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The Robin and Neumann problems for the nonlinear Schrödinger equation on the half-plane

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This work studies the initial-boundary value problem (ibvp) of the two-dimensional nonlinear Schrödinger equation on the half-plane with initial data in Sobolev spaces and Neumann or Robin boundary data in appropriate Bourgain spaces. It establishes well-posedness in the sense of Hadamard by using the explicit solution formula for the forced linear ibvp obtained via Fokas's unified transform, and a contraction mapping argument.

1. Introduction

We study the initial-boundary value problem (ibvp) of the nonlinear Schrödinger (NLS) equation on the half-plane with a Robin boundary condition, that is

$$\begin{aligned} &iu_t + u_{x_1x_1} + u_{x_2x_2} = \pm|u|^{\alpha-1}u, \\ &\qquad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+, \ t \in (0, T), \\ &u(x_1, x_2, 0) = u_0(x_1, x_2), \\ &(u_{x_2} + \gamma u)(x_1, 0, t) = g(x_1, t), \end{aligned} \tag{1.1}$$

where $(\alpha - 1)/2 \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. When $\gamma = 0$, this is the Neumann problem, which we examine at the end of this work. Here, we establish the local well-posedness of ibvp (1.1) for initial data u_0 in the Sobolev space of the half-plane $H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})$ and boundary data in the Bourgain-type space B^s_T suggested by the solution estimate of the reduced pure linear ibvp (see (2.4)).

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We recall that $H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})$ is defined as the restriction of the Sobolev space $H^s(\mathbb{R}^2)$ to $\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2}$ with norm

$$||f||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}=\inf\{||F||_{H^s(\mathbb{R}^2)}:F\in H^s(\mathbb{R}^2)\text{ and }F|_{\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2}}=f\}. \tag{1.2}$$

The boundary data space B^s_T, which can be thought as expressing the time regularity of the linear homogeneous problem in two dimensions, is defined by

$$B^s_T=X^{0,((2s-1)/4)}_T\cap X^{s,-(1/4)}_T, \tag{1.3}$$

where $X^{0,((2s-1)/4)}_T$ and $X^{s,-(1/4)}_T$ are the Bourgain-type spaces defined via the norm

$$||g||^2_{X^{\sigma,b}_T}=\int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^\sigma||\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{g}^{x_1}(k_1,t)||^2_{H^b(0,T)}\,\mathrm{d}k_1 \tag{1.4}$$

with $\widehat{g}^{x_1}$ denoting the Fourier transform of g with respect to x_1, i.e.

$$\widehat{g}^{x_1}(k_1,t)=\int_{x_1\in\mathbb{R}}\mathrm{e}^{-\mathrm{i}k_1x_1}g(x_1,t)\,\mathrm{d}x_1,\quad k_1\in\mathbb{R},\ t\in(0,T). \tag{1.5}$$

Furthermore, we note that the spaces $X^{\sigma,b}_T$ can be regarded as restrictions on $\mathbb{R}\times(0,T)$ of the celebrated Bourgain spaces $X^{\sigma,b}(\mathbb{R}_x\times\mathbb{R}_t)$, which are defined via the norm [1]

$$\begin{aligned}||g||^2_{X^{\sigma,b}}&=\int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^\sigma\left(1+\left|\tau+k_1^2\right|^2\right)^b|\widehat{g}(k_1,\tau)|^2\,\mathrm{d}k_1\\&=\int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^\sigma\left\|\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{g}^{x_1}(k_1,t)\right\|^2_{H^b(\mathbb{R}_t)}\,\mathrm{d}k_1.\end{aligned} \tag{1.6}$$

In addition, for $\frac{3}{2}<s<\frac{5}{2}$ the initial and boundary data satisfy the compatibility condition

$$(\partial_{x_2}u_0+\gamma u_0)(x_1,0)=g(x_1,0),\quad x_1\in\mathbb{R}. \tag{1.7}$$

Now, we are able to state the main result of this work more precisely as follows.

Theorem 1.1 (Local well-posedness). *Suppose $1<s<\frac{5}{2}$ with $s\neq\frac{3}{2}$. Then, for initial data $u_0\in H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})$ and Robin ($\gamma\neq0$) or Neumann ($\gamma=0$) boundary data $g\in B^s_T$ satisfying the compatibility condition (1.7) for $\frac{3}{2}<s<\frac{5}{2}$, the NLS ibvp (1.1) is locally well-posed in the sense of Hadamard. More precisely, for*

$$T^*=\min\left\{T,c_{s,\gamma,\alpha}\left(||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}+||g||_{B^s_T}\right)^{-2(\alpha-1)}\right\},\quad c_{s,\gamma,\alpha}>0, \tag{1.8}$$

there exists a unique solution $u\in C([0,T^*];H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2}))$, which satisfies the estimate

$$\sup_{t\in[0,T^*]}||u(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\le 2c_{s,\gamma}\left(||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}+||g||_{B^s_T}\right),\quad c_{s,\gamma}>0. \tag{1.9}$$

In addition, the data-to-solution map $\{u_0,g\}\mapsto u$ is locally Lipschitz continuous.

Theorem 1.1 completes the picture of the NLS ibvp well-posedness on the half-plane for smooth data initiated in our earlier work [2] for the case of Dirichlet data. In one spatial dimension and on the half-line, the well-posedness of the NLS ibvp by using the Fokas unified transform method was established in [3] for Dirichlet data and in [4] and [5] for Neumann and Robin data, respectively. Furthermore, concerning the ibvp for the Korteweg-de Vries (KdV) equation on the half-line, this approach for proving well-posedness has been implemented in [6] for Dirichlet data and in [7] for Neumann and Robin data. We also mention that there are two other approaches in the literature for studying the well-posedness of the ibvp for KdV and NLS on the half-line with Dirichlet data. The first approach is due to Bona *et al*. [8–10] and uses the Laplace transform in the temporal variable for analysing the linear problem (see also [11]). The second approach was developed by Colliander & Kenig for the generalized KdV on the half-line [12], and later on by Holmer for the NLS and KdV on the half-line [13,14], and is based on expressing the linear ibvp as a superposition of initial value problems. Moreover, besides [2], two other works in the literature on the well-posedness of the two-dimensional NLS equation on the half-plane are those by Ran

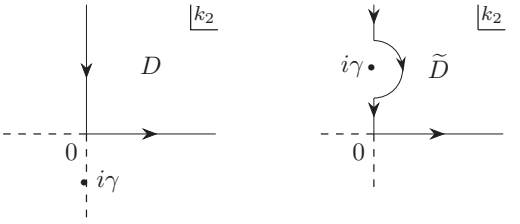

Figure 1. The regions D and $\tilde{D}$ with their positively oriented boundaries ∂D and $\partial\tilde{D}$, which assume the role of the contour $\mathcal{C}$ in formula (1.11) for $\gamma \leq 0$ and $\gamma > 0$, respectively.

et al. [15] and Audiard [16] (the latter work includes the case of Neumann data with a function space similar to the one obtained in the present work).

The well-posedness of the nonlinear ibvp (1.1) will be established via a contraction mapping argument using the unified transform solution formula and the estimates obtained for the forced linear version of that ibvp. Therefore, the first step of our approach is to derive the Fokas unified transform solution for the forced linear ibvp

$$\begin{aligned} &\mathrm{i}u_t + u_{x_1x_1} + u_{x_2x_2} = f(x_1,x_2,t) \in C([0,T];H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})),\\ &u(x_1,x_2,0) = u_0(x_1,x_2) \in H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2}),\\ &(u_{x_2} + \gamma u)(x_1,0,t) = g(x_1,t) \in B^s_T. \end{aligned} \tag{1.10}$$

This formula is given by (see §6 for an outline of its derivation)

$$\begin{aligned} u(x_1,x_2,t) &= S\big[u_0,g;f\big](x_1,x_2,t)\\ &= \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}_0(k_1,k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1\\ &\quad+\frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathcal{C}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2+\mathrm{i}\gamma}{k_2-\mathrm{i}\gamma}\widehat{u}_0(k_1,-k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1\\ &\quad-\frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathcal{C}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\int_{t'=0}^{t}\mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1\\ &\quad-\frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathcal{C}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2+\mathrm{i}\gamma}{k_2-\mathrm{i}\gamma}\int_{t'=0}^{t}\mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,-k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1\\ &\quad-\frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathcal{C}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{2k_2}{k_2-\mathrm{i}\gamma}\tilde{g}(k_1,k_1^2+k_2^2,T)\,\mathrm{d}k_2\,\mathrm{d}k_1, \end{aligned} \tag{1.11}$$

where the complex contour $\mathcal{C}$ is either ∂D (for $\gamma \leq 0$) or $\partial\tilde{D}$ (for $\gamma > 0$), as shown in figure 1, the terms $\widehat{u}_0$ and $\widehat{f}$ denote the *half-plane* Fourier transforms of u_0 and f defined according to

$$\widehat{\varphi}(k_1,k_2) = \int_{x_1\in\mathbb{R}}\int_{x_2=0}^{\infty}\mathrm{e}^{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2}\varphi(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1, \tag{1.12}$$

the transform $\tilde{g}$ is defined in terms of the boundary data g by

$$\tilde{g}(k_1,k_1^2+k_2^2,T) = \int_{t=0}^{T}\mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t}\int_{x_1\in\mathbb{R}}\mathrm{e}^{-\mathrm{i}k_1x_1}g(x_1,t)\,\mathrm{d}x_1\mathrm{d}t. \tag{1.13}$$

We note that the indented contour $\partial\tilde{D}$ as the path of integration $\mathcal{C}$ in (1.11) appears only for $\gamma > 0$. In particular, that contour is not present in the analysis of the Dirichlet problem given in [2].

The second step of our approach consists in estimating the solution (1.11) of the forced linear ibvp (1.10). The estimate derived is described in the following result.

Theorem 1.2 (Linear ibvp). *Suppose* $1 \leq s < \frac{5}{2}$ *with* $s \neq \frac{3}{2}$. *Then, the Fokas unified transform formula (1.11) defines a solution* $u = S[u_0,g;f]$ *to the forced linear Schrödinger ibvp (1.10) with initial data* $u_0 \in$

$H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})$, Robin ($\gamma\neq 0$) or Neumann ($\gamma=0$) boundary data $g\in B^s_T$ satisfying the compatibility condition (1.7) for $\frac{3}{2}<s<\frac{5}{2}$, and forcing $f\in C([0,T];H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2}))$, which satisfies the estimate

$$\sup_{t\in[0,T]}\left\|S[u_0,g;f](t)\right\|_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\leq c_{s,\gamma}\left(||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}+||g||_{B^s_T}+\sqrt{T}\sup_{t\in[0,T]}||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\right). \tag{1.14}$$

The unified transform providing the solution formula (1.11) for the forced linear ibvp studied in this work was introduced in 1997 by Fokas [17] (see also the monograph [18]). The method was originally motivated through an effort to develop an ibvp counterpart for the inverse scattering transform used for studying completely integrable *nonlinear* equations in the initial value problem setting. However, it was immediately realized that Fokas's transform had significant implications also at the level of *linear* ibvps, in particular, taking into account its applicability to linear evolution equations of arbitrary spatial order and dimension, formulated with any kind of admissible boundary conditions. In this regard, the unified transform provides the direct, *natural* analogue in the linear ibvp setting of the classical Fourier transform used for solving linear initial value problems. For additional results on the ibvp of NLS, KdV and related equations via the Fokas method; see, for example, [19–30] as well as the review articles [31,32].

The NLS equation has an extensive literature. Concerning its physical significance, it arises as a universal model in mathematical physics, e.g. in nonlinear optics [33], water waves [34,35], plasmas [36] and Bose-Einstein condensates [37]. Moreover, the cubic NLS in one spatial dimension is a prime example of a completely integrable system and can be studied via the inverse scattering transform [38]. Finally, concerning the well-posedness of the initial value problem for NLS in Sobolev spaces, we refer the reader to [1,39–48] and the references therein.

Organization. In §2, we estimate the solution to the reduced pure linear Robin problem, which has zero forcing, zero initial data and boundary data compactly supported in time. Section 3 is devoted to the estimation of the linear Schrödinger initial value problem. In §4, we combine the results of the previous two sections to prove theorem 1.2 for the forced linear ibvp (1.10) and, in turn, theorem 1.1 for the well-posedness of the nonlinear ibvp (1.1). Section 5 provides the modifications required in the proofs in the case of the Neumann problem. Finally, in §6, we give a brief derivation of the Fokas unified transform solution formula (1.11).

