A Unified Model of Arithmetic with Whole Numbers, Fractions, and Decimals

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This study was not preregistered. Simulation code, output, and analysis scripts are available at https://github.com/baixiwei/UMA_PR02.git.

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Abstract

This article describes UMA (Unified Model of Arithmetic), a theory of children's arithmetic implemented as a computational model. UMA builds on FARRA (Braithwaite, Pyke, & Siegler, 2017), a model of children's fraction arithmetic. Whereas FARRA—like all previous models of arithmetic—focused on arithmetic with only one type of number, UMA simulates arithmetic with whole numbers, fractions, and decimals. The model was trained on arithmetic problems from the first to sixth grade volumes of a math textbook series; its performance on tests administered at the end of each grade was compared to the performance of children in prior empirical research. In whole number arithmetic (Study 1), fraction arithmetic (Study 2), and decimal arithmetic (Study 3), UMA displayed types of errors, effects of problem features on error rates, and individual differences in strategy use that resembled those documented in previous studies of children. Further, UMA generated correlations between individual differences in basic and advanced arithmetic skills similar to those observed in longitudinal studies of arithmetic development (Study 4). The results support UMA's main theoretical assumptions regarding arithmetic development: (1) most errors reflect small deviations from standard procedures via two mechanisms, overgeneralization and omission; (2) between-problem variations in error rates reflect effects of intrinsic difficulty and differential amounts of practice; and (3) individual differences in strategy use reflect underlying variation in parameters governing learning and decision making.

Keywords: Numeric development; Arithmetic; Strategy choice; Cognitive model; Individual differences

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Arithmetic—combining numbers using addition, subtraction, multiplication, and division—is among the most critical math skills that children learn. It is a foundation for more advanced skills. Algebra, which is viewed as a gatekeeper for success in secondary and higher education, would be impossible to learn without knowledge of arithmetic. Indeed, algebra errors often reflect poor understanding of arithmetic (Booth et al., 2014; Herscovics & Linchevski, 1994; Linchevski & Livneh, 1999). Arithmetic also has practical value outside of formal education. In a nationally representative sample of over 2000 working adults in the US, 86% reported using addition and/or subtraction, and 78% reported using multiplication and/or division, in their jobs (Handel, 2016).

Children's acquisition of arithmetic is protracted and complex. Usually beginning before school entry, children learn to add and subtract whole numbers using varied strategies, including counting and fact retrieval. These skills continue to develop in early primary education, during which children also learn symbolic algorithms for calculating with multidigit whole numbers and single-digit multiplication and division. In fourth to sixth grade, children are taught algorithms for decimal arithmetic, which are based on multidigit whole number algorithms but involve new procedures for dealing with decimal points. Over the same period, children are also taught fraction arithmetic algorithms, which are entirely different from those used for decimal arithmetic despite the conceptual similarity between fractions and decimals. Children also learn multi-digit multiplication and division during this period. The variety of methods used for arithmetic with different types of numbers suggests that arithmetic is not one skill, but many.

Reflecting this complexity, existing cognitive process models of arithmetic focus on arithmetic with only one type of number, such as single-digit whole numbers (Aubin et al., 2017;

Campbell & Graham, 1985; Shrager & Siegler, 1998; Verguts & Fias, 2005), multidigit whole numbers (J. S. Brown & VanLehn, 1980), decimals (Hiebert & Wearne, 1985), or fractions (Braithwaite et al., 2017). The difficulty of modeling even one type of arithmetic may have discouraged efforts to account for multiple types of arithmetic within a single model. However, the limited scope of existing models leaves unclear whether their theoretical assumptions can account for arithmetic performance outside the domains for which the models were designed, as well as rendering impossible tests of whether the models explain relations that exist at the level of individual differences among arithmetic skills with different types of numbers.

To address these challenges, we created UMA (Unified Model of Arithmetic). UMA is a theory of children's arithmetic, implemented as a computational model. UMA builds on FARRA (Fraction Arithmetic Reflects Rules and Associations), a model of children's fraction arithmetic (Braithwaite et al., 2017). Unlike FARRA, UMA accounts for arithmetic phenomena involving different types of numbers, including whole numbers, fractions, and decimals, as well as relations among individual differences in these different arithmetic skills.

Below, we describe UMA's theoretical assumptions, organizing our description around three fundamental questions about arithmetic development: (1) Where do incorrect answers come from? (2) Why are errors more common on some problems than others? (3) What causes individual differences in strategy use? Next, we explain how these assumptions were implemented as a computational model. Then, four studies are presented that evaluate the model by comparing its performance to that of children. Studies 1, 2, and 3 evaluated the model on arithmetic with whole numbers, fractions, and decimals respectively. Study 4 investigated the model's predictions regarding relations between basic and advanced arithmetic skills.

UMA's Theoretical Assumptions

Where Do Incorrect Answers Come From?

UMA is based on the assumption that most incorrect answers reflect relatively small deviations from standard correct procedures. This assumption is shared with many previous models, including models of whole number arithmetic (J. S. Brown & VanLehn, 1980), fraction arithmetic (Braithwaite et al., 2017), and decimal arithmetic (Hiebert & Wearne, 1985). The assumption implies that standard correct procedures are a good starting point for understanding incorrect performance.

Although many idiosyncratic deviations from correct procedures can and do occur, UMA posits that most errors result from two types of deviation: *overgeneralization* and *omission*. Overgeneralization refers to responses that are inappropriate for the problem at hand but would be appropriate for a different problem. Omission refers to omitting one or more parts of an otherwise correct procedure. Examples of overgeneralization are shown in Table 1; examples of omission are shown in Table 2. These and other specific arithmetic errors are discussed in more detail in Studies 1, 2, and 3.

===== Table 1 =====

===== Table 2 =====

Why Are Errors More Common on Some Problems Than Others?

Children err more often on some problems than others. UMA proposes that two factors contribute to such variations in error rates: *intrinsic difficulty* and *amount of practice*.

Intrinsic difficulty refers to sources of difficulty that are inherent to the features of the problem and the methods available to learners for solving the problem. Analysis of these solution methods, and the cognitive processes involved in executing them can reveal sources of intrinsic difficulty (see, e.g., Hiebert & Wearne, 1985). UMA assumes that procedures with more opportunities for error—via overgeneralization and/or omission—have higher intrinsic difficulty. For example, adding fractions is intrinsically harder with unequal than equal denominators (e.g., 3/5+1/4 is harder than 3/5+1/5), because only the former requires conversion to a common denominator, which increases opportunities for error.

Difficulty due to amount of practice depends on the problems that students receive in classrooms, homework, textbooks, and other contexts. UMA assumes that learners associate correct procedures most strongly with types of problems that they encounter frequently, and therefore tend to err more often on types of problems that they encounter rarely. For the types of problems presented in formal education, analysis of textbooks and homework assignments can reveal which types of problems are rare and therefore more likely to elicit errors than their intrinsic difficulty alone would suggest. For example, in math textbooks, fraction multiplication problems rarely involve equal denominators (Braithwaite & Siegler, 2018); despite the fraction multiplication procedure being identical for operands with equal and unequal denominators, children err considerably more often when multiplying with equal than unequal denominators (e.g., they err more on 3/5×1/5 than 3/5×1/4; Siegler & Pyke, 2013). The same types of problems that are scarce in textbooks are also scarce in homework assignments by teachers, which likely reinforces the effects of frequency of different types of textbook problems (Tian et al., 2022).

Many cognitive models of arithmetic assume that problems differ in intrinsic difficulty (e.g., Campbell, 1995; Hiebert & Wearne, 1985; Rickard, 2005; Verguts & Fias, 2005; Widaman et al., 1989). In contrast, few models incorporate effects of amount of practice on difficulty (for exceptions, see Braithwaite et al., 2017 and Siegler, 1988). However, the assumption that children learn from distributional characteristics of the input they receive is consistent with research on learning and development in many other domains, such as research on statistical learning in language development (e.g., Perruchet & Pacton, 2006; Saffran et al., 1996).

What Causes Individual Differences in Strategy Use?

Children differ not only with respect to how quickly or accurately they perform mathematical tasks, but also with respect to the strategies they use. In arithmetic, individual differences in strategy use have been observed with whole numbers (Rhodes et al., 2019; Siegler, 1988a), fractions (Braithwaite et al., 2019), and decimals (Braithwaite et al., 2021). Such differences in strategy use often affect performance more than individual differences in fluency with particular strategies do. For example, a child who retrieves the answer to 9×8 from memory is likely to solve the problem more quickly than a child who uses repeated addition, regardless of how skilled the latter child is with repeated addition.

Variations in strategy use sometimes reflect children having different strategy repertoires. In early arithmetic development, children discover strategies at different times (e.g., Svenson & Sjöberg, 1983). Thus, at a given time, some strategies are known to some but not all children. However, in later arithmetic development, the strategies to which different children have been exposed are likely to be more uniform, due to the influence of formal math instruction.

UMA focuses on arithmetic development from the beginning of formal instruction (first grade) onward. The theory therefore assumes that different children have access to the same set of strategies. Whereas this simplifying assumption is shared with nearly all existing cognitive process models of arithmetic, UMA further assumes that children vary with respect to parameters governing learning and decision making, and that such variation generates distinct patterns of strategy use (see, e.g., Siegler, 1988a). UMA's specific assumptions about these parameters are presented in our detailed description of the model in the next section; its predictions regarding individual differences in strategy use are presented in Studies 1, 2, and 3.

Limits of the Theory

UMA is a unified theory in the sense that it explains arithmetic phenomena with different types of numbers (whole numbers, fractions, and decimals) by appealing to a common set of theoretical assumptions and using the same mechanisms. However, the theory is limited in two ways. First, UMA deals only with symbolic arithmetic calculation. Other arithmetic tasks, such as estimation and solving story problems, are outside the theory's purview. Second, UMA deals only with procedural knowledge, or knowledge of action sequences for solving problems. The theory does not account for how children represent and use conceptual knowledge (knowledge of mathematical categories, relations, and principles; Crooks & Alibali, 2014), such as the associative, commutative, and distributive properties of arithmetic.

There were two reasons for the above limitations. First, symbolic arithmetic calculation is a broader area than might be initially evident, especially when different types of numbers and arithmetic operations are considered. A unified theory of this area would constitute a substantial advance over prior theory and provide a foundation for future theorizing about other aspects of arithmetic. Second, despite the importance of conceptual knowledge, parsimony demands that assumptions about such knowledge be made only when needed to explain the phenomena under investigation. We hypothesized that explicit conceptual knowledge would not be needed to explain the phenomena that UMA was designed to explain—namely, the types of errors that children commit, the relative difficulties of different problems, and the strategy use patterns of different children. The model and simulations described in the following sections constitute a test of that hypothesis.

UMA's Implementation as a Computational Model

To implement the theory, we constructed a computational model, UMA. UMA is a production system in the tradition of ACT-R (Anderson et al., 2004; Lebiere, 1999), although it is not an ACT-R model. This modeling approach was chosen because it offers a natural way to represent arithmetic procedures involving complex sequences of steps with hierarchical goal structures, a type of knowledge that is difficult to represent in some other modeling frameworks.

UMA's architecture is shown in Figure 1. Procedure and answer memory correspond to procedural and declarative long-term memory in a recently proposed "standard model of the mind" (Laird et al., 2017). The workspace and central executive roughly correspond to parts of working memory in the standard model, except that the workspace represents not only internal but also external information storage (e.g., paper). We describe each component in detail below.

