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CENTRAL MOMENTS OF THE FREE ENERGY OF THE STATIONARY O'CONNELL–YOR POLYMER

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Seppäläinen and Valkó showed in (*ALEA Lat. Am. J. Probab. Math. Stat.* **7** (2010) 451–476) that for a suitable choice of parameters, the variance growth of the free energy of the stationary O'Connell–Yor polymer is governed by the exponent $2/3$, characteristic of models in the KPZ universality class.

We develop exact formulas based on Gaussian integration by parts to relate the cumulants of the free energy, $\log Z^{\theta}_{n,t}$, to expectations of products of quenched cumulants of the time of the first jump from the boundary into the system, s_0. We then use these formulas to obtain estimates for the kth central moment of $\log Z^{\theta}_{n,t}$ as well as the kth annealed moment of s_0 for $k > 2$, with nearly optimal exponents $(1/3)k + \epsilon$ and $(2/3)k + \epsilon$, respectively.

As an application, we derive new high probability bounds for the distance between the polymer path and a straight line connecting the origin to the endpoint of the path.

1. Introduction. The semidiscrete polymer in a Brownian environment was introduced by O'Connell and Yor in [19]. It is one of only a few known examples of integrable polymer models. To define it, let $n \geq 1$, $t > 0$, and $B_n(t)$, $n = 1, 2, \dots$ be independent Brownian motions started at 0. Introduce the energy

$$\mathcal{E}_{n,t}(s_1, \dots, s_{n-1}) = \sum_{j=1}^{n} \big(B_j(s_j) - B_j(s_{j-1})\big),$$

where we set $s_0 := 0$ and $s_n := t$. The semidiscrete (point-to-point) polymer partition function from $(0, 0)$ to (t, n) is given by

$$Z_{n,t} = \int_{0<s_1<\cdots<s_{n-1}<t} e^{\mathcal{E}_{n,t}(s_1,\dots,s_{n-1})}\, \mathrm{d}s_1 \dots \mathrm{d}s_{n-1}.$$

The probabilistic interpretation of the right-hand side is as a Gibbs ensemble of up-right paths between $(0, 0)$ and (t, n). Each path consists of $n - 1$ Poisson-distributed successive jumps at times $0 < s_1 < s_2, \dots < s_{n-1} < t$ of height one between discrete levels $j = 1, \dots, n$. For each j, the path remains on level j for time $s_j - s_{j-1}$. See [3], Definition 1.1, for a precise description of the path interpretation. The path interpretation justifies the name *polymer*, and reveals $Z_{n,t}$ as the partition function of the Gibbs ensemble described above.

In this paper, we consider a family of stationary versions of the polymer partition function, also studied in [19]. To define it, we introduce an extra two-sided Brownian motion $B_0(s)$, $s \in \mathbb{R}$, independent of $B_1, \dots, B_n$ and also extend the Brownian motions $B_1, \dots, B_n$ to two-sided Brownian motions. For $\theta > 0$, define

$$\mathcal{E}^{\theta}_{n,t}(s_0, \dots, s_{n-1}) := \theta s_0 - B_0(s_0) + \sum_{j=1}^{n} \big(B_j(s_j) - B_j(s_{j-1})\big).$$

Here s_0 is allowed to vary.

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The stationary partition function is then

$$Z_{n,t}^{\theta} = \int_{-\infty < s_0 < s_1 < \cdots < s_{n-1} < t} e^{\mathcal{E}_{n,t}^{\theta}(s_0,\dots,s_{n-1})} \, \mathrm{d}s_0 \, \mathrm{d}s_1 \dots \mathrm{d}s_{n-1}.$$

For $n = 0$, we let

$$Z_{0,t}^{\theta} = e^{-B_0(t)+\theta t}.$$

Note that now the s_j can range over the entire real line. Following Seppäläinen and Valkó [21], the Gibbs distribution of the initial jump s_0 plays a key role in the analysis in this paper, because it is a dual variable to the parameter $\theta > 0$.

The main result in [19], the Burke property for this model, implies that the free energy, $\log Z_{n,t}^{\theta}$, equals a combination of a sum of i.i.d. random variables and the Brownian motion $B_0(t)$. The following statement is adapted from [21], Theorem 3.3.

PROPOSITION 1.1 ([19]). *For each $n \geq 1$ and $t \geq 0$, write*

$$\log Z_{n,t}^{\theta} = \sum_{j=1}^{n} r_j^{\theta}(t) - B_0(t) + \theta t, \tag{1}$$

where

$$r_j^{\theta}(t) := \log Z_{j,t}^{\theta} - \log Z_{j-1,t}^{\theta}.$$

Then $\{r_j^{\theta}(t)\}_{j=1,\dots,n}$ are independent and identically distributed, with law equal to that of the random variable

$$\log \frac{1}{X_\theta},$$

where X_θ is gamma-distributed with parameter θ:

$$\mathbb{P}(X_\theta \in \mathrm{d}x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} \, \mathrm{d}x,$$

where Γ denotes the Gamma function, see (11).

O'Connell and Moriarty [13] used the representation (1) of Proposition 1.1, to compute the first order asymptotics of $\log Z_{n,t}$. Since its introduction in [19], the semidiscrete polymer has been the subject of much investigation, revealing a rich algebraic structure far beyond the invariant measure statement contained in Proposition 1.1. See, for example, [2, 3, 8–10, 12–14, 17, 19, 21]. Here we mention only a few of the many existing results about the semidiscrete polymer. In [17], O'Connell embedded the processes $\log Z_{j,t}$, $j = 1, \dots, n$, $t > 0$ in a triangular array of solutions to stochastic differential equations. He identified $\log Z_{n,t}$, as the first coordinate of an n-dimensional diffusion, the h-transform of a Brownian motion by a certain Whittaker function. O'Connell used this connection to obtain an explicit formula for the Laplace transform of $\log Z_{n,t}$. Borodin, Corwin, and Ferrari [3] used a modification of O'Connell's formula to show that the centered and rescaled free energy $\log Z_{n,t}$ converges in distribution to a Tracy–Widom GUE random variable.

Closer to the spirit of this paper, Seppäläinen and Valkó adapted an argument from Seppäläinen's work on the discrete log-gamma polymer [20] to obtain upper and lower bounds for the fluctuation exponents associated with the polymer. Predictions from physics [11] have led to the expectation that, for a broad family of $1 + 1$-dimensional polymer models in random environments, there exist exponents χ, ξ such that the variance of the free energy is of order $n^{2\chi}$, while the typical deviation of the polymer paths from a straight line is of order n^{ξ}. For the stationary semidiscrete polymer, the paper [21] contains a proof of the estimates

$$\mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t}) \asymp n^{2\chi}, \quad \mathbb{E}[E^{\theta}_{n,t}[|s_0|]] \asymp n^{\xi}, \tag{2}$$

with $\xi = 2\chi = \frac{2}{3}$, where $E^{\theta}_{n,t}[\cdot]$ denotes the expectation with respect to the (random) polymer measure (see Definition 4). See also Moreno–Flores, Seppäläinen, and Valkó [12] for a derivation of the fluctuation and wandering exponents in the so-called *intermediate disorder regime* where the partition function $Z^{\theta}_{n,t}$ has an additional n-dependent temperature parameter. In Section 4, we reprove the upper bounds of (2) by an alternative argument using the convexity of the free energy, $\log Z^{\theta}_{n,t}$ in the parameter θ.

Our main result complements the upper bounds in (2) with nearly optimal (up to n^{ϵ}) estimates for all central moments of $\log Z^{\theta}_{n,t}$ and all annealed moments of s_0, implying strong concentration on an almost optimal scale. As explained in Section 6, the proof relies on inequalities that appear closely related to the predicted Kardar–Parisi–Zhang scaling relations [6, 11].

To the best of our knowledge, our results are the first bounds for higher central moments of the partition function in any model in the KPZ class. In follow-up work [15], we build on the technique introduced here to obtain concentration for several discrete integrable polymer models: the log-gamma polymer [20], the strict-weak polymer [7, 18], the beta polymer [2], and the inverse-beta polymer [22]. Those four models are treated simultaneously using the Mellin-transform framework in [4]. Together with the present paper, these are the only instances of estimates for higher moments in the KPZ class.

It may be possible to extend our argument to integrable zero-temperature models analogous to the polymer models we treat here such as the Brownian last passage percolation and last passage percolation with exponential weights. We also expect that the argument given here extends without the need for serious modifications to the intermediate disorder regime considered, for example, in [12]. We leave such questions to later work.

1.1. *Main results.* To state our results, we introduce some notation for expectations with respect to the Gibbs measure associated with $\log Z^{\theta}_{n,t}$. Let $\theta > 0$, $n \geq 1$, $t > 0$, and $f = f(s_0, \dots, s_{n-1})$ be a real-valued function on $\mathbb{R}^n$ such that

$$\bigl|f(s_0, \dots, s_{n-1})\bigr| \leq e^{-\nu \min(s_0, 0)} \qquad \text{for all } s_0 \in \mathbb{R} \tag{3}$$

with some $\nu < \theta$. The assumption (3) will guarantee integrability with respect to the random measure defined below.

We define the *quenched expectation* by

$$E^{\theta}_{n,t}[f] := \frac{1}{Z^{\theta}_{n,t}} \int_{-\infty < s_0 < s_1 < \cdots < s_{n-1} < t} e^{\mathcal{E}^{\theta}_{n,t}(s_0, \dots, s_{n-1})} f(s_0, s_1 \dots, s_{n-1}) \, \underline{\mathrm{d}s}, \tag{4}$$

where

$$\underline{\mathrm{d}s} = \mathrm{d}s_0 \, \mathrm{d}s_1 \dots \mathrm{d}s_{n-1}.$$

The *annealed expectation* is defined by

$$\mathbb{E}^{\theta}_{n,t}[f] := \mathbb{E}[E^{\theta}_{n,t}[f]].$$

In many instances below, n and t are fixed throughout a section or computation, and we omit these variables from the notation: $\mathbb{E}^{\theta}[f] = \mathbb{E}^{\theta}_{n,t}[f]$.

Let $\mathbb{1}_A$ be the indicator of a set $A \subset \mathbb{R}^n$:

$$\mathbb{1}_A(s_0, \dots, s_{n-1}) = \begin{cases} 1 & \text{if } (s_0, \dots, s_{n-1}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We use the suggestive notation

$$P^\theta_{n,t}(A) := E^\theta_{n,t}[\mathbb{1}_A] \quad \text{and} \quad \mathbb{P}^\theta(A) := \mathbb{E}^\theta[\mathbb{1}_A].$$

We refer to the first quantity as the quenched probability of the event A, and the second quantity as its annealed probability.

Our main result provides near-optimal estimates for *any* moment of the centered free energy and *any* annealed moment of the time of first jump:

THEOREM 1. *Let $\psi_1(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\Gamma'(\theta)/\Gamma(\theta))$ denote the trigamma function, and suppose that*

$$\big|t - n\psi_1(\theta)\big| \le An^{2/3} \tag{5}$$

for some constant $0 \le A < \infty$. Then, for every $\epsilon > 0$, $\theta \in (0, \infty)$, and $p \in (0, \infty)$, there exists a constant $C = C(A, \epsilon, \theta, p) > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\big[\big|\overline{\log Z^\theta_{n,t}}\big|^p\big] \le Cn^{(1/3)p+\epsilon} \quad \textit{and} \tag{6}$$

$$\mathbb{E}^\theta_{n,t}\big[|s_0|^p\big] \le Cn^{(2/3)p+\epsilon}, \tag{7}$$

where $\overline{X} = X - \mathbb{E}[X]$ denotes the centered random variable.

