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Conformal fields and the structure of the space of solutions of the Einstein constraint equations

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The drift method, introduced in [22], provides a new formulation of the Einstein constraint equations, either in vacuum or with matter fields. The natural of the geometry underlying this method compensates for its slightly greater analytic complexity over, say, the conformal or conformal thin sandwich methods. We review this theory here and apply it to the study of solutions of the constraint equations with non-constant mean curvature. We show that this method reproduces previously known existence results obtained by other methods, and does better in one important regard. Namely, it can be applied even when the underlying metric admits conformal Killing (but not true Killing) vector fields. We also prove that the absence of true Killing fields holds generically.

1. Introduction

Let (M, g, K) denote a triplet consisting of an n-dimensional manifold M, a metric g on M, and an auxiliary symmetric 2-tensor K. The vacuum Einstein constraint equations for this triplet are

$$R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 = 0 \qquad \text{[Hamiltonian constraint]} \tag{1.1a}$$

$$\operatorname{div}_g K - \mathbf{d}(\operatorname{tr}_g K) = 0. \qquad \text{[momentum constraint]} \tag{1.1b}$$

We typically assume that M is compact, or at least that (M, g) is complete. Solutions correspond to space-like hypersurfaces in a Lorentzian spacetime (X, G), i.e., solutions of the vacuum Einstein equations $\operatorname{Ric}(G) = 0$, so g is the induced metric and K the second fundamental form of this hypersurface. Solutions to system (1.1) serve as Cauchy data for the Einstein evolution problem (which of course must be supplemented by some choice of gauge to make the problem hyperbolic). The interest in finding solutions of the constraint equations is directly tied in this way to the study of the general Einstein equations. More general versions of these equations include a cosmological constant and source terms, and will be recalled below.

The set of pairs (g, K) which solve (1.1a, 1.1b) is infinite dimensional, and in a suitable topology constitutes a Banach manifold (at least away from the solutions for which the linearized operator has cokernel). To turn the search for these solutions into a less underdetermined and hence more tractable problem, it is customary to decompose the space of all pairs (g, K) into 'slices' and consider the constraint equations as an equation within each slice. If done correctly, this leads to a family of semilinear elliptic equations, one for each slice, to which one can apply a vast panoply of known techniques. The traditional slicing is known as the conformal method, originally proposed by Lichnerowicz and Choquet-Bruhat, and studied by them and many others over the past 60 years. Another common method appearing in the intervening years is called the conformal thin sandwich method. Although apparently different, it was proved by the second author [23] that this is completely equivalent to the older conformal method.

In the conformal method, the data for the slices consist of triplets $(g, \tau, \sigma; N)$ where g dictates the conformal class $[g]$ of the solution metric $\overline{g}$, τ is the 'mean curvature function', i.e., $\tau = \operatorname{tr}_{\overline{g}} K$ for the eventual solution, σ is a transverse-traceless (i.e. trace-free and divergence-free) tensor with respect to g, and N is a positive function that plays the role of a gauge choice and is related to the so-called lapse associated with a coordinate system on the spacetime generated by the solution of the constraint equations. A comprehensive description of solutions to the conformal method is known in the special case when τ is constant [13], and this led to perturbative results shortly thereafter [15]. Significant breakthroughs were obtained by the first and later the second authors [11, 20] concerning existence for 'far-from-CMC' data, where the mean-curvature function is allowed to be variable and seemingly nowhere close to constant, with a price of requiring the transverse-traceless tensor to be very small. This led to several new developments, and extensions and refinements of these ideas in various other standard settings. It was pointed out recently, however, by Gicquaud and his collaborators [9] that upon recasting the setup in certain way, all of these results are still fundamentally perturbative and hence should be regarded as 'near-CMC'.

In recent years limitations of the conformal method in the far-from-CMC setting have appeared. We point to [21] [24] along with the very nice results in [26] (based on the original blowup analysis of [7]) for examples where there exist either no or multiple solutions of the constraint equations corresponding to a given set of conformal data $(g, \sigma, \tau; N)$, and there is nothing apparent in the geometry of this data set which allows one to a priori predict what happens. Motivated by these difficulties, the second author here proposed [22] a different idea to slice up the space of pairs (g, K). This is known as the *drift* method, and is based on an invariant geometric interpretation of the dynamics of spacelike hypersurfaces evolving in a Lorentzian Einstein manifold. We review these methods carefully below. For now let us note one key difference. In the drift method, the mean curvature function τ is replaced by a pair (τ_*, V), where τ_* is a certain average V is a vector field which represents a 'drift' equivalence class. The equations in this formulation are more nonlinear and more complicated than for the older methods, but the key motivation is that this new framework should make it easier to handle various well-known obstructions and subtleties in the conformal method. More specifically, it is not clear how to make the conformal method work when the conformal class $[g]$ admits conformal Killing fields, and indeed, we show here that there is a fundamental breakdown in that procedure. That method is also less tractable when τ has zeros. In fact, there are no general a priori estimates for solutions of these equations, and there are examples of families of solutions which blow up. The hope remains that better methods may predict the data sets near which a priori estimates fail.

The goal of the present paper is to show that drift method does at least as well as the conformal method, and in a certain sense, much better. More specifically, we prove a set of existence results for the drift formulation of the constraint equations, both without and with source terms, which include the far-from-CMC results cited above. All of this is done perturbatively around the CMC case. The major improvement is that these results also hold when the conformal class $[g]$ admits conformal Killing fields, so long as the the metric we are perturbing from has no Killing fields.

This paper is organized as follows. We begin by reviewing the standard conformal method and introducing the notion of conformal momentum, and then describe the precise way by which conformal Killing fields present an obstruction in the conformal method. We finally present the drift method in §4, and in §5 the adaptations necessary to incorporate matter fields. Section 6 then proves the existence of near-CMC solutions using the drift method, and also establishes that the hypotheses needed to apply this theorem hold generically.

1.1. Notation and conventions

In this paper we assume that M is a manifold of dimension $n \geq 3$. We assume M is compact, and occasionally do not say this explicitly in the statements of results, etc. Solutions to various equations are found in a Sobolev space $W^{k,p}$, where $k \in \mathbb{N}$, $k \geq 2$, and $p > 1$ are chosen so that

$$\frac{1}{p} - \frac{k-1}{n} < 0;$$

this ensures that $W^{k,p}$ functions have Hölder continuous first derivatives. If E is any smooth vector bundle over M, we write $W^{k,p}(M,E)$ for the space of sections of E which are in $W^{k,p}$ with respect to any local trivialization. In particular, we have the bundles TM of vector fields, T^*M of covector fields, S_2M of symmetric $(0,2)$ tensors and its subbundle $S_{\mathrm{tt}}(g)$ of transverse-traceless tensors with respect to the metric g. Function spaces of positive functions are denoted by a subscript $+$, e.g. $W^{k,p}_+(M)$.

We henceforth set the constants

$$q = \frac{2n}{n-2}, \qquad \kappa = \frac{n-1}{n}, \qquad a = 2\kappa q,$$

so q is a critical Sobolev exponent and κ and a are dimensional constants which appear in various equations below.

We also consider the conformal Killing operator, whose action on vector fields is

$$(\mathbf{L}\, X)_{ab} = \nabla_a X_b + \nabla_b X_a - \frac{2}{n} \operatorname{div} X\, g_{ab}.$$

Its adjoint $\mathbf{L}^*$ acts on symmetric, trace-free $(0,2)$ tensor A_{ab} by

$$(\mathbf{L}^* A)_b = -2\nabla^a A_{ab}$$

The kernel of $\mathbf{L}$ is the finite dimensional space $\mathcal{Q}$ of conformal Killing fields.

2. The standard conformal method and conformal momentum

The conformal method appears in the literature in two forms. The original conformal method was introduced by Lichnerowicz [19] and substantially extended by York, O'Murchadha, and Choquet-Bruhat among others in the 1970s. Some decades later York introduced the conformal thin-sandwich method [30], and later, with Pfeiffer, also gave an equivalent Hamiltonian formulation [27]. It turns out that the original conformal method and the conformal thin-sandwich method are really the same parameterization of the constraint equations [23]; we describe them here in a unified fashion that will also be helpful for describing the drift formulations of the constraint equations. For simplicity, we focus for now on the vacuum constraint equations; Section 5 below describes an approach for incorporating matter fields into both the standard conformal method and the drift formulation.

A metric and second fundamental form (g, K) canonically determine

- a conformal class $[g]$, and
- a mean curvature $\tau = g^{ab}K_{ab}$.

These are two of the parameters of the conformal method. The third and final parameter is not completely canonical and depends on a choice of volume form α. We will call α a *volume gauge*. Once this has been fixed, the final parameter is

- the conformal momentum of (g, K) measured by α,

which we define in Definition 2.3 below. We refer to [23] and [22] for the geometric and physical motivation behind this terminology.

Definition 2.1 (Conformal Momentum). A *conformal momentum* is an equivalence class of pairs (g, σ) where g is a metric, σ is transverse traceless with respect to g (i.e., σ is trace-free and divergence-free) and where we identify pairs

$$(g,\sigma) \sim (\phi^{q-2} g, \phi^{-2}\sigma) \tag{2.1}$$

for any conformal factor $\phi \in W^{k,p}_+(M)$.

To complete the description of the measurement of conformal momentum, we first recall a variation of York splitting [29].

Lemma 2.2. *Suppose that $A \in W^{k-1,p}(M, S_2 M)$ be trace-free and fix any $N \in W^{k,p}_+(M)$. Then there is a unique transverse-traceless $\sigma \in W^{k-1,p}(M, S_2 M)$ and a vector field $W \in W^{k,p}(M, TM)$ such that*

$$A = \sigma + \frac{1}{2N} \mathbf{L}\, W. \tag{2.2}$$

This formulation uses that M is compact. The vector field W here is uniquely determined up to addition with a conformal Killing field.

The special case $N \equiv 1/2$ is more commonly known as York splitting, but the result for arbitrary N is a consequence of the $N \equiv 1/2$ case [23], or alternatively, can be proved directly by applying Lemma 6.10 below to solve

$$\mathbf{L}^* \frac{1}{2N} \mathbf{L}\, W = \mathbf{L}^* A. \tag{2.3}$$

Definition 2.3 (Measurement of Conformal Momentum). Suppose that α is a fixed $W^{k,p}$ volume form on M. The conformal momentum of (g, K) measured by α, denoted $[g, K]_\alpha$, is the equivalence class of the pair (g, σ), where σ is computed as follows. Write $K = A + \frac{\tau}{n} g$ where A is trace-free, and let $N = dV_g/\alpha$; then apply York splitting to decompose

$$A = \sigma + \frac{1}{2N} \mathbf{L}\, W. \tag{2.4}$$

Briefly, the aim of the conformal method is to use the conformal class, conformal momentum measured by α, and mean curvature as the ‘seed data’ for solutions of the constraint equations. Fixing α, we prescribe a conformal class $\mathbf{g}$, a conformal momentum $\boldsymbol{\sigma}$, and a mean curvature τ, and seek a solution $(\overline{g}, \overline{K})$ of the vacuum constraints with

$$[\overline{g}] = \mathbf{g}, \quad [\overline{g}, \overline{K}]_\alpha = \boldsymbol{\sigma}, \quad \overline{g}^{ab}\overline{K}_{ab} = \tau. \tag{2.5}$$

To cast this as a PDE, pick an arbitrary representative g of $\mathbf{g}$, let σ be the unique g-transverse-traceless tensor such that (g, σ) is a representative of $\boldsymbol{\sigma}$, and define the lapse $N = dV_g/\alpha$. We call $(g, \sigma, \tau;\ N)$ a *conformal data set*, with the lapse segregated from the other terms to reflect its role as a gauge choice. Starting from a conformal data set we seek a conformal factor ϕ and a vector field W solving

$$-a\Delta\phi + R\phi - \left|\sigma + \frac{1}{2N}\,\mathbf{L}\,W\right|^2 \phi^{-q-1} + \kappa\tau^2\phi^{q-1} = 0 \tag{2.6a}$$

[CTS-H Hamiltonian constraint]

$$\frac{1}{2}\,\mathbf{L}^*\left[\frac{1}{2N}\,\mathbf{L}\,W\right] + \kappa\phi^q\,\mathbf{d}\,\tau = 0 \tag{2.6b}$$

[CTS-H momentum constraint]

which we call the conformal thin-sandwich equations in their Hamiltonian formulation (the CTS-H equations). If (ϕ, W) solves these equations then the pair

$$\overline{g} = \phi^{q-2}g, \quad \overline{K} = \phi^{-2}\left(\sigma + \frac{1}{2N}\,\mathbf{L}\,W\right) + \frac{\tau}{n}\overline{g} \tag{2.7}$$

solves (2.5), and all solutions of problem (2.5) are obtained this way.

We observe that (2.5) is intrinsically conformally covariant, and hence the CTS-H equations must also be. Concretely, the solutions determined by $(g, \sigma, \tau;\ N)$ and

$$(\hat{g}, \hat{\sigma}, \hat{\tau};\ \hat{N}) = (\psi^{q-2}g, \psi^{-2}\sigma, \tau;\ \psi^q N) \tag{2.8}$$

are the same. In other words, we are expressing the same problem (2.5) using two different, but conformally related, sets of data. The standard conformal method corresponds to using the conformal representative of $\mathbf{g}$ with volume form $dV_g = \alpha/2$, so that $N \equiv 1/2$ in (2.6); we are thus restricting ourselves to an inflexible choice for the background metric to represent the problem, whereas if we allow an arbitrary background metric in the conformal class, we must introduce the lapse function N into (2.6), which then gives the Hamiltonian conformal thin-sandwich method of [27].

