2.7 gives nonlinear stability of the diffusive contact wave $(\bar{v}, 1)$ if the wave strengthAs implied by (2.25), (2.39) and (2.41), in the decomposition (2.34) the$|v_- - v_+|$ is weak. L_2 decay rates for v^* and V_x are $(t+1)^{-\frac{1}{4}}$ and $(t+1)^{-\frac{1}{2}}$, respectively, and the corresponding L^∞ rates are $(t+1)^{-\frac{1}{2}}$ and $(t+1)^{-\frac{3}{4}}$. The rates indicate that in both L^2 and L^∞, while $\bar{v}$ is the primary wave, v^* is a secondary wave in the v-component.

Similarly, in the u-component the primary wave upon the equilibrium stste $u = 1$ is $\bar{u}$, which is a heat kernel and hence has decay rates $(t+1)^{-\frac{1}{4}}$ and $(t+1)^{-\frac{1}{2}}$ in L^2 and L^∞, respectively. On the other hand, u^* and U in (2.34) are of higher order in time-decay. The rates are $(t+1)^{-\frac{3}{4}}$ and $(t+1)^{-1}$ in L^2 and $(t+1)^{-1}$ and $(t+1)^{-\frac{5}{4}}$ in L^∞, respectively, see

(2.25), (2.40) and (2.41). This also justifies that u^* is a secondary wave in the u-component.

In summary, in the special case $u_- = u_+ = 1$ the global solution (v,u) of (1.1)– (1.3) is time-asymptotically approximated by $(\bar{v}, 1+\bar{u})(x+x_0,t)$, with $(v^*,u^*)(x+x_0,t)$ as the leading term is the error. The approximation is in both L^2 and L^∞.

In the general case $u_\pm > 0$ but $\{u_-, u_+\} \neq \{1\}$, there is a correction term $(\tilde{v}, \tilde{u})$ in the asymptotic error. While $(\tilde{v},\tilde{u})$ decays exponentially in time in L^∞, $\tilde{u}$ is not in L^2, see Lemma 2.3. Therefore, for the general case the solution (v,u) of (1.1)–(1.3) is time asymptotically approximated by $(\bar{v}, 1+\bar{u})(x+x_0,t)$ in L^∞ and

$(v^*,u^*)(x+x_0,t)$ is the leading term is the error. We comment that in the general case although $|u_- - u_+|$ is small, $u_\pm$ themselves do not need to be close to the equilibrium state $u = 1$.

The rates for V, U and their derivatives in (2.30)–(2.32) can be improved to optimal ones by a different set of analytic tools. It is left to a future work since the main purpose of this paper is the global existence of solution, the nonlinear stability of and the convergence to diffusive contact wave, and the identification of the asymptotic ansatz.

3. Global Existence of Solution

In this section, we prove Theorem 2.4 to establish the existence of a solution global in time for (1.1)–(1.3). First, we rewrite (1.1) in terms of V and U as defined in (2.21) and (2.22). Substituting (2.23) into (1.1) and applying (2.1), (2.5) and (2.17) give us

$$\begin{cases} V_{xt} + U_x = 0, \\ U_t + (1+\bar u + u^* + \tilde u + U)V_{xx} + (\bar v + v^* + \tilde v + V_x)U_x = F_1 + \tilde F, \end{cases} \tag{3.1}$$

where

$$F_1 = -\bar u_t - u^*_t - \tilde v_x - [(\bar u + u^* + \tilde u)(\bar v + v^* + \tilde v)]_x + \bar u_x \bar v - r(\bar u + u^*)^2$$
$$- 2r(\bar u + u^*)\tilde u,$$

$$\tilde F = -rU - rU^2 - (\bar v_x + v^*_x + \tilde v_x)U - (\bar u_x + u^*_x + \tilde u_x)V_x$$
$$-2r(\bar u + u^* + \tilde u)U. \tag{3.2}$$

Here in (3.1), (3.2) and for the rest of the paper, $(\bar v,\bar u)$ and (v^*,u^*) are understood as $(\bar v,\bar u)(x+x_0,t)$ and $(v^*,u^*)(x+x_0,t)$, respectively.

Using matrix notation we write (3.1) as

$$\begin{pmatrix} V_x \\ U \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1+\bar u + u^* + \tilde u + U & \bar v + v^* + \tilde v + V_x \end{pmatrix} \begin{pmatrix} V_x \\ U \end{pmatrix}_x = \begin{pmatrix} 0 \\ F_1 + \tilde F \end{pmatrix}, \tag{3.3}$$

which is for the unknown function $(V_x,U)^t$. It is clear that if

$$u = 1 + \bar u + u^* + \tilde u + U > 0, \tag{3.4}$$

the coefficient matrix on the left-hand side of (3.3) has two real, distinct eigenvalues

$$\lambda_{\pm} = \frac{1}{2}(v \pm \sqrt{v^2 + 4u}).$$

In this case, (3.3) is strictly hyperbolic. Here for small solutions under consideration, (3.4) is guaranteed by our assumptions $u_{\pm} > 0$ and $|v_- - v_+| \ll 1$, together with the estimates on $\bar{u}$, u^* and $\tilde{u}$ given in Lemmas 2.1–2.3.

The existence of a unique solution local in time for the Cauchy problem of a hyperbolic system is classical and via standard iterations [1]. Also see [12, 10]. With appropriate *a priori* estimates, global existence of solution follows, using a standard continuity argument. Our goal in this section is to prove the *a priori* estimates given in the following proposition, which then implies Theorem 2.4.

Proposition 3.1. *Let $m \geq 2$ be an integer, $V_0 \in H^{m+1}(\mathbb{R})$ and $U_0 \in H^m(\mathbb{R})$. Suppose that (V,U) is a solution of* (3.1) *with Cauchy data $V(x,0) = V_0(x)$, $U(x,0) = U_0(x)$, satisfying for some $T > 0$ the following regularity: $V \in C^i(0,T;H^{m+1-i}(\mathbb{R}))$, $0 \leq i \leq 2$,*

$$U \in C^j(0,T;H^{m-j}(\mathbb{R})), \qquad 0 \leq j \leq 1.$$

Let

$$N_m^2(t) = \sup_{0\le\tau\le t}\{\|V(\tau)\|_{m+1}^2 + \|U(\tau)\|_m^2\} + \int_0^t [\|V_x(\tau)\|_m^2 + \|U(\tau)\|_m^2]\,d\tau, \quad t \in [0,T]. \tag{3.5}$$

Then there exist constants $\delta_0, \delta_1 > 0$, such that if $|v_- - v_+| + |u_- - u_+| \leq \delta_0$ and $N_m(T) \leq \delta_1$, the following a priori estimate holds:

$$N_m^2(T) \le C(\|V_0\|_{m+1}^2 + \|U_0\|_m^2 + |v_- - v_+|^2 + |u_- - u_+|^2), \tag{3.6}$$

where $C > 0$ is a constant.

Proof. In the following C denotes a generic positive constant. In particular, it is independent of T. We first rewrite (3.1) as

$$\begin{cases} V_{xt} + U_x = 0, \\ U_t + [(1+\bar{u}+u^*+\tilde{u}+U)V_x]_x + rU = F, \end{cases} \tag{3.7}$$

where with F_1 defined in (3.2), we have

$$F = F_1 + F_2 + F_3,$$

$$F_2 = -rU^2, \tag{3.8}$$

$$F_3 = -[(\bar{v} + v^* + \tilde{v})U]_x - 2r(\bar{u} + u^* + \tilde{u})U.$$

For $0 \leq l \leq m$, $m \geq 2$, we apply ∂_x^l to $(3.7)_2$ to have

$$\partial_x^l U_t + \partial_x^{l+1}[(1+\bar{u}+u^*+\tilde{u}+U)V_x] + r\partial_x^l U = \partial_x^l F. \tag{3.9}$$