===== Figure 1 =====

Procedure Memory

Production Rules

UMA represents arithmetic procedures using production rules. A production rule consists of a condition, which specifies when the rule can be used, and an action, which specifies what to do if the rule is used. Every rule's condition specifies the type of goal (e.g., solve a problem, convert fractions to a common denominator) that the rule is meant to achieve. Thus, the model's current goal constrains which rules can be selected.

Most procedures are represented by multiple production rules. For example, the repeated addition procedure for multiplying whole numbers is represented by three rules: one that initiates the procedure, one that performs a single "step" of repeated addition (executing the procedure usually involves using this rule multiple times), and one that terminates the procedure. Some production rules create goals that are subsequently achieved by other rules. For example, the rule that implements the standard procedure for multiplying fractions creates goals to multiply the operands' numerators and set the answer's numerator equal to their product, and to multiply the operands' denominators and set the answer's denominator equal to that product.

UMA's production rules implement arithmetic procedures that have been described in previous research or curricular materials such as math textbooks. Our goal was to include the most common and broadly applicable procedures, rather than all strategies that children use. For example, the model implements column addition but not a compensation strategy for addition (e.g., solving 38+45 by rounding 38 to 40, calculating 40+45, and subtracting 2 from the result), because the former procedure is more general and, we suspect, more widely used. This decision was based on the hypothesis that focusing on common procedures would allow UMA to generate outcomes resembling those of children.

UMA represents incorrect procedures as well as correct ones. Incorrect procedures are represented by *mal-rules*, which were generated by modifying correct rules via overgeneralization or omission. Both overgeneralization and omission involved deleting part of a correct rule. Overgeneralization involved deleting part of the rule's condition side, resulting in a rule that could overextend the original procedure to situations in which that procedure was not appropriate. Omission involved removing part of the rule's action side, resulting in a rule that would execute some but not all of the original correct procedure.

Studies 1, 2, and 3 provide further detail regarding UMA's production rules, including correct rules and mal-rules for whole number, fraction, and decimal arithmetic.

Feature-Rule Associations

Effects of learning by practicing procedures are represented by weights associating problem features with production rules. Weights are initialized to zero and modified during practice, as described below. Features are properties of arithmetic problems that we assume are salient to nearly all children. These include the arithmetic operation (+, –, ×, ÷) and the type of operands: WW (two whole numbers); WF (a whole number and a fraction or mixed number); MF (a mixed number and a fraction or mixed number); FF (two fractions); WD (a whole number and a decimal); and DD (two decimals). In the case of problems with a fraction or mixed number operand, the set of salient features also includes whether the operands' denominators are equal (ED) or unequal (UD).

Answer Memory

UMA can solve some problems by retrieving answers from memory rather than by executing procedures. This capacity relies on answer memory: a set of weights associating specific problems (e.g., "3+2") with answers (e.g., "5"). Weights are initialized to zero and modified during practice. Children retrieve answers from memory primarily for small whole number arithmetic problems, especially those involving addition or multiplication[1]. Thus, the problems represented in answer memory are the whole number addition and multiplication problems with operands from 1 to 10, and the answers included therein are the numbers from 1 to 100. In principle, UMA could include common fraction and decimal arithmetic problems, such as 1/2+1/2, in its answer memory, but this capacity was not implemented in the present model, because few fraction and decimal arithmetic problems appear sufficiently often for answers to be memorized.

[1] Children retrieve answers for whole number subtraction and division problems considerably less often than for addition and multiplication, and it is not clear that memory retrieval ever becomes the dominant strategy for either subtraction or division (for subtraction, see Barrouillet et al., 2008; for division, see Robinson et al., 2006).

Workspace

UMA's workspace stores information relating to the current problem-solving episode. This storage is temporary, like the contents of working memory. However, the workspace represents not only mental but also written storage, because writing is often required for arithmetic calculation. Thus, unlike working memory, the workspace has no capacity limit.

The primary contents of the workspace are chunks, which represent items such as individual operands, problems, and answers. When presented a problem, UMA adds a chunk representing the problem to its workspace. Production rules can add more chunks, such as sub-problems (e.g., when solving "3/5+1/5," UMA might create a chunk representing "3+1"), and modify existing chunks, such as by adding an answer to a problem (e.g., changing "3/5+1/5" into "3/5+1/5 = 4/5"). The workspace includes a retrieval buffer, which stores the answer most recently retrieved from answer memory (if any exists).

The workspace also includes two control structures, a goal stack and a problem stack. The current goal is the one on top of the goal stack; the current problem is the one on top of the problem stack. These are not necessarily identical, because solving a problem often requires achieving several smaller goals (e.g., solving "3/5+1/4" requires converting the operands to a common denominator, adding the numerators, and passing the common denominator to the answer). The current goal constrains which production rules can be selected, as mentioned above. The current problem affects the model's probabilities of selecting different production rules and retrieving different answers, as described in the next section.

Central Executive

UMA's central executive simulates two cognitive processes: solving arithmetic problems and learning from practice.

Solving Arithmetic Problems

Production Rule Loop. When presented a problem, UMA adds a chunk representing the problem to its workspace, adds a goal to solve the problem to its goal stack, and adds the problem to its problem stack. UMA then identifies all production rules whose conditions are compatible with the contents of its workspace, selects one of those rules, and fires it. Firing a rule modifies the workspace, possibly including the goal stack and problem stack, which usually changes which rules' conditions are met. UMA continues iteratively selecting and firing rules until the problem has been solved.

If only one rule's condition is compatible with UMA's current workspace, UMA selects and fires that rule. If there are multiple candidates—such as a correct rule and one or more mal-rules—UMA selects among them stochastically according to Equations 1 and 2:

$$A(r_j|X) = \sum_{x_i \in X} w_{ij} / \sum_{x_i \in X} 1 \quad (1)$$

$$P(r_j|X) = e^{gA(r_j|X)} / \sum_k e^{gA(r_k|X)} \quad (2)$$

Equation 1 indicates that the activation $A(r_j|X)$ of rule r_j in the context of problem X equals the average of the feature-rule weights w_{ij} connecting the features x_i of problem X with r_j. Equation 2 indicates the probability $P(r_j|X)$ of selecting r_j in the context of problem X is given by a softmax function of the activations of r_j and the other candidate rules r_k, a variant of a choice rule proposed by Luce (1959) that is widely used in cognitive modeling (e.g., Scheibehenne & Pachur, 2015).

The variable *g* (Equation 2) is a free parameter called "decision determinism." If *g* is 0, all candidate rules have equal probability of being selected. Positive values of *g* make UMA prefer rules with higher activations, and this preference becomes stronger as *g* increases. Thus, *g* governs the degree to which rule activations affect choices among candidate rules. Equivalently, because activations depend on past experience solving arithmetic problems (see "Learning from Practice" below), g governs the degree to which such past experience affects present decisions.

To implement the capacity to solve problems by retrieving answers from memory, UMA includes a production rule whose condition tests whether the retrieval buffer (see "Workspace" above) contains a number. If this condition is satisfied, the rule can be selected as described by Equations 1 and 2; if selected and fired, the rule's action sets the current problem's answer to the number in the retrieval buffer. Whether the retrieval buffer contains a number depends on the operation of UMA's answer retrieval mechanism, described below.

Answer Retrieval. When UMA has a goal to solve a problem for which there is a corresponding entry in answer memory, UMA attempts to retrieve an answer. If an answer is retrieved, it is placed in the retrieval buffer, where UMA's production rules can access it as described above. The following process determines whether an answer is retrieved.

First, UMA determines the activation of each answer in answer memory according to Equations 3 and 4:

$$A(a|X) = \sum_Y v_{Ya} S(X,Y) \quad (3)$$

$$S(X,Y) = e^{-cD(X,Y)} \quad (4)$$

The activation of answer a is the sum of weights v_{Ya} associating the problems Y in answer memory with a, multiplied by the similarities of these problems to the current problem X. The similarity between problems X and Y is an exponentially decreasing function of the distance $D(X,Y)$ between them (Kruschke, 1992; Nosofsky, 1986). The rate of decrease is governed by the "specificity" parameter c. $D(X,Y)$ is the square of the number of differences between X and Y. Three differences are possible (different operation, one operand different or both operands different[2]), so $D(X, Y)$ ranges from 0 to 9.

Retrieval succeeds when at least one answer's activation exceeds a threshold. To enable retrieval to sometimes succeed and sometimes fail for the same problem, as in previous models of arithmetic fact retrieval (Shrager & Siegler, 1998; Siegler & Shrager, 1984), each time retrieval is attempted, the threshold is determined randomly from a normal distribution. The mean (*rt_mu*) and standard deviation (*rt_sd*) of this distribution are free parameters of the model. If multiple answers' activations exceed the threshold, the probability of a given answer being selected follows a softmax function of the activations of that answer and all others whose activations exceed the threshold:

$$P(a) = e^{gA(a)} / \sum_b e^{gA(b)} \tag{5}$$

Learning From Practice

When in learning mode, after solving a problem, UMA modifies its procedure memory and answer memory. For each feature of the problem and each production rule that was used while solving it, the corresponding feature-rule association is adjusted based on Equation 6.

$$\Delta w = 1 - err * (0.5 + d) \tag{6}$$

If answer memory includes an association between the problem and the answer that was obtained for it, that association is adjusted based on Equation 7.

$$\Delta v = 1 - err * (0.5 + 0.5 * d) \tag{7}$$

In Equations 6 and 7, err is an error signal indicating whether the model's answer to the problem was correct (0) or incorrect (1), and d is a free parameter termed "error discount," whose value is constrained to [0, 1]. Thus, after correct answers, $\Delta w = \Delta v = 1$; after incorrect

[2] Operands are compared without regard to order. For example, "2+1" is considered to have no operands different from "1+2" and one operand different from "1+3."

answers, Δw ranges from -0.5 to 0.5 (Equation 6) and Δv ranges 0 to 0.5 (Equation 7).

UMA often solves multiple problems during one problem-solving episode. For example, if presented "13/25+12/25," UMA might generate the sub-problem "13+12," leading in turn to sub-problems "1+1" and "3+2." In such cases, for each problem that was solved during the episode, UMA adjusts associations of all features of the problem with the production rules used while solving it[3], according to Equation 6. Similarly, for each problem that was solved such that answer memory includes an association between that problem and the answer that was obtained for it, UMA adjusts that association according to Equation 7. The error signal (err) used for these adjustments is based on the final answer obtained for the main problem.

How the Computational Model Implements UMA's Theoretical Assumptions

Where Do Incorrect Answers Come From?

All of UMA's errors result from overgeneralization and/or omission. These mechanisms can generate errors directly, when mal-rules are selected and fired, and indirectly, when using mal-rules leads to reinforcement of incorrect answers in UMA's answer memory, making those incorrect answers more likely to be retrieved from memory subsequently. UMA's answer retrieval mechanism creates additional opportunities for overgeneralization, in that answers previously generated for one problem may later be retrieved for a different problem, especially if the two are similar (e.g., "3×4" and "3×5").

Because UMA only generates errors by overgeneralization and omission, our hypothesis that these mechanisms can explain most of children's errors can be tested by comparing UMA's errors to those of children. UMA's mal-rules were created manually, a fact that may seem to afford excessive modeler freedom. However, this freedom was constrained by the requirement

[3] "While solving it" means "while the problem was at the top of the problem stack in UMA's workspace."

that mal-rules be created only by overgeneralization or omission. Further, once created, the mal-rules determine—without modeler input—the specific incorrect answers that UMA can generate for infinitely many particular problems. Also, UMA can combine multiple mal-rules, such as an overgeneralized strategy followed by an omission during execution, which can lead to unexpected outcomes. Such complexities limited the possibility of engineering specific outcomes, while avoiding undesired ones, via manual coding of mal-rules. Automatic generation of mal-rules is a goal for the future development of UMA.