A conformal data set $(g, \sigma, \tau;\ N)$ determines the volume gauge α by $N = dV_g/\alpha$. Thus, fixing the background metric, the choice of lapse is equivalent to the choice of a volume gauge. It is important to note that the lapse transforms conformally by $\hat{N} = \psi^q N$, cf. [30]; we say that the conformal method involves a *densitized lapse*. On the other hand, the volume gauge α is a fixed object, and applies to all representatives of a conformal class.

3. Conformal Killing fields and the conformal method

Suppose $(\overline{g}, K)$ is a vacuum initial data set and that $\overline{g}$ admits a conformal Killing field Q. The momentum constraint implies

$$-\overline{\nabla}^a (K_{ab} - \tau\, \overline{g}_{ab}) = 0 \tag{3.1}$$

where, as usual, $\tau = \overline{g}^{ab} K_{ab}$. Multiplying equation (3.1) by the conformal Killing field Q, integrating by parts, and using the conformal Killing equation

$$\overline{\nabla}_a Q_b + \overline{\nabla}_b Q_a = \frac{2}{n} \overline{\nabla}_c Q^c g_{ab}, \tag{3.2}$$

we find

$$\begin{aligned} 0 &= \int_M [K_{ab} - \tau\, \overline{g}_{ab}]\, \frac{1}{2} (\overline{\nabla}^a Q^b + \overline{\nabla}^b Q^a)\, dV_{\overline{g}} \\ &= \int_M [K_{ab} - \tau\, \overline{g}_{ab}]\, \frac{1}{n} (\overline{\nabla}_c Q^c) \overline{g}^{ab}\, dV_{\overline{g}} \\ &= \frac{1-n}{n} \int_M \tau\, \overline{\nabla}_c Q^c\, dV_{\overline{g}}. \end{aligned} \tag{3.3}$$

Integrating by parts one more time gives the CKF compatibility condition

$$\int_M Q(\tau)\, dV_{\overline{g}} = 0 \tag{3.4}$$

between mean curvature and conformal Killing fields.

Suppose $(g, \sigma, \tau;\ N)$ is a CTS-H conformal data set where g admits a conformal Killing field Q, and let (ϕ, W) be a solution of the corresponding CTS-H equations. The CKF compatibility condition (3.4) then becomes

$$\int_M Q(\tau)\, \phi^q\, dV_g = 0. \tag{3.5}$$

Since (3.5) involves the unknown ϕ, it is not obvious whether the CKF compatibility condition imposes a genuine restriction on allowable conformal data sets; conceivably, the conformal method might always manage to find a conformal factor ϕ satisfying (3.5), regardless of the choice of τ. Nevertheless, equation (3.5) presents an obstacle in current solution techniques for the CTS-H equations. Typically one generates a sequence $(\phi_{(n)}, W_{(n)})$ of approximate solutions iteratively; each iteration involves solving a variation of the momentum constraint such as

$$\nabla_b \left[\frac{1}{2N} (\mathbf{L}\, W_{(n+1)})^{ab} \right] = \frac{n-1}{n} (\phi_{(n)})^q \, \nabla^a \tau. \tag{3.6}$$

Equation (3.6) is solvable for $W_{(n+1)}$ if and only if

$$\int_M Q(\tau) \, \phi_{(n)}^q \, dV_g = 0 \tag{3.7}$$

for all conformal Killing fields Q. Any standard method does not ensure that the successive functions $\phi_{(n)}$ still satisfy (3.7), so it may not be possible to continue the iteration procedure.

Nevertheless, for certain conformal data sets, conformal Killing fields are not an obstruction to solving the CTS-H equations. Most importantly, if τ is constant then $Q(\tau) \equiv 0$ for any conformal Killing field, and condition (3.5) is satisfied trivially for every conformal factor. In other words, the presence of conformal Killing fields plays no role in the CMC theory as described in, for example, [13]. A minor generalization is that if τ is constant on the integral curves of every conformal Killing field Q, then we still have that $Q(\tau) \equiv 0$, hence (3.5) is satisfied trivially regardless of the conformal factor. This observation was exploited in [4] to construct near-CMC solutions under this hypothesis on the mean curvature function. This is a strong hypothesis, of course (and amounts to assuming that τ is constant for metrics conformal to the flat torus or the round sphere). Moreover, this hypothesis is not necessary: [21] and [24] contain examples of non-CMC conformal data sets where the background metric is a flat torus (hence not covered by [4]) and where there exist solutions. The current theory for the CTS-H equations does not exclude the possibility that conformal Killing fields are irrelevant to solvability.

We now give a simple example which shows that at least in certain situations, the existence theory is sensitive to the presence of conformal Killing fields. The argument stems from the observation that (3.4) is analogous to the Pohozaev constraint

$$\int_M Q(R) \, dV_g = 0, \tag{3.8}$$

which relates the scalar curvature function R and conformal Killing fields Q on a compact manifold [2]. Its proof is a straightforward adaptation of ideas from [17] and [2] concerning obstructions to the existence of solutions for the

Nirenberg problem of finding metrics in a conformal class with prescribed scalar curvature.

Proposition 3.1. *Let g be the round metric on the sphere S^n, $\sigma \not\equiv 0$ a smooth transverse traceless tensor, and τ_0 a constant. There exists a smooth function T such that for every $\epsilon \in \mathbb{R}$, the conformal data set $(g, \sigma, \tau_0 + \epsilon T; N)$ admits a solution of the vacuum CTS-H equations if and only if $\epsilon = 0$.*

Proof. Fix $p \in S^n$ and let T be the distance function from p, and Q the conformal Killing field $\operatorname{grad} T$. Define $\tau_\epsilon = \tau_0 + \epsilon T$. Since (S^n, g) is Yamabe positive and $\sigma \not\equiv 0$, the CMC case of existence theory for the conformal method implies there exists a solution of the CTS-H equations when $\epsilon = 0$. On the other hand, if $\epsilon \neq 0$, then $Q(\tau_\epsilon) = \epsilon Q(T)$ has a single sign (except at the antipodal points), and hence $\int_{S^n} Q(\tau)\phi^q \, dV_g \neq 0$ for any choice of conformal factor ϕ. This violates the CKF compatibility condition (3.5) for every possible conformal factor and hence there exists no solution of the CTS-H equations for this conformal data when $\epsilon \neq 0$. The lack of smoothness of T at the antipodal points is not relevant here since we could replace T by a smooth nonnegative function of $\operatorname{dist}(p, \cdot)$ which is smooth on S^n and satisfies the same conclusion. $\square$

Proposition 3.1 shows that there exist CMC solutions of the constraint equations such that, replacing the mean curvature function by certain arbitrarily small perturbations of it in the conformal data set, then the CTS-H equations no longer have a solution. This means that the standard hypothesis in the near-CMC theory that the metric does not admit nontrivial conformal Killing fields cannot be dropped completely. It is not at all clear if there is some natural and easily apparent geometric condition that distinguishes when one should expect there to exist solutions or not.

We shall take an alternate course and give up on prescribing the mean curvature function specifically. The drift method described in the next section involves the prescription of different sets of data, and implicitly shows how to adjust the mean curvature to account for the CKF compatibility condition.

4. Drift variations of the conformal method

In this section we give a brisk description of the drift formulations of the conformal method [22]. Before getting into details, we observe that the principal distinction between the drift and standard conformal methods is that while the conformal method prescribes the mean curvature τ of the solution directly, the drift techniques involve a decomposition

$$\tau = \tau_* + \frac{1}{N}\operatorname{div} V \tag{4.1}$$

where $\tau_* \in \mathbb{R}$, N is a positive function (the same lapse appearing in the CTS-H equations) and V is a vector field. The mean curvature determined by τ_* and V changes as we move between representatives in a given conformal class for several reasons. First, the divergence operator depends on the choice of representative. Second, the lapse transforms as a densitized lapse, as described at the end of Section 2. Finally, if the metric admits conformal Killing fields, we cannot prescribe V directly, but must add a suitable conformal Killing field Q that changes as we change the conformal class representative. So in general,

$$\tau = \tau_* + \frac{1}{N}\operatorname{div}(V + Q) \tag{4.2}$$

where Q is a conformal Killing field determined by V. In short, the actual mean curvature function determined by the data in the drift formulations naturally adapts to the presence of conformal Killing fields. This allows one to prove slightly more general results.

As discussed next in Section 4.1, the constant τ_* in equation (4.2) represents a certain dynamical quantity called the volumetric momentum. To interpret the vector field V, we first note that the mean curvature is unchanged by adding a divergence-free vector field to V. Moreover, V is prescribed only up to adjustment by a suitable conformal Killing field, so V is an element of the space of vector fields modulo both conformal Killing fields and divergence-free vector fields. This quotient space is the space of so-called drifts. We discuss them further in Section 4.2, before giving the equations for the drift formulations in Section 4.3.

4.1. Volumetric momentum

The first step toward the drift parameterization of the constraint equations involves the identification of a parameter, volumetric momentum. This plays a role somewhat analogous to the transverse-traceless tensor in the standard conformal method, and represents a cotangent vector to the one-dimensional space of volume forms modulo diffeomorphisms, so the volumetric momentum is just a number. It arises in the following analog of York splitting.

Lemma 4.1. *Let $\tau \in W^{k-1,p}(M)$ and let $N \in W^{k,p}_{+}(M)$. There is a unique constant τ_* and a vector field $V \in W^{k,p}$ such that*

$$\tau = \tau_* + \frac{1}{N} \operatorname{div} V. \tag{4.3}$$

Moreover, V is uniquely determined up to addition of a divergence-free vector field, and

$$\tau_* = \frac{\int_M N\tau \, dV_g}{\int_M N \, dV_g}. \tag{4.4}$$

This is proved in [22] when the data is smooth, but the same proof works for metrics and data with the regularity stated here.

Definition 4.2 (Measurement of Volumetric Momentum). Let α be a $W^{k,p}$ volume form on M. The *volumetric momentum* of (g, K) measured by α is computed as follows. First, let $N = dV_g/\alpha$ and define $\tau = g^{ab}K_{ab}$. By Lemma 4.1, $\tau = \tau_* + \frac{1}{N} \operatorname{div} V$ for a unique constant τ_*. The volumetric momentum of (g, K) measured by α is

$$[g, \tau]_\alpha = -2\kappa\tau_*, \quad \kappa = (n-1)/n. \tag{4.5}$$

Volumetric momentum is already an interesting parameter in the standard conformal method. Examples in [24] exhibit the development of certain one-parameter families of non-CMC solutions of the constraint equations generated by the standard conformal method, and $\tau_* = 0$ is among the several necessary conditions needed to generate these families. Curiously, $\tau_* = 0$ is not easily detected from the usual conformal data; in effect, one must solve the equations of the conformal method to determine if τ_* vanishes or not. These examples motivate finding a parameterization in which τ_* is explicitly prescribed.

4.2. Drift

Fixing a volume gauge α, the conformal momentum and volumetric momentum of (g, K) measured by α drop out of the momentum constraint. Indeed, using Lemmas 2.2 and 4.1 to decompose

$$K = \sigma + \frac{1}{2N} \mathbf{L}\, W + \frac{1}{n} \left[\tau_* + \frac{1}{N} \operatorname{div} V \right] g, \tag{4.6}$$

then the vacuum momentum constraint becomes

$$-\frac{1}{2}\,\mathbf{L}^*\,\frac{1}{2N}\,\mathbf{L}\,W = \kappa\,\mathbf{d}\left[\frac{1}{N}\operatorname{div} V\right]. \tag{4.7}$$

The momentum equation in this formulation has interesting symmetries. The vector fields W and V appearing in it each represent a certain geometric object, coined a drift in [22].

Definition 4.3 (Drift). Let g be a $W^{k,p}$ metric. A *drift* at g is an element of

$$W^{k,p}(M,TM)/(\operatorname{Ker}\mathbf{L}_g + \operatorname{Ker}\operatorname{div}_g). \tag{4.8}$$

We write $[W]_g^{\mathrm{drift}}$ for the drift at g determined by the vector field W and Drift_g for the space of drifts at g.

Remark 4.4. *The spaces* $\operatorname{Ker}\mathbf{L}_g$ *and* $\operatorname{Ker}\operatorname{div}_g$ *intersect in the space of Killing fields, but since* $\operatorname{Ker}\operatorname{div}_g$ *is closed in* $W^{k,p}(M,TM)$ *and* $\operatorname{Ker}\mathbf{L}_g$ *is finite dimensional,* $\operatorname{Ker}\mathbf{L}_g + \operatorname{Ker}\operatorname{div}_g$ *is also a closed subspace and the quotient* Drift_g *inherits a Banach space topology.*

As elaborated in [22], a drift represents an infinitesimal motion in the space of metrics, modulo diffeomorphisms, that preserves the conformal class up to diffeomorphism and the volume. Note that such a motion need not preserve the diffeomorphism class of the metric.