2. The reduced pure linear ibvp

The basis for proving the nonlinear well-posedness theorem 1.1 is provided by the linear estimate of theorem 1.2 for the forced linear ibvp (1.10). In order to establish this crucial estimate, we begin our analysis from a simplified version of problem (1.10) which involves zero forcing, zero initial data and compactly supported in time boundary data. We call this problem the *reduced pure linear ibvp*, as its non-boundary components are both zero and, furthermore, its boundary datum is reduced to the class of functions with compact support in t.

More precisely, for the Robin problem ($\gamma\neq 0$), the reduced pure linear ibvp is given by

$$\begin{aligned}&iv_t+v_{x_1x_1}+v_{x_2x_2}=0,\quad (x_1,x_2)\in\mathbb{R}\times\mathbb{R}^+,\ t\in(0,2),\\ &v(x_1,x_2,0)=0,\\ &(v_{x_2}+\gamma v)(x_1,0,t)=g(x_1,t),\quad \operatorname{supp}(g)\subset\mathbb{R}_{x_1}\times(0,2),\end{aligned} \tag{2.1}$$

where $g(x_1,t)$ is a globally defined function with compact support in t. For the Neumann problem ($\gamma=0$), the analysis of the reduced pure linear ibvp is provided in §5. We note that, since $T<1$, the interval $(0,2)$ for the t-support of g could be replaced by any fixed interval of the form $(0,a)$, $a>1$. In the case of problem (2.1), the Fokas unified transform formula (1.11) simplifies to

$$v(x_1,x_2,t)=-\frac{i}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathcal{C}}e^{ik_1x_1+ik_2x_2-i(k_1^2+k_2^2)t}\frac{2k_2}{k_2-i\gamma}\widehat{g}(k_1,-k_1^2-k_2^2)\,dk_2\,dk_1, \tag{2.2}$$

where the transform $\tilde{g}$ defined by (1.13) has now been replaced by the Fourier transform $\widehat{g}$ of g in x_1 and t since, thanks to the compact support of g in t,

$$\tilde{g}(k_1, k_1^2 + k_2^2, 2) = \int_{x_1 \in \mathbb{R}} \int_{t \in \mathbb{R}} e^{-ik_1 x_1 + i(k_1^2 + k_2^2)t} g(x_1, t)\, dt\, dx_1 = \widehat{g}(k_1, -k_1^2 - k_2^2). \tag{2.3}$$

Next, we will use the Fokas formula (2.2) in order to estimate the solution of the reduced pure linear ibvp (2.1) in the Hadamard space $C([0,2]; H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2}))$. Through this process, we will discover the correct function space for the boundary datum $g(x_1, t)$. In particular, our analysis will reveal the Bourgain spaces $X^{0,((2s-1)/4)}$ and $X^{s,-(1/4)}$ as the global analogues of the spaces $X_T^{0,((2s-1)/4)}$ and $X_T^{s,-(1/4)}$ given in the introduction for the boundary data of the non-reduced problem (1.10). The precise statement of this result is as follows.

Theorem 2.1 (Basic linear estimate for the Robin problem). *Let $s \geq 0$ and $\gamma \neq 0$. Then, the solution $v(x_1, x_2, t)$ of the reduced pure linear ibvp (2.1), as given by the Fokas formula (2.2), satisfies the Hadamard space estimate*

$$\sup_{t \in [0,2]} ||v(t)||_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})} \leq c_{s,\gamma} \left(||g||_{X^{0,((2s-1)/4)}} + ||g||_{X^{s,-(1/4)}} \right), \tag{2.4}$$

where the Bourgain spaces $X^{\sigma,b}$ are defined by (1.6).

In the remaining of this section, we prove theorem 2.1. We start from the case $\gamma < 0$, for which we provide the proof in detail, and continue to the case $\gamma > 0$, for which we give the modifications required due to the presence of the simple pole at $i\gamma$ along the positive imaginary k_2-axis.

Proof of theorem 2.1 for $\gamma < 0$. Parameterizing the contour $\mathcal{C} = \partial D$ (figure 1), we write $v = v_1 + v_2$ with

$$v_1(x_1, x_2, t) = -\frac{1}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} e^{ik_1 x_1 - k_2 x_2 - i(k_1^2 - k_2^2)t} \frac{2k_2}{k_2 - \gamma} \widehat{g}(k_1, -k_1^2 + k_2^2)\, dk_2\, dk_1 \tag{2.5}$$

and

$$v_2(x_1, x_2, t) = -\frac{i}{(2\pi)^2} \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} e^{ik_1 x_1 + ik_2 x_2 - i(k_1^2 + k_2^2)t} \frac{2k_2}{k_2 - i\gamma} \widehat{g}(k_1, -k_1^2 - k_2^2)\, dk_2\, dk_1, \tag{2.6}$$

and estimate v_1 and v_2 individually. We begin with v_2, which involves a purely oscillatory exponential and hence can be handled as a globally defined function via Plancherel's theorem for the Fourier transform. Then, we proceed to v_1, which does not make sense for $x_2 < 0$ and hence requires a different treatment via the L^2 boundedness of the Laplace transform. ■

Estimation of v_2. Since v_2 makes sense for all $(x_1, x_2) \in \mathbb{R}^2$, by the definition of the $H^s(\mathbb{R}^2)$ norm and the fact that $(1 + k_1^2 + k_2^2)^s \lesssim (1 + k_1^2)^s + (k_2^2)^s$ for any $s \in \mathbb{R}$ (we write $a \lesssim b$ if there exists $C > 0$ such that $a \leq Cb$), we have

$$\begin{aligned} ||v_2(t)||^2_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})} &\leq \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left(1 + k_1^2 + k_2^2\right)^s \frac{4k_2^2}{|k_2 - i\gamma|^2} \left|\widehat{g}(k_1, -k_1^2 - k_2^2)\right|^2 dk_2\, dk_1 \\ &\lesssim \int_{k_1 \in \mathbb{R}} \int_{k_2 = 0}^{\infty} \left[\left(1 + k_1^2\right)^s + \left(k_2^2\right)^s \right] \frac{k_2^2}{k_2^2 + \gamma^2} \left|\widehat{g}(k_1, -k_1^2 - k_2^2)\right|^2 dk_2\, dk_1. \end{aligned}$$

Furthermore, making the change of variable $k_2 = (-\tau - k_1^2)^{(1/2)}$ and breaking the resulting τ integral near and away from $-k_1^2$, we obtain

$$||v_2(t)||^2_{H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})} \lesssim \int_{k_1 \in \mathbb{R}} \int_{\tau = -\infty}^{-1-k_1^2} \left[\left(1 + k_1^2\right)^s + \left|\tau + k_1^2\right|^s \right] \frac{\left|\tau + k_1^2\right|^{(1/2)} \left|\widehat{g}(k_1, \tau)\right|^2}{\left|\tau + k_1^2\right| + \gamma^2} d\tau\, dk_1 \tag{2.7a}$$

$$+ \int_{k_1 \in \mathbb{R}} \int_{\tau = -1-k_1^2}^{-k_1^2} \left[\left(1 + k_1^2\right)^s + \left|\tau + k_1^2\right|^s \right] \frac{\left|\tau + k_1^2\right|^{(1/2)} \left|\widehat{g}(k_1, \tau)\right|^2}{\left|\tau + k_1^2\right| + \gamma^2} \Big| d\tau\, dk_1. \tag{2.7b}$$

For $|\tau+k_1^2|\geq 1$, we have $(|\tau+k_1^2|^{(1/2)})/(|\tau+k_1^2|+\gamma^2)\lesssim (1+|\tau+k_1^2|)^{-(1/2)}$ and $|\tau+k_1^2|^s\lesssim (1+|\tau+k_1^2|)^s$, $s\in\mathbb{R}$. Thus, noting also that $1+|\tau+k_1^2|\simeq (1+|\tau+k_1^2|^2)^{(1/2)}$ (we write $a\simeq b$ if $a\lesssim b$ and $b\lesssim a$), the first of the above integrals becomes

$$(2.7a)\lesssim \int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^s\int_{\tau=-\infty}^{-1-k_1^2}(1+|\tau+k_1^2|^2)^{-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1 + \int_{k_1\in\mathbb{R}}\int_{\tau=-\infty}^{-1-k_1^2}(1+|\tau+k_1^2|^2)^{(2s-1)/4}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1,\quad s\in\mathbb{R}. \tag{2.8}$$

For $|\tau+k_1^2|\leq 1$ and $\gamma\neq 0$, we have

$$\left(|\tau+k_1^2|+\gamma^2\right)^{-1}\leq \max\left\{\tfrac{1}{\gamma^2},1\right\}\left(1+|\tau+k_1^2|\right)^{-1} \tag{2.9}$$

(this is not true for $\gamma=0$, which is why the Neumann problem is treated separately in §5) and $|\tau+k_1^2|^{s+(1/2)}\leq (1+|\tau+k_1^2|)^{s+(1/2)}$ for $s\geq -\frac{1}{2}$. Thus, since $1+|\tau+k_1^2|\simeq (1+|\tau+k_1^2|^2)^{(1/2)}$, for $s\geq -\frac{1}{2}$ we obtain

$$(2.7b)\lesssim \int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^s\int_{\tau=-1-k_1^2}^{-k_1^2}\left(1+|\tau+k_1^2|^2\right)^{-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1 + \int_{k_1\in\mathbb{R}}\int_{\tau=-1-k_1^2}^{-k_1^2}\left(1+|\tau+k_1^2|^2\right)^{(2s-1)/4}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1. \tag{2.10}$$

Combining estimates (2.8) and (2.10), for $s\geq -\frac{1}{2}$, $\gamma<0$ and $t\in[0,2]$, we deduce

$$||v_2(t)||^2_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\lesssim \int_{k_1\in\mathbb{R}}\left(1+k_1^2\right)^s\int_{\tau=-\infty}^{-k_1^2}\left(1+|\tau+k_1^2|^2\right)^{-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1 + \int_{k_1\in\mathbb{R}}\int_{\tau=-\infty}^{-k_1^2}\left(1+|\tau+k_1^2|^2\right)^{(s/2)-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1. \tag{2.11}$$

Estimation of v_1. As the expression (2.5) for v_1 only makes sense for $x_2>0$, we will estimate it by employing the definition of the Sobolev norm in terms of derivatives in L^2. In particular, restricting $s\geq 0$, we have

$$||v_1(t)||^2_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}=\sum_{|\mu|\leq\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}+\sum_{|\mu|=\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_{\beta} \tag{2.12}$$

where for $x=(x_1,x_2)$, $\mu=(\mu_1,\mu_2)$, we denote $\partial_x^\mu=\partial_{x_1}^{\mu_1}\partial_{x_2}^{\mu_2}$, $|\mu|=\mu_1+\mu_2$, and for $\beta=s-\lfloor s\rfloor\in(0,1)$, we define the fractional norm

$$||v_1(t)||^2_{\beta}=\int_{x\in\mathbb{R}\times\mathbb{R}^+}\int_{y\in\mathbb{R}\times\mathbb{R}^+}\frac{|v_1(x,t)-v_1(y,t)|^2}{|x-y|^{2(1+\beta)}}\,\mathrm{d}y\,\mathrm{d}x. \tag{2.13}$$

We begin with the integer part of the norm (2.12) and, more specifically, with $||\partial_x^\mu v_1(t)||_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}$ for $|\mu|=\mu_1+\mu_2\in\mathbb{N}\cup\{0\}$ and $|\mu|\leq\lfloor s\rfloor$. Differentiating the unified transform expression (2.5) for v_1, we find

$$\partial_x^\mu v_1(x_1,x_2,t)\simeq\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty}\mathrm{e}^{\mathrm{i}k_1x_1-k_2x_2-\mathrm{i}(k_1^2-k_2^2)t}k_1^{\mu_1}k_2^{\mu_2}\frac{k_2}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1.$$

Hence, Plancherel's theorem between the x_1 and k_1 integrals implies

$$||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\simeq\int_{k_1\in\mathbb{R}}(k_1^2)^{\mu_1}\left\|\int_{k_2=0}^{\infty}\mathrm{e}^{-k_2x_2+\mathrm{i}k_2^2t}\frac{k_2^{\mu_2+1}}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\,\mathrm{d}k_2\right\|^2_{L^2(\mathbb{R}^+_{x_2})}\mathrm{d}k_1.$$

