REAL AND SYMMETRIC MATRICES

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Abstract. We construct a family of involutions on the space $\mathfrak{gl}'_n(\mathbb{C})$ of $n \times n$ matrices with real eigenvalues interpolating the complex conjugation and the transpose. We deduce from it a stratified homeomorphism between the space $\mathfrak{gl}'_n(\mathbb{R})$ of $n \times n$ real matrices with real eigenvalues and the space $\mathfrak{p}'_n(\mathbb{C})$ of $n \times n$ symmetric matrices with real eigenvalues, which restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-adjoint orbits and $\mathrm{O}_n(\mathbb{C})$-adjoint orbits. We also establish similar results in more general settings of Lie algebras of classical types and quiver varieties. To this end, we prove a general result about involutions on hyper-Kähler quotients of linear spaces. We provide applications to the (generalized) Kostant-Sekiguchi correspondence, singularities of real and symmetric adjoint orbit closures, and Springer theory for real groups and symmetric spaces.

Contents

1. Introduction

1.1. **Main results.** A key structural result in Lie theory is Cartan's classification of real forms of a complex reductive Lie algebra $\mathfrak{g}$ in terms of holomorphic involutions. It amounts to a bijection

$$\{\text{complex conjugations } \eta \text{ of } \mathfrak{g}\}/\text{isom} \longleftrightarrow \{\text{holomorphic involutions } \theta \text{ of } \mathfrak{g}\}/\text{isom} \tag{1.1}$$

between isomorphism classes of complex conjugations and holomorphic involutions of $\mathfrak{g}$. For example, in the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, the complex conjugation $\eta(M) = \overline{M}$ with real form consisting of real matrices $\mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}_n(\mathbb{R})$ corresponds to the involution $\theta(M) = -M^t$ with $(-\theta)$-fixed points consisting of symmetric matrices $\mathfrak{p} = \mathfrak{p}_n(\mathbb{C})$. The interplay between the real $\mathfrak{g}_{\mathbb{R}}$ and symmetric $\mathfrak{p}$ pictures plays a fundamental role in the structure and representation theory of real groups, going back at least to Harish-Chandra's formulation of the representation theory of real groups in terms of $(\mathfrak{g}, K)$-modules.

One of the goals of the paper is to get a better understanding of Cartan's bijection and also the real and symmetric pictures for real groups from the geometric point of view. To this end, let η be a conjugation on $\mathfrak{g}$ and let θ be the corresponding involution under (1.1). For simplicity, we assume η is the split conjugation. Then the subspace $\mathfrak{g}'$ of $\mathfrak{g}$ consisting of elements with real eigenvalues[1] is preserved by both η and $-\theta$ and the first main result in the paper, Theorem 1.4, is a construction of a real analytic family of involutions on $\mathfrak{g}'$

$$\alpha_a : \mathfrak{g}' \longrightarrow \mathfrak{g}' \quad a \in [\,0, 1] \tag{1.2}$$

interpolating the conjugation η and the holomorphic involution $-\theta$, that is, we have $\alpha_0 = \eta$ and $\alpha_1 = -\theta$, in the case when $\mathfrak{g}$ is of classical type. Using the family of involutions above we prove the second main result of the paper, Theorem 1.3, which says that there exists a stratified homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}' \tag{1.3}$$

[1]Elements $x \in \mathfrak{g}$ such that the the adjoint action $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues.

between the η and $(-\theta)$-fixed points on $\mathfrak{g}'$ compatible with various structures.[2] The family of involutions in (1.2) and the homeomorphism (1.3) can be thought as geometric refinements of Cartan's bijection (1.1).

We deduce several applications from the main results. Assume $\mathfrak{g}$ is of classical types. In Corollary 1.9, We show that there exists a stratified homeomorphism

$$\mathcal{N}_{\mathbb{R}} \xrightarrow{\sim} \mathcal{N}_{\mathfrak{p}}$$

between the real nilpotent cone $\mathcal{N}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ and the symmetric nilpotent cone $\mathcal{N}_{\mathfrak{p}} \subset \mathfrak{p}$ providing a lift of the celebrated Kostant-Sekiguchi correspondence between real and symmetric nilpotent orbits. In particular, it implies that $\mathcal{N}_{\mathbb{R}}$ and $\mathcal{N}_{\mathfrak{p}}$ have the same singularities, answering an open question (see, e.g., [He, p354]). In Corollary 1.16, we show that Grinberg's nearby cycles sheaf on $\mathcal{N}_{\mathfrak{p}}$ is isomorphic to the real Springer sheaf given by the push-forward of the constant sheaf along the real Springer map, establishing a conjecture of Vilonen-Xue and the first author.

The key ingredients in the proof are the hyper-Kähler SU(2)-action on the space matrices arising from the quiver variety description in [KP, KS, M, MV, Nak1], and a general result about involutions on hyper-Kähler quotients of linear spaces (see Theorem 1.6). The techniques used in the proof are not specific to matrices and are applicable to a more general setting. For example, we also establish a quiver variety version of the main results.

We now describe the paper in more details.

1.1.1. *Real-symmetric homeomorphisms for matrices.* Let us first illustrate our main results with a notable case accessible to a general audience.

Let $\mathfrak{gl}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ denote the space of $n \times n$ complex matrices. Let $\mathfrak{gl}_n(\mathbb{R}) \subset \mathfrak{gl}_n(\mathbb{C})$ denote the real matrices, i.e. those with real entries, and $\mathfrak{p}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ the symmetric matrices, i.e. those equal to their transpose. Introduce the following subspaces

$$\mathfrak{gl}'_n(\mathbb{R}) = \{x \in \mathfrak{gl}_n(\mathbb{R}) | \text{ eigenvalues of } x \text{ are real}\}$$

$$\mathfrak{p}'_n(\mathbb{C}) = \{x \in \mathfrak{p}_n(\mathbb{C}) | \text{ eigenvalues of } x \text{ are real}\}.$$

The real general linear group $\mathrm{GL}_n(\mathbb{R})$ and complex orthogonal group $\mathrm{O}_n(\mathbb{C})$ naturally act by conjugation on $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$ respectively. The real orthogonal group $\mathrm{O}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{O}_n(\mathbb{C})$ acts on both $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$. We also have the natural linear $\mathbb{R}^\times$-actions on both $\mathfrak{gl}'_n(\mathbb{R})$ and $\mathfrak{p}'_n(\mathbb{C})$. Consider the adjoint quotient map $\chi : \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}^n$ which associates to each matrix $x \in \mathfrak{gl}_n(\mathbb{C})$ the coefficients of its characteristic polynomial. Equivalently, one can think of it as giving the eigenvalues of the matrix (with multiplicities).

Here is a notable case of our general results.

Theorem 1.1. *There is an* $\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times$*-equivariant homeomorphism*

$$\mathfrak{gl}'_n(\mathbb{R}) \xrightarrow{\sim} \mathfrak{p}'_n(\mathbb{C}) \tag{1.4}$$

[2]It is necessary to consider the subspace $\mathfrak{g}' \subset \mathfrak{g}$ but not the whole Lie algebra $\mathfrak{g}$ in the main results because in general the fixed points $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^\eta$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$ have different dimensions and hence can't be homeomorphic to each other.

which is compatible with the adjoint quotient map. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$*-orbits and* $\mathrm{O}_n(\mathbb{C})$*-orbits.*

We deduce Theorem 1.1 from the following more fundamental structure of linear algebra. Consider the subspace

$$\mathfrak{gl}'_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) | \text{ eigenvalues of } x \text{ are real}\}.$$

Let $\chi' : \mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$ be the restriction of the adjoint quotient map to $\mathfrak{gl}'_n(\mathbb{C})$.

Theorem 1.2. *There is a continuous one-parameter family of* $\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times$*-equivariant maps*

$$\alpha_a : \mathfrak{gl}'_n(\mathbb{C}) \longrightarrow \mathfrak{gl}'_n(\mathbb{C}) \qquad a \in [0,1] \tag{1.5}$$

satisfying the properties:

(1) α_a^2 *is the identity, for all* $a \in [0,1]$*.*
(2) α_a *commutes with* $\chi' : \mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$*.*
(3) α_a *takes each* $\mathrm{GL}_n(\mathbb{C})$*-orbit real analytically to a* $\mathrm{GL}_n(\mathbb{C})$*-orbit, for all* $a \in [0,1]$*.*
(4) At $a = 0$*, we recover conjugation:* $\alpha_0(A) = \bar{A}$*.*
(5) At $a = 1$*, we recover transpose:* $\alpha_1(A) = A^t$

1.1.2. *Real-symmetric homeomorphisms for Lie algebras.* To state a general version of our main results, we next recall some standard constructions in Lie theory, in particular in the study of real reductive groups.

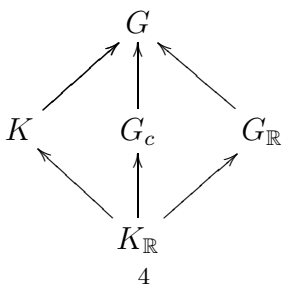
Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{c} = \mathfrak{g}//G$ be the categorical quotient with respect to the adjoint action of G on $\mathfrak{g}$. The adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{c}$ is the *Chevalley map*.

Let $G_\mathbb{R} \subset G$ be a real form, defined by a conjugation $\eta : G \to G$, with Lie algebra $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$. Choose a Cartan conjugation $\delta : G \to G$ that commutes with η, and let $G_c \subset G$ be the corresponding maximal compact subgroup.

Introduce the Cartan involution $\theta = \delta \circ \eta : G \to G$, and let $K \subset G$ be the fixed subgroup of θ with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. The subgroup K is called the symmetric subgroup. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p} \subset \mathfrak{g}$ is the -1-eigenspace of θ. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace contained in $\mathfrak{p}$ and let $\mathfrak{t} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra containing $\mathfrak{a}$. Let $\mathrm{W}_G = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ be the Weyl group of G and $\mathrm{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the little Weyl group of the symmetric pair pair (G, K). We denote by $\mathfrak{p}_\mathbb{R} = \mathfrak{p} \cap \mathfrak{g}_\mathbb{R}$, $\mathfrak{k}_\mathbb{R} = \mathfrak{k} \cap \mathfrak{g}_\mathbb{R}$, $\mathfrak{a}_\mathbb{R} = \mathfrak{a} \cap \mathfrak{g}_\mathbb{R}$, etc.

One can organize the above groups into the diagram:

(1.6)



Here $K_{\mathbb{R}}$ is the fixed subgroup of θ, δ, and η together (or any two of the three) and the maximal compact subgroup of $G_{\mathbb{R}}$ with complexification K.

Let $\mathfrak{g}'_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ (resp. $\mathfrak{p}' \subset \mathfrak{p}$) be the subspace consisting of elements $x \in \mathfrak{g}_{\mathbb{R}}$ (resp. $x \in \mathfrak{p}$) such that the eigenvalues of the adjoint map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ are real. The real from $G_{\mathbb{R}}$ and the symmetric subgroup K act naturally on $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{p}'$ by the adjoint action. The compact subgroup $K_{\mathbb{R}} = G_{\mathbb{R}} \cap K$ and $\mathbb{R}^\times$ act both on $\mathfrak{g}'_{\mathbb{R}}$ and $\mathfrak{p}'$.

Theorem 1.3 (Theorem 4.1)**.** *Suppose $\mathfrak{g}$ is of classical type. There is a $K_{\mathbb{R}} \times \mathbb{R}^\times$-equivariant homeomorphism*

$$\mathfrak{g}'_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}' \tag{1.7}$$

which is compatible with the adjoint quotient map. Furthermore, it restricts to a real analytic isomorphism between individual $G_{\mathbb{R}}$-orbits and K-orbits.

We deduce Theorem 1.3 from the following. Let $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ be the image of the natural map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{c} = \mathfrak{t}//\mathrm{W}_G$. Introduce $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ and let $\chi' : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ be the projection map.

Theorem 1.4 (Theorem 4.2)**.** *Under the same assumption as Theorem 1.3, there is a continuous one-parameter family of $K_{\mathbb{R}} \times \mathbb{R}^\times$-equivariant maps*

$$\alpha_a : \mathfrak{g}' \longrightarrow \mathfrak{g}' \qquad a \in [0,1] \tag{1.8}$$

satisfying the properties:

(1) α_a^2 is the identity, for all $s \in [0,1]$.
(2) α_a commutes with $\chi' : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$.
(3) α_a takes each G-orbit real analytically to a G-orbit, for all $a \in [0,1]$.
(4) At $a = 0$, we recover the conjugation: $\alpha_0 = \eta$.
(5) At $a = 1$, we recover the anti-symmetry: $\alpha_1 = -\theta$.

Remark 1.5. The special case of Theorems 1.3 and 1.4 stated in Theorems 1.1 and 1.2 is when $G = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} \simeq \mathfrak{gl}_n(\mathbb{C})$, $G_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{R})$, $K = \mathrm{O}_n(\mathbb{C})$, and $K_{\mathbb{R}} = \mathrm{O}_n(\mathbb{R})$.

1.1.3. *Involutions on hyper-Kähler quotients.* We deduce Theorem 1.3 and Theorem 1.4 from a general result about involutions on hyper-Kähler quotients of linear spaces.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions and let $\mathrm{Sp}(1) \subset \mathbb{H}$ be the group consisting of elements of norm one. Let $\mathbf{M}$ be a be a finite dimensional quaternionic representation of a compact Lie group H_u. We assume that the quaternionic representation is unitary, that is, there is a H_u-inner product $(,)$ on $\mathbf{M}$ which is hermitian with respect to the complex structures I, J, K on $\mathbf{M}$ given by multiplication by i, j, k respectively. We have the hyper-Kähler moment map

$$\mu : \mathbf{M} \to \mathrm{Im}\,\mathbb{H} \otimes \mathfrak{h}_u^*$$

vanishing at the origin. Using the isomorphism $\mathrm{Im}\,\mathbb{H} = \mathbb{R} \oplus \mathbb{C}$ sending $x_1 i + x_2 j + x_3 k$ to $(x_1, x_2 + x_3 i)$, we can identify $\mathrm{Im}\,\mathbb{H} \otimes \mathfrak{h}_u^* = \mathfrak{h}_u^* \oplus \mathfrak{h}^*$ and hence obtain a decomposition of the moment map

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}_u^* \oplus \mathfrak{h}^*$$

of μ into real and complex components. We consider the hyper-Kähler quotient

$$\mathfrak{M}_0 = \mu^{-1}(0)/H_u \simeq \mu_{\mathbb{C}}^{-1}(0)//H$$

where the right hand side is the categorial quotient of $\mu_{\mathbb{C}}^{-1}(0)$ by the complexification H of H_u, and the second isomorphism follows from a result of Kempf-Ness [KN].

The hyper-Kähler quotient $\mathfrak{M}_0$ has the following structures: (1) it has a orbit type stratification

$$\mathfrak{M}_0 = \bigsqcup_{(L)} \mathfrak{M}_{0,(L)}$$

where a stratum $\mathfrak{M}_{0,(L)}$ consists of orbits through points x whose stabilizer in H_u is conjugate to L, (2) there is a hyper-Kähler $\mathrm{SU}(2) = \mathrm{Sp}(1)$-action on $\mathfrak{M}_0$, denoted by $\phi(q) : \mathfrak{M}_0 \to \mathfrak{M}_0$, $q \in \mathrm{Sp}(1)$, coming from the $\mathbb{H}$-module structure on $\mathbf{M}$.

In Section 2, we prove the following general results about involutions on hyper-Kähler quotients.

Theorem 1.6 (Proposition 2.9 and Example 2.13)**.**

(1) Let η_H and $\eta_{\boldsymbol{M}}$ be complex conjugations on H and $\boldsymbol{M}$ which are compatible with the unitary-quaternionic representation of H_u on $\boldsymbol{M}$ (see Definition 2.4 for the precise definition). Then η_H and $\eta_{\boldsymbol{M}}$ induce an anti-holomorphic involution

$$\eta : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_0 \tag{1.9}$$

such that the composition of η with the hyper-Kähler $\mathrm{SU}(2)$-action of $q_a = \cos(\frac{a\pi}{2})i + \sin(\frac{a\pi}{2})k \in \mathrm{Sp}(1)$ on $\mathfrak{M}_0$, $a \in [0,1]$, gives rise to a continuous family of involutions

$$\alpha_a : \mathfrak{M}_0 \longrightarrow \mathfrak{M}_0 \qquad a \in [0,1] \tag{1.10}$$

interpolating the anti-holomorphic involution $\alpha_0 = \phi(i) \circ \eta$ and the holomorphic involution $\alpha_1 = \phi(k) \circ \eta$.

(2) Let $\mathfrak{M}_0(\mathbb{R})$ and $\mathfrak{M}_0^{sym}(\mathbb{C})$ be the fixed points of α_0 and α_1 on $\mathfrak{M}_0$ respectively. Then the intersection of the stratum $\mathfrak{M}_{0,(L)}$ with $\mathfrak{M}_0(\mathbb{R})$ (resp. $\mathfrak{M}_0^{sym}(\mathbb{C})$) define a stratification of $\mathfrak{M}_0(\mathbb{R})$ (resp. $\mathfrak{M}_0^{sym}(\mathbb{C})$) and there exists a stratified homeomorphism

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{sym}(\mathbb{C}) \tag{1.11}$$

which is real analytic on each stratum.

Remark 1.7. Let $G, G_{\mathbb{R}}, G_u, K_{\mathbb{R}}$ be as in Section 1.1.2. Suppose that $\mathbf{M}$ is a unitary quaternionic representation of the larger group $H_u \times G_u$ and the conjugations $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ on $H \times G$ and $\mathbf{M}$ are compatible with the unitary-quaternionic representation. Then the hyper-Kähler quotient $\mathfrak{M}$ carries an action of $K_{\mathbb{R}}$ such that the involutions (1.9), (1.10), and homeomorphism (1.11) are $K_{\mathbb{R}}$-equivariant.

It is well-known that the complex nilpotent cone $\mathcal{N}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ is an example of hyper-Kähler quotients known as Nakajima's quiver varieties (see [KP], [KS], [M], [Nak1]). Applying Theorem 1.6 to this particular example, we obtain a family of $\mathrm{O}_n(\mathbb{C})$-equivariant involutions

$$\alpha_a : \mathcal{N}_n(\mathbb{C}) \longrightarrow \mathcal{N}_n(\mathbb{C}) \qquad a \in [0,1] \tag{1.12}$$

interpolating the complex conjugation $\alpha_0(M) = \overline{M}$ and the transpose $\alpha_1(M) = M^t$, and a $\mathrm{O}_n(\mathbb{C})$-equivariant homeomorphism

$$\mathcal{N}_n(\mathbb{R}) \xrightarrow{\sim} \mathcal{N}_n^{sym}(\mathbb{C}) \tag{1.13}$$

between real and symmetric nilpotent cone which restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-orbits and $\mathrm{O}_n(\mathbb{C})$-orbits. This establishes a special case of Theorems 1.3 and 1.4 for the fiber of the adjoint quotient map $\chi' : \mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$ over $0 \in \mathbb{C}^n$, that is, matrices with zero eigenvalues. To extend the results to matrices with real eigenvalues, we prove a version of Theorem 1.6 for the family of hyper-Kähler quotients

$$\mathfrak{M}_{Z_\mathbb{C}} = \mu_\mathbb{R}^{-1}(0) \cap \mu_\mathbb{C}^{-1}(Z_\mathbb{C})/H_u \to Z_\mathbb{C}$$

where $Z_\mathbb{C} \subset \mathfrak{h}^*$ is the dual of the center of $\mathfrak{h}$ and then deduce the results using the description of general adjoint orbits closures as quiver varieties in [MV]. Finally, we check that the constructions are compatible with inner automorphisms and Cartan involutions and then deduce the case of Lie algebras of classical types from the case of $\mathfrak{gl}_n(\mathbb{C})$.