Equation (4.7) represents a relationship between two drifts. To see this, suppose $\mathbf{V}$ is a drift at g, with V any representative. Equation (4.7) can be regarded as a PDE in W. If g admits conformal Killing fields, there is no solution unless the right-hand side of (4.7) is orthogonal to $\mathbf{L}_g$. Assuming this orthogonality, hypothesis Theorem 10.1 of [22] shows (in the smooth category) that there is a conformal Killing field Q and a vector field W such that

$$-\frac{1}{2}\,\mathbf{L}^*\,\frac{1}{2N}\,\mathbf{L}\,W = \kappa\,\mathbf{d}\left[\frac{1}{N}\operatorname{div}(V+Q)\right]. \tag{4.9}$$

Here Q is uniquely determined up to a true Killing field, W is uniquely determined up to a conformal Killing field, and $[W]_g^{\mathrm{drift}}$ is independent of the choice of representative of the drift $\mathbf{V}$.

This process can be reversed. Suppose $\mathbf{W} \in \mathrm{Drift}_g$ and let W be any representative. We now wish to solve (4.7) for V, but to do so, the left-hand side of (4.7) must be orthogonal to the space of divergence-free vector fields. Theorem 10.6 of [22] shows (in the smooth category) that there is a divergence-free vector field E and a vector field V such that

$$-\frac{1}{2}\mathbf{L}^*\frac{1}{2N}\mathbf{L}(W+E)=\kappa\,\mathbf{d}\left[\frac{1}{N}\operatorname{div}(V)\right]. \tag{4.10}$$

Now E is uniquely determined up to a true Killing field, V is uniquely determined up to a divergence-free vector field, and $[V]_g^{\text{drift}}$ is independent of the choice of representative of the drift $\mathbf{W}$.

Motivated by this discussion, we assign a pair of drifts to a pair (g,K) as follows.

Definition 4.5 (Measurement of Drift). Suppose g is a $W^{k,p}$ metric, $K\in W^{k-1,p}(M,S_2M)$ and α is a $W^{k,p}$ volume form. Set $N=dV_g/\alpha$ and decompose

$$K=A+\frac{\tau}{n}g, \tag{4.11}$$

where A is trace-free. Now use Lemmas 2.2 and 4.1 to write

$$A=\sigma+\frac{1}{2N}\mathbf{L}\,W \tag{4.12}$$

and

$$\tau=\tau_*+\frac{1}{N}\operatorname{div}V. \tag{4.13}$$

The *volumetric drift* of (g,K) measured by α is $[V]_g^{\text{drift}}$, and the *conformal drift* measured by α is $[W]_g^{\text{drift}}$.

In our application of drifts to the construction of near-CMC solutions of the Einstein constraint equations we shall specify the conformal class of the solution metric and, among other parameters, the volumetric drift. Since drift is defined in terms of a metric rather than the conformal class, one needs to be able to specify a drift at an unknown solution metric $\overline{g}=\phi^{q-2}g$ starting from a given representative g of the conformal class. One can always specify the vector field V and let it determine the drift $[V]_{\overline{g}}^{\text{drift}}$, but unless one knows the conformal factor ϕ, it is impossible to know *a priori* whether V is divergence-free with respect to the solution metric and hence $[V]_{\overline{g}}^{\text{drift}}=0$.

To address this difficulty, suppose for the moment that the $W^{k,p}$ metric g admits no (nontrivial) conformal Killing fields. The Helmholtz decomposition implies

$$W^{k,p}(M,TM) = \mathcal{E} \oplus \mathcal{E}^{\perp}$$

where $\mathcal{E}$ is the set of $W^{k,p}$ divergence-free vector fields and $\mathcal{E}^{\perp}$ is the image of grad acting on $W^{k+1,p}$ functions. The factors in the direct sum are L^2 orthogonal, and the projection of a vector field X onto $\mathcal{E}^{\perp}$ is $\operatorname{grad} u$ where $\Delta\, u = \operatorname{div} X$. Because g admits no conformal Killing fields, the drifts at g can be identified with $\mathcal{E}^{\perp}$. Moreover, for a conformally related metric $\overline{g} = \phi^{q-2} g$ the conformal transformation rule for gradients implies

$$\mathcal{E}^{\perp}_{\overline{g}} = \phi^{2-q}\mathcal{E}^{\perp}_{g}.$$

Hence in absence of conformal Killing fields we have a mechanism for parameterizing drifts within a conformal class: drifts can be represented by elements of $\mathcal{E}^{\perp}_{g}$ and $V \in \mathcal{E}^{\perp}_{g}$ corresponds to $\phi^{2-q}V \in \mathcal{E}^{\perp}_{\overline{g}}$.

In the event that g admits conformal Killing fields the representation of drift within a conformal class is less straightforward because divergence-free vector fields and conformal Killing fields obey different conformal transformation laws. In this case the drifts at g can be identified with any one of a number of subspaces of $\mathcal{E}^{\perp}$, and it seems natural to use the L^2 orthogonal complement of $P(\mathcal{Q})$, where P is the L^2 projection of $W^{k,p}(M,TM)$ onto $\mathcal{E}^{\perp}$ discussed above.

Definition 4.6. A *canonical drift representative* at a $W^{k,p}$ metric g is a vector field $V \in \mathcal{E}^{\perp}$ satisfying

$$\int_M g(V, P(Q))\, dV_g = 0$$

for all conformal Killing fields Q. The set of canonical drift representatives at g is denoted by $\mathcal{D}_g$.

It is easy to see that the map $V \mapsto [V]^{\mathrm{drift}}_g$ from $\mathcal{D}_g$ to Drift_g is a Banach space isomorphism and hence Drift_g can be identified with a subspace of $\mathcal{E}^{\perp}_g$ with codimension equal to $\dim P(\mathcal{Q})$. This codimension need not be constant among all representatives of a conformal class, however. Indeed, a conformal Killing field Q is a true Killing field exactly when it is divergence-free, i.e. when $P(Q) = 0$. Thus $\dim P(\mathcal{Q}) \le \dim \mathcal{Q}$ with strict inequality whenever the metric admits nontrivial Killing fields. The non-constant codimension of $\mathcal{D}_g$ in $\mathcal{E}_g^\perp$ poses an obstacle to the universal representation of drift for a fixed conformal class. Nevertheless, our main application of drifts to the conformal method is perturbative, and the following lemma shows that we can use $\mathcal{D}_g$ to identify drifts at nearby representatives of the conformal class so long as g does not admit any true Killing fields.

Lemma 4.7. *Let g be a $W^{k,p}$ metric. Given a conformal factor $\phi \in W^{k,p}$ let $\overline{g} = \phi^{q-2} g$. The map*

$$V \mapsto [\phi^{2-q} V]_{\overline{g}}^{\mathrm{drift}}$$

from $\mathcal{D}_g$ to $\mathrm{Drift}_{\overline{g}}$ is an isomorphism if either

- *g admits no (nontrivial) conformal Killing fields, or*
- *g admits no (nontrivial) Killing fields and ϕ is sufficiently close to 1 in $W^{k,p}$.*

Proof. Let $\mathcal{E}_\phi$, $\mathcal{E}_\phi^\perp$, $\mathcal{D}_\phi$ and so forth represent objects associated with the metric $\overline{g} = \phi^{q-2} g$, let P_ϕ be the $\overline{g}$-L^2 projection of $W^{k,p}(M, TM)$ onto $\mathcal{E}_\phi^\perp$, and let D_ϕ be the $\overline{g}$-L^2 projection of $W^{k,p}(M, TM)$ onto $\mathcal{D}_\phi$. One readily verifies that these maps are continuous, in part using the standing hypotheses on k and p (which ensure that $W^{k,p} \subset L^2$), along with the fact that $P(\mathcal{Q})$ is finite dimensional. Given a vector field V, the projections $P_\phi(V)$ and $(D_\phi \circ P_\phi)(V)$ differ from V by linear combinations of conformal Killing fields and $\overline{g}$-divergence-free vector fields. Hence

$$[V]_{\overline{g}}^{\mathrm{drift}} = [P_\phi V]_{\overline{g}}^{\mathrm{drift}} = [(D_\phi \circ P_\phi)(V)]_{\overline{g}}^{\mathrm{drift}}.$$

In particular, if $V \in \mathcal{D}_1$, then $\phi^{2-q} V \in \mathcal{E}_\phi^\perp$ and

$$[\phi^{2-q} V]_{\overline{g}}^{\mathrm{drift}} = [D_\phi(\phi^{2-q} V)]_{\overline{g}}^{\mathrm{drift}}.$$

Since the projection from $\mathcal{D}_{\overline{g}}$ onto $\mathrm{Drift}_{\overline{g}}$ is an isomorphism it is therefore enough to show that $F_\phi : \mathcal{D}_1 \to \mathcal{D}_\phi$ defined by

$$F_\phi(V) = D_\phi(\phi^{q-2} V)$$

is an isomorphism under the given hypotheses on g and ϕ.

Suppose first that g admits no conformal Killing fields. In this case $\mathcal{D}_1 = \mathcal{E}_1^\perp$, $\mathcal{D}_\phi = \mathcal{E}_\phi^\perp$, and the result follows from the previously discussed isomorphism $\mathcal{E}_1^\perp \to \phi^{2-q} \mathcal{E}_1^\perp = \mathcal{E}_\phi^\perp$.

Now consider the case where g admits conformal Killing fields, but no Killing fields. Let G_ϕ be the $\overline{g}$-L^2 projection of $W^{k,p}(M,TM)$ onto $P_\phi(\mathcal{Q})$; we claim that G_ϕ is continuous in ϕ when thought of as a map with codomain $W^{k,p}(M,TM)$. Indeed first note that the maps P_ϕ, defined previously in terms of solving a Poisson problem for the metric $\overline{g}$, are continuous in ϕ. Hence, fixing a basis $\{Q_i\}$ for $\mathcal{Q}$, the vectors $P_\phi(Q_i)$ also depend continuously on ϕ. Moreover, since g has no Killing fields the map $P_1|_{\mathcal{Q}}$ is injective, and the continuity of P_ϕ with respect to ϕ (along with the fact that $\mathcal{Q}$ is finite dimensional) ensures that $P_\phi|_{\mathcal{Q}}$ is injective for ϕ sufficiently close to 1. Hence the vectors $\{P_\phi(Q_i)\}$ are linearly independent. The map taking a frame (in this case $\{P_\phi(Q_i)\}$) to an orthonormal frame via the Gram-Schmidt algorithm is continuous in the frame and the inner product jointly, and writing the projection G_ϕ with respect to the orthonormal frame it readily follows that G_ϕ is continuous in ϕ, as is $D_\phi = \mathrm{Id} - \mathrm{G}_\phi$ with codomain $W^{k,p}(M,TM)$.

Now consider the maps

$$B_\phi = (D_1 \circ S_\phi^{-1}) \circ (D_\phi \circ S_\phi) : \mathcal{D}_1 \to \mathcal{D}_1$$

where $S_\phi(V) = \phi^{2-q}V$. From our observation that D_ϕ is continuous in ϕ, so are the maps B_ϕ. And since $B_1 = \mathrm{Id}$, we conclude that B_ϕ is an isomorphism for ϕ sufficiently close to 1. Noting that

$$B_\phi = (D_1 \circ S_\phi^{-1}|_{\mathcal{D}_\phi}) \circ F_\phi, \tag{4.14}$$

to show that F_ϕ is an isomorphism for ϕ close to 1 it is therefore enough to establish the same fact for $D_1 \circ S_\phi^{-1}|_{\mathcal{D}_\phi} : \mathcal{D}_\phi \to \mathcal{D}_1$. Moreover, the factorization (4.14) already implies that $D_1 \circ S_\phi^{-1}|_{\mathcal{D}_\phi}$ is surjective for all conformal factors sufficiently near 1, and we need only establish injectivity.

The kernel of D_1 is $P_1(\mathcal{Q})$ and hence

$$\operatorname{Ker} D_1 \circ S_\phi^{-1}|_{\mathcal{D}_\phi} = (S_\phi \circ P_1)(\mathcal{Q}) \cap \mathcal{D}_\phi.$$

Now $(S_\phi \circ P_1)(\mathcal{Q}) \subseteq \mathcal{E}_\phi^\perp$, and since $\operatorname{Ker} G_\phi|_{\mathcal{E}_\phi^\perp} = \mathcal{D}_\phi$, to show that the subspace $(S_\phi \circ P_1)(\mathcal{Q}) \cap \mathcal{D}_\phi$ is trivial (for ϕ close to 1) it is enough to show that

$$G_\phi \circ S_\phi \circ P_1|_{\mathcal{Q}} : \mathcal{Q} \to W^{k,p}(M,TM)$$

is injective. But this follows from the fact that this family of linear maps has finite-dimensional domain, is continuous in ϕ, and is injective at $\phi \equiv 1$. $\square$

4.3. Parametrizations of the constraints using conformal deformation, expansion and drift

Recall that in the standard conformal method we prescribe the conformal class, conformal momentum, and mean curvature of the solution. In the drift formulation, we replace the mean curvature with the combination of volumetric momentum and either volumetric or conformal drift.

Consider a solution $(\overline{g}, K)$ of the vacuum constraint equations, and let α be a volume gauge. From Lemmas 2.2 and 4.1 the solution uniquely determines

- a conformal class $[\overline{g}]$,
- a conformal momentum measured by α represented by $(\overline{g}, \overline{\sigma})$ where $\overline{\sigma}$ is transverse-traceless with respect to $\overline{g}$,
- a volumetric momentum $-2\kappa\tau_*$ measured by α
- and a volumetric drift $[\overline{V}]_{\overline{g}}^{\mathrm{drift}}$ measured by α.