Identifying the k_2 integral as the Laplace transform of the function $\mathrm{e}^{\mathrm{i}k_2^2t}k_2^{\mu_2}(k_2/(k_2-\gamma))\widehat{g}(k_1,-k_1^2+k_2^2)$, we estimate the L^2 norm of that integral by using the fact (see [49] and lemma 3.2 in [3])

that the Laplace transform $\mathcal{L}:\varphi(k_)\mapsto\int_{k=0}^{\infty}\mathrm{e}^{-kx}\varphi(k_)\,\mathrm{d}k$ is bounded from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ with $||\mathcal{L}\{\varphi\}||_{L^2(\mathbb{R}^+)}\leq\sqrt{\pi}||\varphi||_{L^2(\mathbb{R}^+)}$. Thus, we obtain

$$||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\lesssim\int_{k_1\in\mathbb{R}}(k_1^2)^{\mu_1}\left\|\mathrm{e}^{\mathrm{i}k_2^2t}k_2^{\mu_2}\frac{k_2}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\right\|^2_{L^2(\mathbb{R}^+_{k_2})}\mathrm{d}k_1$$

and, since $(k_2-\gamma)^2\geq k_2^2+\gamma^2$ for $k_2\geq 0$ and $\gamma\leq 0$ (note that this is not true when $\gamma>0$),

$$||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\lesssim\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty}(k_1^2)^{\mu_1}(k_2^2)^{\mu_2}\frac{k_2^2}{k_2^2+\gamma^2}\left|\widehat{g}(k_1,-k_1^2+k_2^2)\right|^2\mathrm{d}k_2\,\mathrm{d}k_1.\tag{2.14}$$

Inserting this estimate in the integer part of the Sobolev norm (2.12) and recalling that $|\mu|=\mu_1+\mu_2\in\mathbb{N}\cup\{0\}$ and $|\mu|\leq\lfloor s\rfloor$, we find

$$\sum_{|\mu|\leq\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\lesssim\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty}\frac{(1+k_1^2+k_2^2)^{\lfloor s\rfloor}k_2^2\left|\widehat{g}(k_1,-k_1^2+k_2^2)\right|^2}{k_2^2+\gamma^2}\mathrm{d}k_2\,\mathrm{d}k_1,\tag{2.15}$$

where we have used the binomial theorem twice in order to compute the two sums over μ_2 and $|\mu|$. Noting that $(1+k_1^2+k_2^2)^{\lfloor s\rfloor}\simeq(1+k_1^2)^{\lfloor s\rfloor}+(k_2^2)^{\lfloor s\rfloor}$ and making the change of variable $k_2=(\tau+k_1^2)^{(1/2)}$, we handle the right-hand side of (2.15) similarly to (2.7) for v_2, i.e. by splitting the range of the τ integral near and away from $-k_1^2$. Eventually, this yields

$$\begin{aligned}\sum_{|\mu|\leq\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}&\lesssim\int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty}\left(1+k_1^2\right)^{\lfloor s\rfloor}(1+\left|\tau+k_1^2\right|^2)^{-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1\\&\quad+\int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty}\left(1+\left|\tau+k_1^2\right|^2\right)^{(2\lfloor s\rfloor-1)/4}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1\end{aligned}\tag{2.16}$$

for all $s\geq 0$ and $\gamma<0$ (as for v_2, the Neumann case $\gamma=0$ is treated separately in §5).

Having completed the estimation of the integer part of the Sobolev norm (2.12), we turn our attention to the fractional norms $||\partial_x^\mu v_1(t)||_\beta$ with $\mu_1+\mu_2=\lfloor s\rfloor\in\mathbb{N}\cup\{0\}$. Note that (2.13) can be expressed in the convenient form $||v_1(t)||^2_\beta\simeq\int_{x\in\mathbb{R}\times\mathbb{R}^+}\int_{z\in\mathbb{R}\times\mathbb{R}^+}((|v_1(x+z,t)-v_1(x,t)|^2)/(|z|^{2(1+\beta)}\,\mathrm{d}z\,\mathrm{d}x))$. Then, differentiating (2.5) and employing Plancherel's theorem for the integrals with respect to x_1 and k_1 as well as (once again) the Laplace transform boundedness in $L^2(\mathbb{R}^+)$ for the integrals with respect to x_2 and k_2, we obtain

$$||\partial_x^\mu v_1(t)||^2_\beta\lesssim\int_{k_1\in\mathbb{R}}(k_1^2)^{\mu_1}\int_{k_2=0}^{\infty}\left|k_2^{\mu_2}\frac{k_2}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\right|^2I(k_1,k_2,\beta)\,\mathrm{d}k_2\,\mathrm{d}k_1,\tag{2.17}$$

where $I(k_1,k_2,\beta)=\int_{z_1\in\mathbb{R}}\int_{z_2=0}^{\infty}(|\mathrm{e}^{\mathrm{i}k_1z_1-k_2z_2}-1|^2/(z_1^2+z_2^2)^{1+\beta})\,\mathrm{d}z_2\,\mathrm{d}z_1$. By lemma 2.2 of [2], for $\beta\in(0,1)$, we have $I(k_1,k_2,\beta)\lesssim(k_1^2+k_2^2)^\beta$. Hence, (2.17) becomes

$$||\partial_x^\mu v_1(t)||^2_\beta\lesssim\int_{k_1\in\mathbb{R}}(k_1^2)^{\mu_1}\int_{k_2=0}^{\infty}\left(k_1^2+k_2^2\right)^\beta\left|k_2^{\mu_2}\frac{k_2}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\right|^2\mathrm{d}k_2\,\mathrm{d}k_1$$

and, using the inequality $(k_2-\gamma)^2\geq k_2^2+\gamma^2$ together with the binomial theorem, we find

$$\sum_{|\mu|=\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_\beta\lesssim\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty}\left(k_1^2+k_2^2\right)^s\frac{k_2^2}{k_2^2+\gamma^2}\left|\widehat{g}(k_1,-k_1^2+k_2^2)\right|^2\mathrm{d}k_2\,\mathrm{d}k_1.$$

The term on the right-hand side can be handled like the one in (2.15) to yield

$$\begin{aligned}\sum_{|\mu|=\lfloor s\rfloor}||\partial_x^\mu v_1(t)||^2_\beta&\lesssim\int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty}\left(1+k_1^2\right)^s\left(1+\left|\tau+k_1^2\right|^2\right)^{-(1/4)}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1\\&\quad+\int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty}\left(1+\left|\tau+k_1^2\right|^2\right)^{(2s-1)/4}\left|\widehat{g}(k_1,\tau)\right|^2\mathrm{d}\tau\,\mathrm{d}k_1,\quad s\geq 0,\ \gamma<0.\end{aligned}\tag{2.18}$$

Overall, estimates (2.16) and (2.18) combined with the Sobolev norm definition (2.12) imply

$$||v_1(t)||^2_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} \lesssim \int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty} \left(1+k_1^2\right)^s \left(1+|\tau+k_1^2|^2\right)^{-(1/4)} |\widehat{g}(k_1,\tau)|^2 \,\mathrm{d}\tau\,\mathrm{d}k_1 + \int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty} \left(1+|\tau+k_1^2|^2\right)^{(2s-1)/4} |\widehat{g}(k_1,\tau)|^2 \,\mathrm{d}\tau\,\mathrm{d}k_1 \tag{2.19}$$

for all $s\geq 0$, $\gamma<0$ and $t\in[0,2]$, which together with estimate (2.11) for v_2 yields the desired Hadamard estimate (2.4) in the case $\gamma<0$.

Proof of theorem 2.1 for $\gamma>0$. We now provide the modifications necessary for proving estimate (2.4) when $\gamma>0$. Recall that the difference between the cases $\gamma<0$ and $\gamma>0$ is that, in the latter case, the complex contour of integration $\mathcal{C}$ in the unified transform formula (2.2) is given by $\partial\tilde{D}$ instead of ∂D (figure 1), so that the singularity at $k_2=\mathrm{i}\gamma$ (which for $\gamma>0$ lies along ∂D) is avoided by means of $C_{\gamma/2}(\mathrm{i}\gamma)$, which denotes the right half of the negatively oriented circle of radius $\gamma/2$ and centre at $\mathrm{i}\gamma$. Hence, for $\gamma>0$, the solution of the reduced pure linear ibvp (2.1) consists of three parts, $v=v_{1,1}+v_{1,2}+v_2$, where

$$v_{1,1}(x_1,x_2,t)\simeq \int_{k_1\in\mathbb{R}}\left[\int_{k_2=0}^{\gamma/2}+\int_{k_2=(3\gamma/2)}^{\infty}\right] \mathrm{e}^{\mathrm{i}k_1x_1-k_2x_2-\mathrm{i}(k_1^2-k_2^2)t}\frac{k_2}{k_2-\gamma}\widehat{g}(k_1,-k_1^2+k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1, \tag{2.20}$$

$$v_{1,2}(x_1,x_2,t)\simeq \int_{k_1\in\mathbb{R}}\int_{k_2\in C_{\gamma/2}(\mathrm{i}\gamma)} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2}{k_2-\mathrm{i}\gamma}\widehat{g}(k_1,-k_1^2-k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1, \tag{2.21}$$

$$v_2(x_1,x_2,t)\simeq \int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2}{k_2-\mathrm{i}\gamma}\widehat{g}(k_1,-k_1^2-k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1. \tag{2.22}$$

The term (2.22) can be estimated exactly like the term (2.6) from the case $\gamma<0$. Also, the first integral in (2.20) is similar to the term (2.5). Indeed, since $k_2\in[0,\gamma/2]$, we have $(k_2-\gamma)^2\geq\frac{1}{8}(k_2^2+\gamma^2)$. This inequality plays the role of inequality $(k_2-\gamma)^2\geq k_2^2+\gamma^2$ that was valid for $k_2\geq 0$ and $\gamma\leq 0$, and hence allows us to handle the first integral in (2.20) just like v_1 from the case $\gamma<0$, eventually obtaining estimate (2.14) and, in turn, estimate (2.16). For the second integral in (2.20), since $|k_2-\mathrm{i}\gamma|\geq\gamma/2$, we have $k_2^2/(k_2-\gamma)^2\leq 2+(2\gamma^2/(k_2-\gamma)^2)\leq 10$. We remark that this crude bound can be used for the integral over $k_2\in[3\gamma/2,\infty)$ but not for the one over $k_2\in[0,\gamma/2]$. This is because, in the former range, we can handle the term $|\tau+k_1^2|^{-(1/2)}$ arising from the change of variable $k_2=(-\tau-k_1^2)$ via the inequality $|\tau+k_1^2|^{-(1/2)}\lesssim\max\{\frac{1}{\gamma},1\}(1+|\tau+k_1^2|^2)^{-(1/4)}$, which follows from the fact that $k_2\geq 3\gamma/2$. Overall, we conclude that the term (2.20) admits the same estimate with the term v_1 appearing in the case $\gamma<0$.

The remaining component $v_{1,2}$ of the reduced pure ibvp solution, which is defined by (2.21), has not appeared before, since it involves for the first time the semicircular contour $C_{\gamma/2}(\mathrm{i}\gamma)$. This is a finite contour that stays a fixed distance $\gamma/2$ away from the singularity at $k_2=\mathrm{i}\gamma$ and so, in principle, the estimation of $v_{1,2}$ should go through without any issues. However, it turns out that some technical details are needed. The main reason for this is that the changes of variable $\tau=-k_1^2\pm k_2^2$, which have been used in all of the previous estimations, would now result in complex values for τ, thus making it difficult to relate the relevant integrals to the Bourgain norms (1.6). Thus, instead of making these changes of variables, we will exploit the boundedness of the contour $C_{\gamma/2}(\mathrm{i}\gamma)$ together with the compact support in t of the reduced pure linear ibvp datum $g(x_1,t)$ in order to estimate $v_{1,2}$ in a different way. More specifically, parametrizing $C_{\gamma/2}(\mathrm{i}\gamma)$ by $k_2=k_2(\theta)=\mathrm{i}\gamma+(\gamma/2)\,\mathrm{e}^{\mathrm{i}\theta}$, we have

$$v_{1,2}(x_1,x_2,t)\simeq \int_{k_1\in\mathbb{R}}\int_{\theta=-(\pi/2)}^{(\pi/2)} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2(\theta)x_2-\mathrm{i}(k_1^2+k_2(\theta)^2)t}k_2(\theta)\widehat{g}(k_1,-k_1^2-k_2(\theta)^2)\,\mathrm{d}\theta\,\mathrm{d}k_1.$$