This result should be compared to that in [21], Theorems 2.2 and 2.3 and equation (4.12), where the following bounds were obtained for the corresponding moments

$$\begin{aligned} \mathbb{E}\big[\big|\overline{\log Z^\theta_{n,t}}\big|^p\big] &\le C(\theta, p)n^{(1/3)p}, \quad p = 2, \\ \mathbb{E}^\theta_{n,t}\big[|s_0|^p\big] &\le C(\theta, p)n^{(2/3)p}, \quad p \in (0, 3). \end{aligned} \tag{8}$$

These authors also obtain the lower bound

$$\mathbb{E}\big[\big|\overline{\log Z^\theta_{n,t}}\big|^2\big] \ge cn^{2/3}.$$

By Jensen's inequality, one sees that (6), (7) are indeed optimal up to an $O(n^\epsilon)$ factor. The n-dependence in (8) is optimal with no ϵ-loss, but only low moments are controlled.

Theorem 1 is based on an inductive argument involving two inequalities. A crucial tool is an expression for the kth cumulant of $\log Z^\theta_{n,t}$ as a sum of multilinear expressions in expectations of products of quenched cumulants of s_0^+, the positive part of s_0, as well as lower order powers of $\log Z^\theta_{n,t}$. This relation between the free energy and the first jump in the system leads to a "scaling relation" which allows us to simultaneously control s_0^+ (or s_0^-) and $\log Z^\theta_{n,t}$.

In order to state the expression for the kth cumulant of $\log Z^\theta_{n,t}$, let $H_{n,\sigma^2}(x)$ denote the nth Hermite polynomial with respect to a Gaussian random variable of variance σ^2, defined in (20), and $\psi_k(\theta)$ be the kth derivative of the digamma function (12). Let $\kappa_k(X)$ denote the kth cumulant of the random variable X. The kth cumulant of a function f with respect to the quenched measure in (4) is denoted by $\kappa^\theta_k(f)$. See Section 2.1 for details.

THEOREM 2. *For integers $k \geq 2$,*

$$\kappa_k(\log Z_{n,t}^\theta) + n(-1)^{k-1}\psi_{k-1}(\theta) + t \cdot \delta_{k,2} \tag{9}$$
$$= \sum_{\pi \in \mathcal{P}} (|\pi|-1)!(-1)^{|\pi|} \sum_{j=1}^{k-1} \binom{k}{j} \prod_{B \in \pi} \mathbb{E}\big[\big(\overline{\log Z_{n,t}^\theta}\big)^{a_{j,B}} H_{b_{j,B},t}\big(B_0(t)\big)\big],$$

where the first sum ranges over partitions π of $\{1,\dots,k\}$, $a_{j,B} = |B \cap \{1,\dots,j\}|$, $b_{j,B} = |B \cap \{j+1,\dots,k\}| = |B| - a_{j,B}$, and $\delta_{i,j}$ is the Kronecker delta function. We can omit any product of blocks that has a block B completely contained inside $\{j+1,\dots,k\}$, as well as any partition that contains a singleton.

Moreover, each factor in the products appearing in (9) has an expression in terms of quenched cumulants of s_0^+:

$$\mathbb{E}\big[\big(\overline{\log Z_{n,t}^\theta}\big)^a H_{b,t}\big(B_0(t)\big)\big] = (-1)^b \sum_{\substack{\ell_1+\cdots+\ell_a=b \\ \ell_i \geq 0}} \frac{b!}{\ell_1!\cdots\ell_a!} \mathbb{E}\Bigg[\prod_{i=1}^a \kappa_{\ell_i}^\theta(s_0^+)\Bigg],$$

where we use the convention $\kappa_0^\theta(s_0^+) := \overline{\log Z_{n,t}^\theta}$.

The case $k=2$ was previously obtained by Seppäläinen and Valkó in [21]. For explicit expressions when $k=3$ or $k=4$, see Lemma 5.2 and Corollary 5.5 respectively.

1.2. *Application*: *Localization of the polymer paths*. The strong estimates implied by (7) give deviation estimates for the polymer path away from the right endpoint: with very high probability, the entire polymer path lies within $O(n^{\frac{2}{3}+})$ of the line through $(0,0)$ and (t,n):

PROPOSITION 1.2. *For any $\theta, \epsilon > 0$ and $n \geq 1$ such that condition (5) holds. Let $k \geq 1$ positive integer, there are constants $C(A,\theta,k,\epsilon) > 0$ such that*

$$\mathbb{P}_{n,t}^\theta\left(\max_{0 \leq j \leq n}\left|s_j - \frac{j}{n}t\right| \geq n^{\frac{2}{3}+\epsilon}\right) \leq C(A,\theta,k,\epsilon)n^{-k}.$$

PROOF. We use the Burke property of the O'Connell–Yor polymer [19] in the form of the identity [21], equation (6.4):

$$P_{n,t}^\theta\big(|s_j - (j/n)t| > n^{\frac{2}{3}+\epsilon}\big) = P_{n-j,(1-j/n)t}^\theta\big(|s_0| > n^{\frac{2}{3}+\epsilon}\big).$$

For the second quenched probability, we have the following relation between the parameters:

$$\begin{aligned}\big|(1-(j/n)) \cdot t - (n-j)\psi_1(\theta)\big| &\leq \left(1 + \frac{j}{n}\right)\big|t - n\psi_1(\theta)\big| \\ &\leq 2An^{2/3}.\end{aligned}$$

Next we use (7) to obtain

$$\begin{aligned}\mathbb{P}_{n-j,(1-j/n)t}^\theta\big(|s_0| > n^{\frac{2}{3}(K+\epsilon)}\big) &\leq n^{-\frac{2}{3}(K+\epsilon)} \cdot \mathbb{E}_{n-j,(1-j/n)t}^\theta\big[|s_0|^K\big] \\ &\leq C(2A,\epsilon,\theta,K)n^{-(\frac{2}{3}+\epsilon)K}n^{\frac{2}{3}K+\epsilon} \\ &= O\big(n^{-(K-1)\epsilon}\big).\end{aligned}$$

Choosing $K \geq \frac{k+1+\epsilon}{\epsilon}$, we have

$$\mathbb{P}_{n,t}^\theta\big(|s_j - (j/n)t| > n^{\frac{2}{3}+\epsilon}\big) \leq C(\theta,k,\epsilon)n^{-k-1}.$$

The result then follows by writing

$$\mathbb{P}^{\theta}_{n,t}\left(\max_{0\le j\le n}\left|s_j-\frac{j}{n}t\right|\ge n^{\frac{2}{3}+\epsilon}\right)\le\sum_{j=1}^{n}\mathbb{P}^{\theta}_{n,t}\left(\left|s_j-(j/n)t\right|>n^{\frac{2}{3}+\epsilon}\right).$$

□

1.3. *Outline of paper.* In Section 2, we introduce some basic definitions, and review elementary properties of the stationary polymer which appeared in previous literature. We also introduce the notation used throughout the paper.

In Section 3, we use the Cameron–Martin–Girsanov theorem to derive formulas of "integration by parts" type, relating the positive part of the first jump, s_0^+, to the free energy, $\log Z^{\theta}_{n,t}$, by perturbing the path $B_0(t)$, $t\ge 0$. These formulas are generalizations of a relation in [21], which was used to derive the variance estimate

$$cn^{2/3}\le\mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t})\le Cn^{2/3} \tag{10}$$

for some n-independent constants c, $C>0$.

Section 4 serves as an illustration of the general methodology used to derive Theorem 1, exploiting the reciprocal relation between s_0^+ and $\log Z^{\theta}_{n,t}$. Using convexity of the free energy of the stationary polymer, we give an alternate, shorter proof of the upper bound of the variance estimate (10), first obtained in [21].

In Section 5, we exploit Gaussian integration by parts to derive a formula for the cumulants of $\log Z^{\theta}_{n,t}$ in terms of multilinear expressions in expectations of lower moments of $\log Z^{\theta}_{n,t}$ and quenched cumulants of s_0^+. The formula, which appears in Theorem 2, is a generalization of the variance identity in [21], and it facilitates an inductive analysis of the moments of $\log Z^{\theta}_{n.t}$: higher central moments of the free energy are estimated by lower moments, as well as lower moments of s_0^+.

In Section 6, we use the formula in Theorem 2 to obtain near-optimal bounds on the central moments of the free energy of the stationary polymer, as well as annealed moments of the first jump in the system. Our proof is iterative, combining two inequalities to improve bounds on $\log Z^{\theta}_{n,t}$ using estimates on the tail of s_0^+, and vice versa, with a "fixed point" at the optimal values of the exponents $(\chi,\xi)=(1/3,2/3)$. An important observation here is that a high probability bound of the form $s_0^+\ll\tau$ implies that $\log Z^{\theta}_{n,t}$ is insensitive to perturbations of the boundary path $B_0(s)$, $0\le s\le t$ that affect it only for $s\gg\tau$.

2. Preliminaries and notation. In this paper, we denote by $\mathbb{P}$ and $\mathbb{E}$ the probability measure, resp. expectation on the common probability space Ω where the two-sided Brownian motions $(B_n(t))_{t\in\mathbb{R}}$, $n=0,1,2,\dots$ are defined. For a random X on Ω, we denote the centered random variable as follows:

$$\overline{X}:=X-\mathbb{E}[X].$$

The covariance and variance with respect to $\mathbb{E}$ are respectively denoted by

$$\mathbb{C}\mathrm{ov}(X,Y):=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\quad\text{and}\quad\mathbb{V}\mathrm{ar}(X):=\mathbb{C}\mathrm{ov}(X,X)=\mathbb{E}\big[(\overline{X})^2\big].$$

2.1. *Cumulants.* The main input for the computations presented in this paper is Proposition 1.1. That result provides explicit formulas for the cumulants of s_0, the first jump in the system. To explain this, introduce the gamma function, defined for $\theta>0$ by

$$\Gamma(\theta)=\int_0^{\infty}s^{\theta-1}e^{-s}\,\mathrm{d}s. \tag{11}$$

The *digamma* function is the logarithmic derivative of Γ

$$\psi_0(\theta)=\frac{\Gamma'(\theta)}{\Gamma(\theta)}. \tag{12}$$

The higher derivatives are denoted by ψ_k, $k = 1, 2, \dots$

$$\psi_k(\theta) = \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}\psi_0(\theta).$$

We have $(-1)^k\psi_k(s) < 0$ for any $k \in \mathbb{N}$ and $s > 0$, [20]. By taking expectations in equation (1), we find

$$\mathbb{E}\big[\log Z^{\theta}_{n,t}\big] = -n\psi_0(\theta) + \theta t. \tag{13}$$

As we discuss below, the relation (13) gives an expression for the expected cumulant generating function of s_0, the first jump in the system.

