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Linear-Quadratic-Gaussian Control with Time-Varying Disturbance Forecast

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***Abstract*— Linear-quadratic-Gaussian (LQG) control is a classical optimal control problem where the disturbance in the system dynamics is traditionally treated as random noise. Motivated by the possibility of forecasting future disturbance in some relevant works for linear-quadratic regulator (LQR) systems where the disturbance distribution is arbitrary, we introduce a time-varying disturbance forecast model in the LQG problem. Our model characterizes the Gaussianity of the disturbances and thus enables us to give theoretical results including optimal average cost even though the forecast error can be unbounded. Numerical examples are provided to illustrate the theoretical results.**

I. INTRODUCTION

In the control community, the linear-quadratic-Gaussian (LQG) control problem is a classical and fundamental control problem governed by $x_{t+1} = Ax_t + Bu_t + w_t$, where x_t, u_t and w_t represent state, control and Gaussian disturbance, respectively. Under the assumption that w_t follows Gaussian distribution, it is well-known that the optimal control law is a combination of Kalman filter and linear-quadratic regulator (LQR) according to the separation principle.

Traditionally, the disturbance w_t in LQG is treated as a non-predictable system noise. On the other hand, recently, there have been a variety of relevant works making assumptions on the predictability of the future disturbance w_t and targeting at designing an optimal or near-optimal control for an LQR system, with the assumption of arbitrary disturbance distribution. For example, references [1] and [2] give the optimal open-loop control and the optimal feedback control when all the future disturbances are perfectly known, respectively; And reference [3] presents the optimal feedback control when only the future disturbances within a lookahead window can be perfectly predicted. However, these works all assume perfect forecast, while in reality a forecast is usually imperfect and even time-varying. It is common that more relevant information becomes available during the passage of time, making it possible to reduce the forecast inaccuracy when the far future becomes closer. For example, in electric power load forecasting, the next-day load forecast may achieve an inaccuracy level of less than 3%, while the same accuracy is not achievable for the next-year load forecast due to the unavailability of accurate long-term weather forecast [4]. Some other references [5]–[7] characterize the forecast error of the future disturbance, yet only upper bounds of the associated optimal cost as unbounded functions of the maximum forecast error are provided, due to the assumption of arbitrary disturbance distribution. On the other hand, in the LQG problem, the assumption of Gaussian disturbance implies that the forecast error can also be unbounded but the computation of optimal *average* cost is possible.

In this paper, we introduce a time-varying disturbance forecast model in the LQG problem, which considers the Gaussianity of the disturbances and hence is more concrete compared to [5]–[7]. Moreover, Gaussianity in LQG makes it possible for us to give the optimal average cost even though the forecast error can be unbounded. The rest of this paper is organized as follows: In Sec. II we briefly review the classical LQG problem; In Sec. III we present the main theoretical results including optimal state estimate, optimal control law and optimal average cost associated with the proposed time-varying disturbance forecast model; Sec. IV illustrates the theoretical results through two numerical examples, and Sec. V concludes the paper.

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II. CLASSICAL LQG PROBLEM

In this section we introduce the classical LQG problem with partial state observation [8], [9]. Consider a discrete-time stochastic linear system over finite horizon T:

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad t = 0, \cdots, T-1 \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control, $w_t \in \mathbb{R}^n$ is the disturbance, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are fixed matrices. Moreover, at time t, we have a disturbed partial observation of state x_t, given by:

$$y_t = Cx_t + v_t, \quad t = 0, \cdots, T \tag{2}$$

where $y_t \in \mathbb{R}^k$ is the observation, $v_t \in \mathbb{R}^k$ is the disturbance, and $C \in \mathbb{R}^{k\times n}$ is a fixed matrix. In LQG we assume $x_0 \sim \mathcal{N}(0, X)$, $w_t \sim \mathcal{N}(0, W_t), \forall t$ and $v_t \sim \mathcal{N}(0, V_t), \forall t$ are all independent Gaussian random vectors, where $\mathcal{N}(\mu, \Sigma)$ denotes a multivariate Gaussian distribution with mean μ and covariance Σ. The overall objective is to determine the optimal control u_t at each time t, such that the cost

$$J = \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) + x_T^\top Q x_T \tag{3}$$

is minimized, where $Q \in \mathbb{R}^{n\times n} \succeq \mathbf{0}$ and $R \in \mathbb{R}^{m\times m} \succ \mathbf{0}$ are fixed matrices.