The first three parameters can be prescribed in a conformally invariant fashion by choosing a representative g of the conformal class, along with a g-transverse-traceless tensor σ and a constant τ_*. Then $\overline{g} = \phi^{q-2} g$ for some conformal factor ϕ and $\overline{\sigma} = \phi^{-2}\sigma$. As for the drift, recall from Lemma 4.7 that the map $\mathcal{D}_g \to \mathrm{Drift}_{\overline{g}}$ given by $V \mapsto [\phi^{2-q} V]_{\overline{g}}^{\mathrm{drift}}$ is an isomorphism so long as g has no conformal Killing fields, or so long as g has no Killing fields and $\overline{g}$ is sufficiently close to g. Hence we will select $V \in \mathcal{D}_g$ and set $\overline{V} = \phi^{2-q} V$, and the aim of the drift method is to recover the solution of the constraint equation from these parameters.

More precisely, we prescribe the following conformal data:

$$(g, \sigma, \tau_*, V;\ N) \tag{4.15}$$

where σ is transverse-traceless, τ_* is a constant, $V \in \mathcal{D}_g$ is a canonical drift representative at g, and N is lapse specifying a volume gauge α according to the relationship $N = dV_g/\alpha$. We seek a solution (ϕ, W, Q) of the following variation of equations from [22] Section 12:

$$\begin{aligned} &- a\,\Delta\,\phi + R\phi + \left|\sigma + \frac{1}{2N}\,\mathbf{L}\,W\right|^2 \phi^{-q-1} \\ &\qquad + \kappa\left(\tau_* + \frac{1}{N\phi^q}\,\mathrm{div}_\phi(\phi^{2-q} V + Q))\right)^2 \phi^{q-1} = 0 \\ &\frac{1}{2}\,\mathbf{L}^*\left(\frac{1}{2N}\,\mathbf{L}\,W\right) - \kappa\,\mathrm{div}_\phi^*\left(\frac{1}{N}\,\mathrm{div}_\phi(\phi^{2-q} V + Q)\right) = 0 \end{aligned} \tag{4.16}$$

where W is an arbitrary vector field and Q is a conformal Killing field. Here we are using the notation

$$\operatorname{div}_\phi = \phi^{-q} \operatorname{div} \phi^q \tag{4.17}$$

for the divergence operator of the metric $\phi^{q-2} g$, while

$$\operatorname{div}_\phi^* = -\phi^q \;\mathbf{d}\; \phi^{-q} \tag{4.18}$$

is the adjoint of div_ϕ with respect to the background metric g. The conformal Killing field Q is determined by the CKF compatibility condition

$$\int \frac{1}{N} \operatorname{div}_\phi(\phi^{2-q} V + Q) \operatorname{div}_\phi P \, dV_g = 0 \tag{4.19}$$

for all conformal Killing fields P, which can be added to system (4.16) to make the number of equations match the number of unknowns.

Supposing (ϕ, W, Q) solves system (4.16), let

$$\begin{aligned} \overline{g} &= \phi^{q-2} g \\ \tau &= \tau_* + \frac{1}{N\phi^q} \operatorname{div}_\phi(\phi^{2-q} V + Q) \\ \overline{K} &= \phi^{-2} \left[\sigma + \frac{1}{2N} \mathbf{L}\, W \right] + \frac{\tau}{n} \overline{g}. \end{aligned} \tag{4.20}$$

Following arguments from [22] it follows that $(\overline{g}, \overline{K})$ is a solution of the constraints with conformal class $[g]$, conformal momentum represented by (g, σ), volumetric momentum $-2\kappa\tau_*$, and volumetric drift $[\phi^{2-q} V]_{\overline{g}}^{\text{drift}}$ as desired. We will call system (4.16) together with (4.19) the vacuum CED-V equations, short for conformal deformation, expansion, and (volumetric) drift.

Alternatively, we can prescribe the conformal drift instead of the volumetric drift. Starting with conformal data

$$(g, \sigma, \tau_*, W;\; N) \tag{4.21}$$

with $W \in \mathcal{D}_g$ we seek a solution (ϕ, V, E) of

$$
\begin{aligned}
&- a\,\Delta\,\phi + R\phi + \left|\sigma + \frac{1}{2N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-q}E)\right|^2 \phi^{-q-1} \\
&\qquad + \kappa\left(\tau_* + \frac{1}{N\phi^q}\,\mathrm{div}_\phi(V+Q))\right)^2 \phi^{q-1} = 0 \\
&\frac{1}{2}\,\mathbf{L}^*\left(\frac{1}{2N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-q}E)\right) - \kappa\,\mathrm{div}_\phi^*\left(\frac{1}{N}\,\mathrm{div}_\phi(V)\right) = 0
\end{aligned} \tag{4.22}
$$

where V is an arbitrary vector field and E is divergence free. The vector field E is determined by the compatibility condition

$$
-\int \frac{1}{4N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-q}E)\,\mathbf{L}(\phi^{-q}F)\;dV_g = 0 \tag{4.23}
$$

for all divergence-free vector fields F. Given (ϕ, V, E) solving system (4.22), let

$$
\begin{aligned}
\overline{g} &= \phi^{q-2}g \\
\tau &= \tau_* + \frac{1}{N\phi^q}\,\mathrm{div}_\phi(V) \\
\overline{K} &= \phi^{-2}\left[\sigma + \frac{1}{2N}(\mathbf{L}\,W + \phi^{-q}E)\right] + \frac{\tau}{n}\overline{g}.
\end{aligned} \tag{4.24}
$$

We find $(\overline{g}, \overline{K})$ is a solution of the constraints as before, except that we have prescribed conformal drift $[\phi^{2-q}W]_{\overline{g}}^{\mathrm{drift}}$ rather than volumetric drift, and we will call system (4.22) the vacuum CED-C equations.

5. Conformal description of matter

Section 2 described the conformal method in terms of natural geometric parameters such as conformal momentum. By contrast, the current literature for including matter in the conformal method is somewhat ad hoc, and is guided by finding formulations that make the problem mathematically tractable [3]. We note, for example, the methods of scaling and unscaling sources, in the vocabulary of [5]. It has long been understood that in the CMC case the conformal method is compatible with scaling sources, whereas unscaling sources lead to undesirable non-uniqueness properties [28]. We also point to [16], which enunciates a fundamental guiding principle that leads to to the method of scaling sources; in effect we specify the configuration and momentum of matter independent of the metric.[1] Given our interest in constructing near-CMC solutions, we employ scaling sources in the framework laid out in [14]. This is described briefly here without any focus on the underlying principle of [16].

We represent matter fields as sections $\mathcal{F}$ of a smooth vector bundle over M. The energy and momentum densities of the matter fields are functions jointly of $\mathcal{F}$ and the metric g,

$$\mathcal{E}(\mathcal{F}, g) \text{ and } \mathcal{J}(\mathcal{F}, g) \tag{5.1}$$

respectively, and with this notation the full Einstein constraint equations read

$$R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 = 16\pi\mathcal{E}(\mathcal{F}, g) + 2\Lambda \tag{5.2a}$$

$$\operatorname{div}_g K - \mathbf{d}(\operatorname{tr}_g K) = -J(\mathcal{F}, g) \tag{5.2b}$$

where Λ is the cosmological constant.

We assume that $\mathcal{F}$ obeys a conformal transformation law. Specifically, if the metric changes from g to $\hat{g} = \phi^{q-2} g$ then the fields transform according to $\widehat{\mathcal{F}} = \Phi(\mathcal{F}, \phi)$ where Φ is a group action of the conformal factors on the matter fields, i.e., $\Phi(\mathcal{F}, 1) = \mathcal{F}$ and $\Phi(\Phi(\mathcal{F}, \phi_1), \phi_2) = \Phi(\mathcal{F}, \phi_1\phi_2)$. We assume moreover that any necessary compatibility conditions on the matter fields (e.g. the divergence-free condition for magnetic fields) are preserved as we transform from g to $\hat{g}$ and $\mathcal{F}$ to $\widehat{\mathcal{F}}$. The key hypothesis for scaling sources is that

$$\mathcal{J}(\Phi(\mathcal{F}, \phi), \phi^{q-2} g) = \phi^{-q} \mathcal{J}(\mathcal{F}, g). \tag{5.3}$$

This perhaps unmotivated scaling occurs naturally in practice and for CMC conformal data leads to a momentum constraint that is semi-decoupled from the Hamiltonian constraint. Fixing $\mathcal{F}$ at the metric g, the transformation law (5.3) amounts to assuming that the momentum density is described by a one form j that conformally transforms according to

$$\hat{j} = \psi^{-q} j. \tag{5.4}$$

[1] In light of [16], the term ‘scaling sources’ is a misnomer. In the method of scaling sources the configuration of matter is conformally invariant, and only the metric used to measure it changes.

Turning to the energy density, again fix $\mathcal{F}$ at g and define

$$\rho(\phi) = \mathcal{E}(\Phi(\mathcal{F}, \phi), \phi^{q-2} g). \tag{5.5}$$

The details of this map depend strongly on the specific type of matter, and we make the following minimal hypothesis.

Definition 5.1. A smooth map $\rho : W^{k,p}_{+}(M) \to W^{k-2,p}(M)$ satisfies the *energy scaling condition* if:

1) The linearization of ρ at ϕ in the direction $\dot{\phi}$ can be written in the form

$$D\rho_{\phi}[\dot{\phi}] = r\dot{\phi} \tag{5.6}$$

where $r \in W^{k-2,p}(M)$ depends on ϕ.

2) Either
   - $\rho(\phi) \equiv 0$ for all $\phi \in W^{k,p}_{+}(M)$, or
   - for all $\phi \in W^{k,p}_{+}(M)$ the $W^{k-2,p}(M)$ function that is the linearization of

$$\phi \mapsto \phi^{q-2}\rho(\phi) \tag{5.7}$$

   is non-positive and not identically zero.

As with hypothesis (5.4) for the momentum density, Definition 5.1 is somewhat unmotivated, but admits the following loose interpretation: energy density measured by the metric is a local property, depending on the value of ϕ but not its derivatives, and it grows at least as fast as ϕ^{2-q} as $\phi \to 0$, and decays at least as fast as ϕ^{2-q} as $\phi \to \infty$. We will use the notation $\rho(\cdot)$ for the map ρ as a reminder that it is a function taking a conformal factor as an argument, rather than simply a function defined on M.

The framework for matter described here is broad enough to include a number of important matter models including electromagnetism (and Yang-Mills fields more generally), perfect fluids (including dust), and Vlasov models. These details were treated in [14], where the energy scaling condition appears in a somewhat obscured form as hypothesis N1.[2] This framework notably excludes scalar fields, however, where condition 2 of Definition 5.1

---

[2]Condition N1 of [14] is equivalent to

$$\phi \mapsto \phi^{q-2}\rho(\phi) \tag{5.8}$$

fails, and we refer to [10] for alternate techniques needed to include scalar fields in the conformal method. As a concrete example, consider electromagnetism in 3-dimensions without charged sources. The matter fields consist of divergence-free one-forms E and B representing the electric and magnetic fields and we have energy and momentum densities

$$\begin{aligned} \mathcal{E}(E,B,g) &= |E|_g^2 + |B^2|_g^2 \\ \mathcal{J}(E,B,g) &= *_g(E \wedge B) \end{aligned} \tag{5.9}$$

where $*_g$ is the Hodge-star operator. We conformally transform the fields according to $\Phi((E,B),\phi) = (\phi^{-2}E, \phi^{-2}B)$ which preserve the conditions that these one-forms must be divergence-free. One readily verifies that

$$\begin{aligned} \mathcal{J}(\phi^{-2}E, \phi^{-2}B, \phi^{q-2}g) &= \phi^{-2} *_g (\phi^{-2}E \wedge \phi^{-2}B) \\ &= \phi^{-q}\mathcal{J}(E,B,g) \end{aligned} \tag{5.10}$$

since $q = 6$ when $n = 3$. Thus we can take $j = *_g(E \wedge B)$. For the Hamiltonian constraint we have

$$\mathcal{E}(\phi^{-2}E, \phi^{-2}B, \phi^{q-2}g) = \phi^{-8}|E|^2 + \phi^{-8}|B|^2 \tag{5.11}$$

and hence

$$\rho(\phi) = \left[|E|^2 + |B|^2\right]\phi^{-8}. \tag{5.12}$$

Noting that $q - 2 = 4$ when $n = 3$,

$$\phi^{q-2}\rho(\phi) = \left[|E|^2 + |B|^2\right]\phi^{-4} \tag{5.13}$$

which evidently satisfies the energy scaling condition.