We will estimate this integral by using the norm (2.12). For the integer part of that norm, we have $\partial_x^\mu=\partial_{x_1}^{\mu_1}\partial_{x_2}^{\mu_2}$ with $|\mu|=\mu_1+\mu_2\in\mathbb{N}\cup\{0\}$ and $|\mu|\leq\lfloor s\rfloor$. Thus, differentiating the above expression for $v_{1,2}$ and then using Plancherel's theorem in x_1 and k_1, Minkowski's integral inequality for the

x_2 and θ integrals, and the fact that $\int_{x_2=0}^{\infty} |e^{ik_2(\theta)x_2}|^2\,dx_2 = [\gamma(2+\sin\theta)]^{-1} = c_\gamma < \infty$, we can bound $||\partial_x^\mu v_{1,2}(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}$ by

$$\int_{k_1\in\mathbb{R}} (k_1^2)^{\mu_1}\left[\int_{\theta=-(\pi/2)}^{(\pi/2)} \left| e^{-ik_2(\theta)^2 t} k_2(\theta)^{\mu_2+1}\widehat{g}(k_1,-k_1^2-k_2(\theta)^2)\right|(2+\sin\theta)^{-\frac{1}{2}}\,d\theta\right]^2 dk_1. \tag{2.23}$$

Since $0\le\mu_1\le\lfloor s\rfloor$, the right-hand side of the above inequality will have to be controlled by the norm of the Bourgain space $X^{s,-(1/4)}$ (as opposed to $X^{0,((2s-1)/4)}$). In order to accomplish this, we will introduce a multiplier $\simeq 1$ in the above θ integral. More specifically, let

$$\psi(t) = \left.\begin{cases} e^{-t}, & t\in(0,2)\\ 0, & t\in[0,2]^c\end{cases}\right\} \Rightarrow \widehat{\psi}(\tau) = \frac{1-e^{-2(1+i\tau)}}{1+i\tau}$$

and observe that, for $k_2\in C_{\gamma/2}(i\gamma)$, we have $|\widehat{\psi}(-k_2^2)| = |(1-e^{2(ik_2^2-1)})/(1-ik_2^2)|^2 \simeq c_\gamma \simeq 1$. Indeed, since $\mathrm{Im}(k_2^2)\ge 0$ along $C_{\gamma/2}(i\gamma)$, it follows that $|1-e^{2(ik_2^2-1)}|\simeq 1$. Moreover, $|1-ik_2^2| = c_\gamma \simeq 1$ since $(1+(\gamma^4/16))^{(1/2)} \le |1-ik_2^2| \le 1+(9\gamma^2/4)$. Therefore,

$$\begin{aligned}(2.23) &\lesssim \int_{k_1\in\mathbb{R}} (k_1^2)^{\mu_1}\left[\int_{\theta=-(\pi/2)}^{\pi/2} \left|e^{-ik_2(\theta)^2 t}k_2(\theta)^{\mu_2+1}\widehat{\psi}(-k_2(\theta)^2)\right.\right.\\ &\qquad \left.\left.\cdot\,\widehat{g}(k_1,-k_1^2-k_2(\theta)^2)\right|(2+\sin\theta)^{-\frac{1}{2}}\,d\theta\right]^2 dk_1\\ &\lesssim \int_{k_1\in\mathbb{R}} (k_1^2)^{\mu_1} \sup_{\theta\in[-(\pi/2),\pi/2]} |\widehat{\psi}(-k_2(\theta)^2)\cdot\widehat{g}(k_1,-k_1^2-k_2(\theta)^2)|^2\,dk_1,\end{aligned} \tag{2.24}$$

since the θ integral remaining after taking the above supremum is bounded by some $c_{s,\gamma}<\infty$.

The supremum in (2.24) can be bounded after noting that, by the convolution property of the Fourier transform and the compact support in t of both ψ and g,

$$\widehat{\psi}(-k_2^2)\cdot\widehat{g}(k_1,-k_1^2-k_2^2) = \int_{t=0}^{4} e^{i(k_1^2+k_2^2)t}\left\{e^{-ik_1^2\cdot}\psi(\cdot) * \widehat{g}^{x_1}(k_1,\cdot)\right\}(t)\,dt.$$

Hence, by the Cauchy–Schwarz inequality, the fact that $\mathrm{Im}(k_2^2)\ge 0$ along $C_{\gamma/2}(i\gamma)$, and Plancherel's theorem, we obtain

$$|\widehat{\psi}(-k_2^2)\cdot\widehat{g}(k_1,-k_1^2-k_2^2)| \lesssim \left\|\mathcal{F}_t\left\{e^{-ik_1^2 t}\psi(t)\right\}(\tau)\cdot\widehat{g}(k_1,\tau)\right\|_{L^2(\mathbb{R}_\tau)}.$$

Then, substituting $\mathcal{F}_t\{e^{-ik_1^2 t}\psi(t)\}(\tau) = (1-e^{-2(1+i(\tau+k_1^2))})/(1+i(\tau+k_1^2))$ and noting that for $\tau, k_1\in\mathbb{R}$, we have $|1-e^{-2(1+i(\tau+k_1^2))}|\le 1+e^{-2}$, we find

$$|\widehat{\psi}(-k_2^2)\cdot\widehat{g}(k_1,-k_1^2-k_2^2)|^2 \lesssim \int_{\tau\in\mathbb{R}} (1+(\tau+k_1^2)^2)^{-1}|\widehat{g}(k_1,\tau)|^2\,d\tau.$$

Inserting this estimate in (2.24), we deduce

$$\begin{aligned}||\partial_x^\mu v_{1,2}(t)||^2_{L^2(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} &\lesssim \int_{k_1\in\mathbb{R}} (k_1^2)^{\mu_1}\int_{\tau\in\mathbb{R}} (1+(\tau+k_1^2)^2)^{-1}|\widehat{g}(k_1,\tau)|^2\,d\tau\,dk_1\\ &\simeq ||g||^2_{X^{\mu_1,-1}} \le ||g||^2_{X^{\lfloor s\rfloor,-1}} \le ||g||^2_{X^{s,-(1/4)}}\end{aligned} \tag{2.25}$$

for all $t\in(0,2)$ and $|\mu| = \mu_1+\mu_2\in\mathbb{N}\cup\{0\}$ with $|\mu|\le\lfloor s\rfloor$, which gives the desired estimate for the integer part of the norm (2.12).

The fractional norm $||\partial_x^\mu v_{1,2}(t)||_\beta$ also satisfies (2.25) for each $|\mu| = \mu_1+\mu_2 = \lfloor s\rfloor\in\mathbb{N}\cup\{0\}$. This can be shown via the same steps with the integer part above together with the bound

$$\int_{z_1\in\mathbb{R}}\int_{z_2=0}^{\infty} \frac{\left|e^{ik_1 z_1 + ik_2 z_2}-1\right|^2}{(z_1^2+z_2^2)^{1+\beta}}\,dz_2\,dz_1 \lesssim \max\left\{1,(k_1^2)^\beta\right\},$$

which holds for all $k_1 \in \mathbb{R}$, $k_2 \in C_{\gamma/2}(\mathrm{i}\gamma)$ and $\beta \in (0,1)$. This bound can be proved similarly to lemma 2.2 in [2] by considering the cases $|k_1| \geq 1$ and $|k_1| \leq 1$ separately and then employing the inequality $|\mathrm{e}^z - 1| \leq |z|\,\mathrm{e}^{|z|}$, $z \in \mathbb{C}$. This completes the proof of theorem 2.1 for $\gamma > 0$. ■

3. Estimates for the linear Schrödinger equation on the plane

In this section, we establish various estimates for the linear Schrödinger initial value problem (ivp) on the plane. These results will be combined in §4 with those of §2 on the reduced pure linear ibvp in order to prove theorem 1.2 for the forced linear ibvp (1.10), which is the basis for proving the well-posedness theorem 1.1 for the NLS ibvp (1.1). In particular, besides the Hadamard solution space, we will obtain estimates for the linear ivp in the Bourgain-type spaces (1.4) motivated by theorem 2.1.

We begin with the homogeneous ivp

$$\begin{aligned} &\mathrm{i}U_t + U_{x_1x_1} + U_{x_2x_2} = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ &U(x,0) = U_0(x) \in H^s(\mathbb{R}^2), \end{aligned} \tag{3.1}$$

whose solution is given by

$$U(x,t) = S\big[U_0;0\big](x,t) = \frac{1}{(2\pi)^2}\int_{k\in\mathbb{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x - \mathrm{i}k^2 t}\widehat{U}_0(k)\,\mathrm{d}k, \tag{3.2}$$

where $\widehat{U}_0$ denotes the Fourier transform of U_0 on the whole plane, $\widehat{U}_0(k) = \int_{x\in\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k\cdot x}U_0(x)\,\mathrm{d}x$, with $k = (k_1,k_2) \in \mathbb{R}^2$, $k\cdot x = k_1x_1 + k_2x_2$ and $k^2 = k_1^2 + k_2^2$.

Theorem 3.1 (Homogeneous linear ivp). *The solution $U = S[U_0;0]$ to the linear Schrödinger ivp (3.1), given by the Fourier transform formula (3.2), satisfies the isometry relation*

$$\sup_{t\in[0,T]} ||U(t)||_{H^s(\mathbb{R}^2_x)} = ||U_0||_{H^s(\mathbb{R}^2_x)}, \quad s \in \mathbb{R}, \tag{3.3}$$

as well as the Bourgain-type estimates

$$\sup_{x_2\in\mathbb{R}} ||(U_{x_2} + \gamma U)(x_2)||_{X_T^{0,((2s-1)/4)}} \leq c_s||U_0||_{H^s(\mathbb{R}^2_x)}, \quad s \geq 1, \tag{3.4}$$

$$\sup_{x_2\in\mathbb{R}} ||(U_{x_2} + \gamma U)(x_2)||_{X_T^{s,-(1/4)}} \leq c_{s,\gamma}||U_0||_{H^s(\mathbb{R}^2_x)}, \quad s \geq 0. \tag{3.5}$$

Proof. The isometry relation (3.3) follows directly from formula (3.2) and the definition of the Sobolev norm. Estimate (3.4) can be deduced by estimate (3.5) in [2], according to which $\sup_{x_2\in\mathbb{R}} ||U(x_2)||_{X_T^{0,(2s+1)/4}} \leq c_s||U_0||_{H^s(\mathbb{R}^2_x)}$, $s \geq 0$. In particular, observe that $U_{x_2} = S[\partial_{x_2}U_0;0]$ so it suffices to employ that estimate with U_{x_2} in place of U and with s replaced by $s-1$.

Concerning estimate (3.5), we note that (3.5) in [2] additionally implies the estimate $\sup_{x_2\in\mathbb{R}} ||U(x_2)||_{X_T^{s,(1/4)}} \leq c_s||U_0||_{H^s(\mathbb{R}^2_x)}$ for $s \geq 0$, which takes care of the second term in the norm on the left-hand side of (3.5). Furthermore, denoting by $\dot{H}^{-(1/4)}$ the homogeneous counterpart of the Sobolev space $H^{-(1/4)}$ and using the fact that $\dot{H}^{-(1/4)} \subset H^{-(1/4)}$, we have

$$||U_{x_2}(x_2)||^2_{X_T^{s,-(1/4)}} \leq \int_{k_1\in\mathbb{R}} \big(1+k_1^2\big)^s ||\mathrm{e}^{\mathrm{i}k_1^2\cdot}\widehat{U}^{x_1}_{x_2}(k_1,x_2,\cdot)||^2_{\dot{H}^{-(1/4)}(\mathbb{R}_t)}\,\mathrm{d}k_1. \tag{3.6}$$

Then, we use the solution formula (3.2) together with the change of variable $\tau = -k_2^2$ to write

$$\mathrm{e}^{\mathrm{i}k_1^2 t}\widehat{U}^{x_1}_{x_2}(k_1,x_2,t) = \frac{1}{4i\pi}\int_{\tau=-\infty}^{0} \mathrm{e}^{\mathrm{i}\tau t}\left[\mathrm{e}^{-\mathrm{i}\sqrt{-\tau}x_2}\widehat{U}_0\left(k_1,-\sqrt{-\tau}\right) - \mathrm{e}^{\mathrm{i}\sqrt{-\tau}x_2}\widehat{U}_0\left(k_1,\sqrt{-\tau}\right)\right]\mathrm{d}\tau.$$

Hence, by the usual definition of the $\dot{H}^{-(1/4)}(\mathbb{R})$ norm, we find

$$||\mathrm{e}^{\mathrm{i}k_1^2\cdot}\widehat{U}^{x_1}_{x_2}(k_1,x_2,\cdot)||^2_{\dot{H}^{-(1/4)}(\mathbb{R}_t)} \lesssim \int_{\tau=-\infty}^{0} |\tau|^{-(1/2)}\left[\left|\widehat{U}_0(k_1,-\sqrt{-\tau})\right|^2 + \left|\widehat{U}_0(k_1,\sqrt{-\tau})\right|^2\right]\mathrm{d}\tau$$

and, changing variable from τ back to k_2, we obtain

$$||\mathrm{e}^{\mathrm{i}k_1^2\cdot}\widehat{U}_{x_2}^{x_1}(k_1,x_2,\cdot)||^2_{\dot{H}^{-(1/4)}(\mathbb{R}_t)} \lesssim \int_{k_2\in\mathbb{R}} \left|\widehat{U}_0(k_1,k_2)\right|^2 \mathrm{d}k_2.$$

In turn, (3.6) yields $||U_{x_2}(x_2)||^2_{X_T^{s,-(1/4)}} \lesssim \int_{k_1\in\mathbb{R}}(1+k_1^2)^s \int_{k_2\in\mathbb{R}} |\widehat{U}_0(k_1,k_2)|^2 \,\mathrm{d}k_2\,\mathrm{d}k_1 \leq ||U_0||^2_{H^s(\mathbb{R}^2_x)}$ with the second inequality due to the fact that $s \geq 0$. ■

We proceed to the forced linear ivp with zero initial data,

$$\mathrm{i}W_t + W_{x_1x_1} + W_{x_2x_2} = F(x,t), \quad x=(x_1,x_2)\in\mathbb{R}^2, \quad t\in\mathbb{R}, \tag{3.7a}$$

$$W(x,0)=0, \quad x\in\mathbb{R}^2, \tag{3.7b}$$

whose solution is given by

$$W(x,t) = S\big[0;F\big](x,t) = -\frac{\mathrm{i}}{(2\pi)^2}\int_{t'=0}^{t}\int_{k\in\mathbb{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x - \mathrm{i}k^2(t-t')}\widehat{F}^x(k,t')\,\mathrm{d}k\,\mathrm{d}t', \tag{3.8}$$

where $\widehat{F}^x(k,t) = \int_{x\in\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}k\cdot x}F(x,t)\,\mathrm{d}x$ is the spatial Fourier transform of F on the whole plane.

