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Concentration for integrable directed polymer models

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Abstract. In this paper, we consider four integrable models of directed polymers for which the free energy is known to exhibit KPZ fluctuations. A common framework for the analysis of these models was introduced in (*ALEA Lat. Am. J. Probab. Math. Stat.* **15** (2018) 509–547).

We derive estimates for the central moments of the partition function, of any order, on the near-optimal scale $N^{1/3+\epsilon}$, using the iterative method we applied to the semi-discrete polymer in (Noack and Sosoe (2020)). Among the innovations exploiting the integrable structure, we develop formulas for correlations between functions of the free energy and the boundary weights that replace the Gaussian integration by parts appearing in our previous paper (Noack and Sosoe (2020)).

Résumé. Dans cet article, nous considérons quatre modèles intégrables de polymères dirigés pour lesquels on sait démontrer que l'énergie libre a des fluctuations de type KPZ. Un cadre d'analyse commun pour ces modèles est présenté dans (*ALEA Lat. Am. J. Probab. Math. Stat.* **15** (2018) 509–547).

Nous obtenons des estimées pour les moments centraux de la fonction de partition, d'ordre quelconque, à l'échelle quasi-optimale $N^{\frac{1}{3}+\epsilon}$, à l'aide d'une méthode itérative déjà appliquée au polymère semi-discret dans (Noack and Sosoe (2020)). Parmi les nouveautés qui tirent profit de la structur structure intégrable, nous développons des formules pour les corrélations entre des fonctions de l'énergie libre et les poids au bord. Ces formulent remplacent l'intégration par partie gaussienne qui apparait dans notre précédent travail (Noack and Sosoe (2020)).

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1. Introduction

In this paper, we consider four models for $1+1$ dimensional integrable polymers in random environment, and study the higher moments of the centered free energy: the log-gamma polymer, introduced by Seppäläinen [14]; the strict-weak polymer, which was simultaneously introduced and analyzed by Corwin–Seppäläinen–Shen [10] and O'Connell–Ortmann [12]; the beta random walk of Barraquand and Corwin [5]; and the inverse beta model introduced by Thiery and Le Doussal [15].

The models in question are distinguished because they each possess algebraic structure that has enabled the verification of several predictions regarding their fluctuations. These include upper and lower bounds for the variance of the free energy, of order $O(N^{2/3})$ (see [14] for log-gamma and [9] for the three other models) as well as asymptotic Tracy–Widom distribution (see [6] for the log-gamma polymer, the original papers [5,10,12] for the strict-weak polymer and beta random walk models, as well a formal argument for the inverse beta model in [15]). Results of this type are characteristic of the KPZ universality class [8], and are expected to hold in a more general setting where the integrable structure is not available, but proving this is out of reach using current methods.

We note that the techniques used to prove asymptotic Tracy–Widom distribution by relating the free energy to a Fredholm determinant are markedly different from those that have been used to obtain variance bounds starting with the work of Seppäläinen [14]. The ideas in that work have their origins in earlier work of Seppäläinen and co-authors on fluctuations of one-dimensional interacting particle systems [1–4]. Despite their power, the Bethe ansatz methods used to obtain the asymptotic distribution are not easily adapted to estimating the size of the central moments.

Here, we build on our previous paper [11] on the O'Connell–Yor polymer, a semi-discrete $1+1$ dimensional polymer model introduced in [13], to obtain bounds of nearly optimal order for all the central moments of the free energy in the stationary version of each of the four models mentioned above. Our main result, Theorem 1, states that for each $k \geq 1$, the kth central moment of the free energy in a system of size $O(N^2)$ is bounded by $O(N^{k/3+\epsilon})$, where the implicit constant depends on ϵ.

The proof in [11] proceeded by deriving a pair of inequalities which appear related to the physicists' *KPZ scaling relations* and which enable an iterative proof of the bound for the order of fluctuations of the free energy by successive improvements starting from the trivial $O(N^{1/2})$ bound. A crucial idea was the repeated application of Gaussian integration by parts to relate cross-terms involving the "boundary Brownian motion" component of the free energy and the free energy itself to quenched cumulants of the first vertical jump of the polymers paths. This tool is not available in the discrete models we consider here. Nevertheless, we develop a substitute for it by introducing a sequence of polynomials which play a role analogous to that of Hermite polynomials for the O'Connell–Yor polymer, and allow us to derive formulas for the cumulants of the partition function in terms of quenched cumulants of the time of the first jump. Here, the Mellin transform framework introduced in [7] plays a central role. See Section 4.4.

1.1. *The polymer model*

To each edge e of the $\mathbb{Z}_+^2$ lattice we assign a positive random weight. The superscripts 1 and 2 are used to denote horizontal and vertical edge weights, respectively. For $z \in \mathbb{N}^2$, let Y_z^1 and Y_z^2 denote the horizontal and vertical incoming edge weights, see Figure 1. We assume that the collection of pairs $\{(Y_z^1, Y_z^2)\}_{z\in\mathbb{N}^2}$ is independent and identically distributed with common distribution (Y^1, Y^2), but do not insist that Y_z^1 is independent of Y_z^2. $\{(Y_z^1, Y_z^2)\}_{z\in\mathbb{N}^2}$ are the *bulk weights*. For $x \in \mathbb{N}\times\{0\}$, let R_x^1 denote the horizontal incoming edge weight, and for $y \in \{0\}\times\mathbb{N}$, let R_y^2 denote the vertical incoming edge weight. We take the collections $\{R_x^1\}_{x\in\mathbb{N}\times\{0\}}$ and $\{R_y^2\}_{y\in\{0\}\times\mathbb{N}}$ to be independent and identically distributed, with common distributions R^1 and R^2. We refer to these as the *horizontal* and *vertical boundary weights*, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 1. We call

$$\omega = \left\{R_x^1, R_y^2, \left(Y_z^1, Y_z^2\right) : x \in \mathbb{N}\times\{0\}, y \in \{0\}\times\mathbb{N}, z \in \mathbb{N}^2\right\} \tag{1}$$

the *polymer environment*. We use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability measure and corresponding expectation of the polymer environment.

The weight of a path is given by the product of the weights along its edges. For $(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ we define a probability measure on all up-right paths from $(0,0)$ to (m,n). See Figure 2 for an example of an up-right path. Let $\Pi_{m,n}$ denote the collection of all such paths. We identify paths $x_\bullet = (x_0, x_1, \ldots, x_{m+n})$ either with their sequence of vertices or with their sequence of edges $(e_1, \ldots, e_{m+n})$, where $e_i = \{x_{i-1}, x_i\}$, as convenient. Define the quenched polymer measure on $\Pi_{m,n}$,

$$Q_{m,n}(x_\bullet) := \frac{1}{Z_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i},$$

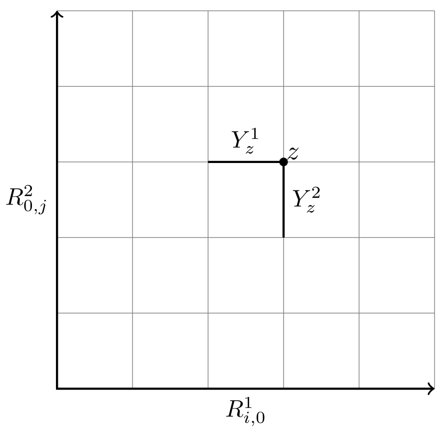

Fig. 1. Assignment of edge weights.

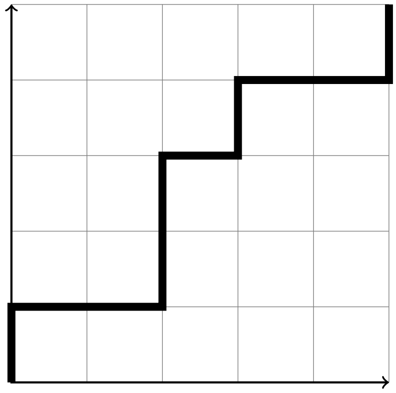

Fig. 2. An up-right path from (0, 0) to (5, 5).

where ω_e is the weight associated to the edge e and

$$Z_{m,n} := \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i}$$

is the associated partition function. At the origin, define $Z_{0,0} := 1$. Taking the expectation $\mathbb{E}$ of the quenched measure with respect to the edge weights gives the annealed measure on $\Pi_{m,n}$,

$$P_{m,n}(x_\bullet) := \mathbb{E}\big[Q_{m,n}(x_\bullet)\big]. \tag{2}$$

The annealed expectation will be denoted by $E_{m,n}$.

We specify the edge weight distributions for the four stationary polymer models. The notation $X \sim \mathrm{Ga}(\alpha, \beta)$ is used to denote that a random variable is gamma(α, β) distributed, i.e. has density

$$\frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x}$$

supported on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, \mathrm{d}x$ is the gamma function. $X \sim \mathrm{Be}(\alpha, \beta)$ is used to say that X is beta(α, β) distributed, i.e. has density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},$$

supported on $(0, 1)$. We then use $X \sim \mathrm{Ga}^{-1}(\alpha, \beta)$ and $X \sim \mathrm{Be}^{-1}(\alpha, \beta)$ to denote that $X^{-1} \sim \mathrm{Ga}(\alpha, \beta)$ and $X^{-1} \sim \mathrm{Be}(\alpha, \beta)$, respectively. We also use $X \sim (\mathrm{Be}^{-1}(\alpha, \beta) - 1)$ to denote that $X + 1 \sim \mathrm{Be}^{-1}(\alpha, \beta)$.

Each of the four models we consider is obtained by choosing the distribution of the boundary and bulk weights according to one of the four following specifications:

- *Inverse-gamma (IG)*: This is also known as the log-gamma model. Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{aligned} &R^1 \sim \mathrm{Ga}^{-1}(\mu - \theta, \beta), \qquad R^2 \sim \mathrm{Ga}^{-1}(\theta, \beta), \\ &\big(Y^1, Y^2\big) = (X, X) \quad \text{where } X \sim \mathrm{Ga}^{-1}(\mu, \beta). \end{aligned} \tag{3}$$

- *Gamma (G)*: This is also known as the strict-weak model. Assume $\theta, \mu, \beta > 0$ and

$$\begin{aligned} &R^1 \sim \mathrm{Ga}(\mu + \theta, \beta), \qquad R^2 \sim \mathrm{Be}^{-1}(\theta, \mu), \\ &\big(Y^1, Y^2\big) = (X, 1) \quad \text{where } X \sim \mathrm{Ga}(\mu, \beta). \end{aligned} \tag{4}$$

- *Beta (B)*: Assume $\theta, \mu, \beta > 0$ and

$$\begin{aligned} &R^1 \sim \mathrm{Be}(\mu + \theta, \beta), \qquad R^2 \sim \mathrm{Be}^{-1}(\theta, \mu), \\ &\big(Y^1, Y^2\big) = (X, 1 - X) \quad \text{where } X \sim \mathrm{Be}(\mu, \beta). \end{aligned} \tag{5}$$

- *Inverse-beta (IB)*: Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{aligned} &R^1 \sim \mathrm{Be}^{-1}(\mu-\theta,\beta), \qquad R^2 \sim \bigl(\mathrm{Be}^{-1}(\theta,\beta+\mu-\theta)-1\bigr), \\ &\bigl(Y^1,Y^2\bigr)=(X,X-1) \quad \text{where } X\sim \mathrm{Be}^{-1}(\mu,\beta). \end{aligned} \tag{6}$$

Note that each of these four choices in fact generates a family of models by choosing different values of the parameters μ, θ, β. The name of each model refers to the distribution of the bulk weights. We call these models the *four basic beta-gamma models*.

1.2. *Main result*

Having defined the models we will consider, we are now ready to state our main result. Given a path $x_\bullet \in \Pi_{m,n}$, define the *exit points* of the path from the horizontal and vertical axes by

$$t_1 := \max\bigl\{i : (i,0) \in x_\bullet\bigr\} \quad \text{and} \quad t_2 := \max\bigl\{j : (0,j) \in x_\bullet\bigr\}. \tag{7}$$

Theorem 1. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of (3) through (6), and let $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ be a sequence such that*

$$\bigl|m_N - N\,\mathbb{V}\mathrm{ar}\bigl[\log R^2\bigr]\bigr| \le \gamma N^{2/3} \quad \textit{and} \quad \bigl|n_N - N\,\mathbb{V}\mathrm{ar}\bigl[\log R^1\bigr]\bigr| \le \gamma N^{2/3} \tag{8}$$

for some fixed $\gamma > 0$. Then for every $\epsilon > 0$ and $p > 0$, there exists a constant $C = C(\epsilon, p) > 0$ such that for any $N \in \mathbb{N}$,

$$\mathbb{E}\bigl[|\overline{\log Z_{m,n}}|^p\bigr] \le C N^{\frac{1}{3}p+\epsilon} \quad \textit{and} \tag{9}$$

$$E_{m,n}\bigl[(t_j)^p\bigr] \le C N^{\frac{2}{3}p+\epsilon} \quad \textit{for both } j = 1, 2. \tag{10}$$

Note that by [7, Theorem 1.2], we have

$$\mathbb{E}\bigl[|\overline{\log Z_{m,n}}|^2\bigr] \ge c N^{\frac{2}{3}}$$

in the regime considered in Theorem 1. By Jensen's inequality this implies

$$\mathbb{E}[|\overline{\log Z_{m,n}}|^p| \ge c^p N^{\frac{p}{3}},$$

so that the bound we obtain is indeed near-optimal. We have not quantified the dependence of the constants on ϵ and p. In particular, the estimates in the Theorem are obtained by an iterative process, where the number of iterations depends on ϵ. With our current method, the implicit constants grow without bounds as the number of iterations increases. It would be of great interest to prove the Theorem with $\epsilon = 0$, but it is not clear to us whether this can be easily achieved by extending our current methods. We leave these questions to further work.

We also obtain exact formulas for the cumulants of the free energy, see Corollary 4.8.

1.3. *Outline of the paper*

In Section 2, after establishing some basic notation, we recall the Mellin transform framework introduced in [7], where it was noticed that the four basic beta-gamma models can be treated simultaneously.

In Section 3, we recall the "down-right" property shared by the four basic beta-gamma models. This is a consequence of the stronger Burke property, and implies in particular that the free energy can be written as the sum of two i.i.d. sums of order $O(N)$ see (13). Understanding the fluctuations of the free energy becomes equivalent to understanding the correlation between these two sums, or equivalently the correlation between one of them and the free energy. This is manifested in the expansion for the cumulants of the free energy appearing in Lemma 3.1.

In Section 4, we develop a formula of "integration by parts" type which expresses certain correlations appearing in the expansion for the cumulant in terms of derivatives of expectations of moments of the free energy with respect to the parameter in the boundary weights. See Lemma 4.1. We use this to obtain formulas relating the the cumulants of the free energy to expectations of productions of quenched cumulants of t_1, the first jump in the system. See Corollary 4.8.

In Section 5, we prove our main result. The key idea here is that the formulas obtained in Section 4 allow one to get improved estimates (compared to the trivial $O(N^{1/2})$ bound) for the moments of the free energy given estimates for the annealed moments of t_1. See Lemma 5.7. Conversely, an inequality due essentially to Seppäläinen [14] relates the moments of t_1 to moments of the centered free energy. Iterating through these two inequalities a finite number of times, we can obtain bounds that are arbitrarily close to order $N^{1/3}$.

2. Preliminaries and notation

2.1. *Notation*

We let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, \ldots\}$, while $\mathbb{R}$ denotes the real numbers.

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $\vee$ and $\wedge$ denote maximum and minimum, respectively:

$$a \vee b = \max\{a, b\},$$

$$a \wedge b = \min\{a, b\}.$$

Given a real valued function f, let $\operatorname{supp}(f) = \{x : f(x) \neq 0\}$ denote the support of the function f (note that we do not insist on taking the closure of this set).

Given a random variable X with finite expectation, we let

$$\overline{X} = X - \mathbb{E}[X].$$

The symbol $\otimes$ is used to denote (independent) product distribution.

2.2. *The Mellin transform framework*

Here we introduce a framework, developed in [7], which allows us to treat the four basic beta-gamma models simultaneously.