We would like to emphasize that the keys in the proof of Theorems 1.3 and 1.4 are the symmetries on adjoint orbit closures (or rather, the symmetries on the whole family $\mathfrak{gl}'_n(\mathbb{C}) \to \mathbb{C}^n$) coming from the hyper-Kähler SU(2)-action. Those symmetries are not immediately visible in their original definitions as algebraic varieties.

Remark 1.8. The use of hyper-Kähler SU(2)-actions in the study of geometry of nilpotent orbits goes back to the celebrated work of Kronheimer [Kr] where he used those symmetries to gave a differential-geometric interpretation of Brieskorn's theorem on sub-regular singularities.

1.2. **Applications.** We discuss here applications to the Kostant-Sekiguchi correspondence, singularities of real and symmetric adjoint orbit closures, and Springer theory for symmetric spaces.

In the rest of the section, we assume $\mathfrak{g}$ is of classical type.

1.2.1. *Generalized Kostant-Sekiguchi homeomorphisms.* The celebrated Kostant-Sekiguchi correspondence is an isomorphism between real and symmetric nilpotent orbit posets

$$|G_\mathbb{R}\backslash\mathcal{N}_\mathbb{R}| \longleftrightarrow |K\backslash\mathcal{N}_\mathfrak{p}|. \tag{1.14}$$

The bijection was proved by Kostant (unpublished) and Sekiguchi [S]. Vergne [V], using Kronheimer's instanton flow [Kr], showed the corresponding orbits are diffeomorphic. Schmid-Vilonen [SV] gave an alternative proof and further refinements using Ness' moment map. Barbasch-Sepanski [BaSe] deduced the bijection is a poset isomorphism from Vergne's results.

We shall state a lift/generalization of the Kostant-Sekiguchi correspondence to stratified homeomorphisms between adjoint orbits closures in the real Lie algebra $\mathfrak{g}_\mathbb{R}$ and symmetric subspace $\mathfrak{p}$ whose eigenvalues are real but not necessarily zero.

Denote by $\mathcal{N}_\xi = \chi^{-1}(\xi)$ the fiber of the Chevalley map $\chi : \mathfrak{g} \to \mathfrak{c}$ over $\xi \in \mathfrak{c}$. In [Ko1], Kostant proved that there are finitely many G-orbits in $\mathcal{N}_\xi$ and there is a unique closed

orbit $\mathcal{O}_\xi^{\mathfrak{s}}$ consisting of semisimple elements and a unique open orbit $\mathcal{O}_\xi^{\mathfrak{r}}$ consisting of regular elements. Moreover, we have $\mathcal{N}_\xi = \overline{\mathcal{O}_\xi^{\mathfrak{r}}}$.

Assume $\xi \in \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$. Then ξ is fixed by the involutions on $\mathfrak{c}$ induced by η and $-\theta$ and hence the fiber $\mathcal{N}_\xi$ is stable under η and $-\theta$. We write

$$\mathcal{N}_{\xi,\mathbb{R}} = \mathcal{N}_\xi \cap \mathfrak{g}_\mathbb{R} \qquad \mathcal{N}_{\xi,\mathfrak{p}} = \mathcal{N}_\xi \cap \mathfrak{p}.$$

for the fixed points. There are finitely many $G_\mathbb{R}$-orbits and K-orbits on $\mathcal{N}_{\xi,\mathbb{R}}$ and $\mathcal{N}_{\xi,\mathfrak{p}}$

$$\mathcal{N}_{\xi,\mathbb{R}} = \bigsqcup_l \mathcal{O}_{\mathbb{R},l} \qquad \mathcal{N}_{\xi,\mathfrak{p}} = \bigsqcup_l \mathcal{O}_{\mathfrak{p},l}$$

Corollary 1.9. *There is a $K_\mathbb{R}$-equivaraint stratified homeomorphism*

$$\mathcal{N}_{\xi,\mathbb{R}} \xrightarrow{\sim} \mathcal{N}_{\xi,\mathfrak{p}} \tag{1.15}$$

which restricts to real analytic isomorphisms between individual $G_\mathbb{R}$-orbits and K-orbits. The homeomorphism induces an isomorphism between $G_\mathbb{R}$-orbits and K-orbits posets

$$|G_\mathbb{R}\backslash\mathcal{N}_{\xi,\mathbb{R}}| \longleftrightarrow |K\backslash\mathcal{N}_{\xi,\mathfrak{p}}|. \tag{1.16}$$

Proof. It follows immediately from Theorem 1.3. □

Remark 1.10. Thanks to the work of Vergne [V], it is known that under the Kostant-Sekiguchi bijection the correspondence orbits are diffeomorphic. It is an open question whether the corresponding orbit closures have the same singularities (see, e.g., [He, Introduction]). Corollary 1.9 gives a positive answer in the case of classical Lie algebras.

Remark 1.11. In [Bie] and [Biq], the authors proved an extended Kostant-Sekiguchi correspondence for certain adjoint orbits. We expect that their correspondence is compatible with the one in (1.16).

Remark 1.12. In Theorem 3.2, we also establish a Kostant-Sekiguchi correspondence between real and symmetric symplectic leaves for quiver varieties.

1.2.2. *Derived categories.* Let $D_{G_\mathbb{R}}(\mathcal{N}_{\xi,\mathbb{R}})$, $D_K(\mathcal{N}_{\xi,\mathfrak{p}})$ denote the respective equivariant derived categories of sheaves (over any commutative ring). Since $K_\mathbb{R} \to G_\mathbb{R}$, $K_\mathbb{R} \to K$ are homotopy equivalences, the forgetful functors $D_{G_\mathbb{R}}(\mathcal{N}_\mathbb{R}) \to D_{K_\mathbb{R}}(\mathcal{N}_\mathbb{R})$, $D_K(\mathcal{N}_\mathfrak{p}) \to D_{K_\mathbb{R}}(\mathcal{N}_\mathfrak{p})$ to $K_\mathbb{R}$-equivariant complexes are fully faithful with essential image those complexes constructible along the respective orbits of $G_\mathbb{R}$ and K.

Transport along the homeomorphism of Theorem 1.9 immediately provides:

Corollary 1.13. *Pushforward along the homeomorphism* (1.15) *provides an equivalence of equivariant derived categories*

$$D_{G_\mathbb{R}}(\mathcal{N}_{\xi,\mathbb{R}}) \simeq D_K(\mathcal{N}_{\xi,\mathfrak{p}}) \tag{1.17}$$

1.2.3. *Vanishing of odd dimensional intersection cohomology.* Theorem 1.9 implies that the singularities of symmetric nilpotent orbit closures $\bar{\mathcal{O}}_{\mathfrak{p}} \subset \mathcal{N}_{\mathfrak{p}}$ are homeomorphic to the singularities of the corresponding real nilpotent orbit closures $\bar{\mathcal{O}}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$. Thus we can deduce results about one from the other.

Here is a notable example. Let $\mathrm{IC}(\mathcal{O}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ be the intersection cohomology sheaf of a real nilpotent orbit $\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$ with coefficients in a $G_{\mathbb{R}}$-equivariant local system $\mathcal{L}_{\mathbb{R}}$. (Recall that all nilpotent orbits $\mathcal{O} \subset \mathcal{N}$ have even complex dimension, so all real nilpotent orbits $\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}}$ have even real dimension, hence middle perversity makes sense.)

Corollary 1.14. *The cohomology sheaves $\mathcal{H}^i(\mathrm{IC}(\mathcal{O}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}))$ vanish for $i - \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{R}}/2$ odd.*

Proof. Using the equivalence (1.17), it suffices to prove the asserted vanishing for the intersection cohomology sheaf $\mathrm{IC}(\mathcal{O}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ of a symmetric nilpotent orbit $\mathcal{O}_{\mathfrak{p}} \subset \mathcal{N}_{\mathfrak{p}}$ with coefficients in a K-equivariant local system $\mathcal{L}_{\mathfrak{p}}$, and $i - \dim_{\mathbb{C}} \mathcal{O}_{\mathfrak{p}}$ odd. This is proved in [LY, Theorem 14.10].[3] □

Remark 1.15. The proof of [LY, Theorem 14.10] makes use of Deligne's theory of weights and the theory of canonical bases, and hence does not have an evident generalization to a real algebraic setting.

1.2.4. *Formula for the sheaf of symmetric nearby cycles.* Consider the quotient map $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}} = \mathfrak{p}//K$. According to [KR], the generic fiber of $\chi_{\mathfrak{p}}$ is a single K-orbit through a semisimple element in $\mathfrak{p}$ and the special fiber over the base point $\chi_{\mathfrak{p}}(0) \in \mathfrak{c}_{\mathfrak{p}}$ is the symmetric nilpotent cone $\mathcal{N}_{\mathfrak{p}}$. Following Grinberg [G1] (see also [G2, GVX]), we consider the sheaf $\mathcal{F}_{\mathfrak{p}} \in D_K(\mathcal{N}_{\mathfrak{p}})$ of nearby cycles along the special fiber $\mathcal{N}_{\mathfrak{p}}$ in the family $\chi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{c}_{\mathfrak{p}}$ (see Section 5.3 for the precise definition). We will call $\mathcal{F}_{\mathfrak{p}}$ the *sheaf of symmetric nearby cycles*.

Let $B_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a minimal parabolic subgroup with Lie algebra $\mathfrak{b}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$ where $\mathfrak{m}_{\mathbb{R}} = Z_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ and $\mathfrak{n}_{\mathbb{R}}$ is the nilpotent radical. Consider the real Springer map

$$\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}$$

where $\widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}$ and $\pi_{\mathbb{R}}(g, v) = \mathrm{Ad}_g\, v$.

We have the following formula for the sheaf of symmetric nearby cycles:

Corollary 1.16 (Theorem 5.3)**.** *Under the equivalence $D_K(\mathcal{N}_{\mathfrak{p}}) \simeq D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}})$ (1.17), the sheaf of symmetric nearby cycles $\mathcal{F}_{\mathfrak{p}}$ becomes the real Springer sheaf $\mathcal{S}_{\mathbb{R}} := (\pi_{\mathbb{R}})_! \mathbb{C}[\dim_{\mathbb{R}} \mathcal{N}_{\mathbb{R}}/2]$. In particular, the real Springer map $\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}$ is a semi-small map and the real Springer sheaf $\mathcal{S}_{\mathbb{R}}$ is a perverse sheaf.*

In fact, Theorem 5.3 is slightly stronger than the one stated here. We also prove a formula for the sheaf of symmetric nearby cycles with coefficients in K-equivariant local systems and we show that, for any $\mathfrak{g}_{\mathbb{R}}$ (not just for classical types), the real Springer sheaf is isomorphic to the *sheaf of real nearby cycles* $\mathcal{F}_{\mathbb{R}}$ introduced in Section 5.2.

Remark 1.17. The formula above for symmetric nearby cycles was originally conjectured by Vilonen-Xue and the first author. It can be viewed as a symmetric space version of the

[3] In fact, [LY] establishes the odd vanishing in the more general setting of graded Lie algebras.

well-known result that the sheaf of nearby cycles along the special fiber $\mathcal{N}$ in the family $\chi : \mathfrak{g} \to \mathfrak{c}$ is isomorphic to the Springer sheaf.

Remark 1.18. In [CVX], the authors used the sheaves of symmetric nearby cycles (with coefficients) to produce all cuspidal complexes on $\mathcal{N}_\mathfrak{p}$ and use them to establish a Springer correspondence for the split symmetric pair of type A (see [VX] for the cases of classical symmetric pairs). The formula established in Corollary 1.16 provides new insights and methods into the study of Springer theory for general symmetric pairs and real groups. We will give one example below. The details will be discussed in a sequel [CN2].

1.2.5. *Real Springer theory and Hecke algebras at roots of unity.* In [G1], Grinberg gave a generalization of Springer theory using nearby cycles. One of the main results in *loc. cit.* is a description of the endomorphism algebra $\mathrm{End}(\mathcal{F}_\mathfrak{p})$ of the sheaf of symmetric nearby cycles as a certain Hecke algebra at roots of unity.[4] To explain his result, let $(\Phi, \mathfrak{a}_\mathbb{R}^*)$ be the root system (possible non-reduced) of $(\mathfrak{g}_\mathbb{R}, \mathfrak{a}_\mathbb{R})$. For each $\alpha \in \Phi$ we denote by $\mathfrak{g}_{\mathbb{R},\alpha} \subset \mathfrak{g}_\mathbb{R}$ the corresponding α-eigenspace. Choose a system of simple roots $\Delta \subset \Phi$ and let $\mathrm{S} \subset \mathrm{W}$ be the set of simple reflections of the little Weyl group associated to Δ. Consider the following algebra

$$\mathcal{H}_{G_\mathbb{R}} := \mathbb{C}[B_\mathrm{W}]/(T_s - 1)(T_s + (-1)^{d_s})_{s\in \mathrm{S}}$$

where $\mathbb{C}[B_\mathrm{W}]$ is the group algebra of the braid group B_W of W with generators T_s, $s \in \mathrm{S}$, and d_s is the integer given by

$$d_s = \sum_{\alpha\in\Delta, s_\alpha = s} \dim_\mathbb{R}(\mathfrak{g}_{\mathbb{R},\alpha}),$$

where s_α denotes the reflection corresponding to the simple root $\alpha \in \Delta$[5]. For examples, if $G_\mathbb{R}$ is a split real from, then we have $d_s = 1$ for all $s \in \mathrm{S}$ and $\mathcal{H}_{G_\mathbb{R}}$ is isomorphic to the Hecke algebra associated to W at $q = -1$. On the other hand, if $G_\mathbb{R}$ is a complex group, then we have $d_s = 2$ and $\mathcal{H}_{G_\mathbb{R}}$ is isomorphic to the group algebra $\mathbb{C}[\mathrm{W}]$.

In [G1, Theorem 6.1], Grinberg showed that there is a canonical isomorphism of algebras

$$\mathrm{End}(\mathcal{F}_\mathfrak{p}) \simeq \mathcal{H}_{G_\mathbb{R}}. \tag{1.18}$$

Since the algebra $\mathcal{H}_{G_\mathbb{R}}$ is in general not semi-simple, as an interesting corollary of (1.18), we see that the sheaf of symmetric nearby cycles $\mathcal{F}_\mathfrak{p}$ is not semi-simple in general.

Now combining Corollary 1.16 with Grinberg's theorem, we obtain the the following result in real Springer theory:

Corollary 1.19. *We have a canonical isomorphism of algebras*

$$\mathrm{End}(\mathcal{S}_\mathbb{R}) \simeq \mathcal{H}_{G_\mathbb{R}}.$$

In particular, the real Springer sheaf $\mathcal{S}_\mathbb{R}$ is in general not semi-simple and, for any $x \in \mathcal{N}_\mathbb{R}$, the cohomologies $\mathrm{H}^*(\mathcal{B}_x, \mathbb{C})$ of the real Springer fiber $\mathcal{B}_x = \pi_\mathbb{R}^{-1}(x)$ carry a natural action of the algebra $\mathcal{H}_{G_\mathbb{R}}$.

[4] In fact, he works in a more general setting of polar representations.

[5] Since the root system might not be reduced, there might be more than one simple root α such that $s_\alpha = s$.

Remark 1.20. In [CN2], we will give an alternative proof of Corollary 1.19 (for all types) following the classical arguments in Springer theory. In particular, combining with Corollary 1.16, we obtain a new proof of Grinberg's theorem on the endomorphism algebra of $\mathcal{F}_{\mathfrak{p}}$.

1.3. **Previous work.** In our previous work [CN1], we establish Corollary 1.9 in the case of nilpotent cone using the geometry of moduli space of quasi-maps associated to a symmetric pair (G, K). In more detail, we use the factorization properties of the moduli space of quasi-maps to establish a real-symmetric homeomorphism in the setting of Beilinson-Drinfeld Grassmannians (for any reductive group G) and then deduce Corollary 1.9 using the Lusztig embedding of the nilpotent cone for $\mathfrak{gl}_n(\mathbb{C})$ into the affine Grassmannian for $\mathrm{GL}_n(\mathbb{C})$. The result in the present paper suggests that there should be a hyper-Kähler geometry interpretation of the results in [CN1]. This will be discussed in detail in a sequel.

We conclude the introduction with the following conjecture.

Conjecture 1.21. *Theorems 1.3 and 1.4 remain true when $\mathfrak{g}$ is of exceptional type.*

1.4. **Organization.** We briefly summarize here the main goals of each section. In Sect. 2 immediately to follow, we study involutions on hyper-Kähler quotients of linear spaces. In Sect. 3, we apply the results established in the previous section to the case of quiver varieties. In Sect. 4, we establish our main results Theorems 4.1 and 4.2. In Sect. 5, we discuss applications to Springer theory for real groups and symmetric spaces.

1.5. **Acknowledgements.** The authors would like to thank Marco Gualtieri for inspiring discussions about symmetries of hyper-Kähler quotients and Jeffrey Adams for useful discussions about Kostant-Sekiguchi correspondence. We also would like to thank Kari Vilonen and Ting Xue for useful comments.

The research of T.H. Chen is supported by NSF grant DMS-1702337 and that of D. Nadler by NSF grant DMS-1802373.

2. A family of involutions on hyper-Kähler quotients

In this section we introduce a family of involutions on hyper-Kähler quotients of linear spaces with remarkable properties. The main references for hyper-Kähler quotients are [Hi] and [HKLR].

2.1. **Quaternions.** Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternions. For any $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ we denote by $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$. Then the paring $(x, x') = \mathrm{Re}(x\bar{x}')$ defines a real-valued inner product on $\mathbb{H}$. We denote by $\mathrm{Im}(\mathbb{H}) = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$, the pure imaginary quaternions, and $\mathrm{Sp}(1) = \{x \in \mathbb{H} | (x, x) = 1\}$ the group of quaternions of norm one.

2.2. **Hyper-Kähler quotient of linear spaces.** Let H be a complex reductive group with compact real form H_u. Let $\mathbf{M}$ be a quaternionic representation of H_u, that is, $\mathbf{M}$ is a finite dimensional quaternionic vector space together with a $\mathbb{H}$-linear action of H_u. We assume that the quaternionic representation is unitary, that is, there is a H_u-inner product $(,)$ on $\mathbf{M}$ which is hermitian with respect to the complex structures I, J, K on $\mathbf{M}$ given by

multiplication by i, j, k respectively. We have a natural complex representation of H on $\mathbf{M}$ preserving the complex symplectic form $\omega_{\mathbb{C}}(v, v') = (Jv, v') + i(Kv, v')$ on $\mathbf{M}$.