Theorem 3.2 (Forced linear ivp with zero initial data). *The solution $W = S[0;F]$ to the forced linear Schrödinger ivp (3.7), given by the Fourier transform formula (3.8), satisfies the Hadamard space estimate*

$$\sup_{t\in[0,T]} ||W(t)||_{H^s(\mathbb{R}^2_x)} \leq T \sup_{t\in[0,T]} ||F(t)||_{H^s(\mathbb{R}^2_x)}, \quad s\in\mathbb{R}, \tag{3.9}$$

and the Bourgain-type estimates

$$\sup_{x_2\in\mathbb{R}} ||(W_{x_2}+\gamma W)(x_2)||_{X_T^{0,((2s-1)/4)}} \leq c_s\sqrt{T} \sup_{t\in[0,T]} ||F(t)||_{H^s(\mathbb{R}^2_x)}, \quad 1\leq s\leq \tfrac{5}{2},\ s\neq\frac{3}{2}, \tag{3.10}$$

$$\sup_{x_2\in\mathbb{R}} ||(W_{x_2}+\gamma W)(x_2)||_{X_T^{s,-(1/4)}} \leq c_{s,\gamma}\sqrt{T} \sup_{t\in[0,T]} ||F(t)||_{H^s(\mathbb{R}^2_x)}, \quad s\geq 0. \tag{3.11}$$

Before giving the proof of the above results, we remark that estimates (3.4), (3.5), (3.10) and (3.11) along with estimate (2.4) for the reduced pure linear ibvp confirm that the space B_T^s defined by (1.3) is the correct space for the boundary data g of the forced linear ibvp (1.10) (and hence of the nonlinear ibvp (1.1)).

Proof. Estimate (3.9) is (3.21) from [2]. Estimate (3.10) follows from (3.25) in [2], which reads

$$\sup_{x_2\in\mathbb{R}} ||W(x_2)||_{X_T^{0,(2s+1)/4}} \leq c_s\sqrt{T} \sup_{t\in[0,T]} ||F(t)||_{H^s(\mathbb{R}^2_x)}, \quad 0\leq s\leq \frac{3}{2},\ s\neq\frac{1}{2}, \tag{3.12}$$

after noting that $W_{x_2} = S[0;F_{x_2}]$ and employing (3.12) for W and W_{x_2} with $s-1$ instead of s.

Concerning estimate (3.11), we first note that (3.26) in [2] yields

$$\sup_{x_2\in\mathbb{R}} ||W(x_2)||_{X_T^{s,-(1/4)}} \leq \sup_{x_2\in\mathbb{R}} ||W(x_2)||_{X_T^{s,1/4}} \lesssim \sqrt{T} \sup_{t\in[0,T]} ||F(t)||_{H^s(\mathbb{R}^2_x)}, \quad s\geq 0. \tag{3.13}$$

Moreover, as in the proof of theorem 3.1, we have

$$||W_{x_2}(x_2)||^2_{X_T^{s,-(1/4)}} \leq \int_{k_1\in\mathbb{R}} \left(1+k_1^2\right)^s ||\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{W}_{x_2}^{x_1}(k_1,x_2,t)||^2_{\dot{H}^{-(1/4)}(0,T)}\,\mathrm{d}k_1. \tag{3.14}$$

For $k_1,x_2\in\mathbb{R}$ and $t\in[0,T]$, formula (3.8) implies $\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{W}_{x_2}^{x_1}(k_1,x_2,t) = R(k_1,x_2,t)$, where

$$R(k_1,x_2,t) = -\frac{\mathrm{i}}{2\pi}\int_{t'=0}^{t} \chi_{[0,T]}(t')\,\mathrm{e}^{\mathrm{i}k_1^2t'} \int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_2x_2 - \mathrm{i}k_2^2(t-t')}\widehat{\partial_{x_2}F}^x(k_1,k_2,t')\,\mathrm{d}k_2\,\mathrm{d}t' \tag{3.15}$$

where $\chi_{[0,T]}$ denotes the characteristic function of the interval $[0,T]$. The important observation is that, for each $k_1 \in \mathbb{R}$, $R(k_1,x_2,t)$ satisfies the one-dimensional ivp

$$\begin{aligned} &iR_t + R_{x_2x_2} = \chi_{[0,T]}(t)\, e^{ik_1^2 t}\widehat{\partial_{x_2}F^{x_1}}(k_1,x_2,t), \quad x_2 \in \mathbb{R},\ t \in \mathbb{R}, \\ &R(k_1,x_2,0) = 0, \quad x_2 \in \mathbb{R}. \end{aligned} \tag{3.16}$$

In this connection, from the proof of lemma 11 in [13], we have

$$\sup_{x_2\in\mathbb{R}} ||R(k_1,x_2)||_{\dot{H}^{-(1/4)}(\mathbb{R}_t)} \le c_s ||\chi_{[0,T]}(t)\, e^{ik_1^2 t}\widehat{\partial_{x_2}F^{x_1}}(k_1,x_2,t)||_{L^1(\mathbb{R}_t;\dot{H}^{-1}(\mathbb{R}_{x_2}))}. \tag{3.17}$$

Combining this bound with (3.14) and Minkowski's integral inequality, we find

$$\sup_{x_2\in\mathbb{R}} ||W_{x_2}(x_2)||_{X_T^{s,-(1/4)}} \lesssim \int_{t=0}^{T} \left(\int_{k_1\in\mathbb{R}} (1+k_1^2)^s \int_{k_2\in\mathbb{R}} \left|\widehat{F^x}(k_1,k_2,t)\right|^2 dk_2\, dk_1 \right)^{\frac{1}{2}} dt,$$

which completes the proof of estimate (3.11) in view of $(1+k_1^2)^s \le (1+k_1^2+k_2^2)^s$ for $s \ge 0$. ■

4. The forced linear ibvp and proof of theorems 1.2 and 1.1

In this section, we combine the estimates for the reduced pure linear ibvp (theorem 2.1) and the homogeneous and forced linear ivps (theorems 3.1 and 3.2) in order to prove theorem 1.2 for the forced linear ibvp (1.10) and the Hadamard well-posedness theorem 1.1 for the nonlinear ibvp (1.1). For this purpose, we decompose ibvp (1.10) into component problems by using the superposition principle and suitable extensions of the initial and boundary data.

Decomposition into simpler problems. Let $U_0 \in H^s(\mathbb{R}_x^2)$ and $F(t) \in H^s(\mathbb{R}_x^2)$ be, respectively, extensions of the initial datum $u_0 \in H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)$ and the forcing $f(t) \in H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)$ of ibvp (1.10) such that

$$\begin{aligned} &||U_0||_{H^s(\mathbb{R}_x^2)} \le 2||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)}, \\ &||F(t)||_{H^s(\mathbb{R}_x^2)} \le 2||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)}, \quad t \in [0,T]. \end{aligned} \tag{4.1}$$

Then, thanks to linearity, we can express the solution $S[u_0,g;f]$ of ibvp (1.10) in the form

$$S[u_0,g;f] = S[U_0;0]\big|_{x_2>0} + S[0;F]\big|_{x_2>0} + S[0,\psi_1;0] + S[0,\psi_2;0] \tag{4.2}$$

where, for U_0 and F as chosen above, $S[U_0;0]$ solves the homogeneous linear ivp (3.1), $S[0;F]$ solves the forced ivp with zero data (3.7) and, for boundary data ψ_1 and ψ_2 given by

$$\begin{aligned} &\psi_1(x_1,t) = g(x_1,t) - (\partial_{x_2}S[U_0;0] + \gamma S[U_0;0])(x_1,0,t), \\ &\psi_2(x_1,t) = -(\partial_{x_2}S[0;F] + \gamma S[0;F])(x_1,0,t), \end{aligned} \tag{4.3}$$

and $S[0,\psi_1;0]$ and $S[0,\psi_2;0]$ solve the *pure linear ibvp*

$$\begin{aligned} &iu_t + u_{x_1x_1} + u_{x_2x_2} = 0, \quad (x_1,x_2) \in \mathbb{R}\times\mathbb{R}^+,\ t \in (0,T), \\ &u(x_1,x_2,0) = 0, \\ &(u_{x_2} + \gamma u)(x_1,0,t) = \psi(x_1,t), \end{aligned} \tag{4.4}$$

with $\psi = \psi_1$ and $\psi = \psi_2$, respectively. Note that, thanks to theorems 3.1 and 3.2 and the extension inequalities (4.1), the boundary data ψ_1 and ψ_2 belong to the space B_T^s with

$$||\psi_1||_{B_T^s} \lesssim ||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)} + ||g||_{B_T^s} \quad \text{and} \quad ||\psi_2||_{B_T^s} \lesssim \sqrt{T} \sup_{t\in[0,T]} ||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}^+)}. \tag{4.5}$$

Also, by the compatibility condition (1.7) and the time regularity of the linear ivp established in theorems 3.1 and 3.2, for $\frac{3}{2} < s < \frac{5}{2}$, we have

$$\psi_1(x_1,0) = \psi_2(x_1,0) = 0, \quad x_1 \in \mathbb{R}. \tag{4.6}$$

In view of (3.3), (3.9) and (4.1), the decomposition (4.2) implies

$$\sup_{t\in[0,T]}||S\big[u_0,g;f\big](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}\le||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}+T\sup_{t\in[0,T]}||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}$$
$$+\sup_{t\in[0,T]}||S\big[0,\psi_1;0\big](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}$$
$$+\sup_{t\in[0,T]}||S\big[0,\psi_2;0\big](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}. \tag{4.7}$$

Next, by relating the pure linear ibvp (4.4) with the *reduced* pure linear ibvp (2.1), we will deduce estimates for the Hadamard norms of $S[0,\psi_1;0]$ and $S[0,\psi_2;0]$.

