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Growth bound and nonlinear smoothing for the periodic derivative nonlinear Schrödinger equation

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Abstract
A polynomial-in-time growth bound is established for global Sobolev $H^s(\mathbb{T})$ solutions to the derivative nonlinear Schrödinger equation on the circle with $s > 1$. These bounds are derived as a consequence of a nonlinear smoothing effect for an appropriate gauge-transformed version of the periodic Cauchy problem, according to which a solution with its linear part removed possesses higher spatial regularity than the initial datum associated with that solution.

Mathematics Subject Classification Primary 35Q55 · 35B65 · 42B37

1 Introduction and results

We consider the Cauchy problem for the derivative nonlinear Schrödinger (dNLS) equation on the circle

$$u_t - iu_{xx} = \partial_x(|u|^2 u), \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \tag{1.1a}$$
$$u(x, 0) = u_0(x) \in H^s(\mathbb{T}), \tag{1.1b}$$

where $u = u(x, t)$ is a complex-valued function, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional torus (circle), and $H^s(\mathbb{T})$ is the L^2-based Sobolev space on the circle.

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The dNLS equation was derived as a model in plasma physics in the 1970s, see [39]. As shown in [33], it is a completely integrable system, possessing a Lax pair formulation and an infinite number of conserved quantities, including the following:

$$
\begin{aligned}
M(u) &= \int_{\mathbb{T}} |u|^2 \, dx, \quad P(u) = \int_{\mathbb{T}} \Big[\mathrm{Im}(u\overline{u_x}) + \tfrac{1}{2}|u|^4 \Big] \, dx, \\
E(u) &= \int_{\mathbb{T}} \Big[|u_x|^2 + \tfrac{3}{2}|u|^2 \mathrm{Im}(u\overline{u_x}) + \tfrac{1}{2}|u|^6 \Big] \, dx,
\end{aligned} \tag{1.2}
$$

where $M(u)$, $P(u)$ and $E(u)$ correspond to the mass, momentum and energy, respectively, of the solution. Note that $P(u)$ is the Hamiltonian for (1.1).

Concerning the well-posedness of the Cauchy problem (1.1), Fukuda and Tsutsumi [19] showed local well-posedness in H^s, $s > 3/2$, on both the line and the circle using the method of parabolic regularization. Furthermore, in [20] they demonstrated global well-posedness of solutions in H^2 with sufficiently small norm $\|u_0\|_{H^1}$. Hayashi and Ozawa [27–29] improved upon this result in the Euclidean setting by showing global well-posedness of solutions in $H^1(\mathbb{R})$ with sufficiently small norm $\|u_0\|_{L^2(\mathbb{R})}$. In particular, their result was obtained by first performing a gauge transformation of Eq. (1.1a), which removed the term $|u|^2 u_x$ from the nonlinearity. Takaoka [52] combined the gauge transformation of Hayashi and Ozawa and the Fourier restriction norm method introduced by Bourgain in the breakthrough paper [4] to establish local well-posedness in $H^{1/2}(\mathbb{R})$. This result was shown to be sharp by Biagioni and Linares [3] in the sense that the data-to-solution map fails to be uniformly continuous for $s < 1/2$. Thus, $s = 1/2$ is the optimal result attainable for the well-posedness of (1.1) using a fixed point argument on the gauge equation, although the critical regularity for scaling in the Euclidean setting is at the level of $s = 0$. Under the assumption of a sufficiently small $\|u_0\|_{L^2(\mathbb{R})}$ norm, Colliander, Keel, Staffilani, Takaoka and Tao [8] obtained global well-posedness for $s > 1/2$. Global well-posedness for $s = 1/2$ was shown by Miao, Wu and Xu [38] and later by Guo and Wu [26], with the latter work improving the restriction on the initial data from $\|u_0\|_{L^2(\mathbb{R})} < \sqrt{2\pi}$ to $\|u_0\|_{L^2(\mathbb{R})} < \sqrt{4\pi}$. Such mass restrictions come from the sharp Gagliardo–Nirenberg inequalities. Finally, it is worth mentioning that Jenkins, Liu, Perry and Sulem [32] and, more recently, Bahouri and Perelman [2] proved global well-posedness of dNLS with initial data u_0 in the weighted Sobolev space $H^{2,2}(\mathbb{R})$ and in $H^{1/2}(\mathbb{R})$, respectively, without a mass restriction.

The majority of the above results concern the Cauchy problem on the line. Regarding the periodic problem (1.1), local well-posedness in $H^{1/2}(\mathbb{T})$ was established by Herr [30] by adapting the gauge transformation of Hayashi and Ozawa to the periodic setting. The same article gives global well-posedness for $u_0 \in H^1(\mathbb{T})$ such that $\|u_0\|_{L^2(\mathbb{T})} < 2/3$. This mass threshold was improved by Mosincat and Oh in [41], where they show global well-posedness in $H^1(\mathbb{T})$ for $\|u_0\|_{L^2(\mathbb{T})} < \sqrt{4\pi}$. Using the I-method, Win [54] obtained global well-posedness in $H^s(\mathbb{T})$ for $s > 1/2$ under the assumption of a sufficiently small $\|u_0\|_{L^2(\mathbb{T})}$ norm. Finally, Mosincat [40] established global well-posedness in $H^{1/2}(\mathbb{T})$ provided that $\|u_0\|_{L^2(\mathbb{T})} < \sqrt{4\pi}$, and also proved failure of uniform continuity of the data-to-solution map for $s < 1/2$. At the time of writing of the present article, the mass restriction condition for well-posedness in the periodic case had not been removed. Further well-posedness results on the periodic dNLS in the low regularity setting can be found in Fukaya, Hayashi and Inui [18], Grünrock and Herr [24], Nahmod, Oh, Rey-Bellet and Staffilani [43], and Deng, Nahmod and Yue [12].

Notation. In order to state the main results of this work, we introduce the following notation.

- For $a, b > 0$, we write $a \lesssim b$ if there exists $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$ then we write $a \sim b$. Furthermore, if $C \geq 10^6$ and $a < \frac{1}{C} b$ with $a \nsim b$ then we write $a \ll b$.
- For $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, we define the spatial Fourier transform of f, denoted by $\mathcal{F}_x(f) = \widehat{f}$, as

$$\mathcal{F}_x(f)(\xi) = \widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-i\xi x} f(x)\, dx, \quad \xi \in \mathbb{Z}. \tag{1.3}$$

  Furthermore, for $f \in L^2(\mathbb{T})$, we have the inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\xi \in \mathbb{Z}} e^{i\xi x} \widehat{f}(\xi). \tag{1.4}$$

  For $f \in \mathcal{S}(\mathbb{R})$, the space of Schwartz functions, we define the temporal Fourier transform of f, denoted by $\mathcal{F}_t(f)$, as

$$\mathcal{F}_t(f)(\tau) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\tau} f(t)\, dt, \quad \tau \in \mathbb{R}. \tag{1.5}$$

  Finally, for $f \in \mathcal{S}(\mathbb{R}_t; L^p(\mathbb{T}_x))$ we denote the spatiotemporal Fourier transform of f by

$$\mathcal{F}_t \mathcal{F}_x(f)(\xi, \tau) = \widetilde{f}(\xi, \tau). \tag{1.6}$$

- We define the Bessel potential J_x^s via Fourier transform as

$$\widehat{J_x^s f}(\xi) := \langle \xi \rangle^s \widehat{f}(\xi), \quad \langle \cdot \rangle := \left(1 + |\cdot|^2\right)^{\frac{1}{2}}. \tag{1.7a}$$

  Then, for any $s \geq 0$ and $p \geq 1$, we define the Bessel potential space

$$H^{s,p}(\mathbb{T}) := \left\{ f \in L^p(\mathbb{T}) : \|f\|_{H^{s,p}(\mathbb{T})} := \left\| J_x^s f \right\|_{L^p(\mathbb{T})} < \infty \right\}, \tag{1.7b}$$

  In the special case $p = 2$, the above space reduces to the Sobolev space $H^s(\mathbb{T})$.

- For any $s, b \in \mathbb{R}$, we define the Bourgain space $X^{s,b}$ as the closure of $\mathcal{S}(\mathbb{R}_t; C^\infty(\mathbb{T}_x))$ under the norm

$$\|f\|_{X^{s,b}} := \left\| \langle\xi\rangle^s \left\langle \tau + \xi^2 \right\rangle^b \widetilde{f}(\xi, \tau) \right\|_{\ell^2_\xi L^2_\tau}. \tag{1.8}$$

Similarly, the space $Y^{s,b}$ is defined via the norm

$$\|f\|_{Y^{s,b}} := \left\| \langle\xi\rangle^s \langle \tau + \xi^2 \rangle^b \widetilde{f}(\xi, \tau) \right\|_{\ell^2_\xi L^1_\tau}. \tag{1.9}$$

In addition, we define the Banach space $Z^s := X^{s,\frac{1}{2}} \cap Y^{s,0}$ with norm

$$\|f\|_{Z^s} := \|f\|_{X^{s,\frac{1}{2}}} + \|f\|_{Y^{s,0}}. \tag{1.10}$$

Finally, the restriction of Z^s on $\mathbb{T} \times [0, T]$ with $T > 0$ is denoted by Z^s_T and is defined via the norm

$$\|f\|_{Z^s_T} := \inf \left\{ \|g\|_{Z^s} : g|_{[0,T]} = f \right\}. \tag{1.11}$$

- We define the Littlewood–Paley-type projection operator P_k by

$$\widehat{P_k(f)}(\xi) := \begin{cases} \chi_{\{\xi=0\}} \widehat{f}(0), & k = 0 \\ \chi_{\{2^{k-1} \le |\xi| < 2^k\}} \widehat{f}(\xi), & k \in \mathbb{N}, \end{cases} \tag{1.12}$$

where χ_A is the characteristic function of the set A. We will often denote $P_k(f)$ simply by f_k. By this definition, it follows that

$$\sum_{k=0}^{\infty} \widehat{f_k}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{Z}. \tag{1.13}$$

- Following [30], we introduce the periodic gauge transformation of a solution u to (1.1) by

$$v(x, t) = \mathcal{G}(u)(x, t) := e^{-i\mathcal{I}(u)(x,t)} u(x, t), \tag{1.14}$$

where $\mathcal{I}(u)(x, t)$ is the mean-zero spatial primitive of $|u(x, t)|^2 - \frac{1}{2\pi} \|u(t)\|^2_{L^2(\mathbb{T})}$ given by

$$\mathcal{I}(u)(x, t) := \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x \left[|u(y, t)|^2 - \frac{1}{2\pi} \|u(t)\|^2_{L^2(\mathbb{T})} \right] dy d\theta.$$

Let

$$\mu := \frac{1}{2\pi} \|u_0\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi} \|u(t)\|^2_{L^2(\mathbb{T})}, \quad t \in \mathbb{R}, \tag{1.15}$$

where the second equality is due to the conservation of mass in (1.2). In fact, we further have $\mu = \frac{1}{2\pi} \|v(t)\|^2_{L^2(\mathbb{T})}$. A straightforward computation then shows that v satisfies the equation

$$v_t - i v_{xx} - 2\mu v_x = -v^2 \overline{v}_x + \frac{i}{2}|v|^4 v - i\mu|v|^2 v + i\psi(v)v, \tag{1.16}$$

where

$$\psi(v)(t) := \frac{1}{2\pi}\int_0^{2\pi} \left[2\mathrm{Im}(\overline{v}_x v)(\theta, t) - \frac{1}{2}|v|^4(\theta, t)\right] d\theta + \mu^2.$$

The term $2\mu v_x$ can be removed from (1.16) by means of the transformation

$$w(x,t) = \tau_{-\mu} v(x,t) := v(x - 2\mu t, t). \tag{1.17}$$

Indeed, since $\tau_{-\mu}$ commutes with ψ and is an isometry on $L^2(\mathbb{T})$, we find that w satisfies

$$w_t - i w_{xx} = -w^2 \overline{w}_x + \frac{i}{2}|w|^4 w - i\mu|w|^2 w + i\psi(w)w. \tag{1.18}$$

Finally, we introduce a second gauge transformation,

$$z(x,t) := e^{-ig(t)} w(x,t), \tag{1.19}$$

where

$$g(t) := \frac{1}{4\pi}\int_0^t \|w(t')\|^4_{L^4(\mathbb{T})} dt' - \mu^2 t.$$

We note that z is related to u as follows:

$$z(x,t) = e^{-ig(t)} e^{-i\mathcal{I}(u)(x-2\mu t, t)} u(x - 2\mu t, t) = e^{-ig(t)} \tau_{-\mu} \mathcal{G}(u)(x,t).$$

Also, it will be shown in Sect. 2 that z satisfies the Cauchy problem (2.12) as well as the integral equation (2.27).

The need for the second gauge transform (1.19) can be appreciated once Eq. (1.18) for w is further analyzed. In particular, this equation can be put in the form (2.11), where the last two terms are unfavorable for the purpose of showing nonlinear smoothing. The second gauge transform (1.19) removes those two problematic terms apart from a "leftover" term—the last one in Eq. (2.12a) for z—which is kept intentionally in order to match a problematic term arising from the first term of Eq. (2.12a). In this regard, we also note that the coefficient $i/(4\pi)$ in front of the last term in (2.12a) is chosen so that this term cancels with the problematic term involved in $NR(-z^2\bar{z}_x)$ (see computation leading to (2.26)).

With the above notation in place, we now state some essential previous results and then introduce the main results of this work. We begin with the well-posedness of the gauge-equivalent Cauchy problem (2.12), which follows from Theorem 5.1 of [30].

Theorem 1.1 (Well-posedness of the gauge equation—[30, Theorem 5.1]) *Suppose $z_0 \in H^s(\mathbb{T})$ with $s \geq 1/2$. Then, there exists a non-increasing function $T : [0, \infty) \to [0, \infty)$ with $T = T(\|z_0\|_{H^s(\mathbb{T})})$ and a unique $z \in Z^s_T$ satisfying the gauge-equivalent Cauchy problem (2.12) in the Duhamel sense with the estimate*

$$\|z\|_{Z^s_T} \leq c\, \|z_0\|_{H^s}\,. \tag{1.20}$$

Furthermore, the data-to-solution map is Lipschitz from bounded subsets of $H^s(\mathbb{T})$ to bounded subsets of Z^s_T.

Remark 1.1 In [30], it is stated that the local time of existence for the solution z can be taken to depend only on $\|z_0\|_{H^{1/2}(\mathbb{T})}$ instead of $\|z_0\|_{H^s(\mathbb{T})}$, namely, $T = T(\|z_0\|_{H^{1/2}(\mathbb{T})})$.

Next, we recall the well-posedness of the dNLS Cauchy problem (1.1) as guaranteed by Theorem 1.1 of [30].

Theorem 1.2 (Well-posedness of dNLS on $\mathbb{T}$— [30, Theorem 1.1]) *Suppose $u_0 \in H^s(\mathbb{T})$ with $s \geq 1/2$. If $z \in Z^s_T$ is the solution to the gauge-equivalent Cauchy problem (2.12) as guaranteed by Theorem 1.1, then $u = e^{ig(t)}\mathcal{G}^{-1}(\tau_\mu z) \in C([0, T]; H^s(\mathbb{T}))$ is the unique solution satisfying the dNLS Cauchy problem (1.1) in the sense of Duhamel. Furthermore, u is a limit of smooth solutions.*

Remark 1.2 Lemma 5.1 implies that there exists a non-increasing function $\tilde{T} : [0, \infty) \to [0, \infty)$ such that $\tilde{T} = \tilde{T}(\|u_0\|_{H^s(\mathbb{T})})$ and $\tilde{T}(\|u_0\|_{H^s(\mathbb{T})}) \leq T(\|z_0\|_{H^s(\mathbb{T})})$. Therefore, for $s \geq 1/2$ the time of existence in Theorem 1.2 may be taken to depend on $\|u_0\|_{H^{1/2}(\mathbb{T})}$ instead of $\|u_0\|_{H^s(\mathbb{T})}$.

We also state the following global existence result from [30].