Why Are Errors More Common on Some Problems Than Others?

The assumption that problems vary in intrinsic difficulty is implemented via the range of mal-rules in the model. The model can only commit errors on problems for which applicable mal-rules exist, and the larger the number of applicable mal-rules, the more likely the model is to commit an error. Further, incorrect answers that are initially generated via mal-rules may be subsequently retrieved from memory, so problems that elicit more errors when solved using procedures will also tend to elicit more errors when answers are retrieved from memory.

The assumption that practice experience affects problem difficulty is implemented via UMA's rule selection and answer retrieval mechanisms (Equations 2 and 5) operating through activations (Equations 1 and 3) that are derived from practice experience (Equations 6 and 7). These mechanisms imply that UMA is more likely to select correct procedures on types of problems, and to retrieve correct answers for specific problems, that it has practiced frequently in the past. Conversely, UMA is more likely to err on problems that it has encountered rarely.

What Causes Individual Differences in Strategy Use?

UMA assumes that individual differences in strategy use reflect differences in parameters governing learning and decision making. This assumption is implemented and tested by varying

UMA's free parameters and investigating whether this variation yields strategy use patterns consistent with empirical data. In the present study, we focus on effects of varying decision determinism (*g*), error discount (*d*), and mean retrieval threshold (*rt_mu*), as well as a simulation parameter called "initial counting experience" (*ice*), which is described in Study 1. These parameters were chosen because prior research suggested that they correspond to meaningful dimensions of individual differences among children (Braithwaite et al., 2019; Siegler, 1988a). Future research will explore whether the same is true of UMA's other free parameters (*c* and *rt_sd*), which were not varied in the present simulations.

Simulation Method

We evaluated UMA by using it to simulate arithmetic learning and performance from first to sixth grade. First, we created a cohort of simulated students. Then, we trained each simulated student on a learning set consisting of arithmetic problems from the first to sixth grade volumes of a math textbook series. Each simulated student's arithmetic performance was assessed after it completed the learning problems for each grade.

This section provides a general overview of the simulations. Details specific to whole number arithmetic, fraction arithmetic, and decimal arithmetic are presented as Studies 1, 2, and 3, respectively. Relations among different arithmetic skills are analyzed in Study 4. Simulation code, output, and analysis scripts are available at https://github.com/baixiwei/UMA_PR02.git.

Simulated Cohort

Our simulated cohort consisted of 1,000 instances of UMA, one for each combination of 10 values for *decision determinism* (*g*: {.01, .02, .03, .04, .05, .06, .07, .08, .09, .10}), 5 values for *error discount* (*d*: {0.1, 0.3, 0.5, 0.7, 0.9}), 4 values for *mean retrieval threshold* (*rt_mu*: {3, 4, 5, 6}), and 5 values for *initial counting experience* (*ice*: {0, 25, 50, 75, 100}). These varying parameter values were intended to simulate individual differences among children. Fixed values were used for UMA's other free parameters: specificity (*c*: 5) and standard deviation of the retrieval threshold (*rt_sd*: 1).

The above parameter values were selected because they yielded reasonable results in initial tests; the values were not based on theory or prior research. Systematic parameter fitting was not conducted, due to the complexity of the outcomes that were used to evaluate the model. In cognitive modeling, it is common practice to use different model parameters to simulate different empirical datasets. In contrast, we used the same parameters for UMA throughout Studies 1-4, which afforded a particularly stringent test of the model.

Learning Set

The learning set was created by extracting arithmetic problems from the first to sixth grade student textbooks of GO MATH! (Dixon et al., 2015, 2018). This textbook series was chosen because it was used in the schools attended by participants in a prior empirical study (Braithwaite et al., 2021) whose data were used to evaluate UMA’s performance in decimal arithmetic (Study 3), and because its problem distributions closely resembled those in the two other textbook series analyzed by Braithwaite et al. (2017). The distribution of fraction and decimal arithmetic problems in math homework assigned by teachers has been found to be similar to the distribution in math textbooks (Tian et al., 2022). Thus, we considered these textbook problems to be a reasonable proxy for children’s arithmetic practice experience.

The learning set consisted of arithmetic problems with two operands, which could be whole numbers, fractions, mixed numbers, or decimals. Such problems were included if they were in symbolic form (not story problems), were not solved or partially solved in the textbook, and requested an exact (not estimated) answer in open-response (not multiple choice) format. All problems meeting these criteria were included except for division problems involving whole numbers or decimals, which were excluded because the current version of UMA does not implement long division. Fraction division problems were included because for fractions, unlike whole numbers and decimals, the standard division procedure is quite similar to, and easily confused with, the procedures for other arithmetic operations.

The resulting learning set included 3,222 arithmetic problems. The types of problems included in the set, and the number of problems of each type, are shown in Table 3. Problems appeared in the learning set in the same order as in the textbook.

===== Table 3 =====

Assessment

End-of-year assessments were created to assess UMA's learning after each grade. Whole number addition problems were included in every grade's assessment. Whole number multiplication problems were included in the assessments starting from grade 3, which was the earliest grade in which whole number multiplication problems appeared in the learning set. Fraction and decimal arithmetic problems were included in the assessment for grade 6, which was the earliest grade included in the empirical data to which we compared our simulation results for fraction and decimal arithmetic. Further details regarding the assessment problems are provided in Studies 1, 2, and 3.

Simulation Procedure

Each simulated student first solved the problems from the grade 1 textbook in learning mode, meaning that after solving each problem, the model adjusted its procedure and answer memories according to Equation 6. After solving the grade 1 textbook problems, the simulated students completed the grade 1 assessment with learning mode off, meaning that the model did not modify its memory during the test. Then, the simulated students followed the same procedure to complete the learning problems and assessment for each subsequent grade.

Except where otherwise noted, UMA was prevented from using a given procedure until the first problem on which the procedure was appropriate appeared in the learning set. For example, the model could not access the invert-and-multiply procedure for fractions until it encountered the first fraction division problem. This prevented the model from displaying unrealistic behaviors such as using the invert-and-multiply procedure to solve fraction addition problems before that procedure would have been taught in school.

Evaluation of Simulations

We evaluated UMA by comparing its performance to that of children with respect to types of errors, variables that influenced problem difficulty, and individual differences in strategy use. Overall accuracy was also analyzed to ensure that generating the aforementioned phenomena did not depend on producing unrealistically high or low accuracy.

Study 1: Whole Number Arithmetic

UMA's predecessor FARRA (Braithwaite et al., 2017) was a theory of children's fraction arithmetic, whereas UMA also encompasses whole number arithmetic. Whole number arithmetic differs from fraction arithmetic, in that high frequency repetition of problems with operands between 1 and 9 makes retrieval of answers from memory a common solution strategy in whole number arithmetic, which it is not in fraction arithmetic. For children in the first few grades, counting and repeated addition are also common strategies with whole number arithmetic, which they are not for fraction arithmetic. Therefore, it was far from obvious that a model developed to simulate fraction arithmetic would also effectively simulate whole number arithmetic.

Our evaluation of UMA in this context focuses on addition and multiplication with single-digit whole numbers. The rationale for this focus is that there is extensive empirical research on the types of errors that children commit, and on factors that influence their error rates, for whole number addition and multiplication. In contrast, there is much less research on these topics for whole number subtraction and division.

Below, we briefly review key phenomena in children's whole number addition and multiplication. Then, we assess whether these phenomena are generated by UMA.

Types of Errors

Children's errors on singledigit whole number addition problems are usually close to the correct answer. For example, children are much more likely to claim 2+1 = 4 than 2+1 = 9 (Ashcraft & Fierman, 1982; Muthukrishnan et al., 2019; Siegler & Shrager, 1984). When multiplying single-digit whole numbers, the most common errors are "operand errors"—answers that would be correct for a problem differing by one operand, such as 6×9 = 48 (Buwalda et al., 2016; Campbell & Graham, 1985; Siegler, 1988b). Operand errors are usually close to correct answers in the multiplication table; for example, 6×9 = 48 is more likely than 6×9 = 30.

Influences on Problem Difficulty

Children have greater difficulty with problems involving larger operands, a phenomenon known as the "problem size effect" (Zbrodoff & Logan, 2005). This effect has been observed for both addition (Ashcraft & Guillaume, 2009; Moore & Ashcraft, 2015) and multiplication (Hofman et al., 2018; Mabbott & Bisanz, 2003; van der Ven et al., 2015). Although many factors affect problem difficulty, we focus on problem size, because that effect is large and well documented.

Individual Differences in Strategy Use

Among school age children, the most common strategy for adding and multiplying single-digit whole numbers is retrieval from memory. When unable to retrieve the answer, children use backup strategies such as adding by counting (e.g., solving 5+3 by counting "6, 7, 8"; Siegler, 1987; Svenson & Sjöberg, 1983) and multiplying by repeated addition (e.g., solving 5×3 by saying "5, 10, 15"; Mabbott & Bisanz, 2003; Siegler, 1988b). Use of retrieval increases over development (Ashcraft & Fierman, 1982; Cooney et al., 1988; Koshmider & Ashcraft, 1991; Lemaire & Siegler, 1995; Mabbott & Bisanz, 2003; Siegler, 1987; Svenson & Sjöberg, 1983).

Our analyses of UMA's behavior focused on two aspects of individual differences. First, some individuals shift from backup strategies to memory retrieval more quickly than others (Geary et al., 2004; Hofman et al., 2018; Lemaire & Siegler, 1995; Rhodes et al., 2019). Second, although many students display either high accuracy and high reliance on retrieval or low accuracy and low reliance on retrieval, some students display high accuracy but low reliance on retrieval; Siegler (1988a) dubbed these three types "good students," "not-so-good students," and "perfectionists."

How UMA Simulates Whole Number Arithmetic

UMA can add and multiply single-digit whole numbers using retrieval from memory, mathematical principles, or backup strategies. Retrieval occurs as described in the Introduction. Principles include the facts that for any number n, $0+n = n+0 = n$, $n\times0 = 0\times n = 0$, and $n\times1 = 1\times n = n$. Backup strategies include adding by counting and multiplying by repeated addition.

At the beginning of each simulation run, weights in answer memory are initialized to zero, which prevents any answer from achieving sufficient activation to be retrieved. Thus, when presented a problem that cannot be solved by a principle, UMA must use a backup strategy. Generating answers via backup strategies increases these answers' weights in answer memory, eventually making retrieval possible.

UMA uses two backup strategies: adding by counting and multiplying by repeated addition. Both rely on an accumulation procedure, which involves repeatedly increasing a running total, while also incrementing a counter, until the counter reaches a target. To solve 5+3 by counting, UMA would set the total to 5 and the counter to 0, then repeatedly increment the total and the counter until the counter reached 3. (Preschoolers often add by counting from one, but by first grade, the point at which the simulation starts, children more often add by counting on from the larger addend (Siegler, 1987.)) To solve 5×3 by repeated addition, UMA would set the total to 5 and the counter to 1, then repeatedly add 5 to the total and increment the counter until the counter reached 3. Practice with addition by counting yields some transfer to multiplication by repeated addition, because both use the production rules that represent the accumulation procedure.