Recall that for a random variable X with exponential moments, the kth cumulant, denoted by $\kappa_k(X)$, is equal to the kth derivative at zero of the log-moment generating function [5]:

$$\log E\big[e^{\delta X}\big] = \sum_{k=0}^{\infty}\frac{\delta^k}{k!}\kappa_k(X),$$

where δ is small enough for the left side to converge. To define the *quenched cumulants*, let $0 < \delta < 1$ and let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy (3). The cumulant generating function of f is given by

$$\log Z^{\theta,\delta f}_{n,t} := \log \int_{-\infty < s_0 < \cdots < s_{n-1} < t} e^{\delta f(s_0,\dots,s_{n-1}) + \mathcal{E}^{\theta}_{n,t}(s_0,\dots,s_{n-1})}\, \underline{\mathrm{d}s}.$$

For $k \geq 1$, the kth *quenched cumulant* with respect to $E^{\theta}_{n,t}[\cdot]$ is then

$$\kappa^{\theta}_k(f) = \frac{\mathrm{d}^k}{\mathrm{d}\delta^k}\log E^{\theta}_{n,t}\big[e^{\delta f}\big]\bigg|_{\delta=0} = \frac{\mathrm{d}^k}{\mathrm{d}\delta^k}\log Z^{\theta,\delta f}_{n,t}\bigg|_{\delta=0}.$$

For example,

$$\kappa^{\theta}_1(f) = E^{\theta}_{n,t}[f] \quad \text{and} \quad \kappa^{\theta}_2(f) = E^{\theta}_{n,t}\big[f^2\big] - \big(E^{\theta}_{n,t}[f]\big)^2.$$

Note that we suppress the dependence on n and t from the notation for simplicity.

Differentiating (13) with respect to θ, we have

$$\mathbb{E}\big[\kappa^{\theta}_k(s_0)\big] = t\delta_{k,1} + n\psi_k(\theta). \tag{14}$$

Thus, Proposition 1.1 implies that all *expected* quenched cumulants of s_0 for $k \geq 2$ are of order n. Similarly, Proposition 1.1 implies that for each $t > 0$, $k \geq 1$, and $1 \leq j \leq n$:

$$\kappa_{k+1}\big(r^{\theta}_j(t)\big) = (-1)^{k+1}\psi_k(\theta). \tag{15}$$

2.2. *A priori bounds*. In this section, we collect a few basic bounds on the quantities we will be interested in under the condition (5). For $x, y \in \mathbb{R}$, we denote the minimum and maximum of x and y by

$$x \wedge y = \min\{x, y\} \quad \text{and} \quad x \vee y = \max\{x, y\}.$$

The positive and negative parts of x are denoted by

$$x^+ = \max\{0, x\} \quad \text{and} \quad x^- = \max\{0, -x\}.$$

An immediate consequence of Proposition 1.1 is that $\log Z^{\theta}_{n,t}$ has finite exponential moments. Moreover, if we define

$$R := \sum_{j=1}^{n} r^{\theta}_j(t), \tag{16}$$

we see that for $p \geq 1$, the centered free energy $\overline{\log Z_{n,t}^{\theta}}$ satisfies

$$\mathbb{E}\big[\big|\overline{\log Z_{n,t}^{\theta}}\big|^{p}\big]^{1/p} \leq \mathbb{E}\big[\big|B_0(t)\big|^{p}\big]^{1/p} + \mathbb{E}\big[|\overline{R}|^{p}\big]^{1/p} \leq C'(\theta, p)(\sqrt{t} + \sqrt{n}). \tag{17}$$

From [21], Lemma 4.4, we also have

$$\mathbb{E}^{\theta}\big[|s_0|^{p}\big] \leq C(\theta, p)n^{p} \quad \text{for every } p > 0. \tag{18}$$

Expressing cumulants in terms of moments, we have

$$\big|\kappa_k^{\theta}(s_0^{+})\big| \leq C(k)E_{n,t}^{\theta}\big[(s_0^{+})^{k}\big].$$

Combining this with (18) and using Jensen's inequality gives

$$\mathbb{E}\big[\big|\kappa_k^{\theta}(s_0^{+})\big|^{p}\big]^{\frac{1}{p}} \leq C(\theta, p, k)n^{k} < \infty \quad \text{for every } k \in \mathbb{N} \text{ and } p \geq 1.$$

3. Gaussian integration by parts. The Hermite polynomials are defined by the formula

$$H_k(x) = (-1)^{k} e^{-\frac{x^2}{2}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{\frac{x^2}{2}}, \quad k = 0, 1, 2, \dots.$$

The polynomials are orthogonal with respect to the standard Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. The Hermite generating function is [16], equation (1.1),

$$e^{\lambda x - \frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x). \tag{19}$$

For $t > 0$, we also define the generalized Hermite polynomials, with variance t by

$$H_{k,t}(x) := t^{\frac{k}{2}} H_k\left(\frac{x}{\sqrt{t}}\right). \tag{20}$$

Rescaling (19), we have

$$e^{\lambda x - \frac{\lambda^2 t}{2}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_{n,t}(x). \tag{21}$$

Recall that the cumulants of s_0^{+} with respect to the quenched measure $P_{n,t}^{\theta}$ are given by

$$\kappa_k^{\theta}(s_0^{+}) = \frac{\mathrm{d}^k}{\mathrm{d}\delta^k} \log Z_{n,t}^{\theta, \delta s_0^{+}} \bigg|_{\delta=0} \quad \text{for } k \geq 1. \tag{22}$$

For $k = 0$, we use the convention:

$$\kappa_0^{\theta}(s_0^{+}) := \overline{\log Z_{n,t}^{\theta}}. \tag{23}$$

LEMMA 3.1. *For $t > 0$, $j, k \geq 1$,*

$$\mathbb{E}\big[\big(\overline{\log Z_{n,t}^{\theta}}\big)^{j} H_{k,t}\big(B_0(t)\big)\big] = (-1)^{k} \sum_{\substack{\ell_1 + \cdots + \ell_j = k \\ \ell_i \geq 0}} \frac{k!}{\ell_1! \cdots \ell_j!} \mathbb{E}\left[\prod_{i=1}^{j} \kappa_{\ell_i}^{\theta}(s_0^{+})\right]. \tag{24}$$

PROOF. Let $0 < \delta < \min\{\theta, 1\}$. The expectation

$$\mathbb{E}\big[\big(\log Z_{n,t}^{\theta, -\delta s_0^{+}} - \mathbb{E}[\log Z_{n,t}^{\theta}]\big)^{j}\big]$$

equals

$$\mathbb{E}\left[\left(\log \int_{-\infty<s_0<\cdots<s_{n-1}<t} e^{\theta s_0 - B_0(s_0) - \delta s_0^+ + \mathcal{E}_{n,t}(s_0,\ldots,s_{n-1})}\,\underline{\mathrm{d}s} - \mathbb{E}[\log Z_{n,t}^{\theta}]\right)^j\right].$$

By the Cameron–Martin–Girsanov Theorem ([16], Proposition 4.1.2), this equals

$$\mathbb{E}\big[e^{\delta B_0(t) - \frac{\delta^2}{2}t}(\overline{\log Z_{n,t}^{\theta}})^j\big].$$

The exponential factor in the expectation is the generating function of the generalized Hermite polynomials (21) with variance t, so (24) follows by repeated differentiation with respect to δ.

To justify the use of differentiation under the expectation, we show the difference quotients are dominated independently of δ. The derivative

$$\frac{\mathrm{d}^k}{\mathrm{d}\delta^k}(\overline{\log Z_{n.t}^{\theta,-\delta s_0^+}})^j \tag{25}$$

is a linear combination of products of the form

$$\prod_{i=1}^{j} \kappa_{\ell_i}^{\theta,-\delta}(s_0^+),$$

where $\sum \ell_i = k$, and $\kappa_k^{\theta,-\delta}$ is the kth cumulant with respect to the measure

$$E_{n,t}^{\theta,-\delta}[\,\cdot\,] := \frac{E_{n,t}^{\theta}[e^{-\delta s_0^+}\cdot\,]}{E_{n,t}^{\theta}[e^{-\delta s_0^+}]}.$$

Using the trivial estimate

$$E_{n,t}^{\theta,-\delta}[f] \le e^t E_{n,t}^{\theta}[f]$$

and expressing the cumulants in terms of moments, we see that (25) is bounded up to a constant by a sum of terms of the form

$$\big|E_{n,t}^{\theta}\big[(s_0^+)^k\big]\big| \cdot \big|\log Z_{n,t}^{\theta,-\delta s_0^+}\big|^b,$$

where $b = \#\{i : \ell_i = 0\}$. Since

$$\log Z_{n,0}^{\theta} + B_n(t) = \log Z_{n,0}^{\theta,-\delta s_0^+} + B_n(t) \le \log Z_{n,t}^{\theta,-\delta s_0^+} \le \log Z_{n,t}^{\theta},$$

and all moments of s_0^+ and $\log Z_{n,t}^{\theta}$ are finite, we find that the derivative (25) is dominated by an integrable function, so the lemma now follows from the dominated convergence theorem. □

The next proposition is a generalization of (24) to "stopped" Brownian motions.

PROPOSITION 3.2. *Let* $0 < \tau \le t$, *and* $j, k \ge 0$. *We have*

$$\mathbb{E}\big[(\overline{\log Z_{n,t}^{\theta}})^j H_{k,\tau}(B_0(\tau))\big] = (-1)^k \sum_{\substack{\ell_1+\cdots+\ell_j=k\\ \ell_i\ge 0}} \frac{k!}{\ell_1!\cdots\ell_j!}\mathbb{E}\left[\prod_{i=1}^{j}\kappa_{\ell_i}^{\theta}(s_0^+\wedge\tau)\right].$$

PROOF. We apply the Cameron–Martin–Girsanov theorem to the Brownian motion $B_0(s_0)$, $s_0 \geq 0$ in the form

$$\mathbb{E}\left[F\left(B_0(s_0) + \delta \int_0^{s_0} b(s)\,\mathrm{d}s\right)\right] = \mathbb{E}\left[e^{\delta \int_0^t b(s)\,\mathrm{d}B_0(s) - \frac{\delta^2}{2}\|b\|^2_{L^2([0,t])}} F(B_0|_{[0,t]})\right] \tag{26}$$

with F with $F = (\overline{\log Z^\theta_{n,t}})^j$ and $b(s) = -\mathbb{1}_{[0,\tau]}(s)$, so

$$\int_0^{s_0} b(s)\,\mathrm{d}B_0(s) = -B_0(s_0^+ \wedge \tau),$$

and proceed as in the proof of Lemma 3.1. Differentiation inside the expectation is justified as in that proof. □

3.1. *Application*: *Seppäläinen and Valkó's variance identity.* Recall the notation from (16): $R = \sum_{j=1}^n r_j^\theta(t)$. By Proposition 1.1,

$$\overline{R} = \overline{\log Z^\theta_{n,t}} + B_0(t).$$

Squaring both sides, taking expectations, and using (15), we obtain

$$\mathbb{E}[(\overline{R})^2] = n\psi_1(\theta) = \mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}) + t + 2\mathbb{E}[\log Z^\theta_{n,t} B_0(t)]. \tag{27}$$

Applying the integration by parts formula (24) with $j = k = 1$, we obtain the identity

$$\mathbb{E}[\log Z^\theta_{n,t} B_0(t)] = -\mathbb{E}[E^\theta_{n,t}[s_0^+]].$$

Plugging this into (27) and rearranging yields the key variance identity

$$\mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}) = n\psi_1(\theta) - t + 2\mathbb{E}^\theta_{n,t}[s_0^+]. \tag{28}$$

Similar identities relating the variance of a free energy to transversal fluctuations have appeared in several works of Seppäläinen and collaborators on studying anomalous fluctuations in KPZ models. See [21], Theorem 3.6 and [20], Theorem 3.7. One of our main results yields higher order versions of (28).