Corollary 1.1 (Global existence—[30, Corollary 1.2]) *For $s \geq 1$ let $u \in C([0, T]; H^s(\mathbb{T}))$ be the solution to the Cauchy problem (1.1) from Theorem 1.2. Then, for $\|u_0\|_{L^2(\mathbb{T})}$ sufficiently small,*

$$\|u(t)\|_{H^1(\mathbb{T})} \leq C(\|u_0\|_{H^1(\mathbb{T})}), \quad t \in [0, T]. \tag{1.21}$$

Consequently, the time of existence for the solution u can be taken arbitrarily large.

Remark 1.3 The above global result follows from the observation that the local time of existence T is bounded below by a function of $\|u_0\|_{H^1(\mathbb{T})}$. While global well-posedness in $H^{1/2}(\mathbb{T})$ has been obtained in [40], an estimate of the form (1.21) is not readily available when $H^1(\mathbb{T})$ is replaced by $H^{1/2}(\mathbb{T})$.

The main goal of the present work is to establish a polynomial-in-time bound on the growth of global solutions to the periodic dNLS Cauchy problem (1.1). Key to demonstrating this bound is the discovery of a local nonlinear smoothing effect for the gauge problem (2.12), according to which the solution z of (2.12) with the linear part removed possesses higher spatial regularity than the initial data z_0. The effect is more readily seen by first recasting (2.12) into the Duhamel form (2.27) via a method known as differentiation by parts (see Sect. 2). The precise statement of our first result is the following.

Theorem 1.3 (Nonlinear smoothing) *Suppose $s > 1/2 + \varepsilon$ with $0 < \varepsilon \ll 1/2$ and let $z_0 \in H^s(\mathbb{T})$. Then, for $T = T(\|z_0\|_{H^{1/2+\varepsilon}(\mathbb{T})})$, the solution $z \in Z_T^s$ of the Cauchy problem (2.12) from Theorem 1.1 satisfies the integral equation (2.27).*

Moreover, for $0 < a < \min\{s - 1/2 - \varepsilon, 1/2 - \varepsilon\}$ and $\sigma = \min\{s, 1\}$ we have $z - e^{it\partial_x^2} z_0 \in C([0, T]; H^{s+a}(\mathbb{T}))$ with

$$\left\| z - e^{it\partial_x^2} z_0 \right\|_{C([0,T];H^{s+a}(\mathbb{T}))} \leq C(s, \|z_0\|_{H^\sigma(\mathbb{T})}) \|z_0\|_{H^s(\mathbb{T})} . \tag{1.22}$$

Remark 1.4 Corollary 1.1 and Lemma 5.1 imply that z satisfies (2.27) globally.

We note that the dispersion on the circle is weaker than on the line in the sense that no Kato smoothing or maximal inequalities are available on the circle. Thus, proving the nonlinear smoothing effect (1.22) requires a careful treatment of resonant frequencies in addition to the differentiation by parts mentioned above.

The nonlinear smoothing estimate (1.22) allows us to demonstrate the following polynomial-in-time bound, which is the main result of this work.

Theorem 1.4 (Polynomial bound) *Let $s \geq 1$. Then, the global solution u to the periodic dNLS Cauchy problem (1.1) given by Corollary 1.1 satisfies*

$$\|u(t)\|_{H^s(\mathbb{T})} \leq C(\varepsilon, s, \|u_0\|_{H^s(\mathbb{T})}) \langle t \rangle^{2(s-1)+\varepsilon}, \tag{1.23}$$

for all $t \in \mathbb{R}$ and $0 < \varepsilon \ll 1/2$.

Bourgain [5, 6] was the first to demonstrate the connection between nonlinear smoothing and polynomial bounds for Hamiltonian equations. By employing Fourier truncation operators in conjunction with smoothing estimates, he obtained the following local-in-time inequality for solutions of various dispersive equations:

$$\|u(t+\delta)\|_{H^s} \leq \|u(t)\|_{H^s} + C \|u(t)\|_{H^s}^{1-\delta} \tag{1.24}$$

for some $\delta \in (0, 1)$. Local time iterations using the above inequality resulted in the polynomial growth bound $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{1/\delta}$. Staffilani [50, 51] used further multilinear smoothing estimates to obtain (1.24), which led to polynomial bounds of H^s solutions, $s > 1$, for Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations. Colliander, Keel, Staffilani, Takaoka and Tao [9] developed a new method using modified energy called the "upside-down I-method" to produce polynomial bounds in low Sobolev norms, $s \in (0, 1)$, for the NLS equation. Sohinger [48, 49] further developed the upside-down I-method to obtain polynomial bounds for high Sobolev norms for NLS. We also refer the reader to [10] and the references therein for further developments in this direction. In addition, Oh and Stefanov [46] determined a nonlinear smoothing effect for periodic, generalized KdV equations that gave rise to a polynomial bound in H^s, $s > 1$. Finally, in [31], Oh and the authors of the present work identified a nonlinear smoothing effect for a periodic, gauge-transformed Benjamin–Ono equation which led to a polynomial bound on solutions to the periodic Benjamin–Ono equation for $1/2 < s \leq 1$.

Several recent works have established uniform-in-time bounds for a number of completely integrable dispersive equations using inverse scattering techniques. In particular, Killip, Visan and Zhang [34] showed that the H^s-norm of solutions to the KdV and NLS equations is uniformly bounded in time for $-1 \leq s < 1$ and $-1/2 < s < 1$, respectively, both on the line and on the circle. Similarly, Koch and Tataru [37] showed that there exists a conserved energy equivalent to the H^s-norm for $s > -1/2$ in the case of the NLS and mKdV equations and for $s \geq -1$ in the case of the KdV equation. For the Benjamin–Ono equation, Talbut [53] proved a uniform-in-time bound in H^s for $-1/2 < s < 0$ on the line and the circle. Gérard, Kappeler and Topalov [21] then established this uniform bound for the periodic Benjamin–Ono equation with $s > 0$. In the case of dNLS, uniform-in-time bounds were obtained on the line and the circle by Klaus and Schippa [36] for $0 < s < 1/2$. Furthermore, Bahouri and Perelman [2] showed boundedness of $H^{1/2}(\mathbb{R})$ solutions.

Nonlinear smoothing properties analogous to the one of Theorem 1.3 have been previously established for several important dispersive equations. Indicatively, we mention the works of Erdogan and Tzirakis on the periodic KdV equation [15] as well as on the fractional NLS equation on the circle and line [14], the NLS equation on the half-line [17], the dNLS equation on the line and half-line [13], and the Zakharov system on the circle [16]. The main technique used in the proof of these results is known as the normal form method or, as previously mentioned, the differentiation by parts method. It was first introduced by Shatah [47] in the context of the Klein–Gordon equation with a quadratic nonlinearity and was further developed by Germain, Masmoudi and Shatah for two-dimensional quadratic Schrödinger equations [22] and for the gravity water waves equation [23]. Babin, Ilyin and Titi [1] applied this method to obtain unconditional well-posedness results for the periodic KdV equation. Chung, Guo, Kwon and Oh [7] also obtained unconditional well-posedness of the quadratic dNLS using normal form reductions. An alternative formulation of the normal form method, which involves a multilinear, pseudo-differential operator in place of differentiation by parts, was provided by Oh and Stefanov [44, 45] for establishing smoothing estimates and well-posedness. It should be pointed out that [15] is the first work that employed the normal form method in the framework of $X^{s,b}$ spaces.

Finally, we note that in the recent preprint [11] Correia and Silva suggest a unified approach for showing nonlinear smoothing for dispersive equations on $\mathbb{R}$. This approach relies on the use of infinite iterations of normal form/differentiation by parts reductions and, among other equations, has been employed for dNLS on $\mathbb{R}$ (see Corollary 5 in [11]). An interesting question is whether the nonlinear smoothing effect of Theorem 1.3 (and hence the polynomial bound of Theorem 1.4) established in the present work could also be obtained by adapting the unified approach of [11] to the periodic setting. Estimates for the periodic problem shown by Kishimoto [35] could be relevant in this direction (see also Guo, Kwon and Oh [25] and Mosincat and Yoon [42]).

Structure of the paper. In Sect. 2, we employ the gauge transformation (1.19) to remove the resonant frequencies from the w-equation (1.18) and perform differentiation by parts on the resulting gauge problem (2.12) for z in order to establish (formally) the Duhamel form (2.27). In Sect. 3, we prove a number of useful a priori estimates for (2.27) which are key to establishing Theorem 1.3. In Sect. 4, we utilize the aforementioned estimates to complete the proof of Theorem 1.3. Finally, in Sect. 5, we employ the nonlinear smoothing effect from Theorem 1.3 in order to prove the polynomial bound of Theorem 1.4.

2 Removal of resonances and differentiation by parts

We begin with Eq. (1.18) for w and proceed with formal computations whose purpose is to remove the resonant terms present in that equation. This procedure eventually takes us to the Cauchy problem (2.12) for z, which we then rewrite in the form of the integral equation (2.27). This is the equation used for proving the a priori estimates leading to nonlinear smoothing in Sect. 3.

First, note that

$$-w^2\overline{w}_x = \frac{i}{(2\pi)^{3/2}} \sum_{\xi_1,\xi_2,\xi_3} e^{i(\xi_1-\xi_2+\xi_3)x} \xi_2 \widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3). \tag{2.1}$$

The resonant frequencies in (2.1) are associated with $\{\xi_1 = \xi_2\} \cup \{\xi_2 = \xi_3\}$. Hence, we split (2.1) into resonant and nonresonant frequencies as follows:

$$\begin{aligned}
-w^2\overline{w}_x &:= NR(-w^2\overline{w}_x) + \frac{2i}{(2\pi)^{3/2}} \sum_{\xi_1=\xi_2} e^{i(\xi_1-\xi_2+\xi_3)x} \xi_2 \widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3) \\
&\quad - \frac{i}{(2\pi)^{3/2}} \sum_{\xi_0} e^{i\xi_0 x} \xi_0 \widehat{w}(\xi_0)|\widehat{w}(\xi_0)|^2 \\
&:= NR(-w^2\overline{w}_x) + \frac{2iw}{2\pi} \sum_{\xi} \xi |\widehat{w}(\xi)|^2 + \mathcal{R}_1(w),
\end{aligned}$$

where we have used the symmetry in ξ_1 and ξ_3 and

$$NR(-w^2\overline{w_x}) = \frac{i}{(2\pi)^{3/2}} \sum_{\xi_1\neq\xi_2,\ \xi_2\neq\xi_3} e^{i(\xi_1-\xi_2+\xi_3)x} \xi_2 \widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3), \tag{2.2}$$

$$\mathcal{R}_1(w) = -\frac{i}{(2\pi)^{3/2}} \sum_{\xi_0} e^{i\xi_0 x} \xi_0 \widehat{w}(\xi_0)|\widehat{w}(\xi_0)|^2. \tag{2.3}$$

Next, we claim that

$$\sum_{\xi} \xi |\widehat{w}(\xi)|^2 = -\int_0^{2\pi} \mathrm{Im}(\overline{w}_x w)(\theta, t)\, d\theta.$$

Indeed, we have

$$\begin{aligned} -\int_0^{2\pi} \mathrm{Im}(\overline{w}_x w)(\theta, t)\, d\theta &= \mathrm{Im} \int_0^{2\pi} \frac{i}{2\pi} \sum_{\xi_1, \xi_2} e^{i(\xi_1 - \xi_2)\theta} \xi_2 \widehat{w}(\xi_1) \overline{\widehat{w}}(\xi_2)\, d\theta \\ &= \mathrm{Im} \sum_{\xi_1, \xi_2} \int_0^{2\pi} \frac{i}{2\pi} e^{i(\xi_1 - \xi_2)\theta} \xi_2 \widehat{w}(\xi_1) \overline{\widehat{w}}(\xi_2)\, d\theta \\ &= \mathrm{Im} \sum_{\xi_1 = \xi_2} i \xi_2 \widehat{w}(\xi_1) \overline{\widehat{w}}(\xi_2) = \mathrm{Im} \sum_{\xi} i \xi |\widehat{w}(\xi)|^2 \\ &= \sum_{\xi} \xi |\widehat{w}(\xi)|^2. \end{aligned}$$

Therefore, the one of the resonant terms associated with $-w^2 \overline{w}_x$ cancels out and Eq. (1.18) for w becomes

$$\begin{aligned} w_t - i w_{xx} = NR(-w^2 \overline{w}_x) + \mathcal{R}_1(w) + \frac{i}{2} |w|^4 w - i\mu |w|^2 w \\ + iw \left(-\frac{1}{4\pi} \int_0^{2\pi} |w|^4(\theta, t)\, d\theta + \mu^2 \right). \end{aligned} \tag{2.4}$$

It will be shown that the additional resonant term $\mathcal{R}_1(w)$ possesses smoothing.

We now examine the quintic term

$$\begin{aligned} \frac{i}{2} |w|^4 w = \frac{i}{2} w \overline{w} w \overline{w} w = \frac{i}{2(2\pi)^{5/2}} \sum_{\xi_1, \dots, \xi_5} e^{i(\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5)x} \\ \times \widehat{w}(\xi_1) \overline{\widehat{w}(\xi_2)} \widehat{w}(\xi_3) \overline{\widehat{w}(\xi_4)} \widehat{w}(\xi_5). \end{aligned}$$

Here, the undesirable frequencies we wish to isolate are $\{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0\} \cup \{-\xi_2 + \xi_3 - \xi_4 + \xi_5 = 0\} \cup \{\xi_1 - \xi_2 - \xi_4 + \xi_5 = 0\}$. We note that these sets are not necessarily resonant. Thus, we will rewrite the above quintic term as

$$\begin{aligned} \frac{i}{2} |w|^4 w &= \frac{i}{2(2\pi)^{5/2}} \sum_{\xi_1, \dots, \xi_5} e^{i(\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5)x} \widehat{w}(\xi_1) \overline{\widehat{w}(\xi_2)} \widehat{w}(\xi_3) \overline{\widehat{w}(\xi_4)} \widehat{w}(\xi_5) \\ &:= \mathcal{A}(w) + \frac{3iw}{2(2\pi)^2} \sum_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} \widehat{w}(\xi_1) \overline{\widehat{w}(\xi_2)} \widehat{w}(\xi_3) \overline{\widehat{w}(\xi_4)} \\ &\quad - \frac{i}{(2\pi)^{5/2}} \sum_{\substack{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0 \\ \xi_1 - \xi_2 - \xi_4 + \xi_5 = 0 \\ -\xi_2 + \xi_3 - \xi_4 + \xi_5 = 0}} e^{i\xi_1 x} \widehat{w}(\xi_1) \overline{\widehat{w}(\xi_2)} \widehat{w}(\xi_3) \overline{\widehat{w}(\xi_4)} \widehat{w}(\xi_5) \end{aligned}$$

$$
\begin{aligned}
&- \frac{3i}{2(2\pi)^{5/2}} \sum_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\ \xi_1-\xi_2-\xi_4+\xi_5=0\\ -\xi_2+\xi_3-\xi_4+\xi_5\neq 0}} e^{i\xi_5 x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}\widehat{w}(\xi_5)\\
&:= \mathcal{A}(w) + \frac{3iw}{2(2\pi)^2}\sum_{\xi_1-\xi_2+\xi_3-\xi_4=0}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)} + \mathcal{R}_1^*(w) + \mathcal{R}_2^*(w)
\end{aligned} \tag{2.5}
$$

where we have once again used symmetry and let

$$
\mathcal{A}(w) = \frac{i}{2(2\pi)^{5/2}} \sum_{\substack{\xi_2+\xi_4\neq\xi_1+\xi_3\\ \xi_2+\xi_4\neq\xi_1+\xi_5\\ \xi_2+\xi_4\neq\xi_3+\xi_5}} e^{i(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5)x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}\widehat{w}(\xi_5), \tag{2.6}
$$

$$
\begin{aligned}
\mathcal{R}_1^*(w) &= -\frac{i}{(2\pi)^{5/2}} \sum_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\ \xi_1-\xi_2-\xi_4+\xi_5=0\\ -\xi_2+\xi_3-\xi_4+\xi_5=0}} e^{i\xi_1 x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}\widehat{w}(\xi_5)\\
&= -\frac{i}{(2\pi)^{5/2}} \sum_{2\xi_1=\xi_2+\xi_4} e^{i\xi_1 x}\,[\widehat{w}(\xi_1)]^3\,\overline{\widehat{w}(\xi_2)\widehat{w}(\xi_4)} \qquad (2.7)\\
\mathcal{R}_2^*(w) &= -\frac{3i}{2(2\pi)^{5/2}} \sum_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\ \xi_1-\xi_2-\xi_4+\xi_5=0\\ -\xi_2+\xi_3-\xi_4+\xi_5\neq 0}} e^{i\xi_3 x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}\widehat{w}(\xi_5)\\
&= -\frac{3i}{2(2\pi)^{5/2}} \sum_{\substack{\xi_1-\xi_2+\xi_3-\xi_4=0\\ \xi_2+\xi_4\neq 2\xi_3}} e^{i\xi_3 x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\,[\widehat{w}(\xi_3)]^2\,\overline{\widehat{w}(\xi_4)} \qquad (2.8)
\end{aligned}
$$