UMA's implementation of accumulation includes two mal-rules created via omission. One increments the counter without adding to the total; the other adds to the total without incrementing the counter. These mal-rules enable UMA to simulate errors involving counting or adding too few or too many times, such as 5+3 = 7 or 9, or 5×3 = 10 or 20.

UMA generates retrieval errors in two ways that were previously described by Lebiere (1999). First, incorrect answers generated by the above mal-rules leave traces in answer memory, enabling these answers subsequently to be retrieved. Second, answers that were previously generated for one problem can later be retrieved for a different problem—a form of overgeneralization.

To reflect the fact that most children begin first grade with some counting skill, at the start of each simulation, we increased the weights associating the correct accumulation rule (i.e., the one that increments the counter and the total) with the problem features representing "two whole numbers" and "addition." The amount of this increase was a free parameter called "initial counting experience" (*ice*), which assumed the values {0, 25, 50, 75, 100} equally often. This variation was intended to reflect individual differences in children's early counting skill, which have been found to predict differences in arithmetic development (Bartelet et al., 2014; Moore & Ashcraft, 2015).

Assessment Problems

Following the precedent of other models of small whole number arithmetic (Campbell, 1995; Shrager & Siegler, 1998; Verguts & Fias, 2005), the problems used to assess UMA's whole number arithmetic were those that cannot be solved by principles—that is, the 81 addition problems with operands from 1-9 and the 64 multiplication problems with operands from 2-9.

Simulation Results

Accuracy

Accuracy increased with grade for both whole number addition (grade 1: 92.5%, grade 6: 98.3%) and multiplication (grade 3: 89.8%, grade 6: 94.2%; Figure 2A). This increase mainly reflected increased use of retrieval (Figure 2B), which was associated with higher accuracy than using accumulation (i.e., counting or repeated addition). Increased use of retrieval, in turn, reflected effects of practice, in that the more times UMA solved a problem, the more likely that an answer for that problem would reach the retrieval threshold on subsequent trials.

===== Figure 2 =====

Types of Errors

Across grades, UMA erred on 2.9% of whole number addition problems. These errors mainly reflected overcounting, undercounting, or retrieval of incorrect answers previously generated in these ways. Because UMA is less likely to overcount or undercount many times than to do so few times, the model's incorrect answers—like those of children—were usually close to the correct answer (Figure 3A). Incorrect answers that were within 1, 2, and 3 of the correct answer were 57.4%, 79.5%, and 88.7%, respectively, of total incorrect answers.

UMA erred on 7.3% of multiplication problems across grades. As with children, most errors (59.0%) were operand errors—answers that would be correct for a problem differing by one operand. For example, the five most common errors generated by UMA for 6×9 were (in order) 45, 36, 72, 63, and 48. For comparison, the five most common errors on this problem among 5- to 13-year-olds in Buwalda et al. (2016) were (in order) 45, 56, 63, 36, and 53. All of UMA's top 5 errors appeared in children's data and vice versa.

UMA generates operand errors in multiplication by adding too many or too few times when doing repeated addition or by retrieving answers previously generated in these ways. Because UMA is more likely to make few than many mistakes when repeatedly adding, its operand errors—like children's—tended to be close to the correct answer (Figure 3B). Among these errors, the operand for which UMA's answer would have been correct was within 1, 2, and 3 of the actual operand on 54.4%, 78.6%, and 88.4% of trials, respectively.

Influences on Problem Difficulty

To assess whether UMA generated problem size effects, accuracy was calculated for each problem across grades and simulated students. Then, separately for addition and multiplication, these accuracies were regressed on the sum of each problem's operands, a measure of problem size (Ashcraft & Fierman, 1982). Both regressions found that accuracies were lower for problems with larger operand sums, $B = -.004$, $t(79) = 9.7$, $p < .001$ for addition and $B = -.006$,

$t(62) = 8.2$, $p < .001$ for multiplication[4]. Thus, UMA displayed problem size effects for both operations.

Problem size effects in UMA mainly reflect variation in intrinsic difficulty, in that problems with larger operands present more opportunities for errors when solved using backup strategies. For example, solving 3+1 by counting from the smaller addend requires only one correct count and therefore presents only one chance to count incorrectly, whereas calculating 3+2 by such counting requires two counts (“4, 5”) and therefore presents two chances to err. Repeatedly adding larger numbers to solve multiplication problems is also more error-prone than repeatedly adding smaller numbers (Siegler, 1988b).

Individual Differences in Strategy Use

Like children, our simulated students shifted from backup strategies to retrieval at different rates. Consistent with UMA’s theoretical assumptions, these differences were related to differences in UMA’s parameter values (Figure 4). For both addition and multiplication, the shift to retrieval occurred more quickly when decision determinism was high (Figure 4A), when mean retrieval threshold was low (Figure 4C), and when initial counting experience was high (Figure 4D). Error discount (Figure 4B) had little effect on the shift to retrieval, except that high error discount slightly slowed the shift to retrieval in later grades for multiplication.

===== Figure 4 =====

The reason why low mean retrieval thresholds increase retrieval frequency is obvious, but the reasons why high decision determinism and high initial counting experience do so are a little more complex. Both of these parameter values tend to strengthen UMA’s preference for the

[4] Analogous effects of operand sum were found in linear mixed models with operand sum, grade, and their interaction as fixed effects and problem as a random effect.

production rules that correctly implement the accumulation procedure[5]. This, in turn, causes UMA to generate correct answers more frequently when using that procedure, which increases the strength of correct answers in memory. Thus, correct answers reach the memory retrieval threshold earlier, enabling a faster shift from counting to retrieval.

Next, to test whether UMA generated individual difference patterns like those observed by Siegler (1988a), we classified our simulated students into four groups based on whether they were above or below average with respect to accuracy and use of retrieval on the whole number addition assessment in first grade, the same grade as Siegler's (1988a) participants. As shown in Table 4, the three groups that were analogous to Siegler's (1988a) types together included 99% of simulated students, with each including at least 12% of simulated students. Simulated students who were above average on both dimensions (Siegler's "good students") had moderate to high decision determinism, low retrieval thresholds, and high initial counting experience; those who were below average on both dimensions (Siegler's "not-so-good students") showed the opposite tendencies. Simulated students with above average accuracy but below average use of retrieval (Siegler's "perfectionists") displayed moderate to high decision determinism accompanied by high retrieval thresholds and low initial counting experience. High retrieval thresholds are consistent with Siegler's (1988a) interpretation of this type of students as "perfectionists," but low initial counting experience suggests an alternate cause for the same behavior pattern.

Discussion

UMA generated a number of phenomena that have been observed in children's addition and multiplication of single-digit whole numbers. The model showed (1) increasing accuracy

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[5] High decision determinism causes UMA more strongly to prefer production rules that have yielded correct answers in the past. High initial counting experience increases the initial activation of the production rules referred to here, making those rules more likely to be used and thereby reinforced.

with grade and mathematical experience, (2) increasing reliance on retrieval with grade and mathematical experience, (3) most incorrect answers for addition problems being close to the correct answer, (4) most incorrect answers for multiplication problems being correct answers for a problem differing by one operand, (5) problem size effects for addition and multiplication, (6) individuals shifting from backup strategies to retrieval at differing rates, and (7) individual differences in accuracy and strategy use analogous to those previously observed among children. Thus, although UMA's theoretical assumptions were originally devised to explain children's fraction arithmetic, they also enabled the model to simulate children's performance and development in whole number arithmetic.

Simulating whole number arithmetic in UMA involved adding a new architectural component that was not included in UMA's predecessor FARRA: answer memory. This new component reflects an observation and a theoretical assumption. The observation is that individual single-digit whole number problems are encountered sufficiently often to allow strong associations between problems and specific answers (Siegler, 1988b; Siegler & Shrager, 1984). The theoretical assumption is that answer retrieval, which is simulated by UMA's answer memory, involves different cognitive mechanisms from strategy choice, which is simulated by procedure memory, as in FARRA. One difference between answer memory and procedure memory is that answer memory can only retrieve answers whose activations reach a threshold, which prevents implausible answers such as 1+1 = 80 from being retrieved. Procedure memory includes no such threshold.

Study 2: Fraction Arithmetic

UMA extends FARRA, whose ability to simulate children's fraction arithmetic was demonstrated by Braithwaite et al. (2017, 2019). It was important to test whether UMA could replicate these previous results for at least two reasons. First, unlike FARRA, UMA generates whole number arithmetic errors, which could affect the model's fraction arithmetic performance. Second, testing UMA on fraction arithmetic within a simulation that also included whole number and decimal arithmetic would reveal whether the same values of UMA's free parameters could yield realistic performance in all three domains.

As with whole number arithmetic, we first review key phenomena in children's fraction arithmetic performance, all of which were previously generated by FARRA. Then, we test whether UMA also generates these phenomena.

Types of Errors

Most of children's fraction arithmetic errors involve using inappropriate strategies; a substantial minority involve incorrect execution of appropriate strategies (G. Brown & Quinn, 2006; Byrnes & Wasik, 1991; Gabriel et al., 2013; Hecht, 1998; Newton et al., 2014; Siegler et al., 2011; Siegler & Pyke, 2013). Examples of common strategy errors are shown in Table 1; examples of common execution errors are shown in Table 2.

Influences on Problem Difficulty

Children are less accurate when dividing fractions than when adding, subtracting, or multiplying them (Siegler et al., 2011; Siegler & Pyke, 2013). When adding or subtracting, children are more accurate on problems involving equal than unequal denominators, whereas the reverse is true for multiplying fractions (Gabriel et al., 2013; Siegler et al., 2011; Siegler & Pyke, 2013).

Individual Differences in Strategy Use

UMA's predecessor, FARRA, predicted that children would display several qualitatively distinct patterns of strategy use in fraction arithmetic (Braithwaite et al., 2019). These individual difference patterns were: (1) “Correct Strategies,” in which a correct strategy is used on most or all problems; (2) “Addition/Subtraction Perseveration,” in which a strategy that would be correct for addition and subtraction is used on most or all problems; (3) “Multiplication Perseveration,” in which a strategy that would be correct for multiplication is used on most or all problems; and (4) “Variable Strategies,” in which multiple strategies are used for each arithmetic operation, with no one of them dominant. The predicted patterns all emerged in children’s data from Siegler and Pyke (2013), and jointly accounted for 90% of children (Braithwaite et al., 2019).

How UMA Simulates Fraction Arithmetic

UMA’s production rules for fraction arithmetic are essentially identical to FARRA’s. We describe them briefly here and refer readers to Braithwaite et al. (2017) for further detail.

UMA implements three correct strategies for fraction arithmetic. The fraction addition/subtraction strategy involves converting the operands to a common denominator if necessary, then performing the specified operation on the numerators and passing the common denominator into the answer. The fraction multiplication strategy involves performing the given operation separately on the operands’ numerators and denominators. The fraction division strategy involves inverting the second operand, changing the operation to multiplication, then performing the operation separately on the operands’ numerators and denominators.

UMA includes two versions of each of the above strategies, the correct version and an overgeneralized version. The correct version can only be used on problems involving the appropriate arithmetic operation, whereas the overgeneralized version can be used on problems involving any arithmetic operation. These overgeneralized strategies enable UMA to generate common strategy errors. For example, UMA generates 3/5+1/4 = 4/9 via the overgeneralized version of the multiplication strategy (perform the operation separately on the numerators and the denominators), and it generates 4/5×3/5 = 12/5 via the overgeneralized version of the addition/subtraction strategy (perform the operation on the numerator and pass through the denominator to the answer).