4. Convexity proof of Seppäläinen and Valkó's fluctuation estimate.

In this section, we present an alternative proof of the estimate

$$\mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}) \leq C(\theta)n^{2/3} \tag{29}$$

given the following *characteristic direction* condition

$$|t - n\psi_1(\theta)| \leq An^{2/3}. \tag{30}$$

The estimate (29) and the corresponding lower bound were originally obtained by Seppäläinen and Valkó [21]. We replace the key step in their proof by the convexity of the free energy.

LEMMA 4.1. *Almost surely, the function*

$$\theta \mapsto \log Z^\theta_{n,t}$$

is convex for all t. The first derivative with respect to θ equals

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log Z^\theta_{n,t} = E^\theta_{n,t}[s_0], \tag{31}$$

while the second derivative with respect to θ equals

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log Z^\theta_{n,t} = \mathrm{Var}^\theta(s_0) := E^\theta_{n,t}[(s_0 - E^\theta_{n,t}[s_0])^2] \geq 0. \tag{32}$$

In particular, for $\eta < \theta < \lambda$, almost surely, we have

$$\frac{\log Z^{\theta}_{n,t} - \log Z^{\eta}_{n,t}}{\theta - \eta} \le E^{\theta}_{n,t}[s_0] \le \frac{\log Z^{\lambda}_{n,t} - \log Z^{\theta}_{n,t}}{\lambda - \theta}. \tag{33}$$

PROOF. The expresions for the derivatives (31) and (32) follow by direct computation, and the remaining statements are a consequence of the nonnegativity of the second derivative. □

The following computation relates the quenched second moment and variance of s_0, to those of s_0^+. For simplicity, in the rest of this section, we write $E = E^{\theta}_{n,t}$.

LEMMA 4.2. *Almost surely,*

$$E[(s_0 - E[s_0])^2] = E[(s_0^+ - E[s_0^+])^2] + E[(s_0^- - E[s_0^-])^2] + 2E[s_0^+]E[s_0^-]. \tag{34}$$

In particular,

$$\begin{aligned} \mathbb{E}[E[s_0^+]^2] &\le \mathbb{E}[E[s_0]^2] + 2\mathbb{E}[E[s_0^+]E[s_0^-]] \\ &\le \mathbb{E}[E[s_0]^2] - n\psi_2(\theta). \end{aligned} \tag{35}$$

PROOF. By direct computation,

$$\begin{aligned} E[(s_0 - E[s_0])^2] &= E[((s_0^+ - E[s_0^+]) - (s_0^- - E[s_0^-]))^2] \\ &= E[(s_0^+ - E[s_0^+])^2] + E[(s_0^- - E[s_0^-])^2] \\ &\quad - 2E[(s_0^+ - E[s_0^+])(s_0^- - E[s_0^-])]. \end{aligned}$$

Since s_0^+ and s_0^- have disjoint support,

$$E[(s_0^+ - E[s_0^+])(s_0^- - E[s_0^-])] = -E[s_0^+]E[s_0^-],$$

which yields (34). All terms in (34) are nonnegative, so

$$0 \le E[s_0^+]E[s_0^-] \le \frac{1}{2}E[(s_0 - E[s_0])^2] = \frac{1}{2}\kappa_2^{\theta}(s_0).$$

Taking expectations and using (14),

$$\mathbb{E}[E[s_0^+]E[s_0^-]] \le -\frac{n}{2}\psi_2(\theta). \tag{36}$$

Finally, after expanding, we get

$$E[s_0]^2 = E[s_0^+]^2 + E[s_0^-]^2 - 2E[s_0^+]E[s_0^-].$$

Taking expectations and applying (36) yields (35). □

The following property regarding the map $\theta \mapsto \mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t})$ was already used by Seppäläinen and Valkó. See [21], Lemma 4.3.

LEMMA 4.3. *For $\theta, \lambda > 0$,*

$$|\mathbb{V}\mathrm{ar}(\log Z^{\lambda}_{n,t}) - \mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t})| \le n|\psi_1(\lambda) - \psi_1(\theta)|. \tag{37}$$

PROOF OF ESTIMATE (29). By (33) with $\lambda - \theta = \theta - \eta = n^{-1/3}$,

$$n^{-1/3}|E[s_0]| \le |\log Z^\theta_{n,t} - \log Z^\lambda_{n,t}| + |\log Z^\eta_{n,t} - \log Z^\theta_{n,t}|.$$

By a Taylor series expansion of $\psi_0(\lambda)$ about $\lambda = \theta$,

$$|\psi_0(\lambda) - \psi_0(\theta) - (\lambda - \theta)\psi_1(\theta)| \le C(\theta)(\lambda - \theta)^2. \tag{38}$$

Combined with (13), (38), and (30), we can center the free energies to obtain

$$\begin{aligned}
&|\log Z^\theta_{n,t} - \log Z^\lambda_{n,t}| + |\log Z^\eta_{n,t} - \log Z^\theta_{n,t}| \\
&\quad \le An^{\frac{2}{3}}(|\theta - \lambda| + |\eta - \theta|) + Cn((\lambda - \theta)^2 + (\eta - \theta)^2) \\
&\qquad + |\overline{\log Z^\theta_{n,t}} - \overline{\log Z^\lambda_{n,t}}| + |\overline{\log Z^\eta_{n,t}} - \overline{\log Z^\theta_{n,t}}| \\
&\quad \le C(\theta)n^{1/3} + |\overline{\log Z^\theta_{n,t}} - \overline{\log Z^\lambda_{n,t}}| + |\overline{\log Z^\eta_{n,t}} - \overline{\log Z^\theta_{n,t}}|.
\end{aligned}$$

We will continue to use this simplification for the remainder of this section. Squaring, taking expectations, and using (37), we have the bound

$$\begin{aligned}
n^{-2/3}\mathbb{E}[E[s_0]^2] &\le C(\theta)(n^{2/3} + \mathbb{E}[|\overline{\log Z^\theta_{n,t}} - \overline{\log Z^\lambda_{n,t}}|^2] + \mathbb{E}[|\overline{\log Z^\eta_{n,t}} - \overline{\log Z^\theta_{n,t}}|^2]) \\
&\le C(\theta)(n^{2/3} + \mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}) + n|\lambda - \theta| + n|\eta - \theta|).
\end{aligned} \tag{39}$$

Using (35), we find

$$\mathbb{E}[E[s_0^+]] \le \mathbb{E}[E[s_0]^2]^{1/2} + C(\theta)n^{1/2}.$$

Finally, (28), (30), and (39) give

$$\mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}) \le Cn^{1/3}(n^{2/3} + \mathbb{V}\mathrm{ar}(\log Z^\theta_{n,t}))^{1/2},$$

a quadratic relation which implies (29). □

5. Formulas for $\kappa_k(\log Z^\theta_{n,t})$. In order to give exact formulas for $\kappa_k(\log Z^\theta_{n,t})$ we first discuss joint cumulants and their connection to Hermite polynomials. The joint cumulant of the random variables $X_1, \dots, X_k$ is defined by

$$\kappa(X_1, \dots, X_k) := \frac{\partial^k}{\partial \xi_1 \dots \partial \xi_k} \log \mathbb{E}[e^{\sum_{j=1}^k \xi_j X_j}]|_{\xi_i = 0}. \tag{40}$$

Alternatively, it can be written as a combination of products of expectations of the underlying random variables:

$$\kappa(X_1, \dots, X_k) = \sum_{\pi \in \mathcal{P}} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}\Big[\prod_{i \in B} X_i\Big], \tag{41}$$

where $\mathcal{P}$ ranges over partitions π of $\{1, \dots, k\}$ and $|A|$ stands for the size of the set A. This expression is commonly used as the definition of joint cumulants, we show that it is equivalent to (41) in the Appendix.

Note that the joint cumulant is multilinear. In the case where $X_1 = X_2 = \cdots = X_k = X$, the joint cumulant reduces to the kth cumulant of X, $\kappa_k(X)$. Two important properties of cumulants that we will take advantage of are shift-invariance:

$$\kappa_k(X + c) = \kappa_k(X) \quad \text{for } k \ge 2, \text{ where } c \text{ is constant},$$

and additivity for independent random variables:

$$\kappa_k(X + Y) = \kappa_k(X) + \kappa_k(Y) \quad \text{for any } k, \text{ if } X \text{ and } Y \text{ are independent}.$$

The following lemma relates the kth cumulant of the free energy to a sum of joint cumulants involving the centered free energy Brownian motion B_0.

LEMMA 5.1. *Let $\theta > 0$, $t > 0$, and $n \in \mathbb{N}$. Then for any integer $k \geq 2$,*

$$\kappa_k(\log Z_{n,t}^{\theta}) = n(-1)^k \psi_{k-1}(\theta) - \sum_{j=0}^{k-1} \binom{k}{j} \kappa(\underbrace{\overline{\log Z_{n,t}^{\theta}}, \ldots, \overline{\log Z_{n,t}^{\theta}}}_{j\text{-times}}, \underbrace{B_0(t), \ldots, B_0(t)}_{k-j \text{ times}}). \tag{42}$$

Note that the 0th term in the summation is $\kappa_k(B_0(t))$ which equals 0 when $k \neq 2$, and t when $k = 2$.

PROOF. For convenience, put $A := \overline{\log Z_{n,t}^{\theta}}$, $B_0 := B_0(t)$, and $R := \sum_{j=1}^{n} r_j^{\theta}(t)$, so $\overline{R} = A + B_0$. The shift-invariance of the cumulant along with the mulitlinearity of the joint cumulant gives

$$\kappa_k(R) = \kappa_k(\overline{R}) = \kappa(\underbrace{A + B_0, A + B_0, \ldots, A + B_0}_{k\text{-times}}) = \sum_{j=0}^{k} \binom{k}{j} \kappa(\underbrace{A, \ldots, A}_{j\text{-times}}, \underbrace{B_0, \ldots, B_0}_{k-j \text{ times}}).$$

The left-hand side simplifies to $\kappa_k(R) = n\kappa_k(r_j^{\theta}(t)) = n(-1)^k \psi_{k-1}(\theta)$ by equation (15), as R is a sum of n i.i.d. random variables, while the kth entry in the sum on the right-hand side gives $\kappa_k(\log Z_{n,t}^{\theta})$. Rearranging yields the desired result. □

5.1. *Estimate for $\kappa_3(\log Z_{n,t}^{\theta})$.* To motivate computations in the upcoming sections we use Lemma 5.1 and [21], equation (4.13), to obtain a bound of the optimal order, $n^{(1/3)\cdot 3}$, for the third centered moment of $\log Z_{n,t}^{\theta}$.

The joint cumulants simplify when the random variables are centered. For example, if X, Y, Z are centered, then

$$\kappa(X, Y, Z) = \mathbb{E}[XYZ]. \tag{43}$$

Therefore the third cumulant of a random variable agrees with its third central moment. We now use (43) to obtain an exact formula for the third cumulant/central moment of the free energy.