We have the hyper-Kähler moment map

$$\mu : \mathbf{M} \to \operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{h}_u^*$$

satisfying

$$\langle \xi, \mu(\phi) \rangle = (I\xi\phi, \phi)i + (J\xi\phi, \phi)j + (K\xi\phi, \phi)k \in \operatorname{Im}\mathbb{H}$$

where $\xi \in \mathfrak{h}_u$, $\phi \in \mathbf{M}$, and $\langle , \rangle$ is the paring between $\mathfrak{h}_u^*$ and $\mathfrak{h}_u$. The map μ has the following equivariant properties: (1) it intertwines the $\mathrm{Sp}(1) \times H_u$ action on $\mathbf{M}$ and the one on $\operatorname{Im}(\mathbb{H}) \otimes_{\mathbb{R}} \mathfrak{h}_u^*$ given by $(q, h)(w, u) = (\mathrm{Ad}_q\, w, \mathrm{Ad}_{h^{-1}}\, u)$ (2) we have $\mu(tv) = t^2\mu(v)$ for $t \in \mathbb{R}^\times, v \in \mathbf{M}$.

Using the isomorphism $\operatorname{Im}\mathbb{H} = \mathbb{R} \oplus \mathbb{C}$ sending $x_1 i + x_2 j + x_3 k$ to $(x_1, x_2 + x_3 i)$, we can identify $\operatorname{Im}\mathbb{H} \otimes \mathfrak{h}_u^* = \mathfrak{h}_u^* \oplus \mathfrak{h}^*$ and hence obtain a decomposition of the moment map

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}_u^* \oplus \mathfrak{h}^*$$

of μ into real and complex components. The map $\mu_{\mathbb{C}} : \mathbf{M} \to \mathfrak{h}^*$ is holomorphic with respect to the complex structure I on $\mathbf{M}$ and satisfies

$$\langle \xi, \mu_{\mathbb{C}}(\phi) \rangle = \omega_{\mathbb{C}}(\xi\phi, \phi)$$

where $\xi \in \mathfrak{h}$ and $\phi \in \mathbf{M}$. Moreover, it is H-equivariant with respect to the complex representation of H on $\mathbf{M}$ and the the inverse of the adjoint representation on $\mathfrak{h}^*$.

Let $Z = \{v \in \mathfrak{h}_u^* | \, \mathrm{Ad}_h(v) = v \text{ for all } h \in H_u\}$ and $Z_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} Z$. Then we have $\operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} Z = Z \oplus Z_{\mathbb{C}}$. For any $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$, we can consider the hyper-Kähler quotient

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})/H_u. \tag{2.1}$$

We have the holomorphic description

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} \simeq \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})//H$$

where the right hand side is the categorial quotient of $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ by H. One can form a perturbed hyper-Kähler quotient

$$\mathfrak{M}_{(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})} = \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})/H_u$$

with not necessarily zero real component $\zeta_{\mathbb{R}}$. The composition $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})//H$ gives rise to a map

$$\pi : \mathfrak{M}_{(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})} \to \mathfrak{M}_{\zeta_{\mathbb{C}}} \tag{2.2}$$

which is holomorphic with respect to the complex structure I.

From now on we will fix a real parameter $\zeta_{\mathbb{R}}$. For any subset $S \subset Z_{\mathbb{C}}$ we can consider the following family of hyper-Kähler quotients

$$\chi_S : \mathfrak{M}_S = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-S)/H_u \to S$$

$$\tilde{\chi}_S : \mathfrak{M}_{(\zeta_{\mathbb{R}}, S)} = \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(-S)/H_u \to S$$

Then the map (2.2) gives rise to a map

$$\pi_S : \mathfrak{M}_{(\zeta_{\mathbb{R}}, S)} \to \mathfrak{M}_S \tag{2.3}$$

compatible with the projection maps to S.

2.3. **A stratification.** Let $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$. Let L be a subgroup of H_u. We denote by $\mathbf{M}_{(L)}$ be the set of all points in $\mathbf{M}$ whose stabilizer is conjugate to L. A point in $\mathfrak{M}_{\zeta_{\mathbb{C}}}$ is said to be of stabilizer type (L) if it has a representative in $\mathbf{M}_{(L)}$. The set of all points of stabilizer type (L) is denoted by $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$. We have a *orbit type* stratification

$$\mathfrak{M}_{\zeta_{\mathbb{C}}} = \bigsqcup_{(L)} \mathfrak{M}_{\zeta_{\mathbb{C}},(L)} \tag{2.4}$$

where the union runs over the set of all conjugacy classes of subgroups of U. Each stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ is a smooth hyper-Kähler manifold, moreover, it is an affine symplectic variety with respect to the complex structure I.

2.4. **Symmetries of hyper-Kähler quotients.** Let G be another complex reductive group with a compact real form G_u. Consider a unitary representation of G_u on $\mathbf{M}$ commuting with the H_u-action on $\mathbf{M}$. Then for any $S \subset Z_{\mathbb{C}}$, the action of the complexification G on $\mathbf{M}$ descends to an action on the hyper-Kähler quotient $\mathfrak{M}_S$ which is compatible with the projection map to S and holomorphic with respect to the complex structure I.

Assume $S \subset Z_{\mathbb{C}}$ is a $\mathbb{R}$-linear subspace. Then the action of $\mathbb{R}^\times$ on $\mathbf{M}$ descends to a G-equivaraint $\mathbb{R}^\times$-action on $\mathfrak{M}_S$:

$$\phi(t) : \mathfrak{M}_S \to \mathfrak{M}_S, \quad t \in \mathbb{R}^\times. \tag{2.5}$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_S & \xrightarrow{\phi(t)} & \mathfrak{M}_S \\ \downarrow & & \downarrow \\ S & \xrightarrow{t^2(-)} & S \end{array}$$

where the bottom arrow is the multiplication by t^2.

Let $q \in \mathrm{Sp}(1)$ such that $\mathrm{Ad}_q(S) \subset S$. Here $\mathrm{Ad}_q : \mathrm{Im}\,\mathbb{H} \otimes Z \to \mathrm{Im}\,\mathbb{H} \otimes Z$ is the map $\mathrm{Ad}_q(w, u) = (\mathrm{Ad}_q\, w, u)$ and we identify $Z_{\mathbb{C}} = (\mathbb{R}j \oplus \mathbb{R}k) \otimes Z$, and hence S, as a subspace of $\mathrm{Im}\,\mathbb{H} \otimes Z$ with zero i-component. The action of $q \in \mathrm{Sp}(1)$ on $\mathbf{M}$ gives rise to a G_u-equivaraint map

$$\phi(q) : \mathfrak{M}_S \to \mathfrak{M}_S \tag{2.6}$$

commuting with the $\mathbb{R}^\times$-actions. In addition, we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_S & \xrightarrow{\phi(q)} & \mathfrak{M}_S \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \end{array}$$

where the bottom arrow is $\mathrm{Ad}_q : S \to S$.

It is straightforward to check that the stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ in (2.4) is stable under the G and $\mathbb{R}^\times$-actions. Moreover, for any $q \in \mathrm{Sp}(1)$ (resp. $t \in \mathbb{R}^\times$) and S as above the map $\phi(q)$ (resp.

$\phi(t)$) is compatible with the stratifications in the sense that it maps the stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ in the fiber $\chi_S^{-1}(\zeta_\mathbb{C}) = \mathfrak{M}_{\zeta_\mathbb{C}}$ to the corresponding stratum $\mathfrak{M}_{\zeta'_\mathbb{C},(L)}$ in the fiber $\chi_S^{-1}(\zeta'_\mathbb{C}) = \mathfrak{M}_{\zeta'_\mathbb{C}}$ where $\zeta'_\mathbb{C} = \mathrm{Ad}_q\, \zeta_\mathbb{C}$ (resp. $\zeta'_\mathbb{C} = t^2\zeta_\mathbb{C}$).

Example 2.1. Let $S = 0$. Then we have $\mathrm{Ad}_q(0) = 0$ for all $q \in \mathrm{Sp}(1)$ and the family of maps $\phi(q)$ in (2.6) gives rise to a $(G_u \times \mathbb{R}^\times)$-equivarint $\mathrm{Sp}(1)$-action on $\mathfrak{M}_0$, called the *hyper-Kähler* $\mathrm{Sp}(1)$*-action*. Moreover the stratum $\mathfrak{M}_{0,(L)}$ is stable under the $\mathrm{Sp}(1)$-action.

2.5. **Conjugations on $\mathfrak{M}_{Z_\mathbb{C}}$.**

Definition 2.2. Let η_H and $\eta_\mathbf{M}$ be conjugations on H and $\mathbf{M}$ respectively. We say that η_H and $\eta_\mathbf{M}$ are compatible with the symplectic representation of H on $\mathbf{M}$ if the following holds:

(1) we have $\eta_\mathbf{M}(hv) = \eta_H(h)\eta_\mathbf{M}(v)$ for all $h \in H$ and $v \in \mathbf{M}$.
(2) we have $\omega_\mathbb{C}(\eta_\mathbf{M}(v), \eta_\mathbf{M}(v')) = \overline{\omega_\mathbb{C}(v, v')}$ for all $v, v' \in \mathbf{M}$.

Lemma 2.3. *Let η_H and $\eta_{\boldsymbol{M}}$ be conjugations on H and $\boldsymbol{M}$ compatible with the symplectic representation of H on $\boldsymbol{M}$. Then the complex moment map $\mu_\mathbb{C} : \boldsymbol{M} \to \mathfrak{h}^*$ intertwines $\eta_{\boldsymbol{M}}$ and η_H.*

Proof. For any $\xi \in \mathfrak{h}, v \in \mathbf{M}$, we have

$$\langle \xi, \mu_\mathbb{C}(\eta_\mathbf{M}(v))\rangle = \omega_\mathbb{C}(\xi\eta_\mathbf{M}(v), \eta_\mathbf{M}(v)) = \omega_\mathbb{C}(\eta_\mathbf{M}(\eta_H(\xi)v), \eta_\mathbf{M}(v)) = \overline{\omega_\mathbb{C}(\eta_H(\xi)v, v)} =$$

$$= \overline{\langle \eta_H(\xi), \mu_\mathbb{C}(v)\rangle} = \langle \xi, \eta_H(\mu_\mathbb{C}(v))\rangle.$$

This implies $\mu_\mathbb{C}(\eta_\mathbf{M}(v)) = \eta_H(\mu_\mathbb{C}(v))$ for all $v \in \mathbf{M}$. The lemma follows. □

Let η_H and $\eta_\mathbf{M}$ be as in Lemma 2.3. Then the center of $\mathfrak{h}$, and hence $Z_\mathbb{C}$, is stable under η_H. It follows that, for any $\zeta_\mathbb{C} \in Z_\mathbb{C}$, the conjugation $\eta_\mathbf{M}$ on $\mathbf{M}$ descends to a map

$$\mathfrak{M}_{\zeta_\mathbb{C}} = \mu_\mathbb{C}^{-1}(-\zeta_\mathbb{C})//H \to \mathfrak{M}_{\eta_H(\zeta_\mathbb{C})} = \mu_\mathbb{C}^{-1}(-\eta_H(\zeta_\mathbb{C}))//H \tag{2.7}$$

which is anti-holomorphic with respect to the complex structure I. Moreover, it maps the stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ to the corresponding stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$. As $\zeta_\mathbb{C}$ varies over $Z_\mathbb{C}$, the map (2.7) organize into a map

$$\eta_{Z_\mathbb{C}} : \mathfrak{M}_{Z_\mathbb{C}} \to \mathfrak{M}_{Z_\mathbb{C}} \tag{2.8}$$

making the following diagram commute

$$\begin{array}{ccc} \mathfrak{M}_{Z_\mathbb{C}} & \xrightarrow{\eta_{Z_\mathbb{C}}} & \mathfrak{M}_{Z_\mathbb{C}} \\ \downarrow & & \downarrow \\ Z_\mathbb{C} & \xrightarrow{\eta_H} & Z_\mathbb{C} \end{array} . \tag{2.9}$$

We will call $\eta_{Z_\mathbb{C}}$ the *conjugation* on $\mathfrak{M}_{Z_\mathbb{C}}$ associated to the conjugations η_H and $\eta_\mathbf{M}$.

2.6. **Compatibility with symmetries.** Recall the $\mathbb{R}$-subspace $Z \subset Z_{\mathbb{C}}$. For any $s \in [0, 2\pi]$, let

$$q_s = \cos(s)i + \sin(s)k \in \mathrm{Sp}(1).$$

A direct computation shows that Ad_{q_s} preserves the subspace $Z = \mathbb{R}j \otimes_{\mathbb{R}} Z \subset \mathrm{Im}\,\mathbb{H} \otimes_{\mathbb{R}} Z$[6] and its restriction to Z is given by $-\mathrm{id}_Z$. Consider the family of hyper-Kähler quotients

$$\mathfrak{M}_Z = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-Z)/H_u \tag{2.10}$$

over Z. Then the discussion in the previous section shows that there is a family of maps

$$\phi_s = \phi(q_s) : \mathfrak{M}_Z \to \mathfrak{M}_Z \qquad s \in [0, 2\pi] \tag{2.11}$$

making the following diagram commute

$$\begin{array}{ccc} \mathfrak{M}_Z & \xrightarrow{\phi_s} & \mathfrak{M}_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{-\mathrm{id}_Z} & Z \end{array} . \tag{2.12}$$

Consider $j \in \mathrm{Sp}(1)$. Since $\mathrm{Ad}_j = \mathrm{id}_Z$ on Z we have a map

$$\phi(j) : \mathfrak{M}_Z \to \mathfrak{M}_Z \tag{2.13}$$

making the following diagram commute

$$\begin{array}{ccc} \mathfrak{M}_Z & \xrightarrow{\phi(j)} & \mathfrak{M}_Z \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\mathrm{id}_Z} & Z \end{array} . \tag{2.14}$$

Note that

$$\phi_s^2 = \phi(j)^2 = \phi(-1). \tag{2.15}$$

In particular, if $\phi(-1)$ is equal to the identity map then ϕ_s and $\phi(j)$ are involutions on $\mathfrak{M}_Z$.

Our next goal is to study the compatibility between the maps ϕ_s, $\phi(j)$, and the conjugation $\eta_{Z_{\mathbb{C}}}$ introduced in Section 2.5.

Definition 2.4. Let η_H and $\eta_{\mathbf{M}}$ be conjugations on H and $\mathbf{M}$ respectively. We say that η_H and $\eta_{\mathbf{M}}$ are compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$ if the following holds:

(1) the pair (η_H,$\eta_{\mathbf{M}}$) is compatible with the symplectic representation of H on $\mathbf{M}$ (see Definition 2.2)
(2) $\eta_{\mathbf{M}}$ preserves the inner product $(,)$, that is, we have $(\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = (v, v')$ for $v, v' \in \mathbf{M}$.
(3) η_H commutes with the Cartan conjugation δ_H.

[6]Note that $Z_{\mathbb{C}} = (\mathbb{R}j + \mathbb{R}k) \otimes_{\mathbb{R}} Z$ is not stable under the family of maps Ad_{q_s}.

Proposition 2.5. *Let η_H and $\eta_{\boldsymbol{M}}$ be conjugations on H and $\boldsymbol{M}$ compatible with the unitary quaternionic representation of H_u on $\boldsymbol{M}$. Let $\eta_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{M}_{Z_{\mathbb{C}}}$ be the conjugation in (2.8)[7]. Then the subspace $\mathfrak{M}_Z \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ is stable under $\eta_{Z_{\mathbb{C}}}$. Moreover if we denote by*

$$\eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z \tag{2.16}$$

the resulting map, we have the following equality of maps on $\mathfrak{M}_Z$

$$\phi_s \circ \eta_Z = \phi(-1) \circ \eta_Z \circ \phi_s \quad \phi_s \circ \phi(j) = \phi(-j) \circ \phi_s \quad \phi(j) \circ \eta_Z = \eta_Z \circ \phi(j) \tag{2.17}$$

Proof. Since η_H commutes with the Cartan conjugation δ_H, the center of $\mathfrak{h}_u$, and hence its real dual $Z \subset \mathfrak{h}_u^*$, is stable under η_H and (2.9) implies that $\mathfrak{M}_Z$ is preserved by the conjugation $\eta_{Z_{\mathbb{C}}}$.

We claim that conditions (1) and (2) in Definition 2.4 imply that $\eta_{\mathbf{M}}$ commutes with J and preserves $\mu_{\mathbb{R}}^{-1}(0)$. Assume the claim for the moment. Then using the equality $I \circ \eta_{\mathbf{M}} = -\eta_{\mathbf{M}} \circ I$ and $K = IJ$, a direct computation shows that we have the following equality of maps on $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-Z)$:

$$(\cos(s)I + \sin(s)K) \circ \eta_{\mathbf{M}} = -\eta_{\mathbf{M}} \circ (\cos(s)I + \sin(s)K)$$

$$(\cos(s)I + \sin(s)K) \circ J = -J \circ (\cos(s)I + \sin(s)K)$$

$$J \circ \eta_{\mathbf{M}} = \eta_{\mathbf{M}} \circ J$$

compatible with the H_u-action. The desired equality (2.17) follows.

Proof of the claim. For any $\xi \in \mathfrak{h}_u$ and $v \in \mathbf{M}$, we have

$$\langle \xi, \mu_{\mathbb{R}}(\eta_{\mathbf{M}}(v)) \rangle = (I\xi\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v)) = -(\eta_{\mathbf{M}}(I\eta_H(\xi)v), \eta_{\mathbf{M}}(v)) = -(I\eta_H(\xi)v, v) = \langle -\eta_H(\xi), \mu_{\mathbb{R}}(v) \rangle =$$

$$= \langle \xi, -\eta_H(\mu_{\mathbb{R}}(v)) \rangle$$

Thus we have $\mu_{\mathbb{R}}(\eta_{\mathbf{M}}(v)) = -\eta_H(\mu_{\mathbb{R}}(v))$ and it follows that $\mu_{\mathbb{R}}^{-1}(0)$ is stable under the conjugation $\eta_{\mathbf{M}}$. Recall that $\omega_{\mathbb{C}}(v, v') = (Jv, v') + i(Kv, v')$. Thus the equality $\omega_{\mathbb{C}}(\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = \overline{\omega_{\mathbb{C}}(v, v')}$ is equivalent to

$$(J\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) + i(K\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = (Jv, v') - i(Kv, v')$$

which implies

$$(J\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = (Jv, v').$$

Since $\eta_{\mathbf{M}}$ preserves $(,)$, the above equality implies

$$(J\eta_{\mathbf{M}}(v), \eta_{\mathbf{M}}(v')) = (\eta_{\mathbf{M}} J(v), \eta_{\mathbf{M}}(v'))$$

and it follows that $J \circ \eta_{\mathbf{M}} = \eta_{\mathbf{M}} \circ J$. This finishes the proof of the claim. □

Remark 2.6. The proof above shows that condition (2) in Definition 2.4 is equivalent to the condition that $\eta_{\mathbf{M}}$ commutes with J.