Solution. The classical LQG problem has known solution according to separation principle, which dictates that the optimal feedback control law has the following form:

$$u_t^* = K_t \hat{x}_t, \quad t = 0, \cdots, T-1 \tag{4}$$

where $K_t \in \mathbb{R}^{m \times n}$ is a feedback matrix depends only on A, B, Q, R; and $\hat{x}_t \triangleq \mathbb{E}[x_t | y_{0:t}]$ is the minimum mean squared error (MMSE) state estimate depends only on $A, B, C, X, W_{0:t-1}, V_{0:t}$. For K_t, we have

$$K_t = -(R + B^\top P_{t+1} B)^{-1} B^\top P_{t+1} A, \tag{5}$$

where $P_t \in \mathbb{R}^{n \times n}$ is determined recursively by algebraic Riccati equation

$$P_{t-1} = A^\top P_t A + Q - A^\top P_t B (R + B^\top P_t B)^{-1} B^\top P_t A, \tag{6}$$

with terminal value $P_T = Q$. For $\hat{x}_t$, we have the following recursive equations according to Kalman filtering algorithms

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L_{t+1} e_{t+1}, \tag{7}$$

$$e_{t+1} = y_{t+1} - C(A\hat{x}_t + Bu_t), \tag{8}$$

with initial estimate $\hat{x}_0 = L_0 y_0$, and $L_t \in \mathbb{R}^{n \times k}$ is the Kalman gain. If we use Σ_t to denote the covariance matrix of current state estimate error $x_t - \mathbb{E}[x_t | y_{0:t}]$, and $\Sigma_{t+1|t}$ to denote the covariance matrix of next state estimate error $x_{t+1} - \mathbb{E}[x_{t+1} | y_{0:t}]$, then we have recursive equations

$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1} C^\top (C \Sigma_{t|t-1} C^\top + V_t)^{-1} C \Sigma_{t|t-1}, \tag{9}$$

$$\Sigma_{t+1|t} = A \Sigma_t A^\top + W_t, \tag{10}$$

with initial value $\Sigma_{0|=-1} = X$, and Kalman gain L_t is given by

$$L_t = \Sigma_{t|t-1} C^\top (C \Sigma_{t|t-1} C^\top + V_t)^{-1}. \tag{11}$$

The optimal average cost J^* corresponding to the optimal control law has the following form:

$$J^* = \underbrace{\mathrm{Tr}(P_0 X) + \sum_{t=1}^{T} \mathrm{Tr}(P_t W_{t-1})}_{J_{\text{lqr}}} + \underbrace{\sum_{t=0}^{T} \mathrm{Tr}((Q - P_t)\Sigma_t) + \sum_{t=1}^{T} \mathrm{Tr}(P_t A \Sigma_{t-1} A^\top)}_{J_{\text{est}}} \tag{12}$$

where J_{lqr} is the optimal cost for LQR in which the observation $y_t = x_t$ is perfect; and J_{est} is the additional cost due to imperfect state estimation $\hat{x}_t$.

III. MAIN RESULTS

In this section, we present the main theoretical results. In Sec. III-A, we introduce the time-varying disturbance forecast model in the classical LQG problem. In Sec. III-B, we discuss the associated MMSE state estimate. And in Sec. III-C, we give corresponding optimal control law and optimal average cost.

A. Time-varying Disturbance Forecast

Suppose the disturbance w_t in the system dynamics Eq. (1) can be partially forecast, and the forecast can be updated at any time $\tau \le t$. More specifically, we assume

$$w_t = \hat{w}_t + \tilde{w}_t, \quad t = 0, \cdots, T-1 \tag{13}$$

$$\hat{w}_t = \sum_{\tau=0}^{t} \hat{w}_t^{(\tau)}, \quad t = 0, \cdots, T-1 \tag{14}$$

