# Nonconvex, Fully Distributed Optimization Based CAV Platooning Control Under Nonlinear Vehicle Dynamics

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***Abstract*— CAV platooning technology has received considerable attention, driven by the next generation smart transportation systems. This paper considers nonlinear vehicle dynamics and develops fully distributed optimization based CAV platooning control schemes via the platoon centered MPC approach for a possibly heterogeneous CAV platoon. The nonlinear vehicle dynamics leads to major difficulties in distributed algorithm development and control analysis. Specifically, the underlying MPC optimization problem is nonconvex and densely coupled. Further, the closed loop dynamics becomes a time-varying nonlinear system with non-vanishing external perturbations, making stability analysis rather complicated. To overcome these difficulties, we formulate the underlying MPC optimization problem as a locally coupled, albeit nonconvex, optimization problem and develop a sequential convex programming based fully distributed scheme for a general MPC horizon. Such a scheme can be effectively implemented for real-time computing using operator splitting methods. To analyze the closed loop stability, we apply various tools from global implicit function theorems, stability of linear time-varying systems, and Lyapunov theory for input-to-state stability to show that the closed loop system is locally input-to-state stable uniformly in all small coefficients pertaining to the nonlinear dynamic effects. Numerical tests on a heterogeneous CAV platoon in a real traffic condition illustrate the effectiveness of the proposed method.**

***Index Terms*— Connected and autonomous vehicle, car following control, distributed algorithm, nonconvex optimization, input-to-state stability, Lyapunov stability theory.**

## I. Introduction

INSPIRED by the next generation smart transportation systems, connected and autonomous vehicle (CAV) technologies emerge and offer tremendous opportunities to reduce traffic congestion and improve road safety and traffic efficiency, through innovative traffic flow control and operations. Particularly, vehicle platooning technology links a group of CAVs through cooperative acceleration or speed control to improve system efficiency and safety. This technology allows adjacent group members of a CAV platoon to travel safely at a higher speed with smaller spacing. It will increase lane capacity, improve traffic flow efficiency, and reduce congestion, emission, and fuel consumption [1], [13].

Extensive research on CAV platooning control has been conducted, and many approaches have been proposed, e.g., adaptive cruise control (ACC) [14], [16], [18], [29], [36], cooperative adaptive cruise control (CACC) [25], [26], [28], [33], and platoon centered vehicle platooning control [5], [6], [30], [31]. The ACC and CACC approaches aim to improve an individual vehicle's safety and mobility as well as string stability instead of system performance of the entire platoon. On the other hand, the recently developed platoon centered approach seeks to optimize the platoon's transient traffic dynamics for a smooth traffic flow and to achieve stability and other desired long-time dynamical behaviors. This approach can significantly improve system performance and efficiency of the entire platoon [6], [30]. Despite this advantage, the platoon centered platooning approach often encounters large-scale optimization or optimal control problems that require efficient numerical solvers for real-time computation [30]. Distributed optimization techniques provide a favorable solution for the platoon centered approach. Supported by portable computing capability of each vehicle and vehicle-to-vehicle communication [32], distributed computation can handle high computation load efficiently, is more flexible to communication network topologies, and is more robust to communication delays or network malfunctions [32]. In this paper, we focus on platoon centered CAV platooning via distributed optimization. It is worth noting that a platoon centered CAV platooing control is a centralized control method although its computation is distributed, i.e., each CAV computes its own control input in a distributed manner [23]. Hence, this method is different from decentralized control in control engineering [2], [3], [34], [37]. Especially, the platoon centered method focuses on stability of the entire platoon instead of stability of individual vehicles and their interactions, e.g., string stability [2], [37].

Various distributed control or optimization schemes have been proposed for CAV platooning [30], [32], [33], [37]. These schemes can be classified into two types: partially distributed schemes, and fully distributed schemes. Partially distributed schemes are referred to as those schemes that either require all vehicles to exchange information with a central component for centralized data processing or perform centralized computation in at least one step [15], whereas fully distributed

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schemes do not require centralized data processing or carry out centralized computation through the entire schemes [23]. The former type includes [5], [6]. The second type includes the recent paper [23], which develops fully distributed schemes for CAV platooning under the linear vehicle dynamics. Compared with partially distributed schemes, fully distributed schemes impose less restrictions on vehicle communication networks and can be easily implemented on a wide range of networks [23].

In spite of the abovementioned progress, most of the existing research in the platoon centered approach considers the linear vehicle dynamics [5], [6], [23], [30]. Although the linear vehicle dynamics is suitable for small vehicles, nonlinear dynamic effects, e.g, aerodynamic drag, friction, and rolling resistance, play a non-negligible role in trucks, heavy duty vehicles, and other types of CAVs. Motivated by the lack of research for nonlinear vehicle dynamics in the platoon centered approach, this paper aims to develop fully distributed optimization based, platoon centered CAV platooning under nonlinear vehicle dynamics over a general vehicle communication network. To achieve this goal, we propose a $p$-horizon MPC model subject to the nonlinear vehicle dynamics of the CAVs and various physical or safety constraints. Several new challenges arise for the MPC horizon $p \geq 2$ when the nonlinear vehicle dynamics is considered. First, the underlying MPC model gives rise to a *densely coupled*, *nonconvex* optimization problem, where both the objective function and constraints are nonconvex. This is very different from the linear vehicle dynamics treated in [23], for which a convex MPC model is obtained so that various convex distributed optimization schemes can be used. Second, a local optimal solution to the MPC is characterized by a highly sophisticated nonlinear equation and does not attain a closed form expression. Hence, the closed loop system is defined by a time-varying nonlinear dynamical system, whose right-hand side has no closed form expression, subject to non-vanishing external disturbances. These pose a difficulty in closed loop stability analysis and design. To address these challenges, we exploit various new techniques for distributed algorithm development and control analysis and design.

The major novelties and main contributions of this paper are summarized as follows:

(1) Distributed algorithm development. To develop fully distributed schemes for the nonconvex MPC optimization problem when $p \geq 2$, we first formulate the underlying densely coupled MPC optimization problem as a locally coupled, albeit nonconvex, optimization problem using a decomposition method recently developed for the linear CAV dynamics [23]. Furthermore, we propose a sequential convex programming (SCP) [17] based distributed scheme to solve the locally coupled optimization problem. This SCP based scheme solves a sequence of convex, quadratically constrained quadratic programs (QCQPs) that approximate the original nonconvex program at each iteration; such a convex QCQP can be efficiently solved using (generalized) Douglas-Rachford method or other operator splitting methods [4] in the fully distributed manner. The SCP based distributed scheme converges to a stationary point, which often coincides or is close to an optimal solution, under mild assumptions.

(2) Closed loop stability analysis. To analyze the closed loop dynamics, we first formulate the closed loop system as a tracking system defined by a time-varying, nonlinear dynamical system subject to non-vanishing external disturbances. The right-hand side of this nonlinear dynamical system depends on a local optimal solution to the underlying MPC optimization problem, which does not attain a closed-form expression. By exploiting global implicit function theorems, we show that this (local) optimal solution is an implicit smooth function of state variables for all sufficiently small parameters pertaining to the nonlinear dynamic effects. We then apply stability theory of linear time-varying systems and Lyapunov theory for input-to-state stability to show that for all sufficiently small parameters pertaining to the nonlinear dynamic effects, the closed loop system is locally input-to-state stable provided that the corresponding linear closed loop dynamics under the linear vehicle dynamics (or equivalently when the abovementioned parameters are zero) is Schur stable.

(3) Numerical implementation for real-time computation. For real-time implementation of the proposed fully distributed schemes, initial guess warm-up techniques are developed. Besides, a further analysis shows that steady state errors of spacing exist in the close loop dynamics but can be made small by choosing suitable weights in the MPC model while ensuring the input-to-state stability and satisfactory performance of transient dynamics. Numerical tests have been carried out for a heterogeneous CAV platoon in a real traffic condition. The numerical results illustrate the effectiveness of the proposed distributed scheme and CAV platooning control under the nonlinear vehicle dynamics.

The paper is organized as follows. Section II introduces the nonlinear vehicle dynamics and constraints, and vehicle communication networks. Sequential feasibility and the constraint sets are discussed in Section III. A MPC model is proposed in Section IV and is formulated as a nonconvex constrained optimization problem. Section V develops sequentially convex programming based fully distributed schemes for the densely coupled nonconvex MPC optimization problem. Control design and closed loop stability analysis is carried out in Section VI, and numerical results are given in Section VII with conclusions made in Section VIII. Due to the paper length limit, most of the proofs and some technical details are omitted and can be found in the online version [22] of the paper.

## II. Vehicle Dynamics, Constraints, and Communication Topology

Consider a platoon consisting of heterogeneous vehicles (e.g., cars and trucks) on a roadway, where the (uncontrolled) leading vehicle is labeled by the index 0 and its $n$ following CAVs are labeled by the indices $i = 1, \ldots, n$, respectively. Let $x_i, v_i$ denote the longitudinal position and speed of the $i$th vehicle, respectively. Let $\tau > 0$ be the sampling time, and each time interval is $[k\tau, (k+1)\tau)$ for $k \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$.

We first introduce the following nonlinear vehicle dynamical model which captures aerodynamic drag, friction, and rolling resistance [34]:

$$x_i(k+1) = x_i(k) + \tau v_i(k) + \frac{\tau^2}{2} a_i(k), \tag{1a}$$
$$v_i(k+1) = v_i(k) + \tau a_i(k), \tag{1b}$$

where $a_i(k) := u_i(k) - c_{2,i} \cdot v_i^2(k) - c_{3,i} \cdot g$, $u_i(k)$ denotes the desired driving/braking acceleration treated as the control input, $c_{2,i} \cdot v_i^2(k)$ characterizes the deceleration due to aerodynamic drag with the coefficient $c_{2,i} > 0$, and $c_{3,i} \cdot g$ characterizes friction and rolling resistance with $g = 9.8\, m/s^2$ being the gravitational constant and $c_{3,i} > 0$ being the rolling friction coefficient. For different vehicles, the coefficients $c_{2,i}, c_{3,i}$ can be different.

The coefficients $c_{2,i}$ and $c_{3,i}$ in model (1) are usually small for certain types of vehicles or road conditions. For example, $c_{2,i}$ typically ranges from $2.5 \times 10^{-4}/m$ to $4.5 \times 10^{-4}/m$, and $c_{3,i}$ typically ranges from 0.006 to 0.015 [34]. Since these coefficients are small, the nonlinear terms in (1) are often neglected. This yields the following double-integrator model for the linear vehicle dynamics:

$$\begin{aligned} x_i(k+1) &= x_i(k) + \tau v_i(k) + \frac{\tau^2}{2} u_i(k), \\ v_i(k+1) &= v_i(k) + \tau u_i(k). \end{aligned} \tag{2}$$

The model (2) is suitable for small-size passenger cars, while model (1) can be used for medium-size or large-size vehicles, e.g., trucks and heavy-duty vehicles.

**State and control constraints.** Each vehicle is subject to important state and control constraints. For any $i = 1, \ldots, n$,

(i) Control constraint: $a_{i,\min} \le u_i \le a_{i,\max}$, where $a_{i,\min} < 0$ and $a_{i,\max} > 0$ are pre-specified acceleration or deceleration bounds for the $i$th vehicle;

(ii) Speed constraint: $v_{\min} \le v_i \le v_{\max}$, where $0 \le v_{\min} < v_{\max}$ are pre-specified bounds on longitudinal speed for the $i$th vehicle;

(iii) Safety distance constraint: this constraint guarantees sufficient spacing between neighboring vehicles to avoid collision. The safety distance constraint is given by:

$$x_{i-1} - x_i \ \ge\ L_i + r_i \cdot v_i - \frac{(v_i - v_{\min})^2}{2 a_{i,\min}}, \tag{3}$$

where $L_i > 0$ is a constant depending on vehicle length, and $r_i > 0$ is the reaction time of vehicle $i$.

In the above constraints, the acceleration/decelerations bounds as well as the vehicle length $L_i$ and the reaction time $r_i$ can be different for different types of vehicles. Note that constraints (i) and (ii) are decoupled across the vehicles, whereas the safety distance constraint (iii) is state-control coupled. This yields challenges to distributed computation. Further, the leading vehicle is subject to the similar acceleration and speed constraints, i.e., $a_{0,\min} \le u_0 \le a_{0,\max}$ and $v_{\min} \le v_0 \le v_{\max}$, where $a_{0,\min} < 0 < a_{0,\max}$.

**Communication network topology.** We consider a general communication network whose topology is modeled by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of nodes with the $i$th node corresponding to the $i$th CAV, and $\mathcal{E}$ is the set of edges connecting two nodes in $\mathcal{V}$. Let $\mathcal{N}_i$ denote the set of neighbors of node $i$, i.e., $\mathcal{N}_i = \{j \mid (i, j) \in \mathcal{E}\}$. The following assumption is made throughout the paper:

**A.1** The graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is undirected and connected. Further, two neighboring vehicles form a bidirectional edge of the graph, i.e., $(1, 2), (2, 3), \ldots, (n-1, n) \in \mathcal{E}$.

The setting given by **A.1** includes many widely used communication networks of CAV platoons, e.g., immediate-preceding, multiple-preceding, and preceding-and-following [34]. We also assume that the first CAV can receive $x_0$, $v_0$ and $u_0$ from the leading vehicle.

## III. Sequential Feasibility and Properties of Constraint Sets

The constraint set of the underlying MPC optimization problem at time $k$ (cf. Section IV) depends on the position and speed of the vehicles at times $0, 1, \ldots, k-1$. A fundamental question is whether the constraint set is nonempty at each $k$ along a system trajectory for any feasible initial condition, provided that the leading vehicle satisfies the acceleration and speed constraints for all $k \in \mathbb{Z}_+$. If the answer is affirmative, the system is *sequentially feasible* [6], which has been established for the linear vehicle dynamics [6]. We show it for the nonlinear vehicle dynamics (1) below.

Given $(x_i, v_i)_{i=0}^n$ and $u_0$, we introduce the following constraint set on the control $u$ subject to the nonlinear vehicle dynamics and the state and control constraints:

$$\mathcal{W}((x_i, v_i)_{i=0}^n, u_0) := \Big\{ u \in \mathbb{R}^n \mid a_{i,\min} \le u_i \le a_{i,\max}, \\ v_{\min} \le v_i + \tau a_i(u_i) \le v_{\max},\ h_i(u) \le 0,\ i = 1, \ldots, n \Big\},$$

where the function $h_i$ is given by

$$h_i(u) := L_i + r_i(v_i + \tau a_i(u_i)) - \frac{(v_i + \tau a_i(u_i) - v_{\min})^2}{2 a_{i,\min}} \\ + (x_i - x_{i-1}) + \tau(v_i - v_{i-1}) + \frac{\tau^2}{2}[a_i(u_i) - a_{i-1}(u_{i-1})],$$

and $a_i(u_i) := u_i - c_{2,i} v_i^2 - c_{3,i} g$ for each $i = 0, 1, \ldots, n$. The sequential feasibility holds if $\mathcal{W}((x_i, v_i)_{i=0}^n, u_0)$ is nonempty for any feasible $(x_i, v_i)_{i=0}^n$ and $u_0$, i.e., $a_{0,\min} \le u_0 \le a_{0,\max}$, $v_{\min} \le v_0 \le v_{\max}$, $v_{\min} \le v_0 + \tau u_0 \le v_{\max}$, $v_{\min} \le v_i \le v_{\max}$ and $p_i((x_i, v_i)_{i=0}^n) := L_i + r_i v_i - \frac{(v_i - v_{\min})^2}{2 a_{i,\min}} + (x_i - x_{i-1}) \le 0$ for each $i = 1, \ldots, n$.

*Proposition 1: [22, Proposition 3.1] Consider the nonlinear vehicle dynamics given by (1). Suppose the nonnegative constants $c_{2,i}, c_{3,i}$ are such that $c_{2,i} v_{\max}^2 + c_{3,i} g \le a_{i,\max}$ and $r_i \ge \tau$ for each $i = 1, \ldots, n$. Then the system is sequentially feasible for an arbitrary feasible initial condition.*

It is also shown in [22, Proposition 3.2] that under mild assumptions, the constraint set has nonempty interior. This property is critical for the Slater's constraint qualification in optimization. In light of this result, we make the following assumption throughout the rest of the paper:

**A.2** For each $i = 1, \ldots, n$, the nonnegative constants $c_{2,i}, c_{3,i}$ satisfy $c_{2,i} v_{\max}^2 + c_{3,i} g < a_{i,\max}$ and the reaction time $r_i$ satisfies $r_i \ge \tau$. Further, $(v_0(k), u_0(k))$ is feasible with $v_0(k) > v_{\min}$ for all $k \in \mathbb{Z}_+$.