For convenience, we treat the cosmological constant Λ as an additional form of matter, and we will call a triple $(\rho(\cdot), j, \Lambda)$ where $\rho(\cdot)$ satisfies the conditions of Definition 5.1 a *conformal matter distribution.* The CTS-H

---

being decreasing in ϕ. The hypothesis of Definition 5.1 that it is *strictly* decreasing somewhere (except in vacuum) does not appear explicitly in [14], but is easy enough to verify from the expressions computed in that paper that this additional condition is satisfied for all the specific matter fields treated in that work.

equations include a conformal matter distribution according to
(5.14)

$$
\begin{aligned}
-a\Delta\phi + R\phi - \left|\sigma + \frac{1}{2N}\,\mathbf{L}\,W\right|^2 \phi^{-q-1} + \kappa\tau^2\phi^{q-1} &= 2\left[8\pi\rho(\phi)\phi^{q-1} + \Lambda\phi^{q-1}\right] \\
\frac{1}{2}\,\mathbf{L}^*\left[\frac{1}{2N}\,\mathbf{L}\,W\right] + \kappa\phi^q d\tau &= 8\pi j.
\end{aligned}
$$

If (ϕ, W) is a solution of these equations then $(\overline{g}, \overline{K})$ defined by equations (2.7) solve the constraint equations for matter fields $\overline{\mathcal{F}} = \Phi(\mathcal{F}, \phi)$ giving an energy density $\overline{\rho} = \rho(\phi)$ and a momentum density $\overline{j} = \phi^{-q} j$. We will call $(\overline{\rho}, \overline{j}, \Lambda)$ a physical matter distribution. An easy computation using the fact that Φ is group action shows that the CTS-H equations with matter are conformally covariant as well, so long as when we conformally transform to $\hat{g} = \psi^{q-2} g$ we also transform to the field $\hat{\mathcal{F}} = \Phi(\mathcal{F}, \hat{g})$ to obtain $\hat{\rho}(\cdot) = \rho(\psi\,\cdot)$ and $\hat{j} = \psi^{-q} j$.

Because the drift formulations differ from the CTS-H equations only in their treatment of the mean curvature, a conformal matter distribution $(\rho(\cdot), j, \Lambda)$ appears in the drift formulations of the constraint equations in exactly the same way as for the CTS-H equations. Simply replace the zeros on the right-hand sides of equations (4.16) or (4.22) with the right-hand sides of equations (5.14), but observe that the associated compatibility conditions need to account for the momentum density.

The CED-V equations in their final form, extending system (4.16) to include matter, are

$$
\begin{aligned}
&- a\,\Delta\,\phi + R\phi + \left|\sigma + \frac{1}{2N}\,\mathbf{L}\,W\right|^2 \phi^{-q-1} \\
&\qquad + \kappa\left(\tau_* + \frac{1}{N\phi^q}\,\mathrm{div}_\phi(\phi^{2-q}V + Q)\right)^2 \phi^{q-1} \\
&\qquad = 2\left[8\pi\rho(\phi)\phi^{q-1} + \Lambda\phi^{q-1}\right]\,; \\
&\frac{1}{2}\,\mathbf{L}^*\left(\frac{1}{2N}\,\mathbf{L}\,W\right) - \kappa\,\mathrm{div}_\phi^*\left(\frac{1}{N}\,\mathrm{div}_\phi(\phi^{2-q}V + Q)\right) = 8\pi j
\end{aligned}
\tag{5.15}
$$

where the CKF compatibility condition (4.19) becomes

$$
\kappa\int\frac{1}{N}\,\mathrm{div}_\phi(\phi^{2-q}V + Q)\,\mathrm{div}_\phi\,P\,dV_g = -8\pi\int_M j_a P^a dV_g \tag{5.16}
$$

for all conformal Killing fields P.

Analogously, CED-C equations with matter, generalizing system (4.22), are

$$
\begin{aligned}
& - a\,\Delta\,\phi + R\phi + \left|\sigma + \frac{1}{2N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-}qE)\right|^{2}\phi^{-q-1} \\
& \quad + \kappa\left(\tau_{*} + \frac{1}{N\phi^{q}}\,\mathrm{div}_{\phi}(V + Q))\right)^{2}\phi^{q-1} \\
& \qquad = 2\left[8\pi\rho(\phi)\phi^{q-1} + \Lambda\phi^{q-1}\right]\,; \\
& \frac{1}{2}\,\mathbf{L}^{*}\left(\frac{1}{2N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-q}E)\right) - \kappa\,\mathrm{div}^{*}_{\phi}\left(\frac{1}{N}\,\mathrm{div}_{\phi}(V)\right) = 8\pi j,
\end{aligned}
\tag{5.17}
$$

with compatibility condition

$$
\int \frac{1}{4N}\,\mathbf{L}(\phi^{2-q}W + \phi^{-q}E)\,\mathbf{L}(\phi^{-q}F)\;dV_{g} = 8\pi\int j_{a}\phi^{-q}F^{a}dV_{g}
\tag{5.18}
$$

for all divergence-free vector fields F.

6. Near-CMC solutions on compact manifolds using drifts

In this section we prove the main result that, loosely stated, the drift method provides a good parameterization of solutions of the constraint equations on compact manifolds near CMC solutions, even when the metric admits conformal Killing fields.

To begin, we characterize the CMC solutions with respect to drift parameters.

Lemma 6.1. *Suppose* $(\overline{g}, \overline{K})$ *is a solution of the constraints equations* (1.1) *with matter fields* $(\overline{\rho}, \overline{j}, \Lambda)$*, and let* α *be an arbitrary volume gauge. The solution is CMC if and only if*

1) *the solution has zero volumetric drift measured by* α*, and*

2) *for all conformal Killing fields* P*,*

$$
\int \overline{j}_{a}P^{a}dV_{\overline{g}} = 0.
\tag{6.1}
$$

Proof. Write $\overline{K} = \overline{A} + \frac{\tau}{n}\overline{g}$, where $\overline{A}$ is trace-free with respect to $\overline{g}$, and then, applying Lemmas 2.2 and 4.1 with $N = dV_{g}/\alpha$, decompose further as

$$
\overline{A} = \overline{\sigma} + \frac{1}{2N}(\mathbf{L}\,\overline{W}),\quad \tau = \tau_{*} + \frac{1}{N}\,\mathrm{div}\,\overline{V},
\tag{6.2}
$$

where $\overline{\sigma}$ is transverse-traceless, τ_* is constant, and $\overline{W}$ and $\overline{V}$ are vector fields. With respect to this decomposition, the momentum constraint reads

$$\frac{1}{2}\,\mathbf{L}^*\,\frac{1}{2N}\,\mathbf{L}\,\overline{W}+\kappa\,\mathbf{d}\,\frac{1}{N}\operatorname{div}\overline{V}=\overline{j}. \tag{6.3}$$

First suppose that the solution is CMC. The expression for τ in (6.2) implies that $\int N(\tau-\tau_*)\,dV_{\overline{g}}=\int \operatorname{div}\overline{V}\,dV_{\overline{g}}=0$, and since $N>0$ everywhere, we see first that $\tau=\tau_*$ and then that $\operatorname{div}\overline{V}=0$. Recall now from Definition 4.5 that the volumetric drift measured by α is $\overline{V}+(\operatorname{Ker}\mathbf{L}+\operatorname{Ker}\operatorname{div})$. Since $\overline{V}\in\operatorname{Ker}\operatorname{div}$, the solution has zero volumetric drift. Moreover, multiplying equation (6.3) by a conformal Killing field and integrating by parts on the left-hand side yields (6.1).

Conversely, suppose the solution has zero volumetric drift measured by α and that (6.1) holds. Since the solution has zero volumetric drift we can write $\overline{V}=E+Q$ where E is divergence-free and Q is a conformal Killing field. Observe that $\operatorname{div}\overline{V}=\operatorname{div}Q$. Now multiply the momentum constraint (6.3) by Q and integrate by parts to get

$$0=\int \overline{j}_a Q^a dV_g=-\kappa\int\frac{1}{N}(\operatorname{div}\overline{V})(\operatorname{div}Q)\,dV_{\overline{g}}=-\kappa\int\frac{1}{N}(\operatorname{div}Q)^2\,dV_{\overline{g}}, \tag{6.4}$$

i.e., $\operatorname{div}Q=0$. Finally,

$$\tau=\tau_*+\frac{1}{N}\operatorname{div}\overline{V}=\tau_*+\frac{1}{N}\operatorname{div}Q=\tau_*, \tag{6.5}$$

so the solution is CMC. □

Lemma 6.1 suggests that the volumetric form of the drift method is nicely adapted to generate CMC solutions. Indeed, if a volumetric drift conformal data $(g,\sigma,\tau_*,V;\ N)$ generates a solution metric $\overline{g}=\phi^{q-2}g$, then the corresponding volumetric drift is $[\phi^{2-q}V]^{\text{drift}}_{\overline{g}}$. This means that $V=0$ suffices; furthermore, at least when the metric admits no conformal Killing fields, Lemma 4.7 implies that $V=0$ is the only choice in $\mathcal{D}_g$ which results in a zero volumetric drift. The only potential difficulty is that (6.1) involves the unknown solution metric and the solution momentum density $\overline{j}$. Fortunately, the scaling law (5.4) for momentum density ensures that (6.1) is conformally invariant, so this is not a real problem.

Corollary 6.2. *Consider a CED-V data set $(g,\sigma,\tau_*,V;\ N)$ with $V\equiv 0$ and a conformal matter distribution $(\rho(\cdot),j,\Lambda)$ that generates a solution*

(ϕ, W, Q) of the CED-V equations. The associated solution of the constraint equations is CMC if and only if

$$\int j_a P^a \, dV_g = 0 \tag{6.6}$$

for all conformal Killing fields P. Moreover, when the solution is CMC, then Q is a true Killing field for the solution metric, $(\phi, W, \tilde{Q} \equiv 0)$ is also a solution of the CED-V equations that generates the same solution of the constraints, and (ϕ, W) solves the standard CTS-H equations (5.14) *with constant mean curvature $\tau = \tau_*$.*

Proof. Let (ϕ, W, Q) be the solution of (5.15) corresponding to a solution $(\overline{g}, \overline{K})$ of the constraints. Since $V = 0$ the volumetric drift of the solution is $[\phi^{2-q}V]^{\text{drift}}_{\overline{g}} = 0$, and Lemma 6.1 implies that the solution is CMC if and only if

$$\int \overline{j}_a P^a dV_{\overline{g}} = 0 \tag{6.7}$$

for all conformal Killing fields P; here $\overline{j}$ is the physical momentum density given by

$$\overline{j} = \phi^{-q} j. \tag{6.8}$$

The physical volume form is $dV_{\overline{g}} = \phi^q dV_g$, and hence $\overline{j} dV_{\overline{g}} = j dV_g$. So (6.7) holds for all conformal Killing fields P if and only if the same is true for (6.6).

Supposing now that the solution is CMC, equation (6.6) along with the choice $V \equiv 0$ implies that the CKF compatibility condition (5.16) reduces to

$$\int \frac{1}{N} (\operatorname{div}_\phi Q)(\operatorname{div}_\phi P) \, dV_g = 0 \tag{6.9}$$

for all conformal Killing fields P. In particular, $\int (\operatorname{div}_\phi Q)^2 N^{-1} \, dV_g = 0$. But $\operatorname{div}_\phi = \operatorname{div}_{\overline{g}}$, so Q is a divergence-free conformal Killing field for $\overline{g}$, i.e. it is a true Killing field for $\overline{g}$. Since Q appears in the CED-V equations only via $\operatorname{div}_\phi Q$, we may as well take it to be zero and arrive at the same solution of the constraints. Finally, since $V \equiv 0$ as well, a quick inspection verifies that the CED-V equations (5.15) reduce to the CTS-H equations (5.14) with $\tau \equiv \tau_*$. $\square$

Corollary 6.2 shows that the CMC theory for the CTS-H equations transfers directly to the CED-V equations. In vacuum we have the tidy and complete classification of CMC solutions completed in [13]. Conversely, for a positive cosmological constant Λ with $2\Lambda > \kappa\tau_*^2$ we have all the complexity demonstrated in, e.g, [6].

6.1. Near-CMC solutions parametrized by small volumetric drift

Given the natural connection between CMC solutions and volumetric drift, we first examine the construction of near-CMC solutions by perturbing to small volumetric drift. We use the implicit function theorem in a fashion parallel to that of [9], but with some technical features to handle conformal Killing fields. Indeed, in the presence of conformal Killing fields, the vector Laplacian $\frac{1}{2}\mathbf{L}^* \frac{1}{2N}\mathbf{L} : W^{k,p}(M,TM) \to W^{k-2,p}(M,T^*M)$ is Fredholm and has kernel equal to $\mathcal{Q}$ and cokernel

$$\mathcal{Q}^\perp := \Big\{ \eta \in W^{k-2,p}(M,T^*M) : \\ \int_M \eta_a Q^a \, dV_g = 0 \text{ for all conformal Killing fields } Q \Big\}.$$

By modifying the domain and range slightly we can make this into an isomorphism

$$\mathbf{L}^* \frac{1}{2N}\mathbf{L} : W^{k,p}(M,TM)/\mathcal{Q} \to W^{k-2,p}(M,T^*M) \cap \mathcal{Q}^\perp. \tag{6.10}$$

Indeed, writing $[W]_\mathcal{Q}$ for the projection of a $W^{k,p}$ vector field W to the quotient space $W^{k,p}(M,TM)/\mathcal{Q}$, it is clear that $\mathbf{L}[W]_\mathcal{Q} = \mathbf{L}\,W$ is well-defined. We also set

$$\mathcal{P} : W^{k-2,p}(M,T^*M) \to W^{k-2,p}(M,T^*M) \cap \mathcal{Q}^\perp, \tag{6.11}$$

to be the projection with kernel consisting of the conformal Killing covector fields; this is defined since $W^{k-2,p}(M,T^*M) \cap \mathcal{Q}^\perp$ has finite codimension in $W^{k-2,p}(M,T^*M)$.