Given a function $f : (0, \infty) \to [0, \infty)$, write M_f for its Mellin transform

$$M_f(a) := \int_0^\infty x^{a-1} f(x)\,\mathrm{d}x$$

for any $a \in \mathbb{R}$ such that the integral converges. For more information about the Mellin transform and its relation to other classical integral transforms see [16], especially Sections 1.5, 1.29 and 7.7; note however that we use only elementary properties of the transform in this work. Define

$$D(M_f) := \operatorname{interior}\big(\{a \in \mathbb{R} : 0 < M_f(a) < \infty\}\big).$$

Definition 2.1. Given a function $f : (0, \infty) \to [0, \infty)$ such that $D(M_f)$ is non-empty, we define a family of densities on $(0, \infty)$ parametrized by $a \in D(M_f)$:

$$\rho_{f,a}(x) := M_f(a)^{-1} x^{a-1} f(x). \tag{11}$$

We write $X \sim m_f(a)$ to denote that the random variable X has density $\rho_{f,a}$.

Remark 2. If $f : (0, \infty) \to [0, \infty)$ is such that $D(M_f)$ is non-empty, then M_f is C^∞ throughout $D(M_f)$. Furthermore, if $X \sim m_f(a)$, then

(1) $\log X$ has finite exponential moments. That is, since $D(M_f)$ is open, there exists some $\epsilon > 0$ such that

$$\mathbb{E}\big[e^{\epsilon|\log X|}\big] \le \mathbb{E}\big[X^\epsilon\big] + \mathbb{E}\big[X^{-\epsilon}\big] = \frac{M_f(a+\epsilon) + M_f(a-\epsilon)}{M_f(a)} < \infty.$$

(2) For all $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial a^k} M_f(a) = M_f(a)\mathbb{E}\big[(\log X)^k\big].$$

(3) The kth cumulant of $\log X$, which we denote by $\kappa_k(\log X)$ following the notation introduced in (16), equals $\psi_k^f(a)$, where

$$\psi_k^f(a) := \frac{\partial^{k+1}}{\partial a^{k+1}} \log M_f(a) \quad \text{for } k \in \mathbb{Z}_+.$$

Table 1
Density factors and distribution for the four basic beta-gamma polymer models with parameters a and b

$f(x)$	$m_f(a)$
e^{-bx}	$\mathrm{Ga}(a,b)$
$e^{-b/x}$	$\mathrm{Ga}^{-1}(-a,b)$
$(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$	$\mathrm{Be}(a,b)$
$(1-\frac{1}{x})^{b-1}\mathbb{1}_{\{x>1\}}$	$\mathrm{Be}^{-1}(-a,b)$
$(\frac{x}{x+1})^{b}$	$\mathrm{Be}^{-1}(-a,b+a)-1$

Table 2
Functions and parameters to fit the four basic beta-gamma models into the Mellin framework

Model	$f^1(x)$	$f^2(x)$	(a_1,a_2,a_3)	
IG	$e^{-\beta/x}$	$e^{-\beta/x}$	$(\theta-\mu,-\theta,-\mu)$	$\theta\in(0,\mu)$
G	$e^{-\beta x}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta,-\theta,\mu)$	$\theta\in(0,\infty)$
B	$(1-x)^{\beta-1}\mathbb{1}_{\{0<x<1\}}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta,-\theta,\mu)$	$\theta\in(0,\infty)$
IB	$(1-\frac{1}{x})^{\beta-1}\mathbb{1}_{\{x>1\}}$	$(\frac{x}{x+1})^{(\beta+\mu)}$	$(\theta-\mu,-\theta,-\mu)$	$\theta\in(0,\mu)$

2.3. *The four basic beta-gamma models are Mellin-type*

The random variables appearing in each of the four basic beta-gamma models have densities of the form (11), for various choices of f, which we specify here. In Table 1, we assume $b>0$ and $a\in D(M_f)$.

To express the distribution of the polymer environment in each of the four models given in (3) through (6) within the above framework, we let

$$\big(R^1,R^2,X\big)\sim m_{f^1}(a_1)\otimes m_{f^2}(a_2)\otimes m_{f^1}(a_3), \tag{12}$$

where the functions f^1, f^2 and parameters a_j, $j=1,2,3$ are given in Table 2. Recall that in each of the models, (Y^1,Y^2) are given in terms of X. For Table 2 we assume $\mu,\beta>0$.

When the polymer environment is as in (12) with parameters (a_1,a_2), we use $\mathbb{P}^{(a_1,a_2)}$, $\mathbb{E}^{(a_1,a_2)}$, $\mathbb{V}\mathrm{ar}^{(a_1,a_2)}$, $\mathbb{C}\mathrm{ov}^{(a_1,a_2)}$ in place of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$, $\mathbb{C}\mathrm{ov}$ respectively.

Remark 3. For each fixed value of the bulk parameter a_3, we obtain a family of models with boundary parameters a_1 and a_2 satisfying $a_1+a_2=a_3$.

3. The down-right property

Write $\alpha_1=(1,0)$, $\alpha_2=(0,1)$. For $k=1,2$ define ratios of partition functions

$$R^k_x:=\frac{Z_x}{Z_{x-\alpha_k}}\quad\text{for all } x \text{ such that } x-\alpha_k\in\mathbb{Z}^2_+.$$

Note that these extend the definitions of $R^1_{i,0}$ and $R^2_{0,j}$, since for example $Z_{i,0}=\prod_{k=1}^{i}R^1_{k,0}$. We say that $\pi=\{\pi_k\}_{k\in\mathbb{Z}}$ is a down-right path in $\mathbb{Z}^2_+$ if $\pi_k\in\mathbb{Z}^2_+$ and $\pi_{k+1}-\pi_k\in\{\alpha_1,-\alpha_2\}$ for each $k\in\mathbb{Z}$. To each edge along a down-right path we associate the random variable

$$\Lambda_{\{\pi_{k-1},\pi_k\}}:=\begin{cases}R^1_{\pi_k} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is horizontal,}\\ R^2_{\pi_{k-1}} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is vertical.}\end{cases}$$

The following definition is a weaker form of the Burke property, see [14, Theorem 3.3].

Definition 3.1. Say the polymer model has the *down-right property* if for all down-right paths $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$, the random variables

$$\Lambda(\pi) := \{\Lambda_{\{\pi_{k-1},\pi_k\}} : k \in \mathbb{Z}\}$$

are independent and each $R^1_{\pi_k}$ and $R^2_{\pi_k}$ appearing in the collection are respectively distributed as R^1 and R^2.

Proposition 3.1. *Each of the four basic beta-gamma models*, (3) *through* (6), *possesses the down-right property.*

See Proposition 2.3 of [7].

3.1. *Consequences of the down-right property*

The free energy has two useful expressions.

$$\log Z_{m,n} = W + N = S + E \tag{13}$$

where

$$W_n := \sum_{j=1}^{n} \log R^2_{0,j}, \qquad E_n := \sum_{j=1}^{n} \log R^2_{m,j}, \qquad N_m := \sum_{i=1}^{m} \log R^1_{i,n}, \qquad S_m := \sum_{i=1}^{m} \log R^1_{i,0}. \tag{14}$$

Notice that $W_n = \log Z_{0,n}$ and $S_m = \log Z_{m,0}$. If the model possesses the down-right property (see Definition 3.1), then W_n, E_n, N_m, S_m are each sums of i.i.d. random variables. The subscripts n and m on W_n, E_n, N_m, S_m indicate the length of the sums.

Recall that if the random variables $X_1, \dots, X_k$ have finite exponential moments, then their joint cumulant is defined by

$$\kappa(X_1, \dots, X_k) := \frac{\partial^k}{\partial \xi_1 \dots \partial \xi_k} \log \mathbb{E}\big[e^{\sum_{j=1}^k \xi_j X_j}\big]\Big|_{\xi_i=0}. \tag{15}$$

Alternatively, the joint cumulant can be written as a combination of products of expectations of the underlying random variables:

$$\kappa(X_1, \dots, X_k) = \sum_{\pi\in\mathcal{P}} \big(|\pi| - 1\big)!(-1)^{|\pi|-1} \prod_{B\in\pi} \mathbb{E}\Big[\prod_{i\in B} X_i\Big] \tag{16}$$

where $\mathcal{P}$ ranges over partitions π of $\{1, \dots, k\}$ and $|A|$ stands for the size of the set A. In the case where $X_1 = X_2 = \dots = X_k = X$, the joint cumulant reduces to the k-th cumulant of X which we denote by $\kappa_k(X)$. The identity (16) is classical and indeed is often used as the definition of joint cumulants. For the reader's convenience, we check its equivalence with the definition (15) in Appendix A.

Lemma 3.1. *Assume the polymer environment is such that* $|\log R^1|$, $|\log R^2|$, $|\log Y^1|$, *and* $|\log Y^2|$ *all have finite exponential moments. Let* N_m, E_n, S_m *and* W_n *be as in* (14), *Then, for any positive integer* k,

$$\kappa_k(\log Z_{m,n}) = \kappa_k(E_n) - (-1)^k \kappa_k(S_m) - \sum_{j=1}^{k-1} \binom{k}{j} (-1)^{k-j} \kappa\big(\underbrace{\log Z_{m,n}, \dots, \log Z_{m,n}}_{j\ times}, \underbrace{S_m, \dots, S_m}_{k-j\ times}\big) \quad \textit{and} \tag{17}$$

$$\kappa_k(\log Z_{m,n}) = \kappa_k(N_m) - (-1)^k \kappa_k(W_n) - \sum_{j=1}^{k-1} \binom{k}{j} (-1)^{k-j} \kappa\big(\underbrace{\log Z_{m,n}, \dots, \log Z_{m,n}}_{j\ times}, \underbrace{W_n, \dots, W_n}_{k-j\ times}\big). \tag{18}$$

Moreover, if the polymer model also possesses the down-right property, then

$$\kappa_k(E_n) = \kappa_k(W_n) = n\kappa_k\big(R^1\big) \quad \textit{and}$$
$$\kappa_k(N_m) = \kappa_k(S_m) = m\kappa_k\big(R^2\big).$$

Proof. By Lemma C.1, $\log Z_{m,n}$, N_m, S_m, E_n, W_n all have finite exponential moments, so their cumulants and joint cumulants exist. By (13), $E_n = \log Z_{m,n} - S_n$. Since the joint cumulant is multi-linear,

$$\kappa_k(E_n) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \kappa(\underbrace{\log Z_{m,n}, \dots, \log Z_{m,n}}_{j \text{ times}}, \underbrace{S_m, \dots, S_m}_{k-j \text{ times}}).$$

The $j = k$ term in the summand is $\kappa_k(\log Z_{m,n})$ and the $j = 0$ term is $(-1)^k \kappa_k(S_m)$. Rearranging yields equation (17). To obtain equation (18), apply the same argument with (N_m, W_n) in place of (E_n, S_m). For the last part of the Lemma use the fact that $\kappa_k(X+Y) = \kappa_k(X) + \kappa_k(Y)$ if X and Y are independent. □

Remark 4. Each of the four basic beta-gamma models satisfy the moment conditions of Lemma 3.1.

4. Formulas for the central moments

In the next two sections, we give an exact formula for the terms appearing in the summands on the right-hand side of (17). The same arguments can be used to obtain analogous estimates for (18), but as will be apparent in Section 5 the estimates in (17) will be sufficient to obtain Theorem 1.

4.1. *Integration by parts type formula*

Referring to the notation in Section 2.2, fix an integer $r \geq 1$ and let $f_k : (0, \infty) \to [0, \infty)$ for $k = 1, \dots, r$ and a, a_0, a_1 be real numbers such that $a_0 < a < a_1$ and $[a_0, a_1] \subset \bigcap_{k=1}^r D(M_{f_k})$. Consider a collection of independent random variables $\{X_k\}_{k=1}^r$ where $X_k \sim m_{f_k}(a)$ for all $1 \leq k \leq r$, and let $\mathbb{E}^a$ correspond to the expectation over these random variables. Finally, define

$$T := \sum_{k=1}^{r} \log X_k.$$

We introduce a sequence $\{p_n(t, a; r)\}_{n \geq 0}$ of n-th degree polynomials in t defined recursively by

$$\begin{aligned} p_0(t, a; r) &= 1, \\ p_n(t, a; r) &= \frac{\partial}{\partial a} p_{n-1}(t, a; r) + p_{n-1}(t, a; r)\big(t - \mathbb{E}^a[T]\big) \quad \text{for } n \geq 1. \end{aligned} \tag{19}$$

Note that the dependence of $p_n(\cdot, \cdot; r)$ on r is polynomial, a fact we use explicitly later in our argument (see Proposition 5.1).

The following lemma extends Lemma B.2 from [7].

Lemma 4.1. *$A : \mathbb{R}^r \to \mathbb{R}$ be a measurable function such that $\mathbb{E}^a[A(X_1, \dots, X_r)^2] < \infty$ for all $a \in [a_0, a_1]$. Then*

$$\frac{\partial^n}{\partial a^n} \mathbb{E}^a\big[A(X_1, \dots, X_r)\big] = \mathbb{E}^a\big[A(X_1, \dots, X_r) p_n(T, a; r)\big]. \tag{20}$$

Proof. The joint density of $(\log X_1, \log X_2, \dots, \log X_r)$ is given by

$$g(x_1, \dots, x_r) = \frac{e^{a \sum_{k=1}^r x_k}}{\prod_{k=1}^r M_{f_k}(a)} \prod_{k=1}^{r} f_k\big(e^{x_k}\big).$$

Thus the density of $T = \sum_{k=1}^r \log X_k$ is

$$h_a(t) = \frac{e^{at}}{\prod_{k=1}^r M_{f_k}(a)} \int_{\mathbb{R}^{r-1}} f_1\big(e^{x_1}\big) f_2\big(e^{x_2 - x_1}\big) \dots f_r\big(e^{t - x_{r-1}}\big) \, \mathrm{d}x_1, \dots, x_{r-1}. \tag{21}$$

Therefore, the joint density of $(\log X_1, \log X_2, \dots, \log X_r)$ given that $T = t$ is given by

$$\frac{g(x_1, \dots, x_r) \mathbb{1}_{\{\sum_{k=1}^r x_k = t\}}}{h_a(t)} = \frac{\prod_{k=1}^r f_k(e^{x_k})}{\int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2 - x_1}) \dots f_r(e^{t - x_{r-1}}) \, \mathrm{d}x_1, \dots, x_{r-1}}, \tag{22}$$

which has no a-dependence. Recursion (19) and $\frac{\partial}{\partial a} h_a(t) = h_a(t)(t - \mathbb{E}^a[T])$ inductively imply

$$\frac{\partial^n}{\partial a^n} h_a(t) = h_a(t) p_n(t, a; r) \quad \text{for all } n \in \mathbb{Z}_+. \tag{23}$$

By (22) and (23),

$$\begin{aligned}
\frac{\partial^n}{\partial a^n} \mathbb{E}^a\big[A(X_1, \dots, X_k)\big] &= \frac{\partial^n}{\partial a^n} \int_{\mathbb{R}} \mathbb{E}^a\big[A(X_1, \dots, X_k) \mid T = t\big] h_a(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \mathbb{E}^a\big[A(X_1, \dots, X_k) \mid T = t\big] \frac{\partial^n}{\partial a^n} h_a(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \mathbb{E}^a\big[A(X_1, \dots, X_k) \mid T = t\big] h_a(t) p_n(t, a; r)\,\mathrm{d}t \\
&= \mathbb{E}^a\big(A(X_1, \dots, X_k) p_n(T, a; r)\big).
\end{aligned}$$

The interchanging of the n-th derivative and the integral will be justified by the bound:

$$\int_{\mathbb{R}} \mathbb{E}\big[\big|A\big(\{X_k\}_{k=1}^r\big)\big| \mid T = t\big] \sup_{a \in [a_0, a_1]} \left|\frac{\partial^n}{\partial a^n} h_a(t)\right| \mathrm{d}t < \infty. \tag{24}$$

To obtain this bound first notice that recursion (19) implies that $p_n(t, a)$ are degree n polynomials in t with coefficients that are smooth in a. Thus, by (23), there exist constants $0 < C_n < \infty$ independent of t such that

$$\sup_{a \in [a_0, a_1]} \left|\frac{\partial^n}{\partial a^n} h_a(t)\right| \le C_n\big(1 + |t|\big)^n \sup_{a \in [a_0, a_1]} h_a(t). \tag{25}$$

Once we show that there is a constant C depending only on a_0 and a_1 such that

$$\sup_{a \in [a_0, a_1]} h_a(t) \le h_{a_0}(t) + C h_{a_1}(t) \quad \text{for all } s \in \mathbb{R}, \tag{26}$$