Under this assumption, the constraint set of a $p$-horizon MPC model has nonempty interior; see Corollary 1.

## IV. Model Predictive Control for CAV Platooning

We consider the model predictive control (MPC) [23, Section 3] approach for CAV platooning under the nonlinear vehicle dynamics. Let $\Delta$ be the desired constant spacing between two adjacent vehicles, and $(x_0, v_0, u_0)$ be the position, speed, and control input of the leading vehicle, respectively. Define (i) the relative spacing error $z(k) := \big(x_0 - x_1 - \Delta, \dots, x_{n-1} - x_n - \Delta\big)(k) \in \mathbb{R}^n$; (ii) the relative speed between adjacent vehicles $z'(k) := \big(v_0 - v_1, \dots, v_{n-1} - v_n\big)(k) \in \mathbb{R}^n$; and (iii) the control input $u(k) := \big(u_1, \dots, u_n\big)(k) \in \mathbb{R}^n$. Further, let $w_i(k) := u_{i-1}(k) - u_i(k)$ for each $i = 1, \dots, n$, and $w(k) := \big(w_1, \dots, w_n\big)(k) \in \mathbb{R}^n$, representing the difference of control input between adjacent vehicles. Hence, for any $k \in \mathbb{Z}_+$, $u(k) = -S_n w(k) + u_0(k) \cdot \mathbf{1}$, where $\mathbf{1}$ is the vector of ones, and $S_n$ is an $n \times n$ lower triangular matrix with $(S_n)_{i,j} = 1$ for all $i \le j$. Hence $S_n^{-1}$ is such that $(S_n^{-1})_{i,i} = 1, \forall i$, $(S_n^{-1})_{i,i+1} = -1$ for all $i = 1, \dots, n-1$, and the other elements of $S_n^{-1}$ are zero.

Given $p \in \mathbb{N}$, the $p$-horizon MPC control is determined by solving the following constrained optimization problem at each $k \in \mathbb{Z}_+$, involving all vehicles' control inputs for given feasible state $(x_i(k), v_i(k))_{i=1}^n$ and $(v_0(k), u_0(k))$ at time $k$ subject to the nonlinear vehicle dynamics (1):

$$\begin{aligned} \text{minimize} \quad & J(u(k), \dots, u(k+p-1)) := \\ & \frac{1}{2} \sum_{s=1}^{p} \Big( \tau^2 u^T(k+s-1) S_n^{-T} Q_{w,s} S_n^{-1} u(k+s-1) \\ & + z^T(k+s) Q_{z,s} z(k+s) + (z'(k+s))^T Q_{z',s} z'(k+s) \Big) \end{aligned} \tag{4}$$

subject to: for each $i = 1, \dots, n$ and $s = 1, \dots, p$, $a_{i,\min} \le u_i(k+s-1) \le a_{i,\max}$, $v_{\min} \le v_i(k+s) \le v_{\max}$, and

$$x_{i-1}(k+s) - x_i(k+s) \ge L_i + r_i \cdot v_i(k+s) - \frac{(v_i(k+s) - v_{\min})^2}{2a_{i,\min}},$$

where $Q_{z,s}$, $Q_{z',s}$ and $Q_{w,s}$ are $n \times n$ symmetric positive semidefinite weight matrices. We assume that $u_0(k+s) = u_0(k)$ for all $s = 1, \dots, p-1$ and use these $u_0(k+s)$'s and the vehicle dynamics model (1) to predict $(x_0(k+s+1), v_0(k+s+1))$ for $s = 1, \dots, p-1$. See [23, Remark 3.1] for the interpretation of the three terms in the objective function $J$.

To develop fully distributed schemes for general vehicle network topologies and to facilitate control design and analysis, we make the following assumption on the weight matrices $Q_{z,s}$, $Q_{z',s}$, and $Q_{w,s}$ through the rest of the paper:

**A.3** For each $s = 1, \dots, p$, $Q_{z,s}$ and $Q_{z',s}$ are diagonal and positive semidefinite (PSD), and $Q_{w,s}$ is diagonal and positive definite (PD).

It is shown below that the constraint set of the $p$-horizon MPC model has nonempty interior at each $k$ for any $p$.

*Corollary 1: [22, Corollary 4.1] Suppose* **A.2** *holds. Then for any $p \in \mathbb{N}$, the constraint set of the $p$-horizon MPC optimization problem (4) has nonempty interior at each $k$.*

### *A. Constrained Optimization Model Under the Nonlinear Vehicle Dynamics*

We discuss the MPC model (4) under the nonlinear vehicle dynamics (1) with the parameters $c_{2,i}$ and $c_{3,i}$. Define the parameter vectors $\boldsymbol{\varphi}_d := (c_{2,1}, \dots, c_{2,n}) \in \mathbb{R}^n_+$ and $\boldsymbol{\varphi}_f := (c_{3,1}, \dots, c_{3,n}) \in \mathbb{R}^n_+$, where the subscripts $d$ and $f$ denote the drag and friction respectively. Let $\boldsymbol{\varphi} := (\boldsymbol{\varphi}_d, \boldsymbol{\varphi}_f) \in \mathbb{R}^{2n}_+$. We set $c_{2,0} = c_{3,0} = 0$ as $u_0(k)$ is the actual acceleration of the leading vehicle.

Consider the MPC model (4) at a fixed time $k \in \mathbb{Z}_+$. Let $\mathbf{u}(k) := (\mathbf{u}_1(k), \dots, \mathbf{u}_n(k)) \in \mathbb{R}^{np}$ with $\mathbf{u}_i(k) := (u_i(k), \dots, u_i(k+p-1)) \in \mathbb{R}^p$. Recall that for each $i = 1, \dots, n$ and $j = 0, \dots, p-1$,

$$a_i\big(k+j, u_i(k), \dots, u_i(k+j)\big) = u_i(k+j) - c_{2,i} v_i^2(k+j) - c_{3,i} g,$$

where we note that $v_i(k+j)$ depends on $u_i(k), \dots, u_i(k+j-1)$ for $j \ge 1$. Specifically, for $p > 1$,

$$\begin{aligned} & a_i(k, u_i(k)) = u_i(k) - c_{2,i} v_i^2(k) - c_{3,i} g, \\ & a_i(k+1, u_i(k), u_i(k+1)) = u_i(k+1) \\ & \qquad - c_{2,i}\big[v_i(k) + \tau a_i(k, u_i(k))\big]^2 - c_{3,i} g, \\ & \qquad \vdots \qquad \vdots \qquad \vdots \\ & a_i\big(k+p-1, u_i(k), \dots, u_i(k+p-1)\big) = u_i(k+p-1) \\ & \qquad - c_{2,i}\Big[v_i(k) + \tau \sum_{s=0}^{p-2} a_i(k+s, u_i(k), \dots, u_i(k+s))\Big]^2 \\ & \qquad - c_{3,i} g. \end{aligned}$$

By slightly abusing the notation, we may denote each $a_i\big(k+j, u_i(k), \dots, u_i(k+j)\big)$ by $a_i(k+j, \mathbf{u}_i(k))$.

Define for each $i = 1, \dots, n$ and $j = 0, 1, \dots, p-1$,

$$b_i(k+j, \mathbf{u}_{i-1}(k), \mathbf{u}_i(k)) := a_{i-1}(k+j, \mathbf{u}_{i-1}(k)) - a_i(k+j, \mathbf{u}_i(k)),$$

where $a_0(k+j, \mathbf{u}_0(k)) := u_0(k)$ for all $j = 0, 1, \dots, p-1$ due to $\mathbf{u}_0(k) := u_0(k) \cdot \mathbf{1}$. It follows from the nonlinear vehicle dynamics (1) that for each $i = 1, \dots, n$ and $j = 1, \dots, p$,

$$z_i(k+j) = z_i(k) + j\tau z_i'(k) + \tau^2 \sum_{s=0}^{j-1} \frac{2(j-s)-1}{2} b_i(k+s, \mathbf{u}_{i-1}(k), \mathbf{u}_i(k)), \tag{5}$$

$$z_i'(k+j) = z_i'(k) + \tau \sum_{s=0}^{j-1} b_i(k+s, \mathbf{u}_{i-1}(k), \mathbf{u}_i(k)). \tag{6}$$

For a fixed $k \in \mathbb{Z}_+$, define for each $i = 1, \dots, n$, $\mathbf{a}_i(\mathbf{u}_i(k)) := \big(a_i\big(k, u_i(k)\big), a_i\big(k+1, u_i(k), u_i(k+1)\big), \dots, a_i\big(k+p-1, u_i(k), \dots, u_i(k+p-1)\big)\big)$. In what follow, we often omit $k$ in $\mathbf{u}_i(k)$ when $k$ is fixed. Further, define the function $\mathbf{a} : \mathbb{R}^{np} \to \mathbb{R}^{np}$ as $\mathbf{a}(\mathbf{u}) := \big(\mathbf{a}_1(\mathbf{u}_1), \dots, \mathbf{a}_n(\mathbf{u}_n)\big)$. Note that if $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_d, \boldsymbol{\varphi}_f) = (c_{2,i}, c_{3,i})_{i=1}^n = 0$, then $\mathbf{a}(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^{np}$. We introduce more notation. Define the following matrices: $\overline{Q}_w := \mathrm{diag}\big(Q_{w,1}, \dots Q_{w,p}\big) \in \mathbb{R}^{np \times np}$, and $\mathbf{S}^{-1} := \mathrm{diag}\big(\underbrace{S_n^{-1}, \ldots, S_n^{-1}}_{p-\text{copies}}\big) \in \mathbb{R}^{np\times np}$. Further, let $E \in \mathbb{R}^{np\times np}$ be the permutation matrix such that

$$\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+p-1) \end{bmatrix} = E \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = E\mathbf{u}.$$

Using these matrices, it is easy to verify that the following term in the objective function $J$ in (4) satisfies $\left(\mathbf{S}^{-1}\begin{bmatrix} u(k) \\ \vdots \\ u(k+p-1) \end{bmatrix}\right)^T \overline{Q}_w \left(\mathbf{S}^{-1}\begin{bmatrix} u(k) \\ \vdots \\ u(k+p-1) \end{bmatrix}\right) = \mathbf{u}^T \Psi\, \mathbf{u}$, where $\Psi := E^T \mathbf{S}^{-T} \overline{Q}_w \mathbf{S}^{-1} E \in \mathbb{R}^{np\times np}$ is symmetric PD under **A.3**. Thus the objective function $J$ becomes

$$\begin{aligned} J(\mathbf{u}) &= \frac{\tau^2}{2}\mathbf{u}^T \Psi\, \mathbf{u} \\ &\quad + \frac{1}{2}\Big[\sum_{s=1}^{p} z^T(k+s) Q_{z,s} z(k+s) \\ &\quad + (z'(k+s))^T Q_{z',s} z'(k+s)\Big] \\ &= \frac{\tau^2}{2}\mathbf{a}^T(\mathbf{u})\Psi\, \mathbf{a}(\mathbf{u}) + \frac{\tau^2}{2}\Big(\mathbf{u}^T\Psi\mathbf{u} - \mathbf{a}^T(\mathbf{u})\Psi\mathbf{a}(\mathbf{u})\Big) \\ &\quad + \frac{1}{2}\Big[\sum_{s=1}^{p} z^T(k+s) Q_{z,s} z(k+s) \\ &\quad + (z'(k+s))^T Q_{z',s} z'(k+s)\Big]. \end{aligned}$$

In light of the expressions for $z(k+j)$ and $z'(k+j)$ in (5)-(6), we have, via the similar argument in [23, Section 3.1],

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{a}^T(\mathbf{u}) W \mathbf{a}(\mathbf{u}) + c^T \mathbf{a}(\mathbf{u}) + \gamma + \frac{\tau^2}{2}\Big(\mathbf{u}^T\Psi\mathbf{u} - \mathbf{a}^T(\mathbf{u})\Psi\mathbf{a}(\mathbf{u})\Big),$$

where $W \in \mathbb{R}^{np\times np}$, $c \in \mathbb{R}^{np}$, and $\gamma \in \mathbb{R}$. In fact, $W = E^T \mathbf{S}^{-T} \Theta \mathbf{S}^{-1} E$ for a symmetric PSD matrix $\Theta$ whose blocks are diagonal, and $W$ is PD under **A.3**; see [23, Section 3.1] for details. Besides, the linear term in $J(\mathbf{u})$ can be written as $c^T\mathbf{a}(\mathbf{u}) = \sum_{i=1}^n c_{\mathcal{I}_i}^T \mathbf{a}_i(\mathbf{u}_i)$, where $c_{\mathcal{I}_i}$ is the subvector of $c$ corresponding to $\mathbf{a}_i(\mathbf{u}_i)$. By [23, Lemma 3.2], the subvector $c_{\mathcal{I}_i}$ depends only on $z_i(k), z_i'(k), z_{i+1}(k), z_{i+1}'(k)$'s for $i = 1, \ldots, n-1$, $c_{\mathcal{I}_n}$ depends only on $z_n(k), z_n'(k)$, and only $c_{\mathcal{I}_1}$ depends on $u_0(k)$. These properties are important for developing fully distributed schemes later on.

To characterize the constraints, let the matrix $S_p \in \mathbb{R}^{p\times p}$ be defined in the same way as is $S_n$ with $n$ replaced by $p$, and $(S_p\mathbf{u}_i)_0 := 0$. Recall that for each $i = 1, \ldots, n$ and $j = 1, \ldots, p$, $v_i(k+j) = v_i(k) + \tau\sum_{s=0}^{j-1} a_i(k+s, \mathbf{u}_i(k)) = v_i(k) + \tau\big(S_p\,\mathbf{a}_i(\mathbf{u}_i)\big)_j$. Further, $x_{i-1}(k+j) - x_i(k+j) = z_i(k+j) + \Delta$ depends only on $\mathbf{u}_i(k)$ and $\mathbf{u}_{i-1}(k)$ as shown in (5). Hence, we see that for each $i = 1, \ldots, n$ and each $j = 1, \ldots, p$, the safety distance constraint is given by:

$$\begin{aligned} \big(H_i(\mathbf{u}_{i-1}(k), \mathbf{u}_i(k))\big)_j &:= L_i + r_i \cdot v_i(k+j) \\ &\quad - \frac{(v_i(k+j) - v_{\min})^2}{2\, a_{i,\min}} - [x_{i-1}(k+j) - x_i(k+j)] \le 0. \end{aligned}$$

Note that $H_1(\cdot)$ depends only on $\mathbf{u}_1(k)$ although it is written in the above form for notational convenience. Combining the above results and setting $\gamma \equiv 0$, the MPC model (4) is formulated as the following optimization problem:

$$\begin{aligned} \min J(\mathbf{u}) &= \frac{1}{2}\mathbf{a}^T(\mathbf{u})\big(W - \tau^2\Psi\big)\mathbf{a}(\mathbf{u}) + c^T\mathbf{a}(\mathbf{u}) + \frac{\tau^2}{2}\mathbf{u}^T\Psi\mathbf{u}, \\ \text{s.t.} \quad & \mathbf{u}_i \in \mathcal{X}_i, \quad v_{\min} \le v_i(k) + \tau\big(S_p\mathbf{a}_i(\mathbf{u}_i)\big)_s \le v_{\max}, \\ & (H_i(\mathbf{u}_{i-1}, \mathbf{u}_i))_s \le 0, \forall\, i = 1, \ldots, n,\ s = 1, \ldots, p, \end{aligned} \tag{7}$$

where $\mathcal{X}_i := \{\mathbf{u}_i \in \mathbb{R}^p \,|\, a_{i,\min}\mathbf{1} \le \mathbf{u}_i \le a_{i,\max}\mathbf{1}\}$ for each $i = 1, \ldots, n$. It can be shown via the expressions of $W$ and $\Psi$ given in [23, Section 3.1] that $W - \tau^2\Psi$ is PSD. When $p = 1$, (7) is a convex optimization problem. whereas when $p > 1$, (7) yields a nonconvex optimization problem. Since $J$ is continuous, each $\mathcal{X}_i$ is compact, and the other constraints are defined by continuous functions, (7) has a solution. Moreover, the objective function $J$ is densely coupled, and the safety distance constraint function $\big(H_i(\mathbf{u}_{i-1}, \mathbf{u}_i)\big)_j$ is locally coupled with its neighboring vehicles. This coupling structure, together with the nonconvexity of (7), leads to many challenges in developing fully distributed schemes.

## V. Fully Distributed Algorithms for Coupled Nonconvex MPC Optimization Problem

In this section, we develop fully distributed algorithms for solving the underlying coupled, nonconvex optimization problem (7) at each time $k \in \mathbb{Z}_+$ when $p > 1$. To achieve this goal, various new techniques are exploited: the formulation of locally coupled, albeit nonconvex, optimization, sequential convex programming, and operator splitting methods.