Next, we observe that

$$
\begin{aligned}
-\frac{1}{4\pi}\int_0^{2\pi}|w|^4(\theta,t)\,d\theta &= -\frac{1}{4\pi}\int_0^{2\pi}\frac{1}{(2\pi)^2}\sum_{\xi_1,\dots,\xi_4} e^{i(\xi_1-\xi_2+\xi_3-\xi_4)\theta}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}\,d\theta\\
&= -\frac{1}{2(2\pi)^2}\sum_{\xi_1-\xi_2+\xi_3-\xi_4=0}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3)\overline{\widehat{w}(\xi_4)}.
\end{aligned}
$$

Thus, we may write (2.4) as

$$
\begin{aligned}
w_t - iw_{xx} &= NR(-w^2\overline{w}_x) + \mathcal{R}_1(w) + \mathcal{A}(w) + \mathcal{R}_1^*(w) + \mathcal{R}_2^*(w)\\
&\quad - i\mu|w|^2 w + \frac{iw}{2\pi}\int_0^{2\pi}|w(\theta,t)|^4\,d\theta + i\mu^2 w.
\end{aligned}
$$

Finally, we recognize that $|w|^2 w$ contains the same resonant frequencies as $-w^2\overline{w}_x$. Thus, writing

$$\begin{aligned}|w|^2w &= NR(|w|^2w) + \frac{2w}{2\pi}\sum_{\xi}|\widehat{w}(\xi)|^2 - \frac{1}{(2\pi)^{3/2}}\sum_{\xi_0} e^{i\xi_0 x}\widehat{w}(\xi_0)|\widehat{w}(\xi_0)|^2\\ &:= NR(|w|^2w) + \frac{2w}{2\pi}\|w\|^2_{L^2(\mathbb{T})} + \frac{i}{\mu}\mathcal{R}_2(w)\\ &= NR(|w|^2w) + 2\mu w + \frac{i}{\mu}\mathcal{R}_2(w)\end{aligned}$$

where, analogously to (2.2) and (2.3),

$$NR(|w|^2w) = \frac{1}{(2\pi)^{3/2}}\sum_{\xi_1\neq\xi_2,\ \xi_2\neq\xi_3} e^{i(\xi_1-\xi_2+\xi_3)x}\widehat{w}(\xi_1)\overline{\widehat{w}(\xi_2)}\widehat{w}(\xi_3), \tag{2.9}$$

$$\mathcal{R}_2(w) = \frac{i\mu}{(2\pi)^{3/2}}\sum_{\xi_0} e^{i\xi_0 x}\widehat{w}(\xi_0)|\widehat{w}(\xi_0)|^2, \tag{2.10}$$

we overall conclude that w formally satisfies the equation

$$\begin{aligned}w_t - iw_{xx} &= NR(-w^2\overline{w}_x) + \mathcal{R}_1(w) + \mathcal{R}_2(w) + \mathcal{A}(w) + \mathcal{R}_1^*(w) + \mathcal{R}_2^*(w)\\ &\quad - i\mu NR(|w|^2w) + \frac{iw}{2\pi}\int_0^{2\pi}|w(\theta,t)|^4\,d\theta - i\mu^2 w.\end{aligned} \tag{2.11}$$

In turn, recalling the second gauge transformation (1.19), we deduce that z satisfies

$$\begin{aligned}z_t - iz_{xx} &= NR(-z^2\overline{z}_x) + \mathcal{A}(z) + \sum_{\ell=1}^{2}\left(\mathcal{R}_\ell(z) + \mathcal{R}_\ell^*(z)\right)\\ &\quad - i\mu NR(|z|^2z) + \frac{i}{4\pi}\|z(t)\|^4_{L^4(\mathbb{T})}\,z, \quad x\in\mathbb{T},\ t\in\mathbb{R},\end{aligned} \tag{2.12a}$$

$$z(x,0) = z_0(x) \in H^s(\mathbb{T}). \tag{2.12b}$$

Next, we proceed with differentiation by parts at the interaction representation level $e^{it\partial_x^2}z$ of Eq. (2.12a) in order to determine a nonlinear smoothing effect for the solution z emanating from Theorem 1.1.

First, we isolate the more troublesome terms associated with (2.12a) and then we apply differentiation by parts to these terms. Note that

$$\begin{aligned}\partial_t(e^{it\xi^2}\widehat{z}(\xi)) &= \frac{1}{2\pi}\sum_{\substack{\xi_1-\xi_2+\xi_3=\xi\\ \xi_1\neq\xi_2,\,\xi_2\neq\xi_3}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + e^{it\xi^2}\widehat{\mathcal{A}(z)}(\xi)\\ &\quad+\sum_{\ell=1}^{2} e^{it\xi^2}\left(\widehat{\mathcal{R}_\ell(z)}(\xi)+\widehat{\mathcal{R}^*_\ell(z)}(\xi)\right)\\ &\quad- i\mu e^{it\xi^2}\mathcal{F}_x\left(NR(|z|^2z)\right)(\xi) + \frac{i}{4\pi}\|z(t)\|^4_{L^4(\mathbb{T})} e^{it\xi^2}\widehat{z}(\xi). \end{aligned} \tag{2.13}$$

Next, letting $S_\xi := \{\xi_1-\xi_2+\xi_3=\xi\}\cap\{\xi_1\neq\xi_2, \xi_2\neq\xi_3\}$, we write

$$\begin{aligned}&\sum_{S_\xi} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &= \sum_{\substack{S_\xi\\ |\xi_1|\gg|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + \sum_{\substack{S_\xi\\ |\xi_1|\lesssim|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &= \sum_{\substack{S_\xi\\ |\xi_1|\gg|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + \sum_{\substack{S_\xi\\ |\xi_1|\sim|\xi_2|\gg|\xi_3|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &\quad+\sum_{\substack{S_\xi\\ |\xi_1|\sim|\xi_2|\sim|\xi_3|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + \sum_{\substack{S_\xi\\ |\xi_1|\sim|\xi_2|\ll|\xi_3|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &\quad+\sum_{\substack{S_\xi\\ |\xi_1|\ll|\xi_2|\ll|\xi_3|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + \sum_{\substack{S_\xi\\ |\xi_1|\ll|\xi_2|\sim|\xi_3|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &\quad+\sum_{\substack{S_\xi\\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\ &=: 2\pi e^{it\xi^2}\sum_{\ell=1}^{6}\widehat{\mathcal{B}_\ell(z)}(\xi) + \sum_{\substack{S_\xi\\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3),\end{aligned} \tag{2.14}$$

where

$$\widehat{\mathcal{B}_1(z)}(\xi) = \frac{1}{2\pi}\sum_{\substack{S_\xi\\ |\xi_1|\gg|\xi_2|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.15a}$$

$$\widehat{\mathcal{B}_2(z)}(\xi) = \frac{1}{2\pi}\sum_{\substack{S_\xi\\ |\xi_1|\sim|\xi_2|\gg|\xi_3|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.15b}$$

$$\widehat{\mathcal{B}_3(z)}(\xi) = \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|\sim|\xi_2|\sim|\xi_3|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.15c}$$

$$\widehat{\mathcal{B}_4(z)}(\xi) = \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|\sim|\xi_2|\ll|\xi_3|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.15d}$$

$$\widehat{\mathcal{B}_5(z)}(\xi) = \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|\ll|\xi_2|\ll|\xi_3|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.15e}$$

$$\widehat{\mathcal{B}_6(z)}(\xi) = \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|\ll|\xi_2|\sim|\xi_3|}} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3). \tag{2.15f}$$

Thus, it follows that

$$\begin{aligned}\partial_t(e^{it\xi^2}\widehat{z}(\xi)) &= e^{it\xi^2}\sum_{\ell=1}^{6}\widehat{\mathcal{B}_\ell(z)}(\xi) + e^{it\xi^2}\widehat{\mathcal{A}(z)}(\xi) \\ &+ \sum_{\ell=1}^{2} e^{it\xi^2}\left(\widehat{\mathcal{R}_\ell(z)}(\xi) + \widehat{\mathcal{R}_\ell^*(z)}(\xi)\right) - i\mu e^{it\xi^2}\mathcal{F}_x\left(NR(|z|^2 z)\right)(\xi) \\ &+ \frac{i}{4\pi}e^{it\xi^2}\,\|z(t)\|_{L^4(\mathbb{T})}^4\,\widehat{z}(\xi) + \frac{1}{2\pi}\sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3).\end{aligned} \tag{2.16}$$

We will apply differentiation by parts on the last term of (2.16), as this term does not necessarily gain derivatives via the Fourier restriction norm method. First, observe that

$$\begin{aligned}&\sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) \\ &\quad = \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{2it(\xi_2-\xi_1)(\xi_2-\xi_3)} i\xi_2(e^{it\xi_1^2}\widehat{z}(\xi_1))\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3)).\end{aligned}$$

Let

$$\Psi = 2(\xi_2 - \xi_1)(\xi_2 - \xi_3). \tag{2.17}$$

Applying differentiation by parts and symmetry in ξ_1 and ξ_3 yields

$$\begin{aligned}&\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{2it(\xi_2-\xi_1)(\xi_2-\xi_3)} i\xi_2 (e^{it\xi_1^2}\widehat{z}(\xi_1))\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3))\\ &=: \partial_t\left[e^{it\xi^2}\widehat{NF(z)}(\xi)\right]-2\widehat{\mathcal{N}_1(z)}(\xi)-\widehat{\mathcal{N}_2(z)}(\xi),\end{aligned}$$

where

$$\widehat{NF(z)}(\xi)=\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} \frac{\xi_2}{\Psi}\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3), \tag{2.18}$$

$$\widehat{\mathcal{N}_1(z)}(\xi)=\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\Psi}\frac{\xi_2}{\Psi}\partial_t(e^{it\xi_1^2}\widehat{z}(\xi_1))\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3)), \tag{2.19}$$

$$\widehat{\mathcal{N}_2(z)}(\xi)=\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\Psi}\frac{\xi_2}{\Psi}(e^{it\xi_1^2}\widehat{z}(\xi_1))\partial_t\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3)). \tag{2.20}$$

Inserting (2.13) into $\mathcal{N}_1(z)$, we obtain

$$\begin{aligned}\widehat{\mathcal{N}_1(z)}(\xi)&=\frac{1}{4\pi^2}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\Psi}\frac{\xi_2}{\Psi}\Bigg(\sum_{\substack{\xi_4-\xi_5+\xi_6=\xi_1\\ \xi_4\neq\xi_5\neq\xi_6}} e^{it\xi_1^2} i\xi_5\widehat{z}(\xi_4)\overline{\widehat{z}(\xi_5)}\widehat{z}(\xi_6)\Bigg)\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3))\\
&+\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2}\frac{\xi_2}{\Psi}\widehat{\mathcal{A}(z)}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\
&-\frac{i\mu}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2}\frac{\xi_2}{\Psi}\mathcal{F}_x\Big(NR(|z|^2z)\Big)(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\
&+\frac{1}{2\pi}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2}\frac{\xi_2}{\Psi}\sum_{\ell=1}^{2}\Big(\widehat{\mathcal{R}_\ell(z)}(\xi_1)+\widehat{\mathcal{R}^*_\ell(z)}(\xi_1)\Big)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\\
&+\frac{i}{(2\pi)(4\pi)}\|z(t)\|^4_{L^4(\mathbb{T})}\sum_{\substack{S_\xi\\|\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\Psi}\frac{\xi_2}{\Psi}(e^{it\xi_1^2}\widehat{z}(\xi_1))\overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))}(e^{it\xi_3^2}\widehat{z}(\xi_3))\\
&=: e^{it\xi^2}\sum_{\ell=1}^{4}\widehat{\mathcal{N}_{1,\ell}(z)}(\xi)+\frac{i}{4\pi}\|z(t)\|^4_{L^4(\mathbb{T})}e^{it\xi^2}\widehat{NF(z)}(\xi).\end{aligned} \tag{2.21}$$

Similarly, for $\mathcal{N}_2(z)$ we find

$$\begin{aligned}
\widehat{\mathcal{N}_2(z)}(\xi) &= \frac{1}{4\pi^2} \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} e^{it\Psi} \frac{\xi_2}{\Psi} (e^{it\xi_1^2}\widehat{z}(\xi_1)) \overline{\Bigg(\sum_{\substack{\xi_4-\xi_5+\xi_6=\xi_2 \\ \xi_4 \neq \xi_5 \neq \xi_6}} e^{it\xi_2^2} i\xi_5 \widehat{z}(\xi_4)\overline{\widehat{z}(\xi_5)}\widehat{z}(\xi_6) \Bigg)} (e^{it\xi_3^2}\widehat{z}(\xi_3)) \\
&\quad + \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} e^{it\xi^2} \frac{\xi_2}{\Psi} \widehat{z}(\xi_1) \overline{\widehat{\mathcal{A}(z)}(\xi_2)} \widehat{z}(\xi_3) \\
&\quad + \frac{i\mu}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} e^{it\xi^2} \frac{\xi_2}{\Psi} \widehat{z}(\xi_1) \overline{\mathcal{F}_x\left(NR(|z|^2 z)\right)(\xi_2)} \widehat{z}(\xi_3) \\
&\quad + \frac{1}{2\pi} \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} e^{it\xi^2} \frac{\xi_2}{\Psi} \widehat{z}(\xi_1) \sum_{\ell=1}^{2} \overline{\left(\widehat{\mathcal{R}_\ell(z)}(\xi_1) + \widehat{\mathcal{R}_\ell^*(z)}(\xi_1)\right)} \widehat{z}(\xi_3) \\
&\quad - \frac{i}{(2\pi)(4\pi)} \|z(t)\|_{L^4(\mathbb{T})}^4 \sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} e^{it\Psi} \frac{\xi_2}{\Psi} (e^{it\xi_1^2}\widehat{z}(\xi_1)) \overline{(e^{it\xi_2^2}\widehat{z}(\xi_2))} (e^{it\xi_3^2}\widehat{z}(\xi_3)) \\
&=: e^{it\xi^2} \sum_{\ell=1}^{4} \widehat{\mathcal{N}_{2,\ell}(z)}(\xi) - \frac{i}{4\pi} \|z(t)\|_{L^4(\mathbb{T})}^4 e^{it\xi^2} \widehat{NF(z)}(\xi).
\end{aligned} \tag{2.22}$$