[7] $\eta_{Z_{\mathbb{C}}}$ is well-defined since η_H and $\eta_{\mathbf{M}}$ are compatible with the symplectic representation of H on $\mathbf{M}$.

2.7. **A family of involutions.** Let η_H and $\eta_{\mathbf{M}}$ be conjugations on H and $\mathbf{M}$ compatible with the unitary quaternionic representation of H_u on $\mathbf{M}$. Let G be another complex reductive group with a compact real from G_u and let η_G be a conjugation on G with real form $G_{\mathbb{R}}$. Suppose that $\mathbf{M}$ is a unitary quaternionic representation of the larger group $H_u \times G_u$ and the conjugations $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ are compatible with the unitary quaternionic representation. Then the maps η_Z, ϕ_s, $\phi(j)$ in Proposition 2.5 are $K_{\mathbb{R}}$-equivaraint where $K_{\mathbb{R}} = G_{\mathbb{R}} \cap G_u$ is a maximal compact subgroup of $G_{\mathbb{R}}$.

Definition 2.7. Consider the following maps

(1) $\alpha_a = \phi_{\frac{a\pi}{2}} \circ \eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z, \quad a \in [0,1]$
(2) $\beta = \phi(j) \circ \eta_Z : \mathfrak{M}_Z \to \mathfrak{M}_Z$.

Proposition 2.8. *We have $\alpha_a \circ \beta = \beta \circ \alpha_a$ for all $a \in [0,1]$.*

Proof. Set $s = \frac{a\pi}{2}$. By Proposition 2.5, we have

$$\alpha_a \circ \beta = \phi_s \circ \eta_Z \circ \phi(j) \circ \eta_Z = \phi_s \circ \phi(j)$$

and

$$\alpha_a \circ \beta = \phi(j) \circ \eta_Z \circ \phi_s \circ \eta_Z = \phi(j) \circ \phi(-1) \circ \phi_s = \phi_s \circ \phi(j).$$

The result follows. □

Proposition 2.9. *The continuous family of maps*

$$\alpha_a : \mathfrak{M}_Z \longrightarrow \mathfrak{M}_Z, \qquad a \in [0,1]$$

satisfies the following:

(1) α_a^2 is equal to identity, for all $a \in [0,1]$.
(2) α_a is $K_{\mathbb{R}}$-equivariant and commutes with the $\mathbb{R}^\times$-action.
(3) α_a commutes with the projection map $\mathfrak{M}_Z \to Z$ and induces involutions on the fibers $\alpha_a : \mathfrak{M}_{\zeta_{\mathbb{C}}} \to \mathfrak{M}_{\zeta_{\mathbb{C}}}, \zeta_{\mathbb{C}} \in Z$ preserving the stratification $\mathfrak{M}_{\zeta_{\mathbb{C}}} = \bigsqcup_{(L)} \mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$.
(4) At $a = 0$, we have $\alpha_0 = \phi(i) \circ \eta_Z$ which is an anti-holomorphic involution.
(5) At $a = 1$, we have $\alpha_1 = \phi(k) \circ \eta_Z$ which is a holomorphic involution.

Proof. Note that $q_s^2 = -1$ and hence $\phi_s^2 = \phi(q_s^2) = \phi(-1)$. By Proposition 2.5 we have

$$(\phi_s \circ \eta_Z)^2 = \phi_s \circ \eta_Z \circ \phi_s \circ \eta_Z = \phi_s^2 \circ \phi(-1) \circ \eta_Z^2 = \mathrm{id}.$$

Part (1) follows. Part (2),(3),(4),(5) follow from the construction. □

Proposition 2.10. *The map*

$$\beta : \mathfrak{M}_Z \longrightarrow \mathfrak{M}_Z$$

satisfies the following:

(1) $\beta^2 = \phi(-1)$.
(2) β is $K_{\mathbb{R}}$-equivariant and commutes with the $\mathbb{R}^\times$-action.
(3) β induces a holomorphic map between fibers $\beta : \mathfrak{M}_{\zeta_{\mathbb{C}}} \to \mathfrak{M}_{-\zeta_{\mathbb{C}}}$ which takes the stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ to the stratum $\mathfrak{M}_{-\zeta_{\mathbb{C}},(L)}$.

Proof. Since $\beta^2 = \phi(j) \circ \eta_Z \circ \phi(j) \circ \eta_Z = \phi(j)^2 = \phi(-1)$, part (1) follows. Part (2), (3) follow from the construction. □

Remark 2.11. Unlike the family of involutions α_s, the map β is well-defined on the whole family $\mathfrak{M}_{Z_{\mathbb{C}}}$.

2.8. **A stratified homeomorphism.** Our aim is to trivialize the fixed-point of the family involution α_a. To that end, we will invoke the following lemma.

Recall that a subset S of a real analytic manifold M (resp. real algebraic variety M) is called semi-analytic (resp. semi-algebraic) if any point $s \in S$ has a open neighbourhood U (resp. a Zariski affine open neighbourhood U) such that the intersection $S \cap U$ is a finite union of sets of the form

$$\{x \in U | f_1(x) = \cdots = f_r(x) = 0, g_1(x) > 0, ..., g_l(x) > 0\},$$

where the f_i and g_j are real analytic functions on U (resp. polynomial functions on U).

Lemma 2.12. *Let M and N be two semi-analytic sets and let $f : M \to N$ be a continuous map. Let*

$$\alpha_a : M \to M, \quad a \in [0, 1]$$

be a continuous family of involutions over N.

(1) Assume α_a preserves a semi-analytic stratification of M[8] and restricts to a real analytic map on each stratum. Then the fixed-points of the strata are real analytic manifolds and the α_a-fixed points M^{α_a} is stratified by the fixed-points of the strata.

(2) Assume further that there is a continuous $\mathbb{R}_{>0}$-action on M (resp. N) real analytic on strata and a proper continuous map $||-||: M \to \mathbb{R}_{\geq 0}$ such that (i) $f : M \to N$ is $\mathbb{R}_{>0}$-equivariant (ii) the $\mathbb{R}_{>0}$-action on M has a unique fixed point $o_M \in M$, which is also a stratum (iii) $||tm|| = t||m||$ and $||\alpha_a(m)|| = ||m||$ for $t \in \mathbb{R}_{>0}, a \in [0, 1], m \in M$. Then for any $a, a' \in [0, 1]$ there is a $\mathbb{R}_{>0}$-equivariant stratified homeomorphism

$$M^{\alpha_s} \simeq M^{\alpha_{s'}} \tag{2.18}$$

which is real analytic on each stratum and compatible with the natural maps to N.

(3) Assume further that there is a continuous action of a compact group L on M satisfying (i) the action commutes with the map $f : M \to N$, the involutions α_a, and the $\mathbb{R}_{>0}$-action, and is real analytic on each stratum (ii) the map $||-||: M \to \mathbb{R}_{\geq 0}$ is L-invariant. Then the homeomorphism in (2.18) *is L-equivariant.*

Proof. Proof of (1). Only the first claim requires a proof and it follows from the general fact that the fixed points M^α of a real analytic involution α on a real analytic manifold M is again a real analytic manifold.

Proof of (2). Step 1. Let $M_0 = M \setminus \{o_M\}$ and $C = \{m \in M_0 | ||m|| = 1\}$. Since $||-||: M \to \mathbb{R}_{\geq 0}$ is α_a-invariant and proper, C is compact and stable under the α_a-action. Since $\mathbb{R}_{>0}$ acts freely on M_0 and $||-||$ is $\mathbb{R}_{>0}$-equivariant, the restriction $||-||_{M_0} : M_0 \to \mathbb{R}_{>0}$ is a stratified submersion (where $\mathbb{R}_{>0}$ is equipped with the trivial stratification). It follows that $C = ||-||^{-1}(1) \subset M_0$ is stratified by the intersection of the strata with C.

[8]A stratification of a semi-analytic set is called semi-analytic if each stratum is a real analytic manifold.

Step 2. We shall show that there exists a stratified homeomorphism

$$\nu : C^{\alpha_a} \simeq C^{\alpha_{a'}} \tag{2.19}$$

which is real analytic on each stratum and is compatible with natural maps to N. Consider the involution $\alpha : [0,1] \times C \to [0,1] \times C, \alpha(a,m) = (a, \alpha_a(m))$. Let w be the average of the vector field $\partial_a \times 0$ on $[0,1] \times C$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action given by the involution α. Since $[0,1] \times C$ is compact and the $\mathbb{Z}/2\mathbb{Z}$-action is real analytic on each stratum, the vector field w is complete and the integral curves of w defines the desired stratified homeomorphism $\nu : C^{\alpha_a} \simeq C^{\alpha_{a'}}$, $a, a' \in [0,1]$ between the fibers of the α-fixed point $([0,1] \times C)^{\alpha}$ along the projection map to $[0,1]$.

Step 3. We have a natural map $M_0^{\alpha_a} \to C^{\alpha_a}$ sending m to $\frac{m}{||m||}$. Consider the following map

$$M_0^{\alpha_a} \to M_0^{\alpha_{a'}} \quad m \to ||m|| \nu(\frac{m}{||m||}). \tag{2.20}$$

Note that M^{α_a} is homeomorphic to the cone $C(M_0^{\alpha_a}) = M_0^{\alpha_a} \cup \{o_M\}$ of $M_0^{\alpha_a}$. Thus by the functoriality of cone, the map (2.20) extends to a homeomorphism

$$M^{\alpha_a} \to M^{\alpha_{a'}} \tag{2.21}$$

sending o_M to o_M. It is straightforward to check that (2.21) is a $\mathbb{R}_{>0}$-equivariant stratified homeomorphism which are real analytic on each stratum and compatible with the natural maps to N. This finishes the proof of part (2). Part (3) is clear from the construction of (2.21). □

Example 2.13. We preserve the set-up in Section 2.7. The map $||-||$: $\mathfrak{M}_Z = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(Z)/H_u \to \mathbb{R}_{\geq 0}$ given by $||m|| = (\tilde{m}, \tilde{m})^{\frac{1}{2}}$, where $\tilde{m} \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(Z)$ is a lift of m, is a $K_{\mathbb{R}} \times \alpha_a$-invariant proper real analytic map satisfying $||\phi(t)m|| = t||m||, t \in \mathbb{R}_{>0}$. Let $\mathfrak{M}_0 = \mu^{-1}(0)/H_u$, $\alpha_a : \mathfrak{M}_0 \to \mathfrak{M}_0$ be the family of involutions in Proposition 2.9, and $\phi(t) : \mathfrak{M}_0 \to \mathfrak{M}_0$ be the $\mathbb{R}_{>0}$-action in (2.5). Denote by

$$\mathfrak{M}_0(\mathbb{R}) = \mathfrak{M}_0^{\alpha_0} \qquad \mathfrak{M}_0^{sym}(\mathbb{C}) = \mathfrak{M}_0^{\alpha_1}$$

the fixed points of α_0 and α_1 on $\mathfrak{M}_0$ respectively. Applying Lemma 2.12 to the case $M = \mathfrak{M}_0$ with the stratification $\mathfrak{M}_0 = \bigsqcup_{(L)} \mathfrak{M}_{0,(L)}$, $N = 0$, $L = K_{\mathbb{R}}$, and the restriction $||-|||_M : M = \mathfrak{M}_0 \to \mathbb{R}_{\geq 0}$ of the function $||-||$ above to $\mathfrak{M}_0 \subset \mathfrak{M}_Z$, we see that there is a $K_{\mathbb{R}} \times \mathbb{R}_{>0}$-equivariant stratified homeomorphism

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{sym}(\mathbb{C}) \tag{2.22}$$

which are real analytic on each stratum. Note that whereas $\mathfrak{M}_0^{sym}(\mathbb{C})$ is complex analytic $\mathfrak{M}_0(\mathbb{R})$ is not, it is a real form of $\mathfrak{M}_0$.

3. Quiver varieties

In this section we consider the examples when the hyper-Kähler quotients are Nakajima's quiver varieties. We show that any quiver variety has a canonical conjugation called the split conjugation and hence has a canonical family of involutions α_a introduced in Section 2.7. The main reference for quiver varieties is [Nak1].

3.1. **Split conjugations.** Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. For any Q_0-graded hermitian vector space $V = \bigoplus_{k\in Q_0} V_k$, we write $\mathrm{GL}(V) = \prod_{k\in Q_0} \mathrm{GL}(V_i)$ and $\mathrm{U}(V) = \prod_{k\in Q_0} \mathrm{U}(V_k)$ where $\mathrm{U}(V_k)$ is the unitary group associated to the hermitian vector space V_k. We denote by $\mathfrak{gl}(V)$ and $\mathfrak{u}(V)$ be the Lie algebras of $\mathrm{GL}(V)$ and $\mathrm{U}(V)$ respectively.

Let $V = \bigoplus_{k\in Q_0} V_k$ and $W = \bigoplus_{k\in Q_0} W_k$ be two Q_0-graded hermitian vector spaces. Define

(3.1)
$$\mathbf{M} = \mathbf{M}(V,W) = \bigoplus_{h\in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \mathrm{Hom}(V_{i(h)}, V_{o(h)}) \bigoplus_{k\in Q_0} \mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k).$$

Here $o(h)$ and $i(h)$ are the outgoing and incoming vertices of the oriented arrow $h \in Q_1$ respectively.

We consider the $\mathbb{H}$-vector space structure on M given by the original complex structure I together with the new complex structure J given by

$$J(X,Y,x,y) = (-Y^\dagger, X^\dagger, -y^\dagger, x^\dagger) \tag{3.2}$$

where $(X,Y,x,y) \in \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \mathrm{Hom}(V_{i(h)}, V_{o(h)}) \oplus \mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k)$ and $(-)^\dagger$ is the hermitian adjoint.

The hermitian inner products on V_k and W_k induces a hermitian inner product on $\mathrm{Hom}(V_k, W_k)$ (resp. $\mathrm{Hom}(V_k, V_{k'})$) given by $(f,g) = \mathrm{tr}(fg^\dagger)$. We consider the hermitian inner product on $\mathbf{M}$ induced from the ones on V_k and W_k.

Let $H = \mathrm{GL}(V)$ and $G = \mathrm{GL}(W)$ with compact real from $H_u = \mathrm{U}(V)$ and $G_u = \mathrm{U}(W)$. Then action of $H \times G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ on M give by the formula

$$(g,g')(X,Y,x,y) = (gXg^{-1}, gYg^{-1}, gx(g')^{-1}, g'yg^{-1})$$

defines a unitary quaternionic representation of $\mathrm{U}(V) \times \mathrm{U}(W)$ on $\mathbf{M}$. The holomorphic symplectic form $\omega_{\mathbb{C}}$ is given by

$$\omega_{\mathbb{C}}((X,Y,x,y),(X',Y',x',y')) = \mathrm{tr}(XY' - YX') + \mathrm{tr}(xy' - x'y) \tag{3.3}$$

We denote by

$$\mu : \mathbf{M} \to \mathrm{Im}(\mathbb{H}) \otimes \mathfrak{u}(V)^* = \mathrm{Im}(\mathbb{H}) \otimes \mathfrak{u}(V) \tag{3.4}$$

the hyper-Kahler moment map with respect to the $\mathrm{U}(V)$-action. Here we identify $\mathfrak{u}(V)$ with its dual space $\mathfrak{u}(V)^*$ via the above hermitian inner product. We have the following formulas for the real and complex moment maps

$$\mu_{\mathbb{R}}(X,Y,x,y) = \frac{i}{2}(XX^\dagger - Y^\dagger Y + xx^\dagger - y^\dagger y) \in \mathfrak{u}(V),$$
$$\mu_{\mathbb{C}}(X,Y,x,y) = [X,Y] + xy \in \mathfrak{gl}(V) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(V).$$

The hyper-Kähler quotient $\mathfrak{M}_\zeta$ is called the *quiver variety*.

Lemma 3.1. *Let η_V and η_W be conjugations on V and W compatible with the Q_0-grading*[9] *and let η_H, η_G, and $\eta_{\boldsymbol{M}}$ be the induced conjugations on $H = \mathrm{GL}(V)$, $G = \mathrm{GL}(W)$, and $\boldsymbol{M}$ respectively. Assume η_H and η_G commute with the Cartan conjugations on H and G given*

[9]That is, we have $\eta_V(V_k) = V_k, \eta_W(W_k) = W_k$ for all $k \in Q_0$.

by the hermitian adjoint. Then the conjugations $\eta_H \times \eta_G$ and $\eta_{\boldsymbol{M}}$ are compatible with the unitary quaternionic representation of $H_u \times G_u$ on $\boldsymbol{M}$.

Proof. $\eta_H \times \eta_G$ commutes with the Cartan involution on $H \times G$ by assumption. Using (3.2) and (3.3), it is straightforward to check that $\eta_{\mathbf{M}}$ commutes with J and $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ are compatible with the symplectic representation of $H \times G$ on $\mathbf{M}$. In view of Remark 2.6, we see that $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ satisfy (1), (2), (3) in Definition 2.4. The lemma follows. □

Choose $\mathbf{v} = (\mathbf{v}_k)_{k\in Q_0}, \mathbf{w} = (\mathbf{w}_k)_{k\in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ and let $\mathbf{M}(\mathbf{v}, \mathbf{w}) = \mathbf{M}(V, W)$ where $V = \bigoplus_{k\in Q_0} \mathbb{C}^{\mathbf{v}_k}$ and $W = \bigoplus_{k\in Q_0} \mathbb{C}^{\mathbf{w}_k}$ equipped with the standard hermitian inner products. The standard complex conjugations on V and W induce the split conjugations on $H = \mathrm{GL}(V)$ and $G = \mathrm{GL}(W)$ commuting with the Cartan conjugations, and hence give rise to involutions η_H, η_G and $\eta_{\mathbf{M}}$ compatible with the unitary quaternionic representation. We will call the conjugation

$$\eta_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{M}_{Z_{\mathbb{C}}}$$

on the family of quiver varieties $\mathfrak{M}_{Z_{\mathbb{C}}}$ associated to $\eta_H \times \eta_G$ and $\eta_{\mathbf{M}}$ the *split conjugation*.

3.2. **Real-symmetric homoemorphisms for quiver varieties.** Let $\mathrm{O}(W_{\mathbb{R}}) = \mathrm{U}(W) \cap \mathrm{GL}(W_{\mathbb{R}})$ be the real orthogonal group. By Propositions 2.5 and 2.9, the split conjugation $\eta_{Z_{\mathbb{C}}}$ on $\mathfrak{M}_{Z_{\mathbb{C}}}$ preserves the subspace $\mathfrak{M}_Z \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ and gives rise to a family of $\mathrm{O}(W_{\mathbb{R}})$-equivariant involutions

$$\alpha_a : \mathfrak{M}_Z \to \mathfrak{M}_Z \quad a \in [0, 1] \tag{3.5}$$

interpolating the anti-holomorphic involution $\alpha_0 = \phi(i) \circ \eta_Z$ and the holomorphic involution $\alpha_1 = \phi(k) \circ \eta_Z$, and preserving the strata $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ of the fiber $\mathfrak{M}_{\zeta_{\mathbb{C}}}$ for $\zeta_{\mathbb{C}} \in Z$.