where $\hat{w}_t \sim \mathcal{N}(0, \hat{W}_t)$ is a portion of w_t that we can fully forecast at time t, $\tilde{w}_t \sim \mathcal{N}(0, \tilde{W}_t)$ is the remaining portion that we cannot forecast, and $\hat{w}_t^{(\tau)} \sim \mathcal{N}(0, \hat{W}_t^{(\tau)})$ is a portion of $\hat{w}_t$ whose value is revealed at time τ. Moreover, we assume $\hat{w}_t^{(\tau)}, \forall t, \tau$ and $\tilde{w}_t, \forall t$ are all independent, and hence $W_t = \hat{W}_t + \tilde{W}_t = \sum_{\tau=0}^{t} \hat{W}_t^{(\tau)} + \tilde{W}_t, \forall t$. For convenience, we also define $\hat{w}_{t|t'} \triangleq \sum_{\tau=0}^{t'} \hat{w}_t^{(\tau)}$, which is the MMSE estimate of w_t at any time $t' \le t$.

Remarks. A few remarks are in order. First, our formulation is based on the assumption that the Gaussian disturbance w_t is *intrinsically* the summation of multiple independent Gaussian random variables with smaller covariances. Those can be forecast at a time $\tau \le t$ constitute $\hat{w}_t^{(\tau)}$, and those cannot be forecast constitute $\tilde{w}_t$. Second, the mean-squared error $\mathbb{E}[\|\hat{w}_{t|t'} - w_t\|_2^2] = \sum_{\tau=t'+1}^{t} \mathrm{Tr}(\hat{W}_t^{(\tau)}) + \mathrm{Tr}(\tilde{W}_t)$ is monotonically decreasing in t', which means $\hat{w}_{t|t'}$ becomes more accurate on average when time t is getting closer. Third, when only a finite lookahead window H of the disturbances are predictable, which is a common setting in model predictive control, we will have $\hat{w}_t^{(\tau)} \equiv 0, \forall \tau \le t-H$, which means one cannot get a meaningful forecast of w_t at or before time $t-H$. And last, our model differs from the models in [5]–[7], which characterize the maximum value of the forecast error rather than its distribution, and the model in [10], which characterize the forecast of observation disturbance v_t rather than dynamics disturbance w_t.

B. MMSE state estimate

In this subsection we determine the MMSE estimate $\hat{x}_t$ when the disturbance forecast is given. It is clear that we can still obtain $\hat{x}_t$ by Kalman filtering algorithms with some minor changes. Since at time t, the value of $\hat{w}_t$ is known while $\tilde{w}_t$ remains an unknown Gaussian random variable, the MMSE estimate of the next state $\hat{x}_{t+1}$ should include the knowledge of $\hat{w}_t$. That is, we replace Eq. (7), (8) by

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + \hat{w}_t + L_{t+1} e_{t+1}, \tag{15}$$

$$e_{t+1} = y_{t+1} - C(A\hat{x}_t + Bu_t + \hat{w}_t). \tag{16}$$

Moreover, for the computation of Kalman gain L_t, Eq. (10) is replaced by

$$\Sigma_{t+1|t} = A \Sigma_t A^\top + \tilde{W}_t, \tag{17}$$

while Eq. (9) and (11) remain unchanged. From Eq. (15) and (16), we notice that $\hat{x}_t$ depends only on $\hat{w}_t$ and is independent of the distribution of $\hat{w}_t^{(\tau)}$'s, i.e., the MMSE state estimate only depends on how much can be eventually forecast instead of how the forecast varies across the time.

$$J_t^*(\hat{x}_t) = \hat{x}_t^\top P_t \hat{x}_t + 2\sum_{i=t}^{T-1} \hat{w}_i^\top M_{i,t}\hat{x}_t + \sum_{t\le\tau\le i\le T-1} \hat{w}_i^{(\tau)\top} N_t(\hat{w}_i^{(\tau)})\hat{w}_i^{(\tau)} + \tag{18}$$

$$2\sum_{\substack{t\le\tau\le i\le T-1,\\ t\le\tau'\le i'\le T-1,\\ \tau'>\tau \text{ or } i'>i}} \hat{w}_i^{(\tau)\top} N_t(\hat{w}_i^{(\tau)}, \hat{w}_{i'}^{(\tau')})\hat{w}_{i'}^{(\tau')} + \sum_{i=t}^{T-1}\text{Tr}(P_{i+1}(\Sigma_{i+1|i} - \Sigma_{i+1})) + \sum_{i=t}^{T}\text{Tr}(Q\Sigma_i),$$