Further expanding these terms, we see that

$$\begin{aligned}
\widehat{\mathcal{N}_{1,1}(z)}(\xi) &= \frac{1}{4\pi^2} \sum_{N_{1,1}(\xi)} \frac{i\xi_2\xi_4}{\Psi_{1,1}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5), \\
\widehat{\mathcal{N}_{1,2}(z)}(\xi) &= \frac{1}{16\pi^3} \sum_{N_{1,2}(\xi)} \frac{i\xi_6}{\Psi_{1,2}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5)\overline{\widehat{z}(\xi_6)}\widehat{z}(\xi_7) \\
\widehat{\mathcal{N}_{2,1}(z)}(\xi) &= \frac{1}{4\pi^2} \sum_{N_{2,1}(\xi)} \frac{-i(\xi_2-\xi_3+\xi_4)\xi_3}{\Psi_{2,1}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5), \\
\widehat{\mathcal{N}_{2,2}(z)}(\xi) &= \frac{1}{16\pi^3} \sum_{N_{2,2}(\xi)} \frac{-i(\xi_2-\xi_3+\xi_4-\xi_5+\xi_6)}{\Psi_{2,2}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5)\overline{\widehat{z}(\xi_6)}\widehat{z}(\xi_7),
\end{aligned}$$

where the sets $N_{1,1}$, $N_{1,2}$ are given by

$$\begin{aligned}
N_{1,1}(\xi) &= \{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5=\xi\} \cap \{\xi_1-\xi_2+\xi_3 \neq \xi_4 \neq \xi_5\} \\
&\qquad \cap \{\xi_1 \neq \xi_2,\ \xi_2 \neq \xi_3\} \cap \{|\xi_1-\xi_2+\xi_3|, |\xi_5| \ll |\xi_4|\}, \\
N_{1,2}(\xi) &= \{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5-\xi_6+\xi_7=\xi\} \\
&\qquad \cap \{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5 \neq \xi_6 \neq \xi_7\} \\
&\qquad \cap \{\xi_2+\xi_4 \neq \xi_1+\xi_3, \xi_1+\xi_5, \xi_3+\xi_5\} \\
&\qquad \cap \{|\xi_1-\xi_2+\xi_3+\xi_4+\xi_5|, |\xi_7| \ll |\xi_6|\},
\end{aligned}$$

the sets $N_{2,1}$ and $N_{2,2}$ are given by

$$
\begin{aligned}
N_{2,1}(\xi) &= \{\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5 = \xi\} \cap \{\xi_1 \neq \xi_2 - \xi_3 + \xi_4 \neq \xi_5\} \\
&\qquad \cap \{\xi_2 \neq \xi_3 \neq \xi_4\} \cap \{|\xi_1|, |\xi_5| \ll |\xi_2 - \xi_3 + \xi_4|\}, \\
N_{2,2}(\xi) &= \{\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 + \xi_7 = \xi\} \\
&\qquad \cap \{\xi_1 \neq \xi_2 - \xi_3 + \xi_4 - \xi_5 + \xi_6 \neq \xi_7\} \\
&\qquad \cap \{\xi_3 + \xi_5 \neq \xi_2 + \xi_4, \xi_2 + \xi_6, \xi_4 + \xi_6\} \\
&\qquad \cap \{|\xi_1|, |\xi_7| \ll |\xi_2 - \xi_3 + \xi_4 - \xi_5 + \xi_6|\},
\end{aligned}
$$

and $\Psi_{i,j}$ is the relabeling of Ψ according to the set $N_{i,j}$. The analysis of $\mathcal{N}_{1,1}(z)$ and $\mathcal{N}_{2,1}(z)$ covers the analysis of $\mathcal{N}_{1,3}(z)$ and $\mathcal{N}_{2,3}(z)$, respectively; therefore, the analysis of the latter two terms is omitted. Similarly, we do not expand $\mathcal{N}_{1,4}(z)$ or $\mathcal{N}_{2,4}(z)$.

Finally, we note that $\mathcal{N}_{2,1}(z)$ possesses the unfavorable frequency interaction $\xi_2 + \xi_4 = \xi_1 + \xi_5$. To handle this, we decompose $\mathcal{N}_{2,1}(z)$ as

$$
\begin{aligned}
\widehat{\mathcal{N}_{2,1}(z)}(\xi) &= \frac{1}{4\pi^2} \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4 \neq \xi_1+\xi_5}} \frac{-i(\xi_2 - \xi_3 + \xi_4)\xi_3}{\Psi_{2,1}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5) \\
&\quad + \frac{1}{4\pi^2} \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4 = \xi_1+\xi_5}} \frac{-i(\xi_2 - \xi_3 + \xi_4)\xi_3}{\Psi_{2,1}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5) \\
&=: \widehat{\mathcal{N}^*_{2,1}(z)}(\xi) + \frac{1}{4\pi^2} \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4 = \xi_1+\xi_5}} \frac{i(\xi - \xi_1 - \xi_5)\xi}{2(\xi - \xi_1)(\xi - \xi_5)} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5) \\
&= \widehat{\mathcal{N}^*_{2,1}(z)}(\xi) - \frac{1}{4\pi^2} \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4 = \xi_1+\xi_5}} \frac{i\xi_1\xi_5}{2(\xi - \xi_1)(\xi - \xi_5)} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5) \\
&\quad + \frac{i}{8\pi^2} \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4 = \xi_1+\xi_5}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi)\overline{\widehat{z}(\xi_4)}\widehat{z}(\xi_5) =: \sum_{\ell=1}^{3} \widehat{\mathcal{N}^*_{2,\ell}(z)}(\xi)
\end{aligned} \tag{2.23}
$$

and further expand $\mathcal{N}^*_{2,3}(z)$ as follows:

$$
\begin{aligned}
\widehat{\mathcal{N}^*_{2,3}(z)}(\xi) &= \frac{i}{8\pi^2}\widehat{z}(\xi) \sum_{\substack{N_{2,1}(\xi) \\ \xi_1+\xi_5 = \xi_2+\xi_4}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_5)\overline{\widehat{z}(\xi_4)} \\
&= \frac{i}{8\pi^2}\widehat{z}(\xi) \sum_{\substack{\xi_1+\xi_3 = \xi_2+\xi_4 \\ |\xi_1|, |\xi_2|, |\xi_3|, |\xi_4| \ll |\xi|}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}
\end{aligned}
$$

$$+ \frac{i}{8\pi^2}\widehat{z}(\xi) \sum_{\substack{N_{2,1}(\xi) \\ \xi_1+\xi_5=\xi_2+\xi_4 \\ |\xi|\lesssim|\xi_\ell|, \text{ for some } \ell}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_5)\overline{\widehat{z}(\xi_4)}$$

$$=: \frac{i}{16\pi^3}\widehat{z}(\xi)\int_0^{2\pi}\Big|\sum_{|\xi_1|\ll|\xi|} e^{i\xi_1\theta}\widehat{z}(\xi_1)\Big|^4 d\theta + \widehat{\mathcal{E}_1(z)}(\xi)$$

$$= \frac{i}{4\pi}\widehat{z}(\xi)\int_0^{2\pi}\Big|z - \frac{1}{\sqrt{2\pi}}\sum_{|\xi|\lesssim|\xi_1|} e^{i\xi_1\theta}\widehat{z}(\xi_1)\Big|^4 d\theta + \widehat{\mathcal{E}_1(z)}(\xi)$$

$$=: \frac{i}{4\pi}\widehat{z}(\xi)\,\|z\|^4_{L^4(\mathbb{T})} + \widehat{\mathcal{E}_1(z)}(\xi) + \widehat{\mathcal{E}_2(z)}(\xi), \tag{2.24}$$

where

$$\begin{aligned}
-4\pi i\widehat{\mathcal{E}_2(z)}(\xi) &= -2\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|}\overline{\widehat{z}(\xi_1)}\widehat{|z|^2 z}(\xi_1) - 2\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|}\widehat{z}(\xi_1)\overline{\widehat{|z|^2 z}(\xi_1)} \\
&+ \frac{4}{\sqrt{2\pi}}\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|,|\xi_2|}\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\,\overline{\widehat{|z|^2}(\xi_1-\xi_2)} \\
&+ \frac{1}{\sqrt{2\pi}}\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|,|\xi_2|}\widehat{z}(\xi_1)\widehat{z}(\xi_2)\overline{\widehat{z^2}(\xi_1+\xi_2)} \\
&+ \frac{1}{\sqrt{2\pi}}\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|,|\xi_2|}\overline{\widehat{z}(\xi_1)\widehat{z}(\xi_2)}\widehat{z^2}(\xi_1+\xi_2) \\
&- \frac{2}{2\pi}\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|,|\xi_2|,|\xi_3|}\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)\widehat{z}(\xi_3)}\widehat{z}(-\xi_1+\xi_2+\xi_3) \\
&- \frac{2}{2\pi}\widehat{z}(\xi)\sum_{|\xi|\lesssim|\xi_1|,|\xi_2|,|\xi_3|}\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_1-\xi_2+\xi_3)} \\
&+ \frac{1}{2\pi}\widehat{z}(\xi)\sum_{\substack{\xi_1+\xi_3=\xi_2+\xi_4 \\ |\xi|\lesssim|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|}}\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3)\overline{\widehat{z}(\xi_4)}.
\end{aligned} \tag{2.25}$$

Overall, we have

$$\begin{aligned}
\partial_t(e^{it\xi^2}\widehat{z}(\xi)) &= e^{it\xi^2}\Big(\sum_{\ell=1}^{6}\widehat{\mathcal{B}_\ell(z)}(\xi) + \widehat{\mathcal{A}(z)}(\xi) - i\mu\,\mathcal{F}_x\big(NR(|z|^2 z)\big)(\xi) \\
&+ \sum_{\ell=1}^{2}\Big[\widehat{\mathcal{R}_\ell(z)}(\xi) + \widehat{\mathcal{R}^*_\ell(z)}(\xi)\Big]\Big) \\
&+ \frac{1}{2\pi}\sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3|\ll|\xi_2|}} e^{it\xi^2} i\xi_2\widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)}\widehat{z}(\xi_3) + \frac{i}{4\pi}\,\|z(t)\|^4_{L^4(\mathbb{T})}\,e^{it\xi^2}\widehat{z}(\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \partial_t \left[e^{it\xi^2} \widehat{NF(z)}(\xi) \right] \\
&\quad + e^{it\xi^2} \left(\sum_{\ell=1}^{6} \widehat{\mathcal{B}_\ell(z)}(\xi) + \widehat{\mathcal{A}(z)}(\xi) - i\mu \, \mathcal{F}_x \left(NR(|z|^2 z) \right)(\xi) \right) \\
&\quad + e^{it\xi^2} \left(\sum_{\ell=1}^{2} \left[\widehat{\mathcal{R}_\ell(z)}(\xi) + \widehat{\mathcal{R}^*_\ell(z)}(\xi) \right] - 2 \sum_{\ell=1}^{4} \widehat{\mathcal{N}_{1,\ell}(z)}(\xi) - \sum_{\ell=1}^{4} \widehat{\mathcal{N}_{2,\ell}(z)}(\xi) \right) \\
&\quad + \frac{i}{4\pi} \, \|z(t)\|^4_{L^4(\mathbb{T})} \, e^{it\xi^2} \widehat{z}(\xi) - \frac{i}{4\pi} \, \|z(t)\|^4_{L^4(\mathbb{T})} \, e^{it\xi^2} \widehat{NF(z)}(\xi),
\end{aligned}
$$

where

$$
\widehat{\mathcal{N}_{2,1}(z)}(\xi) = \sum_{\ell=1}^{2} \left(\widehat{\mathcal{N}^*_{2,\ell}(z)}(\xi) + \widehat{\mathcal{E}_\ell(z)}(\xi) \right) + \frac{i}{4\pi} \widehat{z}(\xi) \, \|z\|^4_{L^4(\mathbb{T})} .
$$

Hence, we see that

$$
\begin{aligned}
\partial_t (e^{it\xi^2} \widehat{z}(\xi)) &= \partial_t \left[e^{it\xi^2} \widehat{NF(z)}(\xi) \right] \\
&\quad + e^{it\xi^2} \left(\sum_{\ell=1}^{6} \widehat{\mathcal{B}_\ell(z)}(\xi) + \widehat{\mathcal{A}(z)}(\xi) - i\mu \, \mathcal{F}_x \left(NR(|z|^2 z) \right)(\xi) \right) \\
&\quad + e^{it\xi^2} \left(\sum_{\ell=1}^{2} \left[\widehat{\mathcal{R}_\ell(z)}(\xi) + \widehat{\mathcal{R}^*_\ell(z)}(\xi) \right] - 2 \sum_{\ell=1}^{4} \widehat{\mathcal{N}_{1,\ell}(z)}(\xi) - \sum_{\ell=2}^{4} \widehat{\mathcal{N}_{2,\ell}(z)}(\xi) \right) \\
&\quad - e^{it\xi^2} \sum_{\ell=1}^{2} \left(\widehat{\mathcal{N}^*_{2,\ell}(z)}(\xi) + \widehat{\mathcal{E}_\ell(z)}(\xi) \right) - \frac{i}{4\pi} \, \|z(t)\|^4_{L^4(\mathbb{T})} \, e^{it\xi^2} \widehat{NF(z)}(\xi).
\end{aligned}
\tag{2.26}
$$

Finally, we integrate (2.26) on $[0, t]$ and invert the Fourier transform to arrive at

$$
z(x,t) = e^{it\partial_x^2} z_0(x) + NF(z)(x,t) - e^{it\partial_x^2} NF(z_0)(x) + \int_0^t e^{i(t-t')\partial_x^2} N(z)(x,t') \, dt', \tag{2.27}
$$

where

$$
\begin{aligned}
\widehat{N(z)}(\xi) &= \sum_{\ell=1}^{6} \widehat{\mathcal{B}_\ell(z)}(\xi) + \widehat{\mathcal{A}(z)}(\xi) - i\mu \, \mathcal{F}_x \left(NR(|z|^2 z) \right)(\xi) \\
&\quad + \sum_{\ell=1}^{2} \left[\widehat{\mathcal{R}_\ell(z)}(\xi) + \widehat{\mathcal{R}^*_\ell(z)}(\xi) \right] - 2 \sum_{\ell=1}^{4} \widehat{\mathcal{N}_{1,\ell}(z)}(\xi) - \sum_{\ell=2}^{4} \widehat{\mathcal{N}_{2,\ell}(z)}(\xi) \\
&\quad - \sum_{\ell=1}^{2} \left(\widehat{\mathcal{N}^*_{2,\ell}(z)}(\xi) + \widehat{\mathcal{E}_\ell(z)}(\xi) \right) - \frac{i}{4\pi} \, \|z(t)\|^4_{L^4(\mathbb{T})} \, e^{it\xi^2} \widehat{NF(z)}(\xi).
\end{aligned}
$$

3 A priori estimates

We begin by recalling certain linear estimates from the literature which are relevant for our analysis. From Bourgain [4], we have the following useful estimates.

Lemma 3.1 [4] *For any* $\delta > 0$, *let* $u \in X^{0,\frac{3}{8}+\delta}$. *Then, there exists a constant* $c > 0$ *such that*

$$\|u\|_{L^4_{t,x}} \le c\,\|u\|_{X^{0,\frac{3}{8}+\delta}} . \tag{3.1}$$

Furthermore, if $u \in X^{\delta,\frac{1}{2}+\delta}$ *then there exists a constant* $c > 0$ *such that*

$$\|u\|_{L^6_{t,x}} \le c\,\|u\|_{X^{\delta,\frac{1}{2}+\delta}} . \tag{3.2}$$

In what follows, we assume $\eta_T \in C_0^\infty(-2T, 2T)$ is symmetric and $\eta_T \equiv 1$ on $[-T, T]$, for $0 < T \le 1$. The following linear estimates are proved by Herr in [30].

Lemma 3.2 [30, Lemma 3.3] *If* $2 \le p, q < \infty$, $b \ge \frac{1}{2} - \frac{1}{p}$ *and* $s \ge \frac{1}{2} - \frac{1}{q}$, *then*

$$\|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{s,b}} . \tag{3.3}$$

Moreover, for $s \in \mathbb{R}$,

$$\|u\|_{C(\mathbb{R},H^s(\mathbb{T}))} \lesssim \|u\|_{Z^s} . \tag{3.4}$$

Finally, for $b_2 > b_1 + \frac{1}{2}$,

$$\|z\|_{Y^{s,b_1}} \lesssim \|z\|_{X^{s,b_2}} . \tag{3.5}$$

In particular, for all $b > \frac{1}{2}$, $X^{s,b} \hookrightarrow Z^s$.