### A. Formulation of MPC Optimization Problem as a Locally Coupled Optimization Problem

Since the safety distance constraint of each vehicle $i$ is coupled with its neighboring vehicle $(i-1)$ whereas the acceleration and velocity constraints are decoupled, the constraints of the optimization problem (7) are locally coupled [8]. Motivated by distributed computation for locally coupled *convex* optimization [8], [23], we show that (7) can be formulated as a locally coupled *nonconvex* optimization problem.

The framework of a locally coupled optimization problem requires that both its objective function and constraints are expressed in a locally coupled manner satisfying the communication network topology constraint. However, the objective function in the underlying optimization problem (7) is densely coupled. As indicated in [23, Section 4] for convex optimization, this difficulty is overcome via certain matrix decomposition techniques. It is shown in [23, Lemma 4.1] that under **A.3**, the PSD or PD matrix $W \in \mathbb{R}^{np\times np}$ in (7) can be decomposed as $W = \sum_{s=1}^{n} \widetilde{W}^s$, where all $\widetilde{W}^s \in \mathbb{R}^{np\times np}$ are PSD and satisfy the following conditions: $\widetilde{W}^1 = \begin{bmatrix} \widehat{W}^1 & \\ & \mathbf{0}_{(n-2)p\times(n-2)p} \end{bmatrix}$, $\widetilde{W}^n = \begin{bmatrix} \mathbf{0}_{(n-2)p\times(n-2)p} & \\ & \widehat{W}^n \end{bmatrix}$, and for each $s = 2, \ldots, n-1$, $\widetilde{W}^s = \begin{bmatrix} \mathbf{0}_{(s-2)p\times(s-2)p} & & \\ & \widehat{W}^s & \\ & & \mathbf{0}_{(n-s-1)p\times(n-s-1)p} \end{bmatrix}$, where $\widehat{W}^1 := \begin{bmatrix} (\widetilde{W}^1)_{1,1} & (\widetilde{W}^1)_{1,2} \\ (\widetilde{W}^1)_{2,1} & (\widetilde{W}^1)_{2,2} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}$, $\widehat{W}^n := \begin{bmatrix} (\widetilde{W}^n)_{n-1,n-1} & (\widetilde{W}^n)_{n-1,n} \\ (\widetilde{W}^n)_{n,n-1} & (\widetilde{W}^n)_{n,n} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}$, and $\widehat{W}^s := \begin{bmatrix} (\widetilde{W}^s)_{s-1,s-1} & (\widetilde{W}^s)_{s-1,s} & 0 \\ (\widetilde{W}^s)_{s,s-1} & (\widetilde{W}^s)_{s,s} & (\widetilde{W}^s)_{s,s+1} \\ 0 & (\widetilde{W}^s)_{s+1,s} & (\widetilde{W}^s)_{s+1,s+1} \end{bmatrix} \in \mathbb{R}^{3p\times 3p}$, and each $(\widetilde{W}^n)_{i,j} \in \mathbb{R}^{p\times p}$. When $W$ is PD, it is shown in [23, Lemma 4.1] that there exist $\widetilde{W}^s$'s such that each $\widehat{W}^s$ is PD.

Since $\overline{Q}_w$ is diagonal and PD, it follows from the similar argument in [23, Lemma 4.1] that the PD matrix $\Psi \in \mathbb{R}^{np\times np}$ can be decomposed in the similarly way. Specifically, there exist matrices $\widetilde{\Psi}^s$ such that $\Psi = \sum_{s=1}^{n} \widetilde{\Psi}^s$, where $\widetilde{\Psi}^s$'s satisfy the abovementioned conditions with $\widetilde{W}^s$ (resp. $\widehat{W}^s$) replaced by $\widetilde{\Psi}^s$ (resp. $\widehat{\Psi}^s$). Hence, the objective function $J(\mathbf{u})$ in (7) can be decomposed as

$$J(\mathbf{u}) = J_1(\mathbf{u}_1, \mathbf{u}_2) + \sum_{i=2}^{n-1} J_i(\mathbf{u}_{i-1}, \mathbf{u}_i, \mathbf{u}_{i+1}) + J_n(\mathbf{u}_{n-1}, \mathbf{u}_n),$$

where the functions $J_i$'s on the right hand side are given by

$$\begin{aligned} J_1(\mathbf{u}_1, \mathbf{u}_2) &:= \frac{1}{2}\left[\mathbf{a}_1^T(\mathbf{u}_1)\mathbf{a}_2^T(\mathbf{u}_2)\right]\left(\widehat{W}^1 - \tau^2\widehat{\Psi}^1\right)\begin{bmatrix}\mathbf{a}_1(\mathbf{u}_1)\\ \mathbf{a}_2(\mathbf{u}_2)\end{bmatrix} \\ &\quad + c_{\mathcal{I}_1}^T \mathbf{a}_1(\mathbf{u}_1) + \frac{\tau^2}{2}\left[\mathbf{u}_1^T \mathbf{u}_2^T\right]\widehat{\Psi}^1\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}, \end{aligned}$$

and for $i = 2, \ldots, n-1$,

$$\begin{aligned} J_i(\mathbf{u}_{i-1}, \mathbf{u}_i, \mathbf{u}_{i+1}) &:= \frac{\tau^2}{2}\left[\mathbf{u}_{i-1}^T\ \mathbf{u}_i^T\ \mathbf{u}_{i+1}^T\right]\widehat{\Psi}^i\begin{bmatrix}\mathbf{u}_{i-1}\\ \mathbf{u}_i\\ \mathbf{u}_{i+1}\end{bmatrix} \\ &\quad + c_{\mathcal{I}_i}^T \mathbf{a}_i(\mathbf{u}_i) \\ &\quad + \frac{1}{2}\left[\mathbf{a}_{i-1}^T(\mathbf{u}_{i-1})\ \mathbf{a}_i^T(\mathbf{u}_i)\ \mathbf{a}_{i+1}^T(\mathbf{u}_{i+1})\right] \\ &\quad \times\left(\widehat{W}^i - \tau^2\widehat{\Psi}^i\right)\times\begin{bmatrix}\mathbf{a}_{i-1}(\mathbf{u}_{i-1})\\ \mathbf{a}_i(\mathbf{u}_i)\\ \mathbf{a}_{i+1}(\mathbf{u}_{i+1})\end{bmatrix}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} J_n(\mathbf{u}_{n-1}, \mathbf{u}_n) &:= c_{\mathcal{I}_n}^T \mathbf{a}_n(\mathbf{u}_n) + \frac{\tau^2}{2}\left[\mathbf{u}_{n-1}^T\ \mathbf{u}_n^T\right]\widehat{\Psi}^n\begin{bmatrix}\mathbf{u}_{n-1}\\ \mathbf{u}_n\end{bmatrix} \\ &\quad + \frac{1}{2}\left[\mathbf{a}_{n-1}^T(\mathbf{u}_{n-1})\ \mathbf{a}_n^T(\mathbf{u}_n)\right]\left(\widehat{W}^n - \tau^2\widehat{\Psi}^n\right) \\ &\quad \times\begin{bmatrix}\mathbf{a}_{n-1}(\mathbf{u}_{n-1})\\ \mathbf{a}_n(\mathbf{u}_n)\end{bmatrix}. \end{aligned}$$

By **A.1**, the above decomposition of $J$ satisfies the communication network topology constraint.

We use the above decomposition to formulate a locally coupled optimization problem by introducing copies of local variables. We consider the cyclic like network topology through this subsection, although the proposed formulation and schemes can be easily extended to other network topologies satisfying the assumption **A.1**. In this case, $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_n = \{n-1\}$, and $\mathcal{N}_i = \{i-1, i+1\}$ for $i = 2, \ldots, n-1$. Hence, each $J_i$ in the decomposition of $J$ can be written as $J_i(\mathbf{u}_i, (\mathbf{u}_j)_{j\in\mathcal{N}_i})$.

Recall that for each $i = 1, \ldots, n$, $\mathcal{X}_i := \{\mathbf{u}_i \in \mathbb{R}^p \,|\, a_{i,\min}\mathbf{1} \le \mathbf{u}_i \le a_{i,\max}\mathbf{1}\}$. Further, define

$$\begin{aligned} \mathcal{Y}_i &:= \{\mathbf{u}_i \in \mathbb{R}^p \,\big|\, v_{\min} \le v_i(k) + \tau\big(S_p\,\mathbf{a}_i(\mathbf{u}_i)\big)_s \le v_{\max}, \\ &\qquad \forall\, s = 1, \ldots, p\}, \end{aligned} \tag{9}$$

$$\begin{aligned} \mathcal{Z}_i &:= \{(\mathbf{u}_{i-1}, \mathbf{u}_i) \in \mathbb{R}^p \times \mathbb{R}^p \,\big|\, (H_i(\mathbf{u}_{i-1}, \mathbf{u}_i))_s \le 0, \\ &\qquad \forall\, s = 1, \ldots, p\}. \end{aligned} \tag{10}$$

As indicated before, $\mathcal{Z}_1$ depends only on $\mathbf{u}_1$ although it is written in the above form for notational convenience. Let $\boldsymbol{\delta}_S$ denote the indicator function of a closed set $S$. Define, for each $i = 1, \ldots, n$, $\widehat{J}_i(\mathbf{u}_i, (\mathbf{u}_j)_{j\in\mathcal{N}_i}) := J_i(\mathbf{u}_i, (\mathbf{u}_j)_{j\in\mathcal{N}_i}) + \boldsymbol{\delta}_{\mathcal{X}_i}(\mathbf{u}_i) + \boldsymbol{\delta}_{\mathcal{Y}_i}(\mathbf{u}_i) + \boldsymbol{\delta}_{\mathcal{Z}_i}(\mathbf{u}_{i-1}, \mathbf{u}_i)$. For each $i = 1, \ldots, n$, define $\widehat{\mathbf{u}}_i := \big(\mathbf{u}_i, (\mathbf{u}_{i,j})_{j\in\mathcal{N}_i}\big)$, where the new variables $\mathbf{u}_{i,j}$ represent the predicted values of $\mathbf{u}_j$ of vehicle $j$ in the neighbor $\mathcal{N}_i$ of vehicle $i$, and let $\widehat{\mathbf{u}} := (\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_n) \in \mathbb{R}^N$. Define the consensus subspace

$$\mathcal{A} := \Big\{\widehat{\mathbf{u}} \in \mathbb{R}^N \,\big|\, \mathbf{u}_{i,j} = \mathbf{u}_j,\ \forall\, (i,j) \in \mathcal{E}\Big\}.$$

Then the underlying problem (7) can be equivalently written as the following locally coupled optimization problem:

$$\min_{\widehat{\mathbf{u}}} \ \sum_{i=1}^{n} \widehat{J}_i(\widehat{\mathbf{u}}_i), \qquad \text{subject to} \quad \widehat{\mathbf{u}} \in \mathcal{A}, \tag{11}$$

where the functions $\widehat{J}_i$'s are decoupled, and the consensus constraint $\mathcal{A}$ gives rise to the only coupling in this formulation.

### *B. Sequential Convex Programming and Operator Splitting Method Based Fully Distributed Algorithms for the MPC Optimization Problem*

When the MPC horizon $p = 1$, the underlying MPC optimization problem (7) or (11) is a convex quadratically constrained quadratic program (QCQP), for which the fully distributed schemes developed in [23] can be applied. We consider $p > 1$ from now on. In this case, the underlying MPC optimization problem (7) or (11) yields a non-convex minimization problem whose objective function and constraints are non-convex, whereas the coefficients $c_{2,i} > 0$ and $c_{3,i} > 0$ defining the nonlinearities are small. Therefore, it is expected that an optimal solution under the nonlinear vehicle dynamics is "close" to that under the linear vehicle dynamics. The latter solution, which can be obtained using fully distributed schemes [23], can be used as an initial guess for a distributed scheme for the nonlinear vehicle dynamics. We formally discuss this observation as follows.

Let $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ with $i = 1, \ldots, m$ be all continuous functions. Let $\Omega \subset \mathbb{R}^n$ be a compact set, and $\Theta \subseteq \mathbb{R}^q$ be a set of parameter vectors that contains the zero vector. Given $\theta \in \Theta$, define the parameter dependent constraint set $\mathcal{W}_\theta := \{x \in \mathbb{R}^n \mid g_i(x, \theta) \le 0, \ \forall i = 1, \ldots, m\}$. We assume that for each parameter vector $\theta \in \Theta$, the set $\Omega \cap \mathcal{W}_\theta$ is nonempty. Since $g_i(\cdot, \theta)$ is continuous for a given $\theta$, $\Omega \cap \mathcal{W}_\theta$ is a nonempty compact set such that for a fixed $\theta \in \Theta$, the minimization problem

$$P_\theta : \quad \min_{x \in \Omega \cap \mathcal{W}_\theta} f(x, \theta)$$

has a nonempty closed solution set denoted by $\mathcal{S}_\theta$. For each $x \in \Omega \cap \mathcal{W}_0$, define the index set $\mathcal{I}(x) := \{i \mid g_i(x, 0) = 0\} \subseteq \{1, \ldots, m\}$ corresponding to the index set of active inequality constraints. We introduce the following assumption on $\Omega \cap \mathcal{W}_0$.

**A.4** For any $x^\diamond \in \Omega \cap \mathcal{W}_0$ whose corresponding $\mathcal{I}(x^\diamond)$ is nonempty, there exists a sequence $(w^\ell)$ in $\Omega \cap \mathcal{W}_0$ such that: (i) for each $\ell$, $g_i(w^\ell, 0) < 0$ for all $i = 1, \ldots, m$; and (ii) $(w^\ell)$ converges to $x^\diamond$.

The following result, whose proof is given in [22, Proposition 5.1], establishes the closeness of optimization solutions under perturbed parameters.

*Proposition 2: Suppose $P_0$ has the unique minimizer $x_*$, i.e., $\mathcal{S}_0 = \{x_*\}$. Then under the abovementioned assumptions (including* **A.4***), for any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $\theta \in \Theta$ with $\|\theta\| \le \eta$, $\sup_{z \in \mathcal{S}_\theta} \|z - x_*\| < \varepsilon$.*

We apply this proposition to the optimization problem (7). Recall that the parameter vector $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_d, \boldsymbol{\varphi}_f) = (c_{2,i}, c_{3,i})_{i=1}^n \in \mathbb{R}_+^{2n}$. To emphasize the dependence of the objective function $J$ on $\boldsymbol{\varphi}$, we write it as $J(\mathbf{u}, \boldsymbol{\varphi})$. Further, the constraints in (7) can be written as $\mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z}$, where $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is convex and compact, and $\mathcal{Y} \cap \mathcal{Z} = \{\mathbf{u} \mid g_i(\mathbf{u}, \boldsymbol{\varphi}) \le 0, \ i = 1, \ldots, m\}$ for some real-valued functions $g_i$ depending on $\boldsymbol{\varphi}$. When $\boldsymbol{\varphi} = 0$, $J(\mathbf{u}, 0)$ is strongly convex, and each $g_i(\mathbf{u}, 0)$ is an affine or a convex quadratic function [23] such that (7) becomes a convex problem with a unique solution $\mathbf{u}_{*,0}$. Further, when $r_i \ge \tau$ for all $i$ and $v_0(k) > v_{\min}$, this convex problem has non-empty interior [23, Corollary 3.1] such that **A.4** holds. Letting $\mathcal{S}_{\boldsymbol{\varphi}}$ be the solution set of (7) corresponding to the parameter vector $\boldsymbol{\varphi}$, we obtain the following corollary from Proposition 2.