$$\text{where} \quad M_{i,t} = \begin{cases} P_{t+1}A - P_{t+1}B(R + B^\top P_{t+1}B)^{-1}B^\top P_{t+1}A, & i = t \\ M_{i,t+1}A - M_{i,t+1}B(R + B^\top P_{t+1}B)^{-1}B^\top P_{t+1}A, & i > t \end{cases} \tag{19}$$

$$N_t(\hat{w}_i^{(\tau)}) = \begin{cases} P_{t+1} - P_{t+1}B(R + B^\top P_{t+1}B)^{-1}B^\top P_{t+1}, & i = t, \tau \le i \\ N_{t+1}(\hat{w}_i^{(\tau)}) - M_{i,t+1}B(R + B^\top P_{t+1}B)^{-1}B^\top M_{i,t+1}^\top, & i > t, \tau \le t \\ N_{t+1}(\hat{w}_i^{(\tau)}), & i > t, \tau > t, \tau \le i \end{cases} \tag{20}$$

$$\text{and} \quad \pi_t^* = -(R + B^\top P_{t+1}B)^{-1}B^\top (P_{t+1}A\hat{x}_t + P_{t+1}\hat{w}_t + \sum_{i=t+1}^{T-1} M_{i,t+1}^\top \hat{w}_{i|t}). \tag{21}$$

$$J_t^*(\hat{x}_t) = \mathbb{E}_{v_{t+1}}\big[\min_{u_t} \mathbb{E}_{x_t|\hat{x}_t,\tilde{w}_t}[x_t^\top Q x_t + u_t^\top Q u_t + J_{t+1}^*(Ax_t + Bu_t + \hat{w}_t + L_{t+1}e_{t+1})]\big] \tag{22}$$

$$= \hat{x}_t^\top (Q + A^\top P_{t+1}A)\hat{x}_t + 2\hat{w}_t^\top P_{t+1}A\hat{x}_t + \hat{w}_t^\top P_{t+1}\hat{w}_t + 2\sum_{i=t+1}^{T-1} \hat{w}_i^\top M_{i,t+1}(A\hat{x}_t + \hat{w}_t) +$$

$$\sum_{t+1\le\tau\le i\le T-1} \hat{w}_i^{(\tau)\top} N_{t+1}(\hat{w}_i^{(\tau)})\hat{w}_i^{(\tau)} + 2\sum_{\substack{t+1\le\tau\le i\le T-1,\\ t+1\le\tau'\le i'\le T-1,\\ \tau'>\tau \text{ or } i'>i}} \hat{w}_i^{(\tau)\top} N_{t+1}(\hat{w}_i^{(\tau)}, \hat{w}_{i'}^{(\tau')})\hat{w}_{i'}^{(\tau')} +$$

$$\sum_{i=t}^{T-1}\text{Tr}(P_{i+1}(\Sigma_{i+1|i} - \Sigma_{i+1})) + \sum_{i=t}^{T}\text{Tr}(Q\Sigma_i) +$$

$$\underbrace{\min_{u_t} \mathbb{E}[u_t^\top (R + B^\top P_{t+1}B)u_t + 2(A\hat{x}_t + \hat{w}_t)^\top P_{t+1}Bu_t + 2\sum_{i=t+1}^{T-1} \hat{w}_i^\top M_{i,t+1}Bu_t]}_{\Omega},$$

$$\Omega = -(P_{t+1}A\hat{x}_t + P_{t+1}\hat{w}_t + \sum_{i=t+1}^{T-1} M_{i,t+1}^\top \hat{w}_{i|t} + 2\sum_{i=t+1}^{T-1} M_{i,t+1}^\top \sum_{\tau=t+1}^{i} \hat{w}_i^{(\tau)})^\top \tag{23}$$

$$\cdot B(R + B^\top P_{t+1}B)^{-1}B^\top \cdot (P_{t+1}A\hat{x}_t + P_{t+1}\hat{w}_t + \sum_{i=t+1}^{T-1} M_{i,t+1}^\top \hat{w}_{i|t}).$$