Lemma 3.3 [30, Lemma 3.6] *Let* $s \in \mathbb{R}$ *and* $z_0 \in H^s(\mathbb{T})$. *Then,*

$$\left\|\eta_T e^{it\partial_x^2} z_0\right\|_{Z^s} \lesssim_T \|z_0\|_{H^s(\mathbb{T})} . \tag{3.6}$$

Moreover, for $F \in Y^{s,-1} \cap X^{s,-\frac{1}{2}}$,

$$\left\|\eta_T \int_0^t e^{i(t-t')\partial_x^2} F(t')\,dt'\right\|_{Z^s} \lesssim_T \|F\|_{Y^{s,-1}} + \|F\|_{X^{s,-\frac{1}{2}}} . \tag{3.7}$$

We now state and prove the nonlinear estimates needed for establishing Theorem 1.3.

Lemma 3.4 *For any* $\delta > 0$, *let* $z \in H^s$ *with* $s > \frac{1}{2} + \delta$. *Then, the quantity* $NF(z)$ *defined by* (2.18) *satisfies the estimate*

$$\|NF(z)\|_{H^{s+a}(\mathbb{T})} \lesssim \|z\|^2_{H^{\frac{1}{2}+\delta}(\mathbb{T})} \|z\|_{H^s(\mathbb{T})}, \quad 0 < a < 1. \tag{3.8}$$

Proof We have

$$\begin{aligned}\|NF(z)\|_{H^{s+a}(\mathbb{T})} &= \left(\sum_{\xi} \langle \xi \rangle^{2(s+a)} \left| \widehat{NF(z)}(\xi) \right|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\xi} \langle \xi \rangle^{2(s+a)} \left(\sum_{\substack{S_\xi \\ |\xi_1|,|\xi_3| \ll |\xi_2|}} \frac{\langle \xi_2 \rangle}{\langle \xi_2 - \xi_1 \rangle \langle \xi_2 - \xi_3 \rangle} |\widehat{z}(\xi_1)||\widehat{z}(\xi_2)||\widehat{z}(\xi_3)| \right)^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Next, we note that $|\xi - \xi_1| \sim |\xi - \xi_3| \sim |\xi_2| \sim |\xi|$. Applying this result and the Cauchy–Schwarz inequality in ξ_1 and ξ_3, we find

$$\begin{aligned}&\left(\sum_{\xi} \langle \xi \rangle^{2(s+a)} \left(\sum_{\substack{\xi_1,\xi_3 \\ |\xi_1|,|\xi_3| \ll |\xi - \xi_1 - \xi_3|}} \frac{\langle \xi - \xi_1 - \xi_3 \rangle}{\langle \xi - \xi_1 \rangle \langle \xi - \xi_3 \rangle} |\widehat{z}(\xi_1)||\widehat{z}(\xi - \xi_1 - \xi_3)||\widehat{z}(\xi_3)| \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{\xi} \langle \xi \rangle^{2(s+a-1)} \left(\sum_{\xi_1,\xi_3} |\widehat{z}(\xi_1)||\widehat{z}(\xi - \xi_1 - \xi_3)||\widehat{z}(\xi_3)| \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{\xi} \langle \xi \rangle^{2(a-1)} \sum_{\xi_1,\xi_3} \langle \xi_1 \rangle^{2(\frac{1}{2}+\delta)} |\widehat{z}(\xi_1)|^2 \langle \xi - \xi_1 - \xi_3 \rangle^{2s} |\widehat{z}(\xi - \xi_1 - \xi_3)|^2 \langle \xi_3 \rangle^{2(\frac{1}{2}+\delta)} |\widehat{z}(\xi_3)|^2 \right)^{\frac{1}{2}} \\ &\leq \sup_{\xi} \langle \xi \rangle^{(a-1)} \|z\|^2_{H^{\frac{1}{2}+\delta}(\mathbb{T})} \|z\|_{H^s(\mathbb{T})}.\end{aligned}$$

The above supremum is finite for $a < 1$, completing the proof. □

Lemma 3.5 *For any* $\delta > 0$, *let* $z \in C([0, T]; H^s(\mathbb{T}))$ *with* $s > \frac{1}{2} + \delta$, $T > 0$. *Then,*

$$\begin{aligned}&\left\| \int_0^t e^{i(t-t')\partial_x^2} \|z(t')\|^4_{L^4(\mathbb{T})} NF(z)(t')\, dt' \right\|_{H^{s+a}(\mathbb{T})} \\ &\lesssim T \|z\|^6_{C([0,T]; H^{\frac{1}{2}+\delta}(\mathbb{T}))} \|z\|_{C([0,T]; H^s(\mathbb{T}))}\end{aligned} \tag{3.9}$$

for all $0 < a < 1$ *and* $t \in [0, T]$.

Proof We simply note that

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-t')\partial_x^2} \left\| z(t') \right\|_{L^4(\mathbb{T})}^4 NF(z)(t')\, dt' \right\|_{H^{s+a}(\mathbb{T})} \\
&\leq \int_0^t \left\| z(t') \right\|_{L^4(\mathbb{T})}^4 \left\| NF(z)(t') \right\|_{H^{s+a}(\mathbb{T})} \, dt' \\
&\leq T \sup_{t\in[0,T]} \|z(t)\|_{L^4(\mathbb{T})}^4 \|NF(z)(t)\|_{H^{s+a}(\mathbb{T})} .
\end{aligned}$$

The claim then follows from Sobolev embedding and Lemma 3.4. □

Lemma 3.6 *Let $s > \frac{1}{2}$. Then, for all $\delta > 0$ and $0 < a < \min\left\{s - \frac{1}{2} - 15\delta, \frac{1}{2} - 15\delta\right\}$, the quantities $\mathcal{B}_\ell(z)$ and $NR(|z|^2 z)$ defined by (2.15) and (2.9) satisfy the estimate*

$$\sum_{\ell=1}^{6} \left\| \mathcal{B}_\ell(z) \right\|_{X^{s+a,-\frac{1}{2}+\delta}} + \left\| NR(|z|^2 z) \right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|_{Z^\sigma}^2 \|z\|_{Z^s} . \tag{3.10}$$

Proof We will prove the bound for $\mathcal{B}_1(z)$, $\mathcal{B}_2(z)$, and $\mathcal{B}_3(z)$, as the same proofs apply for the other terms. By applying Littlewood–Paley-type projections, Plancherel's theorem and duality, we have

$$\begin{aligned}
\|\mathcal{B}_1(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} &\lesssim \sum_{j,k} 2^{j(s+a)} 2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \sum_{\substack{S_\xi \\ |\xi_1|\gg|\xi_2|}} \\
&\quad\times \int_{\tau_1-\tau_2+\tau_3=\tau} |\widetilde{z}_{\gtrsim j}(\xi_1,\tau_1)||\widetilde{z}_k(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{\varphi}_j(\xi,\tau)|\, d\sigma,
\end{aligned}$$

where we have assumed, without loss of generality, that $|\xi_1| \geq |\xi_3|$. Note that, in the above region of summation and integration, we have

$$(\tau + \xi^2) = \sum_{\ell=1}^{3} (-1)^{\ell+1} (\tau_\ell + \xi_\ell^2) + 2(\xi_2 - \xi_1)(\xi_2 - \xi_3).$$

Thus, for $\tau + \xi^2 := \tau_0 + \xi_0^2$, we have $\max\left\{\left|\tau_m + \xi_m^2\right|\right\}_{m=0}^3 \gtrsim |(\xi_2 - \xi_1)(\xi_2 - \xi_3)|$.

First, suppose that $\max\left\{\left|\tau_m+\xi_m^2\right|\right\}_{m=0}^3=|\tau+\xi^2|$. Then,

$$\sum_{\substack{S_\xi\\|\xi_1|\gg|\xi_2|}}\int_{\tau_1-\tau_2+\tau_3=\tau}|\widetilde{z}_{\gtrsim j}(\xi_1,\tau_1)||\widetilde{z}_k(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{\varphi}_j(\xi,\tau)|\,d\sigma$$

$$\lesssim\sum_{\substack{S_\xi\\|\xi_1|\gg|\xi_2|}}\int_{\tau_1-\tau_2+\tau_3=\tau}\frac{\langle\tau+\xi^2\rangle^{\frac{1}{2}-\delta}}{\langle\xi_2-\xi_1\rangle^{\frac{1}{2}-\delta}\langle\xi_2-\xi_3\rangle^{\frac{1}{2}-\delta}}|\widetilde{z}_{\gtrsim j}$$
$$\times(\xi_1,\tau_1)||\widetilde{z}_k(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{\varphi}_j(\xi,\tau)|\,d\sigma,$$

where $d\sigma$ is the surface measure. Let $z^{\#}=\mathcal{F}_{t,x}^{-1}(|\widetilde{z}|)$. Since $|\xi_1|\gg|\xi_2|$, we may apply Hölder's inequality, Sobolev embedding, and estimates (3.1) and (3.4) to arrive at

$$\sum_{j,k}2^{j(s+a)}2^k\sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1}\sum_{\substack{S_\xi\\|\xi_1|\gg|\xi_2|}}\int_{\tau_1-\tau_2+\tau_3=\tau}|\widetilde{z}_{\gtrsim j}(\xi_1,\tau_1)||\widetilde{z}_k(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{\varphi}_j(\xi,\tau)|\,d\sigma$$
$$\lesssim\sum_{j,k}2^{j(s+a-\frac{1}{2}+\delta)}2^k\left\|(z_{\gtrsim j})^{\#}(z_k)^{\#}z^{\#}\right\|_{L^2_{t,x}}$$
$$\lesssim\sum_{j,k}2^{j(s+a-\frac{1}{2}+\delta)}2^k\left\|z^{\#}\right\|_{L^\infty_{t,x}}\left\|(z_{\gtrsim j})^{\#}\right\|_{L^4_{t,x}}\left\|(z_k)^{\#}\right\|_{L^4_{t,x}}$$
$$\lesssim\sum_{j,k}2^{j(a-\frac{1}{2}+\delta)}2^{k(1-\sigma)}\,\|z\|_{Z^{\frac{1}{2}+\delta}}\,\|z\|_{X^{\sigma,\frac{1}{2}}}\,\|z\|_{X^{s,\frac{1}{2}}}\,.$$

Then

$$\sum_{j,k}2^{j(a-\frac{1}{2}+\delta)}2^{k(1-\sigma)}\lesssim\sum_j 2^{j(a+\frac{1}{2}+\delta-\sigma)},$$

which converges provided that $a<\sigma-\frac{1}{2}-\delta$. If $\max\left\{\left|\tau_m+\xi_m^2\right|\right\}_{m=0}^3=|\tau_1+\xi_1^2|$ or $\max\left\{|\tau_m+\xi_m^2|\right\}_{m=0}^3=|\tau_2+\xi_2^2|$, a similar argument yields the same result.

Next, recall that the interpolation of the Bourgain spaces X^{s_1,b_1} and X^{s_2,b_2} at θ is given by

$$[X^{s_1,b_1},X^{s_2,b_2}]_\theta=X^{s_1(1-\theta)+s_2\theta,b_1(1-\theta)+b_2\theta}.$$

By estimates (3.2) and (3.3), we have $X^{\delta,\frac{1}{2}+\delta}\hookrightarrow L^6(\mathbb{T}\times\mathbb{R})$ and $X^{\frac{1}{3},\frac{1}{3}}\hookrightarrow L^6(\mathbb{T}\times\mathbb{R})$, respectively. Thus,

$$[X^{\frac{1}{3},\frac{1}{3}},X^{\delta,\frac{1}{2}+\delta}]_\theta\hookrightarrow[L^6(\mathbb{T}\times\mathbb{R}),L^6(\mathbb{T}\times\mathbb{R})]_\theta=L^6(\mathbb{T}\times\mathbb{R})$$

which for $\theta=(\frac{1}{6}-\delta)/(\frac{1}{6}+\delta)$ gives

$$X^{5\delta,\frac{1}{2}-\delta}\hookrightarrow[X^{\frac{1}{3},\frac{1}{3}},X^{\delta,\frac{1}{2}+\delta}]_\theta\hookrightarrow L^6(\mathbb{T}\times\mathbb{R}).$$

If $\max\left\{|\tau_m+\xi_m^2|\right\}_{m=0}^3 = |\tau_3+\xi_3^2|$, then we apply $X^{5\delta,\frac{1}{2}-\delta} \hookrightarrow L^6(\mathbb{T}\times\mathbb{R})$ along with Hölder's inequality to yield

$$\begin{aligned}
&\sum_{j,k} 2^{j(s+a)}2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \sum_{\substack{S_\xi \\ |\xi_1|\gg|\xi_2|}} \int_{\tau_1-\tau_2+\tau_3=\tau} |\widetilde{z}_{\gtrsim j}(\xi_1,\tau_1)||\widetilde{z}_k(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{\varphi}_j(\xi,\tau)|\,d\sigma \\
&\quad\lesssim \sum_{j,k} 2^{j(s+a-\frac{1}{2})}2^k \|z\|_{X^{0,\frac{1}{2}}} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \left\|(z_{\gtrsim j})^\#(z_k)^\#(\varphi_j)^\#\right\|_{L^2_{t,x}} \\
&\quad\lesssim \sum_{j,k} 2^{j(a+10\delta-\frac{1}{2})}2^{k(1+5\delta-\sigma)} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}}
\end{aligned}$$

Then

$$\sum_{j,k} 2^{j(a+10\delta-\frac{1}{2})}2^{k(1+5\delta-\sigma)} \lesssim \sum_{j,k} 2^{j(a+15\delta+\frac{1}{2}-\sigma)},$$

which converges for $a<\sigma-\frac{1}{2}-15\delta$. Overall, we have shown that $\mathcal{B}_1(z)\in X^{s+a,-\frac{1}{2}+\delta}$ provided that $0<a<\min\left\{s-\frac{1}{2}-15\delta,\frac{1}{2}-15\delta\right\}$.

Next, we note that by the definition (2.15) in the case of $\widehat{\mathcal{B}_2(z)}(\xi)$ we have $|\xi_1|\sim|\xi_2|\gg|\xi_3|$. Thus, $|\xi_2-\xi_3|\sim|\xi_2|$, which allows us to yield the same estimates as those obtained for $\mathcal{B}_1(z)$ and thereby deduce that $\mathcal{B}_2(z)\in X^{s+a,-\frac{1}{2}+\delta}$ for $a<\min\left\{s-\frac{1}{2}-15\delta,\frac{1}{2}-15\delta\right\}$.

Finally, by applying Littlewood–Paley-type projections, Hölder's inequality and estimate (3.1) once more, we have

$$\begin{aligned}
\|\mathcal{B}_3(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} &\lesssim \sum_j 2^{j(s+a+1)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} [(z_{\gtrsim j})^\#]^3(\varphi_j)^\#\,dxdt \\
&\lesssim \sum_j 2^{j(s+a+1)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \left\|(z_{\gtrsim j})^\#\right\|^3_{L^4_{t,x}} \left\|(\varphi_j)^\#\right\|_{L^4_{t,x}} \\
&\lesssim \sum_j 2^{j(a+1-2\sigma)} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}},
\end{aligned}$$

where the above sum converges for $a<2\sigma-1=\min\{2s-1,1\}$. □

Lemma 3.7 *Let* $s>\frac{1}{2}$ *and* $z\in C([0,T];H^s(\mathbb{T}))$, *for* $T>0$. *Then* $\mathcal{R}_1(z)$ *and* $\mathcal{R}_2(z)$ *given by* (2.3) *and* (2.10) *satisfy*

$$\begin{aligned}
&\sum_{\ell=1}^2 \left\|\int_0^t e^{i(t-t')\partial_x^2}\mathcal{R}_\ell(z)\,dt'\right\|_{C([0,T];H^{s+a}(\mathbb{T}))} \\
&\quad\lesssim T\,\|z\|^2_{C([0,T];H^\sigma(\mathbb{T}))}\,\|z\|_{C([0,T];H^s(\mathbb{T}))}, \qquad (3.11)
\end{aligned}$$

for $0<a<\min\{2s-1,1\}$.