For the remainder of this section, fix a metric g, a lapse N, and a conformal matter distribution $(\rho(\cdot), j, \Lambda)$. Before setting up an implicit function theorem argument, we define the following three functionals:

- the Hamiltonian constraint

$$
\begin{aligned}
&C_H(\sigma, \tau_*, V;\ \phi, [W]_{\mathcal{Q}}, Q) \\
&\qquad = -a\,\Delta\,\phi + R\phi - \left|\sigma + \frac{1}{2N}\,\mathbf{L}[W]_{\mathcal{Q}}\right|^2 \phi^{-q-1} \\
&\qquad\quad + \kappa\left(\tau_* + \frac{1}{N\phi^q}\operatorname{div}_\phi(\phi^{q-2}V + Q)\right)^2 \phi^{q-1} \\
&\qquad\quad - 2(8\pi\rho(\phi) + \Lambda)\phi^{q-1};
\end{aligned}
\tag{6.12}
$$

- the momentum constraint

$$
\begin{aligned}
&C_M(\sigma, \tau_*, V;\ \phi, [W]_{\mathcal{Q}}, Q) \\
&\quad = \frac{1}{2}\,\mathbf{L}^*\,\frac{1}{2N}\,\mathbf{L}[W]_{\mathcal{Q}} - \mathcal{P}\left[\kappa\operatorname{div}^*_\phi\left(\frac{1}{N}\operatorname{div}_\phi(\phi^{q-2}V + Q)\right) + 8\pi j\right]
\end{aligned}
\tag{6.13}
$$

where $\mathcal{P}$ is the projection (6.11);

- the CKF compatability constraint

$$
\begin{aligned}
&C_C(\sigma, \tau_*, V;\ \phi, [W]_{\mathcal{Q}}, Q) \\
&\qquad = P \mapsto \int \kappa\frac{1}{N}\operatorname{div}_\phi(\phi^{q-2}V + Q)\operatorname{div}_\phi(P) + 8\pi j_a P^a\, dV_g,
\end{aligned}
\tag{6.14}
$$

where P is an arbitrary conformal Killing field.

There is a semicolon appearing in the arguments of these maps to separate those variables that are prescribed, the following spaces:

$$
(\sigma, \tau_*, V) \in [\ker \mathbf{L}^* \subseteq W^{k-1,p}(M, S_2M)] \times \mathbb{R} \times [\mathcal{D}_g \subseteq W^{k,p}(M, TM)],
\tag{6.15}
$$

versus those that must be solved for,

$$
(\phi, [W]_{\mathcal{Q}}, Q) \in W^{k,p}_+(M) \times (W^{k,p}(M, TM)/\mathcal{Q}) \times \mathcal{Q}.
\tag{6.16}
$$

The maps C_H, C_M, and C_C take their values in $W^{k-2,p}(M)$, $W^{k-2,p}(M, T^*M) \cap \mathcal{Q}^\perp$ and $\mathcal{Q}^*$, respectively.

Lemma 6.3. *A triple* (ϕ, W, Q) *solves the CED-V equations* (5.15) *for CED-V data* $(g, \sigma, \tau_*, V;\ N)$ *and conformal matter distribution* $(\rho(\cdot), j, \Lambda)$

if and only if

$$
\begin{aligned}
C_H(\phi, [W]_{\mathcal{Q}}, Q;\ \sigma, \tau_*, V) &= 0\\
C_M(\phi, [W]_{\mathcal{Q}}, Q;\ \sigma, \tau_*, V) &= 0\\
C_C(\phi, [W]_{\mathcal{Q}}, Q;\ \sigma, \tau_*, V) &= 0.
\end{aligned} \tag{6.17}
$$

Proof. If the definition of C_M were not to involve the projection $\mathcal{P}$ there would be nothing to do other than to observe that the distinction between W and $[W]_{\mathcal{Q}}$ is immaterial since W only appears as an argument to $\mathbf{L}$. Hence it suffices to show that if $C_C = 0$ then $C_M = 0$ is equivalent to

$$
\frac{1}{2}\mathbf{L}^* \frac{1}{2N}\mathbf{L}[W] - \operatorname{div}_\phi^*\left(\frac{1}{N}\operatorname{div}_\phi(V+Q)\right) - 8\pi j = 0. \tag{6.18}
$$

Indeed, if $C_C = 0$, then integration by parts shows that

$$
-\operatorname{div}_\phi^*\left(\frac{1}{N}\operatorname{div}_\phi V\right) - 8\pi j \in \mathcal{Q}^\perp. \tag{6.19}
$$

Hence

$$
\begin{aligned}
&\mathcal{P}\left[\operatorname{div}_\phi^*\left(\frac{1}{N}\operatorname{div}_\phi(V+Q)\right) + 8\pi j\right]\\
&\qquad = \operatorname{div}_\phi^*\left(\frac{1}{N}\operatorname{div}_\phi(V+Q)\right) + 8\pi j,
\end{aligned} \tag{6.20}
$$

which is (6.18). ☐

Theorem 6.4. *Consider volumetric drift parameters $(g, \hat\sigma, \hat\tau_*, \hat V;\ N)$ and and a conformal matter distribution $(\rho(\cdot), j, \Lambda)$ where g, N, and $\hat V$ have $W^{k,p}$ regularity, $\hat\sigma$ is of class $W^{k-1,p}$, j is of class $W^{k-2,p}$, and where ρ satisfies the energy scaling condition of Definition 5.1.*

Suppose that $\hat V \equiv 0$ leads to a CMC solution of equations (5.15) *and additionally that*

i) *The CMC solution metric does not admit any true Killing fields.*

ii) $\kappa\tau_*^2 \ge 2\Lambda$.

iii) *Either $\kappa\tau_*^2 > 2\Lambda$, or $\sigma \not\equiv 0$, or the solution is not vacuum.*

Then there exists $\epsilon > 0$ *such that all conformal data* $(g, \sigma, \tau_*, V;\ N)$ *satisfying*

$$||\sigma - \hat{\sigma}||_{W^{k-1,p}} + |\tau_* - \hat{\tau}_*| + ||V||_{W^{k,p}} < \epsilon \tag{6.21}$$

generate a solution of system (5.15)*, and the map from* (σ, τ_*, V) *to the associated solution of the constraint equations is smooth and injective.*

Remark 6.5. *As we show below, in the absence of matter fields, hypothesis i) is satisfied generically in the space of CMC solutions.*

Proof. By conformal covariance of the CED-V equations, assume that the background metric g is the CMC solution metric, which means that the solution of the CED-V equations is $(\hat{\phi}, \hat{W}, \hat{Q})$ with $\hat{\phi} \equiv 1$. Moreover Corollary 6.2 implies $\hat{Q}$ must be a true Killing field, hence $\hat{Q} \equiv 0$. This simplifies various expressions later in the proof. Although $\hat{W}$ does not have a simple expression, the momentum constraint implies

$$-\mathbf{L}^* \frac{1}{2N} \mathbf{L} \hat{W} = j, \tag{6.22}$$

which we also use in the sequel.

Define $\mathcal{F} = (C_H, C_M, C_C)$. By Lemma 4.7 there is a neighborhood Φ of 1 in $W^{k,p}_+(M)$ such that $V \mapsto [\phi^{2-q} V]^{\text{drift}}_{\phi^{q-2}\hat{g}}$, from $\mathcal{D}_{\hat{g}}$ to $\text{Drift}_{\phi^{q-2}\hat{g}}$, is an isomorphism for any $\phi \in \Phi$. We restrict the domain of $\mathcal{F}$ to these conformal factors; it remains an open set in the Banach space (6.15), (6.16).

The map $\mathcal{F}$ is continuously differentiable and its derivative with respect to $(\phi, [W]_{\mathcal{Q}}, Q)$ at

$$(\hat{\sigma}, \hat{\tau}_*, \hat{V}; \hat{\phi}, [\hat{W}]_{\mathcal{Q}}, \hat{Q}) = (\hat{\sigma}, \hat{\tau}_*, 0; 1, [\hat{W}]_{\mathcal{Q}}, 0) \tag{6.23}$$

can be written as

$$\begin{aligned} &DF(\delta\phi, \delta[W]_{\mathcal{Q}}, \delta Q) \\ &= \begin{pmatrix} -a\,\Delta + A & -2\left\langle \sigma + \frac{1}{2N}\mathbf{L}\hat{W}, \frac{1}{2N}\mathbf{L}(\cdot)\right\rangle & 2\kappa\tau_* \frac{1}{N}\operatorname{div}(\cdot) \\ 0 & \frac{1}{2}\mathbf{L}^*\left(\frac{1}{2N}\mathbf{L}(\cdot)\right) & \kappa\mathcal{P}(\operatorname{div}^*\left(\frac{1}{N}\operatorname{div}(\cdot)\right)) \\ 0 & 0 & P \mapsto \kappa\int \frac{1}{N}\operatorname{div}(\cdot)\operatorname{div}(P)\,dV \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta[W]_{\mathcal{Q}} \\ \delta Q \end{pmatrix} \end{aligned} \tag{6.24}$$

where

$$\begin{aligned} A = {} & (q+2)\left|\sigma + \frac{1}{2N}\mathbf{L}\hat{W}\right|^2 + (q-2)[\kappa\tau_*^2 - 2\Lambda] \\ & - 16\pi[\rho'(1) + (q-2)\rho(1)]. \end{aligned} \tag{6.25}$$

Note that we have used the Hamiltonian constraint

$$R - \left|\sigma + \frac{1}{2N}\,\mathbf{L}\,\hat{W}\right|^2 + \kappa\tau_*^2 = 16\pi\rho(1) + 2\Lambda \tag{6.26}$$

to replace the scalar curvature that would otherwise have appeared in the expression for A. From the block upper-triangular form of the matrix we conclude that DF is invertible if each diagonal block is, and we treat each in turn.

The operator

$$-a\,\Delta + A : W^{k,p}(M) \to W^{k-2,p}(M), \tag{6.27}$$

is invertible if $A \geq 0$, $A \not\equiv 0$. Looking at the expression (6.25) we have three terms to consider. First,

$$(q-2)[\kappa\tau_*^2 - 2\Lambda] \geq 0 \tag{6.28}$$

since $q > 2$ (for any $n \geq 3$) and since $\kappa\tau_*^2 \geq 2\Lambda$ by hypothesis. Next,

$$[\rho'(1) + (q-2)\rho(1)] \tag{6.29}$$

is the linearization of

$$\phi \mapsto \phi^{q-2}\rho(\phi) \tag{6.30}$$

evaluated at $\phi \equiv 1$. By Definition 5.1, this is non-positive and hence $-16\pi[\rho'(1) + (q-2)\rho(1)] \geq 0$. The final summand of A is

$$\left|\sigma + \frac{1}{2N}\,\mathbf{L}\,\hat{W}\right|^2, \tag{6.31}$$

which is obviously nonnegative. Moreover, multiplying expression (6.31) by N and integrating yields

$$\int N\left|\sigma + \frac{1}{2N}\,\mathbf{L}\,\hat{W}\right|^2 = \int N|\sigma|^2 + \frac{1}{4N}|\,\mathbf{L}\,\hat{W}|^2, \tag{6.32}$$

using that transverse-traceless tensors are L^2 orthogonal to the image of $\mathbf{L}$. Altogether, $A \equiv 0$ means that

$$\kappa\tau_*^2 = 2\Lambda, \quad \sigma \equiv 0, \quad \hat{W} \equiv 0, \text{ and } \rho'(1) + (q-2)\rho(1) \equiv 0 \tag{6.33}$$

Equation (6.22) shows that $\hat{W} \equiv 0$ implies $j \equiv 0$; by Definition 5.1, if $\rho'(1) + (q-2)\rho(1) \equiv 0$ then $\rho(\cdot) \equiv 0$. From these we get that $\kappa\tau_*^2 = 2\Lambda$, $\sigma \equiv 0$ and the solution is vacuum. Hypothesis 6.4 thus ensures that $A \not\equiv 0$.

The middle block of the matrix in (6.24) is invertible by the discussion around (6.10).

Finally, for the last block, the symmetric bilinear form

$$B : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}, \quad B(Q, P) = \int \frac{1}{N} \operatorname{div} Q \operatorname{div} P \, dV_g \tag{6.34}$$

is nonnegative, and positive definite so long as $\mathcal{Q}$ contains no true Killing fields, which are precisely the divergence free elements in $\mathcal{Q}$. This too holds under our assumptions. Therefore, the map

$$Q \mapsto \int \frac{1}{N} \operatorname{div} Q \operatorname{div}(\cdot) \, dV_g \tag{6.35}$$

is an isomorphism from $\mathcal{Q}$ to $\mathcal{Q}^*$.

Taking Lemma 6.3 into account, the implicit function theorem now provides the existence of the solution map for (σ, τ_*, V) sufficiently near $(\hat{\sigma}, \hat{\tau}_*, \hat{V} = 0)$ in $W^{k,p} \times R \times \mathcal{D}_g$. It remains to establish the global injectivity.

Suppose (σ, τ_*, V) determines a solution (ϕ, W, Q) of the CED-V equations, and thereby a solution $(\overline{g}, K)$ of the constraint equations. We demonstrate injectivity by showing that we can recover (σ, τ_*, V) from $(\overline{g}, K)$ under the hypothesis that $\phi \in \Phi$.