C. Optimal control law and optimal average cost

Since disturbance forecast $\{\hat{w}_t^{(\tau)}\}$ brings new knowledge, we can expect that a control law that relies on the new information is able to make the average cost lower. On the other hand, at time t, the control u_t shall be independent of the unrevealed information $\{\hat{w}_{t'}^{(\tau)}|\tau > t\}$ ($\forall t' \in \{0, \cdots, T-1\}$). Hence, we have

$$u_t = \pi_t(\hat{x}_t, \{\hat{w}_{t'}^{(\tau)}|\tau \le t\}), \quad t = 0, \cdots, T-1 \tag{24}$$

where π_t is a control law whose inputs include not only MMSE estimate $\hat{x}_t$ but also revealed forecast $\{\hat{w}_{t'}^{(\tau)}|\tau \le t\}$.

We define the optimal expected cost-to-go at time t with given state estimate $\hat{x}_t$ and disturbance forecast $\{\hat{w}_{t'}^{(\tau)}\}$ as $J_t^*(\hat{x}_t)$ (for convenience, here $\{\hat{w}_{t'}^{(\tau)}\}$ is not explicitly included as an argument). That is,

$$J_t^*(\hat{x}_t) \triangleq \mathbb{E}_{v_{t+1:T}}\big[\min_{u_{t:T-1}} \mathbb{E}_{x_t|\hat{x}_t,\tilde{w}_{t:T-1}}\big[\sum_{i=t}^{T-1}(x_i^\top Q x_i + u_i^\top R u_i) + x_N^\top Q x_N\big]\big].$$

The following theorem shows that J_t^* is essentially a quadratic function of $\hat{x}_t$ and $\{\hat{w}_{t'}^{(\tau)}|\tau \le t\}$, and it also determines the optimal control law π_t^*.

Theorem 3.1: For any time t, $J_t^*(\hat{x}_t)$ has the form of Eq. (18), where matrix P_t is the same matrix given by Eq. (6), matrix $M_{i,t}$ couples $\hat{w}_i$ and $\hat{x}_t$ and is determined recursively by Eq. (19), matrix $N_t(\hat{w}_i^{(\tau)})$ couples $\hat{w}_i^{(\tau)}$ itself and is

$$
\begin{aligned}
J^* =& \mathbb{E}_{x_0, v_0, \{\hat{w}_t^{(\tau)}\}}[J_0^*(\hat{x}_0)] \\
=& \sum_{0\le\tau\le t\le T-1} \mathrm{Tr}(N_0(\hat{w}_t^{(\tau)})\hat{W}_t^{(\tau)}) + \sum_{t=0}^{T} \mathrm{Tr}(Q\Sigma_t) + \sum_{t=0}^{T} \mathrm{Tr}(P_t(\Sigma_{t|t-1} - \Sigma_t)) \\
=& \underbrace{\mathrm{Tr}(P_0 X) + \sum_{0\le\tau\le t\le T-1} \mathrm{Tr}(N_0(\hat{w}_t^{(\tau)})\hat{W}_t^{(\tau)}) + \sum_{t=1}^{T} \mathrm{Tr}(P_t \tilde{W}_{t-1})}_{J_{\text{lqr}}} + \underbrace{\sum_{t=0}^{T} \mathrm{Tr}((Q - P_t)\Sigma_t) + \sum_{t=1}^{T} \mathrm{Tr}(P_t A \Sigma_{t-1} A^\top)}_{J_{\text{est}}}
\end{aligned} \tag{25}
$$

determined recursively by Eq. (20), matrix[1] $N_t(\hat{w}_i^{(\tau)}, \hat{w}_{i'}^{(\tau')})$ couples $\hat{w}_i^{(\tau)}$ and $\hat{w}_{i'}^{(\tau')}$ which are different. The associated optimal control law π_t^* is given by Eq. (21).

Proof: We will prove this theorem by induction.

First, for $t = T$, we have $x_T|\hat{x}_T \sim \mathcal{N}(\hat{x}_T, \Sigma_T)$, hence $J_T^*(\hat{x}_T) = \hat{x}_T^\top Q \hat{x}_T + \mathrm{Tr}(Q\Sigma_T)$ has the form of Eq. (18).