Proof We will only show the estimate for $\mathcal{R}_1(z)$, as the same argument holds for $\mathcal{R}_2(z)$. First, we have

$$\left\| \int_0^t e^{i(t-t')\partial_x^2} \mathcal{R}_1(z)\, dt' \right\|_{C([0,T];H^{s+a}(\mathbb{T}))} \leq T\, \|\mathcal{R}_1(z)\|_{C([0,T];H^{s+a}(\mathbb{T}))}.$$

Note that

$$\widehat{\mathcal{R}_1(z)}(\xi) = -\frac{i}{2\pi} \xi \widehat{z}(\xi) |\widehat{z}(\xi)|^2.$$

Then

$$\begin{aligned} \|\mathcal{R}_1(z)\|_{C([0,T];H^{s+a}(\mathbb{T}))} &\leq \sup_{t\in[0,T]} \left(\sum_\xi \langle \xi \rangle^{2(s+a+1)} |\widehat{z}(\xi)|^6 \right)^{\frac{1}{2}} \\ &= \sup_{t\in[0,T]} \left(\sum_\xi \langle \xi \rangle^{2(a+1-2\sigma)} (\langle \xi \rangle^\sigma |\widehat{z}(\xi)|)^4 (\langle \xi \rangle^s |\widehat{z}(\xi)|)^2 \right)^{\frac{1}{2}} \\ &\leq \sup_\xi \langle \xi \rangle^{a+1-2\sigma}\, \|z\|^2_{C([0,T];H^\sigma(\mathbb{T}))}\, \|z\|_{C([0,T];H^s(\mathbb{T}))}, \end{aligned} \tag{3.12}$$

where the supremum is finite for $0 < a < \min\{2s-1, 1\}$. □

Lemma 3.8 *Let $s > 0$ and $z \in C([0,T]; H^{s+a}(\mathbb{T}))$, for $T > 0$. Then $\mathcal{R}_1^*(z)$ and $\mathcal{R}_2^*(z)$ given by (2.7) and (2.8) satisfy*

$$\begin{aligned} &\sum_{\ell=1}^2 \left\| \int_0^t e^{i(t-t')\partial_x^2} \mathcal{R}_\ell^*(z)\, dt' \right\|_{C([0,T];H^{s+a}(\mathbb{T}))} \\ &\quad \lesssim T\, \|z\|^4_{C([0,T];H^\sigma(\mathbb{T}))}\, \|z\|_{C([0,T];H^s(\mathbb{T}))} \end{aligned} \tag{3.13}$$

for $0 < a < \min\{s, 1\}$.

Proof It suffices to show the proof only for $\mathcal{R}_2^*(z)$. As in Lemma 3.7,

$$\left\| \int_0^t e^{i(t-t')\partial_x^2} \mathcal{R}_2^*(z)\, dt' \right\|_{C([0,T];H^{s+a}(\mathbb{T}))} \leq T\, \left\|\mathcal{R}_2^*(z)\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}.$$

Then,

$$\widehat{R_2^*(z)}(\xi) = -\frac{3i}{2(2\pi)^2} \sum_{\substack{\xi_1-\xi_2-\xi_4=\xi \\ \xi_2+\xi_4\neq 2\xi}} \widehat{z}(\xi_1)\overline{\widehat{z}(\xi_2)\widehat{z}(\xi_4)}[\widehat{z}(\xi)]^2.$$

Thus,

$$\begin{aligned}\left\| R_2^*(z)\right\|_{C([0,T];H^{s+a}(\mathbb{T}))} &\le \sup_{t\in[0,T]}\left(\sum_{\xi}\langle\xi\rangle^{2(s+a)}\Big|\sum_{\substack{\xi_1-\xi_2-\xi_4=\xi\\ \xi_2+\xi_4\neq 2\xi}}\widehat{z}(\xi_1)\widehat{z}(\xi_2)\widehat{z}(\xi_4)[\widehat{z}(\xi)]^2\Big|^2\right)^{\frac12}\\
&\le \sup_{t\in[0,T]}\left(\sum_{\xi}\langle\xi\rangle^{2(a-\sigma)}(\langle\xi\rangle^{2\sigma}|\widehat{z}(\xi)|^2)(\langle\xi\rangle^{2s}|\widehat{z}(\xi)|^2)\Big|\sum_{\substack{\xi_1-\xi_2-\xi_4=\xi\\ \xi_2+\xi_4\neq 2\xi}}\widehat{z}(\xi_1)\widehat{z}(\xi_2)\widehat{z}(\xi_4)\Big|^2\right)^{\frac12}\\
&\lesssim \sup_{\xi}\langle\xi\rangle^{a-\sigma}\,\|z\|_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{s}(\mathbb{T}))}\Big(\sum_{\xi_0}|\widehat{z}(\xi_0)|\Big)^3\\
&\lesssim \sup_{\xi}\langle\xi\rangle^{a-\sigma}\,\|z\|^4_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{s}(\mathbb{T}))}, \end{aligned}\tag{3.14}$$

where the supremum is finite for $0<a<\sigma$. ◻

Lemma 3.9 *Let $s>\frac12$ and $z\in C([0,T];H^s(\mathbb{T}))$, for $T>0$. Then $\mathcal{N}_{1,4}(z)$ and $\mathcal{N}_{2,4}(z)$ defined by (2.21) and (2.22) satisfy*

$$\begin{aligned}&\left\|\int_0^t e^{i(t-t')\partial_x^2}\mathcal{N}_{1,4}(z)\,dt'\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}\\
&\quad\lesssim_T \|z\|^4_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{s}(\mathbb{T}))}\left(1+\|z\|^2_{C([0,T];H^{\sigma}(\mathbb{T}))}\right),\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\left\|\int_0^t e^{i(t-t')\partial_x^2}\mathcal{N}_{2,4}(z)\,dt'\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}\\
&\quad\lesssim_T \|z\|^4_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{s}(\mathbb{T}))}\left(1+\|z\|^2_{C([0,T];H^{\sigma}(\mathbb{T}))}\right),\end{aligned}\tag{3.16}$$

for $0<a<1$.

Proof First,

$$\left\|\int_0^t e^{i(t-t')\partial_x^2}\mathcal{N}_{1,4}(z)\,dt'\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}\le T\left\|\mathcal{N}_{1,4}(z)\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}.$$

From the proof of Lemma 3.4, we see that

$$\begin{aligned}&\left\|\mathcal{N}_{1,4}(z)\right\|_{C([0,T];H^{s+a}(\mathbb{T}))}\\
&\quad\lesssim\sum_{\ell=1}^{2}\left\|\mathcal{R}_\ell(z)+\mathcal{R}_\ell^*(z)\right\|_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{\sigma}(\mathbb{T}))}\,\|z\|_{C([0,T];H^{s}(\mathbb{T}))},\end{aligned}$$

for $0 < a < 1$. Furthermore, from the proof of Lemmas 3.7 and 3.8,

$$\sum_{\ell=1}^{2} \left\| \mathcal{R}_\ell(z) + \mathcal{R}_\ell^*(z) \right\|_{C([0,T];H^\sigma(\mathbb{T}))} \lesssim \|z\|^4_{C([0,T];H^\sigma(\mathbb{T}))} \|z\|_{C([0,T];H^s(\mathbb{T}))} \left(1 + \|z\|^2_{C([0,T];H^\sigma(\mathbb{T}))}\right).$$

The same proof holds for $\mathcal{N}_{2,4}(z)$. □

Lemma 3.10 *Let $s > \frac{1}{2}$. Then, the five-linear form $\mathcal{A}(z)$ given by (2.6) satisfies the estimate*

$$\|\mathcal{A}(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|^4_{Z^\sigma} \|z\|_{Z^s} \tag{3.17}$$

for all $\delta > 0$ and $0 < a < \frac{1}{2} - \delta$.

Proof Let $A(\xi) = \{\xi_2+\xi_4 \neq \xi_1+\xi_3, \xi_1+\xi_5, \xi_3+\xi_5\} \cup \{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5 = \xi\}$. We first apply Littlewood–Paley projections, duality and Plancherel's theorem to infer

$$\|\mathcal{A}(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_j 2^{j(s+a)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \sum_{\ell=1}^{4} R_\ell,$$

where

$$R_1 = \sum_{\substack{A(\xi) \\ |\xi_1^*| \gg |\xi_2^*|^2}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\widetilde{\varphi}_j(\xi,\tau)|\, d\sigma,$$

$$R_2 = \sum_{\substack{A(\xi) \\ |\xi_3^*|^2 \ll |\xi_1^*| \lesssim |\xi_2^*|^2 \\ |\xi_1^*| \gg |\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\widetilde{\varphi}_j(\xi,\tau)|\, d\sigma,$$

$$R_3 = \sum_{\substack{A(\xi) \\ |\xi_1^*| \lesssim |\xi_2^*|^2, |\xi_3^*|^2 \\ |\xi_1^*| \gg |\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\widetilde{\varphi}_j(\xi,\tau)|\, d\sigma$$

$$R_4 = \sum_{\substack{A(\xi) \\ |\xi_1^*| \lesssim |\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\widetilde{\varphi}_j(\xi,\tau)|\, d\sigma,$$

and ξ_j^* denotes the jth frequency among $\xi_1, \ldots, \xi_5$ after ordering all five frequencies with respect to the size of their absolute value, i.e. $|\xi_1^*| \geq |\xi_2^*| \geq |\xi_3^*| \geq |\xi_4^*| \geq |\xi_5^*|$. Next, we note that in the above range of summation and integration, we have

$$\tau + \xi^2 = \sum_{\ell=1}^{5} (-1)^{\ell+1}(\tau_\ell + \xi_\ell^2) + 2\Phi,$$

where

$$\Phi = \xi_2^2 + \xi_4^2 + \xi_2\xi_4 + \xi_1(-\xi_2 + \xi_3 - \xi_4 + \xi_5) - \xi_2(\xi_3 + \xi_5) + \xi_3(-\xi_4 + \xi_5) - \xi_4\xi_5.$$

Thus, we have $\max\left\{|\tau_\ell + \xi_\ell^2|\right\}_{\ell=0}^5 \gtrsim |\Phi|$, where $(\xi_0, \tau_0) = (\xi, \tau)$.

First, we consider the case $|\xi_1^*| \gg |\xi_2^*|^2$ which corresponds to R_1. Suppose that $\xi_1 = \xi_1^*$. Since $\xi_2 + \xi_4 \neq \xi_3 + \xi_5$, we have

$$|\Phi| \geq |\xi_1| - 8|\xi_2^*|^2 \sim |\xi_1|.$$

The same argument holds if $\xi_3 = \xi_1^*$ or $\xi_5 = \xi_1^*$. Next, suppose that $\xi_2 = \xi_1^*$. Then

$$|\Phi| \geq |\xi_2||\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5| - 7|\xi_2^*|^2 \geq \frac{1}{2}|\xi_2|^2 - 7|\xi_2^*|^2 \sim |\xi_2|^2 = |\xi_1^*|^2$$

while the same argument holds if $\xi_4 = \xi_1^*$.

Second, we consider the case $\{|\xi_3^*|^2 \ll |\xi_1^*| \lesssim |\xi_2^*|^2\} \cap \{|\xi_1^*| \gg |\xi_2^*|\}$ which corresponds to R_2. If $\xi_1 = \xi_1^*$, then

$$\begin{aligned}|\Phi| &\geq |\xi_1|| - \xi_2 + \xi_3 - \xi_4 + \xi_5| - 8|\xi_2^*|^2 \\ &\geq \frac{1}{2}|\xi_1||\xi_2^*| - 8|\xi_2^*|^2 = \frac{1}{2}|\xi_2^*|(|\xi_1| - 8|\xi_2^*|) \sim |\xi_1||\xi_2^*|.\end{aligned}$$

The same result is valid if $\xi_3 = \xi_1^*$ or $\xi_5 = \xi_1^*$. Moreover, if $\xi_2 = \xi_1^*$ or $\xi_4 = \xi_1^*$ then the same argument as before yields

$$|\Phi| \gtrsim |\xi_1^*|^2.$$

Thus, we have now shown that within the range of summation and integration associated to R_1 and R_2 we have $|\Phi| \gtrsim |\xi_1^*|$. Therefore, by an application of Hölder's inequality, the Sobolev embedding and estimates (3.1) and (3.4), we obtain

$$\sum_j 2^{j(s+a)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} (R_1 + R_2) \lesssim \sum_j 2^{j(a-\frac{1}{2}+\delta)} \|z\|^3_{Z^{\frac{1}{2}+\delta}} \|z\|_{X^{0,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}},$$

where the sum on the right-hand side converges for $a < \frac{1}{2} - \delta$.

In the region of summation of R_3 we have $|\xi_1^*|^{\frac{1}{2}} \lesssim |\xi_2^*|, |\xi_3^*|$, while in the region of summation of R_4 we have $|\xi_1^*| \sim |\xi_2^*|$. Thus, using once again Hölder's inequality and the Sobolev embedding along with estimates (3.1) and (3.4), we find

$$\sum_j 2^{j(s+a)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} (R_3 + R_4) \lesssim \sum_j 2^{j(a-\sigma)} \|z\|^2_{Z^{\frac{1}{2}+\delta}} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}}, \quad (3.18)$$

where the sum on the right-hand side converges provided that $a < \sigma = \min\{s, 1\}$. □

Lemma 3.11 *Let $s > \frac{1}{2}$. Then, $\mathcal{N}_{1,1}(z)$, $\mathcal{N}_{1,3}(z)$, and $\mathcal{N}_{2,3}(z)$ defined by (2.21) and (2.22) satisfy the estimate*

$$\begin{aligned} \left\|\mathcal{N}_{1,1}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} + \left\|\mathcal{N}_{1,3}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \\ + \left\|\mathcal{N}_{2,3}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|_{Z^\sigma}^4 \|z\|_{Z^s} \end{aligned} \tag{3.19}$$

for all $\delta > 0$ and $0 < a < \min\{s, 1\}$.

Proof We only prove the estimate for $\mathcal{N}_{1,1}(z)$ as the estimate for $\mathcal{N}_{1,3}(z)$ and $\mathcal{N}_{2,3}(z)$ can be established similarly. Note that the range of summation of $\widehat{\mathcal{N}_{1,1}(z)}(\xi)$ implies $|\xi_5|, |\xi_1 - \xi_2 + \xi_3| \ll |\xi_4|$. Thus, $|\Psi_{1,1}| \gtrsim |\xi_4|^2 \sim |\xi|^2$. Then, using Littlewood–Paley-type projections, duality and Plancherel's theorem, we find

$$\left\|\mathcal{N}_{1,1}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_{j,k} 2^{j(s+a-1)} 2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^3 (z_{\sim j})^\# (z_k)^\# (\varphi_j)^\# \, dxdt.$$

First, suppose that $|\xi_2| \ll |\xi_4|$. From Hölder's inequality and estimates (3.1) and (3.4), we have

$$\begin{aligned} &\sum_{j,k} 2^{j(s+a-1)} 2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^3 (z_{\sim j})^\# (z_k)^\# (\varphi_j)^\# \, dxdt \\ &\lesssim \sum_{j,k} 2^{j(s+a-1)} 2^k \left\|z^\#\right\|_{L^\infty_{t,x}}^3 \left\|(z_{\sim j})^\#\right\|_{L^4_{t,x}} \left\|(z_k)^\#\right\|_{L^4_{t,x}} \\ &\lesssim \sum_{j,k} 2^{j(a-1)} 2^{k(1-\sigma)} \|z\|^3_{Z^{\frac{1}{2}+\delta}} \|z\|_{X^{s,\frac{1}{2}}} \|z\|_{X^{\sigma,\frac{1}{2}}}. \end{aligned}$$

Then

$$\sum_{j,k} 2^{j(a-1)} 2^{k(1-\sigma)} \lesssim \sum_j 2^{j(a-\sigma)},$$

which converges for $a < \sigma = \min\{s, 1\}$.