Errors such as 3/5+1/4 = 4/9 could result from overgeneralization of the procedure for multiplying fractions, but such errors could also result from overgeneralization of whole number knowledge, in that such errors could reflect thinking of a fraction as two independent whole numbers. Consistent with the latter possibility, students often commit such errors before being taught fraction multiplication (Byrnes & Wasik, 1991), a phenomenon sometimes described as "whole number bias" (Ni & Zhou, 2005). To reflect this fact, the strategy of operating separately on the numerators and denominators was made available to UMA starting from the first presentation of any fraction problem, rather than from the first fraction multiplication item.

UMA also includes several execution rules, which represent the individual steps required to carry out strategies. These execution rules include several mal-rules created by omission of steps in a correct rule; the mal-rules led UMA to commit execution errors like those of children. For example, UMA includes a correct rule that achieves the goal of converting the operands to a common denominator by multiplying each operand's numerator and denominator by the other operand's denominator; a corresponding mal-rule omits conversion of the numerators, leading to errors such as 3/5+1/4 = 3/20+1/20 = 4/20. Similarly, the rules for executing the invert-and-multiply procedure for fraction division include a mal-rule that omits the "invert" part of the strategy, leading to errors such as 3/5÷1/4 = 3/5×1/4 = 3/20.

Assessment Problems

Fraction arithmetic learning was assessed using 16 problems, four for each arithmetic operation. The problems for each operation involved the same pairs of operands, two with equal denominators (ED: 3/5 __ 1/5, 4/5 __ 3/5) and two with unequal denominators (UD: 3/5 __ 1/4, 2/3 __ 3/5). These problems were presented to sixth and eighth graders in Siegler and Pyke (2013) and were used previously to assess FARRA (Braithwaite et al., 2017, 2019).

Simulation Results

Accuracy

To evaluate UMA, we compared its performance after completing the learning set to that of children in Siegler and Pyke (2013). Overall, UMA answered 52.6% of test problems correctly, close to the 51.6% answered correctly by children.

Types of Errors

UMA generated children's most common incorrect response for every problem and also many of children's less common incorrect responses. Examples are shown in Table 5. On 72.2% of trials that children answered incorrectly, their answers were among those generated by UMA. Because UMA's errors are all generated by overgeneralization and/or omission, the findings demonstrate that at least 72.2% of children's errors can be generated by these mechanisms. On 77.5% of trials that UMA solved incorrectly, its answers were also advanced by children.

===== Table 5 =====

We also evaluated how well the frequencies of different errors in UMA's data corresponded to their frequencies in children's data. For each incorrect answer that was advanced by children, UMA, or both, we determined the answer's frequencies in children's and UMA's data. Then, we calculated the correlations between these frequencies across all incorrect answers, across incorrect answers that were generated by children, and across incorrect answers that were generated by both children and UMA. These correlations were $r = .83$, .85, and .82 respectively, all ps < .001. Thus, the frequencies of different errors in UMA's data corresponded well to their frequencies in children's data.

Influences on Problem Difficulty

As shown in Figure 5, UMA, like children, was more accurate on fraction addition and subtraction problems with equal than unequal denominators (70.7% vs. 55.3%), despite the learning set containing more of the latter problems (Table 3). Unequal denominator addition and subtraction problems are intrinsically harder for UMA, and presumably for children, because such problems require conversion of the operands to a common denominator, thus presenting an opportunity for error that does not occur with equal denominator addition and subtraction.

===== Figure 5 =====

Also like children, UMA was more accurate on fraction multiplication problems with unequal than equal denominators (66.0% vs. 54.7%). Unlike problems involving addition and subtraction of fractions, denominator equality has no obvious relation to intrinsic difficulty of fraction multiplication problems. With UMA, and presumably with children, this difference in accuracy reflects practice experience. Like many math textbooks (Braithwaite & Siegler, 2018), UMA's learning set contained far more unequal than equal denominator multiplication problems (Table 3); this discrepancy caused UMA to associate the correct procedure more strongly with unequal denominator fraction multiplication problems.

Finally, both children and UMA were less accurate with fraction division than with fraction addition, subtraction, and multiplication. This lower accuracy largely reflects the small number of division problems in the learning set (Table 3), which is typical of math textbooks in the US (Braithwaite et al., 2017; Son & Senk, 2010). Low accuracy on division problems likely also reflects division being presented last among the four arithmetic operations, enabling strategies learned for other operations to interfere with learning of the correct division strategy.

Despite equal denominator division problems being rare in the learning set, UMA—like children—was more accurate on equal than unequal denominator division problems (29.0% vs. 19.6%). This result was previously observed in Braithwaite et al. (2017) and reflects the test set including one equal denominator division problem, 3/5÷1/5, that could be solved correctly by overgeneralizing the multiplication strategy of operating separately on the numerators and denominators (i.e., 3/5÷1/5 = (3÷1)/(5÷5) = 3/1). That problem was solved correctly far more often than the other equal denominator division problem in the test set (4/5÷3/5) by both children (35.8% vs. 20.0%) and UMA (41.6% vs. 16.3%).

Individual Differences in Strategy Use

Children and simulated students were classified as in Braithwaite et al. (2019). First, each solution was coded according to whether it displayed a strategy that would be appropriate for addition and subtraction, multiplication, division, or none of these operations. Children's solutions were coded based on their written work; UMA's solutions were coded based on which strategy the model used to solve each problem. Then, children and simulated students were classified based on their strategy use. Individuals were classified as displaying (1) "Correct Strategies" if they used an appropriate strategy on at least 75% of problems, (2) "Addition/Subtraction Perseveration" if they used a strategy appropriate for addition or subtraction on at least 75% of problems (e.g., all 8 addition and subtraction problems and half of the 8 multiplication and division problems), (3) "Multiplication Perseveration" if they used a strategy appropriate for multiplication on at least 10 of the 16 problems (e.g., all 4 multiplication problems and half of the remaining 12 problems), and (4) "Variable Strategies" if, for at least three arithmetic operations, the individual used more than one strategy across the four problems involving that operation.

Percentages of children and simulated students classified into each group are shown in Table 6. As with children, all four patterns appeared in the simulation data; they jointly accounted for nearly all simulated students. Table 6 also shows average values of UMA's parameters among simulated students classified into each pattern. The Variable Strategies pattern was associated with low decision determinism (as would be expected), and the other three patterns with high decision determinism. Error discount was high in the Correct Strategies group, moderate in the Addition/Subtraction Perseveration group, and low in the Multiplication Perseveration group. Mean retrieval threshold and initial counting experience were similar in all groups.

===== Table 6 =====

Discussion

Although UMA's fraction arithmetic components are essentially the same as FARRA's, UMA—unlike FARRA—commits whole number arithmetic errors. For example, UMA erred on 1.7% of addition problems and 7.2% of multiplication problems on the whole number arithmetic assessment in fourth grade, the grade in which fraction arithmetic problems were first presented to UMA. The presence of whole number arithmetic errors could have affected UMA's learning of fraction arithmetic procedures, leading to results differing from those previously obtained with FARRA. Nevertheless, UMA, like FARRA, generated similar errors with similar frequencies, similar effects of problem features on accuracy, and the same patterns of strategy use as those observed among children.

The Variable Strategies pattern was more common in the simulation than among children (Table 6). A closer match could have been obtained by fitting UMA's parameters to children's data. Indeed, Braithwaite et al. (2019) found a closer match to children's data in simulations conducted using FARRA with different parameter values from those used in the present study. However, in the present study, we did not adjust UMA's parameters to fit children's fraction data, because the simulation results were also to be compared to decimal arithmetic data from a different study and to the whole number data. It was not expected that simulations based on one set of parameters would quantitatively match data from different samples of children in all respects. However, the simulations should produce the qualitative phenomena that have been observed in each domain. Studies 1 and 2 demonstrated this to be the case for whole number and fraction arithmetic; Study 3 evaluated whether it was also the case for decimal arithmetic.

Study 3: Decimal Arithmetic

UMA's decimal arithmetic component represents its largest advance over prior models of arithmetic. Only one model of children's decimal arithmetic has been proposed (Hiebert & Wearne, 1985). Like UMA, that model predicted specific errors and influences on problem difficulty, but UMA went further by simulating learning and problem solving, generating predictions about individual differences, and incorporating decimal arithmetic into a unified model that also includes whole number and fraction arithmetic.

Fraction and decimal arithmetic involve different procedures but present learners with similar challenges, such as the challenge of knowing not only how, but also when, to use each procedure (Lortie-Forgues et al., 2015). Braithwaite, Sprague, and Siegler (2021) therefore hypothesized that children's decimal arithmetic would display phenomena analogous to those observed in fraction arithmetic. This hypothesis was tested in an empirical study of sixth and eighth graders' decimal addition and multiplication. We summarize their findings and related findings below, then evaluate UMA's ability to simulate and explain the findings.

Types of Errors

Braithwaite et al. (2021) predicted that as in fraction arithmetic, most decimal arithmetic errors involve overgeneralization of strategies. As predicted, 72% of children's addition errors involved using strategies that would have been correct for multiplication, and 69% of their multiplication errors involved using strategies that would have been correct for addition. Examples of common strategy errors are shown in Table 1. Similar errors were observed in previous studies (Hiebert & Wearne, 1985; Tian et al., 2021), although these studies did not quantify the prevalence of strategy overgeneralization errors. Execution errors, including algorithmic errors such as the one shown in Table 2, as well as errors in retrieval of single-digit whole number arithmetic facts, accounted for a minority of children's errors.

Influences on Problem Difficulty

As with fraction arithmetic, accuracies on different types of decimal arithmetic problems parallel their frequencies in textbooks. Addition problems in textbooks often involve two decimal operands (DD) and rarely involve a whole number and a decimal operand (WD); in contrast, decimal multiplication problems involve WD more often than DD operand pairs (Tian et al., 2021). Paralleling the textbook frequencies, in Braithwaite et al. (2021) and Tian et al. (2021), children were more accurate on DD than WD addition problems, but the reverse was true for multiplication problems. However, despite textbooks containing more decimal multiplication than addition problems, children were more accurate on decimal addition than decimal multiplication problems (Braithwaite et al., 2021).

Individual Differences in Strategy Use

Braithwaite et al. (2021) reasoned that because decimal arithmetic presents opportunities for strategy overgeneralization analogous to those in fraction arithmetic, strategy use patterns in decimal arithmetic should be analogous to those observed in fraction arithmetic—consistent use of correct strategies, using a single strategy (one suitable either for addition and subtraction or for multiplication) on most or all problems, and variable use of multiple strategies. As predicted, each of these individual difference patterns emerged in children's data; the four patterns jointly described the performance of 97% of children (Braithwaite et al., 2021).

How UMA Simulates Decimal Arithmetic

UMA's production rules for decimal arithmetic begin with standard column arithmetic procedures. In the correct procedure for adding or subtracting[6] decimals, UMA arranges the operands vertically, so that their decimal points align. Then, it adds or subtracts the digits in each column and places the decimal point in the answer at the same location as in the aligned operands. In the correct procedure for multiplying decimals, UMA arranges the operands vertically, so that their rightmost digits align. Then, it multiplies each digit of the second operand by the first operand, adds the results, and places the decimal point in the answer so that the answer has as many decimal digits as the sum of the numbers of decimal digits in the operands.