(25) will give the bound (24) since

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}\big[\big|A\big(\{X_k\}_{k=1}^r\big)\big| \mid T = t\big]\big(1 + |t|\big)^n h_{a_j}(t)\,\mathrm{d}t &= \mathbb{E}^{a_j}\big[\big|A\big(\{X_k\}_{k=1}^r\big)\big|\big(1 + |T|\big)^n\big] \\
&\le \mathbb{E}^{a_j}\big[\big(A\big(\{X_k\}_{k=1}^r\big)\big)^2\big]^{\frac{1}{2}} \mathbb{E}^{a_j}\big[\big(1 + |T|\big)^{2n}\big]^{\frac{1}{2}}.
\end{aligned}$$

This is finite since $\mathbb{E}^{a_j}[A(\{X_k\}_{k=1}^r)^2] < \infty$ by assumption, and T is a sum of independent random variables with finite exponential moments. All that is left to do is verify the bound (26). To accomplish this, notice that equation (21) implies

$$\frac{\partial}{\partial a} \log h_a(t) = t - \mathbb{E}^a[T]. \tag{27}$$

Since $\mathbb{E}^a[T] = \sum_{k=1}^r \psi_0^{f_k}(a)$, $a \mapsto \mathbb{E}^a[T]$ is an increasing function (recall that $\frac{d}{da}\psi_0^{f_k}(a) = \psi_1^{f_k}(a) = \mathbb{V}\mathrm{ar}[X_k] > 0$). Therefore, for all $t \le \mathbb{E}^{a_0}[T]$, the function $a \mapsto h_a(t)$ is non-increasing on $[a_0, a_1]$ which gives

$$\sup_{a \in [a_0, a_1]} h_a(t) \le h_{a_0}(t) \quad \text{for all } t \le \mathbb{E}^{a_0}[T].$$

On the other hand, if $t > \mathbb{E}^{a_0}[T]$, then

$$\frac{\partial}{\partial a} \log\big(h_a(t) \exp\big(a\big(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T]\big)\big)\big) = t - \mathbb{E}^a[T] + \mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T] > 0 \tag{28}$$

for all $a \in [a_0, a_1]$. Thus, for all $t > \mathbb{E}^{a_0}[T]$,

$$a \mapsto h_a(t) \exp\big(a\big(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T]\big)\big)$$

is increasing on the interval $[a_0, a_1]$. Therefore

$$\sup_{a\in[a_0,a_1]} h_a(t) \le C_3 h_{a_1}(t) \quad \text{for all } t > \mathbb{E}^{a_0}[T]$$

where $C = \exp\left((a_1 - a_0)(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T])\right)$. Combining (27) and (28) gives the desired result. □

Lemma 4.2. *Assume the polymer environment satisfies $R^1_{i,0} \sim m_f(a)$ for all $i \ge 1$. Let $k \ge 2$ and $1 \le j \le k$. For $r \ge 1$, let*

$$S_r := \sum_{i=1}^{r} R^1_{i,0}.$$

Then,

$$\begin{aligned}
&\kappa(\underbrace{\log Z_{m,n}, \ldots, \log Z_{m,n}}_{j \text{ times}}, \underbrace{S_r, \ldots, S_r}_{k-j \text{ times}}) = \kappa(\underbrace{\overline{\log Z_{m,n}}, \ldots, \overline{\log Z_{m,n}}}_{j \text{ times}}, \underbrace{S_r, \ldots, S_r}_{k-j \text{ times}}) \\
&\quad = \sum_{\pi\in\mathcal{P}} \left(|\pi| - 1\right)!(-1)^{|\pi|-1} \prod_{B\in\pi} \mathbb{E}\left[(\overline{\log Z_{m,n}})^{|B\cap\{1,\ldots,j\}|} p_{|B\cap\{j+1,\ldots,k\}|}(S_r, a; r)\right] + r\psi^f_{k-j}(a)
\end{aligned} \tag{29}$$

where $\mathcal{P}$ ranges over partitions π of $\{1, \ldots, k\}$ such that no block $B \in \pi$ is contained in $\{j + 1, \ldots, k\}$, and $p_k(s, a)$ are polynomials recursively defined by (19) in the case $f_j = f$ for all j.

Proof. Introduce the function

$$g(s, a; r) := e^{as - r \log M_f(a)}.$$

Note that

$$\frac{\partial}{\partial a} g(s, a; r) = \left(s - r\psi^f_0(a)\right) g(s, a; r),$$

so

$$\begin{aligned}
\frac{\partial}{\partial a}\left(g(s, a; r) p_{k-1}(s, a; r)\right) &= g(s, a; r) \frac{\partial}{\partial a} p_{k-1}(s, a; r) + g(s, a; r) p_{k-1}(s, a; r)\left(s - r\psi^f_0(a)\right) \\
&= g(s, a; r) p_k(s, a; r).
\end{aligned}$$

Rearranging, this gives

$$\begin{aligned}
p_k(s, a; r) &= \frac{1}{g(s, a; r)} \frac{\partial}{\partial a}\left(g(s, a; r) p_{k-1}(s, a; r)\right) \\
&= \cdots \\
&= \frac{1}{g(s, a; r)} \frac{\partial^l}{\partial a^l}\left(g(s, a; r) p_{k-l}(s, a; r)\right).
\end{aligned}$$

Letting $l = k$, we have

$$p_k(s, a; r) = \frac{\frac{\partial^k}{\partial a^k} g(s, a; r)}{g(s, a; r)}.$$

By Taylor expansion around a for λ sufficiently small, we have

$$g(s, a + \lambda; r) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\partial^k}{\partial a^k} g(s, a; r),$$

which yields the generating function for the polynomials $p_k(s,a;r)$:

$$\frac{g(s,a+\lambda;r)}{g(s,a;r)} = e^{\lambda s}\left(\frac{M_f(a)}{M_f(a+\lambda)}\right)^r = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}p_k(s,a;r). \tag{30}$$

Using this, we have a formula for joint cumulants, as follows. Recall

$$\kappa(\underbrace{A,\dots,A}_{j \text{ times}},\underbrace{S_r,\dots,S_r}_{k-j \text{ times}}) = \frac{\partial}{\partial\xi_1\cdots\partial\xi_k}\log\mathbb{E}\big[e^{(\xi_1+\cdots+\xi_j)A}e^{(\xi_{j+1}+\cdots+\xi_k)S_r}\big]\Big|_{\xi_i=0}.$$

Inserting the generating function (30), we have

$$e^{(\xi_{j+1}+\cdots+\xi_k)S_r} = \left(\frac{M_f(a+\xi_{j+1}+\cdots\xi_k)}{M_f(a)}\right)^r\sum_{l=0}^{\infty}\frac{(\xi_{j+1}+\cdots+\xi_k)^l p_l(S_r,a;r)}{l!}.$$

Taking expectations, then logarithms, we have

$$\begin{aligned}\log\mathbb{E}\big[e^{(\xi_{j+1}+\cdots+\xi_k)S_r}\big] &= \log\mathbb{E}\left[e^{(\xi_1+\cdots+\xi_j)A}\sum_{l=0}^{\infty}\frac{(\xi_{j+1}+\cdots+\xi_k)^l}{l!}p_l(S_r,a;r)\right]\\ &\quad - r\log M_f(a) + r\log M_f(a+\xi_{j+1}+\cdots+\xi_k).\end{aligned}$$

Setting $A=\overline{\log Z_{m,n}}$ and taking derivates with respect to the ξ_i's, then evaluating them at zero gives (29).

The only part of the statement that still requires comment is the assertion that partitions $\mathcal{P}$ with a block contained in $\{j+1,\dots,k\}$ make a zero contribution. This is because

$$\mathbb{E}\big[p_n(S_r,a;r)\big]=0$$

for $n\ge 1$, as follows from (20) with $A\equiv 1$. □

4.2. *Coupling of polymer environments*

In order to compare polymer environments with different parameters, we use a coupling to express the boundary weights as functions of i.i.d. uniform(0, 1) random variables.

Recall the notation from Section 2.2. Suppose $f:(0,\infty)\to[0,\infty)$ is a smooth function on its support, $\mathrm{supp}(f)$ is open, and $D(M_f)$ is non-empty. Define $F^f: D(M_f)\times(0,\infty)\to[0,1]$ by

$$F^f(a,x) := \frac{1}{M_f(a)}\int_0^x y^{a-1}f(y)\,\mathrm{d}y.$$

For fixed $a\in D(M_f)$, $x\mapsto F^f(a,x)$ is the cdf of a random variable with $X\sim m_f(a)$. Note that $F^f(a,\cdot)$ is a bijection between $\mathrm{supp}(f)$ and $(0,1)$. For $a\in D(M_f)$, let $H^f(a,\cdot)$ be the inverse of $F^f(a,\cdot)$ defined on $(0,1)$. By the implicit function theorem, $H^f: D(M_f)\times(0,1)\to\mathrm{supp}(f)$ is a smooth function in both of its variables satisfying

$$\frac{\partial}{\partial a}\log H^f(a,x) = L^f\big(a,H^f(a,x)\big), \tag{31}$$

where

$$L^f(a,x) := \frac{1}{x^a f(x)}\int_0^x y^{a-1}\big(\psi_0^f(a)-\log y\big)f(y)\,\mathrm{d}y.$$

Another expression for L^f shows that it is a strictly positive function:

$$L^f(a,x) = \frac{-1}{x^a f(x)}\mathbb{C}\mathrm{ov}(\log X, \mathbb{1}_{\{X\le x\}}) > 0, \tag{32}$$

because the right side of (32) is the negative of the correlation between an increasing and a decreasing function of X. Since f is smooth, L^f is smooth as a function on $D(M_f) \times \operatorname{supp}(f)$. Note that if η is a uniform(0, 1) distributed random variable, then $H^f(a,\eta) \sim m_f(a)$ *for every* $a \in D(M_f)$. This gives us a useful coupling as follows.

Fix $m, n \in \mathbb{N}$ and an environment ω on the square with lower-left corner $(0,0)$ and upper-right corner (m,n). Assume the random variables attached to the southern boundary $R^1_{i,0}$ all have $m_f(a)$ distributions. Fix $1 \le r \le m$ and let $\{\eta_i\}_{i=1}^r$ be i.i.d. uniform(0, 1) distributed random variables which are also independent of the original environment ω. Now create a new environment $\tilde{\omega}$ by replacing $R^1_{i,0}$ in the original environment along the southern boundary by

$$\tilde{R}^1_{i,0}(b) := H^f(b,\eta_i)$$

only for $i = 1, \dots, r$. When $b = a$ the new environment is equal in distribution to the old one:

$$\tilde{\omega} \stackrel{d}{=} \omega, \qquad b = a.$$

Write

$$Z_{m,n}(b) := Z^{\tilde{\omega}}_{m,n} = \sum_{x_\cdot \in \Pi_{m,n}} \prod_{i=1}^{m+n} \tilde{\omega}_{(x_{i-1},x_i)} = \sum_{x_\cdot \in \Pi_{m,n}} \prod_{i=1}^{t_1(x_\cdot)\wedge r} \tilde{R}^1_{i,0} \prod_{i=t_1(x_\cdot)\wedge r+1}^{m+n} \omega_{(x_{i-1},x_i)},$$

where $t_1(x_\cdot) := \max\{i \ge 0 : x_i = (i,0)\}$, i.e. the exit time from the southern boundary. By equation (31),

$$\frac{\partial}{\partial b}\bigg|_{b=a} \tilde{R}^1_{i,0} = \tilde{R}^1_{i,0} L^f\bigl(a, \tilde{R}^1_{i,0}\bigr). \tag{33}$$

Therefore

$$\frac{\partial}{\partial b}\bigg|_{b=a} \log Z_{m,n}(b) = \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x_\cdot)\wedge r} L^f\bigl(a, \tilde{R}^1_{i,0}\bigr)\right], \tag{34}$$

where $\tilde{E}^a_{m,n}[F(x_\cdot)]$ is defined to be the expectation of the function F of up-right paths $x_\cdot$ under the quenched probability measure

$$\tilde{Q}^b_{m,n}(x_\cdot) := \frac{1}{Z_{m,n}(b)} \prod_{i=1}^{m+n} \tilde{\omega}_{(x_{i-1},x_i)} = \frac{1}{Z_{m,n}(b)} \prod_{i=1}^{t_1(x_\cdot)\wedge r} \tilde{R}^1_{i,0}(b) \prod_{i=t_1(x_\cdot)\wedge r+1}^{m+n} \omega_{(x_{i-1},x_i)}. \tag{35}$$

We will use $\tilde{\mathbb{E}}$ and $\tilde{\mathbb{P}}$ to denote annealed expectation and probability for the measure in (35).

By (33) and (34),

$$\frac{\partial}{\partial b}\bigg|_{b=a} \log \tilde{Q}^b_{m,n}(x_\cdot) = \sum_{i=1}^{t_1(x_\cdot)\wedge r} L^f\bigl(a, \tilde{R}^1_{i,0}\bigr) - \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x_\cdot)\wedge r} L^f\bigl(a, \tilde{R}^1_{i,0}\bigr)\right]. \tag{36}$$

4.3. *Derivatives with respect to boundary parameters*

We now use equation (36) to provide a recursion which will help determine $\frac{\partial^k}{\partial b^k}|_{b=a} \log Z_{m,n}(b)$.

Definition 4.1. Suppose $G(a,x) : D(M_f) \times \operatorname{supp}(f) \to \mathbb{R}$ is a C^1 function. The action of the operator $\widetilde{\partial}$ on G is given by

$$\tilde{\partial} G(a,x) := \frac{\partial G}{\partial a}(a,x) + x L^f(a,x) \frac{\partial G}{\partial x}(a,x). \tag{37}$$

Lemma 4.3. *Let $G(a,x) : D(M_f) \times \mathrm{supp}(f) \to \mathbb{R}$ be a C^k function. Then the function $\widetilde{\partial} G : D(M_f) \times \mathrm{supp}(f) \to \mathbb{R}$ is C^{k-1} and satisfies the two equations*:

$$\tilde{\partial} G\big(a, \tilde{R}^1_{i,0}(a)\big) = \frac{\partial}{\partial b}\bigg|_{b=a} \big(G\big(b, \tilde{R}^1_{i,0}(b)\big)\big),$$

$$\frac{\partial}{\partial b}\bigg|_{b=a} \tilde{E}^b_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} G\big(b, \tilde{R}^1_{i,0}\big)\right] = \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} \tilde{\partial} G\big(a, \tilde{R}^1_{i,0}\big)\right] + \widetilde{\mathrm{Cov}}^a_{m,n}\left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}(a)\big), \sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}(a)\big)\right), \tag{38}$$

where $\widetilde{\mathrm{Cov}}^a_{m,n}$ stands for the covariance under $\tilde{E}^a_{m,n}$.