If $|\xi_2| \gtrsim |\xi_4|$, then it follows that $|\xi_2| \sim |\xi_1 + \xi_3| \lesssim \max\{|\xi_1|, |\xi_3|\}$. Therefore,

$$\begin{aligned} &\sum_{j,k} 2^{j(s+a-1)} 2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^2 (z_{\gtrsim k})^\# (z_{\sim j})^\# (z_k)^\# (\varphi_j)^\# \, dxdt \\ &\lesssim \sum_{j,k} 2^{j(a-1)} 2^{k(1-2\sigma)} \|z\|^2_{Z^{\frac{1}{2}+\delta}} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}} \\ &\lesssim \sum_j 2^{j(a-2\sigma)} \|z\|^2_{Z^{\frac{1}{2}+\delta}} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}}, \end{aligned}$$

where the above sum converges for $a < 2\sigma = \min\{2s, 2\}$. □

Lemma 3.12 *Let $s > \frac{1}{2}$. Then, $\mathcal{N}_{1,2}(z)$ and $\mathcal{N}_{2,2}(z)$ defined by (2.21) and (2.22) satisfy the estimate*

$$\left\|\mathcal{N}_{1,2}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} + \left\|\mathcal{N}_{2,2}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|^6_{Z^{\frac{1}{2}+\delta}} \|z\|_{X^{s,\frac{1}{2}}} \tag{3.20}$$

for all $\delta > 0$ and $0 < a < 1$.

Proof As in the proof of Lemma 3.11, we have that $|\Psi_{1,2}| \gtrsim |\xi_6|^2 \sim |\xi|^2$ and so applying Littlewood–Paley-type projections, duality, Plancherel's theorem, Hölder's inequality, and Lemma 3.2, we obtain

$$\begin{aligned}\left\|\mathcal{N}_{1,2}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} &\lesssim \sum_j 2^{j(s+a-1)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^6 (z_{\sim j})^\# (\varphi_j)^\# \, dxdt \\ &\lesssim \sum_j 2^{j(a-1)} \|z\|^6_{Z^{\frac{1}{2}+\delta}} \|z\|_{X^{s,\frac{1}{2}}},\end{aligned}$$

where the sum converges for $a < 1$. The same holds for $\mathcal{N}_{2,2}(z)$. □

Lemma 3.13 *Let $s > \frac{1}{2}$. Then, $\mathcal{N}^*_{2,1}(z)$ defined by (2.23) satisfies the estimate*

$$\left\|\mathcal{N}^*_{2,1}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|^4_{Z^\sigma} \|z\|_{Z^s} \tag{3.21}$$

for all $\delta > 0$ and $0 < a < \frac{1}{2}$.

Proof Employing Littlewood–Paley-type projections, duality and Plancherel's theorem, we find

$$\left\|\mathcal{N}^*_{2,1}(z)\right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_{j,k} 2^{j(s+a-1)} 2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \sum_{\ell=1}^4 R_\ell,$$

where

$$R_1 = \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4\neq\xi_1+\xi_5 \\ |\xi_3|\gg|\xi_2^*|^2}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z_k}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\varphi_j(\xi,\tau)| \, d\sigma,$$

$$R_2 = \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4\neq\xi_1+\xi_5 \\ |\xi_3^*|^2\ll|\xi_3|\lesssim|\xi_2^*|^2 \\ |\xi_3|\gg|\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z_k}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\varphi_j(\xi,\tau)| \, d\sigma,$$

$$R_3 = \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4\neq\xi_1+\xi_5 \\ |\xi_3|\lesssim|\xi_2^*|^2,|\xi_3^*|^2 \\ |\xi_3|\gg|\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z_k}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\varphi_j(\xi,\tau)| \, d\sigma,$$

$$R_4 = \sum_{\substack{N_{2,1}(\xi) \\ \xi_2+\xi_4\neq\xi_1+\xi_5 \\ |\xi_3|\lesssim|\xi_2^*|}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z_k}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z}(\xi_5,\tau_5)||\varphi_j(\xi,\tau)|\,d\sigma.$$

From the proof of Lemma 3.10, we recall that in the case of R_1 and R_2 we have $\max\left\{\left|\tau_m+\xi_m^2\right|\right\}_{m=0}^5 \gtrsim |\xi_3|$. Therefore, Hölder's inequality, the Sobolev embedding and estimates (3.1) and (3.4) yield

$$\begin{aligned}\sum_{j,k} 2^{j(s+a-1)}2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} (R_1+R_2) &\lesssim \sum_{j,k} 2^{j(s+a-1)}2^{k(\frac{1}{2}-s)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s} \\ &\lesssim \sum_j 2^{j(a-\frac{1}{2})} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s},\end{aligned}$$

where the above sum is finite provided that $a<\frac{1}{2}$.

Regarding R_3, we note that there exist $\ell, m \neq 3$ so that $|\xi_3| \lesssim |\xi_\ell|^2, |\xi_m|^2$. Employing once again Hölder's inequality, the Sobolev embedding and estimates (3.1) and (3.4), we deduce

$$\begin{aligned}\sum_{j,k} 2^{j(s+a-1)}2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} R_3 &\lesssim \sum_{j,k} 2^{j(s+a-1)}2^{k(1-s-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s} \\ &\lesssim \sum_j 2^{j(a-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s}\end{aligned}$$

and the right-hand side is finite for $a<\sigma=\min\{s,1\}$.

Finally, consider the region of summation for R_4. First, suppose that $\xi_3=\xi_1^*$. Then $|\xi_3|\sim|\xi_2^*|$, and the above estimate for R_3 holds for R_4. If $\xi_3=\xi_2^*$, then there exists $\xi_1^*\neq\xi_3$ where $|\xi_1^*|\gtrsim 2^{\max\{j,k\}}$. For $j>k$, we have

$$\begin{aligned}\sum_{j,k} 2^{j(s+a-1)}2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} R_4 &\lesssim \sum_{j,k} 2^{j(a-1)}2^{k(1-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s} \\ &\lesssim \sum_j 2^{j(a-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s},\end{aligned}$$

while for $j\leq k$,

$$\begin{aligned}\sum_{j,k} 2^{j(s+a-1)}2^k \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} R_4 &\lesssim \sum_{j,k} 2^{j(s+a-1)}2^{k(1-s-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s} \\ &\lesssim \sum_j 2^{j(a-\sigma)} \|z\|_{Z^\sigma}^4 \|z\|_{Z^s},\end{aligned} \tag{3.22}$$

where the sum converges for $a<\sigma$. Lastly, for $|\xi_3|<|\xi_2^*|$, the above estimate holds with $a-\sigma$ replaced by $a-2\sigma$. □

Lemma 3.14 *Let $s > \frac{1}{2}$. Then, $\mathcal{N}^*_{2,2}(z)$ defined by (2.23) satisfies the estimate*

$$\left\| \mathcal{N}^*_{2,2}(z) \right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|^4_{Z^\sigma} \|z\|_{Z^s} \tag{3.23}$$

for all $\delta > 0$ and $0 < a < \min\{2s, 2\}$.

Proof First, note that $(\xi - \xi_1)(\xi - \xi_5) = (\xi - (\xi_1 + \xi_5) + \xi_5)(\xi - (\xi_1 + \xi_5) + \xi_1)$. Since $|\xi - (\xi_1 + \xi_5)| \gg |\xi_1|, |\xi_5|$, we have that $|\xi - (\xi_1 + \xi_5)|^2 \sim |\xi|^2$. By applying Littlewood–Paley-type projections, duality and Plancherel's theorem, we have

$$\left\| \mathcal{N}^*_{2,2}(z) \right\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_{j,k,\ell} 2^{j(s+a-2)} 2^k 2^\ell \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} R^*,$$

where

$$R^* = \sum_{\xi_2+\xi_4=\xi_1+\xi_3} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z_k}(\xi_1, \tau_1)||\widetilde{z}(\xi_2, \tau_2)||\widetilde{z_\ell}(\xi_3, \tau_3)||\widetilde{z}(\xi_4, \tau_4)||\widetilde{z_j}(\xi, \tau_5)||\widetilde{\varphi_j}(\xi, \tau)| \, d\sigma.$$

Noting that

$$\begin{aligned} R^* &\leq \sum_{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5=\xi} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z_k}(\xi_1, \tau_1)||\widetilde{z}(\xi_2, \tau_2)||\widetilde{z_\ell}(\xi_3, \tau_3)||\widetilde{z}(\xi_4, \tau_4)||\widetilde{z_j}(\xi_5, \tau_5)||\widetilde{\varphi_j}(\xi, \tau)| d\sigma \\ &\simeq \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^2 (z_j)^\# (z_k)^\# (z_\ell)^\# (\varphi_j)^\# \, dxdt \end{aligned}$$

and applying Hölder's inequality, the Sobolev embedding and estimates (3.1) and (3.4), we obtain

$$\begin{aligned} \left\| \mathcal{N}^*_{2,2}(z) \right\|_{X^{s+a,-\frac{1}{2}+\delta}} &\lesssim \sum_{j,k,\ell} 2^{j(s+a-2)} 2^k 2^\ell \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^2 (z_j)^\# (z_k)^\# (z_\ell)^\# (\varphi_j)^\# \, dxdt \\ &\lesssim \sum_{j,k,\ell} 2^{j(a-2)} 2^{k(1-\sigma)} 2^{\ell(1-\sigma)} \|z\|^2_{Z^{\frac{1}{2}+\delta}} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}} . \\ &\lesssim \sum_j 2^{j(a-2\sigma)} \|z\|^2_{Z^{\frac{1}{2}+\delta}} \|z\|^2_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}} , \end{aligned} \tag{3.24}$$

where the sum converges for $a < 2\sigma$. □

Lemma 3.15 *Let $s > \frac{1}{2}$. Then, the quantities $\mathcal{E}_1(z)$ and $\mathcal{E}_2(z)$ given by (2.24) and (2.25) satisfy*

$$\|\mathcal{E}_1(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} + \|\mathcal{E}_2(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \|z\|^4_{Z^\sigma} \|z\|_{Z^s} \tag{3.25}$$

for all $\delta > 0$ and $0 < a < \min\{s, 1\}$.

Proof We only show the estimate for $\mathcal{E}_1(z)$ as all of the terms in $\mathcal{E}_2(z)$ possess the same structure necessary to yield the same estimate for $\mathcal{E}_2(z)$. First, by applying Littlewood–Paley-type projections, duality and Plancherel's theorem, we find

$$\|\mathcal{E}_1(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_j 2^{j(s+a)} \sup_{\|\varphi\|_{X^{0,\frac{1}{2}-\delta}}=1} S^*,$$

where

$$S^* = \sum_{\substack{\xi_2+\xi_4=\xi_1+\xi_3 \\ |\xi|\lesssim|\xi_\ell|,\text{ for some } \ell}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z_j}(\xi,\tau_5)||\widetilde{\varphi_j}(\xi,\tau)|\,d\sigma.$$

Without loss of generality, we assume $|\xi_1| \gtrsim |\xi|$. Then,

$$\begin{aligned}
S^* &\lesssim \sum_{\substack{\xi_2+\xi_4=\xi_1+\xi_3 \\ |\xi|\lesssim|\xi_\ell|,\text{ for some } \ell}} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z_{\gtrsim j}}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z_j}(\xi,\tau_5)||\widetilde{\varphi_j}(\xi,\tau)|d\sigma \\
&\le \sum_{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5=\xi} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} |\widetilde{z_{\gtrsim j}}(\xi_1,\tau_1)||\widetilde{z}(\xi_2,\tau_2)||\widetilde{z}(\xi_3,\tau_3)||\widetilde{z}(\xi_4,\tau_4)||\widetilde{z_j}(\xi_5,\tau_5)||\widetilde{\varphi_j}(\xi,\tau)|d\sigma \\
&\simeq \int_{\mathbb{T}\times\mathbb{R}} (z^\#)^3 (z_{\gtrsim j})^\# (z_j)^\# (\varphi_j)^\#\, dxdt.
\end{aligned}$$

Consequently, Hölder's inequality, Sobolev embedding and estimates (3.1) and (3.4) imply

$$\|\mathcal{E}_1(z)\|_{X^{s+a,-\frac{1}{2}+\delta}} \lesssim \sum_j 2^{j(a-\sigma)} \|z\|^3_{Z^{\frac{1}{2}+\delta}} \|z\|_{X^{\sigma,\frac{1}{2}}} \|z\|_{X^{s,\frac{1}{2}}},$$

where the above sum is finite for $a < \sigma = \min\{s, 1\}$. □

4 Nonlinear smoothing: proof of Theorem 1.3

Having established all necessary nonlinear estimates, we now proceed to the proof of the nonlinear smoothing effect given in Theorem 1.3.

Combining Lemmas 3.2–3.15, we deduce that any solution $z \in Z^s_T$, $s > \frac{1}{2} + \varepsilon$, $T > 0$, of the Duhamel equation (2.27) on $[0, T]$ with $z_0 \in H^s(\mathbb{T})$ and $\|z\|_{Z^s_T} \lesssim \|z_0\|_{H^s(\mathbb{T})}$ enjoys the nonlinear smoothing effect $z - e^{it\partial_x^2} z_0 \in C([0, T]; H^{s+a}(\mathbb{T}))$ with the estimate (1.22). Next, we transition to the solution of the Cauchy problem (2.12).

Let $z_0 \in H^s(\mathbb{T})$ with $s > \frac{1}{2} + \varepsilon$ and take $z_0^{(n)} \in H^\infty(\mathbb{T})$ such that $z_0^{(n)} \to z_0$ in $H^s(\mathbb{T})$. From Theorem 1.1, there exist $T = T(\|z_0\|_{H^{1/2+\varepsilon}(\mathbb{T})}) > 0$ and functions $z \in Z_T^{1/2+\varepsilon}$ and, for n sufficiently large, $z^{(n)} \in Z_T^\infty$ that are solutions of the Cauchy problem (2.12) with initial data z_0 and $z_0^{(n)}$, respectively. Furthermore, from the computations of Sect. 2, the smooth solution $z^{(n)}$ satisfies the Duhamel equation (2.27) on $[0, T]$.

In addition, thanks to the Lipschitz continuity of the data-to-solution map, we have $z^{(n)} \to z$ in $Z_T^{1/2+\varepsilon}$. Therefore, using Lemmas 3.4–3.15 and the fact that $z^{(n)}$ is Cauchy in $Z_T^{1/2+\varepsilon}$, we conclude that $z \in Z_T^{1/2+\varepsilon}$ satisfies (2.27) on $[0, T]$. In turn, we can employ the nonlinear smoothing estimate (1.22) for an appropriate value of a to infer

$$\begin{aligned} \|z\|_{Z_T^{\frac{1}{2}+\varepsilon+a}} &\lesssim \|z_0\|_{H^{\frac{1}{2}+\varepsilon+a}(\mathbb{T})} + \left\|z - e^{it\partial_x^2} z_0\right\|_{Z_T^{\frac{1}{2}+\varepsilon+a}} \\ &\lesssim \|z_0\|_{H^{\frac{1}{2}+\varepsilon+a}(\mathbb{T})} + C(s, \|z\|_{Z_T^\sigma}, T)\, \|z\|_{Z_T^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|z_0\|_{H^{\frac{1}{2}+\varepsilon+a}(\mathbb{T})} + C(s, \|z_0\|_{H^\sigma(\mathbb{T})})\, \|z_0\|_{H^{\frac{1}{2}+\varepsilon}(\mathbb{T})} \\ &\lesssim C(s, \|z_0\|_{H^\sigma(\mathbb{T})})\, \|z_0\|_{H^{\frac{1}{2}+\varepsilon+a}(\mathbb{T})}, \end{aligned} \tag{4.1}$$

which shows that $z \in Z_T^{\frac{1}{2}+\varepsilon+a}$ with the estimate (4.1). Iterating this process, we eventually obtain

$$\|z\|_{Z_T^s} \le C(s, \|z_0\|_{H^\sigma(\mathbb{T})})\, \|z_0\|_{H^s(\mathbb{T})}.$$

This estimate shows that $z \in Z_T^{1/2+\varepsilon}$ actually belongs to Z_T^s. Therefore, since z satisfies the Duhamel equation (2.27), it admits the nonlinear smoothing estimate (1.22), completing the proof of Theorem 1.3.