*Corollary 2: Consider the optimization problem (7) with the parameter vector $\boldsymbol{\varphi} \in \mathbb{R}_+^{2n}$ at time $k$. Suppose $r_i \ge \tau$ for all $i$ and $v_0(k) > v_{\min}$. Then for any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $\boldsymbol{\varphi} \in \mathbb{R}_+^{2n}$ with $\|\boldsymbol{\varphi}\| \le \eta$, $\sup_{\mathbf{u} \in \mathcal{S}_{\boldsymbol{\varphi}}} \|\mathbf{u} - \mathbf{u}_{*,0}\| < \varepsilon$.*

To solve the coupled non-convex optimization problem (7) or (11) with $\boldsymbol{\varphi} \neq 0$, we exploit the sequential convex programming (SCP) method [17]. We provide a brief description of the SCP method below. Consider the nonlinear program

$$(P') : \min_{x \in \mathbb{R}^n} f(x), \text{s.t.} x \in \mathcal{P}, \ g_i(x) - r_i(x) \le 0, i = 1, \ldots, \ell, \tag{12}$$

where $\mathcal{P} \subseteq \mathbb{R}^n$ is a closed convex set, $f$ and each $g_i$ are $C^1$ (but not necessarily convex) functions, and each $r_i$ is a convex $C^1$-function. We assume that $\nabla f$ and $\nabla g_i$ are Lipschitz on $\mathcal{P}$, i.e. there exist constants $L_f > 0$ and $L_{g_i} > 0$ such that $\|\nabla f(x) - \nabla f(x')\|_2 \le L_f \|x - x'\|_2$ and $\|\nabla g_i(x) - \nabla g_i(x')\|_2 \le L_{g_i} \|x - x'\|_2$ for all $x, x' \in \mathcal{P}$ and $i = 1, \ldots, \ell$. Let $\widehat{x}$ be a feasible point of $(P')$, i.e., $\widehat{x} \in \mathcal{P}'$ and $g_i(\widehat{x}) - r_i(\widehat{x}) \le 0, \ i = 1, \ldots, \ell$. Consider an approximation of the constraint set of $(P')$ at $\widehat{x}$:

$$\begin{aligned} &\mathcal{C}(\widehat{x}, \{\nabla g_i(\widehat{x})\}_{i=1}^\ell, \{\nabla r_i(\widehat{x})\}_{i=1}^\ell) \\ &\quad := \Big\{ z \in \mathcal{P} \mid g_i(\widehat{x}) + \nabla g_i(\widehat{x})^T (z - \widehat{x}) + \frac{L_{g_i}}{2} \|z - \widehat{x}\|_2^2 \\ &\qquad - [r_i(\widehat{x}) + \nabla r_i(\widehat{x})^T (z - \widehat{x})] \le 0, \ i = 1, \ldots, \ell \Big\}, \end{aligned}$$

which is a nonempty closed convex set [17, Lemma 3.3]. The next lemma gives a simple sufficient condition for the Slater's condition to hold for the approximated constraint set.

*Lemma 1: [22, Lemma 5.2] Given a feasible point $\widehat{x}$ of $(P')$, suppose $\mathcal{C}(\widehat{x}, \{\nabla g_i(\widehat{x})\}_{i=1}^\ell, \{\nabla r_i(\widehat{x})\}_{i=1}^\ell)$ is not singleton. Then the Slater's condition holds for $\mathcal{C}(\widehat{x}, \{\nabla g_i(\widehat{x})\}_{i=1}^\ell, \{\nabla r_i(\widehat{x})\}_{i=1}^\ell)$, i.e., there exists $\widehat{z} \in \mathcal{P}$ such that $g_i(\widehat{x}) + \nabla g_i(\widehat{x})^T (\widehat{z} - \widehat{x}) + \frac{L_{g_i}}{2} \|\widehat{z} - \widehat{x}\|_2^2 - [r_i(\widehat{x}) + \nabla r_i(\widehat{x})^T (\widehat{z} - \widehat{x})] < 0, \forall i = 1, \ldots, \ell$.*

The SCP scheme solves $(P')$ in (12) as follows [17]: let an approximation of the objective function $f$ for a given feasible point $\widehat{x}$ be $\widetilde{f}(z; \widehat{x}) := f(\widehat{x}) + [\nabla f(\widehat{x})]^T (z - \widehat{x}) + \frac{L_f}{2} \|z - \widehat{x}\|_2^2$, which is strongly convex in $z$. At each step, the SCP scheme solves the convex problem at $x^k$ using $\widetilde{f}(\cdot; x^k)$ over the convex constraint set $\mathcal{C}(x^k, \{\nabla g_i(x^k)\}_{i=1}^\ell, \{\nabla r_i(x^k)\}_{i=1}^\ell)$ to generate a unique solution $x^{k+1}$. It then updates the gradients $\nabla f$, $\nabla g_i$, and $\nabla r_i$ using $x^{k+1}$, and formulates another convex problem and solves it again. It is shown in [17, Theorem 3.4] that any accumulation point $x^*$ of the sequence $(x^k)$ generated by the SCP scheme is a KKT point of $(P')$, provided that $x^*$ satisfies the Slater's condition for $\mathcal{C}(x^*, \{\nabla g_i(x^*)\}_{i=1}^\ell, \{\nabla r_i(x^*)\}_{i=1}^\ell)$.

We now apply the SCP scheme to develop a fully distributed scheme for the locally coupled formulation (11) of the MPC optimization problem (7). Recall that $\widehat{\mathbf{u}}_i := (\mathbf{u}_i, (\mathbf{u}_{i,j})_{j \in \mathcal{N}_i})$, and $\widehat{\mathbf{u}} := (\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_n)$. For each $i = 1, \ldots, n$, it follows from the velocity constraint $\mathcal{Y}_i$ in (9) and the safety distance constraint $\mathcal{Z}_i$ in (10) that there are real-valued smooth functions $g_{i,s}$ and convex quadratic functions $r_{i,s}$ for $s = 1, \ldots, 3p$ such that $\widehat{\mathbf{u}}_i \in \mathcal{Y}_i \cap \mathcal{Z}_i$ if and only if $g_{i,s}(\widehat{\mathbf{u}}_i) - r_{i,s}(\widehat{\mathbf{u}}_i) \le 0$ for $s = 1, \ldots, 3p$; specific choices of $g_{i,s}$ and $r_{i,s}$ are given in Section VII-B. In view of the objective function $J(\widehat{\mathbf{u}}) = \sum_{i=1}^n J_i(\widehat{\mathbf{u}}_i)$, the problem (11) becomes

$$\min \sum_{i=1}^n J_i(\widehat{\mathbf{u}}_i), \ \text{s.t.} \ \widehat{\mathbf{u}} \in \mathcal{A}, \ \widehat{\mathbf{u}}_i \in \mathcal{X}_i, \ g_{i,s}(\widehat{\mathbf{u}}_i) - r_{i,s}(\widehat{\mathbf{u}}_i) \le 0, \quad \forall i = 1, \ldots, n, \quad s = 1, \ldots, 3p.$$

Recall that $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is a convex compact set. Since $\mathcal{X}$ is compact and $\mathcal{A}$ is the consensus subspace, it is easy to show that there are positive Lipschitz constants $L_{J_i}$ and $L_{g_{i,s}}$ for the gradients of $J_i$ and $g_{i,s}$ on $\mathcal{A} \cap \mathcal{X}$, i.e., for all $\widehat{\mathbf{u}}, \widehat{\mathbf{u}}' \in \mathcal{A} \cap \mathcal{X}$, all $i = 1, \ldots, n$, and $s = 1, \ldots, 3p$,

$$\begin{aligned} \|\nabla J_i(\widehat{\mathbf{u}}_i) - \nabla J_i(\widehat{\mathbf{u}}'_i)\|_2 &\le L_{J_i} \cdot \|\widehat{\mathbf{u}}_i - \widehat{\mathbf{u}}'_i\|_2, \\ \|\nabla g_{i,s}(\widehat{\mathbf{u}}_i) - \nabla g_{i,s}(\widehat{\mathbf{u}}'_i)\|_2 &\le L_{g_{i,s}} \cdot \|\widehat{\mathbf{u}}_i - \widehat{\mathbf{u}}'_i\|_2. \end{aligned}$$

To develop a SCP based fully distributed scheme, we introduce more notation. Given any $\widehat{\mathbf{u}} = (\widehat{\mathbf{u}}_i)_{i=1}^n \in \mathcal{X}$ and any vectors $d_{J_i}, d_{g_{i,s}}$, and $d_{r_{i,s}}$ for $i = 1, \ldots, n$ and $s = 1, \ldots, 3p$, let the following function be a convex approximation of the original nonconvex $J$, where $y = (y_1, \dots, y_n) \in \mathbb{R}^N$ with each $y_i$ being a suitable subvector of $y$:

$$f(y;\, \widehat{\mathbf{u}}, \{d_{J_i}\}_{i=1}^n) := \sum_{i=1}^n \Big( J_i(\widehat{\mathbf{u}}_i) + d_{J_i}^T(\widehat{\mathbf{u}}_i)(y_i - \widehat{\mathbf{u}}_i) + \frac{L_{J_i}}{2}\|y_i - \widehat{\mathbf{u}}_i\|_2^2 \Big),$$

and the following sets as convex approximations of the original nonconvex constraint sets $\mathcal{Y} \cap \mathcal{Z}$:

$$\begin{aligned}\mathcal{C}\big(\widehat{\mathbf{u}}, \{d_{g_{i,s}}, d_{r_{i,s}},\ i = 1,\dots,n,\ s = 1,\dots,3p\}\big) := \Big\{ & y \in \mathcal{X} \mid g_{i,s}(\widehat{\mathbf{u}}_i) + d_{g_{i,s}}^T(y_i - \widehat{\mathbf{u}}_i) + \frac{L_{g_{i,s}}}{2}\|y_i - \widehat{\mathbf{u}}_i\|_2^2 \\ & -\big[r_{i,s}(\widehat{\mathbf{u}}_i) + d_{r_{i,s}}^T(y_i - \widehat{\mathbf{u}}_i)\big] \le 0, i = 1,\dots,n, s = 1,\dots,3p \Big\}.\end{aligned}$$

Clearly, $f$ is strongly convex in $y$ and decoupled in $y_i$'s, and the convex set $\mathcal{C}\big(\widehat{\mathbf{u}}, \{d_{g_{i,s}}, d_{r_{i,s}},\ i = 1,\dots,n,\ s = 1,\dots,p\}\big)$ is the Cartesian product of $\mathcal{C}_i$'s for $i = 1,\dots,n$, where each $\mathcal{C}_i\big(\widehat{\mathbf{u}}_i, \{d_{g_{i,s}}\}_{s=1}^{3p}, \{d_{r_{i,s}}\}_{s=1}^{3p}\big) := \big\{ y_i \in \mathcal{X}_i \mid g_{i,s}(\widehat{\mathbf{u}}_i) + d_{g_{i,s}}^T(y_i - \widehat{\mathbf{u}}_i) + \frac{L_{g_{i,s}}}{2}\|y_i - \widehat{\mathbf{u}}_i\|_2^2 - \big[r_{i,s}(\widehat{\mathbf{u}}_i) + d_{r_{i,s}}^T(y_i - \widehat{\mathbf{u}}_i)\big] \le 0, s = 1,\dots,3p \big\}$.

Using the above notation, the iterative scheme of the SCP method is: for a feasible initial guess $\widehat{\mathbf{u}}^0$,

$$\begin{aligned}\widehat{\mathbf{u}}^{k+1} = \operatorname*{argmin}_y \Big\{ & f(y; \widehat{\mathbf{u}}^k, \{\nabla J_i(\widehat{\mathbf{u}}_i^k)\}_{i=1}^n) \,\big|\, y \in \mathcal{A}, \quad \text{and} \\ & y \in \mathcal{C}\big(\widehat{\mathbf{u}}^k, \{\nabla g_{i,s}(\widehat{\mathbf{u}}_i^k), \nabla r_{i,s}(\widehat{\mathbf{u}}_i^k), i = 1,\dots,n, \\ & s = 1,\dots,3\ p\}\big)\Big\}. \end{aligned} \tag{13}$$

By virtue of Corollary 2, the initial $\widehat{\mathbf{u}}^0$ can be chosen as a solution to the problem (11) with $\boldsymbol{\varphi} = 0$. An efficient fully distributed scheme has been developed in [23] to compute such $\widehat{\mathbf{u}}^0$. It is shown in [17, Theorem 4.3] that if $\widehat{\mathbf{u}}^0$ is feasible, then $\widehat{\mathbf{u}}^k$ is feasible for all $k$ and the constraint set in each step $k$ is a nonempty closed convex set [17, Lemma 3.3].

The convex minimization problem (13) at each step $k$ can be solved via operator splitting method based fully distributed schemes. Fix $\widehat{\mathbf{u}}^k = (\widehat{\mathbf{u}}_i^k)_{i=1}^n$ and the related gradients evaluated at $\widehat{\mathbf{u}}^k$. We write the objective function $f(y;\, \widehat{\mathbf{u}}, \{d_{J_i}\}_{i=1}^n)$ as $f(y)$ and the constraint sets $\mathcal{C}_i\big(\widehat{\mathbf{u}}_i^k, \{\nabla g_{i,s}(\widehat{\mathbf{u}}_i^k), \nabla r_{i,s}(\widehat{\mathbf{u}}_i^k),\ s = 1,\dots,3\ p\}\big)$ as $\mathcal{C}_i$'s for notational simplicity. Clearly, $\widehat{\mathbf{u}}_i^k \in \mathcal{C}_i$ for each $i$. If $\mathcal{C}_i$ is singleton for some $i$, i.e., $\mathcal{C}_i = \{\widehat{\mathbf{u}}_i^k\}$, then we have $\widehat{\mathbf{u}}_i^{k+1} = \widehat{\mathbf{u}}_i^k$ such that the optimization problem can be reduced to a simpler problem. When $\mathcal{C}_i$ is non-singleton, it follows from Lemma 1 that the Slater's condition holds for that $\mathcal{C}_i$. Let $F(y) := f(y; \widehat{\mathbf{u}}^k, \{\nabla J_i(\widehat{\mathbf{u}}_i^k)\}_{i=1}^n) + \boldsymbol{\delta}\mathcal{C}(y) + \boldsymbol{\delta}\mathcal{A}(y)$. By [19, Corollary 23.8.1], $\partial F(y) = \{\nabla f(y)\} + \mathcal{N}_{\mathcal{C}}(y) + \mathcal{N}_{\mathcal{A}}(y)$. Hence, several operator splitting method based fully distributed algorithms [4], [8] can be applied to solve (13).

Motivated by [23], we consider the (generalized) Douglas-Rachford splitting method based distributed scheme. Specifically, define for each $i = 1,\dots,n$, $f_i(y_i) := J_i(\widehat{\mathbf{u}}_i^k) + d_{J_i}^T(\widehat{\mathbf{u}}_i^k)(y_i - \widehat{\mathbf{u}}_i) + \frac{L_{J_i}}{2}\|y_i - \widehat{\mathbf{u}}_i^k\|_2^2$, and $\widehat{f}_i(y) := f_i(y_i) + \boldsymbol{\delta}\mathcal{C}_i(y_i)$. Hence, the objective function $f(y) = \sum_{i=1}^n f_i(y_i)$. For any constant $0 < \alpha < 1$ and $\rho > 0$, the Douglas-Rachford splitting method based scheme is given by: for $t \in \mathbb{Z}_+$,

$$\begin{aligned} w^{t+1} &= \Pi_{\mathcal{A}}(z^t), \\ z^{t+1} &= z^t + 2\alpha \cdot \Big[\operatorname{Prox}_{\rho \widehat{f}_1 + \cdots + \rho \widehat{f}_n}\big(2w^{t+1} - z^t\big) - w^{t+1}\Big], \end{aligned}$$

where $\operatorname{Prox}_h$ denotes the proximal operator of a proper lower semicontinuous convex function $h$, and $\Pi_{\mathcal{A}}$ denotes the Euclidean projection onto $\mathcal{A}$. Since $\mathcal{A}$ is the consensus subspace, it is shown that [8, Section IV] that for any $\widehat{\mathbf{u}} := (\widehat{\mathbf{u}}_1, \dots, \widehat{\mathbf{u}}_n)$ where $\widehat{\mathbf{u}}_i := \big(\mathbf{u}_i, (\mathbf{u}_{ij})_{j \in \mathcal{N}_i}\big)$, $\overline{\mathbf{u}} := \Pi_{\mathcal{A}}(\widehat{\mathbf{u}})$ is given by:

$$\overline{\mathbf{u}}_j = \overline{\mathbf{u}}_{ij} = \frac{1}{1 + |\mathcal{N}_j|}\Big(\widehat{\mathbf{u}}_j + \sum_{k \in \mathcal{N}_j} \widehat{\mathbf{u}}_{kj}\Big), \forall\, (i, j) \in \mathcal{E}. \tag{14}$$

Furthermore, since $\widehat{f}_i$'s are decoupled, a distributed version of the above algorithm is given by: for each $i = 1,\dots,n$,

$$w_i^{t+1} = \overline{z}_i^t, \tag{15a}$$

$$z_i^{t+1} = z_i^t + 2\alpha \cdot \Big[\operatorname{Prox}_{\rho \widehat{f}_i}\big(2\ w_i^{t+1} - z_i^t\big) - w_i^{t+1}\Big]. \tag{15b}$$

Note that the proximal operator in the 2nd equation of (15) is given by $\operatorname{Prox}_{\rho \widehat{f}_i}(2\ w_i^{t+1} - z_i^t) = \operatorname{argmin}_{\,y_i \in \mathcal{C}_i} f_i(y_i) + \frac{1}{2\rho}\|y_i - (2w_i^{t+1} - z_i^t)\|_2^2$, where $\mathcal{C}_i$ is the intersection of the polyhedral set $\mathcal{X}_i$ and a quadratically constrained convex set. Since $f_i$ is a convex quadratic function, $\operatorname{Prox}_{\rho \widehat{f}_i}(2\ w_i^{t+1} - z_i^t)$ can be formulated as a second-order cone program or QCQP and solved by SeDuMi [27]. See Algorithm 1 for its pseudo-code.