Proof. By equation (33), we have

$$\frac{\partial}{\partial b}\bigg|_{b=a} G\big(b, \tilde{R}^1_{i,0}(b)\big) = \frac{\partial G}{\partial a}\big(a, \tilde{R}^1_{i,0}(a)\big) + \frac{\partial}{\partial b}\bigg|_{b=a} \tilde{R}^1_{i,0} \frac{\partial G}{\partial x}\big(a, \tilde{R}^1_{i,0}\big),$$

$$\tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right] = \sum_{x.\in\Pi_{m,n}} \left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right) \tilde{Q}^a_{m,n}(x.).$$

Therefore,

$$\frac{\partial}{\partial b}\bigg|_{b=a} \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right] = \sum_{x.\in\Pi_{m,n}} \frac{\partial}{\partial b}\bigg|_{b=a} \left(\left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right) \tilde{Q}^a_{m,n}(x.)\right)$$

$$= \sum_{x.\in\Pi_{m,n}} \left(\sum_{i=1}^{t_1(x.)\wedge r} \frac{\partial}{\partial a} G\big(a, \tilde{R}^1_{i,0}\big)\right) \tilde{Q}^a_{m,n}(x.) \tag{39}$$

$$+ \sum_{x.\in\Pi_{m,n}} \left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right) \frac{\partial}{\partial b}\bigg|_{b=a} \tilde{Q}^a_{m,n}(x.). \tag{40}$$

By the first part of the lemma, the right-hand side of (39) equals $\tilde{E}^a_{m,n}[\sum_{i=1}^{t_1(x.)\wedge r} (\tilde{\partial} G)(a, \tilde{R}^1_{i,0})]$. Using equation (36), (40) equals

$$\sum_{x.\in\Pi_{m,n}} \left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right)\left(\sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big) - \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big)\right]\right) \tilde{Q}^a_{m,n}(x.)$$

$$= \tilde{E}^a_{m,n}\left[\left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right)\left(\sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big)\right)\right]$$

$$- \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big)\right] \tilde{E}^a_{m,n}\left[\sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big)\right]$$

$$= \widetilde{\mathrm{Cov}}^a_{m,n}\left(\sum_{i=1}^{t_1(x.)\wedge r} G\big(a, \tilde{R}^1_{i,0}\big), \sum_{i=1}^{t_1(x.)\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big)\right).$$

□

Letting $\tilde{\kappa}^a_k(X_1, \ldots, X_k)$ denote the quenched joint cumulant of k random variables with respect to $\tilde{E}^a_{m,n}$, repeated application of (38) and the chain rule give:

Lemma 4.4. *For* $g_1, \ldots, g_k \in C^\infty(D(M_f) \times \operatorname{supp}(f))$,

$$\frac{\partial}{\partial a}\tilde{\kappa}_k^a\left(\sum_{i=1}^{t_1\wedge r} g_1\big(a, \tilde{R}_{i,0}^1\big), \sum_{i=1}^{t_1\wedge r} g_2\big(a, \tilde{R}_{i,0}^1\big), \ldots, \sum_{i=1}^{t_1\wedge r} g_k\big(a, \tilde{R}_{i,0}^1\big)\right)$$
$$= \sum_{\substack{\delta_1+\cdots+\delta_k=1\\ \delta_i\in\{0,1\}}} \tilde{\kappa}_k^a\left(\sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\delta_1} g_1\big(a, \tilde{R}_{i,0}^1\big), \sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\delta_2} g_2\big(a, \tilde{R}_{i,0}^1\big), \ldots, \sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\delta_k} g_k\big(a, \tilde{R}_{i,0}^1\big)\right)$$
$$+ \tilde{\kappa}_{k+1}^a\left(\sum_{i=1}^{t_1\wedge r} g_1\big(a, \tilde{R}_{i,0}^1\big), \sum_{i=1}^{t_1\wedge r} g_2\big(a, \tilde{R}_{i,0}^1\big), \ldots, \sum_{i=1}^{t_1\wedge r} g_k\big(a, \tilde{R}_{i,0}^1\big), \sum_{i=1}^{t_1\wedge r} L^f\big(a, \tilde{R}_{i,0}^1\big)\right).$$

Proof. Write $X_l(a) := \sum_{i=1}^{t_1\wedge r} g_l(a, \tilde{R}_{i,0}^1(a))$ for $l = 1, \ldots, k$. We will perturb the parameter in the environment and the parameters in the arguments of the cumulant separately. To this end, define

$$F(a,h) := \tilde{\kappa}_k^h\big(X_1(a), X_2(a), \ldots, X_k(a)\big).$$

Using equation (33) and multi-linearity of the joint cumulant gives

$$\frac{\partial}{\partial a}F(a,h) = \sum_{\substack{\delta_1+\cdots+\delta_k=1\\ \delta_i\geq 0}} \tilde{\kappa}_k^h\big(\tilde{\partial}^{\delta_1}X_1(a), \tilde{\partial}^{\delta_2}X_2(a), \ldots, \tilde{\partial}^{\delta_k}X_k(a)\big). \tag{41}$$

Now define $Y(h) := \sum_{i=1}^{t_1\wedge r} L^f(h, H^f(h, \eta_i))$. Then

$$\frac{\partial}{\partial h}\tilde{E}_{m,n}^h\big[e^{\sum_{l=1}^k \xi_l X_l(a)}\big] = \tilde{E}_{m,n}^h\big[e^{\sum_{l=1}^k \xi_l X_l(a)}\overline{Y(h)}\big]$$

where the centering is respect to $\tilde{E}_{m,n}^h$. Thus

$$\begin{aligned}
\frac{\partial}{\partial h}F(a,h) &= \frac{\partial}{\partial\xi_1\ldots\partial\xi_k}\frac{\partial}{\partial h}\log\tilde{E}_{m,n}^h\big[e^{\sum_{l=1}^k \xi_l X_l(a)}\big]\Big|_{\xi_1=\cdots=\xi_k=0}\\
&= \frac{\partial}{\partial\xi_1\ldots\partial\xi_{k+1}}\log\tilde{E}_{m,n}^h\big[e^{\sum_{l=1}^k \xi_l X_l(a)+\xi_{k+1}\overline{Y(h)}}\big]\Big|_{\xi_1=\cdots=\xi_k=\xi_{k+1}=0}\\
&= \tilde{\kappa}_{k+1}^h\left(\sum_{i=1}^{t_1\wedge r} g_1\big(a, \tilde{R}_{i,0}^1\big), \sum_{i=1}^{t_1\wedge r} g_2\big(a, \tilde{R}_{i,0}^1\big), \ldots, \sum_{i=1}^{t_1\wedge r} g_k\big(a, \tilde{R}_{i,0}^1\big), Y(h)\right).
\end{aligned} \tag{42}$$

Combining (41) and (42) gives

$$\frac{\partial}{\partial a}F(a,a) = \left(\frac{\partial F}{\partial a}(a,h) + \frac{\partial F}{\partial b}(a,h)\right)\Big|_{h=a}$$

yielding the desired result. □

Equation (34) and a repeated application of the previous lemma now give

Corollary 4.5. *Let* $k \geq 1$. *There are non-negative constants* $C_{k,j,\vec{\ell}}$ *indexed by* $1 \leq j \leq k$ *and* j*-tuples* $\vec{\ell} = (\ell_1, \ldots, \ell_j)$ *with* $\ell_i \geq 0$ *and* $\sum_{1\leq i\leq j}\ell_i = k - j$ *such that*

$$\frac{\partial^k}{\partial b^k}\Big|_{b=a}\log Z_{m,n}(b) = \sum_{j=1}^k\sum_{\vec{\ell}} C_{k,j,\vec{\ell}}\,\tilde{\kappa}_j^a\left(\sum_{i=1}^{t_1\wedge r}\tilde{\partial}^{\ell_1}L^f\big(a, \tilde{R}_{i,0}^1\big), \ldots, \sum_{i=1}^{t_1\wedge r}\tilde{\partial}^{\ell_j}L^f\big(a, \tilde{R}_{i,0}^1\big)\right).$$

Here $\tilde{\partial}^{\ell}$ *is the operator defined in* (37), *iterated* ℓ *times. The constants satisfy*

$$\sum_{j=1}^{k} \sum_{\vec{\ell}} C_{k,j,\vec{\ell}} \le k!.$$

Note that $C_{k,k,(0,\dots,0)} = 1$, *so that the the* $j = k$*-th term in the sum is the* k*-th quenched cumulant of* $\sum_{i=1}^{t_1\wedge r} L^f(a, \tilde{R}^1_{i,0})$.

Proof. For $k = 1$, we have

$$\frac{\partial}{\partial b}\bigg|_{b=a} \log Z_{m,n}(b) = \tilde{\kappa}_1^a\left(\sum_{i=1}^{t_1\wedge r} L^f\big(a, \tilde{R}^1_{i,0}\big)\right). \tag{43}$$

This has the required form

$$\tilde{\kappa}_j^a\left(\sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\ell_1} L^f\big(a, \tilde{R}^1_{i,0}\big), \dots, \sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\ell_j} L^f\big(a, \tilde{R}^1_{i,0}\big)\right) \tag{44}$$

with $j = 1$ and $\vec{\ell} = (0)$. Using Lemma 4.4, differentiating any term of the form (44) produces $j+1$ terms of the same form where one of the ℓ_i or j are increased by one. Thus, differentiating (43) $k-1$ times, we obtain a sum of at most $k!$ terms, all of the form (44). □

4.4. *Application of integration by parts*

We now use the results obtained thus far to provide exact expressions for the cumulants of the free energy $\log Z_{m,n}$ (see Corollary 4.8).

We now impose the following assumption on f to have control of the moments of $\tilde{\partial}^k L^f$.

Hypothesis 1. *Assume* f *has non-empty, open support* $\mathrm{supp}(f)$, *non-empty* $D(M_f)$, f *is smooth on its support, and for every* $k \in \mathbb{Z}_+$, *there exist* $C_k \in C(D(M_f))$, *continuous functions on* $D(M_f)$, *such that*

$$\big|\tilde{\partial}^k L^f(a,x)\big| \le C_k(a)\big(1 + |\log x|^{k+1}\big) \quad \text{for all } (a,x) \in D(M_f) \times \mathrm{supp}(f).$$

The following theorem says that all functions appearing in the four models (3), (4), (5), (6) satisfy this hypothesis.

Theorem 5. *Let* f *be one of the functions appearing in Table* 1, *meaning* f *corresponds to a gamma, inverse-gamma, beta, inverse-beta, or beta-prime distribution. Then* f *satisfies Hypothesis* 1.

The proof of Theorem 5 is relegated to the Appendix.

Lemma 4.6. *Assume* f *satisfies Hypothesis* 1. *Let* $[a_0, a_1] \subset D(M_f)$, $\eta \sim \mathrm{uniform}(0,1)$, *and* $k \in \mathbb{N}$. *Then the random variable*

$$\sup_{a\in[a_0,a_1]} \big|\widetilde{\partial}^k L^f\big(a, H^f(a,\eta)\big)\big|$$

has finite moments of all orders.

Proof. Put $Y := \sup_{a\in[a_0,a_1]} |\widetilde{\partial}^k L^f(a, H^f(a,\eta))|$. By Hypothesis 1, there exists a constant $C > 0$ such that

$$\big|\widetilde{\partial}^k L^f\big(a, H^f(a,\eta)\big)\big| \le C\big(1 + \big|\log H^f(a,\eta)\big)\big|^{k+1}\big) \tag{45}$$

$$\le C\big(1 + \big|\log H^f(a_0,\eta)\big)\big|^{k+1} + \big|\log H^f(a_1,\eta)\big)\big|^{k+1}\big). \tag{46}$$

The last inequality follows from the monotonicity $a \mapsto H^f(a,x)$ (by equations (32) and (31)) and holds for all $a \in [a_0, a_1]$. Since $H^f(a_j,\eta) \sim m_f(a_j)$ for $j = 0, 1$, and $a_0, a_1 \in D(M_f)$, both $\log H^f(a_0,\eta)$ and $\log H^f(a_1,\eta)$ have finite exponential moments. Thus Y has finite moments of all orders. □

Now write $\tilde{\mathbb{P}}$, $\tilde{\mathbb{E}}$ for the probability measure and expectation corresponding to the environment $\tilde{\omega}$, as defined in Section 4.2, and $\mathbb{P}$, $\mathbb{E}$ for probability measure and expectation corresponding to the environment ω. Write

$$\widetilde{\sigma}_0(r) := \log Z_{m,n}(a) - \widetilde{\mathbb{E}}\big[\log Z_{m,n}(a)\big] \quad \text{and}$$
$$\widetilde{\sigma}_k(r) := \frac{\partial^k}{\partial b^k}\bigg|_{b=a} \log Z_{m,n}(b) \quad \text{for } k \in \mathbb{N}.$$

Similarly, define

$$\sigma_0(r) := \log Z_{m,n} - \mathbb{E}[\log Z_{m,n}],$$
$$\sigma_k(r) := \sum_{j=1}^{k} \sum_{\substack{\ell_1+\cdots+\ell_j=k-j \\ \ell_i \geq 0}} C_{k,j,\vec{\ell}}, \kappa_j^Q\left(\sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\ell_1} L^f\big(a, R^1_{i,0}\big), \dots, \sum_{i=1}^{t_1\wedge r} \tilde{\partial}^{\ell_j} L^f\big(a, R^1_{i,0}\big)\right), \quad k \in \mathbb{N}, \tag{47}$$

where $\kappa_k^Q(X_1, \dots, X_k)$ denotes the joint-cumulant of the random variable $X_1, \dots, X_k$ with respect to the quenched measure $Q_{m,n}$ and $C_{k,j,\vec{\ell}}$ are the constants appearing in Corollary 4.5. By Corollary 4.5, $\sigma_k(r) \stackrel{d}{=} \tilde{\sigma}_k^a(r)$. Recall that our environment $\tilde{\omega}$ has only changed the southern boundary random variables between the origin and the point $(r, 0)$, so $\log Z_{m,n}(b)$ only depends on a through $R^1_{i,0}$, $r < i \leq m$.

Lemma 4.7. *If f satisfies Hypothesis 1 and $S_r = \sum_{i=1}^r \log R^1_{i,0}$, then for any $j, k \geq 0$,*

$$\mathbb{E}\big[(\overline{\log Z_{m,n}})^j p_k(S_r, a; r)\big] = \sum_{\substack{\ell_1+\cdots+\ell_j=k \\ \ell_i \geq 0}} \frac{k!}{\ell_1! \cdots \ell_j!} \mathbb{E}\left[\prod_{i=1}^{j} \sigma_{\ell_i}(r)\right]. \tag{48}$$

Proof. Write $g(a) := \tilde{\mathbb{E}}^a[\log Z_{m,n}]$. Then the left-hand side of equation (48) is equal to

$$\mathbb{E}\big[\big(\log Z_{m,n} - g(b)\big)^j p_k(S_r, a; r)\big]\big|_{b=a}.$$

Fix $b \in D(M_f)$ and let $\mathcal{F}$ be the sigma-algebra generated by the random variables $R^1_{1,0}, \dots, R^1_{r,0}$. Then there exists a measurable function function $A : \mathbb{R}^r \to \mathbb{R}$ such that $A(R^1_{1,0}, \dots, R^1_{r,0}) = \mathbb{E}[(\log Z_{m,n} - g(b))^j \mid \mathcal{F}]$ almost surely. By Lemma A.1 from [7], $A \in L^2(\mathbb{P})$. Since $S_r \in \mathcal{F}$, Lemma 4.1 gives

$$\begin{aligned}
\mathbb{E}\big[\big(\log Z_{m,n} - g(b)\big)^j p_k(S_r, a; r)\big] &= \mathbb{E}^a\big[A\big(R^1_{1,0}, \dots, R^1_{r,0}\big) p_k(S_r, a; r)\big] &(49)\\
&= \frac{\partial^k}{\partial a^k} \mathbb{E}^a\big[A\big(R^1_{1,0}, \dots, R^1_{r,0}\big)\big] \\
&= \frac{\partial^k}{\partial a^k} \mathbb{E}\big[\big(\log Z_{m,n} - g(b)\big)^j\big], &(50)
\end{aligned}$$

where $\mathbb{E}^a$ emphasizes that we are only taking expectations over $\{R^1_{i,0}\}_{i=1}^r$. Now fix a_0 and a_1 such that $a \in [a_0, a_1] \subset D(M_f)$. Using Corollary 4.5, Lemma 4.6, and $t_1 \leq m$, we see that

$$\widetilde{\mathbb{E}}\left[\sup_{a\in[a_0,a_1]} \left| \frac{\partial^k}{\partial a^k} \big(\log Z_{m,n}(a) - g(b)\big)^j \right| \right] < \infty.$$

Thus

$$\begin{aligned}
(50) &= \frac{\partial^k}{\partial a^k} \tilde{\mathbb{E}}\big[\big(\log Z_{m,n}(a) - g(b)\big)^j\big] = \tilde{\mathbb{E}}\left[\frac{\partial^k}{\partial a^k} \big(\log Z_{m,n}(a) - g(b)\big)^j\right] \\
&= \sum_{\substack{\ell_1+\cdots+\ell_j=k \\ \ell_i \geq 0}} \frac{k!}{\ell_1! \cdots \ell_j!} \tilde{\mathbb{E}}\left[\prod_{i=1}^{j} \frac{\partial^{\ell_i}}{\partial a^{\ell_i}} \big(\log Z_{m,n}(a) - g(b)\big)\right].
\end{aligned}$$

Therefore

$$(50)|_{b=a} = \sum_{\substack{\ell_1+\cdots+\ell_j=k\\ \ell_i\ge 0}} \frac{k!}{\ell_1!\cdots\ell_j!}\widetilde{\mathbb{E}}\left[\prod_{i=1}^{j}\widetilde{\sigma}_{\ell_i}(r)\right]$$

$$= \sum_{\substack{\ell_1+\cdots+\ell_j=k\\ \ell_i\ge 0}} \frac{k!}{\ell_1!\cdots\ell_j!}\mathbb{E}\left[\prod_{i=1}^{j}\sigma_{\ell_i}(r)\right].$$

□

Corollary 4.8. *When $r=m$, and k is even,*

$$\kappa_k(\log Z_{m,n}) = \kappa_k(E_n) - \kappa_k(S_m) - \sum_{j=1}^{k-1}\binom{k}{j}(-1)^j\kappa(\underbrace{\log Z_{m,n},\dots,\log Z_{m,n}}_{j\text{ times}},\underbrace{S_m,\dots,S_m}_{k-j\text{ times}}) \quad \text{and}$$

$$= n\kappa_k\big(\log R^2\big) - m\kappa_k\big(\log R^1\big)$$

$$+ \sum_{\pi\in\mathcal{P}}\big(|\pi|-1\big)!(-1)^{|\pi|}\sum_{j=1}^{k-1}\binom{k}{j}(-1)^j\prod_{B\in\pi}\mathbb{E}\big[(\overline{\log Z_{m,n}})^{a_{j,B}}p_{b_{j,B}}(S_m,a;m)\big],$$

where $a_{j,B}=|B\cap\{1,\dots,j\}|$, $b_{j,B}=|B\cap\{j+1,\dots,k\}|=|B|-a_{j,B}$. Moreover,

$$\mathbb{E}\big[(\overline{\log Z_{m,n}})^j p_k(S_m,a;m))\big] = \sum_{\substack{\ell_1+\cdots+\ell_j=k\\ \ell_i\ge 0}} \frac{k!}{\ell_1!\cdots\ell_j!}\mathbb{E}\left[\prod_{i=1}^{j}\sigma_{\ell_i}(t_1)\right].$$

Note that in the case $k=2$, the formula in the previous corollary coincides with the variance representation in Seppäläinen [14, Theorem 3.7].