Multiply (3.9) by $\partial_x^l U$ and integrate with respect to x. After integration by parts and applying $(3.7)_1$, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|\partial_x^l U\|^2 + \frac{1}{2}\int_{\mathbb{R}}(1+\bar{u}+u^*+\tilde{u}+U)(\partial_x^{l+1}V)^2\,dx\right] + r\|\partial_x^l U\|^2 = \sum_{i=1}^{4} I_i, \tag{3.10}$$

where

$$I_i = \int_{\mathbb{R}} \partial_x^l U \partial_x^l F_i \, dx, \quad 1 \le i \le 3,$$

$$I_4 = \frac{1}{2}\int_{\mathbb{R}} (\bar u_t + u_t^* + \tilde u_t + U_t)(\partial_x^{l+1} V)^2 \, dx - \int_{\mathbb{R}} \partial_x^l U \partial_x \{\partial_x^l [(1 + \bar u + u^* + \tilde u + U) V_x]$$

$$-(1 + \bar u + u^* + \tilde u + U)\partial_x^{l+1} V\} dx. \qquad (3.11)$$

From (3.2), (2.1), (2.5) and (2.6), we have

$$F_1 = \frac{1}{r}\bar v_{xt} + \left(v_x^* + \frac{1}{r}\bar u_x \bar v\right)_t - \tilde v_x - (\bar u \bar v)_x - [\bar u(v^* + \tilde v)]_x - [(u^* + \tilde u)(\bar v + v^* + \tilde v)]_x$$
$$+ \bar u_x \bar v - r\bar u^2 - r(2\bar u u^* + u^{*2} + 2(\bar u + u^*)\tilde u]$$

$$= -\frac{1}{r}(\bar u + u^*)_{xx} + \frac{1}{r}(\bar u_x \bar v)_t - [\tilde v + \bar u(v^* + \tilde v) + (u^* + \tilde u)(\bar v + v^* + \tilde v)]_x$$
$$- r[2\bar u u^* + u^{*2} + 2(\bar u + u^*)\tilde u].$$

Applying (2.1) and Lemmas 2.1–2.3, for $l \ge 0$, we have

$$|\partial_x^l F_1| \le C|v_- - v_+|(t+1)^{-\frac{l+3}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}} + |\partial_x^{l+1}\tilde v| + |\partial_x^{l+1}\tilde u||\bar v| + |\partial_x^{l+1}(\tilde u \tilde v)|,$$

where $0 < r' < \frac{r}{4}$. This implies

$$\|\partial_x^l F_1(t)\|_{L^p(\mathbb{R})} \le C|v_- - v_+|(t+1)^{-\frac{l+3}{2} + \frac{1}{2p}} + C|u_- - u_+|e^{-rt}, \quad 1 \le p \le \infty. \qquad (3.12)$$

Thus, (3.11) and (3.12) give us

$$I_1 \le \|\partial_x^l U\| \|\partial_x^l F_1\| \le C\|\partial_x^l U\|[|v_- - v_+|(t+1)^{-\frac{l}{2} - \frac{5}{4}} + |u_- - u_+|e^{-rt}]. \qquad (3.13)$$

Recall Sobolev inequality: If $\psi \in H^1(\mathbb{R})$, then $\psi \in L^\infty(\mathbb{R})$, with

$$\|\psi\|_{L^\infty} \le \sqrt{2}\|\psi\|^{\frac{1}{2}} \|\psi'\|^{\frac{1}{2}} \le \sqrt{2}\|\psi\|_1. \qquad (3.14)$$

Applying (3.14) to F_2 in (3.8) gives us

$$\|\partial_x^l F_2\| = r\|\partial_x^l U^2\| \le C \sum_{j=0}^{l} \|\partial_x^j U \partial_x^{l-j} U\| \le C\|U\|_m \|U\|_l. \qquad (3.15)$$

Therefore, from (3.11), we have

$$I_2 \le \|\partial_x^l U\| \|\partial_x^l F_2\| \le C\|\partial_x^l U\| \|U\|_m \|U\|_l. \qquad (3.16)$$

For I_3 from (3.8) and (3.11), we have

$$I_3 = -\int_{\mathbb{R}} \partial_x^l U \partial_x^{l+1}[(\bar{v} + v^* + \tilde{v})U]\, dx - 2r \int_{\mathbb{R}} \partial_x^l U \partial_x^l [(\bar{u} + u^* + \tilde{u})U]\, dx$$
$$\equiv I_{31} + I_{32}. \tag{3.17}$$

Applying (2.24)–(2.26) and by integration by parts, we have

$$\begin{aligned}I_{31} &= -\int_{\mathbb{R}}(\bar{v}+v^*+\tilde{v})[\frac{1}{2}(\partial_x^l U)^2]_x\,dx\\&\quad-\int_{\mathbb{R}}\partial_x^l U\{\partial_x^{l+1}[(\bar{v}+v^*+\tilde{v})U]-(\bar{v}+v^*+\tilde{v})\partial_x^{l+1}U\}\,dx\\&\le \frac{1}{2}\int_{\mathbb{R}}(\bar{v}_x+v^*_x+\tilde{v}_x)(\partial_x^l U)^2\,dx\\&\quad+\|\partial_x^l U\|C\sum_{j=0}^{l}\|\partial_x^{l+1-j}(\bar{v}+v^*+\tilde{v})\|_{L^\infty(\mathbb{R})}\|\partial_x^j U\|\\&\le C\sum_{j=0}^{l}[|v_--v_+|(t+1)^{-\frac{1}{2}-\frac{l-j}{2}}+|u_--u_+|e^{-rt}]\|\partial_x^j U\|\|\partial_x^l U\|.\end{aligned}\tag{3.18}$$

Similarly,

$$\begin{aligned}I_{32} &\le 2r\|\partial_x^l U\|\|\partial_x^l[(\bar{u}+u^*+\tilde{u})U]\|\\&\le C\sum_{j=0}^{l-1}[|v_--v_+|(t+1)^{-\frac{1}{2}-\frac{l-j}{2}}+|u_--u_+|e^{-rt}]\|\partial_x^j U\|\|\partial_x^l U\|\\&\quad+C[|v_--v_+|(t+1)^{-\frac{1}{2}}+e^{-rt}]\|\partial_x^l U\|^2.\end{aligned}\tag{3.19}$$

Next, we write I_4 in (3.11) as

$$\begin{aligned}I_4 &= \frac{1}{2}\int_{\mathbb{R}}(\bar{u}_t+u^*_t+\tilde{u}_t)(\partial_x^{l+1}V)^2\,dx+\frac{1}{2}\int_{\mathbb{R}}U_t(\partial_x^{l+1}V)^2\,dx\\&\quad-\int_{\mathbb{R}}\partial_x^l U\partial_x\{\partial_x^l[(1+\bar{u}+u^*+\tilde{u})V_x]-(1+\bar{u}+u^*+\tilde{u})\partial_x^{l+1}V\}\,dx\\&\quad-\int_{\mathbb{R}}\partial_x^l U\partial_x[\partial_x^l(UV_x)-U\partial_x^{l+1}V]\,dx\equiv I_{41}+I_{42}+I_{43}+I_{44}.\end{aligned}\tag{3.20}$$

We use (2.2), (2.7) and $(2.17)_2$ to convert derivatives with respect to t into those with respect to x, and apply the estimates (2.24)–(2.26) to have

$$\begin{aligned}I_{41} &\le \frac{1}{2}\|\bar{u}_t+u^*_t+\tilde{u}_t\|_{L^\infty(\mathbb{R})}\|\partial_x^{l+1}V\|^2\\&\le C[|v_--v_+|(t+1)^{-\frac{3}{2}}+e^{-rt}]\|\partial_x^{l+1}V\|^2.\end{aligned}\tag{3.21}$$

Here we have assumed, say, $\delta_0+\delta_1\le 1$. Similarly, applying $(3.1)_2$, we have

$$\begin{aligned}I_{42} &\le \frac{1}{2}\|U_t\|_{L^\infty(\mathbb{R})}\|\partial_x^{l+1}V\|^2\\&\le \frac{1}{2}[\|(1+\bar{u}+u^*+\tilde{u}+U)V_{xx}\|_{L^\infty(\mathbb{R})}+\|(\bar{v}+v^*+\tilde{v}+V_x)U_x\|_{L^\infty(\mathbb{R})}\\&\quad+\|F_1\|_{L^\infty(\mathbb{R})}+\|\tilde{F}\|_{L^\infty(\mathbb{R})}]\|\partial_x^{l+1}V\|^2.\end{aligned}$$

Besides (2.24)–(2.26), we apply (3.14), (3.12) and (3.2), and note $\delta_0 + \delta_1 \leq 1$ to have

$$I_{42} \leq C[\|V_{xx}\|_1 + \|U\|_2 + |v_- - v_+|(t+1)^{-1} + |u_- - u_+|e^{-rt}]\|\partial_x^{l+1}V\|^2. \tag{3.22}$$

It is clear that $I_{43} = 0$ if $l = 0$. On the other hand, if $l \geq 1$,

$$\begin{aligned} I_{43} &\leq C\|\partial_x^l U\| \sum_{j=0}^{l} \|\partial_x^{l+1-j}(\bar u + u^* + \tilde u)\|_{L^\infty(\mathbb{R})}\|\partial_x^j V_x\| \\ &\leq C\|\partial_x^l U\| \sum_{j=0}^{l} [|v_- - v_+|(t+1)^{-1-\frac{l-j}{2}} + |u_- - u_+|e^{-rt}]\|\partial_x^j V_x\|. \end{aligned} \tag{3.23}$$