Setting $\overline{N} = \phi^q N$, apply Lemmas 2.2 and 4.1 to write

$$K = \overline{\sigma} + \frac{1}{2\overline{N}} \mathbf{L}_{\overline{g}} \overline{W} + \frac{\tau}{n}\overline{g}, \qquad \tau = \overline{\tau}_* + \frac{1}{\overline{N}} \operatorname{div}_{\overline{g}}(\overline{V})$$

where $\overline{\sigma}$ is transverse-traceless with respect to $\overline{g}$, $\overline{\tau}_*$ is constant, and $\overline{V}$ is a vector field. On the other hand equations (4.20) and the conformal transformation laws for the divergence and conformal Killing operators imply

$$K = \phi^{-2}\sigma + \frac{1}{2\overline{N}} \mathbf{L}_{\overline{g}} W + \frac{\tau}{n}\overline{g}, \qquad \tau = \tau_* + \frac{1}{\overline{N}} \operatorname{div}_{\overline{g}}(\phi^{2-q} V + Q).$$

Since $\phi^{-2}\sigma$ is transverse-traceless with respect to $\overline{g}$, the uniqueness clauses of Lemmas 2.2 and 4.1 imply $\tau_* = \overline{\tau}_*$, $\sigma = \phi^2\overline{\sigma}$ and that there is a $\overline{g}$ divergence-free vector field E such that

$$\phi^{2-q} V + Q + E = \overline{V}.$$

But this shows that we have agreement of drifts

$$[\phi^{2-q}V]^{\mathrm{drift}}_{\tilde{g}} = [\overline{V}]^{\mathrm{drift}}_{\tilde{g}}.$$

Since $\phi \in \Phi$, the map $\mathcal{D}_g \to \mathrm{Drift}_{\overline{g}}$ given by $V \mapsto [\phi^{2-q}V]^{\mathrm{drift}}_{\overline{g}}$ is an isomorphism and $V \in \mathcal{D}_g$ is uniquely determined by $\overline{V}$. □

Proposition 3.1 shows that given a CMC solution of vacuum constraint equations with a metric conformal to the round sphere, there exist inadmissible perturbations of the mean curvature. By contrast, Theorem 6.4 shows that, so long as the CMC solution has no Killing fields, arbitrary small perturbations of drift and volumetric momentum produce nearby solutions. We now verify that this condition is generic among the CMC solutions within a conformal class.

Proposition 6.6. *In the space of all CMC solutions to the vacuum constraint equations, the subset of pairs (g, K) for which there are no Killing fields is open and dense. In fact, this is true even within a conformal class.*

Proof. Let (g, K) be any CMC solution and denote by $\mathcal{K}_g$ and $\mathcal{Q}_g$ the spaces of Killing and conformal Killing vector fields for g, respectively; thus

$$\mathcal{Q}_g = \{X : \mathbf{L}\, X = 0\}\,, \quad \text{and} \qquad \mathcal{K}_g = \{X \in \mathcal{Q}_g : \mathrm{div}_g\, X = 0\}\,;$$

of course $\mathcal{Q}_{\tilde{g}} = \mathcal{Q}_g$ for any metric $g' = \phi^{q-2}g$.

We first show that if $\mathcal{K}_g = \{0\}$, then the same is true for any metric g' near to g in the $W^{k,p}$ topology. The second part is to prove that if K_g is nontrivial, then there exist metrics g' arbitrarily near g in the $W^{k,p}$ topology such that $\mathcal{K}_{g'} = \{0\}$.

To begin, observe that the $\mathcal{K}_g$ is also characterized as the nullspace of the map

$$T_g : \mathcal{Q}_g \longrightarrow \mathcal{Q}_g, \quad T_g\xi = \mathbb{P} \circ \mathrm{div}^*_g \circ \mathrm{div}_g,$$

where $\mathbb{P}$ is the L^2 orthogonal projection from the space of symmetric two-tensors onto the finite dimensional space $\mathcal{Q}_g$; this follows easily from the identity $0 = \langle T_g\xi, \xi\rangle = ||\,\mathrm{div}_g\,\xi||^2$ if $T_g\xi = 0$ and $\xi \in \mathcal{Q}_g$. We henceforth identify $\mathcal{Q}_g$ with $\mathbb{R}^N$ for some N. Observe also that T_g depends in a real analytic way on g.

For the first assertion, simply note that if g admits no Killing fields, then $\ker T_g = \{0\}$, and this is an open condition in the space of all $W^{k,p}$ metrics, hence also in the space of metrics g' which appear in a pair (g', K') of CMC solutions of the constraint equations.

As for the second assertion, suppose $\mathcal{K}_{g_0}$ is nontrivial for some metric g_0 which appears in a CMC solution pair $(g_0, K_0 = \frac{\tau}{n} g_0 + \sigma_0)$. Without loss of generality we can assume that $\tau \neq 0$ and $\sigma_0 \not\equiv 0$, for otherwise the CMC theory of the conformal method ensures we can perturb to a nearby solution of the constraint equations satisfying this condition. We consider families of solutions which arise by varying σ in $\mathcal{U} = W^{k,p}(M, S_{\mathrm{tt}}) \setminus \{0\}$, but keeping the conformal class fixed. From the CMC theory of the conformal method, since $\tau \neq 0$, for $\sigma \in \mathcal{U}$ there is a well defined conformal factor $\phi(\sigma)$ obtained by solving the Lichnerowicz equation

$$-a\Delta_0 \phi + R_0 \phi - |\sigma|^2_{g_0} \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 0, \tag{6.36}$$

and

$$(g_\sigma, K_\sigma) = \left(\phi^{q-2} g, \phi^{-2}\sigma + \frac{\tau}{n}\phi^{q-2} g\right) \tag{6.37}$$

is a solution of the constraint equations. For simplicity, we write T_σ instead of T_{g_σ}

Consider, for $j = 0, \dots, N$, the subsets $\mathcal{F}_j = \{\sigma \in \mathcal{U} : \operatorname{rank} T_\sigma \leq j\}$. We claim that since ϕ, and hence g, depends real analytically on σ, each $\mathcal{F}_j$ is an analytic subvariety of finite codimension in $\mathcal{U}$. Indeed, σ lies in $\mathcal{F}_j$ if and only if the determinant of every $(j+1)$-by-$(j+1)$ minor of T_σ vanishes, and this is a finite number of polynomial conditions. By analyticity again, if the set $\mathcal{F}_j^o := \mathcal{F}_j \setminus \mathcal{F}_{j-1}$ of TT tensors σ where the rank of T_σ is exactly j has an interior point, then it is an open dense subset in $\mathcal{U}$. Furthermore, $\mathcal{U}$ is the union of the sets $\mathcal{F}_j$, hence some $\mathcal{F}_k^o$ must have interior, and hence is open and dense. The main conclusion follows if we can show that $k = N$, since T_σ has full rank implies that its nullspace is trivial.

Suppose that this is not the case, so $\mathcal{F}_k^o$ is open and dense in $\mathcal{U}$ for some $k < N$. We first show that there exists a submanifold in $\mathcal{U}$ with finite codimension such that the nullspace of T_σ is equal to the *same* k-dimensional subspace for every σ in the submanifold. Indeed, consider the map $G : \mathcal{F}_k^o \to G(k, N)$ into the Grassmanian of k-planes in $\mathbb{R}^N$, which sends σ to the nullspace of T_σ. Let $\mathcal{R}$ be the image of $\mathcal{U}$ under G. By construction, $\mathcal{R}$ is a subanalytic set in $G(k, N)$, and hence itself admits a stratification, $\mathcal{R} = \sqcup \mathcal{R}_j$ where each $\mathcal{R}_j$ is a smooth j-dimensional submanifold. Suppose that J is the maximal dimension of these strata, and let $\mathcal{U}' = G^{-1}(\mathcal{R}_J)$. This is an open dense set in $\mathcal{U}$.

The point of these maneuvers is to obtain a map $G' = G|_{\mathcal{U}'}$ with maximal rank and image in a smooth manifold. We may now apply some familiar tools of differential topology. By the Sard-Smale theorem, there exists a full measure set of regular values of G', and hence we may choose a k-plane $\Pi \subset \mathbb{R}^N$ such that $\hat{Z} := (G')^{-1}(\Pi)$ is a smooth analytic submanifold of finite codimension in $\mathcal{U}'$. In particular, the nullspace of div_{g_σ} is the same k-dimensional subspace $\Pi \subset \mathcal{Q}_g$ for all $\sigma \in \hat{Z}$.

Fix $\hat{\sigma}_1 \in \hat{Z}$ and and write ϕ_1 and g_1 for the corresponding conformal factor and metric. Set $Z = \phi^{-2}\hat{Z}$, so $Z \subseteq W^{k,p}(M, S_{\mathrm{tt}}(g_1))$ is a submanifold with finite codimension, and $\sigma_1 = \phi_1^{-2}\sigma \in Z$. The Lichnerowicz equation with g_1 as background metric is then

$$-a\Delta_1\phi + R_1\phi - |\sigma|^2_{g_1}\phi^{-q-1} + \kappa\tau^2\phi^{q-1} = 0. \tag{6.38}$$

By solving (6.38) for ϕ, each $\sigma \in Z$ determines a metric $g_\sigma = \phi^{q-2}g_1$ and second fundamental form K_σ solving the constraint equations. Moreover, let H denote the connected component of the identity in the isometry group of (M, g_1). This is a compact, connected Lie group of positive dimension, and the quotient M/H is an orbifold of strictly smaller dimension than M. Each g_σ with $\sigma \in Z$ is invariant under H, or equivalently, the conformal factor $\phi(\sigma)$ (where $g_\sigma = \phi^{q-2}g_1$) is invariant under H. This follows since $T_eH = \mathcal{K}_{g_1}$ is actually constant as σ varies in Z. We show now that this leads to a contradiction.

Suppose that $\sigma(\epsilon)$ is a one-parameter family of TT tensors lying in Z with $\sigma(0) = \sigma_1$ and set $\eta = \dot{\sigma}(0)$. Differentiating the Lichnerowicz equation with respect to ϵ gives

$$L\dot{\phi} = 2\langle\sigma_1, \eta\rangle_{g_1} \tag{6.39}$$

where

$$L := -a\Delta_1 + R_1 + (q+1)|\sigma_1|^2_{g_1} + (q-1)\kappa\tau^2$$

is the Frechet derivative of the Lichnerowicz equation at $\phi = 1$. Next differentiate (6.39) with respect to $X \in \mathcal{K}_1$ to obtain

$$LX\phi' = -[X, L]\phi + 2X\langle\sigma_1, \eta\rangle_{g_1}$$

Setting $\sigma = \sigma_1$ in equation (6.38), the solution is $\phi = 1$ and hence $R_1 + \kappa\tau^2 = |\sigma_1|^2_{g_1}$. The left side of this last relation is annihilated by any $X \in \mathcal{K}_1$, hence so is the right, so it follows that all the coefficient functions of L are annihilated by X, and in particular $[X, L] = 0$. Hence

$$LX\phi' = 2X\langle\sigma_1, \eta\rangle_{g_1}.$$

On the other hand, $X(\phi(\sigma)) = 0$ for all $\sigma \in Z$ and therefore $X\phi' = 0$. Since $R_1 = |\sigma_1|^2_{g_1} - \kappa\tau^2$ we can rewrite

$$L = -a\Delta_1 + (q+2)|\sigma_1|^2_{g_1} + (q-2)\kappa\tau^2$$

to see that L is invertible, and we conclude that the pointwise inner product $\langle\sigma_1, \eta\rangle_{g_1}$ is constant along the H-orbits for every η in the finite codimensional subspace $T_{\sigma_1}Z \subset W^{k,p}(M, S_{\mathrm{tt}}(g_1))$.

We now show that this last conclusion is absurd. To this end, we use a construction presented in a neat and general form in [8], but in fact in fact in this finite regularity setting also following from [25]. Namely, we claim that there exist $\eta \in W^{k,p}(M, S_{\mathrm{tt}}(g_1))$ with arbitrarily small support. The basic principle is that the operator div_g is left-elliptic, and under a certain hypothesis can be shown to be surjective acting between symmetric trace-free two-tensors and vector fields (or 1-forms) which vanish to some high order at the boundary of some domain $\mathcal{O}$. (This is proved in [8] using a weight function which vanishes exponentially in the distance to $\partial\mathcal{O}$, but follows from [25] if one is content with weight functions which vanish at any polynomial rate.) We show how to apply this principle: suppose that $\chi \in \mathcal{C}^\infty_0$ equals 1 on an open set $\mathcal{O}'$ which has closure contained in $\mathcal{O}$ and which vanishes outside $\mathcal{O}$. Denote by Ω the annular domain $\mathcal{O} \setminus \overline{\mathcal{O}'}$. If $\xi \in W^{k,p}(M, S_{\mathrm{tt}}(g_1))$ is arbitrary, then $\mathrm{div}_{g_1}(\chi\xi) = \iota(\nabla\chi)\xi \in W^{k-1,p}$ has compact support in $\overline{\Omega}$. By [8, 25], there exists a symmetric trace-free $W^{k,p}$ two-tensor γ supported in $\overline{\Omega}$ with $\mathrm{div}_{g_1}\gamma = \mathrm{div}_{g_1}(\chi\xi)$ if and only if $\iota(\nabla\chi)\xi$ is L^2 orthogonal to every $Y \in \mathcal{Q}_g$, i.e., $\int_M \xi(\nabla\chi, Y)\, dV_{g_1} = 0$. To show that this is satisfied here, observe that since Y is conformal Killing and ξ is trace-free,

$$\begin{aligned}\mathrm{div}_{g_1}(\chi\, \iota(Y)\xi) &= -\nabla^a_{g_1}(\chi\xi_{ab}Y^b)\\ &= -\xi(\nabla\chi, Y) + \chi\xi^{ab}\frac{1}{n}(\delta_{g_1}Y)(g_1)_{ab} = -\xi(\nabla\chi, Y)\end{aligned}$$

Integrating over M yields the desired orthogonality. Hence $\chi\xi - \gamma \in W^{k,p}(M, S_{\mathrm{tt}}(g_1))$ agrees with ξ in $\mathcal{U}$ and has support in $\mathcal{O}$.