The involutions in (3.5) restricts to a family of involutions $\alpha_a : \mathfrak{M}_0 \to \mathfrak{M}_0$. Write $\mathfrak{M}_0(\mathbb{R}) = \mathfrak{M}_0^{\alpha_0}$ and $\mathfrak{M}_0^{sym}(\mathbb{C}) = \mathfrak{M}_0^{\alpha_1}$ for the fixed-points of α_0 and α_1. The intersections of the stratum $\mathfrak{M}_{0,(L)}$ with $\mathfrak{M}_0(\mathbb{R})$ and $\mathfrak{M}_0^{sym}(\mathbb{C})$ are unions of components

$$\mathfrak{M}_{0,(L)} \cap \mathfrak{M}_0(\mathbb{R}) = \bigsqcup \mathcal{O}_l(\mathbb{R}) \qquad \mathfrak{M}_{0,(L)} \cap \mathfrak{M}_0^{sym}(\mathbb{C}) = \bigsqcup \mathcal{O}_l^{sym}(\mathbb{C})$$

In [BeSc, Theorem 1.9], Bellamy-Schedler proved that the strata $\mathfrak{M}_{0,(L)}$ are symplectic leaves of $\mathfrak{M}_0$. We will call the components $\mathcal{O}_l(\mathbb{R})$ and $\mathcal{O}_l^{sym}(\mathbb{C})$ above the *real symplectic leaves* and *symmetric symplectic leaves* respectively.

The following proposition follows from Example 2.13:

Theorem 3.2. *There is a $\mathrm{O}(W_{\mathbb{R}}) \times \mathbb{R}^\times$-equivaraint stratified homeomorphism*

$$\mathfrak{M}_0(\mathbb{R}) \xrightarrow{\sim} \mathfrak{M}_0^{sym}(\mathbb{C}) \tag{3.6}$$

which restricts to real analytic $\mathrm{O}(W_{\mathbb{R}})$-equivariant isomorphisms between individual real and symmetric symplectic leaves. The homeomorphism induces a bijection

$$\{\mathcal{O}_l(\mathbb{R})\}_l \longleftrightarrow \{\mathcal{O}_l^{sym}(\mathbb{C})\}_l \tag{3.7}$$

between real and symmetric leaves preserving the closure relation.

In the next section we shall see that the nilpotent cone $\mathcal{N}_n(\mathbb{C})$ in $\mathfrak{gl}_n(\mathbb{C})$ is an example of quiver variety and the homeomorphism (3.6) in this case becomes an $\mathrm{O}_n(\mathbb{R})\times\mathbb{R}^\times$-equivariant homeomorphism

$$\mathcal{N}_n(\mathbb{R}) \simeq \mathcal{N}_n^{sym}(\mathbb{C})$$

between the real nilpotent cone in $\mathfrak{gl}_n(\mathbb{R})$ and the symmetric nilpotent cone in the space of symmetric matrices $\mathfrak{p}_n(\mathbb{C})$ and the bijection (3.7) is the well-known Kostant-Sekiguchi bijection between $\mathrm{GL}_n(\mathbb{R})$-orbits in $\mathcal{N}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$-orbits in $\mathcal{N}_n^{sym}(\mathbb{C})$. Thus one can view (3.6) as Kostant-Sekiguchi homeomorphisms for quiver varieties.

4. Real-symmetric homeomorphisms for Lie algebras

4.1. **Main results.** Let us return to the Cartan subgroup $T \subset G$, stable under η and θ, and maximally split with respect to η. Let $\mathfrak{t} \subset \mathfrak{g}$ denote its Lie algebra, $\mathrm{W}_G = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ the Weyl group and introduce the affine quotient $\mathfrak{c} = \mathfrak{g}//G = \mathrm{Spec}(\mathcal{O}(\mathfrak{g})^G) \simeq \mathfrak{t}//\mathrm{W}_G = \mathrm{Spec}(\mathcal{O}(\mathfrak{t})^{\mathrm{W}_G})$. Let $\chi : \mathfrak{g} \to \mathfrak{c}$ be the natural map.

Next, let $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ be the -1-eigenspace of θ, and write $\mathfrak{a}_\mathbb{R} = \mathfrak{a} \cap \mathfrak{g}_\mathbb{R}$ for the real form of $\mathfrak{a}$ with respect to η. Let $\mathrm{W} = N_{K_\mathbb{R}}(\mathfrak{a}_\mathbb{R})/Z_{K_\mathbb{R}}(\mathfrak{a}_\mathbb{R}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the "little Weyl group", and introduce the affine quotient $\mathfrak{c}_\mathfrak{p} = \mathfrak{p}//K = \mathrm{Spec}(\mathcal{O}(\mathfrak{p})^K) \simeq \mathfrak{a}//\mathrm{W} = \mathrm{Spec}(\mathcal{O}(\mathfrak{a})^{\mathrm{W}})$. Let $\chi_\mathfrak{p} : \mathfrak{p} \to \mathfrak{c}_\mathfrak{p}$ denote the natural map.

Let $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ be the image of the natural map $\mathfrak{a}_\mathbb{R} \to \mathfrak{c}$. Since the map $\mathfrak{a}_\mathbb{R} \to \mathfrak{c}$ is a polynomial map, by Tarski-Seidenberg's theorem, its image $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is semi-algebraic. For example, if $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}_2(\mathbb{R})$ then $\mathfrak{c} = \mathbb{C}$ and $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} = \mathbb{R}_{\leq 0}$.

Consider the following semi-algebraic subsets of $\mathfrak{g}$, $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{p}$:

$$\mathfrak{g}' = \mathfrak{g} \times_\mathfrak{c} \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \quad \mathfrak{g}'_\mathbb{R} = \mathfrak{g}_\mathbb{R} \times_\mathfrak{c} \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \quad \mathfrak{p}' = \mathfrak{p} \times_\mathfrak{c} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}. \tag{4.1}$$

We have

$$\mathfrak{g}'_\mathbb{R} = \big\{x \in \mathfrak{g}_\mathbb{R} |\ \text{ eigenvalues of } \mathrm{ad}_x \text{ are real}\big\} \tag{4.2}$$

$$\mathfrak{p}' = \big\{x \in \mathfrak{p} |\ \text{ eigenvalues of } \mathrm{ad}_x \text{ are real}\big\} \tag{4.3}$$

Note that G, $G_\mathbb{R}$ and K naturally act on $\mathfrak{g}'$, $\mathfrak{g}'_\mathbb{R}$ and $\mathfrak{p}'$ respectively, and the actions are along the fibers of the natural projections

$$\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \quad \mathfrak{g}'_\mathbb{R} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \quad \mathfrak{p}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \tag{4.4}$$

Theorem 4.1. *Suppose all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a $K_\mathbb{R}$-equivariant homeomorphism*

$$\mathfrak{g}'_\mathbb{R} \xrightarrow{\sim} \mathfrak{p}' \tag{4.5}$$

compatible with the natural projections to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual $G_\mathbb{R}$-orbits and K-orbits.

We deduce the theorem above from the following.

Theorem 4.2. *Suppose all simple factors of the complex reductive Lie algebra $\mathfrak{g}$ are of classical type. There is a continuous one-parameter families of maps*

$$\alpha_a : \mathfrak{g}' \longrightarrow \mathfrak{g}', \ a \in [0,1]$$

satisfying the following:

(1) α_a^2 is the identity, for all $s \in [0,1]$.
(2) At $a = 0$, we have $\alpha_0(M) = \eta(M)$.
(3) At $a = 1$, we have $\alpha_1(M) = -\theta(M)$.
(4) α_a is $K_{\mathbb{R}}$-equivaraint and take a G-orbit real analytically to a G-orbit.
(5) α_a commutes the with projection map $\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$.

4.2. **Quiver varieties of type A and conjugacy classes of matrices.** Consider the type A_n quiver:

$$Q : \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-2}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

Let $\mathbf{v} = (n, n-1, ..., 2, 1) \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{w} = (n, 0, ..., 0, 0) \in \mathbb{Z}^n_{\geq 0}$. Consider the unitary quaternionic representation $\mathbf{M}(\mathbf{v}, \mathbf{w})$ of $H_u = \prod_{k=1}^n \mathrm{U}(k)$ in Section 3.1. A vector in $\mathbf{M}(\mathbf{v}, \mathbf{w})$ can be represented as a diagram

(4.6)
$$\begin{array}{ccccccccccccc} & & \mathbb{C}^n & & & & & & & & & & \\ & & {\scriptstyle y}\uparrow\ \downarrow{\scriptstyle x} & & & & & & & & & & \\ \mathbb{C}^n & \underset{Y}{\overset{X}{\rightleftarrows}} & \mathbb{C}^{n-1} & \underset{Y}{\overset{X}{\rightleftarrows}} & \mathbb{C}^{n-2} & \underset{Y}{\overset{X}{\rightleftarrows}} & \cdots & \underset{Y}{\overset{X}{\rightleftarrows}} & \mathbb{C}^3 & \underset{Y}{\overset{X}{\rightleftarrows}} & \mathbb{C}^2 & \underset{Y}{\overset{Y}{\rightleftarrows}} & \mathbb{C}^1 \end{array}$$

Let $\mathfrak{M}_{Z_{\mathbb{C}}} = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(-Z_{\mathbb{C}})/H_u \to Z_{\mathbb{C}}$ be the family of quiver varieties associated to $\mathbf{M}(\mathbf{v}, \mathbf{w})$.

Denote by $\mathfrak{g}_n = \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{t}_n \subset \mathfrak{g}_n$ the subspace of diagonal matrices, $\mathfrak{c}_n = \mathfrak{g}_n // \mathrm{GL}_n(\mathbb{C})$, and $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n$ the Chevalley map. We will fix an identification $\mathfrak{c}_n = \mathbb{C}^n$ so that the map $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n = \mathbb{C}^n$ is given by $\chi_n(M) = (c_1, ..., c_n)$, where $T^n + c_1 T^{n-1} + \cdots + c_n$ is the characteristic polynomial of M. Consider the following maps

$$\tilde{\phi}_{n,\mathbb{C}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to \mathfrak{g}_n \times \mathfrak{t}_n \quad [X, Y, x, y] \to (yx, \zeta_{\mathbb{C}}) \tag{4.7}$$

$$\iota_{n,\mathbb{C}} : Z_{\mathbb{C}} \to \mathfrak{t}_n \quad \zeta_{\mathbb{C}} \to (c_1, ..., c_n) \tag{4.8}$$

where $\zeta_{\mathbb{C}} = (\zeta_1, ..., \zeta_n)$ is the image of $[X, Y, x, y] \in \mathfrak{M}_{Z_{\mathbb{C}}}$ under the projection map $\chi_{Z_{\mathbb{C}}} : \mathfrak{M}_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ and $c_i = \zeta_1 + \cdots + \zeta_i$, $1 \leq i \leq n$. Note that the map $\tilde{\phi}_{n,\mathbb{C}}$ intertwines the $\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times$-action on $\mathfrak{M}_{Z_{\mathbb{C}}}$ with the one on $\mathfrak{g}_n \times \mathfrak{t}_n$ given by $(g, a)(M, t) = (gMg^{-1}, a^2 t)$.

Proposition 4.3. *Let $\pi_{Z_{\mathbb{C}}} : \mathfrak{M}_{(\zeta_{\mathbb{R}}, Z_{\mathbb{C}})} \to \mathfrak{M}_{Z_{\mathbb{C}}}$ be the map in* (2.3) *and let $\mathfrak{M}'_{Z_{\mathbb{C}}} \subset \mathfrak{M}_{Z_{\mathbb{C}}}$ be its image. Assume $\zeta = (\zeta_{\mathbb{R}}, 0)$ is generic in the sense of* [Nak1, Definition 2.9].

(1) The fiber $\mathfrak{M}'_{\zeta_{\mathbb{C}}}$ of the projection $\mathfrak{M}'_{Z_{\mathbb{C}}} \to Z_{\mathbb{C}}$ over $\zeta_{\mathbb{C}} \in Z_{\mathbb{C}}$ is a union of strata.
(2) $\mathfrak{M}'_{Z_{\mathbb{C}}}$ is connected and invariant under the $\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times$-action.

(3) The map $\tilde{\phi}_{n,\mathbb{C}}$ (4.7) restricts to a $\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times$-equivariant isomorphism

$$\phi_{n,\mathbb{C}} : \mathfrak{M}'_{Z_\mathbb{C}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$$

of complex algebraic varieties making the following diagram commute

$$\begin{array}{ccc} \mathfrak{M}'_{Z_\mathbb{C}} & \xrightarrow{\phi_{n,\mathbb{C}}} & \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \ . \\ \downarrow & & \downarrow \\ Z_\mathbb{C} & \xrightarrow{\iota_{n,\mathbb{C}}} & \mathfrak{t}_n \end{array}$$

Furthermore, the map $\phi_{n,\mathbb{C}}$ induces stratified isomorphisms between individual fibers of the projections $\mathfrak{M}'_{Z_\mathbb{C}} \to Z_\mathbb{C}$ and $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$. Here we equipped the fibers of $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$ with the $\mathrm{GL}_n(\mathbb{C})$-orbits stratification.

Proof. Part (1) follows from [Nak1, Corollary 6.11]. Proof of (2) and (3). Since each stratum $\mathfrak{M}_{\zeta,(L)}$ is invariant under the $\mathrm{GL}_n(\mathbb{C}) \times \mathbb{R}^\times$-action part (1) implies $\mathfrak{M}'_{Z_\mathbb{C}}$ also has this property. Moreover, since the $\mathbb{R}^\times$-action on $\mathfrak{M}'_{Z_\mathbb{C}}$ is a contracting action with a unique fixed point, $\mathfrak{M}'_{Z_\mathbb{C}}$ is connected. By the result of Mirkovic-Vybornov [MV, Theorem 6.1], which is a generalization the earlier results of Kraft-Procesi [KP] and Nakajima [Nak1], the map $\phi_{n,\mathbb{C}}$ induces isomorphisms between individual fibers of the projections $\mathfrak{M}'_{Z_\mathbb{C}} \to Z_\mathbb{C}, \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$, and hence is a bijection. Since $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is normal and $\mathfrak{M}'_{Z_\mathbb{C}}$ is connected it follows that $\phi_{n,\mathbb{C}}$ is an isomorphism algebraic varieties. We claim that $\phi_{n,\mathbb{C}}$ maps each strata $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ isomorphically to a $\mathrm{GL}_n(\mathbb{C})$-orbit. For this we observe that there are only finitely many $\mathrm{GL}_n(\mathbb{C})$-orbits on the fibers of $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathfrak{t}_n$ and the closure of any non-closed orbit is singular. Since each stratum $\mathfrak{M}_{\zeta_\mathbb{C},(L)}$ is smooth and connected it follows that $\phi_{n,\mathbb{C}}(\mathfrak{M}_{\zeta_\mathbb{C},(L)})$ is a single $\mathrm{GL}_n(\mathbb{C})$-orbit. The claim follows and the proofs of (2) and (3) are complete.

□

4.3. **Reflection functors.** Let $\mathrm{C} = (\mathrm{C}_{kl})_{1 \le k,l \le n}$ be the Cartan matrix of type A_n. Identify $Z_\mathbb{C}$ with $\mathbb{C}^n$ and consider the reflection representation of the Weyl group W on $Z_\mathbb{C}$. For any simple reflection $s_k, k \in [1,n]$ and $\zeta_\mathbb{C} = (\zeta_1, ..., \zeta_n) \in \mathbb{Z}_\mathbb{C}$, we have $s_k(\zeta_\mathbb{C}) = \zeta'_\mathbb{C}$ where $\zeta'_l = \zeta_l - \mathrm{C}_{kl}\,\zeta_k$.

In [Nak1], Nakajima associated to each $k \in [1,n]$ a certain hyper-Kähler isometry $S_k : \mathfrak{M}_{\zeta_\mathbb{C}}(\mathbf{v},\mathbf{w}) \simeq \mathfrak{M}_{\zeta'_\mathbb{C}}(\mathbf{v}',\mathbf{w})$ called the *reflection functor*. Here $\zeta'_\mathbb{C} = s_k(\zeta_\mathbb{C})$ and $\mathbf{v}'$ is given by $v'_k = v_k - \sum_l \mathrm{C}_{kl}\, v_l + w_k$, $v'_l = v_l$ if $l \neq k$ for $\mathbf{v} = (v_1, ..., v_n)$, $\mathbf{w} = (w_1, ..., w_n)$. Moreover, it is shown in *loc. cit.* that the reflection functors S_k satisfy the Coxeter relations of the Weyl group.

In the case $\mathbf{v} = (n, n-1, ..., 1)$ and $\mathbf{w} = (n, 0..., 0)$, a direct calculation shows that, for $k \in [2,n]$, we have $\mathbf{v} = \mathbf{v}'$ and hence $S_k : \mathfrak{M}_{\zeta_\mathbb{C}}(\mathbf{v},\mathbf{w}) \simeq \mathfrak{M}_{\zeta'_\mathbb{C}}(\mathbf{v},\mathbf{w})$. Let $\mathrm{S}_n \subset \mathrm{W}$ be the subgroup generated by the simple reflections $s_2, ..., s_n$. As $\zeta_\mathbb{C}$ varies over $Z_\mathbb{C}$, the reflection functors $S_2, .., S_n$ define a S_n-action on $\mathfrak{M}_{Z_\mathbb{C}} = \bigcup_{\zeta_\mathbb{C} \in Z_\mathbb{C}} \mathfrak{M}_{\zeta_\mathbb{C}}(\mathbf{v},\mathbf{w})$ such that the projection map $\mathfrak{M}_{Z_\mathbb{C}} \to Z_\mathbb{C}$ is S_n-equivariant.