Using (3.14), the last term in (3.20) can be estimated similarly. If $l = 0$, $I_{44} = 0$. If $l \geq 1$,

$$\begin{aligned} I_{44} &\leq -\int_{\mathbb{R}} \partial_x^l U(\partial_x^{l+1}U)V_x\,dx + C\|\partial_x^l U\|\sum_{j=1}^{l}\|\partial_x^j U \partial_x^{l-j+1}V_x\| \\ &\leq \frac{1}{2}\int_{\mathbb{R}} V_{xx}(\partial_x^l U)^2\,dx \\ &\quad + C\|\partial_x^l U\|\left(\sum_{j=1}^{l-1}\|\partial_x^j U\|_{L^\infty(\mathbb{R})}\|\partial_x^{l-j+1}V_x\| + \|\partial_x^l U\|\|V_{xx}\|_{L^\infty(\mathbb{R})}\right) \\ &\leq C\left(\|V_{xx}\|_1\|\partial_x^l U\|^2 + \|\partial_x^l U\|\sum_{j=1}^{l-1}\|\partial_x^j U\|_1\|\partial_x^{l-j+1}V_x\|\right) \\ &\leq C\|\partial_x^l U\|\|U\|_l\|V_{xx}\|_{m-1}. \end{aligned} \tag{3.24}$$

Combining (3.10), (3.13) and (3.16)–(3.24), for $0 \leq l \leq m$, we have

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\partial_x^l U\|^2 + \frac{1}{2}\int_{\mathbb{R}}(1+\bar u + u^* + \tilde u + U)(\partial_x^{l+1}V)^2\,dx\right] + r\|\partial_x^l U\|^2 \\ &\leq C[|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}} + |u_- - u_+|e^{-rt}]\|\partial_x^l U\| + C[\|U\|_m + \|V_{xx}\|_{m-1} \\ &\quad + |v_- - v_+|(t+1)^{-1} + |u_- - u_+|e^{-rt}]\|\partial_x^l U\|\|U\|_l \\ &\quad + C|v_- - v_+|(t+1)^{-\frac{1}{2}}\|\partial_x^l U\|^2 + Ce^{-rt}(\|\partial_x^l U\|^2 + \|\partial_x^{l+1}V\|^2) \\ &\quad + C[|v_- - v_+|(t+1)^{-1} + \|V_{xx}\|_1 + \|U\|_2]\|\partial_x^{l+1}V\|^2 \\ &\quad + C|v_- - v_+|(t+1)^{-\frac{3}{2}}\|\partial_x^l U\|\|V_x\|_{l-1}, \end{aligned} \tag{3.25}$$

where the last term on the right-hand side exists only when $l \geq 1$.

Integrating (3.25) with respect to time on $[0,t]$ and noting (3.5) give us

$$\frac{1}{2}\|\partial_x^l U(t)\|^2 + \frac{1}{2}\int_{\mathbb{R}}[(1+\bar u + u^* + \tilde u + U)(\partial_x^{l+1}V)^2](x,t)\,dx + r\int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau$$

$$
\begin{aligned}
&\le \frac{1}{2}\|\partial_x^l U(0)\|^2 + \frac{1}{2}\|(1+\bar u+u^*+\tilde u+U)(0)\|_{L^\infty(\mathbb{R})}\|\partial_x^{l+1}V(0)\|^2 \\
&\quad + C(|v_- - v_+|^2 + |u_- - u_+|^2) + \frac{r}{2}\int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]\int_0^t [\|U(\tau)\|_l^2 + \|V_x(\tau)\|_l^2]\,d\tau \\
&\quad + C\int_0^t e^{-r\tau}[\|\partial_x^l U(\tau)\|^2 + \|\partial_x^{l+1}V(\tau)\|^2]\,d\tau.
\end{aligned}
$$

This can be further simplified to

$$
\begin{aligned}
&\frac{1}{2}\|\partial_x^l U(t)\|^2 + \frac{1}{2}\int_{\mathbb{R}}[(1+\tilde u)(\partial_x^{l+1}V)^2](x,t)\,dx + \frac{r}{2}\int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau \\
&\quad \le C[\|\partial_x^l U(0)\|^2 + \|\partial_x^{l+1}V(0)\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2] \\
&\quad\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t) \\
&\quad\quad + C\int_0^t e^{-r\tau}[\|\partial_x^l U(\tau)\|^2 + \|\partial_x^{l+1}V(\tau)\|^2]\,d\tau.
\end{aligned}
\tag{3.26}
$$

From Lemma 2.3, 1 + ˜− $^{rt}]^{-1}$, which impliesu monotonically connects $[1 + (1/u_- - 1)e^{-rt}]^{-1}$ to [1 +

$(1/u_+$ $1)e^{-}$

$$
1 + \tilde u \ge \min\{u_-, u_+, 1\} \equiv c_0 > 0. \tag{3.27}
$$

Thus, (3.26) is further simplified to

$$
\begin{aligned}
&\|\partial_x^l U(t)\|^2 + \|\partial_x^{l+1}V(t)\|^2 + \int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau \\
&\quad \le C[\|\partial_x^l U(0)\|^2 + \|\partial_x^{l+1}V(0)\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2] \\
&\quad\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t) \\
&\quad\quad + C\int_0^t e^{-r\tau}[\|\partial_x^l U(\tau)\|^2 + \|\partial_x^{l+1}V(\tau)\|^2]\,d\tau.
\end{aligned}
\tag{3.28}
$$

Let

$$
\begin{aligned}
E(t) &= \|\partial_x^l U(t)\|^2 + \|\partial_x^{l+1}V(t)\|^2 + \int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau, \\
p(t) &= C[\|\partial_x^l U(0)\|^2 + \|\partial_x^{l+1}V(0)\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2] \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t).
\end{aligned}
$$

Then (3.28) implies

$$E(t) \leq p(t) + C \int_0^t e^{-r\tau} E(\tau) d\tau.$$

Applying Gro¨nwall's inequality, we have

$$E(t) \le p(t)e^{C\int_0^t e^{-r\tau}\,d\tau} \le Cp(t).$$

Thar is,

$$\begin{aligned}
&\|\partial_x^l U(t)\|^2 + \|\partial_x^{l+1} V(t)\|^2 + \int_0^t \|\partial_x^l U(\tau)\|^2\,d\tau \\
&\quad \le C[\|\partial_x^l U(0)\|^2 + \|\partial_x^{l+1} V(0)\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2] \\
&\qquad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t), \quad 0 \le l \le m. \qquad (3.29)
\end{aligned}$$

From $(3.1)_1$, we have

$$V_t + U = 0, \qquad (3.30)$$

which implies

$$\frac{d}{dt}\left(\frac{1}{2}\|V\|^2\right) = -\int_{\mathbb{R}} VU\,dx. \qquad (3.31)$$

With $(3.7)_1$ we write

$$U = \frac{1}{r}\{-U_t - [(1+\bar{u}+u^*+\tilde{u}+U)V_x]_x + F\}.$$

Thus,

$$\begin{aligned}
-\int_{\mathbb{R}} VU\,dx &= \frac{1}{r}\int_{\mathbb{R}} VU_t\,dx - \frac{1}{r}\int_{\mathbb{R}}(1+\bar{u}+u^*+\tilde{u}+U)V_x^2\,dx - \frac{1}{r}\int_{\mathbb{R}} VF\,dx \\
&\equiv I_5 + I_6 + I_7. \qquad (3.32)
\end{aligned}$$

Applying (3.30), we have

$$I_5 = \frac{d}{dt}\left(\frac{1}{r}\int_{\mathbb{R}} VU\,dx\right) + \frac{1}{r}\|U\|^2. \qquad (3.33)$$

Applying (2.24), (2.25), (3.14) and (3.5), we also have

$$I_6 \le -\frac{1}{r}\int_{\mathbb{R}}(1+\tilde{u})V_x^2\,dx + C[|v_- - v_+| + N_1(t)]\|V_x\|^2. \qquad (3.34)$$

Finally, from (3.8)

$$I_7 = I_{71} + I_{72} + I_{73}, \quad I_{7k} = -\frac{1}{r}\int_{\mathbb{R}} VF_k\,dx, \quad 1 \le k \le 3. \qquad (3.35)$$

Here applying (3.12) gives us

$$I_{71} \le \frac{1}{r}\|V\|\|F_1\| \le C[|v_- - v_+|(t+1)^{-\frac{5}{4}} + |u_- - u_+|e^{-rt}]\|V\|. \qquad (3.36)$$