As with fraction arithmetic, UMA includes mal-rules that represent overgeneralizations of each correct decimal arithmetic strategy. Thus, when adding or subtracting decimals, UMA can align the operands at their rightmost digits as specified by the multiplication strategy, yielding errors such as 0.826+0.12 = 0.838 (Table 1, second to last row) and 0.826+0.12 = 0.00838 (if the model also places the decimal point in the answer as specified by the multiplication strategy). Conversely, when multiplying decimals, UMA can align the operands and place the decimal point in the answer as specified by the addition/subtraction strategy, yielding errors such as 2.4×1.2 = 28.8 (Table 1, last row).

[6] Decimal subtraction was included in the learning set, though not in the test set. Many of the same production rules are used for decimal addition and subtraction, so practice with one operation affects performance on the other.

UMA also includes two mal-rules that omit parts of correct rules. One of these relates to placement of partial products when multiplying multidigit whole numbers or decimals (e.g., in 24×12, the partial products are 24×2 = 48 and 24×1 = 24). In the correct procedure, each partial product after the first is shifted one column left of the previous partial product (Table 1, last row). The mal-rule in question omits this shift, yielding errors such as 2.4×1.2 = 0.72 (Table 2, last row). The other mal-rule involves placement of the decimal point in the answer when multiplying decimals. The correct procedure involves adding the numbers of decimal digits in the operands to determine the location of the decimal point in the product. The mal-rule in question omits this step, instead placing the decimal point by default immediately to the right of the leftmost digit of the answer, yielding errors such as 0.41×0.31 = 1.271 (rather than .1271).

Test Problems

UMA's test set was the 12 decimal arithmetic problems presented to children in Braithwaite et al. (2021): six addition problems (24.45+0.34, 12.3+5.6, 2.46+4.1, 0.826+0.12, 5.61+23, 0.415+52) and six multiplication problems (0.41×0.31, 2.4×1.2, 2.3×0.13, 0.31×2.1, 31×3.2, 14×0.21).

Simulation Results

Accuracy

UMA correctly answered 73.8% of test problems, close to the 64.3% answered correctly by children in Braithwaite et al. (2021).

Types of Errors

UMA generated children's most common error for every problem in the test set, and it generated many of children's less frequent errors as well. Examples are shown in Table 7. Consistent with Braithwaite et al.'s (2021) empirical findings, UMA's most common errors involved strategy overgeneralization. For example, UMA generated 0.826+0.12 = 0.838 or 0.00838 (Table 7, second row) using the overgeneralized version of the multiplication strategy, and it generated 2.4×1.2 = 28.8 using the overgeneralized version of the addition/subtraction strategy (Table 7, fourth row).

===== Table 7 =====

Children's answers were among those generated by UMA on 60.9% of trials that children answered incorrectly. This percentage is therefore a lower bound on the proportion of children's errors that can be generated by overgeneralization and omission. UMA's answers were among those generated by children on 82.2% of trials that UMA solved incorrectly. As for fraction arithmetic, for each incorrect answer that appeared in either children's or UMA's data, we determined the answer's frequencies in both datasets. Then, we calculated correlations between these frequencies across all incorrect answers, incorrect answers that were generated by children, and incorrect answers that were generated by both children and UMA. These correlations were r = .88, .90, and .89 respectively, all ps < .001. Thus, as with fractions, relative frequencies of decimal arithmetic errors committed by children and UMA were closely related.

Influences on Problem Difficulty

Children's and UMA's accuracies on different types of decimal arithmetic problems are shown in Figure 6.

===== Figure 6 =====

Like children, UMA was more accurate on addition problems whose operands were two decimals (DD) rather than a whole number and a decimal (WD; 87.6% vs. 73.7%). Also like children, UMA was more accurate on multiplication problems with WD than DD operands (79.0% vs. 57.4%). These accuracy patterns paralleled the distributions of different types of decimal arithmetic problems in the learning set, which contained more DD than WD addition problems and more WD than DD multiplication problems (Table 3).

Despite the input including more decimal multiplication than decimal addition problems (Table 3), UMA, like children, was more accurate on decimal addition than decimal multiplication (83.0% vs. 64.6%). Decimal multiplication is intrinsically harder than decimal addition, because there are more opportunities for error when multiplying than when adding decimals. Both operations present the opportunity to err by overgeneralizing the procedure that is appropriate for the other operation, but multiplication presents additional opportunities for error that have no analogues with addition. These potential multiplication errors include failing to shift partial products leftward and failing to add the numbers of decimal digits in the operands to determine the placement of the decimal point in the product.

Individual Differences in Strategy Use

In Braithwaite et al. (2021), children's solutions to decimal arithmetic problems were coded as consistent with an addition strategy, a multiplication strategy, both (e.g., if the operands were aligned as in the addition strategy but the decimal point was placed as in the multiplication strategy), or neither. Then, children's patterns of strategy use were categorized into the same four types as with fraction arithmetic in Study 2. Children were categorized as displaying (1) "Correct Strategies" if they used an appropriate strategy on ≥75% of the problems, (2) "Addition/Subtraction Perseveration" if they used a strategy appropriate for addition and subtraction on at least 75% of problems (e.g., all addition problems and half of the multiplication problems), (3) "Multiplication Perseveration" if they used a strategy appropriate for multiplication on at least 75% of problems (e.g., all multiplication problems and half of the addition problems), or (4) "Variable Strategies" if they used both an addition/subtraction strategy and a multiplication strategy at least once on both addition and multiplication problems.

UMA's solutions were classified as employing an addition/subtraction strategy or a multiplication strategy based on which production rule was used. Simulated students' patterns of strategy use were then categorized into the four patterns described above, using the same criteria as for children. As shown in Table 8, UMA generated all four strategy use patterns, and these patterns jointly accounted for all simulated students. As with children, the "Correct Strategies" pattern was most common and "Multiplication Perseveration" the least common in UMA's data.

===== Table 8 =====

Also shown in Table 8 are the average values of UMA's parameters within each group of simulated students. As with fraction arithmetic, decision determinism and error discount varied substantially among groups, whereas mean retrieval threshold and initial counting experience did not. Also as with fraction arithmetic, decision determinism was lower in the Variable Strategies group than in all other groups, and error discount was relatively high in the Correct Strategies group and low in the Addition/Subtraction Perseveration group. In contrast to fraction arithmetic, in decimal arithmetic, the Multiplication Perseveration group was characterized by low (rather than high) decision determinism and by moderate (rather than low) error discount.

Discussion

UMA simulated three aspects of children's decimal arithmetic performance: the most frequent types of errors, effects of problem features on error rates, and individual differences in strategy use. Braithwaite et al. (2021) predicted that in each of these respects, children would display phenomena that had previously been observed in fraction arithmetic, reflecting a hypothesis that fraction and decimal arithmetic involve similar learning mechanisms. Braithwaite et al.'s (2021) empirical findings provided initial evidence for that hypothesis. The present simulations provide further evidence for it by showing that a single computational model can generate these phenomena in both fraction and decimal arithmetic.

In particular, the present findings—together with those of Studies 1 and 2—demonstrate the power of overgeneralization and omission, UMA's error-generating mechanisms, to explain a wide range of children's errors. UMA's production rules for decimal arithmetic contained only four mal-rules—two created by overgeneralization to represent selection of inappropriate strategies and two created by omission to represent incorrect execution of strategies. These mal-rules enabled UMA to generate most of children's incorrect answers, including their most common incorrect answer for every problem in the test set.

The findings also provide insight regarding possible origins of individual differences in strategy use in fraction and decimal arithmetic. In both cases, varying UMA's parameters, especially decision determinism and error discount, caused the model to generate four patterns of strategy use that also appeared in children's data. The Correct Strategies pattern was associated with moderate or high values of these parameters, suggesting that successful learning of fraction and decimal arithmetic requires consistency (represented by decision determinism) and avoiding perseveration on a single strategy, if doing so generates errors (represented by error discount). Learners who do not meet the first criterion (i.e., those with low decision determinism, which causes strategy choices to be relatively random) may instead exhibit the Variable Strategies pattern. Those who do not meet the second criterion (i.e., low error discount) may perseverate on whichever strategy is learned first—typically, the addition/subtraction strategy, resulting in Addition/Subtraction Perseveration. The parameters associated with Multiplication Perseveration were somewhat different in fraction and decimal arithmetic, but this pattern was relatively uncommon in both domains in both children's and UMA's data.

Study 4: Relations Among Basic and Advanced Arithmetic Skills

Longitudinal studies of children have consistently found predictive relations between early individual differences in basic arithmetic skills and later differences in more advanced arithmetic skills. For example, single-digit addition and subtraction fluency in second and third grades predict single-digit multiplication in third and fourth grades (Jordan et al., 2003; Xu et al., 2021). Similarly, single-digit whole number addition fluency in third grade predicts fraction addition and subtraction in fourth grade (Jordan et al., 2013) and fraction arithmetic with all four operations in sixth grade (Hansen et al., 2017).

The above studies also found that controlling for individual differences in domain-general competencies and mathematical knowledge other than arithmetic reduces, but does not eliminate, the predictive relations between basic and more advanced arithmetic skills. For example, in Jordan et al. (2013), third grade whole number addition correlated $r = .305$ with fourth grade fraction arithmetic, implying that without control variables, the former explained 9% of variance in the latter. However, when fourth grade fraction arithmetic was regressed on language, nonverbal reasoning, attention, working memory, reading fluency, and whole number line estimation, adding third grade whole number addition as a predictor only increased the variance explained by the model by 1%.

UMA implies at least two sources of covariation between basic and advanced arithmetic skills. First, advanced skills depend on basic ones. For example, fraction division via the invert-and-multiply procedure requires whole number multiplication, and whole number multiplication initially involves the repeated addition procedure, which depends on whole number addition. Thus, the model's proficiency with basic arithmetic skills should have a causal effect on its learning and performance of more advanced skills. Second, as observed in Studies 1 to 3, variations in UMA's free parameters affect the model's performance with all arithmetic skills. For example, low decision determinism generates poor performance in whole number, fraction, and decimal arithmetic. Thus, consistency in the model's parameters across these domains produces covariation between basic and advanced skills, beyond the covariation produced by causal effects of basic skills on advanced ones.

This analysis suggested two predictions. First, as with children, UMA's early accuracy with basic arithmetic skills should predict later accuracy with more advanced skills. Second, as predictive relations among children are reduced, but not eliminated, by controlling for differences in domain-general competencies and mathematical knowledge other than arithmetic, the analogous predictive relations with UMA should be reduced, but not eliminated, by controlling for differences in the model's free parameters. Confirming these predictions would suggest that UMA is viable not only as a model of individual arithmetic skills, but also as a model of relations among the skills. We report tests of these predictions below.

Analyses

The above predictions were tested by a series of linear regressions, each of which assessed the relation between a predictor that was considered to represent a basic arithmetic skill measured in one grade and an outcome that was considered to represent a more advanced arithmetic skill measured in a later grade. The predictor-outcome pairs for which such analyses were conducted were as follows: whole number addition-whole number multiplication, whole number addition-fraction arithmetic (all four operations), whole number addition-decimal arithmetic (addition and multiplication), whole number multiplication-fraction arithmetic, and whole number multiplication-decimal arithmetic. The two whole number arithmetic measures were based on assessments from the grades in which the largest individual differences in accuracy appeared, namely, grade 1 for addition and grade 3 for multiplication. The measures of fraction and decimal arithmetic came from the grade 6 data.

Simulated students' mean accuracies were calculated for each predictor and outcome. Then, for each predictor-outcome pair, two linear regressions were conducted with the predictor as an independent variable and the outcome as the dependent variable. The first regression included no control variables. The second included linear and quadratic effects of each of UMA's free parameters—decision determinism, error discount, mean retrieval threshold, and initial counting experience—as control variables. Quadratic effects were included to assess potential nonlinear effects of these parameters on outcomes.