**Algorithm 1** Sequential Convex Programming (SCP) and Douglas-Rachford Splitting Method Based Fully Distributed Algorithm for $p \ge 2$

Choose constants $0 < \alpha < 1$ and $\rho > 0$
Solve the problem (11) with $\boldsymbol{\varphi} = 0$ via a fully distributed scheme and obtain a solution $\widehat{\mathbf{u}}^{\text{lin}}$
Initialize $k = 0$, and set an initial point $\widehat{\mathbf{u}}^0 = \widehat{\mathbf{u}}^{\text{lin}}$
**while** the stopping criteria is not met **do**
  Compute $\nabla J_i(\widehat{\mathbf{u}}_i^k)$, $\nabla g_{i,s}(\widehat{\mathbf{u}}_i^k)$, $\nabla r_{i,s}(\widehat{\mathbf{u}}_i^k)$, and set $z^0 = \widehat{\mathbf{u}}^k$ and $t = 0$.
  **repeat**
    **for** $i = 1,\dots,n$ **do**
      Compute $\overline{z}_i^t$ using (14), and let $w_i^{t+1} \leftarrow \overline{z}_i^t$
    **end for**
    **for** $i = 1,\dots,n$ **do**
      $z_i^{t+1} \leftarrow z_i^t + 2\alpha \cdot \Big[\operatorname{Prox}_{\rho \widehat{f}_i}\big(2\ w_i^{t+1} - z_i^t\big) - w_i^{t+1}\Big]$
    **end for**
    $t \leftarrow t + 1$
  **until** an accumulation point is achieved
  Set $\widehat{\mathbf{u}}^{k+1} = w^t$ and $k \leftarrow k + 1$
**end while**
**return** $\widehat{\mathbf{u}}^* = \widehat{\mathbf{u}}^k$

Since $\mathcal{X}$ is a compact set, the numerical sequence $(\widehat{\mathbf{u}}^k)$ generated by Algorithm 1 always has an accumulation point denoted by $\widehat{\mathbf{u}}^*$. It follows from [17, Theorem 3.4] that under very mild conditions, $\widehat{\mathbf{u}}^*$ is feasible and is a KKT point of (7). Our numerical experiences show that $(\widehat{\mathbf{u}}^k)$ converges to $\widehat{\mathbf{u}}^*$

which is a local minimizer of (7). This coincides with the observation made in Corollary 2 when $c_{2,i}$ and $c_{3,i}$ are small.

*Remark 1:* When $p > 1$, the underlying MPC optimization problem (7) and its locally coupled formulation (11) yield non-convex optimization problems with complicated objective functions and constraints due to highly sophisticated closed-form expressions for $\mathbf{a}_i(\mathbf{u}_i)$'s. Since $c_{2,i}$'s and $c_{3,i}$'s are small, we use suitable approximations, which still lead to a non-convex programming, are accurate enough for transportation applications and facilitate numerical computation. The details of these approximations can be found in [22, Section 5.3].

## VI. Control Design and Stability Analysis of Closed Loop Dynamics

In this section, we discuss the design of the weight matrices $Q_{z,s}$, $Q_{z',s}$ and $Q_{w,s}$ to achieve the closed loop stability. We focus on the constraint free case in view of [6, Section 5]. Recall that $\boldsymbol{\varphi} := (\boldsymbol{\varphi}_d, \boldsymbol{\varphi}_f) \in \mathbb{R}^{2n}_+$, where $\boldsymbol{\varphi}_d := (c_{2,1}, \ldots, c_{2,n}) \in \mathbb{R}^n_+$ and $\boldsymbol{\varphi}_f := (c_{3,1}, \ldots, c_{3,n}) \in \mathbb{R}^n_+$. Further, $c_{2,0} = c_{3,0} = 0$ as indicated before.

### A. Review of the Closed Loop Stability Analysis Under Linear Vehicle Dynamics

When $\boldsymbol{\varphi} = 0$, the nonlinear vehicle dynamics reduces to the linear vehicle dynamics given by (2), for which the closed loop stability of the MPC based platooning control has been analyzed [23, Section 5]. We present a brief review of these stability results as they pave a way for studying closed loop stability under nonlinear vehicle dynamics when $\|\boldsymbol{\varphi}\|$ is small.

Let $\mathbf{w}(k) := (w(k), \ldots, w(k+p-1))$. As before, we omit $k$ when $k$ is fixed. It is shown that under the linear vehicle dynamics, the objective function is [23, Section 5]

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T \mathbf{H}\mathbf{w} + \mathbf{w}^T \left( \mathbf{G} \begin{bmatrix} z(k) \\ z'(k) \end{bmatrix} - u_0(k)\mathbf{g} \right) + \widetilde{\gamma},$$

where $\widetilde{\gamma} \in \mathbb{R}$, and the symmetric PD matrix $\mathcal{H}$, the matrix $\mathcal{G}$ and the vector $\mathbf{g}$ are given in [23, Section 5]. Define the matrix $\mathbf{K}$ and the vector $\mathbf{d}$ as

$$\begin{aligned} \mathbf{K} &:= -\begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix} \mathbf{H}^{-1}\mathbf{G} \in \mathbb{R}^{n\times 2n}, \\ \mathbf{d} &:= \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix} \mathbf{H}^{-1}\mathbf{g} \in \mathbb{R}^{n}. \end{aligned} \tag{16}$$

The closed loop dynamics becomes

$$\begin{bmatrix} z(k+1) \\ z'(k+1) \end{bmatrix} = \underbrace{\left\{ \begin{bmatrix} I_n & \tau I_n \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} \frac{\tau^2}{2} I_n \\ \tau I_n \end{bmatrix} \mathbf{K} \right\}}_{A_{\mathrm{C}}} \begin{bmatrix} z(k) \\ z'(k) \end{bmatrix} + \begin{bmatrix} \frac{\tau^2}{2} I_n \\ \tau I_n \end{bmatrix} u_0(k)\cdot \mathbf{d}, \tag{17}$$

where $A_{\mathrm{C}}$ represents the closed loop dynamics matrix for the linear vehicle dynamics. Conditions on $Q_{z,s}$, $Q_{z',s}$ and $Q_{w,s}$ are given in [23, Section 5] such that $A_{\mathrm{C}}$ is Schur stable. Throughout this section, we assume that for each $p$, $Q_{z,s}$, $Q_{z',s}$ and $Q_{w,s}$ satisfying **A.3** are such that $A_{\mathrm{C}}$ is Schur stable.

### B. Reformulation of the Closed Loop Dynamics

Consider the nonlinear vehicle dynamics (1). It follows from the definitions of $z(k)$, $z'(k)$ and $w(k)$ that for $i = 1, \ldots, n$,

$$z_i(k+1) = z_i(k) + \tau z'_i(k) + \frac{\tau^2}{2} w_{a,i}(k), \tag{18a}$$

$$z'_i(k+1) = z'_i(k) + \tau w_{a,i}(k) \tag{18b}$$

where $w_{a,i}(k) := w_i(k) - [c_{2,i-1}v^2_{i-1}(k) - c_{2,i}v^2_i(k)] - [c_{3,i-1} - c_{3,i}]g$. For given $(v_0(k), u_0(k))$, $k \in \mathbb{Z}_+$, the equilibrium of the above discrete-time system is $(z_e, z'_e) = (0, 0)$ such that $v_{e,i}(k) = v_0(k)$ for all $i = 1, \ldots, n$. Hence, let $w_{e,i}(k) = [c_{2,i-1} - c_{2,i}]v_0^2(k) + [c_{3,i-1} - c_{3,i}]g, \forall\ i = 1, \ldots, n$, and $w_e(k) := (w_{e,1}(k), \ldots, w_{e,n}(k))^T$. By shifting $w(k)$ from the time-varying $w_e(k)$, we define $\widehat{w}(k) := w(k) - w_e(k)$. Further, define the following functions:

$$D(\boldsymbol{\varphi}_d) := \begin{bmatrix} -c_{2,1} & & & \\ & c_{2,1} - c_{2,2} & & \\ & & \ddots & \\ & & & c_{2,n-1} - c_{2,n} \end{bmatrix} S_n, \tag{19}$$

$$\widetilde{h}(z') := \widetilde{D}(\boldsymbol{\varphi}_d)\big[(S_n z') \circ (S_n z')\big], \tag{20}$$

where $\widetilde{D}(\boldsymbol{\varphi}_d) := S_n^{-1}\mathrm{diag}(\boldsymbol{\varphi}_d)$, and $\circ$ denotes the Hadamard product of two vectors in $\mathbb{R}^n$. Note that $D(\boldsymbol{\varphi}_d) = \widetilde{D}(\boldsymbol{\varphi}_d) = 0$ when $\boldsymbol{\varphi}_d = 0$. It is shown in [22, Section 6.2] that the nonlinear vehicle dynamics (1) is described by:

$$\begin{bmatrix} z(k+1) \\ z'(k+1) \end{bmatrix} = \left\{ \begin{bmatrix} I_n & \tau I_n \\ 0 & I_n \end{bmatrix} + v_0(k) \cdot \begin{bmatrix} \frac{\tau^2}{2} I_n \\ \tau I_n \end{bmatrix} \begin{bmatrix} 0 & D(\boldsymbol{\varphi}_d) \end{bmatrix} \right\} \times \begin{bmatrix} z(k) \\ z'(k) \end{bmatrix} + \begin{bmatrix} \frac{\tau^2}{2} I_n \\ \tau I_n \end{bmatrix} \Big( \widehat{w}(k) + \widetilde{h}(z'(k)) \Big).$$

We also write $\widetilde{h}$ as $\widetilde{h}_{\boldsymbol{\varphi}_d}(z')$ to emphasize its dependence on $\boldsymbol{\varphi}_d$. Note that $\widetilde{h}_0(z') \equiv 0$ for any given $z' \in \mathbb{R}^n$.

Define the following matrices:

$$A := \begin{bmatrix} I_n & \tau I_n \\ 0 & I_n \end{bmatrix}, B := \begin{bmatrix} \frac{\tau^2}{2} I_n \\ \tau I_n \end{bmatrix}, \Delta A(\boldsymbol{\varphi}_d) := B \begin{bmatrix} 0 & D(\boldsymbol{\varphi}_d) \end{bmatrix},$$
$$\widehat{A}(k) := A + v_0(k) \cdot \Delta A(\boldsymbol{\varphi}_d). \tag{21}$$

We often write $\widehat{A}(k)$ as $\widehat{A}(v_0(k), \boldsymbol{\varphi}_d)$ to stress its dependence on $v_0(k)$ and $\boldsymbol{\varphi}_d$. Let $\mathbf{z} := (z, z') \in \mathbb{R}^n \times \mathbb{R}^n$. We obtain

$$\mathbf{z}(k+1) = \widehat{A}(k)\mathbf{z}(k) + B\Big(\widehat{w}_*(k) + \widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k))\Big), \tag{22}$$

where $\widehat{w}_*(k)$ is an optimal solution to the unconstrained MPC optimization problem (7) which implicitly depends on $\mathbf{z}(k)$, $v_0(k)$ and $u_0(k)$. For any fixed $\boldsymbol{\varphi}_d$, the closed loop system given by (22) yields a time-varying nonlinear dynamical system, since $\widetilde{h}$ is nonlinear in $z'$ and $v_0(k)$ is time varying.

When $p = 1$, the a closed form expression of $\widehat{w}_*(k)$ is derived in [22, Section 6.2]. When $p > 1$, recall that for any fixed $k \in \mathbb{Z}_+$, $u_0(k+s) = u_0(k)$ for all $s = 1, \ldots, p-1$ in the MPC model. Hence, $v_0(k+s) = v_0(k) + \tau s u_0(k)$ for all $s = 1, \ldots, p-1$. Define $\widehat{A}(k+s) := A + v_0(k+s) \cdot \Delta A(\boldsymbol{\varphi}_d)$ for all $s = 0, 1, \ldots, p-1$. Given $\widehat{A}(k+s)$ with $s = 0, \ldots, p-1$, define the state transition matrix for any $s, s' \in \{0, \ldots, p\}$

with $s \le s'$, $\Phi_{\widehat{A}}(k+s,k+s) := I$, and $\Phi_{\widehat{A}}(k+s',k+s) := \widehat{A}(k+s'-1) \times \cdots \times \widehat{A}(k+s), \forall s' > s$. Based upon these results, we obtain, for any $k \in \mathbb{Z}_+$ and $s = 1, \dots, p$,

$$\begin{aligned}\mathbf{z}(k+s) = \Phi_{\widehat{A}}(k+s,k)\mathbf{z}(k) + \sum_{i=0}^{s-1} \Phi_{\widehat{A}}(k+s,k+i+1) \\ \times B\widehat{w}(k+i) + \sum_{i=0}^{s-1} \Phi_{\widehat{A}}(k+s,k+i+1) \\ \times B\widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k+i)). \end{aligned} \tag{23}$$

In light of (20) and (23), the following lemma can be established via an induction argument on $s$; its proof is omitted.

*Lemma 2: Fix an arbitrary $k \in \mathbb{Z}_+$. For each $s = 1, \dots, p$, $\widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k+s))$ is a vector-valued function whose each entry is a multivariate polynomial in $(z'(k), u_0(k), v_0(k), \widehat{w}(k), \dots, \widehat{w}(k+s-1))$ and $\boldsymbol{\varphi}_d$.*

Further, in view of $u_0(k+s) = u_0(k)$ for any $s \ge 0$ and a fixed $k \in \mathbb{Z}_+$, we have for each $s = 0, \dots, p-1$, $\widetilde{w}(k+s) = w(k+s) - u_0(k+s)\mathbf{e}_1 = \widehat{w}(k+s) + w_e(k+s) - u_0(k)\mathbf{e}_1 = \widehat{w}(k+s) + d(k+s)$, where $d(k+s) := w_e(k+s) - u_0(k)\mathbf{e}_1$. Here we recall that $w_{e,i}(k+s) = [c_{2,i-1} - c_{2,i}]v_0^2(k+s) + [c_{3,i-1} - c_{3,i}]g$, for each $i$, where $v_0(k+s) = v_0(k) + s\tau u_0(k)$.

Consider the unconstrained MPC model. Define the following augmented matrices and vector: for $s = 1, \dots, p$,

$$\overline{Q}_{\mathbf{z},s} := \begin{bmatrix} Q_{z,s} & \\ & Q_{z',s} \end{bmatrix}; \ \overline{Q}_w := \begin{bmatrix} Q_{w,1} & & \\ & \ddots & \\ & & Q_{w,p} \end{bmatrix};$$

and $\widetilde{\mathbf{d}}(k) := \begin{bmatrix} d(k) \\ \vdots \\ d(k+p-1) \end{bmatrix}$. For any fixed $k \in \mathbb{Z}_+$, the objective function in the MPC model is written as

$$\begin{aligned} J\big(\underbrace{\widehat{w}(k), \dots, \widehat{w}(k+p-1)}_{:=\widehat{\mathbf{w}}(k)}\big) &= \frac{1}{2}\Big(\sum_{s=1}^{p} \mathbf{z}(k+s)^T \overline{Q}_{\mathbf{z},s} \mathbf{z}(k+s)\Big) \\ &\quad + \frac{\tau^2}{2}[\widehat{\mathbf{w}}(k) + \widetilde{\mathbf{d}}(k)]^T \overline{Q}_w [\widehat{\mathbf{w}}(k) + \widetilde{\mathbf{d}}(k)]. \end{aligned}$$

Substituting the expression for $\mathbf{z}(k+s)$ given by (23) into the objective function $J$, we obtain the objective function written as $J(\widehat{\mathbf{w}})$ for a fixed $k$. It follows from the previous development and Lemma 2 that $J$ is a polynomial function in $(\widehat{\mathbf{w}}, \mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})$. Moreover, the Hessian of the objective function $J$ with respect to $\widehat{\mathbf{w}}$ is given by

$$HJ(\widehat{\mathbf{w}}) = \left[\frac{\partial J^2(\widehat{\mathbf{w}})}{\partial \widehat{\mathbf{w}}_i \partial \widehat{\mathbf{w}}_j}\right]_{i,j} := \widehat{\mathbf{H}}(\widehat{\mathbf{w}}, \mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi}).$$

For a fixed $k$, we write this Hessian as $\widehat{\mathbf{H}}(\widehat{\mathbf{w}}, \mathbf{z}, v_0, u_0, \boldsymbol{\varphi})$ to emphasize its dependence on these variables. Clearly, $\widehat{\mathbf{H}}$ is an analytic, thus a smooth, function, and for any $(\widehat{\mathbf{w}}, \mathbf{z}, v_0, u_0)$, $\widehat{\mathbf{H}}(\widehat{\mathbf{w}}, \mathbf{z}, v_0, u_0, \boldsymbol{\varphi})|_{\boldsymbol{\varphi}=0} = \mathbf{H}$, where $\mathbf{H}$ is the constant PD matrix given in Section VI-A. If $\boldsymbol{\varphi} = 0$, the objective function $J$ reduces to that for the linear vehicle dynamics whose corresponding optimal solution is given in Section VI-A as

$$\widehat{\mathbf{w}}_*(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi})|_{\boldsymbol{\varphi}=0} = -\mathbf{H}^{-1}\big(\mathbf{G} \cdot \mathbf{z} - u_0 \cdot \mathbf{g}\big).$$

However, the closed form expression of a critical point or a local minimizer $\widehat{\mathbf{w}}_*$ is unavailable for $p > 1$. Hence, we consider non-local (or global) implicit functions to express $\widehat{\mathbf{w}}_*$ in term of $\mathbf{z}, v_0, u_0$ and $\boldsymbol{\varphi}$, since the variables $\mathbf{z}, v_0, u_0$ can be non-local. Toward this end, we exploit global implicit function theorems [9], [21]. Easily verified conditions are given in the following theorem; its proof, resembling that of [9, Theorem 5], exploits the covering map argument [22, Theorem 6.2].