Also from (3.8) and (3.5),

$$\begin{aligned}
I_{72} &= \int_{\mathbb{R}} VU^2\,dx \le C\|V\|_1\|U\|^2 \le CN_0(t)\|U\|^2, \\
I_{73} &= -\frac{1}{r}\int_{\mathbb{R}}(\bar{v}+v^*+\tilde{v})V_xU\,dx + 2\int_{\mathbb{R}}(\bar{u}+u^*+\tilde{u})VU\,dx \qquad (3.37)
\end{aligned}$$

$$\leq C\|V_x\|\|U\| - 2\int_{\mathbb{R}} \bar{u}VV_t\,dx + C[|v_- - v_+|(t+1)^{-1} + e^{-rt}]\|V\|\|U\|$$
$$\leq \frac{c_0}{2r}\|V_x\|^2 + C\|U\|^2 - \frac{d}{dt}\int_{\mathbb{R}} \bar{u}V^2\,dx + C|v_- - v_+|(t+1)^{-\frac{3}{2}}\|V\|^2$$
$$+ C[|v_- - v_+|(t+1)^{-1} + e^{-rt}]^2\|V\|^2, \tag{3.38}$$

where we have used (3.30) and (2.2), and c_0 is defined in (3.27). Combining (3.32)–(3.38) we arrive at

$$-\int_{\mathbb{R}} VU\,dx \leq \frac{d}{dt}\left[\frac{1}{r}\int_{\mathbb{R}} VU\,dx - \int_{\mathbb{R}} \bar{u}V^2\,dx\right] - \frac{1}{r}\int_{\mathbb{R}}(1+\tilde{u})V_x^2\,dx + \frac{c_0}{2r}\|V_x\|^2$$
$$+ C\|U\|^2 + C[|v_- - v_+| + N_1(t)]\|V_x\|^2$$
$$+ C(|v_- - v_+| + |u_- - u_+|)(t+1)^{-\frac{5}{4}}\|V\| + Ce^{-2rt}\|V\|^2. \tag{3.39}$$

Substituting (3.39) into (3.31) gives us

$$\frac{d}{dt}\left[\frac{1}{2}\|V\|^2 - \frac{1}{r}\int_{\mathbb{R}} VU\,dx + \int_{\mathbb{R}} \bar{u}V^2\,dx\right] + \frac{1}{r}\int_{\mathbb{R}}(1+\tilde{u})V_x^2\,dx$$
$$\leq \frac{c_0}{2r}\|V_x\|^2 + C\|U\|^2 + C[|v_- - v_+| + N_1(t)]\|V_x\|^2$$
$$+ C(|v_- - v_+| + |u_- - u_+|)(t+1)^{-\frac{5}{4}}\|V\| + Ce^{-2rt}\|V\|^2.$$

Applying (3.27) and integrating the result with respect to time on $[0,t]$ for $0 \leq t \leq T$, we have

$$\frac{1}{2}\|V(t)\|^2 - \frac{1}{r}\int_{\mathbb{R}}(VU)(x,t)\,dx + \int_{\mathbb{R}}(\bar{u}V^2)(x,t)\,dx + \frac{c_0}{2r}\int_0^t \|V_x(\tau)\|^2\,d\tau$$
$$\leq \frac{1}{2}\|V_0\|^2 - \frac{1}{r}\int_{\mathbb{R}}(VU)(x,0)\,dx + \int_{\mathbb{R}}(\bar{u}V^2)(x,0)\,dx + C\int_0^t \|U(\tau)\|^2\,d\tau$$
$$+ C[|v_- - v_+| + N_1(t)]\int_0^t \|V_x(\tau)\|^2\,d\tau + C(|v_- - v_+| + |u_- - u_+|)$$
$$\times \sup_{0\leq\tau\leq t}\|V(\tau)\| + C\int_0^t e^{-2r\tau}\|V(\tau)\|^2\,d\tau,$$

which can be further simplified to

$$\frac{1}{2}\|V(t)\|^2 + \frac{c_0}{2r}\int_0^t \|V_x(\tau)\|^2\,d\tau \leq C(\|V_0\|^2 + \|U_0\|^2) + \frac{1}{4}\|V(t)\|^2$$
$$+ C[\|U(t)\|^2 + |v_- - v_+|N_0^2(t)$$

$$+\int_0^t \|U(\tau)\|^2\, d\tau + N_1(t)N_0^2(t)]$$
$$+C(|v_- - v_+| + |u_- - u_+|) \sup_{0\le\tau\le t} \|V(\tau)\|$$
$$+C\int_0^t e^{-2r\tau}\|V(\tau)\|^2\, d\tau.$$

After simplifying, we apply (3.29) with $l = 0$ for the estimate on $\|U(t)\|^2 + \int_0^t \|U(\tau)\|^2\, d\tau$. This gives us

$$\begin{aligned}
&\|V(t)\|^2 + \int_0^t \|V_x(\tau)\|^2\, d\tau \\
&\quad \le C(\|V_0\|_1^2 + \|U_0\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\qquad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t) \\
&\qquad + C(|v_- - v_+| + |u_- - u_+|) \sup_{0\le\tau\le t} \|V(\tau)\| + C\int_0^t e^{-2r\tau}\|V(\tau)\|^2\, d\tau.
\end{aligned}$$

Similar to the derivation of (3.29), via Gro¨nwall's inequality we arrive at

$$\begin{aligned}
\|V(t)\|^2 + \int_0^t \|V_x(\tau)\|^2\, d\tau &\le C(\|V_0\|_1^2 + \|U_0\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t) \\
&\quad + C(|v_- - v_+| + |u_- - u_+|) \sup_{0\le\tau\le t} \|V(\tau)\| \\
&\le C(\|V_0\|_1^2 + \|U_0\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t) \\
&\quad + \frac{1}{2} \sup_{0\le\tau\le t} \|V(\tau)\|^2.
\end{aligned}$$

(3.40) Therefore,

$$\begin{aligned}
\sup_{0\le\tau\le t} \|V(\tau)\|^2 &\le C(\|V_0\|_1^2 + \|U_0\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t).
\end{aligned}$$

(3.41) Now we substitute (3.41) into (3.40) to have

$$\begin{aligned}
\|V(t)\|^2 + \int_0^t \|V_x(\tau)\|^2\, d\tau &\le C(\|V_0\|_1^2 + \|U_0\|^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t). \qquad (3.42)
\end{aligned}$$

Finally, we sum up (3.42) and (3.29) for $0 \le l \le m$. This gives us the following estimate:

$$\|U(t)\|_m^2 + \|V(t)\|_{m+1}^2 + \int_0^t [\|U(\tau)\|_m^2 + \|V_x(\tau)\|^2]\, d\tau$$
$$\le C[\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2]$$
$$+ C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t). \tag{3.43}$$

We still need to estimate $\int_0^t \|V_{xx}(\tau)\|_{m-1}^2\, d\tau$ in $N_m^2(t)$. For this we multiply (3.9) by $\partial_x^{l+2} V$, $0 \le l \le m - 1$, and integrate the result with respect to x. These give us

$$\int_{\mathbb{R}} (1 + \tilde{u})(\partial_x^{l+2} V)^2\, dx = I_8 + I_9 + I_{10}, \tag{3.44}$$

where

$$I_8 = -\int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l U_t\, dx,$$
$$I_9 = -\int_{\mathbb{R}} \partial_x^{l+2} V \{\partial_x^{l+1}[(1 + \bar{u} + u^* + \tilde{u} + U)V_x]$$
$$- (1 + \tilde{u})\partial_x^{l+2} V + r\partial_x^l U\}\, dx,$$
$$I_{10} = \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F\, dx. \tag{3.45}$$

By integration by parts and (3.30), we have

$$I_8 = \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U_t\, dx = \frac{d}{dt} \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U\, dx + \|\partial_x^{l+1} U\|^2. \tag{3.46}$$

By (2.24)–(2.26) and (3.14), we have

$$
\begin{aligned}
I_9 &\leq \|\partial_x^{l+2}V\| \Bigg[\|\bar{u} + u^* + U\|_{L^\infty(\mathbb{R})} \|\partial_x^{l+2}V\| \\
&\quad + C\sum_{j=1}^{l+1} \|\partial_x^j(\bar{u} + u^* + \tilde{u} + U)\partial_x^{l+1-j}V_x\| + r\|\partial_x^l U\| \Bigg] \\
&\leq C\|\partial_x^{l+2}V\| \Bigg[|v_- - v_+| \sum_{j=0}^{l+1} (t+1)^{-\frac{j+1}{2}} \|\partial_x^{l+1-j}V_x\| + |u_- - u_+| e^{-rt} \|V_x\|_l \\
&\quad + \|U\|_1 \|\partial_x^{l+2}V\| + \|U_x\|_{m-1} \|V_x\|_l + \|\partial_x^l U\| \Bigg].
\end{aligned}
\tag{3.47}
$$

From (3.8), we write

$$I_{10} = \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_1 \, dx + \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_2 \, dx + \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_3 \, dx$$
$$\equiv I_{101} + I_{102} + I_{103}. \tag{3.48}$$

Applying (3.12) gives us

$$I_{101} \le \|\partial_x^{l+2} V\| \|\partial_x^l F_1\| \le C[|v_- - v_+|(t+1)^{-\frac{5}{4}-\frac{l}{2}} + |u_- - u_+|e^{-rt}] \|\partial_x^{l+2} V\|. \tag{3.49}$$