LEMMA 5.2. *For any $t > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}[(\overline{\log Z_{n,t}^{\theta}})^3] = \kappa_3(\log Z_{n,t}^{\theta}) = -n\psi_2(\theta) + 6\mathbb{E}[\overline{\log Z_{n,t}^{\theta}} E_{n,t}^{\theta}[s_0^+]] - 3\mathbb{E}[\mathrm{Var}^{\theta}(s_0^+)]. \tag{44}$$

PROOF. For convenience we write $Z = Z_{n,t}^{\theta}$, $B_0 = B_0(t)$, and $E = E_{n,t}^{\theta}$. By Lemma 5.1,

$$\kappa_3(\log Z) = -n\psi_2(\theta) - 3\kappa(\overline{\log Z}, \overline{\log Z}, B_0) - 3\kappa(\overline{\log Z}, B_0, B_0). \tag{45}$$

We now analyze the joint cumulants individually. Equation (43) and two applications of Lemma 3.1 give

$$\kappa(\overline{\log Z}, \overline{\log Z}, B_0) = \mathbb{E}[\overline{\log Z}^2 B_0] = -2\mathbb{E}[\overline{\log Z} E[s_0^+]], \tag{46}$$

and

$$\kappa(\overline{\log Z}, B_0, B_0) = \mathbb{E}[\overline{\log Z} B_0^2] = \mathbb{E}[\log Z(B_0^2 - t)] = \mathbb{E}[\mathrm{Var}^{\theta}(s_0^+)]. \tag{47}$$

Combining equations (45), (46), and (47) yields the desired result. □

Next, we use Lemma 5.2 to show that $\kappa_3(\log Z_{n,t}^{\theta})$ has order at most n when n and t satisfy (5).

COROLLARY 5.3. *Assume n and t satisfy*

$$\left|t - n\psi_1(\theta)\right| \le An^{2/3}.$$

Then there exists a constant $C = C(\theta) < \infty$ such that for all $n \in \mathbb{N}$,

$$\left|\mathbb{E}\left[\left(\overline{\log Z^{\theta}_{n,t}}\right)^3\right]\right| \le Cn.$$

PROOF. Applying the Cauchy–Schwarz inequality followed by Jensen's inequality, [21], equation (4.12), and the bound (29),

$$\begin{aligned}\left|\mathbb{E}\left[\overline{\log Z^{\theta}_{n,t}} E^{\theta}_{n,t}[s_0^+]\right]\right| &\le \mathbb{E}\left[\left(\overline{\log Z^{\theta}_{n,t}}\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[E^{\theta}_{n,t}\left[\left(s_0^+\right)^2\right]\right]^{\frac{1}{2}} \\ &\le C\left(n^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(n^{\frac{4}{3}}\right)^{\frac{1}{2}} = Cn.\end{aligned}$$

By equation (14), we have

$$0 \le \mathbb{E}\left[\operatorname{Var}^{\theta}\left(s_0^+\right)\right] \le \mathbb{E}\left[\operatorname{Var}^{\theta}(s_0)\right] = -n\psi_2(\theta).$$

Thus all terms on the right side of (44) are of order at most n. □

5.2. *Higher cumulants*: *Proof of Theorem* 2. We now develop a systematic method to deal with higher cumulants. The following lemma expresses the joint cumulants appearing in the sum on the right-hand side of equation (42) as linear combinations of products of expectations which only involve the free energy and Hermite polynomials of the Brownian motion B_0. After multiple Gaussian integration by parts, the remaining expressions will involve expectations of quenched cumulants rather than the Brownian motion B_0, leading to the exact formula in Theorem 2.

LEMMA 5.4. *Let $\theta > 0$, $t > 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, and $1 \le j \le k$. Then*

$$\begin{aligned}&\kappa\big(\underbrace{\overline{\log Z^{\theta}_{n,t}}, \dots, \overline{\log Z^{\theta}_{n,t}}}_{j\text{-times}}, \underbrace{B_0(t), \dots, B_0(t)}_{k-j\text{-times}}\big) \\ &\quad = \sum_{\pi \in \mathcal{P}} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}\left[\left(\overline{\log Z^{\theta}_{n,t}}\right)^{|B \cap \{1,\dots,j\}|} H_{|B \cap \{j+1,\dots,k\}|,t}\left(B_0(t)\right)\right],\end{aligned}$$

where $\mathcal{P}$ ranges over partitions π of $\{1, \dots, k\}$. We can omit any partition π which has a block B contained in $\{j + 1, \dots, k\}$. We can also omit any partition π which contains a singleton set.

PROOF. For convenience, again put $A = \overline{\log Z^{\theta}_{n,t}}$ and $B_0 = B_0(t)$. Recalling the generalized Hermite generating function (21), we have

$$e^{\lambda B_0} = e^{\frac{\lambda^2 t}{2}} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_{n,t}(B_0).$$

Therefore,

$$\begin{aligned}&\log \mathbb{E}\left[e^{(\xi_1+\cdots+\xi_j)A + (\xi_{j+1}+\cdots+\xi_k)B_0}\right] \\ &\quad = \log \mathbb{E}\left[\sum_{n=0}^{\infty} e^{(\xi_1+\cdots+\xi_j)A} \frac{(\xi_{j+1}+\cdots+\xi_k)^n}{n!} H_{n,t}(B_0)\right] + \frac{(\xi_{j+1}+\cdots+\xi_k)^2 t}{2}.\end{aligned}$$

Plugging this into the right-hand side of (40), taking the derivatives $\partial_{\xi_1}, \dots, \partial_{\xi_k}$, evaluating at $\xi_i = 0$, and using $\mathbb{E}[H_{n,t}(B_0)] = 0$ for $n \geq 1$, we obtain the formula

$$\kappa(\underbrace{A, \dots, A}_{j \text{ times}}, \underbrace{B_0, \dots, B_0}_{k-j \text{ times}}) = \sum_{\pi \in \mathcal{P}} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}[A^{|B \cap \{1,\dots,j\}|} H_{|B \cap \{j+1,\dots,k\}|,t}(B_0)],$$

where $\mathcal{P}$ ranges over partitions π of $\{1, \dots, k\}$ such that no block $B \in \pi$ is contained in $\{j+1, \dots, k\}$. Finally, if B is a singleton set that is contained in $\{1, \dots, j\}$, then

$$\mathbb{E}[A^{|B \cap \{1,\dots,j\}|} H_{|B \cap \{j+1,\dots,k\}|,t}(B_0)] = \mathbb{E}[A] = 0. \qquad \square$$

We can now prove Theorem 2.

PROOF OF THEOREM 2. Combine Lemmas 5.1, 5.4, and 3.1. $\square$

One can verify that the formula for $k = 3$ agrees with that in Lemma 5.2. For another concrete exact formula, one can verify that the formula for $k = 4$ gives

COROLLARY 5.5.

$$\begin{aligned} \kappa_4(\log Z_{n,t}^\theta) &= n\psi_3(\theta) + 4\mathbb{E}[\kappa_3^\theta(s_0^+)] + 12\mathrm{Cov}(E_{n,t}^\theta[s_0^+], (\overline{\log Z_{n,t}^\theta})^2) \\ &\quad - 12\mathbb{V}\mathrm{ar}(E[s_0^+]) - 12\mathbb{E}[\mathrm{Var}^\theta(s_0^+)\overline{\log Z_{n,t}^\theta}]. \end{aligned}$$

6. Estimates for the central moments: Proof of Theorem 1. The proof of Theorem 1 is obtained by iterating the two inequalities (50) and (61). These relate the moments of s_0^+ and the central moments of $\log Z_{n,t}^\theta$, successively improving bounds for both. The inequality (50) exploits the relationship between n and t given in (5) to obtain a first order cancellation, see [21], Lemma 4.2. The case $k = 2$ was used by the authors of [21] to estimate the variance of the partition function, and similar bounds appear in works of Seppäläinen [20] and Balázs–Cator–Seppäläinen [1]. The estimate (61) is enabled by the expression in Theorem 2.

The two inequalities can be interpreted as manifestations of the conjectural scaling relations between the fluctuation exponent χ and the transversal fluctuation exponent ξ for models in the Kardar–Parisi–Zhang class [11]:

$$2\xi \leq 1 + \chi$$

(for (50)) and

$$2\chi \leq \xi$$

for (61). When combined, these give the bounds

$$\chi \leq \frac{1}{3}, \qquad \xi \leq \frac{2}{3}.$$

We give a brief sketch of the argument for the reader's convenience.

(1) Assuming the existence of constants $C, \delta > 0$ such that for all $\theta \in [1, L]$, $k \geq 1$,

$$\mathbb{E}[(\overline{\log Z_{n,t}^\theta})^k] \leq C(k) n^{(1/3+\delta)k},$$

we show in Section 6.1 the estimate

$$\mathbb{E}^\theta[(s_0^+)^{2k}] \leq C'(k) n^{(4/3+\delta)k+\epsilon} \tag{48}$$

for $\theta \in [1, L-1]$ and some n-independent constants $C'(k)$. This bound corresponds to the scaling inequality $2\xi \leq 1 + \chi$.

(2) Using Theorem 2, we have an expression for the cumulants of $\log Z^{\theta}_{n,t}$ of the following form:

$$\kappa_k(\log Z^{\theta}_{n,t}) = \sum_{j=1}^{k-1} c_{k,j} \prod_{i \in I_j} \mathbb{E}[(\overline{\log Z^{\theta}_{n,t}})^{\alpha_{j,i}} H_{\beta_{j,i},t}(B_0(t))], \tag{49}$$

where $\sum_{i\in I_j} \alpha_{j,i} + \beta_{j,i} \le k$ and $\alpha_{j.i} \le j$.

(3) Time truncation argument: by Corollary 3.2, we can replace $H_{\beta_{j,i},t}(B_0(t))$ by the smaller quantity $H_{\beta_{j,i},\tau}(B_0(\tau))$ provided $s_0^+ \ll \tau$.

Using (48), we have the truncation

$$\begin{aligned}\mathbb{E}^{\theta}_{n,t}[(s_0^+)^m, s_0^+ > n^{2/3+\delta/2+\epsilon}] &\le n^{-(2k-m)(2/3+\delta/\mathbf{2}+\epsilon)} \mathbb{E}^{\theta}_{n,t}[(s_0^+)^{2k}] \\ &\le 2k \cdot C(k) n^{(2/3)m+(\delta/\mathbf{2})m-(2k-m+1)\epsilon}.\end{aligned}$$

This is of sub-leading order if we choose $k \gg (m\delta)/\epsilon$.

(4) Thanks to the previous truncations, we can now estimate (49) by effectively replacing $B_0(t)$ by $B_0(\tau)$, where $\tau \gg s_0^+$ is the best current bound for the typical size of s_0^+. Similarly, we can replace $H_{k,t}(B_0(t))$ by $H_{k,\tau}(B_0(\tau))$. The moments of the centered free energy $\mathbb{E}[(\overline{\log Z^{\theta}_{n,t}})^k]$ can now be estimated inductively using (49) and

$$B_0(\tau) \lesssim \tau^{1/2}.$$

The last relation plays the role of the scaling inequality $2\chi \le \xi$.

6.1. *Tail bound for s_0^+.* The following is one of the two pivotal inequalities in our proof. As previously stated, the case $k = 2$ appears in [21]. See also [12], Lemma 2.2.