*Theorem 1: Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a connected set, and $\mathcal{V} \subseteq \mathbb{R}^m$ be a closed set. Let $f : \mathcal{U}' \times \mathcal{V}' \to \mathbb{R}^m$ be a $C^r$-function with $r \ge 1$, where $\mathcal{U}' \subseteq \mathbb{R}^n$ and $\mathcal{V}' \subseteq \mathbb{R}^m$ are open sets containing $\mathcal{U}$ and $\mathcal{V}$ respectively. Suppose the following hold:*

(i) *For some $x_* \in \mathcal{U}$, there exists exactly one $y_* \in \mathcal{V}$ such that $f(x_*, y_*) = 0$;*
(ii) *For any $(x, y) \in \mathcal{G}'_f := \{(x, y) \in \mathcal{U}' \times \mathcal{V}' : f(x, y) = 0\}$, $D_y f(x, y)$ is invertible;*
(iii) *There is a positive constant $\rho$ such that $\|(D_y f(x, y))^{-1}\| \cdot \|D_x f(x, y)\| \le \rho$ for all $(x, y) \in \mathcal{G}'_f$.*

*Then there exists a unique $C^r$ function $g : \mathcal{U} \to \mathcal{V}$ such that $f(x, g(x)) = 0, \forall x \in \mathcal{U}$.*

Using the above theorem and [22, Proposition 6.1], we establish a result on global implication function for $\widehat{\mathbf{w}}_*$ below.

*Proposition 3: [22, Proposition 6.2] Let $\mathcal{U}_{\mathbf{z}}$ be a bounded open convex set in $\mathbb{R}^{2n}$, let $\mathcal{U}_0$ be a bounded open convex set containing $[a_{0,\min}, a_{0,\max}]$, and let $\mathcal{V}_0$ be a bounded open convex set containing $[v_{\min}, v_{\max}]$. Let $\mathcal{U}_{\widehat{\mathbf{w}}}$ be a compact set in $\mathbb{R}^{np}$ containing all $\widehat{\mathbf{w}}_*(\mathbf{z}, v_0, u_0, 0)$ for all $\mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, $v_0 \in \mathcal{V}_0$, and $u_0 \in \mathcal{U}_0$. Then there exist a positive constant $\mu_2 > 0$ and a unique smooth function $\mathbf{h} : \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2) \to \mathcal{U}_{\widehat{\mathbf{w}}}$ such that $\widehat{\mathbf{w}}_* = \mathbf{h}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi})$ for all $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$.*

The above proposition implies that the nonconvex optimization problem $\min J(\widehat{\mathbf{w}})$ has a unique local minimizer $\widehat{\mathbf{w}}_*$ in $\mathcal{U}_{\widehat{\mathbf{w}}}$ for any given $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$. Hence, for any $(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0$ at each $k$, $\widehat{\mathbf{w}}_*(k) = \mathbf{h}(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})$, and $\widehat{w}_*(k) = \big[I_n \ 0 \cdots 0\big]\mathbf{h}(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})$. Moreover, note that $\mathbf{h}(\mathbf{z}, v_0, u_0, 0) = -\mathbf{H}^{-1}\big(\mathbf{G} \cdot \mathbf{z} - u_0\mathbf{g}\big)$ for any fixed $(\mathbf{z}, v_0, u_0) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V} \times \mathcal{U}_0$. Define $\Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) := \big[I_n \ 0 \cdots 0\big]\Big(\mathbf{h}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) - \mathbf{h}(\mathbf{z}, v_0, u_0, 0)\Big)$. Since $\mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$ is an open convex set, it follows from the Mean-value Theorem that for any fixed $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$, $\Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) = \int_0^1 D_{\boldsymbol{\varphi}}\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, t\boldsymbol{\varphi})dt \cdot \boldsymbol{\varphi}$. Thus there is a constant $\varkappa > 0$ such that $\|\Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi})\| \le \varkappa\|\boldsymbol{\varphi}\|_\infty$ for all $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$. Substituting the above results to (22), we obtain

$$\begin{aligned}\mathbf{z}(k+1) &= \widehat{A}(k)\mathbf{z}(k) + B\Big(\widehat{w}_*(k) + \widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k))\Big)\\ &= \Big[\underbrace{(A+B\mathbf{K})}_{A_\mathrm{C}} + v_0(k)\cdot\Delta A(\boldsymbol{\varphi}_d)\Big]\mathbf{z}(k)\\ &\quad + B\Big(u_0(k)\cdot\mathbf{d} + \Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) + \widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k))\Big),\end{aligned}$$

where $\mathbf{K}$ and $\mathbf{d}$ are given by (16), and $A_\mathrm{C}$ is given by (17). This leads to the closed loop dynamics for $p > 1$:

$$\begin{aligned}\mathbf{z}(k+1) &= \Big(A_\mathrm{C} + v_0(k)\cdot\Delta A(\boldsymbol{\varphi}_d)\Big)\mathbf{z}(k) \qquad (24)\\ &\quad + B\Big[u_0(k)\mathbf{d} + \Delta\widehat{\mathbf{h}}(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})\Big] + B\widetilde{h}_{\boldsymbol{\varphi}_d}(z'(k))\end{aligned}$$

for all $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_\mathbf{z} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$, where $\mathcal{U}_\mathbf{z}$ is a bounded open convex set in $\mathbb{R}^{2n}$, $\mathcal{U}_0$ is a bounded open convex set containing $[a_{0,\min}, a_{0,\max}]$, and $\mathcal{V}_0$ is a bounded open convex set containing $[v_{\min}, v_{\max}]$.

### *C. Local Input-to-State Stability of the Closed Loop System*

We first give a brief overview of (local) input-to-state stability. Consider the discrete-time system on $\mathbb{R}^n$:

$$x(k+1) = f(x(k), u(k), k), \qquad \forall\, k \in \mathbb{Z}_+, \qquad (25)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z}_+ \to \mathbb{R}^n$, and $f(\cdot,\cdot,k)$ is continuous for any fixed $k \in \mathbb{Z}_+$. Let $\mathbf{u} := (u(0), u(1), \ldots)$ be a sequence of vectors in $\mathbb{R}^m$ that represents an input on $\mathbb{Z}_+$. Assume that $f(0, 0, k) = 0$ for all $k \in \mathbb{Z}_+$ such that $x_e = 0$ is an equilibrium of (25) under the 0-input, i.e., $\mathbf{u} = 0$. Let $\|\mathbf{u}\|_\infty := \sup\{\|u(k)\| : k \in \mathbb{Z}_+\}$. Hence, for any $\mathbf{u} \in \ell^m_\infty$, $\|\mathbf{u}\|_\infty < \infty$. For a given initial condition $\xi \in \mathbb{R}^n$ and an input function $\mathbf{u}$, let $x(k, \xi, \mathbf{u})$ denote the trajectory of the system (25).

*Definition 1:* The time-varying discrete-time system (25) is *locally input-to-state stable* (ISS) if there exist a $\mathcal{KL}$-function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, a $\mathcal{K}$-function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, and two positive constants $\theta_x, \theta_u$ such that for all $\xi$ with $\|\xi\| \le \theta_x$ and $\mathbf{u} \in \ell^m_\infty$ with $\|\mathbf{u}\|_\infty \le \theta_u$, the following holds:

$$\|x(k, \xi, \mathbf{u})\| \le \beta(\|\xi\|, k) + \gamma(\|\mathbf{u}\|_\infty), \qquad \forall\, k \in \mathbb{Z}_+.$$

The above definition follows from [11, Definition 3.1] for global ISS of discrete-time systems. Also see [7], [24]. The following result establishes local input-to-state stability (ISS) for the time-varying system (25) [10, Lemma 2.3], which is extended from the Lyapunov approach for global ISS [12].

*Theorem 2: Consider the time-varying discrete-time system (25) defined by $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z}_+ \to \mathbb{R}^n$. Suppose there exists a local ISS-Lyapunov function $V : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}_+$ for the system (25), i.e., there exist two sets $\mathcal{D}_x := \{x \in \mathbb{R}^n \,|\, \|x\| \le r\}$ and $\mathcal{D}_u := \{u \in \mathbb{R}^m \,|\, \|u\| \le r_u\}$ for some positive constants $r$ and $r_u$, where $r_u$ can be $+\infty$, such that:*

(i) *There exist two $\mathcal{K}_\infty$-functions $\alpha_1$ and $\alpha_2$ such that $\alpha_1(t) \le \alpha_2(t), \forall\, t \ge 0$ and $\alpha_1(\|x\|) \le V(x, k) \le \alpha_2(\|x\|)$ for all $x \in \mathcal{D}_x$ and all $k \in \mathbb{Z}_+$;*

(ii) *There exist a $\mathcal{K}_\infty$-function $\alpha_3$ and a $\mathcal{K}$-function $\sigma$ such that $V(f(x, u, k), k+1) - V(x, k) \le -\alpha_3(\|x\|) + \sigma(\|u\|)$ for all $x \in \mathcal{D}_x$ and $u \in \mathcal{D}_u$ and all $k \in \mathbb{Z}_+$.*

*Then there exist positive constants $\theta_x$ and $\theta_u$ such that:*

(i) *For any $\xi$ with $\|\xi\| \le \theta_x$ and $\mathbf{u} = (u(k))_{k\in\mathbb{Z}_+} \in \ell^m_\infty$ with $\|\mathbf{u}\|_\infty \le \theta_u$, $x(k, \xi, \mathbf{u}) \in \mathcal{D}_x$ for all $k \in \mathbb{Z}_+$;*

(ii) *The system (25) is locally input-to-state stable in terms of the positive constants $\theta_x$ and $\theta_u$ given in Definition 1.*

We show local input-to-state stability of the closed loop dynamics (24) for $p > 1$ below, assuming that the matrix $A_\mathrm{C}$ given in (17) is Schur stable. The case of $p = 1$ is treated in the similar way; see [22, Theorem 6.4] for details.

*Theorem 3:* [22, Theorem 6.4] *Let $p > 1$. Suppose the weight matrices $Q_{z,s}$, $Q_{z',s}$ and $Q_{w,s}$ satisfying* **A.1** *are such that $A_C$ given in (17) is Schur stable. Then there exist positive constants $\mu$ and $\nu$ such that for all $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_\infty \le \mu$, any $v_0(k) \in [v_{\min}, v_{\max}]$ and any $u_0(k)$ with $|u_0(k)| \le \nu$ for all $k \in \mathbb{Z}_+$, the closed loop dynamics given by (24) is locally input-to-state stable.*

*Proof:* For the given bounded open sets $\mathcal{U}_\mathbf{z}$ containing the zero vector, $\mathcal{U}_0$ containing $[a_{0,\min}, a_{0,\max}]$, and $\mathcal{V}_0$ containing $[v_{\min}, v_{\max}]$, the closed loop dynamics is given by (24) as shown by Proposition 3. Since $A_\mathrm{C}$ is Schur stable, there exist constants $\kappa_\mathrm{C} > 0$ and $r \in (0, 1)$ such that $\|(A_\mathrm{C})^k\| \le \kappa_\mathrm{C} \cdot r^k$ for all $k \in \mathbb{Z}_+$. Consider the time-varying discrete linear system on $\mathbb{R}^{2n}$:

$$\mathbf{z}(k+1) = \Big(A_\mathrm{C} + v_0(k)\cdot\Delta A(\boldsymbol{\varphi}_d)\Big)\mathbf{z}(k), \ \ \forall\, k \in \mathbb{Z}_+. \qquad (26)$$

In view of the expressions of $\Delta A(\boldsymbol{\varphi}_d)$ given by (21) and $D(\boldsymbol{\varphi}_d)$ by (19) and $0 \le v_{\min} \le v_0(k) \le v_{\max}$ for all $k \in \mathbb{Z}_+$, we deduce that there exists a positive constant $\kappa_{\Delta A}$ such that $\|v_0 \cdot \Delta A(\boldsymbol{\varphi}_d)\|_2 \le \kappa_{\Delta A} \cdot v_{\max} \cdot \|\boldsymbol{\varphi}_d\|_\infty, \forall\, v_0 \in [v_{\min}, v_{\max}]$. Define the positive constant $\widetilde{\mu}_3 := -\dfrac{r}{\kappa_\mathrm{C}\cdot\kappa_{\Delta A}\cdot v_{\max}}\ln(r) > 0$. Hence, for all $\boldsymbol{\varphi}_d$ with $\|\boldsymbol{\varphi}_d\|_\infty < \widetilde{\mu}_3$, we have $\kappa_{\Delta A}\cdot v_{\max}\cdot\|\boldsymbol{\varphi}_d\|_\infty \le -\frac{r}{\kappa_\mathrm{C}}\ln(r)$. Then it follows from [35, Theorem 3] that the discrete linear system (26) is uniformly exponentially stable for any $v_0(k) \in [v_{\min}, v_{\max}], \forall\, k \in \mathbb{Z}_+$ and all $\boldsymbol{\varphi}_d$ with $\|\boldsymbol{\varphi}_d\|_\infty < \widetilde{\mu}_3$. Define $\widehat{A}_\mathrm{C}(k) := A_\mathrm{C} + v_0(k)\Delta A(\boldsymbol{\varphi}_d)$ for all $k \in \mathbb{Z}_+$. (Rigorously speaking, it should be written as $\widehat{A}_\mathrm{C}(v_0(k), \boldsymbol{\varphi}_d)$. For notational simplicity, we write it in this way.) By [20, Theorem 23.3], there exist a matrix sequence $\{P(k)\}_{k\in\mathbb{Z}_+}$ with $P(k) = P^T(k) \in \mathbb{R}^{2n\times 2n}$ for each $k$ and positive constants $\theta_2 \ge \theta_1 > 0$ and $\theta_3 > 0$ such that for all $v_0(k) \in [v_{\min}, v_{\max}], \forall\, k \in \mathbb{Z}_+$ and all $\boldsymbol{\varphi}_d$ with $\|\boldsymbol{\varphi}_d\|_\infty < \widetilde{\mu}_3$, $\theta_1 I_{2n} \preccurlyeq P(k) \preccurlyeq \theta_2 I_{2n}$ and $\widehat{A}^T_\mathrm{C}(k)P(k+1)\widehat{A}_\mathrm{C}(k) - P(k) \preccurlyeq -\theta_3 I_{2n}$ for all $k \in \mathbb{Z}_+$, where $\preccurlyeq$ denotes the positive semi-definite order. Clearly, $\|P(k)\|_2 \le \theta_2$ for all $k \in \mathbb{Z}_+$.