It is clear from (3.8) and (3.14) that

$$I_{102} = -r \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l (U^2) dx \le r \|\partial_x^{l+2} V\| \|\partial_x^l (U^2)\|$$
$$\le C \|\partial_x^{l+2} V\| \|U\|_{m-1} \|U\|_l. \tag{3.50}$$

Also, from (3.8), (2.24)–(2.26) and (3.14), we have

$$I_{103} \le \|\partial_x^{l+2} V\| \|\partial_x^l F_3\|$$
$$\le \|\partial_x^{l+2} V\| \{ \|\partial_x^{l+1} [(\bar{v} + v^* + \tilde{v}) U]\| + 2r \|\partial_x^l [(\bar{u} + u^* + \tilde{u}) U]\| \}$$
$$\le C \|\partial_x^{l+2} V\| \{ \|\partial_x^{l+1} U\| + [|v_- - v_+|(t+1)^{-\frac{1}{2}} + |u_- - u_+|e^{-rt}] \|U\|_l$$
$$+ e^{-rt} \|\partial_x^l U\| \}. \tag{3.51}$$

Combining (3.48)–(3.51) gives us

$$I_{10} \le C[|v_- - v_+|(t+1)^{-\frac{5}{4}-\frac{l}{2}} + |u_- - u_+|e^{-rt}] \|\partial_x^{l+2} V\|$$
$$+ C \|\partial_x^{l+2} V\| \|\partial_x^{l+1} U\| + C \|\partial_x^{l+2} V\| [\|U\|_{m-1} \|U\|_l$$
$$+ |v_- - v_+|(t+1)^{-\frac{1}{2}} \|U\|_l + e^{-rt} \|U\|_l]. \tag{3.52}$$

Now we substitute (3.46), (3.47) and (3.52) into (3.44) and note (3.27). We arrive at

$$c_0 \|\partial_x^{l+2} V\|^2 \le \frac{d}{dt} \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U \, dx + \|\partial_x^{l+1} U\|^2 + C \|\partial_x^{l+2} V\| \|\partial_x^l U\|_1$$
$$+ C |v_- - v_+|(t+1)^{-\frac{1}{2}} \|\partial_x^{l+2} V\| (\|\partial_x^{l+2} V\| + \|U\|_l)$$
$$+ C(|v_- - v_+| + |u_- - u_+|)(t+1)^{-1} \|\partial_x^{l+2} V\| \|V_x\|_l$$
$$+ C \|\partial_x^{l+2} V\| (\|U\|_1 \|\partial_x^{l+2} V\| + \|U_x\|_{m-1} \|V_x\|_l + \|U\|_{m-1} \|U\|_l)$$
$$+ C[|v_- - v_+|(t+1)^{-\frac{5}{4}-\frac{l}{2}} + |u_- - u_+|e^{-rt}] \|\partial_x^{l+2} V\|$$
$$+ C e^{-rt} \|\partial_x^{l+2} V\| \|U\|_l,$$

which can be further simplified to

$$\begin{aligned}
&\frac{c_0}{2}\|\partial_x^{l+2}V\|^2 \\
&\quad \le \frac{d}{dt}\int_{\mathbb{R}} \partial_x^{l+1}V\partial_x^{l+1}U\,dx + C(\|\partial_x^l U\|_1^2 + \|U_x\|_{m-1}^2\|V_x\|_l^2 + \|U\|_{m-1}^2\|U\|_l^2) \\
&\quad + C|v_- - v_+|(t+1)^{-\frac{1}{2}}\|\partial_x^{l+2}V\|(\|\partial_x^{l+2}V\| + \|U\|_l) \\
&\quad + C(|v_- - v_+| + |u_- - u_+|)(t+1)^{-1}\|\partial_x^{l+2}V\|\|V_x\|_l + C\|U\|_1\|\partial_x^{l+2}V\|^2 \\
&\quad + C[|v_- - v_+|^2(t+1)^{-\frac{5}{2}-l} + |u_- - u_+|^2e^{-2rt}] + Ce^{-2rt}\|U\|_l^2 \qquad (3.53)
\end{aligned}$$

for $0 \le l \le m-1$.

We sum up (3.53) for $0 \le l \le m-1$ and integrate the result with respect to time on $[0,t]$. These give us

$$\begin{aligned}
&\frac{c_0}{2}\int_0^t \|V_{xx}(\tau)\|_{m-1}^2\,d\tau \\
&\quad \le \sum_{l=0}^{m-1}[\|\partial_x^{l+1}V(t)\|\|\partial_x^{l+1}U(t)\| + \|\partial_x^{l+1}V(0)\|\|\partial_x^{l+1}U(0)\|] \\
&\quad + C\int_0^t \|U(\tau)\|_m^2\,d\tau + CN_m^3(t) + C(|v_- - v_+| + |u_- - u_+|)N_m^2(t) \\
&\quad + C(|v_- - v_+|^2 + |u_- - u_+|^2) + C\sup_{0\le\tau\le t}\|U(\tau)\|_{m-1}^2. \qquad (3.54)
\end{aligned}$$

Using (3.43) we further simplify (3.54) to

$$\begin{aligned}
\int_0^t \|V_{xx}(\tau)\|_{m-1}^2\,d\tau &\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t). \qquad (3.55)
\end{aligned}$$

Next we combine (3.43) and (3.55) to have

$$\begin{aligned}
&\|U(t)\|_m^2 + \|V(t)\|_{m+1}^2 + \int_0^t [\|U(\tau)\|_m^2 + \|V_x(\tau)\|_m^2]\,d\tau \\
&\quad \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t).
\end{aligned}$$

With the definition (3.5), this implies

$$\begin{aligned}
N_m^2(t) &\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C[N_m(t) + |v_- - v_+| + |u_- - u_+|]N_m^2(t).
\end{aligned}$$

□

Under the assumption $|v_- - v_+| + |u_- - u_+| \leq \delta_0$ and $N_m(t) \leq \delta_1$, and by choosing δ_0 and δ_1 small, we obtain (3.6). Thus, we have proved Proposition 3.1 and hence Theorem 2.4.

4. Asymptotic Behavior of Solution

In this section, we prove Theorem 2.5, which justifies $(\bar{v}(x+x_0,t)+v^*(x+x_0,t), 1+\bar{u}(x+x_0,t)+u^*(x+x_0,t))$ as an asymptotic solution to (1.1)–(1.3). This is to be done by weighted energy estimate. We continue to use C as a generic positive constant.

For $k = 1,2,3$ and $k - 1 \le l \le m$, we multiply (3.25) in the energy estimate by a weight $(t+1)^k$. This gives us

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}(t+1)^k\|\partial_x^l U\|^2 + \frac{1}{2}(t+1)^k\int_{\mathbb{R}}(1+\bar{u}+u^*+\tilde{u}+U)(\partial_x^{l+1}V)^2\,dx\right] \\
&\qquad + r(t+1)^k\|\partial_x^l U\|^2 \\
&\quad \le \frac{k}{2}(t+1)^{k-1}\|\partial_x^l U\|^2 + \frac{k}{2}(t+1)^{k-1}\int_{\mathbb{R}}(1+\bar{u}+u^*+\tilde{u}+U)(\partial_x^{l+1}V)^2\,dx \\
&\qquad + \frac{r}{2}(t+1)^k\|\partial_x^l U\|^2 + C|v_- - v_+|^2(t+1)^{-\frac{3}{2}} + C|u_- - u_+|e^{-\frac{rt}{2}}\|\partial_x^l U\| \\
&\qquad + C[(t+1)^k(\|U\|_m + \|V_{xx}\|_{m-1}) + |v_- - v_+|(t+1)^{k-1} \\
&\qquad + |u_- - u_+|e^{-\frac{rt}{2}}]\|\partial_x^l U\|\|U\|_l + C|v_- - v_+|(t+1)^{k-\frac{1}{2}}\|\partial_x^l U\|^2 \\
&\qquad + Ce^{-\frac{rt}{2}}(\|\partial_x^l U\|^2 + \|\partial_x^{l+1}V\|^2) \\
&\qquad + C[|v_- - v_+|(t+1)^{k-1} + (t+1)^k(\|V_{xx}\|_1 + \|U\|_2)]\|\partial_x^{l+1}V\|^2 \\
&\qquad + C|v_- - v_+|(t+1)^{k-\frac{3}{2}}\|\partial_x^l U\|\|V_x\|_{l-1}.
\end{aligned}$$