LEMMA 6.1. *Let $k \ge 2$ be an even integer, $0 < \theta \le L$, and suppose*

$$|t - n\psi_1(\theta)| \le An^{2/3}.$$

Then there exist constants $s, c, C, K > 0$, which are uniformly bounded in θ, such that, if

$$n^{2/3} \le u \le Kn \quad \textit{and} \quad \lambda - \theta = c\frac{u}{n} = \theta - \eta,$$

then the following inequalities hold:

$$\mathbb{P}(P^{\theta}_{n,t}(s_0^+ > u) \ge e^{-su^2/n}) \le C\frac{n^k}{u^{2k}}(\mathbb{E}[(\overline{\log Z^{\theta}_{n,t}})^k] + \mathbb{E}[(\overline{\log Z^{\lambda}_{n,t}})^k]), \tag{50}$$

$$\mathbb{P}(P^{\theta}_{n,t}(s_0^- > u) \ge e^{-su^2/n}) \le C\frac{n^k}{u^{2k}}(\mathbb{E}[(\overline{\log Z^{\theta}_{n,t}})^k] + \mathbb{E}[(\overline{\log Z^{\eta}_{n,t}})^k]). \tag{51}$$

PROOF. We first prove (50). Let $r, u > 0$. By Markov's inequality,

$$P^{\theta}_{n,t}(s_0^+ > u) = P^{\theta}_{n,t}(s_0 > u) \le e^{-ru} E^{\theta}_{n,t}[e^{rs_0}] = e^{-ru}\frac{Z^{\theta+r}_{n,t}}{Z^{\theta}_{n,t}}.$$

Thus, for any $\alpha > 0$,

$$\begin{aligned}\mathbb{P}(P^{\theta}_{n,t}(s_0^+ > u) \ge e^{-\alpha}) &\le \mathbb{P}\left(\frac{Z^{\theta+r}_{n,t}}{Z^{\theta}_{n,t}} \ge e^{ru-\alpha}\right) \\ &= \mathbb{P}(\log Z^{\theta+r}_{n,t} - \log Z^{\theta}_{n,t} \ge ru - \alpha) \\ &= \mathbb{P}(\overline{\log Z^{\theta+r}_{n,t}} - \overline{\log Z^{\theta}_{n,t}} \ge n(\psi_0(\theta + r) - \psi_0(\theta)) - rt + ru - \alpha).\end{aligned}$$

The last equality follows from (13). For $c_0 = c_0(\theta)$ small enough and $0 < r < c_0$,

$$\left|\psi_0(\theta + r) - \psi_0(\theta) - r\psi_1(\theta)\right| \le -2r^2\psi_2(\theta).$$

Since $|t - n\psi_1(\theta)| \le An^{2/3}$, we have the lower bound

$$n\bigl(\psi_0(\theta + r) - \psi_0(\theta)\bigr) - rt + ru - \alpha \ge n(ru - \alpha) - rAn^{2/3} + 2r^2\psi_2(\theta). \tag{52}$$

Letting

$$r = \lambda - \theta = c\frac{u}{n} \quad \text{and} \quad \alpha = \frac{su^2}{n}$$

we can ensure the right-hand side of (52) is at least $\frac{cu^2}{2n}$ by first fixing c small enough (depending on θ and A) and then fixing s, K small enough in relation to c. Finally, apply Markov's inequality using the kth moment.

To prove (51), let $r < 0$ and $u > 0$. Then

$$P^{\theta}_{n,t}\bigl(s_0^- > u\bigr) = P^{\theta}_{n,t}(s_0 < -u) \le e^{-ru}E^{\theta}_{n,t}\bigl[e^{rs_0}\bigr].$$

The rest of the argument is the same as in the previous case. □

COROLLARY 6.2. *Let $L > 1$ be positive. Suppose*

$$\left|t - n\psi_1(\theta_0)\right| \le An^{2/3}.$$

Let $k \ge 2$ be an even integer and suppose that

$$\mathbb{E}\bigl[\bigl(\overline{\log Z^{\theta}_{n,t}}\bigr)^k\bigr] \le C(k)n^{(1/3)k+\delta k} \tag{53}$$

for some $\delta > 0$ and all $\theta \in [\theta_0, \theta_0 + L]$.

Then, for any $\epsilon > 0$, there exists a constant $C(\epsilon, k, L, \theta_0)$ such that

$$\mathbb{E}^{\theta}\bigl[\bigl(s_0^{\pm}\bigr)^{2k}\bigr] \le C(\epsilon, k, L, \theta_0)n^{(4/3)k+\delta k+\epsilon},$$

for all $\theta \in [\theta_0, \theta_0 + L - 1]$.

PROOF. Write

$$\begin{aligned}\mathbb{E}^{\theta}\bigl[\bigl(s_0^{\pm}\bigr)^{2k}\bigr] &\le \bigl(n^{2/3}\bigr)^{2k} + (2k)(Kn)^{\epsilon}\int_{n^{2/3}}^{Kn} u^{2k-1-\epsilon}\mathbb{P}^{\theta}\bigl(s_0^{\pm} \ge u\bigr)\,\mathrm{d}u + C(\theta, k)\\ &= Cn^{(4/3)k+\delta k+\epsilon}\int_{n^{2/3}}^{Kn} u^{-1-\epsilon}\,\mathrm{d}u + O\bigl(n^{(4/3)k}\bigr).\end{aligned}$$

In the second step, we have used Lemma 6.1 and the assumption (53). The constant K used here is the same one that is guaranteed to exist by Lemma 6.1. To control the region $\{u \ge Kn\}$, we have applied Lemma [21], Lemma 4.4. Performing the integration, we obtain the result. □

REMARK. The reduction in the upper bound on θ from $\theta_0 + L$ to $\theta_0 + L - 1$ in the conclusion of Corollary 6.2 is due to our application of Lemma 6.1, which requires that the assumption (53) hold for θ and λ, where $\theta < \lambda \ll \theta + 1$.

6.2. *Truncation.*

LEMMA 6.3. *Suppose that $t = O(n)$ and there exist constants $C_k = C(k,\theta)$ for $k \in \mathbb{N}$, which are locally bounded in θ, such that for some $\epsilon, \delta > 0$, and all $k \in \mathbb{N}$,*

$$\mathbb{E}^{\theta}\big[(s_0^+)^{2k}\big] \le C(k,\theta) n^{(4/3)k+\delta k+\epsilon} \quad \textit{for all } n \in \mathbb{N}. \tag{54}$$

For any $j, \ell, K \ge 1$, there exist constants $C(j,\ell,\theta,\epsilon,\delta,K)$ (locally bounded in θ) such that

$$\begin{aligned}&\big|\mathbb{E}\big[(\overline{\log Z_{n,t}^{\theta}})^j H_{\ell,\tau}(B_0(\tau))\big] - \mathbb{E}\big[(\overline{\log Z_{n,t}^{\theta}})^j H_{\ell,t}(B_0(t))\big]\big| \\ &\quad \le C(j,\ell,\theta,\epsilon,\delta,K) n^{-K} \quad \textit{for all } n \in \mathbb{N},\end{aligned}$$

where

$$\tau = n^{2/3+\delta/2+\epsilon}.$$

REMARK. We only require (54) hold for s_0^+. We could equivalently replace s_0^+ with s_0^- in the assumption.

PROOF. By Corollary 3.2, for $0 \le \tau \le t$,

$$\mathbb{E}\big[(\overline{\log Z_{n,t}^{\theta}})^j H_{k,\tau}(B_0(\tau))\big] = (-1)^k \sum_{\substack{\ell_1+\cdots+\ell_j=k \\ \ell_i \ge 0}} \frac{k!}{\ell_1!\cdots\ell_j!} \mathbb{E}\Bigg[\prod_{i=1}^{j} \kappa_{\ell_i}^{\theta}(s_0^+ \wedge \tau)\Bigg],$$

where we interpret $\kappa_0^{\theta}(s_0^+ \wedge \tau) = \overline{\log Z_{n,t}^{\theta}}$. It will therefore suffice to compare expectations of products of quenched cumulants of s_0^+ and $s_0^+ \wedge \tau$. Let $I = \{1, \dots, j\}$. We want to estimate

$$\mathbb{E}\Bigg[\prod_{i\in I} \kappa_{\ell_i}(s_0^+)\Bigg] - \mathbb{E}\Bigg[\prod_{i\in I} \kappa_{\ell_i}(s_0^+ \wedge \tau)|\Bigg], \tag{55}$$

where $\sum_i \ell_i = k$. By a telescoping argument, it is enough to estimate

$$\mathbb{E}\Bigg[\kappa_{\ell_a}(s_0^+) \prod_{i\in I_1} \kappa_{\ell_i}(s_0^+) \prod_{i\in I_2} \kappa_{\ell_i}(s_0^+ \wedge \tau)\Bigg] - \mathbb{E}\Bigg[\kappa_{\ell_a}(s_0^+ \wedge \tau) \prod_{i\in I_1} \kappa_{\ell_i}(s_0^+) \prod_{i\in I_2} \kappa_{\ell_i}(s_0^+ \wedge \tau)\Bigg],$$

where $I = I_1 \cup I_2 \cup \{a\}$ and $\ell_a \neq 0$ (if $\ell_a = 0$, then the difference is zero). By Hölder's estimate, this difference is bounded by

$$\begin{aligned}&\mathbb{E}\big[|\kappa_{\ell_a}(s_0^+) - \kappa_{\ell_a}(s_0^+ \wedge \tau)|^2\big]^{1/2} \prod_{i\in I_1} \mathbb{E}\big[|\kappa_{\ell_i}(s_0^+)|^{2j-2}\big]^{1/(2j-2)} \\ &\quad \times \prod_{i\in I_2} \mathbb{E}\big[|\kappa_{\ell_i}(s_0^+ \wedge \tau)|^{2j-2}\big]^{1/(2j-2)}.\end{aligned} \tag{56}$$

To bound the two products in (56), we use equation (41) to obtain the estimate

$$|\kappa_{\ell_i}(s_0^+ \wedge \tau)| \vee |\kappa_{\ell_i}(s_0^+)| \le (\ell_i - 1)! \sum_{\pi} \prod_{B\in\pi} E\big[(s_0^+)^{|B|}\big], \quad \ell_i \neq 0,$$

where π runs over all partitions of $\{1, \dots, \ell_i\}$ and $E = E_{n,t}^{\theta}$. Taking the L^b-norm, $b \ge 1$ we have

$$\mathbb{E}\big[|\kappa_{\ell_i}(s_0^+ \wedge \tau)|^b\big]^{1/b} \vee \mathbb{E}\big[|\kappa_{\ell_i}(s_0^+)|^b\big]^{1/b} \le \begin{cases} C\sqrt{n} & \ell_i = 0, \\ C^{\ell_i}(\ell_i - 1)!\mathbb{E}^{\theta}\big[(s_0^+)^{\ell_i b}\big]^{1/b} & \ell_i \neq 0. \end{cases} \tag{57}$$

Recall from (23) that $\kappa_0(s_0^+ \wedge \tau) = \kappa_0(s_0^+) = \overline{\log Z_{n,t}^\theta}$, so the case $\ell_i = 0$ follows from (17) and the fact that $t = O(n)$. Now define

$$m_0 := \big|\{i \in I_1 \cup I_2 : \ell_i = 0\}\big|.$$

Proceeding with the estimate (57), we have for $j > 1$

$$\begin{aligned}
&\prod_{i\in I_1} \mathbb{E}\big[\big|\kappa_{\ell_i}(s_0^+)\big|^{2j-2}\big]^{1/(2j-2)} \prod_{i\in I_2} \mathbb{E}\big[\big|\kappa_{\ell_i}(s_0^+ \wedge \tau)\big|^{2j-2}\big]^{1/(2j-2)} \\
&\quad \le (C\sqrt{n})^{m_0} \cdot \prod_{i\in I_1, I_2 : \ell_i \ne 0} C^{\ell_i}(\ell_i - 1)! \mathbb{E}^\theta\big[(s_0^+)^{(2j-2)\ell_i}\big]^{1/(2j-2)} \\
&\quad \le C^{m_0+k-\ell_a}(k-\ell_a)! n^{m_0/2} \mathbb{E}^\theta\big[(s_0^+)^{(2j-2)(k-\ell_a)}\big]^{1/(2j-2)}. \\
&\quad \le C^{j+k} k! C\big((j-1)(k-\ell_a), \theta\big)^{\frac{1}{2j-2}} n^{(j+k-\ell_a)(\frac{2}{3}+\frac{\delta}{2}+\frac{\epsilon}{2})}.
\end{aligned} \tag{58}$$

The last inequality follows from $n^{m_0/2} \le n^{j(2/3+\delta/2+\epsilon/2)}$ and the assumption (54).