5. Estimates for the central moments

Lemma 5.1. *Let $0\le r$ and put $S_n=\sum_{i=1}^n g(a,R_{i,0})$ where $g(a,R_{i,0})$ has finite moments of all orders. Recall the notation (2) for the annealed expectation with respect to the polymer environment. Then, for all $k\in 2\mathbb{N}$ there exist finite constants $C_k=C_k(a)>0$ which are locally bounded in a, such that*

$$E_{m,n}\big[(\overline{S_{t_1}-S_{t_1\wedge r}})^k\big] \le C_k\big(E_{m,n}\big[(t_1-t_1\wedge r)^k\big]+1\big) \quad \textit{for all } (m,n)\in\mathbb{N}^2.$$

Here the centering is with respect to the annealed measure $E_{m,n}$.

Proof.

$$E_{m,n}\big[(\overline{S_{t_1}-S_{t_1\wedge r}})^k\big] = E_{m,n}\left[\sum_{l>r}\mathbb{1}_{\{t_1=l\}}(\overline{S_l-S_{l\wedge r}})^k\right] \tag{51}$$

$$+ (-1)^k P_{m,n}(t_1\le r)E_{m,n}[S_{t_1}-S_{t_1\wedge r}]^k. \tag{52}$$

We now treat (51) and (52) separately.

$$(51) = \mathbb{E}\left[\sum_{l>r}(\overline{S_l-S_{l\wedge r}})^k Q_{m,n}(t_1=l)\right]$$

$$\le \mathbb{E}\left[\sum_{l>r}(\overline{S_l-S_{l\wedge r}})^k\mathbb{1}_{\{\overline{S_l-S_{l\wedge r}}>l-r\}}\right]$$

$$+ \mathbb{E}\left[\sum_{l>r}(l-r)^k Q_{m,n}(t_1=l)\right]$$

$$\leq \mathbb{E}\left[\sum_{l=1}^{\infty}(\overline{S_l})^k \mathbb{1}_{\{\overline{S_l}>l\}}\right] \quad \text{by reindexing}$$
$$+ E_{m,n}\big[(t_1 - t_1 \wedge r)^k\big].$$

The last inequality follows from stationarity. Since $\overline{S_l}$ is an i.i.d. sum of mean zero random variables which have finite moments of all orders,

$$\mathbb{E}\big[(\overline{S_l})^k \mathbb{1}_{\{\overline{S_l}>l\}}\big] \leq \mathbb{E}\big[(\overline{S_l})^{2k}\big]^{\frac{1}{2}} \mathbb{P}\big(|\overline{S_l}| > l\big)^{\frac{1}{2}}$$
$$\leq C_k l^{\frac{k}{2}} \mathbb{E}\left[\left(\frac{\overline{S_l}}{l}\right)^{4k}\right]^{\frac{1}{2}} \leq C_k l^{-k}$$

which is summable over l.

For equation (52), we repeat the proof in [14, Lemma 4.2]:

$$\begin{aligned}
E_{m,n}[S_{t_1} - S_{t_1\wedge r}] &= E_{m,n}\big[(t_1 - t_1 \wedge r)\big]\mathbb{E}\big[g(a, R_{i,0})\big] + \mathbb{E}\left[\sum_{i=\tau+1}^{t_1} \overline{g(a, R_{i,0})}\right] \\
&= C E_{m,n}\big[(t_1 - t_1 \wedge r)\big] + \sum_{k=1}^{m} \mathbb{E}\big[Q_{m,n}(t_1 = k)\overline{S_k}\big] \\
&= C\big(E_{m,n}\big[(t_1 - t_1 \wedge r)\big] + 1\big) + \sum_{k=1}^{m} \mathbb{E}[\mathbf{1}_{\{\overline{S_k}\geq k\}}\overline{S_k}] \\
&\leq C\big(E_{m,n}[t_1 - t_1 \wedge r] + 1\big).
\end{aligned} \tag{53}$$

Taking the k-th powers and using Jensen's inequality completes the proof. □

Given a random variable X and $p \in [1, \infty)$, we write

$$\|X\|_{p,\mathbb{E}} := \mathbb{E}\big[|X|^p\big]^{\frac{1}{p}},$$
$$\|X\|_{p,E_{m,n}} := E_{m,n}\big[|X|^p\big]^{\frac{1}{p}}$$

for the p-th norm with respect to the regular expectation $\mathbb{E}$ and the annealed expectation $E_{m,n}$. When m, n is understood we write $E = E_{m,n}$.

Lemma 5.2. *For every even integer $k \geq 2$ there exists a constant C_k such that whenever $\{X_i\}_{i=1}^k$ are random variables with finite annealed moments, then*

$$\big\|\kappa_k^Q(X_1, \dots, X_k)\big\|_{p,\mathbb{E}} \leq C_k \prod_{i=1}^{k} \|\overline{X_i}\|_{pk,E}$$

where the centering on the right-hand side is with respect to the annealed measure E.

Proof. $E_{m,n}[X_i]$ are constants and therefore

$$\kappa_k^Q(X_1, \dots, X_k) = \kappa_k^Q(\overline{X_1}, \dots, \overline{X_k}),$$
$$\left|E^Q\left[\prod_{i\in B}|\overline{X_i}|\right]\right|^p \leq \prod_{i\in B} E^Q\big[|\overline{X_i}|^{p|B|}\big]^{\frac{1}{|B|}} \leq \prod_{i\in B} E^Q\big[|\overline{X_i}|^{pk}\big]^{\frac{1}{k}}.$$

Using Hölder's generalized inequality again,

$$\mathbb{E}\left[\prod_{i=1}^{k} E^Q\big[|\overline{X_i}|^{pk}\big]^{\frac{1}{k}}\right] \leq \prod_{i=1}^{k}\big(\mathbb{E}[E^Q\big[|\overline{X_i}|^{pk}\big])^{\frac{1}{k}}.$$

Taking the p-th root and plugging this into (16) yields the desired result with $C_k = (k-1)!2^k$. □

The following allows us to control moments of $\sigma_k(r)$ in terms of annealed moments of the exit time t_1.

Lemma 5.3. *For any $k \in 2\mathbb{N}$, $p \in [1,\infty)$, there exist positive constants $C(k,p)$ such that the following two conditions hold for all $r, M \in \mathbb{N}$:*

$$\|\sigma_k(r)\|_{p,\mathbb{E}} \le C(k,p)\big(1 + \|(t_1 \wedge r)^k\|_{p,E}\big), \tag{54}$$

$$\|\sigma_k(m) - \sigma_k(r)\|_{p,\mathbb{E}} \le C(k,p)\big(1 + \|(t_1)^k\|_{2p,E}\big)\frac{\|(t_1)^M\|_{2pk,E}}{r^M}. \tag{55}$$

Proof. For $\ell, m \in \mathbb{N}$ define

$$X_\ell(m) := \sum_{i=1}^m \widetilde{\partial}^\ell L^f\big(a, R^1_{i,0}\big).$$

Taking L_p norms of (47), gives

$$\|\sigma_k(r)\|_{p,\mathbb{E}} \le \sum_{j=1}^k \sum_{\substack{\ell_1+\cdots\ell_j=k-j\\ \ell_i\ge 0}} C_{k,j,\vec{\ell}} \big\|\kappa_j^Q\big(X_{\ell_1}(t_1\wedge r),\dots,X_{\ell_j}(t_1\wedge r)\big)\big\|_{p,\mathbb{E}}, \tag{56}$$

and

$$\begin{aligned}&\|\sigma_k(m) - \sigma_k(r)\|_{p,\mathbb{E}}\\ &\quad\le \sum_{j=1}^k \sum_{\substack{\ell_1+\cdots\ell_j=k-j\\ \ell_i\ge 0}} C_{k,j,\vec{\ell}} \big\|\kappa_j^Q\big(X_{\ell_1}(t_1),\dots,X_{\ell_j}(t_1)\big) - \kappa_j^Q\big(X_{\ell_1}(t_1\wedge r),\dots,X_{\ell_j}(t_1\wedge r)\big)\big\|_{p,\mathbb{E}}.\end{aligned} \tag{57}$$

Lemma 5.2, Lemma 5.1, equation (56), and Jensen's inequality give (54). By (57) and a telescoping argument, to obtain (55) it suffices to bound $\|\kappa_j^Q(Y_1,\dots,Y_j)\|_{p,E}$ where

$$Y_i = \begin{cases} X_i(t_1\wedge r) & \text{for } 1 \le i < s,\\ X_s(t_1) - X_s(t_1\wedge r) & \text{for } i = s,\\ X_i(t_1) & \text{for } s < i \le j\end{cases}$$

and $s \in \{1,\dots,j\}$ is fixed. By Lemma 5.1 and Jensen's inequality, for $i \ne s$,

$$\|\overline{Y}_i\|_{pj,\mathbb{E}} \le \|\overline{Y}_i\|_{pk,\mathbb{E}} \le C_{2p,k}\big(1 + \|t_1\|_{pk,E}\big). \tag{58}$$

By Jensen's inequality, the Cauchy–Schwarz inequality, Lemma 5.1, and Markov's inequality,

$$\begin{aligned}\|\overline{Y}_s\|_{pj,\mathbb{E}} &\le \|\overline{Y}_s\|_{pj,\mathbb{E}} = \big\|\overline{X_s(t_1) - X_s(t_1\wedge r)}\big\|_{pk,\mathbb{E}}\\ &\le \big\|\overline{X_s(t_1) - X_s(t_1\wedge r)}\mathbb{1}_{\{t_1>r\}}\big\|_{pk,\mathbb{E}} + \mathbb{E}\big[\big|X_s(t_1) - X_s(t_1\wedge r)\big|\mathbb{1}_{\{t_1>r\}}\big]\\ &\le \big\|\overline{X_s(t_1) - X_s(t_1\wedge r)}\mathbb{1}_{\{t_1>r\}}\big\|_{pk,\mathbb{E}} + \mathbb{E}\big[\big|\overline{X_s(t_1) - X_s(t_1\wedge r)}\big|\mathbb{1}_{\{t_1>r\}}\big]\\ &\quad + \big|\mathbb{E}\big[X_s(t_1) - X_s(t_1\wedge r)\big]\big|P_{m,n}(t_1>r)\\ &\le 2\big\|\overline{X_s(t_1) - X_s(t_1\wedge r)}\big\|_{2pk,\mathbb{E}}\|\mathbb{1}_{\{t_1>r\}}\|_{2pk,\mathbb{E}} + \big|\mathbb{E}\big[X_s(t_1) - X_s(t_1\wedge r)\big]\big|P_{m,n}(t_1>r)\\ &\le C_{2p,k}\big(1 + \|(t_1 - t_1\wedge r)\|_{2pk,E}\big)P_{m,n}(t_1>r)^{\frac{1}{2pk}}\\ &\le C_{2p,k}\big(1 + \|(t_1 - t_1\wedge r)\|_{2pk,E}\big)\frac{\|(t_1)^M\|_{2pk,E}}{r^M}.\end{aligned}$$

In the third to last inequality we again used a slight modification of [14, Lemma 4.2] as in (53). Another application of Jensen's inequality along with (58) gives (55). □

For the following lemma, recall the notation $\mathbb{P}^{(a_1,a_2)}$ and $\mathbb{E}^{(a_1,a_2)}$ defined in Section 2.3.

Lemma 5.4. *Assume the polymer environment is as in* (12) *and the sequence* $(m,n)=(m_N,n_N)_{N=1}^{\infty}$ *satisfies*

$$\left|m-N\psi_1^{f^2}(a_2)\right|\vee\left|n-N\psi_1^{f^1}(a_1)\right|\le\gamma N^{\frac{2}{3}}$$

where γ *is some positive constant. Then there exist finite positive constants* C_1, C_2, C_3, δ, δ_1, b (*uniformly bounded in* (a_1,a_2) *such that for all* $N\in\mathbb{N}$ *the following two bounds hold simultaneously for* $j=1,2$: *for all* $C_1N^{\frac{2}{3}}\le u\le\delta N$,

$$\mathbb{P}^{(a_1,a_2)}\big[Q_{m,n}(t_j\ge u)\ge e^{-\frac{\delta u^2}{N}}\big]\le C_2\bigg(\frac{N^k}{u^{2k}}\big(\mathbb{E}^{(a_1,a_2)}\big[(\overline{\log Z_{m,n}})^k\big]+\mathbb{E}^{(a_1(\lambda_j),a_2(\lambda_j))}\big[(\overline{\log Z_{m,n}})^k\big]\big)\bigg)$$

where $a_1(\lambda):=a_1+\lambda$, $a_2(\lambda)=a_2-\lambda$, $\lambda_1=\frac{bu}{N}$, *and* $\lambda_2=-\frac{bu}{N}$, *while for* $u\ge\delta N$,

$$\mathbb{P}^{(a_1,a_2)}\big[Q_{m,n}(t_j\ge u)\ge e^{-\delta_1u}\big]\le 2e^{-C_3}.$$

Proof. Follow the proof of Proposition 4.3 in [7] verbatim up to the displayed inequality

$$(4.8)\text{ in [7]}\le\tilde{\mathbb{P}}\bigg[\overline{\log Z_{m,n}\big(a_1(\lambda_j),a_2(\lambda_j)\big)-\log Z_{m,n}(a_1,a_2)}\ge C'''\frac{u^2}{N}\bigg],$$

where (4.8) refers to the corresponding equation in [7].