Lemma 4.4. *The subset $\mathfrak{M}'_{Z_\mathbb{C}} \subset \mathfrak{M}_{Z_\mathbb{C}}$ is invariant under the S_n-action and the isomorphism $\phi_{n,\mathbb{C}} : \mathfrak{M}'_{Z_\mathbb{C}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ is S_n-equivariant.*

Proof. We first claim that the map $\tilde{\phi}_{n,\mathbb{C}} : \mathfrak{M}_{Z_\mathbb{C}} \to \mathfrak{g}_n \times \mathfrak{t}_n$ (4.7) is S_n-equivariant. Recall the isomorphism $\iota_{n,\mathbb{C}} : Z_\mathbb{C} \simeq \mathfrak{t}_n$ in (4.8). A direct computation shows that $\iota_{n,\mathbb{C}}$ intertwines the action of s_k and the simple reflection $\sigma_{k-1,k} \in \mathrm{S}_n$ for $k \geq 2$. On the other hand, the formula for the reflection functors in [Nak2, Section 3(i)] implies that, for any $[X, Y, x, y] \in \mathfrak{M}_{Z_\mathbb{C}}$, we have $S_k([X, Y, x, y]) = [\tilde{X}, \tilde{Y}, x, y]$ for $k \geq 2$. All together we see that

$$\tilde{\phi}_{n,\mathbb{C}}(S_k([X, Y, x, y])) = \tilde{\phi}_{n,\mathbb{C}}([X', Y', x, y]) = (yx, \sigma_{k-1,k}(\iota_{n,\mathbb{C}}(\zeta_\mathbb{C}))) = \sigma_{k-1,k}(yx, \iota_{n,\mathbb{C}}(\zeta_\mathbb{C})) =$$

$$= \sigma_{k-1,k}(\tilde{\phi}_{n,\mathbb{C}}([X, Y, x, y])).$$

The claim follows. To complete the proof of the lemma, we need to show that $\mathfrak{M}'_{Z_\mathbb{C}}$ is S_n-invariant. Let $Z^0_\mathbb{C} \subset Z_\mathbb{C}$ (resp. $\mathfrak{t}^0_n \subset \mathfrak{t}_n$)be the open dense subset consisting of vectors with trivial stabilizers in S_n. The isomorphism $\phi_{n,\mathbb{C}}$ induces an isomorphism $\mathfrak{M}'_{Z^0_\mathbb{Z}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}^0_n$, where $\mathfrak{M}'_{Z^0_\mathbb{Z}} = \mathfrak{M}'_{Z_\mathbb{C}} \times_{Z_\mathbb{C}} Z^0_\mathbb{C}$, and it follows that $\mathfrak{M}'_{Z^0_\mathbb{Z}}$ is open dense in $\mathfrak{M}'_{Z_\mathbb{C}}$ and the fibers of the projection $\mathfrak{M}'_{Z^0_\mathbb{Z}} \to Z^0_\mathbb{C}$ are smooth. According to [Nak2, Theorem 4.1], the map $\pi_{Z_\mathbb{C}} : \mathfrak{M}_{\tilde{Z}_\mathbb{C}} \to \mathfrak{M}_{Z_\mathbb{C}}$ is an isomorphism over $\mathfrak{M}_{Z^0_\mathbb{Z}} = \mathfrak{M}_{Z_\mathbb{Z}} \times_{Z_\mathbb{C}} Z^0_\mathbb{C}$ and it follows that $\mathfrak{M}'_{Z^0_\mathbb{Z}} = \mathfrak{M}_{Z_\mathbb{C}} \times_{Z_\mathbb{C}} Z^0_\mathbb{C}$, which is S_n-invariant. On the other hand, the same argument as in the proof of [Nak2, Theorem 4.1(1)] shows that the map $\pi_{Z_\mathbb{C}} : \mathfrak{M}_{(\zeta_\mathbb{R}, Z_\mathbb{C})} \to \mathfrak{M}_{Z_\mathbb{C}}$ is proper and hence its image $\mathfrak{M}'_{Z_\mathbb{C}} = \pi_{Z_\mathbb{C}}(\mathfrak{M}_{(\zeta_\mathbb{R}, Z_\mathbb{C})}) \subset \mathfrak{M}_{Z_\mathbb{C}}$ is a closed subset. Thus $\mathfrak{M}'_{Z_\mathbb{C}}$ is equal to the closure of $\mathfrak{M}'_{Z^0_\mathbb{C}}$ in $\mathfrak{M}_{Z_\mathbb{C}}$ and, as $\mathfrak{M}'_{Z^0_\mathbb{C}}$ is S_n-invariant, it implies $\mathfrak{M}'_{Z_\mathbb{C}}$ is S_n-invariant. The lemma follows.

□

4.4. **Involutions on the spaces of matrices with real eigenvalues.** Let $\mathfrak{M}'_Z \subset \mathfrak{M}_Z$ be the image of $\pi_Z : \mathfrak{M}_{\tilde{Z}} \to \mathfrak{M}_Z$ and let $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}}$ where $i\mathfrak{t}_{n,\mathbb{R}} \subset \mathfrak{t}_n$ is the $\mathbb{R}$-subspace consisting of diagonal matrices with pure imaginary entries. Then the isomorphisms $\phi_{n,\mathbb{C}}$ and $\iota_{n,\mathbb{C}}$ above restricts to isomorphisms

$$\mathfrak{M}'_Z \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}} \qquad Z \simeq i\mathfrak{t}_{n,\mathbb{R}} \tag{4.9}$$

Consider the family of involutions $\alpha_a : \mathfrak{M}_Z \to \mathfrak{M}_Z$ in Proposition 2.9 associated to the split conjugations in Section 3.1 and the map $\beta : \mathfrak{M}_Z \to \mathfrak{M}_Z$ in Proposition 2.10. Note that the action of $-1 \in \mathbb{R}^\times$ on $\mathfrak{M}_Z$ is trivial (it becomes the action of $1 = (-1)^2$ on $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}}$) thus, by Proposition 2.10 (1), β is an involution. Note also that the fibers of the projection $\mathfrak{M}'_Z \to Z$ are union of strata (Proposition 4.3 (1)), thus Proposition 2.9 (3) and Proposition 2.10 (3) imply that $\mathfrak{M}'_Z$ is invariant under the involutions α_a and β.

To relate $\mathfrak{M}'_Z$ with matrices with real eigenvalues let us consider the following composition

$$\phi_n : \mathfrak{M}'_Z \overset{(4.9)}{\simeq} \mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \tag{4.10}$$

$$\iota_n : Z \overset{(4.9)}{\simeq} i\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{t}_{n,\mathbb{R}} \tag{4.11}$$

where the second isomorphisms are given by $\mathfrak{g}_n \times_{\mathfrak{c}_n} i\mathfrak{t}_{n,\mathbb{R}} \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}, (x, v) \to (ix, iv)$ and $i\mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{t}_{n,\mathbb{R}}, v \to iv$. Note that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{M}'_Z & \xrightarrow{\phi_n} & \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\iota_n} & \mathfrak{t}_{n,\mathbb{R}} \end{array} \tag{4.12}$$

where the vertical arrows are the natural projections.

Now the isomorphism $\phi_n : \mathfrak{M}'_Z \simeq \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$ gives rise to involutions on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$:

$$\tilde{\alpha}_{n,a} = \phi_n \circ \alpha_a \circ \phi_n^{-1} : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \quad a \in [0, 1] \tag{4.13}$$

$$\tilde{\beta}_n = \phi_n \circ \beta \circ \phi_n^{-1} : \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \tag{4.14}$$

Lemma 4.5.

(1) The involutions $\tilde{\beta}_n$ is given by $\tilde{\beta}_n(M, v) = (-M^t, -v)$. In particular, $\tilde{\beta}_n$ commutes with the action of the symmetric group S_n on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$.

(2) The involution $\tilde{\alpha}_{n,a}$ commutes with the action of the symmetric group S_n on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$.

Proof. Let $(M, v) \in \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$. Choose $[X, Y, x, y] \in \mathfrak{M}'_{\zeta_{\mathbb{C}}}$ such that

$$\phi_n([X, Y, x, y]) = i(yx, \iota_n(\zeta_{\mathbb{C}})) = (M, v).$$

According to Definition 2.7 and Proposition 2.10, we have

$$\beta([X, Y, x, y]) = \phi(j) \circ \eta_Z([X, Y, x, y]) = [-\bar{Y}^\dagger, \bar{X}^\dagger, -\bar{y}^\dagger, \bar{x}^\dagger] \in \mathfrak{M}'_{-\zeta_{\mathbb{C}}}$$

It follows that

$$\tilde{\beta}_n((M, v)) = \phi_n([-\bar{Y}^\dagger, \bar{X}^\dagger, -\bar{y}^\dagger, \bar{x}^\dagger]) = i((\bar{x}^\dagger)(-\bar{y}^\dagger), -\iota_n(\zeta_{\mathbb{C}})) = (-M^t, -v).$$

Part (1) follows.

According to Definition 2.7, we have

$$\alpha_a([X, Y, x, y]) = (\cos(s)\phi(i) + \sin(s)\phi(k)) \circ \eta_Z([X, Y, x, y]) = [X', Y', x', y'] \in \mathfrak{M}'_{\zeta_{\mathbb{C}}}.$$

where

$$x' = i\cos(s)\bar{x} - i\sin(s)\bar{y}^\dagger, \qquad y' = i\cos(s)\bar{y} + i\sin(s)\bar{x}^\dagger.$$

On the other hand, we have $S_k([X, Y, x, y]) = [\tilde{X}, \tilde{Y}, x, y]$. Thus

$$\alpha_a \circ S_k([X, Y, x, y]) = [(\tilde{X})', (\tilde{Y})', x', y'], \qquad S_k \circ \alpha_a([X, Y, x, y]) = [(\tilde{X'}), (\tilde{Y'}), x', y']. \tag{4.15}$$

Since ϕ_n commutes with the S_n-action (Lemma 4.5), we obtain

$$\tilde{\alpha}_{n,a} \circ S_k((M, v)) = \tilde{\alpha}_{n,a} \circ S_k \circ \phi_n([X, Y, x, y]) = \phi_n \circ \alpha_a \circ S_k([X, Y, x, y]) = i(y'x', s_k(v))$$

$$S_k \circ \tilde{\alpha}_{n,a}((M, v)) = S_k \circ \alpha_a \circ \phi_n([X, Y, x, y]) = \phi_n \circ S_k \circ \alpha_a([X, Y, x, y]) = i(y'x', s_k(v)).$$

Part (2) follows. The proof is complete.

□

Let $\mathfrak{c}_{n,\mathbb{R}} \subset \mathfrak{c}_n$ be the image of the map $\mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{c}_n$ and let $\mathfrak{g}'_n = \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}} \subset \mathfrak{g}_n$. Note that both $\mathfrak{c}_{n,\mathbb{R}}$ and $\mathfrak{g}'_n$ are semi-algebraic sets. We have

$$\mathfrak{g}'_n = \big\{x \in \mathfrak{g}_n |\ \text{ eigenvalues of } x \text{ are real}\big\}. \tag{4.16}$$

Since the natural map $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}'_n = \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}}$ is S_n-equivariant (where S_n-acts trivially on $\mathfrak{g}'_n$), Lemma 4.5 implies that the involutions $\tilde{\alpha}_{n,a}$ and $\tilde{\beta}_n$ in (4.13) and (4.14) descend to a continuous family of involutions on $\mathfrak{g}'_n$:

$$\alpha_{n,a} : \mathfrak{g}'_n \to \mathfrak{g}'_n \tag{4.17}$$

compatible with projections to $\mathfrak{c}_{n,\mathbb{R}}$ and an involution

$$\beta_n : \mathfrak{g}'_n \to \mathfrak{g}'_n. \tag{4.18}$$

Moreover, β_n is equal to the restriction of the Cartan involution on $\mathfrak{g}_n$ to $\mathfrak{g}'_n$:

$$\beta_n(M) = -M^t \tag{4.19}$$

Theorem 4.6. *The continuous one-parameter families of maps*

$$\alpha_{n,a} : \mathfrak{g}'_n \longrightarrow \mathfrak{g}'_n,\ a \in [0,1]$$

satisfying the following:

(1) $\alpha_{n,a}^2$ is equal to the identity map, for all $a \in [0,1]$.
(2) At $a = 0$, we have $\alpha_{n,0}(M) = \overline{M}$.
(3) At $a = 1$, we have $\alpha_{n,1}(M) = M^t$.
(4) $\alpha_{n,a}$ is $\mathrm{O}_n(\mathbb{R})$-equivaraint and take a $\mathrm{GL}_n(\mathbb{C})$-orbit real analytically to itself.
(5) $\alpha_{n,a}$ commutes both with the Cartan involution β_n and with the projection map $\mathfrak{g}'_n \to \mathfrak{c}_{n,\mathbb{R}}$, for all $a \in [0,1]$.

Proof. Part (1) follows from the construction and Part (5) follows from the commutative diagram (4.12). Let $\phi'_n : \mathfrak{M}'_Z \overset{\phi_n}{\simeq} \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}'_n$ where the last map is given by $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}} \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}} = \mathfrak{g}'_n$. Let $M \in \mathfrak{g}'_n$. Choose $[X,Y,x,y] \in \mathfrak{M}'_Z$ such that

$$M = \phi'_n([X,Y,x,y]) = iyx$$

By Definition 2.7, we have

$$\alpha_0([X,Y,x,y]) = \phi(i) \circ \eta_Z([X,Y,x,y]) = [i\overline{X}, i\overline{Y}, i\bar{x}, i\bar{y}]$$
$$\alpha_1([X,Y,x,y]) = \phi(k) \circ \eta_Z([X,Y,x,y]) = [-i\bar{Y}^\dagger, i\bar{X}^\dagger, -i\bar{y}^\dagger, i\bar{x}^\dagger].$$

It follows that

$$\alpha_{n,0}([X,Y,x,y]) = \phi'_n([i\overline{X}, i\overline{Y}, i\bar{x}, i\bar{y}]) = i(-\bar{y}\bar{x}) = \overline{M}$$
$$\alpha_{n,1}([X,Y,x,y]) = \phi'_n([-i\overline{Y}^\dagger, i\overline{X}^\dagger, -i\bar{y}^\dagger, i\bar{x}^\dagger]) = i(\bar{x}^\dagger\bar{y}^\dagger) = i(\overline{(yx)}^\dagger) = i(yx)^t = M^t.$$

Part (2) and (3) follow.

By Proposition 4.3 (3), the isomorphism $\phi_n : \mathfrak{M}'_Z \to \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_{n,\mathbb{R}}$ maps each stratum $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ real analytically to a $\mathrm{GL}_n(\mathbb{C})$-orbit. Now part (4) follows from the fact the involution α_a on $\mathfrak{M}'_Z$ is $\mathrm{O}_n(\mathbb{R})$-equivariant and $\mathfrak{M}_{\zeta_{\mathbb{C}},(L)}$ is invariant under α_a.

□

Let $\mathfrak{g}'_{n,\mathbb{R}}$ be the space of $n \times n$ real matrices with real eigenvalues. Let $\mathfrak{p}'_n$ be the space of $n \times n$ symmetric matrices with real eigenvalues. It is clear that $\mathfrak{g}'_{n,\mathbb{R}} = (\mathfrak{g}'_n)^{\alpha_0}$ and $\mathfrak{p}'_n = (\mathfrak{g}'_n)^{\alpha_1}$.

Theorem 4.7. *There is an* $\mathrm{O}_n(\mathbb{R}) \times \mathbb{R}^\times$*-equivariant homeomorphism*

$$\mathfrak{g}'_{n,\mathbb{R}} \xrightarrow{\sim} \mathfrak{p}'_n \tag{4.20}$$

compatible with the natural projections to $\mathfrak{c}_{n,\mathbb{R}}$*. Furthermore, the homeomorphism restricts to a real analytic isomorphism between individual* $\mathrm{GL}_n(\mathbb{R})$*-orbits and* $\mathrm{O}_n(\mathbb{C})$*-orbits.*

Proof. Consider the Lusztig stratification of $\mathfrak{g}_n$. The stratum through g with a Jordan decomposition $g = s + u$ consists of all $\mathrm{GL}_n(\mathbb{C})$-orbits through $u + Z_r(\mathfrak{l})$ where $\mathfrak{l} = Z_{\mathfrak{g}_n}(s)$ is the centralizer of s in $\mathfrak{g}_n$ and $Z_r(\mathfrak{l}) = \{x \in Z(\mathfrak{l}) | Z_{\mathfrak{g}_n}(x) = \mathfrak{l}\}$ is the regular part of the center $Z(\mathfrak{l})$ of $\mathfrak{l}$. It is clear that the Lusztig stratification restricts to the orbits stratifications on the fibers of the Chevalley map $\chi_n : \mathfrak{g}_n \to \mathfrak{c}_n$ and a stratification on $\mathfrak{g}'_n = \mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{c}_{n,\mathbb{R}}$.

Recall the $\mathrm{U}(n)$-invariant function $||-||\colon \mathfrak{M}_Z \to \mathbb{R}_{\geq 0}$ in Example 2.13. The restriction of $||-||$ along the closed embedding $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \overset{\phi_n}{\simeq} \mathfrak{M}'_Z \subset \mathfrak{M}_Z$ gives rise to a function $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathbb{R}_{\geq 0}$. Its average respect to the S_n-action on $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n$ defines a S_n-invariant function $\mathfrak{g}_n \times_{\mathfrak{c}_n} \mathfrak{t}_n \to \mathbb{R}_{\geq 0}$ which descends to a function $||-||_{\mathfrak{g}'_n}\colon \mathfrak{g}'_n \to \mathbb{R}_{\geq 0}$. It follows from Theorem 4.6 and the construction of $||-||_{\mathfrak{g}'_n}$ that the function $||-||_{\mathfrak{g}'_n}$ together with the real analytic map $\mathfrak{g}'_n \to \mathfrak{c}_{n,\mathbb{R}}$ and the Lusztig stratification on $\mathfrak{g}'_n$ satisfy the assumption in Lemma 2.12, and hence we obtain a stratified $\mathrm{O}_n(\mathbb{R})$-equivariant homeomorphism

$$\mathfrak{g}'_{n,\mathbb{R}} \to \mathfrak{p}'_n \tag{4.21}$$

which are real analytic on each stratum and compatible with the maps to $\mathfrak{c}_{n,\mathbb{R}}$. Since each stratum in $\mathfrak{g}'_{n,\mathbb{R}}$ (resp. $\mathfrak{p}'_n$) is a finite union of $\mathrm{GL}_n(\mathbb{R})$-orbits (resp. $\mathrm{O}_n(\mathbb{C})$-orbits) and $\mathrm{O}_n(\mathbb{R})$-acts simply transitively on connected components of each orbits, it follows that the homeomorphism (4.28) restricts to a real analytic isomorphism between individual $\mathrm{GL}_n(\mathbb{R})$-orbits and $\mathrm{O}_n(\mathbb{C})$-orbits. □

4.5. **Proof of Theorem 4.2.** We shall deduce Theorem 4.2 from Theorem 4.6.