We now estimate the first factor in (56). Expressing $\kappa_{\ell_a}(s_0^+)$, $\kappa_{\ell_a}(s_0^+ \wedge \tau)$ in terms of moments, we see that it suffices to bound the L^1-norm of the difference

$$\prod_{i=1}^r E\big[(s_0^+ \wedge \tau)^{\alpha_i}\big] - \prod_{i=1}^r E\big[(s_0^+)^{\alpha_i}\big],$$

where $r \le \ell_a$ and $\sum_{i=1}^r \alpha_i = \ell_a$. Observe that

$$\begin{aligned}
&\prod_{i=1}^r E\big[(s_0^+ \wedge \tau)^{\alpha_i}\big] - \prod_{i=1}^r E\big[(s_0^+)^{\alpha_i}\big] \\
&\quad = \sum_{j=1}^r \big(E\big[(s_0^+)^{\alpha_j}\big] - E\big[(s_0^+ \wedge \tau)^{\alpha_j}\big]\big) \prod_{1\le i\le j-1} E\big[(s_0^+ \wedge \tau)^{\alpha_i}\big] \prod_{j+1\le i\le r} E\big[(s_0^+)^{\alpha_i}\big].
\end{aligned}$$

It therefore suffices to bound the expectation of

$$\begin{aligned}
&E\big[(s_0^+)^{\alpha_v}, s_0 > \tau\big] \prod_{i=1}^{v-1} E\big[(s_0^+)^{\alpha_i}\big] \prod_{i=v+1}^r E\big[(s_0^+ \wedge \tau)^{\alpha_i}\big] \\
&\quad \le \tau^{-M+\alpha_v} E\big[(s_0^+)^M\big] \prod_{i=1}^{v-1} E\big[(s_0^+)^{\alpha_i}\big] \prod_{i=v+1}^r E\big[(s_0^+ \wedge \tau)^{\alpha_i}\big]
\end{aligned} \tag{59}$$

where $M \ge \ell_a$. Applying Hölder's inequality to (59) followed by (57) and assumption (54),

$$\begin{aligned}
&\mathbb{E}\big[\big|\kappa_{\ell_a}(s_0^+) - \kappa_{\ell_a}(s_0^+ \wedge \tau)\big|^2\big]^{1/2} \\
&\quad \le C^{\ell_a}(\ell_a - 1)! \cdot \tau^{-M} \max_{r\le \ell_a = \sum \alpha_i} \sum_{v=1}^r \tau^{\alpha_v} \mathbb{E}^\theta\big[(s_0^+)^{2Mr}\big]^{1/(2r)} \prod_{i\ne v} \mathbb{E}^\theta\big[(s_0^+)^{2r\cdot\alpha_i}\big]^{1/(2r)} \\
&\quad \le C^k k! \max_{r\le \ell_a = \sum \alpha_i} \sum_{v=1}^r \tau^{\alpha_v - M} C(Mr, \theta)^{\frac{1}{2r}} \Big(\prod_{i\ne v} C(r\alpha_i, \theta)\Big)^{\frac{1}{2r}} n^{(M+\ell_a-\alpha_v)(2/3+\frac{\delta}{2}+\frac{\epsilon}{2})} \\
&\quad \le C'(k, \ell_a, \theta, M) n^{\ell_a(2/3+\frac{\delta}{2}+\frac{\epsilon}{2})} n^{-M\frac{\epsilon}{2}}.
\end{aligned} \tag{60}$$

The maximum in the second line is over choices of $1 \le r \le \ell_a$ and collections α_i, $1 \le i \le r$ with $\sum \alpha_i = \ell_a$.

Combining (58) and (60) we bound (56) by

$$C''(j,k,\theta,M)n^{(j+k)((2/3)+\delta+\epsilon)}n^{-M\frac{\epsilon}{2}}.$$

Choosing M sufficiently large, depending on ϵ, j, k, δ, and K, we find that the difference (55) is indeed negligible. □

6.3. *Improved estimate for central moments.*

LEMMA 6.4. *Suppose*

$$\left|t - n\psi_1(\theta)\right| \le An^{\frac{2}{3}}.$$

Assuming the moment bounds (54), *there are constants* $C(k,\theta)$, *locally bounded in* θ, *such that, for* $k \ge 2$ *even*

$$\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^k\big] \le C(k,\theta)n^{(1/3)k+(\delta/3)k} \tag{61}$$

for all n sufficiently large.

PROOF. The proof is by induction on k. For $k=2$, (61) holds with $\delta = 0$. Assuming the estimate for even exponents less than k, we use the first expression in Theorem 2 to express the cumulant $\kappa_k(\log Z^{\theta}_{n,t})$ as a sum of a term of order $O(n)$ plus terms of the form

$$\prod_{B\in\pi}\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{a_{j,B}}H_{b_{j,B},t}\big(B_0(t)\big)\big], \tag{62}$$

where π is a partition of $\{1,\dots,k\}$ into $|\pi|$ blocks B, and $a_{j,B}+b_{j,B}=|B|$.

Using (17) and Lemma 6.3 with $K > 2k$, we have, for $\tau = n^{2/3+\delta/2+\epsilon}$,

$$\begin{aligned}&\prod_{B\in\pi}\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{a_{j,B}}H_{b_{j,B},t}\big(B_0(t)\big)\big]\\&\quad=\prod_{B\in\pi}\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{a_{j,B}}H_{b_{j,B},\tau}\big(B_0(\tau)\big)\big]+O\big(n^{-k}\big).\end{aligned}$$

Taking absolute values and applying Hölder's inequality,

$$\begin{aligned}&\big|\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{a_{j,B}}H_{b_{j,B},\tau}\big(B_0(\tau)\big)\big]\big|\\&\quad\le\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{k}\big]^{\frac{a_{j,B}}{k}}\mathbb{E}\big[\big|H_{b_{j,B},\tau}\big(B_0(\tau)\big)\big|^{k'}\big]^{\frac{1}{k'}}\\&\quad\le Cn^{((1/3)+\delta/4+\epsilon/2)b_{j,B}}\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{k}\big]^{\frac{a_{j,B}}{k}},\end{aligned}$$

where $\frac{a_{j,B}}{k}+\frac{1}{k'}=1$. Taking the product over $B\in\pi$, we have, up to a constant factor, the bound:

$$n^{(1/3+\delta/4+\epsilon/2)b_j}\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{k}\big]^{\frac{a_j}{k}}, \tag{63}$$

where

$$a_j := \sum_B a_{j,B} \quad\text{and}\quad b_j := \sum_B b_{j,B},$$

so $\frac{a_j}{k}+\frac{b_j}{k}=1$. Note that for $1\le j\le k-1$, both $a_j, b_j \ge 1$. Applying Young's inequality $xy \le \frac{1}{p}x^p+\frac{1}{q}y^q$ to (63), we find that for $\eta > 0$, any term of the form (62) is bounded by

$$\eta\mathbb{E}\big[\big(\overline{\log Z^{\theta}_{n,t}}\big)^{k}\big]+C(\eta)n^{(1/3+\delta/4+\epsilon/2)k}+O\big(n^{-k}\big).$$

Combining this with Theorem 2, we have

$$\kappa_k(\log Z^\theta_{n,t}) = C(k)\eta\mathbb{E}[(\overline{\log Z^\theta_{n,t}})^k] + C(k)C(\eta)n^{((1/3)+\delta/4+\epsilon/2)k} + O(n). \tag{64}$$

Writing

$$\kappa_k(\log Z^\theta_{n,t}) = \mathbb{E}[(\overline{\log Z^\theta_{n,t}})^k] + \sum_{\substack{|\alpha|=k \\ 0\le\alpha_i<k}} c_\alpha \prod_{i=1}^{|\alpha|} \mathbb{E}[(\overline{\log Z^\theta_{n,t}})^{\alpha_i}], \tag{65}$$

where the sum is over multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$, $\sum_i \alpha_i = k$. If some $\alpha_i = k-1$, then the product must equal zero. Therefore, by the induction assumption, all terms in the sum on the right of (65) are of order $n^{((1/3)+\delta/3)k}$. Choosing η sufficiently small in (64) and absorbing $\epsilon/2$ into $\delta/4$, we obtain the result. $\square$

6.4. *Finishing the argument.* Combining Corollary 6.2 and Lemma 6.4 we obtain the following:

LEMMA 6.5. *Suppose*

$$|t - n\psi_1(\theta_0)| \le An^{2/3}.$$

Assume there exist constants $\delta > 0$, $L > 1$, *and* $C(k) > 0$ *for* $k \in \{2, 4, \dots\}$, *such that for any even* k,

$$\mathbb{E}[(\overline{\log Z^\theta_{n,t}})^k] \le C(k)n^{(1/3)k+\delta k} \quad \textit{for all } n \ge 1 \textit{ and } \theta \in [\theta_0, \theta_0 + L].$$

Then there exist constants $C'(k) > 0$ *for* $k \in \{2, 4, \dots\}$ *such that for any even* k,

$$\mathbb{E}[(\overline{\log Z^\theta_{n,t}})^k] \le C'(k)n^{(1/3)k+(\delta/3)k} \quad \textit{for all } n \ge 1 \textit{ and } \theta \in [\theta_0, \theta_0 + L - 1].$$

Theorem 1 will follow from repeated application of Lemma 6.5 once we prove the following:

PROPOSITION 6.6. *For all* $\theta_0 > 0$ *and* $L > 0$, *there exist constants* $C_k = C_k(\theta_0, L) > 0$ *for* $k \in \{2, 4, \dots\}$ *such that for any even* k,

$$\mathbb{E}[(\overline{\log Z^\theta_{n,t}})^k] \le C_k n^{(1/3)k+(1/6)k} \quad \textit{for all } n \ge 1 \textit{ and } \theta \in [\theta_0, \theta_0 + L].$$