Now rather than bounding by the second moment, bound by the k-th moment to get

$$\begin{aligned}(4.8)\text{ in [7]}&\le\left(\frac{N}{C'''u^2}\right)^k\tilde{\mathbb{E}}\big[\big(\overline{\log Z_{m,n}\big(a_1(\lambda_j),a_2(\lambda_j)\big)-\log Z_{m,n}(a_1,a_2)}\big)^k\big]\\&\le\frac{C_2N^k}{u^{2k}}\big(\mathbb{E}^{(a_1,a_2)}\big[(\overline{\log Z_{m,n}})^k\big]+\mathbb{E}^{(a_1(\lambda_j),a_2(\lambda_j))}\big[(\overline{\log Z_{m,n}})^k\big]\big).\end{aligned}$$

The proof of the second part is just as in Proposition 4.3 of [7]. □

Corollary 5.5. *Let* $k\ge2$. *Suppose there exist* $\delta,\epsilon_0>0$ *such that* $[a_1-\epsilon_0,a_1+\epsilon_0]\times[a_2-\epsilon_0,a_2+\epsilon_0]\subset D(M_{f_1})\times D(M_{f_2})$ *and the following holds for every* $N\in\mathbb{N}$ *and every* $\lambda\in[-\epsilon_0,\epsilon_0]$:

$$\mathbb{E}^{(a_1(\lambda),a_2(\lambda))}\big[(\overline{\log Z_{m_N,n_N}})^k\big]\le CN^{(\frac{1}{3})k+\delta k}\tag{59}$$

where $a_1(\lambda)=a_1-\lambda$, *and* $a_2(\lambda)=a_2+\lambda$. *Then, for all* $\epsilon>0$ *there exists a positive constant* $C'=C'(\epsilon,k,a_1,a_2)$ *such that the following bound holds for every* $N\in\mathbb{N}$ *and every* $\lambda\in[-\frac{\epsilon_0}{2},\frac{\epsilon_0}{2}]$:

$$E^{(a_1(\lambda),a_2(\lambda))}\big[(t_j)^{2k}\big]\le C'N^{(\frac{4}{3})k+\delta k+\epsilon}\quad\textit{for both } j=1,2.$$

Here $E^{(a_1,a_2)}$ *denotes the annealed expectations with respect to the measure on paths in the environment* (12).

Proof. We apply Lemma 5.4 and use the same notation as in that lemma for constants. Fix $\lambda_0\in[\frac{-\epsilon_0}{2},\frac{\epsilon_0}{2}]$ and put $(\tilde{a}_1,\tilde{a}_2)=(a_1(\lambda_0),a_2(\lambda_0))\in D(M_{f^1})\times D(M_{f^2})$. Note that $\tilde{a}_1+\tilde{a}_2=a_1+a_2=a_3$ (see (12)). So by Lemma 5.4 and (59) there exist positive constants $N_0=N_0(\epsilon_0)\in\mathbb{N}$, $C_1=C_1(\epsilon_0)$, $\delta=\delta(\epsilon_0)$ such that for all $N\ge N_0$,

$$\begin{aligned}E^{(\tilde{a}_1,\tilde{a}_2)}\big[(t_j)^{2k}\big]&\le\big(C_1N^{\frac{2}{3}}\big)^{2k}+(2k)(\epsilon N)^{\epsilon}\int_{C_1\wedge N^{\frac{2}{3}}}^{\delta N}u^{2k-1-\epsilon}P^{(\tilde{a}_1,\tilde{a}_2)}(t_j\ge u)\,\mathrm{d}u+C'(\delta,\delta_1,C_3,N_0)\\&\le\big(C_1N^{\frac{2}{3}}\big)^{2k}+(2k)(\delta N)^{\epsilon}CC_2N^{(\frac{4}{3})k+\delta k}\int_{C_1N^{\frac{2}{3}}}^{\delta N}u^{-1-\epsilon}\,\mathrm{d}u\\&\le C(\epsilon,k,\epsilon_0)N^{(\frac{4}{3}k+\delta k+\epsilon)}.\end{aligned}$$

□

Lemma 5.6. *Assume the polymer environment is distributed as in (12) and the sequence $(m,n)=(m_N,n_N)_{N=1}^\infty$ satisfies*

$$\left|m-N\psi_1^{f^2}(a_2)\right|\vee\left|n-N\psi_1^{f^1}(a_1)\right|\le\gamma N^{\frac{2}{3}}$$

where γ is some positive constant. Further, suppose there exist positive constants δ, ϵ_0, $\{C_k\}_{k=1}^\infty$ such that $[a_1-\epsilon_0,a_1+\epsilon_0]\times[a_2-\epsilon_0,a_2+\epsilon_0]\subset D(M_{f^1})\times D(M_{f^2})$ and the following hold for every $k,N\in\mathbb{N}$ and every $\lambda\in[-\epsilon_0,\epsilon_0]$:

$$\mathbb{E}^{(a_1(\lambda),a_2(\lambda))}\big[(\overline{\log Z_{m,n}})^k\big]\le C_kN^{(\frac{1}{3}+\delta)k}. \tag{60}$$

Then for all $\epsilon>0$, $M>0$, there exist positive constants $\{C_{j,l}=C_{j,l}(a_1,a_2,\epsilon,\delta,M)\}_{j,l=1}^\infty$ (locally bounded in a_1,a_2) such that for all $N\in\mathbb{N}$ we have the following:

$$\left|\mathbb{E}\big[(\overline{\log Z_{m,n}})^jp_l(S_m,a_1;m)\big]-\mathbb{E}\big[(\overline{\log Z_{m,n}})^jp_l\big(S_{\lfloor\tau\rfloor},a_1;\lfloor\tau\rfloor\big)\big]\right|\le C_{j,l}N^{-M},$$

where $S_r=\sum_{i=1}^r\log R^1_{i,0}$, and $\tau=N^{(\frac{2}{3}+\frac{\delta}{2}+\epsilon)}$.

Proof. By (48), for all $0\le r\le m$,

$$\mathbb{E}\big[(\overline{\log Z_{m,n}})^jp_k(S_r,a_1;r)\big]=\sum_{\substack{\ell_1+\cdots+\ell_j=k\\ \ell_i\ge0}}\frac{k!}{l_1!\dots l_j!}\mathbb{E}\left[\prod_{i=1}^j\sigma_{\ell_i}(r)\right]$$

where $\sigma_0(r)=\overline{\log Z_{m,n}}$. It will therefore suffice to compare $\sigma_{\ell_i}(m)$ with $\sigma_{\ell_i}(r)$. Specifically, for fixed $\ell_1,\dots,\ell_j$, such that $\sum_{i=1}^j\ell_i=k$, we wish to estimate

$$\mathbb{E}\left[\prod_{i=1}^j\sigma_{\ell_i}(m)-\prod_{i=1}^j\sigma_{\ell_i}(r)\right].$$

By a telescoping argument it suffices to bound

$$\mathbb{E}\left[\sigma_{\ell_a}(m)\prod_{i\in I_1}\sigma_{\ell_i}(m)\prod_{i\in I_2}\sigma(r)\right]-\mathbb{E}\left[\sigma_{\ell_a}(r)\prod_{i\in I_1}\sigma_{\ell_i}(m)\prod_{i\in I_2}\sigma(r)\right]$$

where $a\in\{1,2,\dots,j\}$ is such that $\ell_a\neq0$, $I_1=\{1,\dots,a-1\}$, $I_2=\{a+1,\dots,j\}$, and $\sum_{i=1}^j\ell_i=k$. By the generalized Hölder inequality this is bounded by

$$\left\|\sigma_{\ell_a}(m)-\sigma_{\ell_a}(r)\right\|_{2,\mathbb{E}}\prod_{i\in I_1}\left\|\sigma_{\ell_i}(m)\right\|_{2(j-1),\mathbb{E}}\prod_{i\in I_2}\left\|\sigma_{\ell_i}(r)\right\|_{2(j-1),\mathbb{E}}. \tag{61}$$

Let $I_0=\{1\le i\le j:\ell_i=0\}$. By Lemma 5.3, for any $r,M\in\mathbb{N}$,

$$(61)\le C(\ell_a,2)\big(1+\left\|(t_1)^{\ell_a}\right\|_{4,E}\big)\frac{\|(t_1)^M\|_{4\ell_a,E}}{r^M}\|\overline{\log Z_{m,n}}\|^{|I_0|}_{2(j-1),\mathbb{E}}\prod_{i\notin I_0}C\big(\ell_i,2(j-1)\big)\big(1+\left\|t_1^{\ell_i}\right\|_{2(j-1),E}\big). \tag{62}$$

Using the assumption (60), by Corollary 5.5, for any $\epsilon>0$ there exists a constant $C(\epsilon,p)\ge1$ (uniformly bounded in (a_1,a_2) such that

$$\|\overline{\log Z_{m,n}}\|_{p,\mathbb{E}}\le C(p,\epsilon)N^{(\frac{1}{3}+\delta)}\quad\text{and}$$

$$\left\|(t_1)^\ell\right\|_{p,E}\le C(\ell,p,\epsilon)N^{(\frac{2}{3}+\frac{\delta}{2})\ell+\frac{\epsilon}{p}}.$$

This implies the existence of positive constants $C'=C'(k,j,M)$ such that for all $M\in\mathbb{N}$ and all $N\in\mathbb{N}$,

$$(62)\le C'N^{(\frac{2}{3}+\frac{\delta}{2})\ell_a+\frac{\epsilon}{4}}\cdot N^{(\frac{2}{3}+\frac{\delta}{2})M+\frac{\epsilon}{4\ell_a}}r^{-M}\cdot N^{(\frac{1}{3}+\delta)|I_0|}\cdot\prod_{i\notin I_0}N^{(\frac{2}{3}+\frac{\delta}{2})\ell_i+\frac{\epsilon}{2(j-1)}}.$$

Choosing

$$r = \lfloor \tau \rfloor \geq N^{\frac{2}{3}+\frac{\delta}{2}+\epsilon} - 1,$$

we obtain the bound

$$C' N^{(\frac{1}{3}+\delta)(j-1)+(\frac{2}{3}+\frac{\delta}{2})k+2\epsilon-M\epsilon}.$$

Now fix $M_0 = M_0(\epsilon, \delta, j, k)$ large enough such that such that

$$\left(\frac{1}{3}+\delta\right)(j-1) + \left(\frac{2}{3}+\frac{\delta}{2}\right)k + 2\epsilon - M_0\epsilon \leq -K.$$

□

Before proceeding to Lemma 5.7, we note the following property of the polynomials $p_n(T, a; r)$ introduced in (19):

Proposition 5.1. *For each n,*

$$p_n(t, a; r) = \sum_{j=0}^{n} c_j(a)\big(t - r\psi_0(a)\big)^{a_j} r^{b_j},$$

where $c_j(a)$ are independent of r and $0 \leq a_j, b_j \leq n$ are integers with

$$\frac{a_j}{2} + b_j = \frac{n}{2}. \tag{63}$$

In particular, if $T = \sum_{k=1}^{r} \log X_k$ where $X_k \sim m_f(a)$, then we have for integers $b, k \geq 0$

$$\mathbb{E}\big[\big|p_b(T, a; r)\big|^k\big] \leq C_{b,k} r^{kb/2}. \tag{64}$$

Proof. The result is clearly true for $p_0(T, a; r)$. Next, we note that if a_j, b_j satisfy (63), then

$$\begin{aligned}
&\frac{\partial}{\partial a}\big(t - r\psi_0(a)\big)^{a_j} r^{b_j} + \big(t - r\psi_0(a)\big)^{a_j} r^{b_j} \cdot \big(t - r\psi_0(r)\big) \\
&\quad = -a_j\psi_1(a)\big(t - r\psi_0(a)\big)^{a_j-1} r^{b_j+1} + \big(t - r\psi_0(a)\big)^{a_j+1} r^{b_j}.
\end{aligned}$$

Noting that

$$\frac{a_j - 1}{2} + b_j + 1 = \frac{a_j + 1}{2} + b_j = \frac{n+1}{2},$$

the claim follows by induction from the definition (19). □

Lemma 5.7. *With the same assumptions as in Lemma 5.6, for all $k \in \mathbb{N}$ there exist positive constants $C_k = C_k(a_1, a_2)$ (locally bounded) such that for all even $k \geq 2$.*

$$\mathbb{E}\big[(\overline{\log Z_{m,n}})^k\big] \leq C_k N^{(\frac{1}{3}+\frac{\delta}{3})k} \quad \textit{for all } N \in \mathbb{N}. \tag{65}$$

Proof. The proof is by induction on k. For $k = 2$, (65) holds with $\delta = 0$. Assuming the estimate for even exponents less than k, we use the first expression in Corollary 4.8 to express the cumulant $\kappa_k(\log Z_{m,n})$ as a sum of terms of the form

$$\prod_{B\in\pi} \mathbb{E}\big[(\overline{\log Z_{m,n}})^{a_{j,B}} p_{b_{j,B}}(S_m, a_1; m)\big], \tag{66}$$

where π is a partition of $\{1, \ldots, k\}$ into $|\pi|$ blocks B, and $a_{j,B} + b_{j,B} = |B|$.

Using equation (48) and Lemma 5.6 with $K > 2k$, we have, for $\tau = n^{2/3+\delta/2+\epsilon}$,

$$\begin{aligned}
&\prod_{B\in\pi} \mathbb{E}\big[(\overline{\log Z_{m,n}})^{a_{j,B}} p_{b_{j,B}}(S_m, a_1; m)\big] \\
&\quad = \prod_{B\in\pi} \mathbb{E}\big[(\overline{\log Z_{m,n}})^{a_{j,B}} p_{b_{j,B}}\big(S_{\lfloor\tau\rfloor}, a_1; \lfloor\tau\rfloor\big)\big] + O\big(n^{-k}\big).
\end{aligned}$$

Taking absolute values and applying Hölder's inequality,

$$
\begin{aligned}
&\left|\mathbb{E}\left[(\overline{\log Z_{m,n}})^{a_{j,B}} p_{b_{j,B}}\left(S_{\lfloor\tau\rfloor}, a_1; \lfloor\tau\rfloor\right)\right]\right| \\
&\quad \le \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right]^{\frac{a_{j,B}}{k}} \mathbb{E}\left[\left|p_{b_{j,B}}\left(S_{\lfloor\tau\rfloor}, a_1; \lfloor\tau\rfloor\right)\right|^{k'}\right]^{\frac{b_{j,B}}{k'}} \\
&\quad \le C n^{((1/3)+\delta/4+\epsilon/2) b_{j,B}} \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right]^{\frac{a_{j,B}}{k}},
\end{aligned}
$$

where $\frac{a_{j,B}}{k} + \frac{1}{k'} = 1$. The last inequality follows from equation (64) in Proposition 5.1. Taking the product over $B \in \pi$, we have, up to a constant factor, the bound:

$$
n^{((1/3)+\delta/4+\epsilon/2) b_j} \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right]^{\frac{a_j}{k}}, \tag{67}
$$

where

$$
a_j := \sum_B a_{j,B} \quad \text{and} \quad b_j := \sum_B b_{j,B},
$$

so $\frac{a_j}{k} + \frac{b_j}{k} = 1$. Note that for $1 \le j \le k-1$, we have $a_j \le k-1$. Applying Young's inequality $xy \le \frac{1}{p}x^p + \frac{1}{q}y^q$ to (67), we find that for $\eta > 0$, any term of the form (66) is bounded by

$$
\eta \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right] + C(\eta) n^{((1/3)+\delta/4+\epsilon/2)k} + O\left(n^{-k}\right).
$$

Combining this with Corollary 4.8, we have

$$
\kappa_k(\log Z_{m,n}) = C(k)\eta \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right] + C(k)C(\eta) n^{((1/3)+\delta/4+\epsilon/2)k} + O\left(n^{-k}\right). \tag{68}
$$

Writing

$$
\kappa_k(\log Z_{m,n}) = \mathbb{E}\left[(\overline{\log Z_{m,n}})^k\right] + \sum_{\substack{|\alpha|=k \\ 0\le\alpha_i<k}} c_\alpha \prod_{i=1}^{|\alpha|} \mathbb{E}\left[(\overline{\log Z_{m,n}})^{\alpha_i}\right], \tag{69}
$$

where the sum is over multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$, $\sum_i \alpha_i = k$. If some $\alpha_i = k-1$, then the product must equal zero. Therefore, by the induction assumption, all terms in the sum on the right of (69) are of order $n^{((1/3)+\delta/3)k}$. Choosing η sufficiently small in (68) and absorbing $\epsilon/2$ into $\delta/4$, we obtain the result. □

5.1. *Finishing the argument*

Combining Corollary 5.5 and Lemma 5.7 we obtain the following:

Lemma 5.8. *Assume the polymer environment is distributed as in (12) and the sequence $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies*

$$
\left|m - N\psi_1^{f^2}(a_2)\right| \vee \left|n - N\psi_1^{f^1}(a_1)\right| \le \gamma N^{\frac{2}{3}}
$$

where γ is some positive constant. Further, suppose there exist positive constants δ, ϵ_0, $C(k)$ for $k \in \{2, 4, \dots\}$ such that $[a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subset D(M_{f^1}) \times D(M_{f^2})$ and the following hold for any even k and any $\lambda \in [-\epsilon_0, \epsilon_0]$:

$$
\mathbb{E}^{(a_1(\lambda), a_2(\lambda))}\left[(\overline{\log Z_{m,n}})^k\right] \le C(k) N^{(\frac{1}{3}+\delta)k}.
$$