Results

Table 9 shows the effect of each arithmetic predictor on each arithmetic outcome in each regression (effects of UMA's parameters are reported in the Online Supplemental Materials).

===== Table 9 =====

As predicted, when no control variables were included, individual differences among simulated students on each arithmetic predictor were related to differences on each arithmetic outcome. Also as predicted, relations between predictors and outcomes were reduced, but not eliminated, by controlling for UMA's free parameters. Grade 1 addition explained 79% of variance in grade 3 multiplication when no controls were included, but controlling for UMA's free parameters reduced the variance uniquely explained by grade 1 addition to 26%. In the other regressions, the arithmetic predictors explained from 28% to 37% of variance in arithmetic outcomes; controlling for UMA's free parameters reduced the variance uniquely explained by the arithmetic predictors to between 4% and 7%.

Discussion

UMA generated positive relations between individual differences in basic and advanced arithmetic skills that were similar to analogous relations in longitudinal studies of children. To our knowledge, UMA is the only cognitive process model of arithmetic that has done so. These findings suggest that cognitive mechanisms like those implemented by UMA could account, at least in part, for the empirically observed relations.

In UMA, covariation between basic and advanced arithmetic skills partially reflects causal effects of basic skills, including effects on performance and on learning. Effects on performance occur when basic arithmetic errors committed while solving advanced arithmetic problems result in incorrect answers. For example, UMA might solve 3/5×1/4 using the appropriate strategy of multiplying the numerators and denominators but incorrectly retrieve 5×4 = 25, yielding 3/5×1/4 = 3/25. Effects on learning occur when such errors occur during practice of advanced skills, thus causing the strategies chosen for the advanced skills—even if correct—to receive less positive reinforcement than they would have received following correct answers.

Controlling for variation in UMA's free parameters greatly reduced the variance in advanced skills that was uniquely explained by basic skills. This suggests that consistent individual parametric variation was a major cause of covariation between basic and advanced skills. This result is analogous[7] to recent empirical findings suggesting that when considering individual differences in children's mathematical development, trait effects—that is, influences on individuals' development that are stable over time—are substantially larger than state effects—that is, effects of knowledge at one time point on knowledge at a later time point (Bailey et al., 2014; Bailey, Duncan, et al., 2017).

[7] This analogy is not exact, because UMA's parameter values need not, in principle, be stable within individuals, although they were so in the present simulations.

All regressions left unexplained a substantial portion (17% to 82%) of the variance in UMA's advanced arithmetic performance. Some of this unexplained variance could reflect effects of UMA's parameters (e.g., interactions) that were not included in the regressions. However, randomness is intrinsic to the model due to its stochastic decision rules, which imply that perfectly predicting future performance is impossible even with perfect knowledge of the model's present state. We suspect the same is true of children.

General Discussion

UMA proved able to account for numerous findings regarding performance, learning, and individual differences in children's whole number, fraction, and decimal arithmetic. The model produced these effects using common mechanisms and common parameter values for all three types of arithmetic. Below, we discuss the present findings with respect to the three aspects of arithmetic learning and performance that UMA was designed to explain: origins of errors, sources of relative difficulty of different problems, and causes of individual differences in strategy use. Then, we discuss two aspects of arithmetic that are not included in the current theory: conceptual knowledge and representational format.

Origins of Errors

What causes children's arithmetic errors? To say that errors result from not knowing how to solve problems begs the question of why children make the particular errors that they do. Random guessing cannot explain why a child might claim that 1+1 = 3 or that 4×5 = 16 but is unlikely to claim that 1+1 = 16 or 4×5 = 3. Such regularities might be explained by assuming that children filter guesses based on plausibility of the answer's magnitude, but that assumption cannot account for the high frequency of implausible answers that are observed in fraction and decimal arithmetic, such as 3/5–1/4 = 2/1 or 0.415+52 = 0.467. Another possibility is that children invent idiosyncratic strategies to solve problems for which they do not know a standard procedure. However, most errors are not idiosyncratic, but rather are concentrated on a few alternatives.

To explain the above phenomena, UMA proposes that children have access to a small set of correct procedures and that most errors reflect small deviations from these procedures via overgeneralization and/or omission. Braithwaite et al. (2017) showed that these two mechanisms can generate most of children's errors in fraction arithmetic. The present simulations replicated those findings (Study 2) and extended them in two major ways. Study 1 showed that omission can generate many common errors in single-digit whole number addition and multiplication—either directly, via iterating too many or too few times when adding by counting or multiplying by repeated addition; or indirectly, by retrieving incorrect answers previously generated in the aforementioned ways. Study 3 showed that overgeneralization and omission also can generate most of children's errors in decimal arithmetic. Thus, overgeneralization and omission provide a unifying explanation for many errors in whole number, fraction, and decimal arithmetic.

The generality of this explanation is limited by the fact that the present simulations did not include addition and multiplication of multidigit whole numbers, or subtraction and division of whole numbers and decimals. We hypothesize that the aforementioned mechanisms can also explain most errors in these areas. For example, failing to borrow when subtracting multidigit whole numbers (Brown & VanLehn, 1980) and failing to write 0 in the quotient when performing long division (Voigt, 1938) both are examples of omission. Future research should test whether overgeneralization and omission enable UMA to generate most of children's errors in these other types of arithmetic.

Our approach to explaining errors does not assume that they necessarily reflect incorrect beliefs about concepts, commonly termed “misconceptions” (e.g., Booth et al., 2014). The findings demonstrate that many phenomena relating to errors—including which errors occur, how the frequencies of errors vary among problems, and how consistently individuals display certain errors—can be explained without referring to specific incorrect beliefs. Postulating misconceptions may be necessary to explain other phenomena, such as how overgeneralizations and omissions are generated, how questions about concepts are answered, and how knowledge is transferred to novel situations. The present findings do not imply that misconceptions do not exist or are not important, but the findings do challenge the assumption that errors—even consistent ones—are necessarily evidence of misconceptions.

The current theory does not specify the determinants of which specific overgeneralizations and omissions occur. This limitation could be addressed by specifying constraints on error generation, which could enable mal-rules to be added to the model automatically rather than manually. One possible constraint is that children avoid errors that violate the internal logic of procedures. For example, the procedure for adding fractions with unequal denominators specifies equalizing the operands’ denominators by converting the operands into fractions with a common denominator. Children might omit conversion of the numerators but be unlikely to omit conversion of the denominators because doing so would preclude equalizing the denominators. The proposal that, even when they err, children respect the internal logic of procedures is analogous to several earlier proposals regarding how children acquire procedural knowledge in math (J. S. Brown & VanLehn, 1980; Ohlsson & Rees, 1991) such as “goal sketches” (Shrager & Siegler, 1998).

Sources of Difficulty

UMA assumes that the difficulty of an arithmetic problem for children depends on both the intrinsic difficulty of the problem and the amount of practice children have received with it and similar problems in the past. Are both of these factors necessary to explain the full range of variations in problem difficulty that have been observed?

This question is challenging because intrinsic difficulty and practice frequency are often confounded, as the following examples illustrate. (1a) Single-digit whole number problems with large operands are intrinsically harder than those with small operands to solve by counting or repeated addition, as argued in Study 1, and (1b) some textbook analyses (e.g., Ashcraft & Christy, 1995) have found fewer "large" than "small" single-digit whole number problems in textbooks. Similarly, (2a) the standard procedure for dividing fractions is intrinsically harder than the one for multiplying fractions—because the former includes all steps of the latter but also requires inverting the second operand, which creates additional opportunities for error—and (2b) US math textbooks contain fewer fraction division than multiplication problems (Son & Senk, 2010). Moreover, (3a) Hiebert and Wearne (1985) argued that the standard procedure for decimal addition with WD operands is intrinsically harder than the one for adding DD operands—because the former includes all steps of the latter but also involves appending a decimal point to the whole number operand before aligning the operands vertically—and (3b) US math textbooks contain fewer decimal addition problems with WD than DD operands (Tian et al., 2021).

However, other cases avoid such confounds, thus revealing clear effects of both intrinsic difficulty and practice frequency on error rates. (1) The fact that children err more often on ED than UD fraction multiplication problems (Siegler et al., 2011; Siegler & Pyke, 2013) cannot be explained in terms of intrinsic difficulty because the standard fraction multiplication procedure is identical for both. As described in Study 2, UMA explains the difference in accuracies on ED and UD multiplication problems by appealing to practice frequency—specifically, the low frequency of ED fraction multiplication problems in US math textbooks. (2) US students' lower accuracy when multiplying than when adding decimals (Braithwaite et al., 2021; Hiebert & Wearne, 1985; Tian et al., 2021) is challenging to explain based on practice frequency, because US math textbooks contain more decimal multiplication than decimal addition problems (Tian et al., 2021). As shown in Study 3 of the present work, differences in intrinsic difficulty—specifically, the larger number of opportunities for error when executing the procedure for decimal multiplication than for addition—result in UMA generating this phenomenon. Thus, both intrinsic difficulty and practice frequency appear necessary to explain observed variation in error rates on different types of arithmetic problems.

What makes one problem intrinsically harder than another? UMA assumes that the intrinsic difficulty of a problem depends on the range of errors afforded by the procedures available to learners for solving the problem. Conceptual difficulty plays no role in this definition and is not represented in the model. Of course, some arithmetic problems are conceptually more difficult than others. For example, division of fractions appears harder to conceptualize than other arithmetic operations with fractions (Ma, 1999), and multiplication of decimals is more difficult to understand than addition of decimals (Liu & Braithwaite, 2022). However, the present results suggest that such conceptual considerations are not necessary to explain many between-problem variations in error rates.

Compared to the concept of intrinsic difficulty, effects of practice frequency may seem more straightforward. However, in UMA, effects of practice frequency are mediated by problem representations. More specifically, effects of practice on the probabilities of retrieving answers from answer memory are mediated by the function that determines similarities among different problems, and effects of practice on the probabilities of selecting procedures in procedure memory are mediated by the set of problem features that are represented. A limitation of the present model is that both the similarity function and the set of problem features were determined by the modeler and were not systematically compared to alternatives. Overcoming this limitation will require further empirical research to determine which features of arithmetic problems children encode and how these features are integrated to determine perceived relations among different problems.

Causes of Individual Differences in Strategy Use

Children display large individual differences in the strategies they use to solve arithmetic problems. Coarse measures of performance, such as accuracy and solution times, can conceal diversity in strategy use (Siegler, 1987). For example, among those who are highly accurate at single-digit addition, some children ("good students") rely mainly on memory retrieval, whereas others ("perfectionists") frequently rely on backup strategies (Siegler, 1988a). Similarly, in both fraction and decimal arithmetic, some children who often err do so by consistently using a single strategy for most or all problems, whereas others do so by using multiple strategies for each type of problem (Braithwaite et al., 2019, 2021).

Systematically varying UMA's free parameters enabled the model to generate all of the strategy use patterns mentioned above. This result is consistent with the theoretical assumption that individual differences in strategy use reflect variations in the parameters governing learning and decision making. However, the result also begs the question of what UMA's free parameters represent. Possible interpretations of each parameter are discussed below.