5 Polynomial bound: proof of Theorem 1.4

We shall now exploit the nonlinear smoothing effect of Theorem 1.3 in order to establish the polynomial bound of Theorem 1.4. We begin by proving such a bound for the solution z of the gauged Cauchy problem (2.12).

First, we suppose that $1 \le s \le \frac{3}{2} - \varepsilon$ for ε as in Theorem 1.3. Fix $n \in \mathbb{N}$ and $t \in [nT, (n+1)T]$, where $T = T(\|z_0\|_{H^1(\mathbb{T})})$ is the local time of existence from Theorem 1.1. Then, write z in the form

$$z(t) = Q_{\le n^2} z(t) + Q_{>n^2} z(t),$$

where $\widehat{Q_{\le N} z}(\xi) = \chi_{|\xi|\le N} \widehat{z}(\xi)$ and $Q_{>N} z$ is defined similarly. The term $Q_{\le n^2} z(t)$ satisfies

$$\left\|Q_{\le n^2} z(t)\right\|_{H^s(\mathbb{T})} \le c\, \langle n\rangle^{2(s-1)}\, \|z(t)\|_{H^1(\mathbb{T})} \le C(\|z_0\|_{H^1(\mathbb{T})})\, \langle t\rangle^{2(s-1)}. \tag{5.1}$$

The term $Q_{>n^2} z(t)$ can be handled by taking advantage of the nonlinear smoothing effect (1.22). Indeed,

$$Q_{>n^2} z(t) = Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) + Q_{>n^2} e^{i(t-nT)\partial_x^2} z(nT). \tag{5.2}$$

Thus, since $t \in [nT, (n+1)T]$ and $s \leq \frac{3}{2} - \varepsilon$, we have

$$\begin{aligned} &\left\| Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{H^s(\mathbb{T})} \\ &\quad = \left\| J_x^{s-(\frac{3}{2}-\varepsilon)} J_x^{\frac{3}{2}-\varepsilon} Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{L^2(\mathbb{T})} \\ &\quad \leq c\, \langle n \rangle^{2(s-(\frac{3}{2}-\varepsilon))} \left\| Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})} \\ &\quad \leq \langle n \rangle^{2(s-(\frac{3}{2}-\varepsilon))} C(s, \|z(nT)\|_{H^1(\mathbb{T})})\, \|z(nT)\|_{H^1(\mathbb{T})} \\ &\quad \leq \langle n \rangle^{2(s-(\frac{3}{2}-\varepsilon))} \tilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}). \end{aligned} \tag{5.3}$$

In order to estimate $Q_{>n^2} e^{i(t-nT)\partial_x^2} z(nT)$ in (5.2), we first use the strict inequality

$$\left\| Q_{>n^2} e^{i(t-nT)\partial_x^2} z(nT) \right\|_{H^s(\mathbb{T})} \leq \left\| Q_{>(n-1)^2} z(nT) \right\|_{H^s(\mathbb{T})}. \tag{5.4}$$

Then, writing

$$\begin{aligned} Q_{>(n-1)^2} z(nT) = Q_{>(n-1)^2}\big(z(nT) - e^{i(nT-(n-1)T)\partial_x^2} z((n-1)T)\big) \\ + Q_{>(n-1)^2} e^{iT\partial_x^2} z((n-1)T) \end{aligned}$$

and proceeding similarly to (5.3) and (5.4), we obtain

$$\begin{aligned} \left\| Q_{>(n-1)^2} z(nT) \right\|_{H^s(\mathbb{T})} \leq \langle n-1 \rangle^{2(s-(\frac{3}{2}-\varepsilon))} \tilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}) \\ + \left\| Q_{>(n-2)^2} z((n-1)T) \right\|_{H^s(\mathbb{T})}. \end{aligned}$$

We may inductively continue this process to arrive at

$$\left\| Q_{>n^2} z(t) \right\|_{H^s(\mathbb{T})} \leq \sum_{k=1}^{n} \langle k \rangle^{2(s-(\frac{3}{2}-\varepsilon))} \tilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}) + \|z_0\|_{H^s(\mathbb{T})}.$$

Then, noting that

$$\sum_{k=1}^{n} k^\alpha \leq c_\alpha n^{\alpha+1}, \quad \alpha > -1,$$

and observing that $2(s-(\frac{3}{2}-\varepsilon)) > -1$ since $s \geq 1$, we obtain

$$\begin{aligned}\left\| Q_{>n^2} z(t) \right\|_{H^s(\mathbb{T})} &\leq \sum_{k=1}^{n} \langle k \rangle^{2(s-(\frac{3}{2}-\varepsilon))} \tilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}) + \|z_0\|_{H^s(\mathbb{T})} \\ &\leq \langle n \rangle^{2(s-1+\varepsilon)} \widetilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}) + \|z_0\|_{H^s(\mathbb{T})} \\ &\leq \langle t \rangle^{2(s-1+\varepsilon)} \widetilde{\widetilde{C}}(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})}). \end{aligned} \tag{5.5}$$

Combining (5.1) and (5.5) yields

$$\|z(t)\|_{H^s(\mathbb{T})} \leq \langle t \rangle^{2(s-1+\varepsilon)} C(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})}), \quad 1 \leq s \leq \frac{3}{2} - \varepsilon. \tag{5.6}$$

Next, we consider the range $\frac{3}{2} - \varepsilon \leq s \leq 2 - 2\varepsilon$. We remark that the argument outlined for this range also extends to the range $s > 2 - 2\varepsilon$. Indeed, upon proving the polynomial bound for $1 + (j-1)(\frac{1}{2} - \varepsilon) \leq s \leq 1 + j(\frac{1}{2} - \varepsilon)$, we may always use the same argument to establish the bound for $1 + j(\frac{1}{2} - \varepsilon) \leq s \leq 1 + (j+1)(\frac{1}{2} - \varepsilon)$, for $j > 1$. Once again, let $t \in [nT, (n+1)T]$ and split $z(t)$ as before. For $Q_{\leq n^2} z(t)$, we still have estimate (5.1). Also, as before, we write

$$Q_{>n^2} z(t) = Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) + Q_{>n^2} e^{i(t-nT)\partial_x^2} z(nT).$$

Then,

$$\begin{aligned} &\left\| Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{H^s(\mathbb{T})} \\ &\quad = \left\| J_x^{s-(2-2\varepsilon)} J_x^{2-2\varepsilon} Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{L^2(\mathbb{T})} \\ &\quad \leq c \, \langle n \rangle^{2(s-(2-2\varepsilon))} \left\| Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{H^{2-2\varepsilon}(\mathbb{T})} \\ &\quad \leq \langle n \rangle^{2(s-(2-2\varepsilon))} C(s, \|z(nT)\|_{H^1(\mathbb{T})}) \, \|z(nT)\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})} \\ &\quad \leq \langle n \rangle^{2(s-(2-2\varepsilon))} \tilde{C}(s, \|z_0\|_{H^1(\mathbb{T})}) \, \|z(nT)\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})} . \end{aligned}$$

In addition, the bound (5.6) gives

$$\|z(nT)\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})} \leq \langle nT \rangle \, C(\|z_0\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})}).$$

Consequently,

$$\left\| Q_{>n^2}\big(z(t) - e^{i(t-nT)\partial_x^2} z(nT)\big) \right\|_{H^s(\mathbb{T})} \leq \langle n \rangle^{2(s-(2-2\varepsilon))+1} \tilde{C}(\varepsilon, s, \|z_0\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})}).$$

Repeating the procedure as before yields

$$\begin{aligned}\|Q_{>n^2}z(t)\|_{H^s(\mathbb{T})} &\leq \sum_{k=1}^{n}\langle k\rangle^{2(s-(2-2\varepsilon))+1}\tilde{C}(\varepsilon, s, \|z_0\|_{H^{\frac{3}{2}-\varepsilon}(\mathbb{T})}) + \|z_0\|_{H^s(\mathbb{T})}\\ &\leq \langle n\rangle^{2(s-(2-2\varepsilon))+2}\tilde{C}(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})}) + \|z_0\|_{H^s(\mathbb{T})}\\ &\leq \langle t\rangle^{2(s-(1-2\varepsilon))}\tilde{\tilde{C}}(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})}).\end{aligned}$$

Overall, recalling also (5.6), we have established the bound

$$\|z(t)\|_{H^s(\mathbb{T})} \leq \langle t\rangle^{2(s-(1-2\varepsilon))}C(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})}), \quad 1 \leq s \leq 2-2\varepsilon. \tag{5.7}$$

We may then repeat the above procedure to establish the bound (5.7) for all $s \geq 1$.

In order to extend the result to the solution u of the dNLS Cauchy problem (1.1), we begin by establishing the following product estimate.

Proposition 5.1 *Let $s_1, s_2 \geq s \geq 0$ and $s_1 + s_2 > s + \frac{1}{2}$. Then*

$$\|fg\|_{H^s(\mathbb{T})} \lesssim \|f\|_{H^{s_1}(\mathbb{T})}\|g\|_{H^{s_2}(\mathbb{T})}. \tag{5.8}$$

Proof The argument is standard and resembles the proof of the algebra property for Sobolev spaces. For $r, r_1, r_2 \geq 0$, we claim that there exists a constant $C > 0$ so that

$$(1+|x|+|y|)^r \leq C\left[(1+|x|)^{r+r_1}(1+|y|)^{-r_1} + (1+|y|)^{r+r_2}(1+|x|)^{-r_2}\right]. \tag{5.9}$$

Indeed, first note that

$$(1+|x|+|y|)^r(1+|x|)^{r_2}(1+|y|)^{r_1} \leq (1+|x|+|y|)^{r+r_1+r_2}.$$

Thus, to establish (5.9), it suffices to show that there exists $C = C(t) > 0$ so that

$$(1+|x|+|y|)^t \leq C\left[(1+|x|)^t + (1+|y|)^t\right] \tag{5.10}$$

for all $t \geq 0$. Without loss of generality, assume that $(1+|y|) \leq (1+|x|)$. Then

$$(1+|x|+|y|)^t \leq (1+|x|+1+|y|)^t \leq 2^t(1+|x|)^t \leq 2^t\left[(1+|x|)^t + (1+|y|)^t\right].$$

Therefore, (5.10) holds with $C = 2^t$, for all $t \geq 0$. Consequently, we have (5.9).

Next, we apply (5.9) with $r = s$, $r_1 = s_1 - s$, and $r_2 = s_2 - s$ to see that

$$\langle\xi\rangle^s \leq (1+|\xi|)^s \lesssim (1+|\xi-\eta|)^{s_1}(1+|\eta|)^{s-s_1} + (1+|\xi-\eta|)^{s-s_2}(1+|\eta|)^{s_2}.$$

Thus,

$$\begin{aligned}\|fg\|^2_{H^s(\mathbb{T})} &\le \sum_\xi (1+|\xi|)^{2s}\Big(\sum_\eta |\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\Big)^2\\ &\lesssim \sum_\xi \Big(\sum_\eta (1+|\xi-\eta|)^{s_1}|\widehat{f}(\xi-\eta)|(1+|\eta|)^{s-s_1}|\widehat{g}(\eta)|\Big)^2\\ &\quad+\sum_\xi \Big(\sum_\eta (1+|\xi-\eta|)^{s-s_2}|\widehat{f}(\xi-\eta)|(1+|\eta|)^{s_2}|\widehat{g}(\eta)|\Big)^2,\end{aligned}$$

where we have applied (5.10) for $t=2$ in the previous inequality. It suffices to bound the first sum above, as the second sum is treated the same after a change of variables. We apply Minkowski's inequality to obtain

$$\begin{aligned}&\sum_\xi \Big(\sum_\eta (1+|\xi-\eta|)^{s_1}|\widehat{f}(\xi-\eta)|(1+|\eta|)^{s-s_1}|\widehat{g}(\eta)|\Big)^2\\ &\le \|f\|^2_{H^{s_1}(\mathbb{T})}\Big(\sum_\eta (1+|\eta|)^{s-s_1}|\widehat{g}(\eta)|\Big)^2.\end{aligned}$$

Finally, due to the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_\eta (1+|\eta|)^{s-s_1}|\widehat{g}(\eta)| &= \sum_\eta (1+|\eta|)^{s-s_1-s_2}(1+|\eta|)^{s_2}|\widehat{g}(\eta)|\\ &\le \|g\|_{H^{s_2}(\mathbb{T})}\left\|(1+|\eta|)^{-(s_1+s_2-s)}\right\|_{\ell^2_\eta},\end{aligned}$$

where the above norm is finite due to the fact that $s_1+s_2-s>\frac{1}{2}$. □

Lemma 5.1 *Let $s\ge 0$. Then, there exists $C>0$ such that*

$$\|u(t)\|_{H^s(\mathbb{T})} \le C\Big(1+\|z(t)\|^2_{H^{\frac{1}{4}}(\mathbb{T})}\Big)\|z(t)\|_{H^s(\mathbb{T})}. \tag{5.11}$$

Proof Clearly, $\|z(t)\|_{H^s(\mathbb{T})} = \|w(t)\|_{H^s(\mathbb{T})}$, for w given by (1.17) . Furthermore, since $\widehat{w}(\xi,t) = e^{-2i\xi\mu t}\widehat{v}(\xi,t)$, for v given by (1.14), we have that $\|w(t)\|_{H^s(\mathbb{T})} = \|v(t)\|_{H^s(\mathbb{T})}$. Thus, it suffices to establish (5.11) for $z(t)$ replaced by $v(t)$.

Recall that $u(x,t) = e^{i\mathcal{I}(v)(x,t)}v(x,t)$. For $\delta>0$, we apply Proposition 5.1 with $s_1=s$ and $s_2=\frac{1}{2}+\delta$ to obtain

$$\|u(t)\|_{H^s(\mathbb{T})} = \left\|e^{i\mathcal{I}(v)(t)}v(t)\right\|_{H^s(\mathbb{T})} \le C\left\|e^{i\mathcal{I}(v)(t)}\right\|_{H^{\frac{1}{2}+\delta}(\mathbb{T})}\|v(t)\|_{H^s(\mathbb{T})}.$$

Now, observe that

$$\begin{aligned}\left\| e^{i\mathcal{I}(v)(t)} \right\|_{H^{\frac{1}{2}+\delta}(\mathbb{T})} &\leq \left\| e^{i\mathcal{I}(v)(t)} \right\|_{H^1(\mathbb{T})} = \left\| e^{i\mathcal{I}(v)(t)} \right\|_{L^2(\mathbb{T})} + \|\partial_x \mathcal{I}(v)(t)\|_{L^2(\mathbb{T})} \\ &= \sqrt{2\pi} + \left\| |v(t)|^2 - \mu \right\|_{L^2(\mathbb{T})} \leq C\left(1 + \|v(t)\|_{L^4(\mathbb{T})}^2\right) \\ &=\leq C\left(1 + \|v(t)\|_{H^{\frac{1}{4}}(\mathbb{T})}^2\right),\end{aligned}$$

where the last inequality follows from Sobolev embedding. □

Remark 5.1 It is clear from the proof of Lemma 5.1 that the estimate (5.11) holds with the roles of u and z switched.

Returning to the case that $s \geq 1$ and applying Lemma 5.1 and the bound (5.6) yields

$$\begin{aligned}\|u(t)\|_{H^s(\mathbb{T})} &\leq C\left(1 + \|z(t)\|_{H^{\frac{1}{4}}(\mathbb{T})}^2\right) \|z(t)\|_{H^s(\mathbb{T})} \\ &\leq C(\|z_0\|_{H^1(\mathbb{T})})\, \|z(t)\|_{H^s(\mathbb{T})} \\ &\leq C(\|z_0\|_{H^1(\mathbb{T})})\, \tilde{C}(\varepsilon, s, \|z_0\|_{H^s(\mathbb{T})})\, \langle t\rangle^{2(s-1)+\varepsilon} \\ &\leq C(\varepsilon, s, \|u_0\|_{H^s(\mathbb{T})})\, \langle t\rangle^{2(s-1)+\varepsilon},\end{aligned}$$

which concludes the proof of Theorem 1.4.

Data availability The manuscript has no associated data.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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