Then there exist constants $C'(k) > 0$ for $k \in \{2, 4, \dots\}$ such that for any even k and any $\lambda \in [-\frac{\epsilon_0}{2}, \frac{\epsilon_0}{2}]$:

$$
\mathbb{E}^{(a_1(\lambda), a_2(\lambda))}\left[(\overline{\log Z_{m,n}})^k\right] \le C'(k) N^{(\frac{1}{3}+\frac{\delta}{3})k}.
$$

Theorem 1 will follow from repeated application of Lemma 5.8 once we prove the following:

Proposition 5.2. *Assume the polymer environment is distributed as in* (12) *and the sequence* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies*

$$\left|m - N\psi_1^{f^2}(a_2)\right| \vee \left|n - N\psi_1^{f^1}(a_1)\right| \leq \gamma N^{\frac{2}{3}} \tag{70}$$

where γ *is some positive constant. Then there exists positive constants* ϵ_0 *and* $C(k)$ *for* $k \in \{2,4,\dots\}$ *such that* $[a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subset D(M_{f^1}) \times D(M_{f^2})$ *and the following hold for any even k and any* $\lambda \in [-\epsilon_0, \epsilon_0]$:

$$\mathbb{E}^{(a_1(\lambda),a_2(\lambda))}\left[(\overline{\log Z_{m,n}})^k\right] \leq C(k) N^{(\frac{1}{3}+\frac{1}{6})k}. \tag{71}$$

Proof. Since $(a_1,a_2) \in D(M_{f^1}) \times D(M_{f^2})$, there exists a positive constant ϵ_0 such that $[a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subset D(M_{f^1}) \times D(M_{f^2})$. With notation as in Section 3.1, if we define $A := \overline{\log Z_{m,n}}$, then $A = \overline{S_m} + \overline{E_n}$. Thus, for even k,

$$\mathbb{E}\left[A^k\right] \leq 2^{k-1}\left(\mathbb{E}\left[(\overline{S_m})^k\right] + \mathbb{E}\left[(\overline{E_n})^k\right]\right). \tag{72}$$

By Proposition 3.1, all four models described by (12) have the down-right property. So by the discussion in Section 3.1, S_m and E_n are both sums of i.i.d. random variables whose common distributions continuously depends on a_1 and a_2 respectively. Moreover, by Remark 2, all random inside of the summations have finite exponential moments. Therefore, for every $k \in \{2,4,\dots\}$ there exists a positive constant $C_k = C_k(a_1,a_2)$, which is continuous in (a_1,a_2), such that

$$\mathbb{E}\left[(\overline{S_m})^k\right] \leq C_k m^{k/2} \quad \text{for all } m \geq 1$$

and

$$\mathbb{E}\left[(\overline{E_n})^k\right] \leq C_k n^{k/2} \quad \text{for all } n \geq 1.$$

Using equations (72) and (70) now yields the desired result. □

Proof of Theorem 1. The four basic beta-gamma models (3)–(6) can all be described by equation (12). So let $\epsilon > 0$ and $(a_1,a_2) \in D(M_{f^1}) \times D(M_{f^2})$. Fix even integers k, M such that $p \leq k$ and

$$\frac{(1/6)}{3^M} \leq \epsilon.$$

By Jensen's inequality, it suffices to show the bounds (9) and (10) hold with p replaced by k. Now apply Proposition 5.2 followed by M consecutive applications of Lemma 5.8 to obtain the bound (9). Finally, apply Corollary 5.5 to both t_1 and t_2 to obtain the bound (10). □

Appendix A: Combinatorial formula for cumulants

Here we derive the formula (16) for the joint cumulants. This identity is classical and appears on Wikipedia under *Cumulants* [17], but we could not locate a suitable proof to cite.

We will prove the following by induction. Denote

$$Z := \mathbb{E}\left[e^{\sum_{i=1}^n \xi_i X_i}\right],$$
$$E[\cdot] := \frac{1}{Z}\mathbb{E}\left[e^{\sum_{i=1}^n \xi_i X_i} \cdot\right].$$

Note that for $k \leq n$

$$\kappa_k(X_1,\dots,X_k) = \partial_{\xi_1}\cdots\partial_{\xi_k} \log Z|_{\xi_1=\cdots=\xi_n=0}.$$

We will show by induction that

$$\partial_{\xi_1}\cdots\partial_{\xi_k} \log Z = \sum_{\pi \in \mathcal{P}(1,\dots,k)} \left(|\pi|-1\right)!(-1)^{|\pi|-1} \prod_{B\in\pi} E\left[\prod_{i\in B} X_i\right]. \tag{73}$$

Proof. Note that the result holds for $k = 1$. Indeed, this case

$$\partial_{\xi_1} \log Z = E[X_1].$$

Assume the result for $k \le n-1$. We prove the result for $k+1$. Differentiating (73), we obtain for each $\pi \in \mathcal{P}(1,\dots,k)$ appearing in the sum (73):

$$\partial_{\xi_{k+1}} \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] = \sum_{B'\in\pi} \partial_{\xi_{k+1}} E\Big[\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big].$$

For the derivative, we have

$$\begin{aligned}\partial_{\xi_{k+1}} E\Big[\prod_{j\in B'} X_j\Big] &= \partial_{\xi_{k+1}} \frac{1}{Z}\mathbb{E}\Big[e^{\sum_{i=1}^n \xi_i X_j} \prod_{j\in B'} X_j\Big] \\ &= E\Big[X_{k+1}\prod_{j\in B'} X_j\Big] - E[X_{k+1}]E\Big[\prod_{j\in B'} X_j\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned}\partial_{\xi_{k+1}} \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] &= \sum_{B'\in\pi} E\Big[X_{k+1}\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big] \\ &\quad - \sum_{B'\in\pi} E[X_{k+1}]E\Big[\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big] \\ &= \sum_{B'\in\pi} E\Big[X_{k+1}\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big] \\ &\quad - |\pi| E[X_{k+1}] \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big]. \end{aligned} \tag{74}$$

The first term corresponds to adding a factor X_{k+1} to a single B block of the partition π and the second term corresponds to adding a 1-term block $\{k+1\}$ to π. Summing (74) over $\pi \in \mathcal{P}(1,\dots,k)$, we obtain

$$\begin{aligned}&\partial_{\xi_{k+1}} \sum_{\pi\in\mathcal{P}(1,\dots,k)} \big(|\pi|-1\big)!(-1)^{|\pi|-1} \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] \\ &= \sum_{\pi\in\mathcal{P}(1,\dots,k)} \big(|\pi|-1\big)!(-1)^{|\pi|-1} \sum_{B'\in\pi} E\Big[X_{k+1}\prod_{j\in B'} X_j\Big] \prod_{B\neq B'} E\Big[\prod_{i\in B} X_i\Big] \\ &\quad - \sum_{\pi\in\mathcal{P}(1,\dots,k)} |\pi|!(-1)^{|\pi|} E[X_{k+1}] \prod_{B\in\pi} E\Big[\prod_{i\in B} X_i\Big] \\ &= \sum_{\tilde\pi\in\mathcal{P}(1,\dots,k+1)} \big(|\tilde\pi|-1\big)!(-1)^{|\tilde\pi|-1} \prod_{B\in\tilde\pi} E\Big[\prod_{i\in B} X_i\Big].\end{aligned}$$

To verify the final step, note that any partition of $\{1,\dots,k+1\}$ which contains $\{k+1\}$ as a single element block induces a partition π of $\{1,\dots,k\}$ from the remaining blocks with $|\pi| = |\tilde\pi| - 1$; otherwise, if $\{k+1\}$ does not appear as block in $\tilde\pi$, the partition can be obtained from some $\pi \in \mathcal{P}(1,\dots,k)$ by adding $k+1$ to one of the $|\pi|$ blocks without changing the number of blocks. □

Appendix B: Proof of Theorem 5

The next lemma says that it suffices to verify Hypothesis 1 for $f(x) = e^{-bx}$, $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$, and $f(x) = (\frac{x}{1+x})^b$ where $b > 0$.

For $A \subset \mathbb{R}$ write $-A = \{-a : a \in A\}$ and $A^{-1} = \{a^{-1} : a \in A\}$ assuming that $0 \notin A$.

Lemma B.1. *If the function f satisfies Hypothesis 1, then so does the function $g(x) := f(\frac{1}{x})$ for $x \in (0, \infty)$, with the same constants $C_j(a)$.*

Proof. Recall the notation in Sections 2.2 and 4.2. Clearly $\operatorname{supp}(g) = \operatorname{supp}(f)^{-1}$ and $D(M_f) = -D(M_g)$. As in the proof of [7, Lemma A.1], one can verify that:

$$F^g(a,x) = 1 - F^f\left(-a, \frac{1}{x}\right) \quad \text{for } (a,x) \in D(M_g) \times \operatorname{supp}(g),$$
$$L^g(a,x) = L^f\left(-a, \frac{1}{x}\right) \quad \text{for } (a,x) \in D(M_g) \times \operatorname{supp}(g),$$
$$H^g(a,p) = \frac{1}{H^f(-a, 1-p)} \quad \text{for } (a,p) \in D(M_g) \times (0,1). \tag{75}$$

Combining the last two equalities gives

$$L^g\big(a, H^g(a,p)\big) = L^f\big(-a, H^f(-a, 1-p)\big) \quad \text{for } (a,p) \in D(M_g) \times (0,1). \tag{76}$$

Recall also the definition of the derivative $\tilde{\partial}$ in (37). We write $\tilde{\partial}_f$ and $\tilde{\partial}_g$ to denote the dependence on the underlying function. Recall that

$$\tilde{\partial}_g^k L^g\big(a, H^g(a,p)\big) = \frac{\partial^k}{\partial a^k} L^g\big(a, H^g(a,p)\big) \quad \text{for all } (a,p) \in D(M_g) \times \operatorname{supp}(g).$$

Applying $\frac{\partial^k}{\partial a^k}$ to equation (76) gives

$$\begin{aligned}\frac{\partial^k}{\partial a^k} L^g\big(a, H^g(a,p)\big) &= (-1)^k \frac{\partial^k}{\partial b^k}\big(L^f\big(b, H^f(b, 1-p)\big)\big)\Big|_{b=-a} \\ &= (-1)^k \tilde{\partial}_f^k L^f\big(-a, H^f(-a, 1-p)\big),\end{aligned}$$

so

$$\tilde{\partial}_g^k L^g\big(a, H^g(a,p)\big) = (-1)^k \tilde{\partial}_f^k L^f\big(-a, H^f(-a, 1-p)\big) \quad \text{for all } (a,p) \in D(M_g) \times (0,1).$$

Making the substitution $x = H^g(a,p)$ and using equation (75), we get

$$\tilde{\partial}_g^k L^g(a,x) = (-1)^k \tilde{\partial}_f^k L^f\left(-a, \frac{1}{x}\right) \quad \text{for all } (a,x) \in D(M_g) \times \operatorname{supp}(g).$$

Taking absolute values and using the fact that $|\log x| = |\log \frac{1}{x}|$ completes the proof. □

Write $C^\infty(A)$ for the set of smooth functions defined on a set A. For a fixed f with non-empty $D(M_f)$ and which is smooth on its open support, define the linear transformations T and S on $C^\infty(D(M_f) \times \operatorname{supp}(f))$ by

$$T(h)(a,x) := \frac{1}{x^a f(x)} \int_0^x h(a,y) y^{a-1} f(y)\, \mathrm{d}y,$$
$$S(h)(a,x) := \frac{\partial h}{\partial a}(a,x) + h(a,x) \log x$$

for $h \in C^\infty(D(M_f) \times \operatorname{supp}(f))$ and $(a,x) \in D(M_f) \times \operatorname{supp}(f)$. Notice that when $h(a,x) = \psi_0^f(a) - \log x$, $T(h) = L^f$. Notice that $\tilde{\partial}$ in (37) is also a linear transformation on $C^\infty(D(M_f) \times \operatorname{supp}(f))$. The following lemma gives a useful recursion for $\tilde{\partial}^k L^f$:

Lemma B.2. *Assume* $f:(0,\infty)\to[0,\infty)$ *has non-empty* $D(M_f)$, *open* $\operatorname{supp}(f)$, *and satisfies* $f\in C^\infty(\operatorname{supp}(f))$. *If* $h\in C^\infty(D(M_f)\times\operatorname{supp}(f))$, *then for* $(a,x)\in D(M_f)\times\operatorname{supp}(f)$, *and*

$$\big(\tilde\partial T(h)\big)(a,x)=T\circ S(h)(a,x)-\left[\left(a+x\frac{f'(x)}{f(x)}\right)L^f(a,x)+\log x\right]T(h)(a,x)+h(a,x)L^f(a,x).$$

Moreover, if there exists an integer $k\geq 1$ *and a constant* $C=C(a_0,a_1)>0$ *such that*

$$\sup_{a\in[a_0,a_1]}\left|\frac{\partial h}{\partial a}(a,x)\right|\leq C\big(1+|\log x|^k\big)\quad \text{for all } x\in\mathbb{R},\tag{77}$$

then

$$\int_0^\infty h(a,y)y^{a-1}f(y)\,\mathrm{d}y\equiv 0\quad\Rightarrow\quad\int_0^\infty S(h)(a,y)y^{a-1}f(y)\,\mathrm{d}y\equiv 0.$$

Proof. A computation yields

$$\begin{aligned}\frac{\partial T(h)}{\partial a}(a,x)&=-\log x\cdot T(h)(a,x)+T\circ S(h)(a,x),\\ \frac{\partial T(h)}{\partial x}(a,x)&=\left(-\frac{a}{x}-\frac{f'(x)}{f(x)}\right)T(h)(a,x)+\frac{h(a,x)}{x},\end{aligned}$$

which gives the first part. For the second part, by Remark 2 in Section 2.2, $|\log X|$ has finite exponential moments. We can therefore exchange the derivative with the integral in the expression

$$\frac{\partial}{\partial a}\int_0^\infty h(a,y)y^{a-1}f(y)\,\mathrm{d}y.$$

□

For $a\in D(M_f)$ and $x>0$, recursively define

$$\begin{aligned}&h_1(a,x):=\psi_0^f(a)-\log x\quad\text{and}\\ &h_n(a,x):=S(h_{n-1})(a,x)\quad\text{for } n\geq 2.\end{aligned}\tag{78}$$

Then $h_n(a,x)$ is an n-th degree polynomial in $\log x$ with coefficients that are smooth in a. Thus, there exist constants $C_n>0$ for $n=1,2,\dots$ such that

$$\sup_{a\in[a_0,a_1]}\left|\frac{\partial h_n}{\partial a}(a,x)\right|\leq C_n\big(1+|\log x|^n\big)\quad\text{for all } x>0.$$

By the second part of Lemma B.2,

$$\int_0^\infty h_n(a,x)y^{a-1}f(y)\,\mathrm{d}y=0\quad\text{for all } n\in\mathbb{N}\text{ and } a\in(a_0,a_1).\tag{79}$$

The functions h_n will serve as a basis generating all functions obtainable from L^f through repeated application of the operation $\tilde\partial$. To proceed we define some algebraic structures.

Given a subset $F\subset C^\infty(D(M_f)\times\operatorname{supp}(f))$, define $\mathcal{A}(F)\subset C^\infty(D(M_f)\times\operatorname{supp}(f))$ to be the algebra generated by F over the ring $C^\infty(D(M_f))$. More specifically, $g\in\mathcal{A}(F)\Leftrightarrow$ there exist $c_1,\dots,c_n\in C^\infty(D(M_f))$ and $g_1,\dots,g_n\in C^\infty(D(M_f)\times\operatorname{supp}(f))$ such that

$$g(a,x)=\sum_{i=1}^n c_i(a)g_i(a,x)\quad\text{for all } (a,x)\in D(M_f)\times\operatorname{supp}(f).$$

Now let

$$r(x):=x\frac{f'(x)}{f(x)}\tag{80}$$

for $x \in \mathrm{supp}(f)$ and put

$$F := \{\log x, T(h_n)(a,x), r(x)T(h_n)(a,x) : n \in \mathbb{N}\}.$$

Define the degree function $\deg : F \to \mathbb{Z}_+$ by

$$\deg(\log x) = 1 \quad \text{and} \quad \deg\big(T(h_n)\big) = \deg\big(rT(h_n)\big) = n \quad \text{for } n \in \mathbb{N}.$$

Extend the degree function to $\mathcal{A}(F)$ by defining

$$\deg(c) := 0, \qquad \deg(g \cdot h) := \deg(g) + \deg(h), \quad \text{and} \quad \deg(g+h) := \max\big(\deg(g), \deg(h)\big)$$

for $c \in C^\infty(D(M_f))$, and non-zero $f, g \in C^\infty(D(M_f) \times \mathrm{supp}(f))$. Note that this turns deg into an algebra homomorphism from $\mathcal{A}(F)$ into $\mathbb{Z}_+$ with the $(+, \max)$ algebra. For $n \in \mathbb{N}$, let $A_n := \{g \in \mathcal{A}(F) : \deg(g) \le n\}$. Note that A_n is linear.