Given any $v_0(k) \in [v_{\min}, v_{\max}], \forall\, k \in \mathbb{Z}_+$ and any $\boldsymbol{\varphi}_d$ satisfying $\|\boldsymbol{\varphi}_d\|_\infty < \widetilde{\mu}_3$, define the function $f_{\boldsymbol{\varphi}_d} : \mathbb{R}^{2n} \times \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{2n}$ as: $f_{\boldsymbol{\varphi}_d}(\mathbf{z}, d, k) := \widehat{A}_\mathrm{C}(k)\mathbf{z} + Bd + B\widetilde{h}_{\boldsymbol{\varphi}_d}(z')$, where $\mathbf{z} = (z, z') \in \mathbb{R}^{2n}$. Consider the time-varying Lyapunov function $V : \mathbb{R}^{2n} \times \mathbb{Z}_+ \to \mathbb{R}_+$ given by

$$V(\mathbf{z}, k) := \mathbf{z}^T P(k)\mathbf{z}, \quad k \in \mathbb{Z}_+. \qquad (27)$$

TABLE I
PHYSICAL PARAMETERS FOR A HETEROGENEOUS CAV PLATOON WITH $\Delta = 60\,m$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ | $i=8$ | $i=9$ | $i=10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_i(m)$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $r_i(s)$ | 1.21 | 1.155 | 1.0 | 1.045 | 1.21 | 1.155 | 1.0 | 1.045 | 1.155 | 1.045 |
| $a_{i,\min}(m/s^2)$ | −8.14 | −7.77 | −6.66 | −7.03 | −8.14 | −7.77 | −6.66 | −7.03 | −7.77 | −7.03 |
| $a_{i,\max}(m/s^2)$ | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 | 1.8 |
| $c_{2,i}(\times 10^{-4})$ | 3.85 | 3.675 | 3.15 | 3.325 | 3.85 | 3.675 | 3.15 | 3.325 | 3.675 | 3.325 |
| $c_{3,i}(\times 10^{-2})$ | 1.155 | 1.103 | 0.945 | 0.998 | 1.155 | 1.103 | 0.945 | 0.998 | 1.103 | 0.998 |

In light of $\widehat{A}_{\mathrm{C}}^T(k)P(k+1)\widehat{A}_{\mathrm{C}}(k) - P(k) \preccurlyeq -\theta_3 I_{2n}$, we have that for any $k \in \mathbb{Z}_+$,

$$\begin{aligned}
V(f_{\boldsymbol{\varphi}_d}(\mathbf{z}, d, k), k+1) - V(\mathbf{z}, k) &\\
\le -\theta_3\|\mathbf{z}\|_2^2 + 2\Big[\widehat{A}_{\mathrm{C}}(k)\ \mathbf{z}\Big]^T P(k+1)B\Big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\Big] &\\
+ \Big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\Big]^T B^T P(k+1)B\Big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\Big]. &
\end{aligned}$$

Let $\eta_1 := \sup_{\|\boldsymbol{\varphi}_d\|_\infty \le \widetilde{\mu}_3} \Big(\|A_{\mathrm{C}}\|_2 + v_{\max}\cdot\|\Delta A(\boldsymbol{\varphi}_d)\|_2\Big) > 0$. Hence, $\|\widehat{A}_{\mathrm{C}}(k)\| \le \eta_1$ for all $k \in \mathbb{Z}_+$. Moreover, it follows from (20) that there exists a positive constant $\eta_2$ such that $\|\widetilde{h}_{\boldsymbol{\varphi}_d}(z')\|_2 \le \eta_2\cdot\|\boldsymbol{\varphi}_d\|_\infty\cdot\|z'\|_2^2 \le \eta_2\cdot\|\boldsymbol{\varphi}_d\|_\infty\cdot\|\mathbf{z}\|_2^2$ for all $\boldsymbol{\varphi}_d$ and $\mathbf{z}$. Let $\widetilde{\eta}_2 := \eta_2\big(\sup_{\mathbf{z}\in\mathcal{U}_{\mathbf{z}}}\|\mathbf{z}\|\big)$. Therefore, for all $\mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, we have

$$\|\widetilde{h}_{\boldsymbol{\varphi}_d}(z')\|_2 \le \eta_2\big(\sup_{\mathbf{z}\in\mathcal{U}_{\mathbf{z}}}\|\mathbf{z}\|\big)\cdot\|\boldsymbol{\varphi}_d\|_\infty\cdot\|\mathbf{z}\| = \widetilde{\eta}_2\cdot\|\boldsymbol{\varphi}_d\|_\infty\cdot\|\mathbf{z}\|.$$

Consequently, for all $\mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$, $\boldsymbol{\varphi}_d \in \mathcal{B}(0, \widetilde{\mu}_3)$, and $d \in \mathbb{R}^n$, we have,

$$\begin{aligned}
&\big[\widehat{A}_{\mathrm{C}}(k)\ \mathbf{z}\big]^T P(k+1)B\big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\big]\\
&\quad\le \theta_2\|\widehat{A}_{\mathrm{C}}(k)\ \mathbf{z}\|_2\cdot\|B\|_2\big(\|d\|_2 + \|\widetilde{h}_{\boldsymbol{\varphi}_d}(z')\|_2\big)\\
&\quad\le \theta_2\eta_1\|B\|_2\cdot\|\mathbf{z}\|_2\cdot\big(\|d\|_2 + \widetilde{\eta}_2\|\boldsymbol{\varphi}_d\|_\infty\cdot\|\mathbf{z}\|_2\big),
\end{aligned}$$

and

$$\begin{aligned}
&\big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\big]^T B^T P(k+1)B\big[d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\big]\\
&\quad\le \theta_2\cdot\|B\|_2^2\cdot\big\|d + \widetilde{h}_{\boldsymbol{\varphi}_d}(z')\big\|_2^2\\
&\quad\le 2\theta_2\cdot\|B\|_2^2\cdot\Big(\|d\|_2^2 + \big(\widetilde{\eta}_2\|\boldsymbol{\varphi}_d\|_\infty\big)^2\cdot\|\mathbf{z}\|_2^2\Big).
\end{aligned}$$

Combining the above results, we deduce that there exists a constant $\mu_3$ with $0 < \mu_3 \le \min(\widetilde{\mu}_3, \mu_2)$, where $\mu_2$ is given in Proposition 3, such that for all $\|\boldsymbol{\varphi}\|_\infty \le \mu_3, \mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$, and $d \in \mathbb{R}^n$, $V(f_{\boldsymbol{\varphi}_d}(\mathbf{z}, d, k), k+1) - V(\mathbf{z}, k) \le -\frac{2\theta_3}{3}\|\mathbf{z}\|_2^2 + 2\eta_3\|d\|_2\cdot\|\mathbf{z}\|_2 + \eta_4\|d\|_2^2$, where $\eta_3 := \theta_2\eta_1\|B\|_2/2$, and $\eta_4 := 2\theta_2\|B\|_2^2$. Consequently, for all $\|\boldsymbol{\varphi}\|_\infty \le \mu_3$, $\mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$, and $d \in \mathbb{R}^n$, we have, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}
&V(f_{\boldsymbol{\varphi}_d}(\mathbf{z}, d, k), k+1) - V(\mathbf{z}, k)\\
&\le -\frac{2\theta_3}{3}\|\mathbf{z}\|_2^2 + 2\eta_3\cdot\|d\|_2\cdot\|\mathbf{z}\|_2 + \eta_4\cdot\|d\|_2^2\\
&= -\frac{\theta_3}{6}\|\mathbf{z}\|_2^2 - \frac{\theta_3}{2}\|\mathbf{z}\|_2^2 + 2\eta_3\|d\|_2\cdot\|\mathbf{z}\|_2 + \eta_4\|d\|_2^2\\
&= -\frac{\theta_3}{6}\|\mathbf{z}\|_2^2 - \frac{\theta_3}{2}\Big(\|\mathbf{z}\|_2 - \frac{2\eta_3}{\theta_3}\|d\|_2\Big)^2 + \Big(\frac{2\eta_3^2}{\theta_3} + \eta_4\Big)\|d\|_2^2\\
&\le -\frac{\theta_3}{6}\|\mathbf{z}\|_2^2 + \Big(\frac{2\eta_3^2}{\theta_3} + \eta_4\Big)\|d\|_2^2.
\end{aligned}$$

Define the functions $\sigma_1(t) := \theta_1 t^2$, $\sigma_2(t) := \theta_2 t^2$, $\sigma_3(t) := \frac{\theta_3}{6}t^2$, and $\sigma(t) := \Big(\frac{2\eta_3^2}{\theta_3} + \eta_4\Big)t^2$. Clearly, these function are $\mathcal{K}_\infty$-functions. Let $\mathcal{D}_{\mathbf{z}}$ be the largest closed ball centered at the origin that is contained in $\mathcal{U}_{\mathbf{z}}$ (such the closed ball exists since $\mathcal{U}_{\mathbf{z}}$ is a bounded open set containing 0), and $\mathcal{D}_d = \mathbb{R}^n$. Hence, the function $V$ given in (27) is a local ISS-Lyapunov function on $\mathcal{D}_{\mathbf{z}} \times \mathcal{D}_d$ for the discrete time system $\mathbf{z}(k+1) = f_{\boldsymbol{\varphi}_d}(\mathbf{z}(k), d(k), k)$, for all $\boldsymbol{\varphi}_d \in \mathcal{B}_\infty(0, \mu_3)$ and all $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$. It follows from Theorem 2 that there exist two positive constants $\nu_{\mathbf{z}}$ and $\nu_d$ such that for any $\xi$ with $\|\xi\| \le \nu_{\mathbf{z}}$ and $\overline{\mathbf{d}} = (d(k))_{k\in\mathbb{Z}_+} \in \ell_\infty^m$ with $\|\overline{\mathbf{d}}\|_\infty \le \nu_d$, $\mathbf{z}(k, \xi, \overline{\mathbf{d}}) \in \mathcal{U}_{\mathbf{z}}$ for all $k \in \mathbb{Z}_+$. In view of the right-hand side of the closed loop dynamics given by (24), we see that $d(k) = u_0(k)\cdot\mathbf{d} + \Delta\widehat{\mathbf{h}}(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})$ for all $k \in \mathbb{Z}_+$, where $\mathbf{d}$ is the constant vector given by (16), and $\|\Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi})\| \le \varkappa\|\boldsymbol{\varphi}\|_\infty$ for all $(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi}) \in \mathcal{U}_{\mathbf{z}} \times \mathcal{V}_0 \times \mathcal{U}_0 \times \mathcal{B}_\infty(0, \mu_2)$. For an arbitrary but fixed $\varepsilon \in (0,1)$, define the positive constants $\mu_4 := \min\Big(\mu_3, \frac{\varepsilon\cdot\nu_d}{\varkappa}\Big)$, and $\nu_u := \min\Big(|a_{0,\min}|, a_{0,\max}, \dfrac{(1-\varepsilon)\nu_d}{\|\mathbf{d}\|}\Big)$. Hence, $u_0 \in [-a_{0,\min}, a_{0,\max}] \subset \mathcal{U}_0$ and $\|u_0\mathbf{d}\| \le (1-\varepsilon)\nu_d$ for any $u_0$ with $|u_0| \le \nu_u$. (The condition $u_0 \in \mathcal{U}_0$ is needed to derive the closed loop dynamics as shown in Proposition 3.) Further, for all $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_\infty \le \mu_4$, $u_0$ with $|u_0| \le \nu_u$, $v_0 \in [v_{\min}, v_{\max}]$ and $\mathbf{z} \in \mathcal{U}_{\mathbf{z}}$, it is easy to show that $\|u_0\mathbf{d} + \Delta\widehat{\mathbf{h}}(\mathbf{z}, v_0, u_0, \boldsymbol{\varphi})\| \le \nu_d$. It can be further shown via induction on $k$ that for all $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_\infty \le \mu_4$, $u_0(k)$ with $|u_0(k)| \le \nu_u, \forall k \in \mathbb{Z}_+$, $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$, and any $\xi$ with $\|\xi\| \le \nu_{\mathbf{z}}$, $\mathbf{z}(k, \xi, \overline{\mathbf{d}}) \in \mathcal{U}_{\mathbf{z}}$ and $\|d(k)\| = \|u_0(k)\cdot\mathbf{d} + \Delta\widehat{\mathbf{h}}(\mathbf{z}(k), v_0(k), u_0(k), \boldsymbol{\varphi})\| \le \nu_d$ for all $k \in \mathbb{Z}_+$. In view of Theorem 2 again, we deduce that the closed loop dynamics given by (24) is locally input-to-state stable for all $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_\infty \le \mu_4$, $v_0(k) \in [v_{\min}, v_{\max}], \forall k \in \mathbb{Z}_+$, and $u_0(k)$ with $|u_0(k)| \le \nu_u, \forall k \in \mathbb{Z}_+$. □

*Remark 2:* The above theorem establishes the input-to-state stability of the entire platoon under the proposed platoon centered MPC scheme via new techniques that are different from distributed MPC [34], [37].

## VII. NUMERICAL RESULTS

### A. Numerical Experiment Setup and Weight Matrix Design

Numerical tests are carried out to evaluate the performance of the proposed fully distributed schemes and the platooning control for a possibly heterogeneous CAV platoon. Consider a heterogeneous CAV platoon of an uncontrolled leading vehicle labeled by the index 0 and ten CAVs, i.e., $n = 10$. The sample time $\tau = 1\,s$, and the speed limits $v_{\max} = 27.78\,m/s$ and $v_{\min} = 10\,m/s$. The inhomogeneous values of $c_{2,i}$'s and $c_{3,i}$'s, and other parameters [6], [34], i.e., the vehicle length $L_i$, the reaction time $r_i$, the acceleration and deceleration limits $a_{i,\max}$ and $a_{i,\min}$, and the desired spacing $\Delta$, are given in Table I.

The initial state of each CAV platoon is $z(0) = z'(0) = 0$ and $v_i(0) = 25\,m/s$ for all $i = 0, 1, \ldots, n$. The cyclic-like graph is considered for the vehicle communication network, i.e., the bidirectional edges of the graph are $(1, 2), (2, 3), \ldots, (n-1, n) \in \mathcal{E}$. Following the discussions in [23, Section 6], we choose the MPC horizon $p$ as $1 \le p \le 5$.

The weight matrices are chosen as follows. Let

$$\widetilde{\boldsymbol{\alpha}} := \big(38.85, 40.2, 41.55, 42.90, 44.25, 45.60, 46.95, 48.30, 49.65, 51.00\big) \in \mathbb{R}^{10},$$
$$\widetilde{\boldsymbol{\beta}} := \big(130.61, 136.21, 141.82, 147.42, 153.03, 158.64, 164.24, 169.85, 175.46, 181.06\big) \in \mathbb{R}^{10},$$
$$\widetilde{\boldsymbol{\zeta}} := \big(62, 74, 90, 92, 106, 194, 298, 402, 454, 480\big) \in \mathbb{R}^{10}.$$

Further, let $\boldsymbol{\alpha}^1 = 6\widetilde{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}^1 = \widetilde{\boldsymbol{\beta}}$, and $\boldsymbol{\zeta}^1 = 0.5\widetilde{\boldsymbol{\zeta}}$ when $p = 1$. When $p = 2, 3$, $\boldsymbol{\alpha}^1 = 9(\widetilde{\boldsymbol{\alpha}} - \mathbf{1})$, $\boldsymbol{\beta}^1 = \widetilde{\boldsymbol{\beta}} - \mathbf{1}$, $\boldsymbol{\zeta}^1 = 0.5(\widetilde{\boldsymbol{\zeta}} - \mathbf{1})$, and $\boldsymbol{\alpha}^s = \dfrac{0.1368}{(s-1)^4} \times \widetilde{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}^s = \dfrac{0.044}{(s-1)^4} \times \widetilde{\boldsymbol{\beta}}$, $\boldsymbol{\zeta}^s = \dfrac{0.0013}{(s-1)^4} \times \widetilde{\boldsymbol{\zeta}}$, $s = 2, \ldots, \min(p, 3)$. When $p = 4, 5$, $\boldsymbol{\alpha}^s = \dfrac{0.0228}{(s-1)^4} \times \widetilde{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}^s = \dfrac{0.044}{(s-1)^4} \times \widetilde{\boldsymbol{\beta}}$, $\boldsymbol{\zeta}^s = \dfrac{0.0026}{(s-1)^4} \times \widetilde{\boldsymbol{\zeta}}$, $s = 4, \ldots, p$. The diagonal matrices $Q_{z,s}, Q_{z',s}$ and $Q_{w,s}$ are written as $Q_{z,s} = \mathrm{diag}(\boldsymbol{\alpha}^s)$, $Q_{z',s} = \mathrm{diag}(\boldsymbol{\beta}^s)$, and $Q_{w,s} = \mathrm{diag}(\boldsymbol{\zeta}^s)$, where $\boldsymbol{\alpha}^s, \boldsymbol{\beta}^s \in \mathbb{R}^n_+$ and $\boldsymbol{\zeta}^s \in \mathbb{R}^n_{++}$ for all $s = 1, \ldots, p$ This yields the Schur stabel matrix $A_{\mathrm{C}}$ for each $p = 1, \ldots, 5$.

We consider a real-world traffic condition to test the performance of the proposed distributed algorithm and platooning control in a real traffic environment when the leading vehicle undergoes traffic oscillations. Specifically, we consider Next Generation Simulation (NGSIM) data on eastbound I-80 in San Francisco Bay area in California, and the data of position and speed of a real vehicle, which is treated as a leading vehicle, is used to generate its control input at each $k$. The length of the time window is $45s$. In addition, to further evaluate the proposed platooning control in a more realistic setting, random noise is added to each CAV to simulate dynamical disturbances, model mismatch, signal noise, communication delay, and road condition perturbations. In particular, at each $k$, the random noise with the normal distribution $0.2 \times \mathcal{N}(0, 1)$ is added to the first CAV, and the noise with the normal distribution $0.1 \times \mathcal{N}(0, 1)$ is added to each of the rest of the CAVs. Here a larger noise is added to the first CAV since there are more disturbances between the leading vehicle and the first CAV. See [22, Section 7] for additional numerical results for other scenarios and different CAV platoons.