Now choose disjoint open sets $\mathcal{O}'_j$, $j = 1, \ldots, \ell$ such that

- ℓ is larger than the codimension of Z,
- $\sigma_1 \neq 0$ throughout each $\mathcal{O}'_j$ (this is possible since $\sigma_1 \not\equiv 0$),

- no integral curve of X is contained in $\mathcal{O}'_j$.

We can then apply the above construction to $\eta = \sigma_1$ on each $\mathcal{O}'_j$ to obtain localizations σ_{1j}. Since ℓ is larger than the codimension of Z there is a nontrivial linear combination

$$\eta = \sum_j b_j \sigma_{1j} = 0 \mod T_{\sigma_1} Z.$$

That is, $\eta \in T_{\sigma_1} Z$. Picking some j such that $b_j \neq 0$, there is an integral curve of X which contains a point in $\mathcal{O}'_j$ where $\eta = \sigma_1 \neq 0$. But this same integral curve is not contained in $\mathcal{O}'_j$ and hence also contains a point on $\partial O'_j$ where $\eta = 0$. It is then obvious that $\langle \sigma, \eta \rangle_{g_1}$ is not constant along the integral curve.

This is the contradiction we desired. The proof is complete. $\square$

6.2. Rescaling CED-V conformal parameters

In [9], the authors observe that the far-from CMC solutions of the constraints constructed in [12] and [20] can be considered as perturbations of solutions with $\tau \equiv 0$, together with rescaling. In this section we examine how these arguments translate to the CED-V setting.

Starting from a pair (g, K), consider a length $L > 0$ and a rescaled pair $(\hat{g}, \hat{K}) = (L^2 g, L\, K)$. If (g, K) solves the constraints with physical matter distribution (ρ, j, Λ), then $(\hat{g}, \hat{K})$ solves the constraints with physical matter distribution

$$(\hat{\rho}, \hat{j}, \hat{\Lambda}) = (L^{-2}\rho, L^{-1} j_a, L^{-2}\Lambda). \tag{6.40}$$

A straightforward computation establishes how this homothety scaling extends to CED-V parameters.

Lemma 6.7. *Suppose* (ϕ, W, Q) *is a solution of the CED-V equations* (5.15) *for conformal data* $(g, \sigma, \tau_*, V;\ N)$ *and conformal matter distribution* $(\rho(\cdot), j, \Lambda)$. *For any* $L > 0$,

$$(L^{\frac{n}{2}-1}\phi, L^{n-1} W, L^{n-1} Q) \tag{6.41}$$

is a solution of system (5.15) *for conformal data*

$$(g, L^{n-1}\sigma, L^{-1}\tau_*, L^{n-1} V;\ N) \tag{6.42}$$

and conformal matter distribution

$$(L^{-2}\rho(L^{1-\frac{n}{2}}\,\cdot), L^{n-1}j, L^{-2}\Lambda). \tag{6.43}$$

Lemma 6.7 should be compared with the analogous result for the CTS-H equations, where a solution (ϕ, W) for conformal data $(\sigma, \tau;\ N)$ scales to a solution $(L^{\frac{n}{2}-1}\phi, L^{n-1}W)$ for conformal data $(L^{n-1}\sigma, L^{-1}\tau;\ N)$. So for the CTS-H equations, we can effectively trade small τ for large σ or vice-versa. Furthermore, if a solution with $\tau \equiv 0$ can be found, then nearby perturbatios and rescalings allow for arbitrary mean curvature. The situation is more complicated for the CED-V equations because there is an additional parameter involved, but the principle is the same. If we can find a solution with a parameter equal to zero, then we may hope to perturb off of it and rescale to obtain any value of the chosen parameter. In the CMC case, volumetric drift is zero, and hence we can obtain any desired volumetric drift.

Corollary 6.8. *Let $L > 0$ be a constant and consider drift conformal data $(g, L^{n-1}\sigma, L^{-1}\tau_*, V;\ N)$ with conformal matter distribution $(L^{-2}\rho(L^{\frac{n}{2}-1}\,\cdot)$, $L^{n-1}j, L^{-2}\Lambda)$, all with the regularity hypotheses considered in Theorem 6.4. There exists a solution of the CED-V equations* (5.15) *for this data if L is sufficiently large and if all of the following hold:*

- *There exists a solution for the the CMC conformal data $(g, \sigma, \tau_*, 0;\ N)$ with matter distribution $(\rho(\cdot), j, \Lambda)$.*
- *There are no true Killing fields for the metric at the CMC solution.*
- $\kappa\tau_*^2 \geq \Lambda$
- *Either $\kappa\tau_*^2 > \Lambda$, or $\sigma \not\equiv 0$, or the matter distribution is not vacuum.*

Proof. Consider the rescaled conformal data $(g, \sigma, \tau_*, L^{-1-n}V;\ N)$ with conformal matter distribution $(\rho(\cdot), j, \Lambda)$. From the stated assumptions we can apply Theorem 6.4 to conclude that if L is sufficiently large (and hence $L^{-1-n}V$ is sufficiently small) there exists a solution (ϕ, W, Q) of system (5.15) for this data. Let

$$(\hat{\phi}, \hat{W}, \hat{Q}) = (L^{\frac{n}{2}-1}\phi, L^{n-1}W, L^{n-1}\hat{Q}) \tag{6.44}$$

Lemma 6.7 implies $(\hat{\phi}, \hat{W}, \hat{Q})$ is a solution of system (5.15) for conformal data $(g, L^{n-1}\sigma, L^{-1}\tau_*, V;\ N)$ with matter distribution $(L^{-2}\rho(L^{1-\frac{n}{2}}\cdot), L^{n-1}j$, $L^{-2}\Lambda)$. □

In effect, Corollary 6.8 provides a weak notion of the idea that we can obtain any volumetric drift we please so long as we take the conformal momentum sufficiently large and the volumetric momentum sufficiently small. For maximal CMC solutions ($\tau_* = 0$) an analogous procedure shows that we can perturb to an arbitrary volumetric momentum at the penalty of shrinking both the conformal momentum and the volumetric drift.

Corollary 6.9. *Under the same regularity hypotheses as Theorem 6.4 suppose:*

- *There exists a solution for the maximal slice conformal data* $(g, \sigma, 0, 0;\ N)$ *with matter distribution* $(\rho(\cdot), j, \Lambda)$.
- *There are no true Killing fields for the metric at the CMC solution.*
- $\kappa\tau_*^2 \geq \Lambda$
- *Either* $\kappa\tau_*^2 > \Lambda$, *or* $\sigma \not\equiv 0$, *or the matter distribution is not vacuum.*

If $L > 0$ *is sufficiently small, then there exists a solution of the CED-V equations* (5.15) *with prescribed conformal data* $(g, L^{n-1}\sigma, \tau_*, L^{n+2}V;\ N)$ *and matter distribution* $(L^{-2}\rho(L^{\frac{n}{2}-1}\ \cdot), L^{n-1}j, L^{-2}\Lambda)$.

Proof. Consider the rescaled conformal data $(g, \sigma, L\tau_*, LV;\ N)$ with matter distribution $(\rho(\cdot), j, \Lambda)$. Since we have assumed that there exists a solution for the maximal slice data $(g, \sigma, 0, 0;\ N)$, Theorem 6.4 implies that if L is sufficiently small there exists a solution (ϕ, W, Q) of system (5.15) for this data. Rescaling as in the the proof of Corollary 6.8, we then find that that there exists a solution for conformal data $(g, L^{n-1}\sigma, L^{-1}\tau_*, L^{n+2}V;\ N)$ and matter distribution $(L^{-2}\rho(L^{1-\frac{n}{2}}\cdot), L^{n-1}j, L^{-2}\Lambda)$. $\square$

7. Extension to the AE and AH settings

In this brief final section we indicate the modifications necessary to carry these results over to the two main noncompact settings common in this field, namely to sets of data which are asymptotically Euclidean (AE) or asymptotically hyperbolic (AH), respectively. (Extensions to other cases of interest, such as to compact manifolds with boundary, may be established by following the same overall approach.)

As is well known, in either of these cases, we may take advantage of known solvability results for the various linear operators which appear in this paper, acting between appropriate weighted Sobolev spaces. Our intent here is not to be complete, but rather just to briefly describe those parts of the arguments above that can be modified without further effort. In fact, there are no nontrivial conformal Killing fields vanishing at infinity in these settings, so the situation is somewhat simpler. On the other hand, this absence of conformal Killing fields implies both the standard conformal method and the drift method have perfectly adequate near-CMC theories for AE and AH initial data, and any potential advantages of the drift method is these cases would have to arise for far-from CMC data. In the AH setting there are additional deeper questions concerning the 'shear-free' condition (see, e.g., [1]) but these have not been previously addressed even for the standard conformal method and we leave their resolution for elsewhere.

Asymptotically Euclidean Data. We say that (M, g, K) is an asymptotically Euclidean data set if there exists a compact region $K \subset M$ such that each of the finitely many components E of $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ for some $R > 0$, and using this diffeomorphism to give coordinates on each end, $g|_E = \delta + h$ where δ is the Euclidean metric and $h_{ij} = \mathcal{O}(|x|^{-1})$, along with corresponding estimates for the derivatives up to order $2 + \alpha$. At the same time, $K_{ij} = \mathcal{O}(|x|^{-2})$ along with derivatives. It is equally easy from an analytic standpoint to include the somewhat more general case of asymptotically conic data. Here $M \setminus K$ is a finite union of ends E where each E is diffeomorphic to the 'large end' of a Riemannian cone $C(Y)$, with metric $dr^2 + r^2 k_Y$, where (Y, k_Y) is a compact Riemannian manifold, and so that the corresponding estimates as above hold with this conic metric in place of the Euclidean metric. In either case, we also impose suitable decay conditions on matter fields.

The results that need to be modified in this new geometric setting are those which concern the global solvability of certain elliptic problems. The particular results that require different proofs are the York splitting Lemmas 2.2 and 4.1, and our main Theorem 6.4. In Theorem 6.4, we decompose the conformal factor $\phi = 1 + u$, and because there are no conformal Killing fields vanishing at infinity the map F no longer involves the variable Q. Its linearization from equation (6.24) becomes

$$DF(\delta u, \delta W) = \begin{pmatrix} -a\,\Delta + A & -2\left\langle \sigma + \frac{1}{2N}\,\mathbf{L}\,\hat{W}, \frac{1}{2N}\,\mathbf{L}(\cdot)\right\rangle \\ 0 & \frac{1}{2}\,\mathbf{L}^*\left(\frac{1}{2N}\,\mathbf{L}(\cdot)\right) \end{pmatrix} \begin{pmatrix} \delta u \\ \delta W \end{pmatrix} \tag{7.1}$$

where, in vacuum,

$$A = (q+2)\left|\sigma + \frac{1}{2N}\,\mathbf{L}\,\hat{W}\right|^2 \geq 0. \tag{7.2}$$

For all of these adjustments we require the basic Fredholm properties of elliptic operators on asymptotically conic spaces, which appears, for example, in [25] (and many other places). The main observation is that one needs to let such an operator act between spaces which are weighted by powers of $|x|$ at infinity. This theory is well-known, the elliptic operators involved in our application indeed invertible, and there are no unexpected issues.

Asymptotically Hyperbolic Data. Another main setting in relativity is the asymptotically hyperbolic case; this generalizes the spacelike hyperboloid in Minkowski space, or equivalently, hyperbolic space. The natural generalization of this is the class of conformally compact asymptotically hyperbolic spaces. We say that (M, g, K) is an asymptotically hyperbolic data set if the following holds. First, M is the interior of a smooth compact manifold with boundary $\overline{M}$. The metric g is of the form $\overline{g}/\rho^2$, where $\overline{g}$ is a metric smooth and nondegenerate up to $\partial\overline{M}$, and ρ is a boundary defining function for the boundary which satisfies $|\nabla^{\overline{g}}\rho|_{\overline{g}} = 1$ at $\rho = 0$. The tensor K is again smooth up to $\partial\overline{M}$, and if we write $K = \sigma + (\tau/n)g$, then τ converges to a constant at $\rho = 0$. It is straightforward to relax the regularity assumptions on the metric and second fundamental form.

Here too there is a rich and well-developed analytic theory, again to be found in [25] (parts of which again appear in many other places as well). We let the relevant operators act on function spaces which are weighted by powers of ρ, or equivalently, by powers of e^{-d}, where d is the Riemannian distance function on M, e.g. distance to some fixed compact set in the interior. We again observe that Laplace-type operators are Fredholm when acting between weighted Sobolev spaces and that the three main results mentioned above hold in this geometric setting as well. The monograph [18] works out the indicial roots for the relevant elliptic operators in this setting; these indicial roots determine the precise ranges of weights on the function spaces.

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