PROOF. For convenience, again let $A = \overline{\log Z^\theta_{n,t}}$, $B_0 = B_0(t)$, and $R = \sum_{j=1}^n r^\theta_j(t)$. By Proposition 1.1, $A = \overline{R} - B$. Thus, for even k,

$$\mathbb{E}[A^k] \le 2^{k-1}(\mathbb{E}[\overline{R}^k] + \mathbb{E}[B_0^k]). \tag{66}$$

Since R is a sum of i.i.d. random variables whose common distribution continuously depends on θ, there exist constants $C_k(\theta) > 0$, all of which are continuous in θ, such that

$$\mathbb{E}[(\overline{R})^k] \le C_k(\theta)n^{(k/2)} \quad \text{for all } n \ge 1.$$

The other expectation in (66) satisfies

$$\mathbb{E}[B_0^k] = (k/2-1)!!t^{(k/2)} \le (k/2-1)!!(An^{(2/3)} + n\psi_1(\theta_0))^{(k/2)} \le D_k(\theta_0)n^{(k/2)},$$

for all $n \ge 1$, where $D_k(\theta_0) > 0$ are constants which are continuous in θ_0. Plugging these two inequalities into equation (66) and using the continuity of $C_k(\theta)$ and $D_k(\theta_0)$ on $(0, \infty)$ yields the desired result. $\square$

PROOF OF THEOREM 1. Let $\epsilon > 0$, $\theta_0 \in (0, \infty)$, and $p \in (0, \infty)$. Fix even integers k, M such that $p \le k$ and

$$\frac{(1/6)}{3^M} \le \epsilon.$$

By Jensen's inequality, it suffices to show the bounds (6) and (7) hold with p replaced by k. Now fix $L > M$ and apply Proposition 6.6 followed by M consecutive applications of Lemma 6.5 to obtain the bound (6). Finally, apply Corollary 6.2 to both s_0^+ and s_0^- to obtain the bound (7). □

APPENDIX: COMBINATORIAL FORMULA FOR CUMULANTS

Here we derive the formula (41) for the joint cumulants. This is classical and appears, for example, on Wikipedia under *Cumulants* [5], but we could not locate a suitable proof to cite.

We will prove the following by induction. Denote

$$Z := \mathbb{E}\big[e^{\sum_{i=1}^n \xi_i X_i}\big],$$

$$E[\cdot] := \frac{1}{Z}\mathbb{E}\big[e^{\sum_{i=1}^n \xi_i X_i}\cdot\big].$$

Note that for $k \le n$

$$\kappa_k(X_1, \dots, X_k) = \partial_{\xi_1} \cdots \partial_{\xi_k} \log Z|_{\xi_1 = \cdots = \xi_n = 0}.$$

We will show by induction that

$$\partial_{\xi_1} \cdots \partial_{\xi_k} \log Z = \sum_{\pi \in \mathcal{P}(1,\dots,k)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} E\Big[\prod_{i \in B} X_i\Big]. \tag{67}$$

PROOF. Note that the result holds for $k = 1$. Indeed, in this case

$$\partial_{\xi_1} \log Z = E[X_1].$$

Assume the result for $k \le n - 1$. We prove the result for $k + 1$. Differentiating (67), we obtain for each $\pi \in \mathcal{P}(1, \dots, k)$ appearing in the sum (67)

$$\partial_{\xi_{k+1}} \prod_{B \in \pi} E\Big[\prod_{i \in B} X_i\Big] = \sum_{B' \in \pi} \partial_{\xi_{k+1}} E\Big[\prod_{j \in B'} X_j\Big] \prod_{B \ne B'} E\Big[\prod_{i \in B} X_i\Big].$$

For the derivative, we have

$$\begin{aligned}\partial_{\xi_{k+1}} E\Big[\prod_{j \in B'} X_j\Big] &= \partial_{\xi_{k+1}} \frac{1}{Z}\mathbb{E}\Big[e^{\sum_{i=1}^n \xi_i X_j} \prod_{j \in B'} X_j\Big] \\ &= E\Big[X_{k+1} \prod_{j \in B'} X_j\Big] - E[X_{k+1}]E\Big[\prod_{j \in B'} X_j\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned}\partial_{\xi_{k+1}} \prod_{B \in \pi} E\Big[\prod_{i \in B} X_i\Big] &= \sum_{B' \in \pi} E\Big[X_{k+1} \prod_{j \in B'} X_j\Big] \prod_{B \ne B'} E\Big[\prod_{i \in B} X_i\Big] \\ &\quad - \sum_{B' \in \pi} E[X_{k+1}]E\Big[\prod_{j \in B'} X_j\Big] \prod_{B \ne B'} E\Big[\prod_{i \in B} X_i\Big] \\ &= \sum_{B' \in \pi} E\Big[X_{k+1} \prod_{j \in B'} X_j\Big] \prod_{B \ne B'} E\Big[\prod_{i \in B} X_i\Big] \\ &\quad - |\pi| E[X_{k+1}] \prod_{B \in \pi} E\Big[\prod_{i \in B} X_i\Big].\end{aligned} \tag{68}$$

The first term corresponds to adding a factor X_{k+1} to a single B block of the partition π and the second term corresponds to adding a 1-term block $\{k+1\}$ to π. Summing (68) over $\pi \in \mathcal{P}(1,\dots,k)$, we obtain

$$\begin{aligned}
&\partial_{\xi_{k+1}} \sum_{\pi \in \mathcal{P}(1,\dots,k)} (|\pi|-1)!(-1)^{|\pi|-1} \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] \\
&\quad= \sum_{\pi \in \mathcal{P}(1,\dots,k)} (|\pi|-1)!(-1)^{|\pi|-1} \sum_{B'\in\pi} E\Big[X_{k+1}\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big] \\
&\qquad- \sum_{\pi \in \mathcal{P}(1,\dots,k)} |\pi|!(-1)^{|\pi|} E[X_{k+1}] \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] \\
&\quad= \sum_{\tilde\pi \in \mathcal{P}(1,\dots,k+1)} (|\tilde\pi|-1)!(-1)^{|\tilde\pi|-1} \prod_{B\in\tilde\pi} E\Big[\prod_{i\in B} X_i\Big].
\end{aligned}$$

To verify the final step, note that any partition of $\{1,\dots,k+1\}$ which contains $\{k+1\}$ as a single element block induces a partition π of $\{1,\dots,k\}$ from the remaining blocks with $|\pi| = |\tilde\pi|-1$; otherwise, if $\{k+1\}$ does not appear as block in $\tilde\pi$, the partition can be obtained from some $\pi \in \mathcal{P}(1,\dots,k)$ by adding $k+1$ to one of the $|\pi|$ blocks without changing the number of blocks. □

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REFERENCES

[1] BALÁZS, M., CATOR, E. and SEPPÄLÄINEN, T. (2006). Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.* **11** 1094–1132. MR2268539 https://doi.org/10.1214/EJP.v11-366

[2] BARRAQUAND, G. and CORWIN, I. (2017). Random-walk in beta-distributed random environment. *Probab. Theory Related Fields* **167** 1057–1116. MR3627433 https://doi.org/10.1007/s00440-016-0699-z

[3] BORODIN, A., CORWIN, I. and FERRARI, P. (2014). Free energy fluctuations for directed polymers in random media in 1 + 1 dimension. *Comm. Pure Appl. Math.* **67** 1129–1214. MR3207195 https://doi.org/10.1002/cpa.21520

[4] CHAUMONT, H. and NOACK, C. (2018). Fluctuation exponents for stationary exactly solvable lattice polymer models via a Mellin transform framework. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 509–547. MR3800484 https://doi.org/10.30757/alea.v15-21

[5] Wikipedia contributors, *Cumulant*, Wikipedia, The Free Encyclopedia, 5 December 2020, 06:57 UTC, https://en.wikipedia.org/w/index.php?title=Cumulant&oldid=992433669.

[6] CORWIN, I. (2012). The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.* **1** 1130001, 76. MR2930377 https://doi.org/10.1142/S2010326311300014

[7] CORWIN, I., SEPPÄLÄINEN, T. and SHEN, H. (2015). The strict-weak lattice polymer. *J. Stat. Phys.* **160** 1027–1053. MR3373650 https://doi.org/10.1007/s10955-015-1267-0

[8] IMAMURA, T. and SASAMOTO, T. (2016). Determinantal structures in the O'Connell-Yor directed random polymer model. *J. Stat. Phys.* **163** 675–713. MR3488569 https://doi.org/10.1007/s10955-016-1492-1

[9] JANJIGIAN, C. (2015). Large deviations of the free energy in the O'Connell-Yor polymer. *J. Stat. Phys.* **160** 1054–1080. MR3373651 https://doi.org/10.1007/s10955-015-1269-y

[10] JARA, M. and MORENO FLORES, G. R. (2020). Stationary directed polymers and energy solutions of the Burgers equation. *Stochastic Process*. *Appl*. **130** 5973–5998. MR4140024 https://doi.org/10.1016/j.spa.2020.04.012

[11] KRUG, J. and SPOHN, H. Kinetic roughening of growing surfaces. In *Solids Far from Equilibrium* 1992 (C. Godrèche, ed.) Cambridge Univ. Press, Cambridge.

[12] MORENO FLORES, G. R., SEPPÄLÄINEN, T. and VALKÓ, B. (2014). Fluctuation exponents for directed polymers in the intermediate disorder regime. *Electron*. *J*. *Probab*. **19** no. 89, 28. MR3263646 https://doi.org/10.1214/EJP.v19-3307

[13] MORIARTY, J. and O'CONNELL, N. (2007). On the free energy of a directed polymer in a Brownian environment. *Markov Process*. *Related Fields* **13** 251–266. MR2343849

[14] CORWIN, I. and NICA, M. (2017). Intermediate disorder limits for multi-layer semi-discrete directed polymers. *Electron*. *J*. *Probab*. Paper No. 13, 49.

[15] NOACK, C. and SOSOE, P. (2022). Concentration for integrable directed polymer models. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **58** 34–64. MR4374672 https://doi.org/10.1214/21-aihp1154

[16] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[17] O'CONNELL, N. (2012). Directed polymers and the quantum Toda lattice. *Ann*. *Probab*. **40** 437–458. MR2952082 https://doi.org/10.1214/10-AOP632

[18] O'CONNELL, N. and ORTMANN, J. (2015). Tracy-Widom asymptotics for a random polymer model with gamma-distributed weights. *Electron*. *J*. *Probab*. **20** no. 25, 18. MR3325095 https://doi.org/10.1214/EJP.v20-3787

[19] O'CONNELL, N. and YOR, M. (2001). Brownian analogues of Burke's theorem. *Stochastic Process*. *Appl*. **96** 285–304. MR1865759 https://doi.org/10.1016/S0304-4149(01)00119-3

[20] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann*. *Probab*. **40** 19–73. MR2917766 https://doi.org/10.1214/10-AOP617

[21] SEPPÄLÄINEN, T. and VALKÓ, B. (2010). Bounds for scaling exponents for a $1+1$ dimensional directed polymer in a Brownian environment. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **7** 451–476. MR2741194

[22] THIERY, T. and LE DOUSSAL, P. (2015). On integrable directed polymer models on the square lattice. *J*. *Phys*. *A* **48** 465001, 41. MR3418005 https://doi.org/10.1088/1751-8113/48/46/465001