After simplifying, we integrate both sides with respect to time on $[0,t]$. Applying (3.27), (2.24)–(2.26), (3.14) and (2.29) gives us

$$
\begin{aligned}
&(t+1)^k\|\partial_x^l U(t)\|^2 + (t+1)^k\|\partial_x^{l+1} V(t)\|^2 + \int_0^t (\tau+1)^k\|\partial_x^l U(\tau)\|^2\,d\tau \\
&\quad \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\qquad + C[|v_- - v_+|(t+1)^{k-\frac{1}{2}} + (t+1)^k\|U(t)\|_1]\|\partial_x^{l+1} V(t)\|^2 \\
&\qquad + C\int_0^t (\tau+1)^{k-1}[\|\partial_x^l U(\tau)\|^2 + \|\partial_x^{l+1} V(\tau)\|^2]\,d\tau \\
&\qquad + C\int_0^t (\tau+1)^k[\|U(\tau)\|_m + \|V_{xx}(\tau)\|_{m-1}]\|\partial_x^l U(\tau)\|\|U(\tau)\|_l\,d\tau \\
&\qquad + C|v_- - v_+|\int_0^t [(\tau+1)^{k-1}\|\partial_x^l U(\tau)\|\|U(\tau)\|_l + (\tau+1)^{k-\frac{1}{2}}\|\partial_x^l U(\tau)\|^2]\,d\tau \\
&\qquad + C|v_- - v_+|\int_0^t (\tau+1)^{k-1}\|\partial_x^{l+1} V(\tau)\|^2\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t (\tau+1)^k[\|V_{xx}(\tau)\|_1+\|U(\tau)\|_2]\|\partial_x^{l+1}V(\tau)\|^2\,d\tau\\
&+C|v_--v_+|\int_0^t (\tau+1)^{k-\frac{3}{2}}\|\partial_x^l U(\tau)\|\|V_x(\tau)\|_{l-1}\,d\tau,
\end{aligned}
\tag{4.1}
$$

where the last term on the right-hand side is for $l \geq 1$.

Taking $k = 1$ in (4.1) and summing up for $0 \leq l \leq m$, we have

$$
\begin{aligned}
&(t+1)[\|U(t)\|_m^2+\|V_x(t)\|_m^2]+\int_0^t (\tau+1)\|U(\tau)\|_m^2\,d\tau\\
&\quad\leq C(\|U_0\|_m^2+\|V_0\|_{m+1}^2+|v_--v_+|^2+|u_--u_+|^2)\\
&\qquad+C[|v_--v_+|(t+1)^{\frac{1}{2}}+(t+1)\|U(t)\|_1]\|V_x(t)\|_m^2\\
&\qquad+C(\|V_0\|_{m+1}+\|U_0\|_m+|v_--v_+|+|u_--u_+|)\int_0^t (\tau+1)\|U(\tau)\|_m^2\,d\tau\\
&\qquad+C\left[\int_0^t \|V_x(\tau)\|_m^2\,d\tau\right]^{\frac{1}{2}}\\
&\qquad\times\left\{\int_0^t (\tau+1)^2[\|V_{xx}(\tau)\|_1+\|U(\tau)\|_2]^2\|V_x(\tau)\|_m^2\,d\tau\right\}^{\frac{1}{2}},
\end{aligned}
\tag{4.2}
$$

where we have used (2.29) and Cauchy–Schwarz inequality. With (2.29), the last term on the right-hand side is bounded by

$$
\begin{aligned}
&C(\|V_0\|_{m+1}+\|U_0\|_m+|v_--v_+|+|u_--u_+|)\left\{\sup_{0\leq\tau\leq t}[(\tau+1)\|V_x(\tau)\|_m^2]\right\}^{\frac{1}{2}}\\
&\quad\times\left\{\int_0^t (\tau+1)[\|V_{xx}(\tau)\|_1^2+\|U(\tau)\|_2^2]d\tau\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Thus, with the bound ε_0 on the data as defined in (2.28), (4.2) is simplified to

$$
\begin{aligned}
&(t+1)[\|U(t)\|_m^2+\|V_x(t)\|_m^2]+\int_0^t (\tau+1)\|U(\tau)\|_m^2\,d\tau\\
&\quad\leq C(\|U_0\|_m^2+\|V_0\|_{m+1}^2+|v_--v_+|^2+|u_--u_+|^2)\\
&\qquad+C\varepsilon_0\left\{\sup_{0\leq\tau\leq t}[(\tau+1)\|V_x(\tau)\|_m^2]+\int_0^t (\tau+1)[\|V_{xx}(\tau)\|_1^2+\|U(\tau)\|_m^2]d\tau\right\}.
\end{aligned}
\tag{4.3}
$$

Next we multiply (3.53) by the weight $(t + 1)$ and sum up for $0 \le l \le m - 1$. With (2.29), these give us

$$\frac{c_0}{2}(t+1)\|V_{xx}\|_{m-1}^2$$
$$\le \frac{d}{dt}\left[(t+1)\sum_{l=0}^{m-1}\int_{\mathbb{R}}\partial_x^{l+1}V\partial_x^{l+1}U\,dx\right]$$
$$-\sum_{l=0}^{m-1}\int_{\mathbb{R}}\partial_x^{l+1}V\partial_x^{l+1}U\,dx + C(t+1)\|U\|_m^2 + C\varepsilon_0(t+1)\|V_{xx}\|_{m-1}^2$$
$$+C\|V_x\|_m^2 + C[|v_- - v_+|^2(t+1)^{-\frac{3}{2}} + |u_- - u_+|^2 e^{-rt}].$$

Integrating with respect to time on $[0,t]$ and simplifying with (2.29), we arrive at

$$\int_0^t (\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau$$
$$\le C(t+1)\|V_x(t)\|_{m-1}\|U_x(t)\|_{m-1} + C\|V_0'\|_{m-1}\|U_0'\|_{m-1}$$
$$+C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)$$
$$+C\int_0^t (\tau+1)\|U(\tau)\|_m^2\,d\tau + C\varepsilon_0\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau.$$

Substituting (4.3) into the right-hand side gives us

$$\int_0^t (\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau$$
$$\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)$$
$$+C\varepsilon_0\Big\{\sup_{0\le\tau\le t}[(\tau+1)\|V_x(\tau)\|_m^2]$$
$$+\int_0^t(\tau+1)[\|U(\tau)\|_m^2 + \|V_{xx}(\tau)\|_{m-1}^2]d\tau\Big\}. \tag{4.4}$$

We sum up (4.3) and (4.4) to have

$$(t+1)[\|U(t)\|_m^2 + \|V_x(t)\|_m^2] + \int_0^t(\tau+1)[\|U(\tau)\|_m^2 + \|V_{xx}(\tau)\|_{m-1}^2]\,d\tau$$
$$\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)$$
$$+C\varepsilon_0\Big\{\sup_{0\le\tau\le t}[(\tau+1)\|V_x(\tau)\|_m^2] + \int_0^t(\tau+1)[\|U(\tau)\|_m^2 + \|V_{xx}(\tau)\|_{m-1}^2]d\tau\Big\}.$$

For small $\varepsilon_0 > 0$, this implies

$$\sup_{0\le\tau\le t}\{(\tau+1)[\|U(\tau)\|_m^2+\|V_x(\tau)\|_m^2]\} + \int_0^t(\tau+1)[\|U(\tau)\|_m^2+\|V_{xx}(\tau)\|_{m-1}^2]\,d\tau \le C(\|U_0\|_m^2+\|V_0\|_{m+1}^2+|v_--v_+|^2+|u_--u_+|^2). \tag{4.5}$$

Next, we take $k = 2$ in (4.1) and sum up for $1 \le l \le m$. Similar to the derivation of (4.3) but applying both (2.29) and (4.5), we have

$$\begin{aligned}(t+1)^2[\|U_x(t)\|_{m-1}^2&+(1-C\varepsilon_0)\|V_{xx}(t)\|_{m-1}^2] + (1-C\varepsilon_0)\int_0^t(\tau+1)^2\|U_x(\tau)\|_{m-1}^2\,d\tau\\ &\le C(\|U_0\|_m^2+\|V_0\|_{m+1}^2+|v_--v_+|^2+|u_--u_+|^2)\\ &+C\sup_{0\le\tau\le t}[(\tau+1)\|U_x(\tau)\|_{m-1}]\int_0^t(\tau+1)\\ &\times[\|U(\tau)\|_m+\|V_{xx}(\tau)\|_{m-1}]\|U(\tau)\|_m\,d\tau + C\sup_{0\le\tau\le t}[(\tau+1)\|V_{xx}(\tau)\|_{m-1}]\\ &\times\int_0^t(\tau+1)[\|V_{xx}(\tau)\|_1+\|U(\tau)\|_2]\|V_{xx}(\tau)\|_{m-1}\,d\tau\\ &+C|v_--v_+|\int_0^t(\tau+1)^{\frac12}\|U_x(\tau)\|_{m-1}\|V_x(\tau)\|_{m-1}\,d\tau.\end{aligned} \tag{4.6}$$