Extension of the boundary data. For $1\le s<\frac{3}{2}$, given $\psi\in B^s_T$ define h via its x_1-Fourier transform

$$\widehat{h}^{x_1}(k_1,t)=\begin{cases}\widehat{\psi}^{x_1}(k_1,t), & t\in(0,T),\\ 0, & t\in[0,T]^c,\end{cases}\qquad \text{a.e. } k_1\in\mathbb{R}. \tag{4.8}$$

Note that $\mathrm{supp}(h)\subset\mathbb{R}_{x_1}\times(0,2)$ since for the purpose of local well-posedness we take $T<1$. Also, as we show below, $h\in X^{0,((2s-1)/4)}\cap X^{s,-(1/4)}$. Indeed, for each $k_1\in\mathbb{R}$ let $\varphi(t)=\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{\psi}^{x_1}(k_1,t)$ and observe that $\psi\in B^s_T$ implies $\varphi\in H^{(2s-1)/4}(0,T)$ for a.e. $k_1\in\mathbb{R}$. Then, by theorem 11.4 of [50], the extension Φ_0 of φ by zero outside $(0,T)$ satisfies

$$||\Phi_0||_{H^{(2s-1)/4}(\mathbb{R}_t)}\le c_s||\varphi||_{H^{(2s-1)/4}(0,T)}, \tag{4.9}$$

where $c_s=c(s)$ is independent of φ. Furthermore, since $\varphi\in H^{-(1/4)}(0,T)$, there exists an extension $\Phi_1\in H^{-(1/4)}(\mathbb{R}_t)$ such that

$$||\Phi_1||_{H^{-(1/4)}(\mathbb{R}_t)}\le 2||\varphi||_{H^{-(1/4)}(0,T)}.$$

Then, noting that $\chi_{(0,T)}\Phi_1=\Phi_0$ and employing proposition 3.5 of [51] (see also lemma 4.2 in [14]), we infer

$$||\Phi_0||_{H^{-(1/4)}(\mathbb{R}_t)}=||\chi_{(0,T)}\Phi_1||_{H^{-(1/4)}(\mathbb{R}_t)}\le c||\Phi_1||_{H^{-(1/4)}(\mathbb{R}_t)}\le 2c||\varphi||_{H^{-(1/4)}(0,T)} \tag{4.10}$$

for some universal constant c. Therefore, since $\widehat{h}^{x_1}(k_1,t)=\mathrm{e}^{-\mathrm{i}k_1^2t}\Phi_0(k_1,t)$, by the definition of the Bourgain norms (1.6), we find

$$\begin{aligned}||h||^2_{X^{0,((2s-1)/4)}}&=\int_{k_1\in\mathbb{R}}||\Phi_0(k_1)||^2_{H^{(2s-1)/4}(\mathbb{R}_t)}\,\mathrm{d}k_1\\ &\lesssim\int_{k_1\in\mathbb{R}}||\varphi(k_1)||^2_{H^{(2s-1)/4}(0,T)}\,\mathrm{d}k_1=||\psi||^2_{X^{0,((2s-1)/4)}_T},\\ ||h||^2_{X^{s,-(1/4)}}&=\int_{k_1\in\mathbb{R}}\big(1+k_1^2\big)^s||\Phi_0(k_1)||^2_{H^{-(1/4)}(\mathbb{R}_t)}\,\mathrm{d}k_1\\ &\lesssim\int_{k_1\in\mathbb{R}}\big(1+k_1^2\big)^s||\varphi(k_1)||^2_{H^{-(1/4)}(0,T)}\,\mathrm{d}k_1=||\psi||^2_{X^{s,-(1/4)}_T},\end{aligned} \tag{4.11}$$

thereby concluding that $h\in X^{0,((2s-1)/4)}\cap X^{s,-(1/4)}$.

For $\frac{3}{2}<s<\frac{5}{2}$, suppose $\psi\in B^s_T$ with $\psi(x_1,0)=0$, $x_1\in\mathbb{R}$ (note that this equality holds for ψ_1,ψ_2 in view of (4.6)). Then, $\varphi(t)=\mathrm{e}^{\mathrm{i}k_1^2t}\widehat{\psi}^{x_1}(k_1,t)$ belongs to $H^{(2s-1)/4}(0,T)$ for a.e. $k_1\in\mathbb{R}$ and $\varphi(0)=0$. Let $\phi\in H^{(2s-1)/4}(\mathbb{R})$ be an extension of φ such that $||\phi||_{H^{(2s-1)/4}(\mathbb{R}_t)}\le 2||\varphi||_{H^{(2s-1)/4}(0,T)}$. Then, for $\vartheta\in C^\infty_0(\mathbb{R})$ equal to 1 on $[-1,1]$ and decaying smoothly so that it equals zero on $(-2,2)^c$, the function $\vartheta(t)\phi(t)$ belongs to $H^{(2s-1)/4}_0(0,2)$. Thus, by theorem 11.4 of [50], the extension Φ_0 of $\vartheta\phi$ by zero outside $(0,2)$ satisfies $||\Phi_0||_{H^{(2s-1)/4}(\mathbb{R}_t)}\le c_s||\vartheta\phi||_{H^{(2s-1)/4}(0,2)}\lesssim||\varphi||_{H^{(2s-1)/4}(0,T)}$, where we have also used the fact $||\vartheta f||_{H^b(\mathbb{R})}\le c(\vartheta,b)||f||_{H^b(\mathbb{R})}$, $b\in\mathbb{R}$. Furthermore, letting $\Phi_1\in H^{-(1/4)}(\mathbb{R}_t)$ be an extension of $\vartheta\phi$ outside $(0,2)$ such that $||\Phi_1||_{H^{-(1/4)}(\mathbb{R}_t)}\le 2||\vartheta\phi||_{H^{-(1/4)}(0,2)}$ and noting that $\chi_{(0,2)}\Phi_1=\Phi_0$, we employ lemma 4.2 of [14] to infer similarly to (4.10) that $||\Phi_0||_{H^{-(1/4)}(\mathbb{R}_t)}\le 2c||\vartheta\phi||_{H^{-(1/4)}(0,2)}\lesssim||\varphi||_{H^{-(1/4)}(0,T)}$. Hence, defining h via its x_1-Fourier transform as $\widehat{h}^{x_1}(k_1,t)=\mathrm{e}^{-\mathrm{i}k_1^2t}\Phi_0(t)$ a.e. $k_1\in\mathbb{R}$, we have $h=\psi$ on $(0,T)$, $\mathrm{supp}(h)\subset\mathbb{R}_{x_1}\times(0,2)$ and $h\in X^{0,((2s-1)/4)}\cap X^{s,-(1/4)}$ with inequalities (4.11) in place.

Proof of theorem 1.2. The extension h of ψ defined above meets all of the requirements for serving as boundary datum in the reduced pure linear ibvp (2.1). Thus, we employ estimate (2.4) along with inequalities (4.11) and (4.5) to deduce, for $\psi = \psi_1$ and $\psi = \psi_2$, respectively,

$$\begin{aligned} \sup_{t\in[0,T]} ||S[0,\psi_1;0](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} &\lesssim ||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} + ||g||_{B^s_T}, \\ \sup_{t\in[0,T]} ||S[0,\psi_2;0](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} &\lesssim \sqrt{T} \sup_{t\in[0,T]} ||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}. \end{aligned} \tag{4.12}$$

This completes the estimation of the two pure linear ibvps in (4.7) and hence implies the desired estimate (1.14) of theorem 1.2 for $\gamma \neq 0$. See §5 for the modifications needed when $\gamma = 0$. ■

Proof of theorem 1.1. Along the lines of the argument presented in [2], the forced linear ibvp estimate (1.14) can be combined with the algebra property in $H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2})$ (which is valid for $s > 1$) to show that the iteration map $\Phi : u \mapsto S[u_0, g; \pm|u|^{\alpha-1}u]$ is a contraction in a ball inside the Hadamard space $C([0,T^*]; H^s(\mathbb{R}_{x_1} \times \mathbb{R}^+_{x_2}))$ for $1 < s < \frac{5}{2}$, $s \neq \frac{3}{2}$, with lifespan T^* given by (1.8). This amounts to local Hadamard well-posedness for the NLS ibvp (1.1) as stated in theorem 1.1, with a unique solution to (1.1) understood in the sense of the integral equation $u = S[u_0, g; \pm|u|^{\alpha-1}u]$. We note that the assumption $(\alpha-1)/2 \in \mathbb{N}$ allows us to express the difference $|u_1|^{\alpha-1}u_1 - |u_2|^{\alpha-1}u_2$ in the form $(u_1\bar{u}_1)^{(\alpha-1)/2} - (u_2\bar{u}_2)^{(\alpha-1)/2}$ when proving the contraction inequality. ■

5. The Neumann problem

In this section, we provide the modifications needed in the case of Neumann data ($\gamma = 0$). We begin with the reduced pure linear ibvp and then proceed to the proof of theorem 1.2 for the forced linear ibvp when $\gamma = 0$. We note that the proof of theorem 1.1 for the nonlinear problem does not require any modifications from the case $\gamma \neq 0$, since it is based solely on theorem 1.2, which (as we shall show below) holds for both $\gamma \neq 0$ and $\gamma = 0$.

The reduced pure linear ibvp. In the Neumann case $\gamma = 0$, the reduced pure linear ibvp (2.1) becomes

$$\begin{aligned} &\mathrm{i}v_t + v_{x_1x_1} + v_{x_2x_2} = 0, \quad (x_1,x_2) \in \mathbb{R}\times\mathbb{R}^+, \ t \in (0,2), \\ &v(x_1,x_2,0) = 0, \\ &v_{x_2}(x_1,0,t) = g(x_1,t), \quad \operatorname{supp}(g) \subset \mathbb{R}_{x_1} \times (0,2). \end{aligned} \tag{5.1}$$

For this problem, the unified transform solution formula (1.11) takes the simple form

$$v(x_1,x_2,t) = -\frac{\mathrm{i}}{(2\pi)^2} \int_{k_1\in\mathbb{R}} \int_{k_2\in\partial D} \mathrm{e}^{\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2 - \mathrm{i}(k_1^2+k_2^2)t} 2\widehat{g}(k_1, -k_1^2 - k_2^2)\, \mathrm{d}k_2\, \mathrm{d}k_1, \tag{5.2}$$

where the contour ∂D is the boundary of the first quadrant of the complex k_2-plane as shown in figure 1 and $\widehat{g}$ denotes the Fourier transform (2.3). Using formula (5.2), we will prove the following Neumann analogue of theorem 2.1.

Theorem 5.1 (Basic linear estimate for the Neumann problem). *Let $s \geq 0$. Then, the solution $v(x_1,x_2,t)$ of the reduced pure linear ibvp (5.1), as given by formula (5.2), satisfies the Hadamard estimate*

$$\sup_{t\in[0,2]} ||v(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} \leq c_s\big(||g||_{\dot{X}^{0,((2s-1)/4)}} + ||g||_{\dot{X}^{s,-(1/4)}}\big). \tag{5.3}$$

Remark 5.2. In the above theorem, the homogeneous-in-time Bourgain space $\dot{X}^{\sigma,b}$ is defined via the norm

$$\begin{aligned} ||g||^2_{\dot{X}^{\sigma,b}} &= \int_{k_1\in\mathbb{R}} \left(1+k_1^2\right)^{\sigma} \left|\tau + k_1^2\right|^{2b} |\widehat{g}(k_1,\tau)|^2\, \mathrm{d}k_1 \\ &= \int_{k_1\in\mathbb{R}} \left(1+k_1^2\right)^{\sigma} ||\mathrm{e}^{\mathrm{i}k_1^2 t}\widehat{g^{x_1}}(k_1,t)||^2_{\dot{H}^b(\mathbb{R}_t)}\, \mathrm{d}k_1, \end{aligned}$$

where $\dot{H}^b$ denotes the homogeneous Sobolev space. Importantly, for $s \geq \frac{1}{2}$ the space $\dot{X}^{0,((2s-1)/4)}$ *can* be replaced in (5.3) by the nonhomogeneous space $X^{0,((2s-1)/4)}$. However, this is *not* possible for the space $\dot{X}^{s,-(1/4)}$, since the homogeneous weight is now independent of s.