Let $\mathfrak{g}$ be a simple Lie algebra of classical type with real form $\mathfrak{g}_{\mathbb{R}}$. Recall the classification of real forms of classical types:

Lemma 4.8. [OV, Section 4] *Here is the complete list of all possible quadruple* $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}, \eta, \theta)$ *(up to isomorphism):*

(a) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$*:*

- *(1)* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R}), \mathfrak{k} = \mathfrak{so}_n(\mathbb{C}),\ \eta(g) = \bar{g},\ \theta(g) = -g^t.$
- *(2)* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H}), \mathfrak{k} = \mathfrak{sp}_m(\mathbb{C}),\ \eta(g) = \mathrm{Ad}\, S_m(\bar{g}),\ \theta(g) = -\mathrm{Ad}\, S_m(g^t)\ \ (n = 2m).$
- *(3)* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}_{p,n-p}, \mathfrak{k} = (\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C})) \cap \mathfrak{g},\ \eta(g) = -\mathrm{Ad}\, I_{p,n-p}(\bar{g}^t),\ \ \theta(g) = \mathrm{Ad}\, I_{p,n-p}(g).$

(b) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$*:*

- *(1)* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}_{p,n-p}, \mathfrak{k} = \mathfrak{so}_p(\mathbb{C}) \oplus \mathfrak{so}_{n-p}(\mathbb{C}),\ \eta(g) = \mathrm{Ad}\, I_{p,n-p}(\bar{g}),\ \theta(g) = \mathrm{Ad}\, I_{p,n-p}(g).$
- *(2)* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{u}^*_m(\mathbb{H}), \mathfrak{k} = \mathfrak{gl}_m(\mathbb{C}),\ \eta(g) = \mathrm{Ad}\, S_m(\bar{g}),\ \theta(g) = \mathrm{Ad}\, S_m(g)\ \ (n = 2m).$

(c) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$*,* $n = 2m$*:*

(1) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_{2m}(\mathbb{R}), \mathfrak{k} = \mathfrak{gl}_m(\mathbb{C}),\ \eta(g) = \bar{g},\ \theta(g) = \operatorname{Ad} S_m(g).$
(2) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_{p,m-p}, \mathfrak{k} = \mathfrak{sp}_{2p}(\mathbb{C}) \oplus \mathfrak{sp}_{2m-2p}(\mathbb{C}),\ \eta(g) = -\operatorname{Ad} K_{p,m-p}(\bar{g}^t),\ \theta(g) = \operatorname{Ad} K_{p,m-p}(g).$

Here $S_m = \begin{pmatrix} 0 & -Id_m \\ Id_m & 0 \end{pmatrix}$, $I_{p,n-p} = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_{n-p} \end{pmatrix}$, *and* $K_{p,m-p} = \begin{pmatrix} I_{p,m-p} & 0 \\ 0 & I_{p,m-p} \end{pmatrix}$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\iota_{\mathfrak{g}}} & \mathfrak{g}_n \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi_n} \\ \mathfrak{c} & \xrightarrow{\iota_{\mathfrak{c}}} & \mathfrak{c}_n \end{array} \tag{4.22}$$

where $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}_n$ is the natural embedding and $\iota_{\mathfrak{c}} : \mathfrak{c} = \mathfrak{g}//G \to \mathfrak{c}_n = \mathfrak{g}_n//\mathrm{GL}_n(\mathbb{C})$. We have the following explicit description of χ and $\iota_{\mathfrak{c}}$. For any $M \in \mathfrak{g}_n$, let

$$T^n + c_1 T^{n-1} + c_2 T^{n-1} + \cdots + c_n$$

be the characteristic polynomial of M. In the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we have $c_1 = 0$ and one can identify $\mathfrak{c}$ with $\mathbb{C}^{n-1}$ so that

$$\chi(M) = (c_2, c_3, ..., c_n)$$
$$\iota_{\mathfrak{c}}(c_1, ..., c_n) = (0, c_2, ..., c_n)$$

In the case $\mathfrak{g} = \mathfrak{sp}_{\mathfrak{n}}(\mathbb{C})$ or $\mathfrak{so}_n(\mathbb{C})$ we have $c_1 = c_3 = \cdots = 0$ and one can choose an identification of $\mathfrak{c} = \mathbb{C}^{[n/2]}$ such that $\chi : \mathfrak{g} \to \mathfrak{c} = \mathbb{C}^{[n/2]}$ is given by

$$\chi(M) = (c_2, c_4, ..., c_n) \qquad \text{if } \ \mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$$
$$\chi(M) = (c_2, c_4, ..., c_{n-1}) \qquad \text{if } \ \mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) \quad n = 2m+1$$
$$\chi(M) = (c_2, c_4, ..., c_{n-2}, \tilde{c}_n) \qquad \text{if } \ \mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) \quad n = 2m$$

where $\tilde{c}_n = \mathrm{Pf}(M)$ is the Pfaffian of M satisfying $\mathrm{Pf}(M)^2 = \det(M) = c_n$, and the map $\iota_{\mathfrak{c}}$ is given by

$$\iota_{\mathfrak{c}}(c_2, c_4, ..., c_{n-1}) = (0, c_2, 0, c_4, ...0, c_{n-1}) \qquad \text{if } \ \mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$$
$$\iota_{\mathfrak{c}}(c_2, c_4, ..., c_{n-1}) = (0, c_2, 0, c_4, ...0, c_{n-1}) \qquad \text{if } \ \mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) \quad n = 2m+1, l = m$$

$$\iota_{\mathfrak{c}}(c_2, c_4, ..., c_{n-2}, \tilde{c}_n) = (0, c_2, 0, c_4, ..., 0, \tilde{c}_n^2) \qquad \text{if } \ \mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) \quad n = 2m, l = m. \tag{4.23}$$

Remark 4.9. It follows that the map $\iota_{\mathfrak{c}} : \mathfrak{c} \to \mathfrak{c}_n$ is a closed embedding except the case $\mathfrak{g} = \mathfrak{so}_n$, $n = 2m$.

Recall the semi-algebraic sets $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ and $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{g}$ introduced (4.1). Since for any $x \in \mathfrak{g}'$ the eigenvalues of ad_x are real, the embedding $\mathfrak{g}' \to \mathfrak{g}_n$ factors through $\mathfrak{g}' \to \mathfrak{g}'_n \subset \mathfrak{g}_n$ and diagram (4.22) restricts to a digram

$$\begin{array}{ccc} \mathfrak{g}' & \xrightarrow{\iota_{\mathfrak{g}}} & \mathfrak{g}'_n \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi_n} \\ \mathfrak{c}_{\mathfrak{p},\mathbb{R}} & \xrightarrow{\iota_{\mathfrak{c}}} & \mathfrak{c}_{n,\mathbb{R}} \end{array} \quad . \tag{4.24}$$

Note that Proposition 2.8 implies $\alpha_{n,a} \circ \beta_n = \beta_n \circ \alpha_{n,a}$. On the other hand, since $S_m, I_{p,n-p}, K_{p,m-p} \in \mathrm{O}_n(\mathbb{R})$, Proposition 4.6 (4) implies that the involutions $\mathrm{Ad}\, S_m$, $\mathrm{Ad}\, I_{p,n-p}$, and $\mathrm{Ad}\, K_{p,m-p}$ on $\mathfrak{g}'_n$ commute with both $\alpha_{n,a}$ and β_n. Now a direct computation, using the formula of θ in Lemma 4.8, shows that the compositions

$$\alpha_{n,s} \circ \beta_n \circ \theta : \mathfrak{g}'_n \to \mathfrak{g}'_n \quad s \in [0,1]. \tag{4.25}$$

are involutions. We claim that the subspace $\mathfrak{g}' \subset \mathfrak{g}'_n$ is invariant under the involutions (4.25). Consider the involution σ on $\mathfrak{g}_n$ such that $(\mathfrak{g}_n)^\sigma = \mathfrak{g}$, that is, σ is given by $\sigma = \beta_n$ if $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ and $\sigma = \mathrm{Ad}(S_m) \circ \beta_n$ if $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$. Since the map (4.25) commutes with the involution σ, the σ-fixed points $(\mathfrak{g}'_n)^\sigma$ is invariant under the map (4.25). The claim follows.

The diagram (4.24) implies that $\mathfrak{g}'$ is equal to the base-change

$$\mathfrak{g}' = (\mathfrak{g}'_n)^\sigma \times_{\mathfrak{c}_{n,\mathbb{R}}} \iota_{\mathfrak{c}}(\mathfrak{c}_{\mathfrak{p},\mathbb{R}}). \tag{4.26}$$

of $(\mathfrak{g}'_n)^\sigma$ to the subspace $\iota_{\mathfrak{c}}(\mathfrak{c}_{\mathfrak{p},\mathbb{R}}) \subset \mathfrak{c}_{n,\mathbb{R}}$ and hence the maps (4.25) restrict to a family of involutions

$$\alpha_a : \mathfrak{g}' \to \mathfrak{g}' \quad a \in [0,1]. \tag{4.27}$$

We shall show that the map α_a above satisfies properties (1) to (5) in Theorem 4.2. Properties (1), (2), (3) of $\alpha_{n,a}$ in Theorem 4.6 immediately implies that α_a satisfies properties (1), (2), (3) in Theorem 4.2. Property (4) follows from the fact that the intersection of an adjoint orbit of $\mathfrak{g}_n$ with $\mathfrak{g}$ is a finite disjoint union of G-orbits and each G-orbit is a connected component. We now check property (5). We need to show that α_a preserves the fibers of $\chi : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Assume $\mathfrak{g}$ is not of type D. Then by Remark 4.9, the map $\mathfrak{c}_{\mathfrak{p},\mathbb{R}} \to \mathfrak{c}_{n,\mathbb{R}}$ is a closed embedding and property (5) follows from the one for $\alpha_{n,a}$. Assume $\mathfrak{g} = \mathfrak{so}_{n=2m}$. Then from the diagram (4.24) we see that the involution α_a preserves the fibers of $\iota_{\mathfrak{c}} \circ \chi : \mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \to \mathfrak{c}_{n,\mathbb{R}}$. Let $c = (c_2, c_4, ..., \tilde{c}_n) \in \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. According to (4.23), if $\tilde{c}_n = 0$ then $\chi^{-1}(c) = (\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c))$ and if $\tilde{c}_n = 0$ then $(\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c)) = \chi^{-1}(c) \sqcup \chi^{-1}(c')$ where $c' = (c_2, c_4, ..., c_{n-2}, -\tilde{c}_n)$. In the first case, $\chi^{-1}(c)$ is equal to a fiber of $\iota_{\mathfrak{c}} \circ \chi$ and hence is invariant under α_a. Consider the second case. Since $\chi^{-1}(c)$ contain a vector in $\mathfrak{a}_{\mathbb{R}}$ and $\alpha_0(M) = M$ for $M \in \mathfrak{a}_{\mathbb{R}}$, it follow that $\alpha_0(\chi^{-1}(c)) = \chi^{-1}(c)$. Since $\chi^{-1}(c)$ and $\chi^{-1}(c')$ are connected components of $(\iota_{\mathfrak{c}} \circ \chi)^{-1}(\iota_{\mathfrak{c}}(c))$ we must have $\alpha_a(\chi^{-1}(c)) = \chi^{-1}(c)$ for all $a \in [0,1]$. We are done. This finishes the proof of Theorem 4.2.

4.6. **Proof of Theorem 4.1.** The proof is similar to the one of Theorem 4.7. Since $\mathfrak{g} = (\mathfrak{g}_n)^\sigma$ is the fixed-points subspace of the involution σ on $\mathfrak{g}_n$ and the stratum of the Lusztig stratification of $\mathfrak{g}_n$ are invariant under σ (the stratum are invariant under the adjoint action and transpose), we obtain a stratification of $\mathfrak{g}$ given by the σ-fixed points of the strata. The stratification on $\mathfrak{g}$ induces a stratification on $\mathfrak{g}' = \mathfrak{g} \times_{\mathfrak{c}} \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$, moreover, the intersection of each stratum with the fibers of $\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$, if non-empty, is a finite union of G-orbits.

Let $||-||_{\mathfrak{g}'}: \mathfrak{g}' \to \mathbb{R}_{\geq 0}$ be the restriction of the function $||-||_{\mathfrak{g}'_n}$ to $\mathfrak{g}' \subset \mathfrak{g}'_n$ in the proof of Theorem 4.7. It follows from Theorem 4.2 and the construction of the function $||-||_{\mathfrak{g}'_n}$ that the real analytic map $\mathfrak{g}' \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ together with the stratification of $\mathfrak{g}'$ described above and the function $||-||_{\mathfrak{g}'}$ satisfy the assumption in Lemma 2.12, and hence we obtain a stratified

$K_{\mathbb{R}}$-equivariant homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} = (\mathfrak{g}')^{\alpha_0} \to \mathfrak{p}' = (\mathfrak{g}')^{\alpha_1} \tag{4.28}$$

which are real analytic on each stratum and compatible with the maps to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$. Since each stratum in $\mathfrak{g}'_{\mathbb{R}}$ (resp. $\mathfrak{p}'$)is a finite union of $G_{\mathbb{R}}$-orbits (resp. K-orbits) and $K_{\mathbb{R}}$-acts simply transitively on connected components of each orbits, it follows that the homeomorphism (4.28) restricts to a real analytic isomorphism between individual $G_{\mathbb{R}}$-orbits and K-orbits. The proof of Theorem 4.1 is complete.

5. Real and symmetric Springer theory

5.1. **The real Grothendieck-Springer map.** Let $A_{\mathbb{R}} = \exp \mathfrak{a}_{\mathbb{R}}$ which is a closed, connected, abelian, diagonalizable subgroup of $G_{\mathbb{R}}$. Let $(\Phi, \mathfrak{a}^*_{\mathbb{R}})$ be the root system (possible non-reduced) of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}})$. For each $\alpha \in \Phi$ we denote by $\mathfrak{g}_{\mathbb{R},\alpha} \subset \mathfrak{g}_{\mathbb{R}}$ the corresponding α-eigenspace. Choose a system of simple roots $\Delta = \{\alpha_1, ..., \alpha_r\} \subset \Phi$ and denote by Φ^+ (resp. Φ^-) the corresponding set of positive roots (resp. negative roots). We have the following decomposition:

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}} \oplus \bar{\mathfrak{n}}_{\mathbb{R}}.$$

where $\mathfrak{m}_{\mathbb{R}} = Z_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$, $\mathfrak{n}_{\mathbb{R}} = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_{\mathbb{R},\alpha}$, $\bar{\mathfrak{n}}_{\mathbb{R}} = \oplus_{\alpha\in\Phi^-}\mathfrak{g}_{\mathbb{R},\alpha}$.

Let $\mathfrak{b}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}}$ be a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and we denote by $B_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ the corresponding minimal parabolic subgroup, here $N_{\mathbb{R}} = \exp(\mathfrak{n}_{\mathbb{R}})$ and $M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(A_{\mathbb{R}})$ is a group (possible not connected) with Lie algebra $\mathfrak{m}_{\mathbb{R}}$. We write $F = \pi_0(M_{\mathbb{R}})$.

An element $x \in \mathfrak{g}_{\mathbb{R}}$ is called semi-simple (resp. nilpotent) if ad_x is diagonalizable over $\mathbb{C}$ (resp. nilpotent). An element $x \in \mathfrak{g}_{\mathbb{R}}$ is called hyperbolic (resp. elliptic) if it is semi-simple and the eigenvalues of ad_x are real (resp. purely imaginary). For any $x \in \mathfrak{g}_{\mathbb{R}}$ we have the Jordan decomposition $x = x_e + x_h + x_n$ where x_e is elliptic, x_h is hyperbolic, x_n is nilpotent, and the three elements x_e, x_h, x_n commute.

Consider the adjoint action of $G_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{R}}$. By a result of Richardson and Slodowy [RS], there exists a semi-algebraic set $\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$ whose points are the semi-simple $G_{\mathbb{R}}$-orbits on $\mathfrak{g}_{\mathbb{R}}$. Furthermore, there are maps $\chi_{\mathbb{R}} : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}} \to \mathfrak{c}$, such that the restriction of the Chevalley map $\chi : \mathfrak{g} \to \mathfrak{c}$ to $\mathfrak{g}_{\mathbb{R}}$ factors as

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{R}} & \longrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \chi_{\mathbb{R}}} & & \downarrow{\scriptstyle \chi} \\ \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}} & \longrightarrow & \mathfrak{c} \end{array} .$$

For any $x \in \mathfrak{g}_{\mathbb{R}}$ its image $\chi_{\mathbb{R}}(x)$ is given by the $G_{\mathbb{R}}$-orbit through the semi-simple part $x_e + x_h$ of x. We also have an embedding $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$, whose image consists of hyperbolic $G_{\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}$, such that the restriction of $\chi_{\mathbb{R}}$ to $\mathfrak{a}_{\mathbb{R}}$ factors as $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}$.

Recall the subspace $\mathfrak{g}'_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ consisting of elements in $\mathfrak{g}_{\mathbb{R}}$ with hyperbolic semi-simple parts (4.2). By a result of Kostant [Ko2, Proposition 2.4], any hyperbolic element x in $\mathfrak{g}_{\mathbb{R}}$ is

conjugate to an element in $\mathfrak{a}_{\mathbb{R}}$. Moreover, the set of elements in $\mathfrak{a}_{\mathbb{R}}$ which are conjugate to x is single W-orbit. It follows that the embedding $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ factors through an isomorphism

$$\mathfrak{g}'_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \times_{\mathfrak{g}_{\mathbb{R}}//G_{\mathbb{R}}} \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \tag{5.1}$$

In particular, we have a natural projection map

$$\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \tag{5.2}$$

such that the composition $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}$ is equal to the map $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}} \subset \mathfrak{c}$ in (4.4).

Introduce the *real Grothendiek-Springer map*

$$\widetilde{\mathfrak{g}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{b}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}} \quad (g, v) \to \mathrm{Ad}_g(v). \tag{5.3}$$

Note that unlike the complex case, the real Grothendiek-Springer map (5.3) in general is not surjective. Consider the base change of the real Grothendiek-Springer map to $\mathfrak{g}'_{\mathbb{R}}$:

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \to \mathfrak{g}'_{\mathbb{R}} \tag{5.4}$$

where $\widetilde{\mathfrak{g}}'_{\mathbb{R}} = \widetilde{\mathfrak{g}}_{\mathbb{R}} \times_{\mathfrak{g}_{\mathbb{R}}} \mathfrak{g}'_{\mathbb{R}}$. By [Ko2, Proposition 2.5], an element $x \in \mathfrak{g}_{\mathbb{R}}$ is in $\mathfrak{g}'_{\mathbb{R}}$ if and only if it is conjugate to an element in $\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$[10]. It follows that

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}})$$

and the map (5.4) is surjective. Moreover we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}'_{\mathbb{R}} & \longrightarrow & \mathfrak{g}'_{\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathfrak{a}_{\mathbb{R}} & \longrightarrow & \mathfrak{a}_{\mathbb{R}}//\mathrm{W} \end{array} \tag{5.5}$$

where the map $\widetilde{\mathfrak{g}}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ is given by $(g, v = v_a + v_n) \to v_a$.

Consider the *real Springer map*

$$\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}} \tag{5.6}$$

We have the following cartesian diagrams

$$\begin{array}{ccccc} \widetilde{\mathcal{N}}_{\mathbb{R}} & \longrightarrow & \widetilde{\mathfrak{g}}'_{\mathbb{R}} & \longrightarrow & \widetilde{\mathfrak{g}}_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_{\mathbb{R}} & \longrightarrow & \mathfrak{g}'_{\mathbb{R}} & \longrightarrow & \mathfrak{g}_{\mathbb{R}} \end{array} \tag{5.7}$$

Since (5.4) is surjective, the real Springer map (5.6) is also surjective.

Lemma 5.1. *We have a $K_{\mathbb{R}}$-equivariant isomorphism $\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times \mathfrak{a}_{\mathbb{R}}$ commutes with projections to $\mathfrak{a}_{\mathbb{R}}$.*

[10]In *loc. cit.*, the claim is proved in the setting of adjoint action of $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$. But the same argument works for the case of adjoint action of $G_{\mathbb{R}}$ on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$, and hence $\mathfrak{g}_{\mathbb{R}}$.