### B. Performance of the Proposed Fully Distributed Scheme

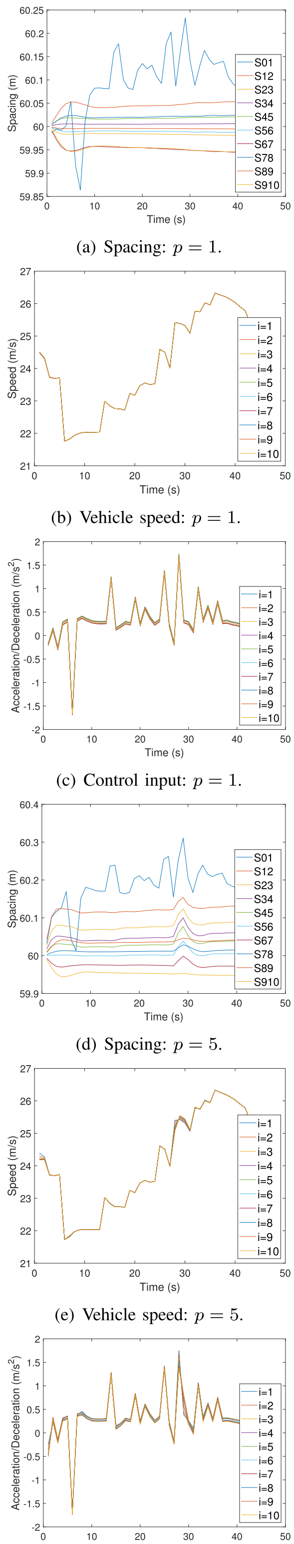
As indicated in Section V-B, when $p = 1$, the underlying MPC optimization problem (11) is a convex QCQP, for which the fully distributed algorithm developed in [23] is used. When $p > 1$, the optimization problem (11) is nonconvex, and the SCP based fully distributed scheme is applied (cf. Algorithm 1). To apply this algorithm, we discuss the choices of the smooth functions $g_{i,s}$ and the convex function $r_{i,s}$ for the (approximate) nonconvex constraint sets $\mathcal{Y}_i$ and $\mathcal{Z}_i$, where $i = 1, \ldots, n$; see Remark 1. For $j = 1, \ldots, p$, define the function $q_{i,j}(\mathbf{u}_i) := v_i(k) + \tau\big((S_p\mathbf{u}_i)_j - j \cdot c_{3,i}g - c_{2,i}\sum_{s=0}^{j-1}\big[v_i(k) + \tau(S_p\mathbf{u}_i)_s\big]^2\big)$. The approximate $\mathcal{Y}_i$ is given by $\mathcal{Y}_i = \{\mathbf{u}_i \,|\, v_{\min} - q_{i,j}(\mathbf{u}_i) \le 0,\ q_{i,j}(\mathbf{u}_i) - v_{\max} \le 0,\ j = 1, \ldots, p\}$. Define $g_{i,s}(\mathbf{u}_i) := v_{\min} - q_{i,j}(\mathbf{u}_i)$, and $r_{i,s}(\mathbf{u}_i) :\equiv 0$ for $s = 1, \ldots, p$; $g_{i,s}(\mathbf{u}_i) :\equiv 0$, and $r_{i,s}(\mathbf{u}_i) := -q_{i,j}(\mathbf{u}_i) + v_{\max}$ for $s = p+1, \ldots, 2p$. Then $\mathcal{Y}_i = \{\mathbf{u}_i \,|\, g_{i,s}(\mathbf{u}_i) - r_{i,s}(\mathbf{u}_i) \le 0,\ s = 1, \ldots, 2p\}$. Similarly, for each $i = 1, \ldots, n$ and $s = 1, \ldots, p$, $\mathcal{Z}_i = \{\widehat{\mathbf{u}}_i \,|\, g'_{i,s}(\widehat{\mathbf{u}}_i) - r'_{i,s}(\widehat{\mathbf{u}}_i) \le 0,\ s = 1, \ldots, p\}$, where $r'_{i,s}(\mathbf{u}_{i-1}, \mathbf{u}_i) \equiv 0$, and

$$\begin{aligned} g'_{i,s}(\mathbf{u}_{i-1}, \mathbf{u}_i) &:= \big(H_i(\mathbf{u}_{i-1}, \mathbf{u}_i)\big)_s \approx \quad L_i + r_i \cdot q_{i,s}(\mathbf{u}_i) \\ &\quad - \frac{1}{2a_{i,\min}}\big[q_{i,s}(\mathbf{u}_i) - v_{\min}\big]^2 - \Big\{ z_i(k) + \Delta + j\tau z'_i(k) \\ &\quad + \tau^2 \sum_{t=0}^{s-1} \frac{2(j-t)-1}{2}\Big[u_{i-1}(k+t) - u_i(k+t) \\ &\quad - \Big(c_{2,i-1}\big[v_{i-1}(k) + \tau\big(S_p\mathbf{u}_{i-1}\big)_t\big]^2 \\ &\quad - c_{2,i}\big[v_i(k) + \tau\big(S_p\mathbf{u}_i\big)_t\big]^2\Big) - \big(c_{3,i-1} - c_{3,i}\big)g\Big]\Big\}. \end{aligned}$$

Furthermore, the Lipschitz constants $L_{J_i}$'s and $L_{g_{i,s}}$'s are given by $\nu_p\|HJ_i(\widehat{\mathbf{u}}_i)\|_2$ and $0.9\|Hg_{i,s}(\widehat{\mathbf{u}}_i)\|_2$, where $\nu_p = 0.8$ for $p = 2, 3$ and $\nu_p = 0.9$ for $p = 4, 5$ respectively, and $Hf$ denotes the Hessian of a real-valued smooth function $f$. The reasons for each Hessian scaled by these factors are twofold: (i) the 2-norm of Hessian is conservative; and (ii) the scaled Hessian leads to faster convergence.

*1) Initial Guess Warm-up:* For real-time implementation of Algorithm 1, we exploit the initial guess warm-up technique for both the linear stage (cf. Line 2) and the inner loop of the SCP-Douglas-Rachford stage (cf. Lines 6-14). For the former, see [23, Section 6.2] for its warm-up scheme. We discuss a warm-up scheme for the latter. Recall that the inner loop solves the following convex problem: $\min_{y=(y_i)\in\mathcal{A}} \sum_{i=1}^n f_i(y_i) + \boldsymbol{\delta}\mathcal{C}_i(y_i)$, where for each $i$, $f_i(y_i) := J_i(\widehat{\mathbf{u}}_i^k) + d_{J_i}^T(\widehat{\mathbf{u}}_i^k)(y_i - \widehat{\mathbf{u}}_i) + \frac{L_{J_i}}{2}\|y_i - \widehat{\mathbf{u}}_i^k\|_2^2$, and $\mathcal{C}_i$ is the intersection of the box-constraint set $\mathcal{X}_i$ and a quadratically constrained convex set; see Section V-B for details. In the warm-up scheme, we replace $\mathcal{C}_i$ by $\mathcal{X}_i$. The generalized Douglas-Rachford scheme (15) is used to solve $\min_{y=(y_i)\in\mathcal{A}} \sum_{i=1}^n f_i(y_i) + \boldsymbol{\delta}\mathcal{X}_i(y_i)$. Since $f_i$ and the box constraint set $\mathcal{X}_i$ are fully decoupled, solving the proximal operator based optimization problem in this scheme becomes solving several decoupled univariate problems of the form: $\min_{t\in[c,d]} at^2 + bt + e$, where $t \in \mathbb{R}$, and $a, b, c, d, e \in \mathbb{R}$ are given constants with $a > 0$. This problem has a simple closed-form solution, which considerably reduces computation load. Numerical tests show that the proposed warm-up scheme significantly improves computation time and solution quality.

*2) Performance of Distributed Schemes:* We implement the proposed fully distributed algorithm via MATLAB on a computer with 4-cores processor: Intel(R) Core(TM) i7-8550U




(a) Spacing: $p = 1$.

(b) Vehicle speed: $p = 1$.

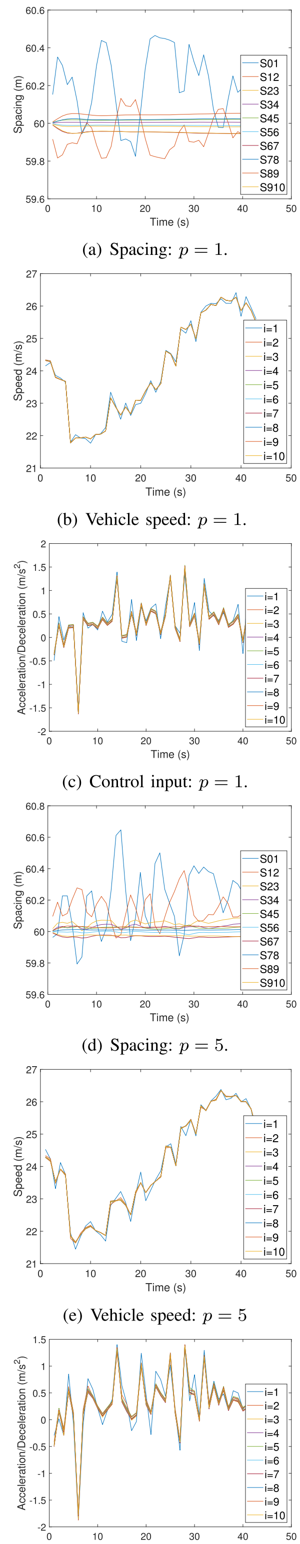
(c) Control input: $p = 1$.

(d) Spacing: $p = 5$.

(e) Vehicle speed: $p = 5$.

(f) Control input: $p = 5$.

Fig. 1. Platooning control with $p = 1$ and $p = 5$.




(a) Spacing: $p = 1$.

(b) Vehicle speed: $p = 1$.

(c) Control input: $p = 1$.

(d) Spacing: $p = 5$.

(e) Vehicle speed: $p = 5$

(f) Control input: $p = 5$.

Fig. 2. Platooning control subject to noises with $p = 1$ and $p = 5$.

CPU @ 1.80 $GHz$ and RAM: 16.0 $GB$. This algorithm is tested for the above-mentioned heterogeneous CAV platoon for different $p$'s. The proposed initial guess warm-up schemes are used with the error tolerance give by $10^{-7}$ for all the cases. Moreover, we choose $\alpha = 0.9$ and $\rho = 0.1$ for the proximal operator based Douglas-Rachford scheme. Further, the stopping criteria are characterized by the minimum of absolute and relative errors of two neighboring iterates for $p = 2, 3$, whereas for $p = 4, 5$, these criteria are characterized by absolute errors of two neighboring iterates. The list of error tolerances for the outer and inner loop at different $p$'s is shown in Table II. Note that there is no inner loop for $p = 1$,

TABLE II
ERROR TOLERANCES FOR OUTER AND INNER LOOPS AT DIFFERENT MPC HORIZONS

| MPC horizon | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|---|---|---|---|---|---|
| Outer loop | $2.5 \times 10^{-3}$ | $6.5 \times 10^{-3}$ | $7.5 \times 10^{-3}$ | $1.0 \times 10^{-2}$ | $1.25 \times 10^{-2}$ |
| Inner loop | NA | $4.0 \times 10^{-3}$ | $5.0 \times 10^{-3}$ | $7.5 \times 10^{-3}$ | $1.0 \times 10^{-2}$ |

TABLE III
COMPUTATION TIME PER CAV ($sec$)

| MPC horizon | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|---|---|---|---|---|---|
| Mean | 0.1408 | 0.2528 | 0.2398 | 0.2883 | 0.2882 |
| Variance | $4.09 \times 10^{-4}$ | $6.3 \times 10^{-3}$ | $4.91 \times 10^{-3}$ | $9.73 \times 10^{-3}$ | 0.0135 |

TABLE IV
MAXIMUM STEADY STATE ERROR OF SPACING ($m$)

| MPC horizon | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|---|---|---|---|---|---|
| Error | 0.0431 | 0.1277 | 0.1369 | 0.1363 | 0.1538 |

since its underlying problem is solved via the fully distributed scheme given in [23]. A summary of mean and variance of computation time per CAV is displayed in Tables III.

The numerical results show that for each $p$, the mean computation time is less than 0.289 $s$, which is less than the reaction time $r_i$ or sample time $\tau = 1s$, with overall small variances. We conclude that the proposed distributed scheme is suitable for real-time computation of a heterogenous CAV platoon with satisfactory numerical precision.

### *C. Performance of CAV Platooning Control*

We evaluate the closed loop performance of the proposed CAV platooning control. Toward this end, we consider the spacing between two neighboring vehicles (i.e., $S_{i-1,i}(k) := x_{i-1}(k) - x_i(k) = z_i(k) + \Delta$), the vehicle speed $v_i(k)$, and the control input $u_i(k)$, $i = 1, \dots, n$.

When $(c_{2,i}, c_{3,i}) \neq 0$ and $u_0(k) = 0$ and $v_0(k) = v_{0,\infty} > 0$ for all large $k$, it is observed from the numerical tests that when the CAV platoon reaches its steady state $(z_{ss}, z'_{ss}) \in \mathbb{R}^n \times \mathbb{R}^n$, i.e., $(z(k), z'(k))$ becomes the constant vector $(z_{ss}, z'_{ss})$ for all large $k$, $z_{ss}$ is nonzero. Physically, the nonzero steady state is due to nonlinear vehicle dynamics and the PD-like control structure of the MPC. An analysis for $p = 1$ shows that $z_{ss} = -2Q_z^{-1}Q_w w_{e,\infty}$, where $w_{e,\infty} \neq 0$ is defined in the same way as that of $w_e(k)$ by setting $v_0(k) \equiv v_{0,\infty}$; see [22, Section 7.3] for details. Similar results are obtained for $p \geq 2$.

We display the closed loop performance only for $p = 1$ and $p = 5$ because of the length limit; see Figure 1 for the noise free case and Figure 2 for the noise case, respectively. Figure 1 shows that $S_{0,1}$ yields the largest spacing variations with the maximum magnitude less than or equal to 0.3 $m$; the other spacings $S_{i-1,i}$, $i = 2, \dots, 10$ demonstrate nearly constant deviations with maximum magnitude less than 0.14 $m$, in spite of the oscillation of $S_{0,1}$. Further, the spacings $S_{i-1,i}$, $i = 2, \dots, 10$ almost reach steady states between $5s$ and $25s$ and after $k = 35$. The maximum steady state errors are given in Table IV. It is seen that the maximum steady state error often appears in $S_{1,2}$ and the largest relative error $\frac{\|z_{ss}\|_\infty}{\Delta} \leq 0.37\%$.

Besides, it is seen from Figure 2 that there are more noticeable spacing deviations from the desired $\Delta$ for all CAVs due to the noises. However, the variation of $S_{0,1}$ is within 1.2 $m$, and the maximum deviation of each $S_{i-1,i}$ with $i \geq 2$ is less than 0.4 $m$. Particularly, the deviations of $S_{i-1,i}$, $i = 3, \dots, 10$ are fairly small starting from $5s$, and the profiles of the CAV speed and control show an almost "coordinated" motion. Other numerical results show that the state or control constraints can be effectively handled by the proposed platooning control.

Consequently, the proposed platooning control effectively mitigates traffic oscillations of the spacing and vehicle speed of the CAV platoon with small steady state errors, even under external perturbations and state or control constraints. Additional numerical studies show that the current control scheme outperforms the linear controller developed in [23] on CAV platoons with non-negligible nonlinear dynamic effects and/or inhomogeneities.

## VIII. CONCLUSION

This paper develops a nonconvex, fully distributed optimization based MPC scheme for platooning control of a heterogeneous CAV platoon under the nonlinear vehicle dynamics. Various new techniques are exploited to address challenges induced by the nonlinear vehicle dynamics, including distributed computation for the coupled nonconvex MPC optimization problem, and stability analysis of time-varying nonlinear closed loop dynamics. We apply locally coupled optimization and sequential convex programming for distributed algorithm development, and global implicit function theorems and Lyapunov theory for input-to-state stability are invoked for stability analysis. Numerical tests illustrate the effectiveness of the proposed scheme and platooning control. Future research topics include extensions of the current distributed scheme and control design to more sophisticated vehicle dynamics and a possibly time-varying or non-uniform communication topology subject to communication delays or missing data. Moreover, directed networks will be considered for distributed scheme development.

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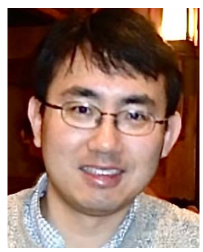

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