Proof. Parameterizing ∂D along the positive imaginary and real axes, we write $v = v_1 + v_2$ with

$$v_1(x_1,x_2,t) = -\frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty} \mathrm{e}^{\mathrm{i}k_1x_1 - k_2x_2 - \mathrm{i}(k_1^2-k_2^2)t}\, 2\widehat{g}(k_1,-k_1^2+k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1,$$

$$v_2(x_1,x_2,t) = -\frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2=0}^{\infty} \mathrm{e}^{\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2 - \mathrm{i}(k_1^2+k_2^2)t}\, 2\widehat{g}(k_1,-k_1^2-k_2^2)\,\mathrm{d}k_2\,\mathrm{d}k_1.$$

These expressions are similar to (2.5) and (2.6) for the Robin problem but without the fractions $k_2/(k_2-\gamma)$ and $k_2/(k_2-\mathrm{i}\gamma)$, respectively. Thus, with the exception of inequality (2.9), which was the key for transitioning from homogeneous to non-homogeneous weights in the Robin case, we can follow the same steps as in the proof of theorem 2.1 to obtain the following analogues of (2.11) and (2.19)

$$\begin{aligned}
||v_2(t)||^2_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} &\lesssim \int_{k_1\in\mathbb{R}}\int_{\tau=-\infty}^{-k_1^2} \left(1+k_1^2\right)^s \left|\tau+k_1^2\right|^{-(1/2)} |\widehat{g}(k_1,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}k_1 \\
&\quad + \int_{k_1\in\mathbb{R}}\int_{\tau=-\infty}^{-k_1^2} \left|\tau+k_1^2\right|^{s-(1/2)} |\widehat{g}(k_1,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}k_1, \quad s\in\mathbb{R}, \\
||v_1(t)||^2_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} &\lesssim \int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty} \left(1+k_1^2\right)^s (\tau+k_1^2)^{-(1/2)} |\widehat{g}(k_1,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}k_1 \\
&\quad + \int_{k_1\in\mathbb{R}}\int_{\tau=-k_1^2}^{\infty} (\tau+k_1^2)^{s-(1/2)} |\widehat{g}(k_1,\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}k_1, \quad s\geq 0.
\end{aligned}$$

These two estimates combine to imply the desired estimate (5.3). ∎

Proof of theorem 1.2 when $\gamma = 0$. Having established theorem 5.1 for the reduced pure linear ibvp in the Neumann case, we shall now combine this result with inequality (4.7) in order to establish estimate (1.14) for the forced linear ibvp (1.10) when $\gamma = 0$ and $1 \leq s < \frac{3}{2}$. Like in the Robin case, this step requires us to define an appropriate extension of the boundary data (4.3), which for $\gamma = 0$ are given by

$$\psi_1(x_1,t) = g(x_1,t) - \partial_{x_2} S\big[U_0;0\big](x_1,0,t) \quad \text{and} \quad \psi_2(x_1,t) = -\partial_{x_2} S\big[0;F\big](x_1,0,t). \tag{5.4}$$

Indeed, the remaining two components $S[U_0;0]$ and $S[0;F]$ that appear in (4.2) concern the homogeneous and forced linear ivp, respectively, and have been estimated in theorems 3.1 and 3.2. Denoting ψ_1 and ψ_2 simply by $\psi \in B^s_T$, for each $k_1 \in \mathbb{R}$ and $1 \leq s < \frac{3}{2}$ we let $\varphi(t) = \mathrm{e}^{\mathrm{i}k_1^2 t}\widehat{\psi}^{x_1}(k_1,t)$ and denote the extension of this function by zero outside $(0,T)$ by $\Phi_0(t)$. Observe that, since $\psi \in B^s_T$, we have $\varphi \in H^{(2s-1)/4}(0,T)$ for a.e. $k_1 \in \mathbb{R}$. Next, define the global function $h(x_1,t)$ via its x_1-Fourier transform as

$$\widehat{h}^{x_1}(k_1,t) = \theta(t)\,\mathrm{e}^{-\mathrm{i}k_1^2 t}\Phi_0(k_1,t), \quad \text{a.e. } k_1\in\mathbb{R},\ t\in\mathbb{R}, \tag{5.5}$$

where $\theta \in C^\infty_0[-1,3]$ is equal to 1 on $[0,2]$ and decays to zero smoothly outside $[0,2]$. Note that this definition implies $\mathrm{supp}(h) \subset \mathbb{R}_{x_1} \times (0,2)$ and $h(t) = \psi(t)$ for $t \in (0,T)$. Also, as we show next, $h \in X^{0,((2s-1)/4)} \cap X^{s,-(1/4)}$. Indeed, the fact $||\theta f||_{H^b(\mathbb{R})} \leq c(\theta,b)||f||_{H^b(\mathbb{R})}$, $b\in\mathbb{R}$, combined with inequalities (4.9) and (4.10) yields

$$||\theta\Phi_0(k_1)||_{H^{(2s-1)/4}(\mathbb{R}_t)} \leq c_{s,\theta}||\varphi(k_1)||_{H^{(2s-1)/4}(0,T)}$$

and

$$||\theta\Phi_0(k_1)||_{H^{-(1/4)}(\mathbb{R}_t)} \leq c_{s,\theta}||\varphi(k_1)||_{H^{-(1/4)}(0,T)}.$$

Thus, by the definition of the Bourgain norms, we find

$$||h||^2_{X^{0,((2s-1)/4)}} = \int_{k_1\in\mathbb{R}} ||\theta\Phi_0(k_1)||^2_{H^{(2s-1)/4}(\mathbb{R}_t)}\,\mathrm{d}k_1 \lesssim \int_{k_1\in\mathbb{R}} ||\varphi(k_1)||^2_{H^{(2s-1)/4}(0,T)}\,\mathrm{d}k_1 = ||\psi||^2_{X_T^{0,((2s-1)/4)}},$$

$$\begin{aligned}||h||^2_{X^{s,-(1/4)}} &= \int_{k_1\in\mathbb{R}} (1+k_1^2)^s ||\theta\Phi_0(k_1)||^2_{H^{-(1/4)}(\mathbb{R}_t)}\,\mathrm{d}k_1\\ &\lesssim \int_{k_1\in\mathbb{R}} (1+k_1^2)^s ||\varphi(k_1)||^2_{H^{-(1/4)}(0,T)}\,\mathrm{d}k_1 = ||\psi||^2_{X_T^{s,-(1/4)}},\end{aligned}$$

thereby deducing that $h\in X^{0,((2s-1)/4)}\cap X^{s,-(1/4)}$. Therefore, h can play the role of the boundary datum in the reduced pure linear ibvp (5.1) and so, for $\psi=\psi_1$, estimate (5.3) yields

$$\sup_{t\in[0,T]} ||S[0,\psi_1;0](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} \lesssim ||h||_{X^{0,((2s-1)/4)}} + ||h||_{\dot{X}^{s,-(1/4)}} \lesssim ||\psi_1||_{X_T^{0,((2s-1)/4)}} + ||h||_{\dot{X}^{s,-(1/4)}} \tag{5.6}$$

after replacing the norm of $\dot{X}^{0,((2s-1)/4)}$ with the one of $X^{0,((2s-1)/4)}$ (since $s>\frac{1}{2}$). In addition, according to lemma 2.8 in [12] (see also lemma 4.1 in [14]), for $0\le b<\frac{1}{2}$ and any $\theta\in C_0^\infty(\mathbb{R})$, we have $||\theta f||_{\dot{H}^{-b}(\mathbb{R})}\le c(\theta,b)||f||_{H^{-b}(\mathbb{R})}$. Thanks to the presence of the function θ in the definition (5.5) of h, we can use this result for $b=\frac{1}{4}$ and $f=\Phi_0$ to obtain $||\theta\Phi_0(k_1)||_{\dot{H}^{-(1/4)}(\mathbb{R}_t)}\lesssim||\Phi_0(k_1)||_{H^{-(1/4)}(\mathbb{R}_t)}\lesssim||\varphi(k_1)||_{H^{-(1/4)}(0,T)}$, where for the last step, we have employed inequality (4.10). Hence,

$$\begin{aligned}||h||^2_{\dot{X}^{s,-(1/4)}} &= \int_{k_1\in\mathbb{R}} (1+k_1^2)^s ||\theta\Phi_0(k_1)||^2_{H^{-(1/4)}(\mathbb{R}_t)}\,\mathrm{d}k_1\\ &\lesssim \int_{k_1\in\mathbb{R}} (1+k_1^2)^s ||\varphi(k_1)||^2_{H^{-(1/4)}(0,T)}\,\mathrm{d}k_1 = ||\psi_1||^2_{X_T^{s,-(1/4)}}.\end{aligned}$$

For $\frac{3}{2}<s<\frac{5}{2}$, the construction of h is similar after making the corresponding modifications for that range as in the Robin case $\gamma\neq 0$. Then, combining (5.6) with (4.5) for ψ_1, we find

$$\sup_{t\in[0,T]} ||S[0,\psi_1;0](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} \lesssim ||g||_{B^s_T} + ||u_0||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}.$$

Along the same lines, we also have

$$\sup_{t\in[0,T]} ||S[0,\psi_2;0](t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})} \lesssim \sqrt{T}\sup_{t\in[0,T]} ||f(t)||_{H^s(\mathbb{R}_{x_1}\times\mathbb{R}^+_{x_2})}.$$

These estimates along with inequality (4.7) imply the desired estimate (1.14), completing the proof of theorem 1.2 in the Neumann case $\gamma=0$. ■

6. Solution of the linear ibvp via the unified transform of Fokas

We provide the derivation of the unified transform formula (1.11) for the forced linear Schrödinger ibvp (1.10) under the assumption of sufficiently smooth initial and boundary values. The case of Sobolev data can be handled via a density argument along the lines of [3]. Taking the half-plane Fourier transform (1.12) of the forced linear Schrödinger equation and using the Robin boundary condition along with the notation $u(x_1,0,t)=g_0(x_1,t)$, we find

$$\widehat{u}_t + \mathrm{i}(k_1^2+k_2^2)\widehat{u} = \int_{x_1\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}k_1x_1}[(k_2+\mathrm{i}\gamma)g_0(x_1,t) - \mathrm{i}g(x_1,t)]\,\mathrm{d}x_1 - \widehat{\mathrm{i}f}, \quad k_1\in\mathbb{R},\ \mathrm{Im}(k_2)\le 0.$$

We note that the domain can be extended to the lower half of the complex k_2-plane thanks to the fact that $x_2>0$. Integrating this expression with respect to t yields the so-called *global relation*

$$\begin{aligned}\mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}(k_1,k_2,t) &= \widehat{u}_0(k_1,k_2) + (k_2+\mathrm{i}\gamma)\tilde{g}_0(k_1,k_1^2+k_2^2,t) - \mathrm{i}\tilde{g}(k_1,k_1^2+k_2^2,t)\\ &\quad - \mathrm{i}\int_{t'=0}^{t} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,k_2,t')\,\mathrm{d}t'\end{aligned} \tag{6.1}$$

with $\tilde{g}_0$ and $\tilde{g}$ defined according to (1.13). Thus, by the inverse Fourier transform, we obtain

$$\begin{aligned} u(x_1,x_2,t) = {} & \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}_0(k_1,k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & - \frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\int_{t'=0}^{t} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & + \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t} \\ & \times\Big[(k_2+\mathrm{i}\gamma)\tilde{g}_0(k_1,k_1^2+k_2^2,t)-\mathrm{i}\tilde{g}(k_1,k_1^2+k_2^2,t)\Big]\,\mathrm{d}k_2\,\mathrm{d}k_1. \end{aligned} \tag{6.2}$$

This is an integral representation for the solution but *not* an explicit formula since $\tilde{g}_0$ involves the unknown Dirichlet boundary value g_0. However, it is possible to eliminate $\tilde{g}_0$ from (6.2) in favour of known quantities. To accomplish this, we begin by observing that for $x_2 \geq 0$ and $t \geq t'$ the exponential $\mathrm{e}^{\mathrm{i}k_2x_2-\mathrm{i}k_2^2(t-t')}$ is bounded for $k_2 \in \{\mathrm{Im}(k_2)\geq 0\}\setminus D$, where D here denotes the first quadrant of the complex k_2-plane (figure 1). Thus, exploiting the analyticity of the half-plane Fourier transform as well as of the transforms $\tilde{g}_0$ and $\tilde{g}$ for all $k_2\in\mathbb{C}$ (which follows via a Paley–Wiener theorem), we apply Cauchy's theorem in the second quadrant of the complex k_2-plane to deform the contour of the k_2 integral in the last term of (6.2) from $\mathbb{R}$ to the positively oriented boundary ∂D of D (figure 1). This deformation is possible thanks to the fact that, for the quartercircle $\gamma_\rho^+ = \{\rho\,\mathrm{e}^{\mathrm{i}\theta} : \frac{\pi}{2}\leq\theta\leq\pi\}$, we can show along the lines of [2] that $\lim_{\rho\to\infty}\int_{k_2\in\gamma_\rho^+}\mathrm{e}^{\mathrm{i}k_2x_2-\mathrm{i}k_2^2t}[(k_2+\mathrm{i}\gamma)\tilde{g}_0(k_1,k_1^2+k_2^2,t)-\mathrm{i}\tilde{g}(k_1,k_1^2+k_2^2,t)]\,\mathrm{d}k_2=0$. Thus, (6.2) becomes

$$\begin{aligned} u(x_1,x_2,t) = {} & \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}_0(k_1,k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & - \frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\int_{t'=0}^{t} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & + \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\partial D} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\Big[(k_2+\mathrm{i}\gamma)\tilde{g}_0(k_1,k_1^2+k_2^2,t)-\mathrm{i}\tilde{g}(k_1,k_1^2+k_2^2,t)\Big]\,\mathrm{d}k_2\,\mathrm{d}k_1. \end{aligned} \tag{6.3}$$

Next, note that under the transformation $k_2\mapsto -k_2$ the global relation (6.1) yields the identity

$$\begin{aligned} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}(k_1,-k_2,t) = {} & \widehat{u}_0(k_1,-k_2)+(-k_2+\mathrm{i}\gamma)\tilde{g}_0(k_1,k_1^2+k_2^2,t)-\mathrm{i}\tilde{g}(k_1,k_1^2+k_2^2,t) \\ & -\mathrm{i}\int_{t'=0}^{t}\mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,-k_2,t')\,\mathrm{d}t', \quad k_1\in\mathbb{R},\ \mathrm{Im}(k_2)\geq 0. \end{aligned} \tag{6.4}$$