Lemma B.3. *Suppose* $\mathrm{supp}(f)$ *is non-empty and open*, $D(M_f)$ *is non-empty*, f *is smooth on its support*, *and the following two statements hold for every* $n \in N$:

(1) $\widetilde{\partial} r \cdot T(h_n) \in A_{n+1}$, *and*
(2) *there exists some* $C_n \in C(D(M_f))$ *such that*

$$\big(1 \vee |r(x)|\big)\big|T(h_n)(a,x)\big| \le C_n(a)\big(1 + |\log x|^n\big) \quad \textit{for all } (a,x) \in D(M_f) \times \mathrm{supp}(f).$$

Then f satisfies Hypothesis 1.

Proof. We first claim that

$$\text{The operator } \tilde{\partial} \text{ maps } A_n \to A_{n+1} \text{ for all } n \in \mathbb{N}. \tag{81}$$

To see this, notice that $\tilde{\partial}$ satisfies a product rule:

$$\tilde{\partial}(g \cdot h) = (\tilde{\partial} g) \cdot h + g \cdot (\tilde{\partial} h),$$

and it maps $C^\infty(D(M_f)) \to C^\infty(D(M_f))$. Thus, to show (81), it suffices to show $\tilde{\partial}(\log x) \in A_1$ and for all $n \in \mathbb{N}$, $\tilde{\partial}(T(h_n))$ and $\tilde{\partial}(r \cdot T(h_n))$ are in A_{n+1}.

Clearly, $\tilde{\partial}(\log x) = L^f = T(h_1) \in F$ has degree 1 by definition, so it is in A_1. By Lemma B.2,

$$\tilde{\partial}\big(T(h_n)\big) = T(h_{n+1}) - \big[\big(a - r(x)\big)L^f + \log x\big]T(h_n) + h_n \cdot L^f \in A_{n+1} \tag{82}$$

since $T(h_{n+1}) \in A_{n+1}$, $T(h_n) \in A_n$, and $r \cdot L^f$, L^f, $\log x \in A_1$. Additionally, using $r \cdot T(h_{n+1}) \in A_{n+1}$ and $r \cdot T(h_n) \in A_n$ we see that $r \cdot \widetilde{\partial}(T(h_n)) \in A_{n+1}$ as well. By assumption, $\widetilde{\partial}(r) \cdot T(h_n) \in A_{n+1}$, so the product rule implies

$$\widetilde{\partial}\big(r \cdot T(h_n)\big) \in A_{n+1},$$

which completes (81).

Now define

$$\mathcal{B} := \big\{g \in \mathcal{A}(F) : \text{there exists } c \in C\big(D(M_f)\big) \text{ for which } \big|g(a,x)\big| \le c(a)\big(1 + |\log x|^{\deg(g)}\big)\big\}.$$

$\mathcal{B}$ is a sub-algebra of $\mathcal{A}(F)$, which clearly contains $\log x$. By assumption 2. in the statement of the Lemma,

$$F \subset \mathcal{B}, \tag{83}$$

which implies $\mathcal{B} = \mathcal{A}(F)$. Now $L^f \in A_1$ and (81) implies $(\widetilde{\partial})^n L^f \in A_{n+1} \subset \mathcal{B}$ which completes the proof. □

We now prove Theorem 5.

Proof of Theorem 5. By Lemma B.1, it suffices to consider only the functions $f(x) = e^{-bx}$, $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$, and $f(x) = (\frac{x}{1+x})^b$ for $b > 0$. We check that the assumptions of Lemma B.3 are satisfied in these three cases.

First note that since $h_n(a,x)$ is an n-th degree polynomial in $\log x$ with coefficients that are smooth in a, for every $n \in \mathbb{N}$ there exists some $\widetilde{C}_n \in C(D(M_f))$ such that

$$\big|h_n(a,x)\big| \le \widetilde{C}_n(a)\big(1+|\log x|^n\big) \quad \text{for all } (a,x) \in D(M_f) \times \operatorname{supp}(f). \tag{84}$$

Moreover, by (79),

$$\int_0^x h_n(a,y)y^{a-1}f(y)\,\mathrm{d}y = \int_x^\infty h_n(a,y)y^{a-1}f(y)\,\mathrm{d}y \quad \text{for all } x \ge 0. \tag{85}$$

Case 1: $f(x) = e^{-bx}$. Here $\operatorname{supp}(f) = (0,\infty) = D(M_f)$, f is clearly smooth on $(0,\infty)$, and $r(x) = -bx$. Notice that

$$\tilde{\partial} r = -bxL^f = r \cdot L^f$$

lies in A_1 by definition. So clearly the first assumption of Lemma B.3 holds. We now check the second assumption of Lemma B.3 is satisfied. When $0 < x \le 1$, using (84) and $a > 0$,

$$\begin{aligned}\big|T(h_j)(a,x)\big| &\le \frac{e^{bx}}{x^a}\int_0^x \big|h_n(a,y)\big|y^{a-1}e^{-y}\,\mathrm{d}y \le \frac{e^b\widetilde{C}_j(a)}{x^a}\int_0^x \big(1+|\log y|^j\big)y^{a-1}\,\mathrm{d}y \\ &\le C_j(a)\big(1+|\log x|^j\big).\end{aligned}$$

When $x > 1$, using (84), (85), followed by the substitution $y \mapsto \frac{y}{x} - 1$, and finally an application of integration by parts yields

$$\begin{aligned}\big|T(h_j)(a,x)\big| &\le \frac{e^{bx}\widetilde{C}_j(a)}{x^a}\int_x^\infty \big(1+|\log y|^j\big)y^{a-1}e^{-by}\,\mathrm{d}y \\ &= \widetilde{C}_j(a)\int_0^\infty \big(1+\big|\log(y+1)+\log x\big|^j\big)y^{a-1}e^{-bxy}\,\mathrm{d}y \\ &= \frac{\widetilde{C}_j(a)}{bx}\big(1+|\log 2|^j+|\log x|^j\big) + O\left(\frac{1}{x^2}\right) \\ &\le \frac{\tilde{C}_j(a)}{bx}\big(1+|\log x|^j\big),\end{aligned}$$

where we increased $\widetilde{C}_j(a)$ in the last step if necessary. Combining these two bounds yields the desired result, completing the proof for Case 1.

Case 2: $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$. Here $\operatorname{supp}(f) = (0,1)$, $D(M_f) = (0,\infty)$, f is clearly smooth on $(0,1)$, and $r(x) = -(b-1)\frac{x}{1-x}$. To see that the first assumption in Lemma B.3 holds, notice that

$$\tilde{\partial} r = -(b-1)\frac{x}{(1-x)^2}L^f = r \cdot \left(1 + \frac{r}{1-b}\right)L^f.$$

Thus $\widetilde{\partial} r \cdot T(h_j) = (r \cdot L^f) \cdot T(h_j) + \frac{1}{b-1}(r \cdot L^f) \cdot (r \cdot T(h_j)) \in A_{j+1}$ since $r \cdot L^f \in A_1$, and $r \cdot T(h_j) \in A_j$ by definition. We now check the second assumption of Lemma B.3 is satisfied. By (84), we have the bounds

$$\big|h_j(a,y)y^{a-1}f(y)\big| \le \begin{cases} \widetilde{C}_j(a)(1+|\log y|^j)y^{a-1} & \text{if } 0 < y < \frac{1}{2}, \\ \widetilde{C}_j(a)(1-y)^{b-1} & \text{if } \frac{1}{2} \le y < 1. \end{cases}$$

Since $a > 0$, for $0 < x < \frac{1}{2}$,

$$\big|T(h_j)(a,x)\big| \le \frac{2^a C_j(a)}{x^a}\int_0^x \big(1+|\log y|^j\big)y^{a-1}\,\mathrm{d}y \le \tilde{C}_j(a)\big(1+|\log x|^j\big). \tag{86}$$

Similarly, using equation (85), for $\frac{1}{2} \le x < 1$,

$$\big|T(h_j)(a,x)\big| \le \frac{2^a\widetilde{C}_j(a)}{(1-x)^{b-1}}\int_x^1 (1-y)^{b-1}\,\mathrm{d}y \le \widetilde{C}_j(a)(1-x) \tag{87}$$

where we increased $\widetilde{C}_j(a)$ if necessary. Thus, for all $0 < x < 1$, $|r(x)| \le |b-1|\frac{1}{1-x}$ implies

$$\left|T(h_j)(a,x)\right| \vee \left|r(x)T(h_j)(a,x)\right| \le C_j(a)\big(1+|\log x|^j\big).$$

This completes the proof for case 2.

Case 3: $f(x) = (\frac{x}{1+x})^b$. Here $\mathrm{supp}(f) = (0,\infty)$, $D(M_f) = (-b,0)$, f is clearly smooth on $(0,\infty)$, and $r(x) = b\frac{1}{1+x}$. To see the first assumption of Lemma B.3 is satisfied, notice that

$$\tilde{\partial} r = -b\frac{x}{(1+x)^2}L^f = -r\cdot\left(1-\frac{r}{b}\right)L^f.$$

Thus $\widetilde{\partial} r\cdot T(h_j) = -(r\cdot L^f)\cdot T(h_j) + \frac{1}{b}(r\cdot L^f)\cdot(r\cdot T(h_j)) \in A_{j+1}$ since $r\cdot L^f \in A_1$, and $r\cdot T(h_j)\in A_j$ by definition. We now check the second assumption of Lemma B.3. By (84), we have the bounds

$$\left|h_j(a,y)y^{a-1}f(y)\right| \le \begin{cases}\widetilde{C}_j(a)(1+|\log y|^j)y^{a+b-1} & \text{if } 0<y<1,\\ \widetilde{C}_j(a)y^{a-1} & \text{if } 1\le y<\infty.\end{cases}$$

Since $a+b>0$, for $0<x<1$,

$$\left|T(h_j)(a,x)\right| \le \frac{2^b C_j(a)}{x^{a+b}}\int_0^x \big(1+|\log y|^j\big)y^{a+b-1}\,\mathrm{d}y \le \tilde{C}_j(a)\big(1+|\log x|^j\big). \tag{88}$$

Similarly, using equation (85), for $1\le x<\infty$,

$$\left|T(h_j)(a,x)\right| \le \frac{2^b \widetilde{C}_j(a)}{x^{a}}\int_x^\infty y^{a-1}\,\mathrm{d}y \le \tilde{C}_j(a) \tag{89}$$

where we increased $\tilde{C}_j(a)$ if necessary. Thus, for all $0<x<\infty$, $|r(x)|\le|b|$ implies

$$\left|T(h_j)(a,x)\right| \vee \left|r(x)T(h_j)(a,x)\right| \le C_j(a)\big(1+|\log x|^j\big).$$

This completes the proof for case 3. □

Appendix C: Finite exponential moments for the free energy

Lemma C.1. *Assume the polymer environment is such that* $|\log R^1|$, $|\log R^2|$, $|\log Y^1|$, *and* $|\log Y^2|$ *all have finite exponential moments. Then*,

$$|\log Z_{m,n}| \textit{ has finite exponential moments for all } (m,n)\in Z_+^2.$$

Proof. Since $\log Z_{0,0}=0$, $\log Z_{k,0} = \sum_{i=1}^k R^1_{i,0}$, and $\log Z_{0,k} = \sum_{j=1}^k R^2_{0,j}$, $\log Z_x$ has finite exponential moments for any $x\in Z_+^2\setminus\mathbb{N}^2$. When $x\in\mathbb{N}^2$, the recursion (19) implies that

$$\big(\log Y_x^1+\log Z_{x-\alpha_1}\big)\wedge\big(\log Y_x^2+\log Z_{x-\alpha_2}\big)\le \log Z_x-\log 2\le \big(\log Y_x^1+\log Z_{x-\alpha_1}\big)\vee\big(\log Y_x^2+\log Z_{x-\alpha_2}\big).$$

Thus

$$|\log Z_x-\log 2|\le\left|\log Y_x^1+\log Z_{x-\alpha_1}\right|\vee\left|\log Y_x^2+\log Z_{x-\alpha_2}\right|.$$

Since $|\log Y_x^1|$ and $|\log Y_x^2|$ have finite exponential moments, and inductive argument finishes the proof. □

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References

[1] M. Balázs, E. Cator and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.* **11** (2006) 1094–1132. MR2268539 https://doi.org/10.1214/EJP.v11-366

[2] M. Balázs, J. Komjáthy and T. Seppäläinen. Microscopic concavity and fluctuation bounds in a class of deposition processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (1) (2012) 151–187. MR2919202 https://doi.org/10.1214/11-AIHP415

[3] M. Balázs, J. Quastel and T. Seppäläinen. Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.* **24** (3) (2011) 683–708. MR2784327 https://doi.org/10.1090/S0894-0347-2011-00692-9

[4] M. Balázs and T. Seppäläinen. Order of current variance and diffusivity in the asymmetric simple exclusion process. *Ann. of Math.* **171** (2) (2010) 1237–1265. MR2630064 https://doi.org/10.4007/annals.2010.171.1237

[5] G. Barraquand and I. Corwin. Random-walk in beta-distributed random environment. *Probab. Theory Related Fields* **167** (3–4) (2017) 1057–1116. MR3627433 https://doi.org/10.1007/s00440-016-0699-z

[6] A. Borodin, I. Corwin and P. Ferrari. Free energy fluctuations for directed polymers in random media in $1+1$ dimension. *Comm. Pure Appl. Math.* **67** (7) (2014) 1129–1214. MR3207195 https://doi.org/10.1002/cpa.21520

[7] H. Chaumont and C. Noack. Fluctuation exponents for stationary exactly solvable lattice polymer models via a Mellin transform framework. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (2018) 509–547. MR3800484 https://doi.org/10.30757/alea.v15-21

[8] I. Corwin. The Kardar–Parisi–Zhang equation and its universality class. *Random Matrices Theory Appl.* **1** (01) (2012). MR2930377 https://doi.org/10.1142/S2010326311300014

[9] I. Corwin and M. Nica. Intermediate disorder limits for multi-layer semi-discrete directed polymers. *Electron. J. Probab.* **22** (2017) 287–322. MR3613706 https://doi.org/10.1214/17-EJP32

[10] I. Corwin, T. Seppäläinen and H. Shen. The strict-weak lattice polymer. *J. Stat. Phys.* **160** (4) (2015) 1027–1053. MR3373650 https://doi.org/10.1007/s10955-015-1267-0

[11] C. Noack and P. Sosoe Central moments of the free energy of the O'Connell–Yor polymer. Preprint, 2020.

[12] N. O'Connell and J. Ortmann. Tracy–Widom asymptotics for a random polymer model with gamma-distributed weights. *Electron. J. Probab.* **20** (18) (2015) 1–18. MR3325095 https://doi.org/10.1214/EJP.v20-3787

[13] N. O'Connell and M. Yor. Brownian analogues of Burke's theorem. *Stochastic Process. Appl.* **96** (2) (2001) 285–304. MR1865759 https://doi.org/10.1016/S0304-4149(01)00119-3

[14] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.* **40** (1) (2009) 19–73. MR2917766 https://doi.org/10.1214/10-AOP617

[15] T. Thiery and P. Le Doussal. On integrable directed polymer models on the square lattice. *J. Phys. A* **48** (**46**) (2015). MR3418005 https://doi.org/10.1088/1751-8113/48/46/465001

[16] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals*. Oxford University Press, London, 1948. MR0942661

[17] Wikipedia contributors. Cumulant. Wikipedia, The Free Encyclopedia, 5 December 2020, 06:57 UTC. Available at https://en.wikipedia.org/w/index.php?title=Cumulant&oldid=992433669.