Next, for $t < T$, suppose $J_{t+1}^*(\hat{x}_{t+1})$ already has the given form. Then we express $J_t^*(\hat{x}_t)$ as in Eq. (22), where the last the equation is obtained by expanding $J_{t+1}^*(Ax_t + Bu_t + \hat{w}_t + L_{t+1}e_{t+1})$ and applying $x_t|\hat{x}_t \sim \mathcal{N}(\hat{x}_t, \Sigma_t)$ and $e_{t+1}|\hat{x}_t \sim \mathcal{N}(0, C\Sigma_{t+1|t}C^\top + V_{t+1})$. Next, we need to determine the value of the last term, which is denoted by Ω for convenience. Since we can only choose u_t from π_t in Eq. (24), and with the knowledge of $\hat{w}_{t'}^{(\tau)}, \forall \tau > t$ is a Gaussian random variable with mean zero, we will have $u_t = \pi_t^*$ in Eq. (21) is the minimizer associated with Ω. Hence Ω has the expression in Eq. (23). Plug it into Eq. (22) we can verify $J_t^*(\hat{x}_t)$ also has the form of Eq. (18).

Hence Eq. (18) holds for any t. ∎

We can therefore compute the optimal average cost J^* by considering the expectation of $J_0^*(\hat{x}_0)$, as illustrated in Eq. (25). By comparing it against Eq. (12), we are able to explain why disturbance forecast can reduce the optimal average cost from three angles:

1) Forecast $\hat{w}_t$ makes the cost of imperfect state estimation J_{est} smaller. Though J_{est} in Eq. (25) has the same form as in Eq. (12), $\tilde{W}_t$ instead of W_t is used in the recursive equation Eq. (17). Since $\tilde{W}_t \preceq W_t, \forall t$, meaning the state estimation $\hat{x}_t$ will be more accurate when forecast $\hat{w}_t$ is available, one can expect J_{est} in Eq. (25) is smaller than the corresponding value in Eq. (12).

2) Forecast $\hat{w}_t$ makes the cost for LQR J_{lqr} smaller. The coefficients of $\hat{W}_t^{(\tau)}$ and $\tilde{W}_t$ are $N_0(\hat{w}_t^{(\tau)})$ and P_{t+1} respectively. According to Eq. (20), we have $N_0(\hat{w}_t^{(\tau)}) \preceq P_{t+1}, \forall \tau \le t$, which demonstrates the benefit of forecasting $\hat{w}_t$ in reducing J_{lqr}.

3) Earlier forecast can further reduce J_{lqr}. According to Eq. (20), we have $N_0(\hat{w}_t^{(\tau-1)}) = N_{\tau-1}(\hat{w}_t^{(\tau-1)}) \preceq N_\tau(\hat{w}_t^{(\tau-1)}) = N_\tau(\hat{w}_t^{(\tau)}) = N_0(\hat{w}_t^{(\tau)}), \forall 0 < \tau < t$, which demonstrates the benefit of having a earlier forecast in reducing J_{lqr}.

[1] Its explicit value is not of our interest in this paper, and later we will show the optimal average cost J^* does not depend on $N_t(\hat{w}_i^{(\tau)}, \hat{w}_{i'}^{(\tau')})$.

IV. EVALUATION

In this section, we illustrate our theoretical results through two numerical examples. In Sec. IV-A, we consider a simple control system where all the random variables are scalars, making it easier for us to compare the coefficients in Eq. (25) since they degenerate into scalars as well. In Sec. IV-B, we consider a widely-used two-dimensional robot control system [5], [6], [11], with small modifications.

A. Simple Control System with Scalars

In this example, we have $n = m = k = 1$, $T = 20$, $A = B = C = Q = 1$, $X = 1$, and $W_t = V_t = 1, \forall t$. Since all the random variables are scalars, we can hence plot coefficients $N_0(\hat{w}_t^{(\tau)})$ and P_t as functions of (t, τ) and t, respectively.