If $\gamma<0$, then $k_2-\mathrm{i}\gamma\neq 0$ for all $k_2\in\partial D$, so we can use (6.4) to substitute for $\tilde{g}_0$ in (6.3) and obtain

$$\begin{aligned} u(x_1,x_2,t) = {} & \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\widehat{u}_0(k_1,k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & + \frac{1}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\partial D} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2+\mathrm{i}\gamma}{k_2-\mathrm{i}\gamma}\widehat{u}_0(k_1,-k_2)\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & - \frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\mathbb{R}} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\int_{t'=0}^{t} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & - \frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\partial D} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{k_2+\mathrm{i}\gamma}{k_2-\mathrm{i}\gamma}\int_{t'=0}^{t} \mathrm{e}^{\mathrm{i}(k_1^2+k_2^2)t'}\widehat{f}(k_1,-k_2,t')\,\mathrm{d}t'\,\mathrm{d}k_2\,\mathrm{d}k_1 \\ & - \frac{\mathrm{i}}{(2\pi)^2}\int_{k_1\in\mathbb{R}}\int_{k_2\in\partial D} \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}(k_1^2+k_2^2)t}\frac{2k_2}{k_2-\mathrm{i}\gamma}\tilde{g}(k_1,k_1^2+k_2^2,t)\,\mathrm{d}k_2\,\mathrm{d}k_1, \end{aligned} \tag{6.5}$$

where we have also used the fact that $\int_{k_2\in\partial D}\mathrm{e}^{\mathrm{i}k_2x_2}((k_2+\mathrm{i}\gamma)/(k_2-\mathrm{i}\gamma))\widehat{u}(k_1,-k_2,t)\,\mathrm{d}k_2=0$ by analyticity and exponential decay of the integrand inside D. Formula (6.5) is true also for $\gamma=0$, since in that case the denominator $k_2-\mathrm{i}\gamma$ cancels out. Moreover, exploiting once again analyticity

and exponential decay in D, we infer $\int_{k_2\in\partial D} e^{ik_2x_2-ik_2^2t}\int_{t'=t}^{T} e^{i(k_1^2+k_2^2)t'}(2k_2/(k_2-i\gamma))\int_{y_1\in\mathbb{R}} e^{-ik_1y_1} g(y_1,t')\,dy_1dt'\,dk_2=0$, which turns (6.5) into the equivalent form (1.11) with $\mathcal{C}=\partial D$.

If $\gamma>0$, then $k_2-i\gamma$ vanishes along the positive imaginary axis, which is part of ∂D. To avoid crossing this singularity, before using identity (6.4) to solve for $\tilde{g}_0$ we locally deform the contour of integration of the last k_2 integral in (6.2) from ∂D to the contour $\partial\tilde{D}$ shown in figure 1. Then, proceeding as for $\gamma<0$, we obtain the unified transform formula (1.11), this time with $\mathcal{C}=\partial\tilde{D}$.

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Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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References

1. Bourgain J. 1993 Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* **3**, 107–156. (doi:10.1007/BF01896020)
2. Himonas A, Mantzavinos D. 2020 Well-posedness of the nonlinear Schrödinger equation on the half-plane. *Nonlinearity* **33**, 5567–5609. (doi:10.1088/1361-6544/ab9499)
3. Fokas A, Himonas A, Mantzavinos D. 2017 The nonlinear Schrödinger equation on the half-line. *Trans. Am. Math. Soc.* **369**, 681–709. (doi:10.1090/tran/6734)
4. Himonas A, Mantzavinos D, Yan F. 2019 The nonlinear Schrödinger equation on the half-line with Neumann boundary conditions. *Appl. Num. Math.* **141**, 2–18. (doi:10.1016/j.apnum.2018.09.018)
5. Himonas A, Mantzavinos D. 2021 The nonlinear Schrödinger equation on the half-line with a Robin boundary condition. *Anal. Math. Phys.* **11**, 1–25. (doi:10.1007/s13324-021-00589-y)
6. Fokas A, Himonas A, Mantzavinos D. 2016 The Korteweg-de Vries equation on the half-line. *Nonlinearity* **29**, 489–527. (doi:10.1088/0951-7715/29/2/489)
7. Himonas A, Madrid C, Yan F. 2021 The Neumann and Robin problems for the Korteweg-de Vries equation on the half-line. *J. Math. Phys.* **62**, 111503. (doi:10.1063/5.0064147)
8. Bona J, Sun S, Zhang B-Y. 2002 A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane. *Trans. Am. Math. Soc.* **354**, 427–490. (doi:10.1090/S0002-9947-01-02885-9)
9. Bona J, Sun S, Zhang B-Y. 2006 Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications. *Dyn. PDE* **3**, 1–69. (doi:10.4310/DPDE.2006.v3.n1.a1)
10. Bona J, Sun S, Zhang B-Y. 2018 Nonhomogeneous boundary value problems of one-dimensional nonlinear Schrödinger equations. *J. Math. Pures Appl.* **109**, 1–66. (doi:10.1016/j.matpur.2017.11.001)
11. Erdogan M, Tzirakis N. 2016 Regularity properties of the cubic nonlinear Schrödinger equation on the half line. *J. Funct. Anal.* **271**, 2539–2568. (doi:10.1016/j.jfa.2016.08.012)
12. Colliander J, Kenig C. 2002 The generalized Korteweg-de Vries equation on the half-line. *Commun. Partial Differ. Equ.* **27**, 2187–2266. (doi:10.1081/PDE-120016157)
13. Holmer J. 2005 The initial-boundary-value problem for the 1D nonlinear Schrödinger equation on the half-line. *Differ. Int. Eq.* **18**, 647–668.
14. Holmer J. 2006 The initial-boundary-value problem for the Korteweg-de Vries equation. *Commun. Partial Differ. Equ.* **31**, 1151–1190. (doi:10.1080/03605300600718503)
15. Ran Y, Sun S, Zhang B-Y. 2018 Nonhomogeneous initial boundary value problems of 2D nonlinear Schrödinger equations in a half-plane. *SIAM J. Math. Anal.* **50**, 2773–2806. (doi:10.1137/17M1119743)
16. Audiard C. 2019 Global Strichartz estimates for the Schrödinger equation with non zero boundary conditions and applications. *Ann. Inst. Fourier (Grenoble)* **69**, 31–80. (doi:10.5802/aif.3238)

17. Fokas A. 1997 A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. A* **453**, 1411–1443. (doi:10.1098/rspa.1997.0077)
18. Fokas A. 2008 *A unified approach to boundary value problems*. Philadelphia, PA: SIAM.
19. Batal A, Fokas A, Özsari T. 2020 Fokas method for linear boundary value problems involving mixed spatial derivatives. *Proc. R. Soc. A* **476**, 1–15. (doi:10.1098/rspa.2020.0076)
20. Caudrelier V. 2018 Interplay between the inverse scattering method and Fokas's unified transform with an application. *Stud. Appl. Math.* **140**, 3–26. (doi:10.1111/sapm.12190)
21. Deconinck B, Fokas A, Lenells J. 2021 The implementation of the unified transform to the nonlinear Schrödinger equation with periodic initial conditions. *Lett. Math. Phys.* **111**, 17. (doi:10.1007/s11005-021-01356-7)
22. Fokas A. 2002 A new transform method for evolution partial differential equations. *IMA J. Appl. Math.* **67**, 559–590. (doi:10.1093/imamat/67.6.559)
23. Fokas A. 2009 The Davey-Stewartson equation on the half-plane. *Commun. Math. Phys.* **289**, 957–993. (doi:10.1007/s00220-009-0809-1)
24. Fokas A, Its A. 2004 The nonlinear Schrödinger equation on the interval. *J. Phys. A* **37**, 6091–6114. (doi:10.1088/0305-4470/37/23/009)
25. Fokas A, Lenells J. 2012 The unified method: I non-linearizable problems on the half-line. *J. Phys. A. Math. Theor.* **45**, 195201. (doi:10.1088/1751-8113/45/19/195201)
26. Fokas A, Lenells J. 2014 Perturbative and exact results on the Neumann value for the nonlinear Schrödinger equation on the half-line. *J. Phys.: Conf. Ser.* **482**, 012015. (doi:10.1088/1742-6596/482/1/012015)
27. Fokas A, Pelloni B. 2005 A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.* **70**, 564–587. (doi:10.1093/imamat/hxh047)
28. Lenells J, Fokas A. 2012 The unified method: II. NLS on the half-line with *t*-periodic boundary conditions. *J. Phys. A. Math. Theor.* **45**, 195202. (doi:10.1088/1751-8113/45/19/195202)
29. Lenells J, Fokas A. 2015 The nonlinear Schrödinger equation with t-periodic data: I. Exact results. *Proc. R. Soc. A* **471**, 20140925. (doi:10.1098/rspa.2014.0925)
30. Özsari T, Yolcu N. 2019 The initial-boundary value problem for the biharmonic Schrödinger equation on the half-line. *Commun. Pure Appl. Anal.* **18**, 3285–3316. (doi:10.3934/cpaa.2019148)
31. Deconinck B, Trogdon T, Vasan V. 2014 The method of Fokas for solving linear partial differential equations. *SIAM Rev.* **56**, 159–186. (doi:10.1137/110821871)
32. Fokas A, Spence E. 2012 Synthesis, as opposed to separation, of variables. *SIAM Rev.* **54**, 291–324. (doi:10.1137/100809647)
33. Talanov V. 1964 Self-focusing of electromagnetic waves in nonlinear media. *Radiophysics* **8**, 254–257.
34. Craig W, Sulem C, Sulem P-L. 1992 Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity* **5**, 497–522. (doi:10.1088/0951-7715/5/2/009)
35. Peregrine D. 1983 Water waves, nonlinear Schrödinger equations and their solutions. *J. Aust. Math. Soc. Ser. B* **25**, 16–43. (doi:10.1017/S0334270000003891)
36. Weiland J, Wiljelmsson H. 1977 *Coherent nonlinear interaction of waves in plasmas*. Oxford, UK: Pergamon Press.
37. Pitaevskii L, Stringari S. 2003 *Bose-Einstein condensation*. Oxford, UK: Clarendon Press.
38. Zakharov V, Shabat A. 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**, 63–69.
39. Bourgain J. 1999 *Global solutions of nonlinear Schrödinger equations*. 46. AMS Colloquium Publications. Providence, RI: AMS.
40. Cazenave T, Weissler F. 1990 The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.* **14**, 807–836. (doi:10.1016/0362-546X(90)90023-A)
41. Colin M, Di Menza L, Saut J-C. 2016 Solitons in quadratic media. *Nonlinearity* **29**, 1000–1035. (doi:10.1088/0951-7715/29/3/1000)
42. Constantin P, Saut J-C. 1989 Local smoothing properties of Schrödinger equations. *Indiana Univ. Math. J.* **38**, 781–810. (doi:10.1512/iumj.1989.38.38037)
43. Ginibre J, Velo G. 1979 On a class of nonlinear Schrödinger equations. II. Scattering theory, general case. *J. Funct. Anal.* **32**, 33–71. (doi:10.1016/0022-1236(79)90077-6)
44. Kenig C, Ponce G, Vega L. 1991 Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.* **40**, 33–69. (doi:10.1512/iumj.1991.40.40003)
45. Kenig C, Ponce G, Vega L. 1993 Small solutions to nonlinear Schrödinger equations. *Ann. Inst. Henri Poincare Anal. Nonlinear* **10**, 255–288. (doi:10.1016/s0294-1449(16)30213-x)

46. Linares F, Ponce G. 2009 *Introduction to nonlinear dispersive equations*. Universitext. New York, NY: Springer.
47. Tao T. 2006 *Nonlinear dispersive equations: local and global analysis*. CBMS Regional Conference Series in Mathematics, vol. 106, 373 pp.
48. Tsutsumi Y. 1987 L^2-solutions for nonlinear Schrödinger equations and nonlinear groups. *Funkc. Ekvacioj* **30**, 115–125.
49. Hardy GH. 1929 Remarks in addition to Dr. Widder's note on inequalities. *J. Lond. Math. Soc.* **4**, 199–202. (doi:10.1112/jlms/s1-4.3.199)
50. Lions J, Magenes E. 1972 *Non-homogeneous boundary value problems and applications*, vol. 1. New York, NY: Springer.
51. Jerison D, Kenig C. 1995 The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.* **130**, 161–219. (doi:10.1006/jfan.1995.1067)