Proof. The Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ gives rise to $K_{\mathbb{R}}$-equivariant isomorphism

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}) \simeq K_{\mathbb{R}} \times^{M_{\mathbb{R}}} (\mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}).$$

Since $M_{\mathbb{R}}$ acts trivially on $\mathfrak{a}_{\mathbb{R}}$, we obtain

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq (K_{\mathbb{R}} \times^{M_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}) \times \mathfrak{a}_{\mathbb{R}}.$$

On the other hand, we have

$$\widetilde{\mathcal{N}}_{\mathbb{R}} = G_{\mathbb{R}} \times^{B_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}} \simeq K_{\mathbb{R}} \times^{M_{\mathbb{R}}} \mathfrak{n}_{\mathbb{R}}.$$

Combining the isomorphisms above we get the desired $K_{\mathbb{R}}$-equivariant trivialization

$$\widetilde{\mathfrak{g}}'_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times \mathfrak{a}_{\mathbb{R}}$$

commutes with projections to $\mathfrak{a}_{\mathbb{R}}$. The proof is complete. □

5.2. **Sheaves of real nearby cycles.** Fix a point $a_{\mathbb{R}} \in \mathfrak{a}_{\mathbb{R}}^{\mathrm{rs}}$ with image $\xi_{\mathbb{R}} \in \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$. Let $\mathcal{O}_{\xi_{\mathbb{R}}}$ be the semi-simple $G_{\mathbb{R}}$-orbit through $a_{\mathbb{R}}$. The centralizer $Z_{G_{\mathbb{R}}}(a_{\mathbb{R}})$ is isomorphic to $M_{\mathbb{R}} A_{\mathbb{R}}$ and it follows that the $G_{\mathbb{R}}$-equivariant fundamental group of $\mathcal{O}_{\xi_{\mathbb{R}}}$ is isomorphic to $\pi_0(M_{\mathbb{R}} A_{\mathbb{R}}) \simeq \pi_0(M_{\mathbb{R}}) = F$. For any one dimensional character χ of F we denote by $\mathcal{L}_{\mathbb{R},\chi}$ the $G_{\mathbb{R}}$-equivariant local system on $\mathcal{O}_{\xi_{\mathbb{R}}}$ corresponding to χ.

Consider the path $\gamma_{\mathbb{R}} : [0,1] \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$ given by $\gamma_{\mathbb{R}}(s) = s\xi_{\mathbb{R}}$ and denote by

$$\mathcal{Z}_{\mathbb{R}} = \mathfrak{g}'_{\mathbb{R}} \times_{\mathfrak{a}_{\mathbb{R}}//\mathrm{W}} [0,1]$$

the base change of $\mathfrak{g}'_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}//\mathrm{W}$ (5.2) along $\gamma_{\mathbb{R}}$. Note that $\gamma_{\mathbb{R}}$ is an embedding and hence $\mathcal{Z}_{\mathbb{R}}$ is closed subvariety of $\mathfrak{g}'_{\mathbb{R}}$. The fibers of the natural projection $f : \mathcal{Z}_{\mathbb{R}} \to [0,1]$ over 0 and 1 are isomorphic to the nilpotent cone $\mathcal{N}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ and semi-simiple orbit $\mathcal{O}_{\xi_{\mathbb{R}}}$ respectively. Moreover the $\mathbb{R}_{>0}$-action on $\mathfrak{g}'_{\mathbb{R}}$ induces a trivialization

$$\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \simeq \mathcal{Z}_{\mathbb{R}}|_{(0,1]} \quad (g,s) \to (sg,s). \tag{5.8}$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \simeq \mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_{\mathbb{R}} & \xleftarrow{v} & \mathcal{N}_{\mathbb{R}} \\ \downarrow & & \downarrow f & & \downarrow \\ (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\} \end{array} \tag{5.9}$$

where u and v are the natural embeddings. Note that all the varieties in the diagram above carry natural $G_{\mathbb{R}}$-actions and all the maps between them are $G_{\mathbb{R}}$-equivariant. Define the nearby cycles functor:

$$\Psi_{\mathbb{R}} : D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \to D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}) \quad \Psi_{\mathbb{R}}(\mathcal{F}) = \psi_f(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = v^* u_*(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}). \tag{5.10}$$

For any character χ of F, consider the sheaf of nearby cycles with coefficient $\mathcal{L}_\chi$

$$\mathcal{F}_{\mathbb{R},\chi} = \Psi_{\mathbb{R}}(\mathcal{L}_\chi) \tag{5.11}$$

We will call $\Psi_{\mathbb{R}}$ the *real nearby cycles functor* and $\mathcal{F}_{\mathbb{R},\chi}$ the *sheaf of real nearby cycles*.

We shall give a formula of the nearby cycles sheaves in terms of the real Springer map $\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}$ (5.6). Since the $G_{\mathbb{R}}$-equivariant fundamental group of $G_{\mathbb{R}}/B_{\mathbb{R}}$, and hence

that of $\widetilde{\mathcal{N}}_{\mathbb{R}}$, is isomorphic to $\pi_0(B_{\mathbb{R}}) = \pi_0(M_{\mathbb{R}}) = F$, any character χ of F gives rise to a $G_{\mathbb{R}}$-equivariant local system $\widetilde{\mathcal{L}}_\chi$ on $\widetilde{\mathcal{N}}_{\mathbb{R}}$. Introduce the *real Springer sheaf*

$$\mathcal{S}_{\mathbb{R},\chi} = (\pi_{\mathbb{R}})_! \widetilde{\mathcal{L}}_\chi. \tag{5.12}$$

Theorem 5.2. *We have* $\mathcal{F}_{\mathbb{R},\chi} \simeq \mathcal{S}_{\mathbb{R},\chi}$.

Proof. Consider the path $\tilde{\gamma}_{\mathbb{R}} : [0,1] \to \mathfrak{a}_{\mathbb{R}}$ given by $\tilde{\gamma}_{\mathbb{R}}(s) = s(a_{\mathbb{R}})$ and let

$$\widetilde{\mathcal{Z}}_{\mathbb{R}} = \tilde{\mathfrak{g}}_{\mathbb{R}} \times_{\mathfrak{a}_{\mathbb{R}}} [0,1]$$

to be the base change of the map $\tilde{\mathfrak{g}}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ along the path $\tilde{\gamma}_{\mathbb{R}}$. The fiber of the projection $\tilde{f} : \widetilde{\mathcal{Z}}_{\mathbb{R}} \to [0,1]$ over 0 and 1 are given by $\widetilde{\mathcal{N}}_{\mathbb{R}}$ and $\mathcal{O}_{\xi_{\mathbb{R}}}$ respectively. Moreover, there is a trivialization

$$\widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} \simeq \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] \quad ((g,v),s) \to (\mathrm{Ad}_g(s^{-1}v), s) \tag{5.13}$$

It follows that the real Grothendieck-Springer map $\widetilde{\mathfrak{g}}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ restricts to a map $\tau_{\mathbb{R}} : \widetilde{\mathcal{Z}}_{\mathbb{R}} \to \mathcal{Z}_{\mathbb{R}}$ which is an isomorphism over $\mathcal{Z}_{\mathbb{R}}|_{(0,1]}$. Consider he following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] & \xrightarrow{(5.13)} & \widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{\tilde{u}} & \widetilde{\mathcal{Z}}_{\mathbb{R}} & \xleftarrow{\tilde{v}} & \widetilde{\mathcal{N}}_{\mathbb{R}} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tau_{\mathbb{R}}} & & \downarrow{\scriptstyle \tau_{\mathbb{R}}} & & \downarrow{\scriptstyle \pi_{\mathbb{R}}} \\
\mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1] & \xrightarrow{(5.8)} & \mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_{\mathbb{R}} & \xleftarrow{v} & \mathcal{N}_{\mathbb{R}} \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\}
\end{array} \tag{5.14}$$

Consider the nearby cycles functor

$$\widetilde{\Psi}_{\mathbb{R}} : D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \to D_{G_{\mathbb{R}}}(\widetilde{\mathcal{N}}_{\mathbb{R}}) \qquad \widetilde{\Psi}_{\mathbb{R}}(\mathcal{F}) = \tilde{v}^* \tilde{u}_* (\mathcal{F} \boxtimes \mathbb{C}_{(0,1]})$$

Since $\tau_{\mathbb{R}}$ is proper and $(\tau_{\mathbb{R}})_!(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) \simeq \mathcal{F} \boxtimes \mathbb{C}_{(0,1]}$, the proper base change for nearby cycles functors implies that there is a canonical isomorphism

$$(\pi_{\mathbb{R}})_! \widetilde{\Psi}_{\mathbb{R}}(\mathcal{F}) = (\pi_{\mathbb{R}})_! \psi_{\tilde{f}}(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) \simeq \psi_f((\tau_{\mathbb{R}})_!(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]})) \simeq \psi_f(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = \Psi_{\mathbb{R}}(\mathcal{F}). \tag{5.15}$$

On the other hand, the $K_{\mathbb{R}}$-equivariant trivialization in Lemma 5.1 gives rise to a $K_{\mathbb{R}}$-equivariant isomorphism

$$\widetilde{\mathcal{Z}}_{\mathbb{R}} \simeq \widetilde{\mathcal{N}}_{\mathbb{R}} \times [0,1] \tag{5.16}$$

commutes with projections to $[0,1]$. In addition, there exits a $K_{\mathbb{R}}$-equivariant isomorphism $q : \widetilde{\mathcal{N}}_{\mathbb{R}} \simeq \mathcal{O}_{\xi_{\mathbb{R}}}$ such that $q^* \mathcal{L}_\chi \simeq \widetilde{\mathcal{L}}_\chi$ and making the following diagram commute

$$\begin{array}{ccc}
\mathcal{Z}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{(5.16)} & \widetilde{\mathcal{N}}_{\mathbb{R}} \times (0,1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle q \times \mathrm{id}} \\
\widetilde{\mathcal{Z}}_{\mathbb{R}}|_{(0,1]} & \xrightarrow{(5.13)} & \mathcal{O}_{\xi_{\mathbb{R}}} \times (0,1]
\end{array}.$$

It follows that

$$\widetilde{\Psi}_{\mathbb{R}}(\mathcal{L}_\chi) \simeq \psi_{\tilde{f}}(\mathcal{L}_\chi \boxtimes \mathbb{C}_{(0,1]}) \simeq \psi_{\tilde{f}}(\widetilde{\mathcal{L}}_\chi \boxtimes \mathbb{C}_{(0,1]}) \simeq \widetilde{\mathcal{L}}_\chi \tag{5.17}$$

as object in $D_{K_\mathbb{R}}(\widetilde{\mathcal{N}}_\mathbb{R})$. Since $D_{G_\mathbb{R}}(\widetilde{\mathcal{N}}_\mathbb{R}) \subset D_{K_\mathbb{R}}(\widetilde{\mathcal{N}}_\mathbb{R})$ is a full subcategory (as $G_\mathbb{R}/K_\mathbb{R}$ is contractible), we conclude that

$$\mathcal{S}_{\mathbb{R},\chi} = (\pi_\mathbb{R})_!\widetilde{\mathcal{L}}_\chi \overset{(5.17)}{\simeq} (\pi_\mathbb{R})_!\widetilde{\Psi}_\mathbb{R}(\mathcal{L}_\chi) \overset{(5.15)}{\simeq} \Psi_\mathbb{R}(\mathcal{L}_\chi) = \mathcal{F}_{\mathbb{R},\chi} \in D_{G_\mathbb{R}}(\mathcal{N}_\mathbb{R})$$

The proof is complete. □

5.3. **Sheaves of symmetric nearby cycles.** The discussion in the previous subsection has a counterpart in the setting of symmetric space. Recall the subspace $\mathfrak{p}' \subset \mathfrak{p}$ consisting of elements x in $\mathfrak{p}$ such that the eigenvalues of ad_x are real. In [KR], Kostant and Rallis proved that for any such x, its semi-simple part $x_s \in \mathfrak{p}$ is conjugate to an element in $\mathfrak{a}_\mathbb{R}$, moreover, the set of elements in $\mathfrak{a}_\mathbb{R}$ which are conjugate to x_s is single W-orbit. It follows that the subspace $\mathfrak{p}'$ is equal to the base change

$$\mathfrak{p}' = \mathfrak{p} \times_{\mathfrak{c}_\mathfrak{p}} \mathfrak{a}_\mathbb{R}//\mathrm{W}$$

of $\chi_\mathfrak{p} : \mathfrak{p} \to \mathfrak{c}_\mathfrak{p}$ along $\mathfrak{a}_\mathbb{R}//\mathrm{W} \subset \mathfrak{c}_\mathfrak{p}$.

Let $a_\mathfrak{p} \in \mathfrak{a}_\mathbb{R}^{\mathrm{rs}}$ with image $\xi_\mathfrak{p} \in \mathfrak{a}_\mathbb{R}//\mathrm{W}$. Let $\mathcal{O}_{\xi_\mathfrak{p}}$ be the K-orbit through $a_\mathfrak{p}$. We have $Z_K(a_\mathfrak{p}) = MA$ and it follows that the K-equivariant fundamental group of $\mathcal{O}_{\xi_\mathfrak{p}}$ is isomorphic to $\pi_0(Z_K(a_\mathfrak{p})) = \pi_0(MA) = \pi_0(M) = F$. For any character χ of F we denote by $\mathcal{L}_{\mathfrak{p},\chi}$ the K-equivariant local system on $\mathcal{O}_{\xi_\mathfrak{p}}$. Consider the path $\gamma_\mathfrak{p} : [0,1] \to \mathfrak{a}_\mathbb{R}//\mathrm{W}$ given by $\gamma_\mathfrak{p}(s) = s\xi_\mathfrak{p}$ and define

$$\mathcal{Z}_\mathfrak{p} = \mathfrak{p}' \times_{\mathfrak{a}_\mathbb{R}//\mathrm{W}} [0,1]$$

The fibers of the natural projection $f_\mathfrak{p} : \mathcal{Z}_\mathfrak{p} \to [0,1]$ over 0 and 1 are isomorphic to the nilpotent cone $\mathcal{N}_\mathfrak{p}$ in $\mathfrak{p}$ and the K-orbit $\mathcal{O}_\mathfrak{p}$. Moreover the $\mathbb{R}_{>0}$-action on $\mathfrak{p}'$ induces a trivialization

$$\mathcal{O}_{\xi_\mathfrak{p}} \times (0,1] \simeq \mathcal{Z}_\mathfrak{p}|_{(0,1]} \quad (g,s) \to (sg,s). \tag{5.18}$$

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{O}_{\xi_\mathfrak{p}} \times (0,1] \simeq \mathcal{Z}_\mathfrak{p}|_{(0,1]} & \xrightarrow{u} & \mathcal{Z}_\mathfrak{p} & \xleftarrow{v} & \mathcal{N}_\mathfrak{p} \\ \downarrow & & \downarrow{\scriptstyle f_\mathfrak{p}} & & \downarrow \\ (0,1] & \longrightarrow & [0,1] & \longleftarrow & \{0\} \end{array} \tag{5.19}$$

where u and v are the natural embeddings. Note that all the varieties in the diagram above carry natural K-actions and all the maps between them are K-equivariant. Introduce the nearby cycles functor:

$$\Psi_\mathfrak{p} : D_K(\mathcal{O}_{\xi_\mathfrak{p}}) \to D_K(\mathcal{N}_\mathfrak{p}) \qquad \Psi_\mathfrak{p}(\mathcal{F}) = \psi_{f_\mathfrak{p}}(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}) = v^*u_*(\mathcal{F} \boxtimes \mathbb{C}_{(0,1]}). \tag{5.20}$$

For any character χ of F, consider the nearby cycles sheaf with coefficient $\mathcal{L}_{\mathfrak{p},\chi}$

$$\mathcal{F}_{\mathfrak{p},\chi} = \Psi_\mathfrak{p}(\mathcal{L}_{\mathfrak{p},\chi}) \tag{5.21}$$

We will call $\Psi_\mathfrak{p}$ the *symmetric nearby cycles functor* and $\mathcal{F}_{\mathfrak{p},\chi}$ the *sheaf of symmetric nearby cycles*.

Recall the $K_{\mathbb{R}}$-equivariant stratified homeomorphism

$$\mathfrak{g}'_{\mathbb{R}} \simeq \mathfrak{p}' \tag{5.22}$$

in Theorem (4.1). Since the homeomorphism (5.22) commutes with projection to $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ and the natural map $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is a finite map[11], for any $\xi_{\mathbb{R}} \in \mathfrak{a}^{\mathrm{rs}}_{\mathbb{R}}/\mathrm{W}$ there exists a unique $\xi_{\mathfrak{p}} \in \mathfrak{a}^{\mathrm{rs}}_{\mathbb{R}}//\mathrm{W}$ such that (5.22) restricts to a $K_{\mathbb{R}}$-equivariant real analytic isomorphism between individual fibers

$$\mathcal{O}_{\xi_{\mathbb{R}}} \simeq \mathcal{O}_{\xi_{\mathfrak{p}}}.$$

Since (5.22) is $\mathbb{R}_{>0}$-equivariant, the isomorphism above and the trivializations (5.8) and (5.18) imply that (5.22) induces a $K_{\mathbb{R}} \times \mathbb{R}_{>0}$-equivariant homeomorphism

$$\mathcal{Z}_{\mathbb{R}} \simeq \mathcal{Z}_{\mathfrak{p}} \tag{5.23}$$

commutes with projections to $[0,1]$. The homeomorphism above gives rise to a canonical commutative square of functors

$$\begin{array}{ccc} D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) & \xrightarrow{\Psi_{\mathbb{R}}} & D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}}) \\ \downarrow & & \downarrow \\ D_K(\mathcal{O}_{\xi_{\mathfrak{p}}}) & \xrightarrow{\Psi_{\mathfrak{p}}} & D_K(\mathcal{N}_{\mathfrak{p}}) \end{array} \tag{5.24}$$

where the upper and lower arrows are the real and symmetric nearby cycles respectively and the vertical arrows are the equivalences in (1.17). Since the equivalence $D_{G_{\mathbb{R}}}(\mathcal{O}_{\xi_{\mathbb{R}}}) \simeq D_K(\mathcal{O}_{\xi_{\mathfrak{p}}})$ maps $\mathcal{L}_{\mathbb{R},\chi}$ to $\mathcal{L}_{\mathfrak{p},\chi}$, the diagram (5.24) and Theorem 5.2 imply the following:

Theorem 5.3. *Assume $\mathfrak{g}$ is of classical type. Under the equivalence $D_K(\mathcal{N}_{\mathfrak{p}}) \simeq D_{G_{\mathbb{R}}}(\mathcal{N}_{\mathbb{R}})$ in* (1.17)*, the sheaf of symmetric nearby cycles $\mathcal{F}_{\mathfrak{p},\chi}$ becomes the the sheaf of real nearby cycles $\mathcal{F}_{\mathbb{R},\chi}$, which is also isomorphic to the real Springer sheaf $\mathcal{S}_{\mathbb{R},\chi}$. In particular, the real Springer map $\pi_{\mathbb{R}} : \widetilde{\mathcal{N}}_{\mathbb{R}} \to \mathcal{N}_{\mathbb{R}}$ is a semi-small map and the real Springer sheaf $\mathcal{S}_{\mathbb{R},\chi}$ is a perverse sheaf.*

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[11]Recall that $\mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is by definition the image of the map $\mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}//\mathrm{W} = \mathfrak{c}_{\mathfrak{p}} \to \mathfrak{c}$. Since the later map $\mathfrak{c}_{\mathfrak{p}} \to \mathfrak{c}$ is in general not a closed embedding, the map $\mathfrak{a}_{\mathbb{R}}//\mathrm{W} \to \mathfrak{c}_{\mathfrak{p},\mathbb{R}}$ is not a closed embedding in general.

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