Fig. 1 shows the numerical results for two different control cost: $R = 1$ (top) and $R = 100$ (bottom). In Fig. 1(a) and Fig. 1(b), we observe $N_0(\hat{w}_t^{(\tau)}) \le 0.62, \forall \tau \le t$ and $P_t \ge 1, \forall t$, which demonstrate the benefit of forecasting a portion of w_t in reducing J_{lqr}. Moreover, we also observe that for a fixed t and different τ's, the value of $N_0(\hat{w}_t^{(\tau)})$ doesn't differ too much when $\tau < t - 1$. This means that knowing the forecast too early brings nearly zero additional advantage under the given setting, and a lookahead forecast window H equals 1 or 2 is already good enough. On the other hand, with a much larger control cost $R = 100$, Fig. 1(c) illustrates a different shape of the plot of $N_0(\hat{w}_t^{(\tau)})$. We observe obvious difference between $N_0(\hat{w}_t^{(\tau-1)})$ and $N_0(\hat{w}_t^{(\tau)})$ for most neighboring $(t, \tau - 1)$ and (t, τ), which demonstrates the importance of knowing the forecast as early as possible due to the high penalty of inaccurate control.

B. Two-dimensional Robot Control System

In this example, we consider a two-dimensional robot control system with $n = 4$, $m = 2$, $k = 4$, $T = 50$. The first and last 2 dimensions of variable x represent the location and the velocity of robot in a 2-d plane, respectively; and the control u represents the acceleration. We assign the following values to the matrices:

$$
A = \begin{bmatrix} 1 & 0 & 0.2 & 0 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \tag{26}
$$

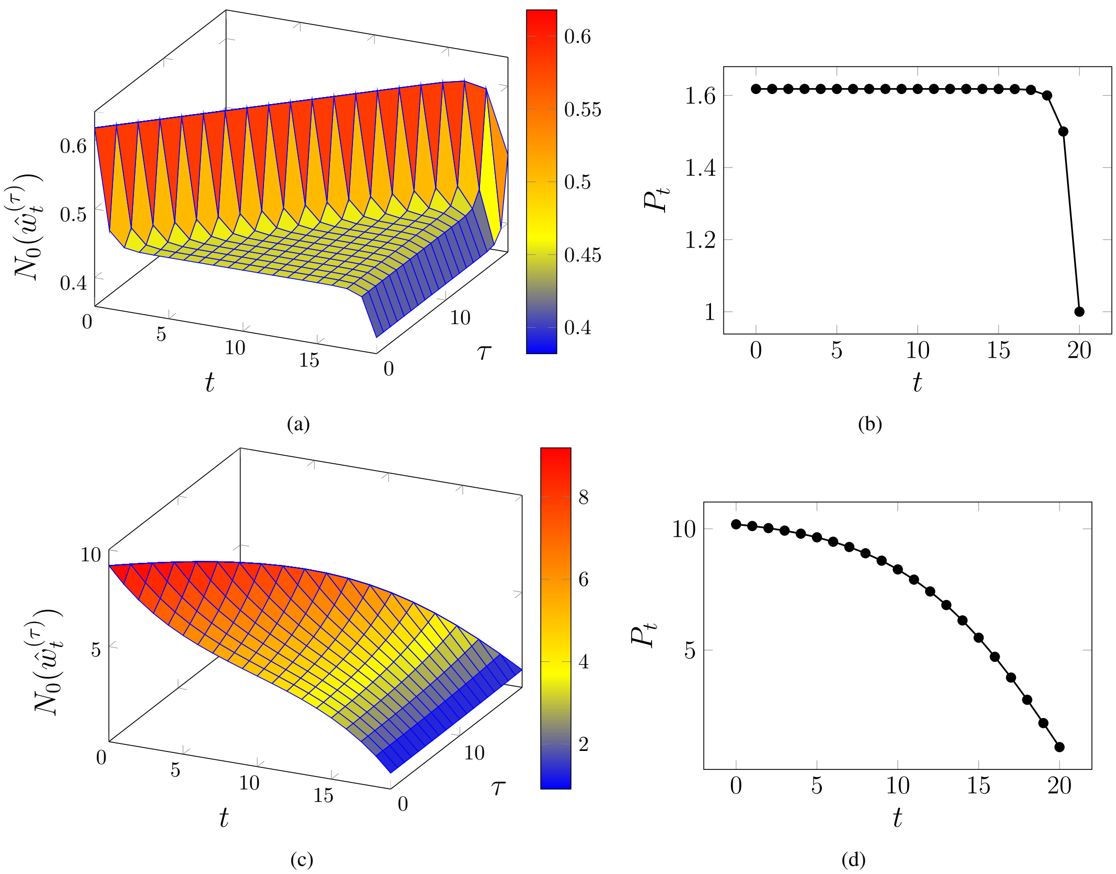

Fig. 1. Numerical results of the simple control system. The upper and lower figures has control cost $R = 1$ and $R = 100$ respectively. The left and right figures plot the values of $N_0(\hat{w}_t^{(\tau)})$ and P_t, respectively.