The second term on the right-hand side of (4.6) is bounded by

$$\begin{aligned}\frac18\left\{\sup_{0\le\tau\le t}[(\tau+1)\|U_x(\tau)\|_{m-1}]\right\}^2 &+ C\left\{\int_0^t(\tau+1)[\|U(\tau)\|_m^2+\|V_{xx}(\tau)\|_{m-1}^2]\,d\tau\right\}^2\\ &\le\frac18\sup_{0\le\tau\le t}[(\tau+1)^2\|U_x(\tau)\|_{m-1}^2]\\ &+C(\|U_0\|_m^2+\|V_0\|_{m+1}^2+|v_--v_+|^2+|u_--u_+|^2)^2,\end{aligned}$$

where we have applied (4.5). The third term on the right-hand side of (4.6) is treated similarly, while the fourth term is bounded by

$$C|v_- - v_+| \int_0^t [(\tau+1)\|U_x(\tau)\|_{m-1}^2 + \|V_x(\tau)\|_{m-1}^2]\, d\tau$$
$$\leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2),$$

using (4.5) and (2.29). Therefore, after simplifying, (4.6) implies

$$
\begin{aligned}
&(1 - C\varepsilon_0) \sup_{0\le\tau\le t} \{(\tau+1)^2[\|U_x(\tau)\|_{m-1}^2 + \|V_{xx}(\tau)\|_{m-1}^2]\} \\
&\quad + (1 - C\varepsilon_0) \int_0^t (\tau+1)^2 \|U_x(\tau)\|_{m-1}^2 \, d\tau \\
&\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + \frac{1}{2} \sup_{0\le\tau\le t} \{(\tau+1)^2[\|U_x(\tau)\|_{m-1}^2 + \|V_{xx}(\tau)\|_{m-1}^2]\}.
\end{aligned}
$$

Taking $\varepsilon_0 > 0$ small, we arrive at

$$
\begin{aligned}
&(t+1)^2[\|U_x(t)\|_{m-1}^2 + \|V_{xx}(t)\|_{m-1}^2] + \int_0^t (\tau+1)^2 \|U_x(\tau)\|_{m-1}^2 \, d\tau \\
&\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2). \qquad (4.7)
\end{aligned}
$$

Next we multiply (3.53) by $(t + 1)^2$, sum up for $1 \le l \le m - 1$, and integrate with respect to time on $[0,t]$. Similar to the derivation of (4.4) but applying (4.5) and (4.7) in addition to (2.29), we have

$$
\begin{aligned}
&\int_0^t (\tau+1)^2 \|V_{xxx}(\tau)\|_{m-2}^2 \, d\tau \\
&\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C \sup_{0\le\tau\le t} [(\tau+1)\|U(\tau)\|_{m-1}^2] \int_0^t (\tau+1)\|U(\tau)\|_{m-1}^2 \, d\tau \\
&\quad + C\varepsilon_0 \left[\int_0^t (\tau+1)^2 \|V_{xxx}(\tau)\|_{m-2}^2 \, d\tau + \int_0^t (\tau+1)\|U(\tau)\|_{m-1}^2 \, d\tau \right. \\
&\quad \left. + \int_0^t \|V_x(\tau)\|_{m-1}^2 \, d\tau \right] \\
&\le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\quad + C\varepsilon_0 \int_0^t (\tau+1)^2 \|V_{xxx}(\tau)\|_{m-2}^2 \, d\tau.
\end{aligned}
$$

Thus for $\varepsilon_0 > 0$ small, we have

$$\int_0^t (\tau+1)^2 \|V_{xxx}(\tau)\|_{m-2}^2 \, d\tau \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2). \tag{4.8}$$

$$\|U(t)\|$$

To improve the decay rate of, we multiply $(3.7)_2$ by U and integrate with respect to x. These give us

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2\right) + r\|U\|^2 \leq \|U\|\{\|[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_x\| + \|F\|\}$$

$$\leq \frac{r}{2}\|U\|^2 + \frac{1}{r}\|[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_x\|^2 + \frac{1}{r}\|F\|^2,$$

which is simplified to

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2\right) + \frac{r}{2}\|U\|^2 \leq \frac{1}{r}\|[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_x\|^2 + \frac{1}{r}\|F\|^2. \tag{4.9}$$

Applying Lemmas 2.1–2.3, (3.14), (2.29), (3.8), (4.7), (3.12) and (4.5), the righthand side is bounded by

$$C(\|\bar{u}_x + u_x^* + \tilde{u}_x + U_x\|_{L^\infty(\mathbb{R})}^2\|V_x\|^2 + \|V_{xx}\|^2 + \|F_1\|^2 + \|U\|_1^2\|U\|^2$$
$$+ \|\bar{v}_x + v_x^* + \tilde{v}_x\|_{L^\infty(\mathbb{R})}^2\|U\|^2 + \|U_x\|^2 + \|\bar{u}+u^*+\tilde{u}\|_{L^\infty(\mathbb{R})}^2\|U\|^2)$$
$$\leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-2}.$$

Thus, (4.9) is simplified to

$$\frac{d}{dt}\|U\|^2 + r\|U\|^2 \leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-2}.$$

Applying Gronwall's inequality we arrive at

$$\|U(t)\|^2 \leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-2}. \tag{4.10}$$

Combining (4.10) and (4.7) gives us

$$\|U(t)\|_m^2 \leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-2}. \tag{4.11}$$

We now go back to (4.1) and take $k = 3$. After summing up for $2 \leq l \leq m$, similar to the derivation of (4.6), we have

$$(t+1)^3[\|U_{xx}(t)\|_{m-2}^2 + (1-C\varepsilon_0)\|V_{xxx}(t)\|_{m-2}^2] + \int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\,d\tau$$
$$\leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)$$
$$+ C\int_0^t (\tau+1)^3[\|U(\tau)\|_m + \|V_{xx}(\tau)\|_{m-1}]\|U_{xx}(\tau)\|_{m-2}\|U(\tau)\|_m\,d\tau$$
$$+ C|v_- - v_+|\int_0^t [(\tau+1)^2\|U_{xx}(\tau)\|_{m-2}\|U(\tau)\|_m + (\tau+1)^{\frac{5}{2}}\|U_{xx}(\tau)\|_{m-2}^2]\,d\tau$$
$$+ C\int_0^t (\tau+1)^3[\|V_{xx}(\tau)\|_1 + \|U(\tau)\|_2]\|V_{xxx}(\tau)\|_{m-2}^2\,d\tau$$
$$+ C|v_- - v_+|\int_0^t (\tau+1)^{\frac{3}{2}}\|U_{xx}(\tau)\|_{m-2}\|V_x(\tau)\|_{m-1}\,d\tau, \tag{4.12}$$

where we have applied (4.7) and (4.8) besides (2.29). For the second and fourth terms on the right-hand side of (4.12) we use (4.11) and bounded them by

$$
\begin{aligned}
&C \sup_{0\le\tau\le t}[(\tau+1)\|U(\tau)\|_m] \int_0^t (\tau+1)^2[\|U(\tau)\|_m + \|V_{xx}(\tau)\|_{m-1}]\|U_{xx}(\tau)\|_{m-2}\, d\tau \\
&\quad + C \sup_{0\le\tau\le t}\{(\tau+1)[\|V_{xx}(\tau)\|_1 + \|U(\tau)\|_2]\} \int_0^t (\tau+1)^2\|V_{xxx}(\tau)\|_{m-2}^2\, d\tau \\
&\le C\varepsilon_0 \left\{ \int_0^t (\tau+1)[\|U(\tau)\|_m^2 + \|V_{xx}(\tau)\|_{m-1}^2]\, d\tau \right. \\
&\quad \left. + \int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\, d\tau \right\} \\
&\quad + C\varepsilon_0(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2) \\
&\le C\varepsilon_0(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2 \\
&\quad + \int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\, d\tau).
\end{aligned}
$$

For the third and fifth terms on the right-hand side of (4.12) we bounded them by

$$
\begin{aligned}
&C|v_- - v_+| \int_0^t [(\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2 + (\tau+1)\|U(\tau)\|_m^2 + \|V_x(\tau)\|_{m-1}^2]\, d\tau \\
&\quad \le C\varepsilon_0 \int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\, d\tau \\
&\quad\quad + C\varepsilon_0(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2).
\end{aligned}
$$

Therefore, (4.12) is simplified as

$$
\begin{aligned}
&(t+1)^3[\|U_{xx}(t)\|_{m-2}^2 + (1-C\varepsilon_0)\|V_{xxx}(t)\|_{m-2}^2] \\
&\quad + (1-C\varepsilon_0)\int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\, d\tau \\
&\quad \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2).
\end{aligned}
$$