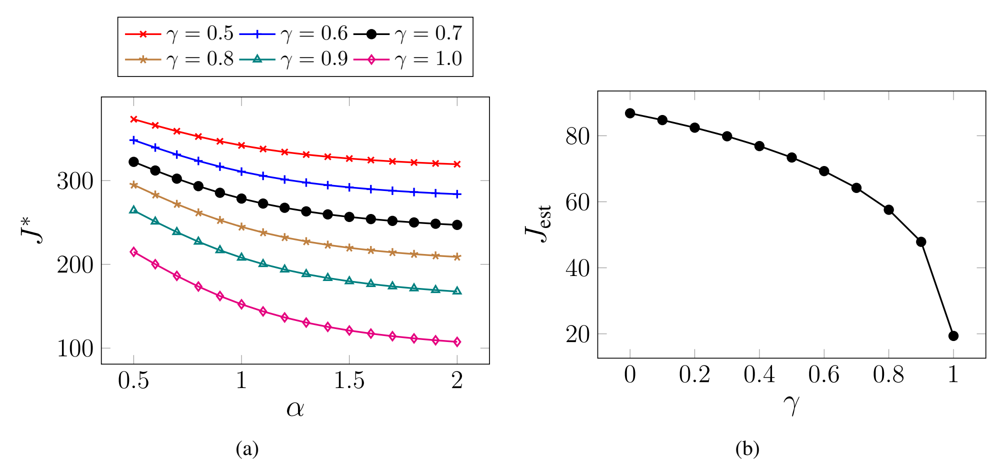

Fig. 2. Numerical results of two-dimensional robot control system. The right and left figures plot the values of J^* and J_{est} respectively.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \tag{27}$$

$$W_t = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.2 \\ 0.5 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.5 \\ 0.2 & 0.2 & 0.5 & 1 \end{bmatrix}, \forall t \tag{28}$$

and $C, X, V_t, \forall t$ are all identity matrices. Moreover, we restrict our attention to a family of forecasts with the following expression: 1) there's a lookahead forecast window $H = 5$; 2) we have $\hat{W}_t = \gamma W_t, \forall t$, where $\gamma \in [0, 1]$ is a coefficient that controls the percentage of w_t that is predictable; and 3) we let $\hat{W}_t^{(\tau-1)} = \alpha \hat{W}_t^{(\tau)}, \forall \max\{0, t-H+1\} \leq \tau \leq t-1$, where $\alpha \in (0, \infty)$ is an another coefficient that controls the allocation of $\hat{W}_t$ among $\hat{W}_t^{(\tau)}$'s.

Fig. 2 shows the numerical results. In Fig. 2(a), we compare the value of J^* for different γ's and α's. For fixed α, a larger γ allows a higher percentage of predictable portion; and for fixed γ, a larger α allows a higher percentage of those earlier forecasts. Therefore, larger γ and larger α both lead to a lower J^*. In Fig. 2(b), we compare the cost of imperfect state estimation J_{est} for different γ's (which is an invariant of the distribution of $\hat{W}_t^{(\tau)}$'s). It demonstrates that a higher percentage of predictable portion can make J_{est} significantly lower.

V. CONCLUSION

In this paper, we introduced a time-varying disturbance forecast model for the LQG problem. We then derived optimal state estimation, optimal control law, and optimal average cost under this formulation. Numerical examples are provided to illustrate the theoretical results.

One line of future work includes further characterization of the cost of obtaining a disturbance forecast timely and accurately, which stems from the fact that many data are not free in practice and may demand human labor or equipment utilization. In particular, inspired by works on rate-cost tradeoff in control [12], [13], where the adopted control is constrained by the bitrate of a communication channel, we plan to investigate the tradeoff between two different costs in a control system: the optimal average cost J^* and the cost for computing disturbance forecast.

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