Taking $\varepsilon_0 > 0$ sufficiently small, we have

$$
\begin{aligned}
&(t+1)^3[\|U_{xx}(t)\|_{m-2}^2 + \|V_{xxx}(t)\|_{m-2}^2] + \int_0^t (\tau+1)^3\|U_{xx}(\tau)\|_{m-2}^2\, d\tau \\
&\quad \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2).
\end{aligned}
\tag{4.13}
$$

Combining (4.5), (4.7), (4.8) and (4.13) we obtain (2.30).

$$\|U_x(t)\|$$

Finally, we consider the improved estimate on. For this we take $l = 1$ in (3.9) and test it with U_x. These give us

$$\frac{d}{dt}\left(\frac{1}{2}\|U_x\|^2\right) + r\|U_x\|^2 = -\int_{\mathbb{R}} U_x[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_{xx}\,dx + \int_{\mathbb{R}} U_xF_x\,dx$$
$$\leq \frac{r}{2}\|U_x\|^2 + \frac{1}{r}\|[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_{xx}\|^2 + \frac{1}{r}\|F_x\|^2. \tag{4.14}$$

Simplifying (4.14) and applying (3.8), we have

$$\frac{d}{dt}\|U_x\|^2 + r\|U_x\|^2 \leq C\{\|[(1+\bar{u}+u^*+\tilde{u}+U)V_x]_{xx}\|^2 + \|F_{1x}\|^2 + \|UU_x\|^2$$
$$+ \|[(\bar{v}+v^*+\tilde{v})U]_{xx}\|^2 + \|[(\bar{u}+u^*+\tilde{u})U]_x\|^2\}.$$

Applying Lemmas 2.1–2.3, (2.29), (2.30), (3.14) and (3.12), one can verify that the right-hand side is bounded by

$$C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-3}.$$

Thus, similar to the derivation of (4.10), Gronwall's inequality gives us

$$\|U_x(t)\|^2 \leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-3}.$$

Together with (2.30), we have

$$\|U_x(t)\|_{m-1}^2 \leq C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2 + |u_- - u_+|^2)(t+1)^{-3}. \tag{4.15}$$

We obtain (2.31) from (4.11) and (4.15).

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Appendix A. Proof of Lemma 2.3

We now prove Lemma 2.3. From (2.13),

$$\varphi_0'(x) = (u_+ - u_-)m_0(x). \tag{A.1}$$

Since $m_0(x) \geq 0$, $\phi_0(x)$ monotonically connects u_- to u_+ on the compact support of m_0

while taking the value u_- on the left and u_+ on the right of the support.

Since $u_\pm > 0$, $\phi_0(x) > 0$ on R. Besides, the denominator in (2.12),

$$\phi_0(x) + [1 - \phi_0(x)]e^{-rt} = \phi_0(x)(1 - e^{-rt}) + e^{-rt},$$

monotonically connects $u_-(1-e^{-rt})+e^{-rt}$ to $u_+(1-e^{-rt})+e^{-rt}$. This implies

$$\begin{aligned}\inf_{\mathbb{R}}\{\phi_0(x)+[1-\phi_0(x)]e^{-rt}\} &\geq \min\{u_-(1-e^{-rt})+e^{-rt}, u_+(1-e^{-rt})+e^{-rt}\}\\ &= \min\{u_-+(1-u_-)e^{-rt}, u_++(1-u_+)e^{-rt}\}\\ &\geq \min\{1,u_-,u_+\}>0. \end{aligned} \tag{A.2}$$

From (2.15) and (2.12),

$$\tilde{u}_x(x,t)=\varphi_x(x,t)=\frac{\varphi_0'(x)e^{-rt}}{\{\varphi_0(x)+[1-\varphi_0(x)]e^{-rt}\}^2}, \tag{A.3}$$

which has the same sign as $\varphi_0'(x)$. Thus for a fixed $t \geq 0$, $\tilde{u}(x,t)$ monotonically connects $\tilde{u}(-\infty,t)$ to $\tilde{u}(\infty,t)$. This justifies the first statement in Lemma 2.3. Also, applying (A.2), we have

$$\|\tilde{u}(t)\|_{L^\infty(\mathbb{R})}\leq \max\left\{\frac{|u_--1|e^{-rt}}{u_-+(1-u_-)e^{-rt}},\frac{|u_+-1|e^{-rt}}{u_++(1-u_+)e^{-rt}}\right\}\leq Ce^{-rt},$$

where $C=\max\{|u_--1|,|u_+-1|\}/\min\{1,u_-,u_+\}$.

Next, for any integer $l \geq 1$, by induction, we have

$$\partial_x^l\tilde{u}(x,t)=e^{-rt}\sum_{k=1}^{l}\frac{(1-e^{-rt})^{k-1}p_{l,k}(x)}{\{\varphi_0(x)+[1-\varphi_0(x)]e^{-rt}\}^{k+1}}, \tag{A.4}$$

where

$$p_{l,k}(x)=\sum_j c_{kj}\varphi_0^{(\alpha_{j1}^k)}(x)\cdots\varphi_0^{(\alpha_{jk_j}^k)}(x), \tag{A.5}$$

c_{kj} are constants, and the orders of derivatives satisfy $1 \leq \alpha^k_{j1},\dots,\alpha^k_{jk_j} \leq l+1-k$, $\alpha_{j1}^k+\cdots+\alpha_{jk_j}^k=l$. In particular, $p_{l,1}(x)=\varphi_0^{(l)}(x)$ and $p_{l,l}(x)=(-1)^{l-1}l![\varphi_0'(x)]^l$. Noting (A.1), we have

$$\|p_{l,k}\|_{L^p(\mathbb{R})}\leq C|u_--u_+|. \tag{A.6}$$

Combining (A.2), (A.4) and (A.6) gives us

$$\|\partial_x^l \tilde{u}(t)\|_{L^p(\mathbb{R})} \leq e^{-rt} \sum_{k=1}^{l} \frac{\|p_{l,k}\|_{L^P(\mathbb{R})}}{\min\{1, u_-^{k+1}, u_+^{k+1}\}} \leq C|u_- - u_+|e^{-rt}, \quad l \geq 1 . \tag{A.7}$$

For the estimate on $\partial_x^l \tilde{v}$ we use (2.16) and (A.3) to write

$$\tilde{v}(x,t) = h(x,t)\tilde{u}_x(x,t),$$

$$h(x,t) = \frac{\varphi_0(x) + [1-\varphi_0(x)]e^{-rt}}{r\varphi_0(x)} = \frac{1 + [\frac{1}{\varphi_0(x)} - 1]e^{-rt}}{r} . \tag{A.8}$$

Thus,

$$h_x(x,t) = \frac{e^{-rt}}{r}\frac{d}{dx}\left[\frac{1}{\varphi_0(x)}\right] = -\frac{e^{-rt}\varphi_0'(x)}{r[\varphi_0(x)]^2}. \tag{A.9}$$

From (A.1), $\varphi_0'(x)$ has a fixed sign on R, so does $h_x(x,t)$ in (A.9). Therefore, for a fixed $t \geq 0$, $h(x,t)$ monotonically connects $h(-\infty,t)$ to $h(\infty,t)$. As discussed above, both the numerator and denominator of h are positive. These give us

$$\begin{aligned}\|h(t)\|_{L^\infty(\mathbb{R})} &= \max\left\{\frac{u_- + (1-u_-)e^{-rt}}{ru_-}, \frac{u_+ + (1-u_+)e^{-rt}}{ru_+}\right\} \\ &\leq \max\left\{\frac{1}{r}, \frac{1}{ru_-}, \frac{1}{ru_+}\right\} = C. \end{aligned}\tag{A.10}$$

Besides, from (A.9) and by induction, for $l \geq 1$, we have

$$\partial_x^l h(x,t) = \frac{e^{-rt}}{r}\frac{d^l}{dx^l}\left[\frac{1}{\varphi_0(x)}\right] = -\frac{e^{-rt}}{r}\sum_{k=1}^{l}\frac{p_{l,k}(x)}{[\varphi_0(x)]^{k+1}},$$

where $p_{l,k}(x)$ are the same as those in (A.5). With (A.6), this implies

$$\|\partial_x^l h(t)\|_{L^\infty(\mathbb{R})} \leq C|u_- - u_+|e^{-rt}, \quad l \geq 1. \tag{A.11}$$

Now by (A.8), (A.10), (A.11) and (A.7), for $l \geq 0$, we have

$$\|\partial_x^l \tilde{v}(t)\|_{L^p(\mathbb{R})} \leq \sum_{k=0}^{l}\tilde{c}_{jk}\|\partial_x^{l-k}h(t)\|_{L^\infty(\mathbb{R})}\|\partial_x^k\tilde{u}_x(t)\|_{L^p(\mathbb{R})} \leq C|u_- - u_+|e^{-rt},$$

where $\tilde{c}_{jk}$ are positive constants.

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