One parameter, decision determinism, was associated with differences in strategy use for all three types of arithmetic that were simulated. Decision determinism governs the degree to which decisions are determined by activations in memory, similar to how threshold parameters govern the degree to which decisions are determined by evidence in drift diffusion models (e.g., Ratcliff et al., 2016). This parameter could correspond to how deliberately or reflectively children make decisions during problem solving. However, the current model, in which decision determinism is multiplied by memory activations in the model's decision rules (Equations 2 and 5), is nearly mathematically equivalent to an alternative formulation in which decision determinism is applied during learning, as a multiplier to the adjustments of connection weights in memory that occur after receiving feedback (Equations 6 and 7). In this alternative interpretation, decision determinism would be more appropriately named "learning rate" and could be used to model variation in how much different children learn from the same experience. Further research is needed to investigate whether individual differences in children's arithmetic strategy use are best explained in terms of variations in decision making, learning rate, or both.

Two parameters, mean retrieval threshold and initial counting experience, were associated with strategy use differences mainly for whole number arithmetic. Both parameters played a role in generating the strategy patterns observed by Siegler (1988a). Specifically, the "good student" and "not-so-good student" patterns differed with respect to initial counting experience but not in their mean retrieval threshold, whereas the "perfectionist" and "not-so-good student" patterns differed with respect to mean retrieval threshold but not initial counting experience (Table 4). The interpretation of the initial counting experience parameter is straightforward. The retrieval threshold parameter reflects an assumption, shared with early theories of arithmetic (e.g., Siegler & Shrager, 1984) and recent general models of memory (e.g., Zhou et al., 2021), that memory retrieval involves a gated process such that only answers whose activation exceeds a threshold can be retrieved. Siegler (1988a) characterized individual differences in the threshold value in attitudinal terms, that is, "perfectionism." Alternatively, such variations could reflect differences in domain-general processes involved in retrieval from long-term memory.

Finally, one parameter, error discount, was associated with differences in strategy use mainly for arithmetic with fractions and decimals. This parameter reflects a theoretical assumption that children differ with respect to how they process corrective feedback. Specifically, UMA assumes that some children learn to avoid procedures that generate negative feedback, as reflected by high error discount, whereas other children persist in using such procedures despite receiving such feedback, as reflected by low error discount. Such differences in learning could reflect differences in conceptual understanding, in that corrective feedback may have a greater impact on children who are able to understand why a solution is incorrect. Alternatively, different responses to corrective feedback might reflect attitudinal differences; for example, persisting in using an incorrect procedure after receiving corrective feedback could reflect learned helplessness (Dweck & Goetz, 2018), which might be especially likely to occur with difficult topics such as fraction and decimal arithmetic.

Conceptual Knowledge

UMA represents an effort to explain as many phenomena as possible in children's arithmetic without reference to conceptual knowledge. Although this approach was productive, several important phenomena were not addressed in UMA. These include the use of conceptual knowledge by some (though not all) children to solve whole number arithmetic problems via shortcuts (Rasmussen et al., 2003; Robinson et al., 2017; Robinson & Dubé, 2009; Siegler & Araya, 2005; Siegler & Stern, 1998), estimate answers to whole number arithmetic problems (Lemaire & Lecacheur, 2002), and interpret and solve word problems. They also include the fact that arithmetic calculation skill is correlated with conceptual knowledge (Bailey, Hansen, et al., 2017; Booth & Siegler, 2008; Fuchs et al., 2010; Hansen et al., 2015; Linsen et al., 2015; Lortie-Forgues & Siegler, 2017; Rittle-Johnson & Koedinger, 2009) and sometimes improved by conceptual interventions (Booth & Siegler, 2008; Dyson et al., 2018; Fuchs et al., 2013, 2014; Rittle-Johnson & Koedinger, 2009; Siegler & Ramani, 2009).

Precise theoretical explanations of the above phenomena have proven elusive. To our knowledge, the only cognitive process model to have simulated any of them is SCADS* (Siegler & Araya, 2005), which simulated children's discovery of a shortcut for solving arithmetic problems of the form $a+b-b$, namely answering "a" without calculation. Because this strategy depends on the principle that addition and subtraction are inverse operations, SCADS* simulated use of a conceptual shortcut in arithmetic, a phenomenon of type (1) above. However, the model had no explicit knowledge of the principle underlying the shortcut.

Part of the challenge involved in developing such models may be that empirical research regarding conceptual knowledge has yielded few insights about the processes involved in using the knowledge. A better understanding of such processes could lead to better understanding of the roles played by conceptual knowledge in children's arithmetic.

Theories of decision making suggest a framework for investigating these issues. Evans (2019) postulated that decision making involves fast, autonomous Type 1 processes, which are associated with intuition; slower, deliberate Type 2 processes, which are associated with deliberation; and Type 3 processes, which determine whether to accept directly the output of Type 1 processes or, instead, to recruit Type 2 processes to evaluate and possibly modify that output (see also Ackerman & Thompson, 2017; Evans & Stanovich, 2013; Kahneman, 2011; Thompson et al., 2011). Similarly, human arithmetic may involve more or less automatic processes such as procedure execution and fact retrieval, analogous to Type 1 processes; deliberate reasoning based on conceptual knowledge, analogous to Type 2 processes; and metacognitive processes that control and mediate between the first two types of processes, analogous to Type 3 processes (see Crowley et al., 1997 for a similar proposal, though one that did not distinguish between the latter two types of processes).

From the perspective of this framework, UMA models only processes of the first type, whereas use of conceptual knowledge is likely also to involve processes of the latter two types. Consistent with this perspective, Braithwaite and Sprague (2021) recently found that adults' use of conceptual knowledge during fraction and decimal arithmetic calculation was correlated with overt displays of metacognitive processes, such as expressions of doubt. Use of conceptual knowledge was rare during calculation, but was much more common in contexts that called for reasoning, such as when participants were asked to explain their procedures.

The above framework suggests that modeling how children use conceptual knowledge in arithmetic, and how conceptual knowledge interacts with procedural knowledge, may require precise models of conceptual reasoning and metacognitive control processes in this context. Research in psychology has yielded formal models of both reasoning (e.g., Khemlani et al., 2018; Oaksford & Chater, 2020) and metacognition (e.g., Jang et al., 2012) but these models have not been employed to simulate mathematics learning. Similarly, research in math education has yielded extensive empirical findings regarding reasoning (e.g., Attridge & Inglis, 2013; Lithner, 2000; Nunes et al., 2007; Simon, 1996) and metacognition (e.g., Garofalo & Lester, 1985; Mevarech & Kramarski, 2003; Schoenfeld, 1992; Veenman & van Cleef, 2019) in mathematical contexts, but these findings have not been included in formal models. Integrating these two lines of research may be a productive approach towards a formal theory of the use of conceptual knowledge in arithmetic, and other mathematical domains as well.

Representational Format

Solving arithmetic problems often involves generating intermediate information, which children might store in various formats including working memory, fingers (when counting or repeatedly adding), or paper (when using symbolic calculation algorithms). UMA's workspace represents all of these formats and does not distinguish among them. We adopted this abstraction because of our focus on children's strategies and errors, which we expected could be explained without modeling effects of information storage format. However, other important phenomena that were not central to the present research may depend critically on this issue. One example is solution times, which likely depend not only on the strategies children use, but also on how information is stored—for example, counting mentally versus on fingers or on paper. Another example is the consistent relation that exists between individual differences in arithmetic and working memory (e.g., Peng et al., 2016). Extending UMA to model explicitly how information is stored and how storage format affects performance could increase the range of phenomena explained by the theory.

Conclusion

To the extent that UMA constitutes a successful model of arithmetic, its success rests on several aspects of its methodology that may be productive for cognitive modeling more generally. First, UMA models not only accuracies but also specific errors; this approach constrains theorizing regarding the processes that generate both correct and incorrect performance. Second, UMA simulates learning using realistic rather than artificial input; doing so encourages the modeler to take seriously the role of input in explaining the outcomes of learning processes. Third, UMA generates predictions regarding not only aggregate performance but also individual differences; besides providing additional criteria for assessing the model, these predictions increase its relevance for educational practice. Finally, rather than modeling a single skill, UMA models a group of related skills, as advocated by Newell (1973); doing so not only constrains the modeling of the individual skills by requiring their integration within a unified framework, but also enables modeling relations among different skills. Future development of UMA, and of alternatives to it, should strive to encompass an even broader range of competencies, especially by explaining the role of conceptual knowledge and use of different representational formats.

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Tables and Figures

Table 1. Overgeneralization Errors in Whole Number, Fraction, and Decimal Arithmetic.

Error	Error Produced By ...	Doing so Would be Appropriate for ...
	Whole Number Arithmetic	
$3 + 3 = 9$	Retrieving "9" from memory	3×3
$2 \times 3 = 5$	Retrieving "5" from memory	$2 + 3$
	Fraction Arithmetic	
$\frac{3}{5} + \frac{1}{4} = \frac{4}{9}$	Performing operation separately on numerators and denominators	$\frac{3}{5} \times \frac{1}{4}$
$\frac{4}{5} \times \frac{3}{5} = \frac{12}{5}$	Passing common denominator of operands into answer	$\frac{4}{5} + \frac{3}{5}$
	Decimal Arithmetic	
$\begin{array}{r} 0.826 \\ \underline{+0.12} \\ 0.838 \end{array}$	Writing operands so their rightmost digits are aligned before calculating	0.826×0.12
$\begin{array}{r} 2.4 \\ \underline{\times 1.2} \\ 4.8 \\ \underline{24.0} \\ 28.8 \end{array}$	Bringing decimal point down from operands into answer	$2.4 + 1.2$

Table 2. Omission Errors in Whole Number, Fraction, and Decimal Arithmetic.

Error	Description of Procedure From Which a Part Was Omitted	Description of Omission
	Whole Number Arithmetic	
$3 + 3 = ?$ "Four, five. The answer is five."	Count from first addend. On each step, increment [total] and [steps done]. Stop when [steps done] equals second addend.	Incremented [steps done] once without incrementing [total].
$2 \times 3 = ?$ "Two, four, six, eight. The answer is eight."	Do repeated addition. On each step, add first operand to [total] and increment [steps done]. Stop when [steps done] equals second operand.	Incremented [total] once without incrementing [steps done].
	Fraction Arithmetic	
$\frac{3}{5} + \frac{1}{4} = \frac{3}{20} + \frac{1}{20} = \frac{4}{20}$	Multiply numerator and denominator of each operand by denominator of other operand	Failed to multiply numerators of operands
$\frac{3}{5} \div \frac{1}{4} = \frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$	Invert second operand, change division to multiplication, and perform operation separately on numerators and denominators	Failed to invert second operand
	Decimal Arithmetic	
$\begin{array}{r} 2.4 \\ \underline{\times\ 1.2} \\ 48 \\ \underline{24} \\ 0.72 \end{array}$	When beginning to write a new partial product (e.g., 24), shift one column to the left	Failed to shift second partial product (i.e., 24) one column to the left

Table 3. Frequencies of Different Problem Types in the Learning Set.

	Arithmetic Operation			
Operands	Addition	Subtraction	Multiplication	Division
Two whole numbers				
Both ≤10	605	223	286	--
At least one >10	458	591	347	--
At least one fraction/mixed number				
Equal denominators (ED)	32	50	6	6
Unequal denominators (UD)	57	73	145	36
At least one decimal				
Two decimals (DD)	65	53	78	--
One whole, one decimal (WD)	1	3	107	--

Table 4. Classification of Simulated Students Based on Accuracy and Use of Retrieval on the First Grade Whole Number Addition Assessment.

Accuracy ≥ or < Average	Use of Retrieval ≥ or < Average	Analogous Pattern in Siegler (1988a)	% of Simulated Students (*N*=1000)	Mean (*SD*) of *g*	Mean (*SD*) of *rt_mu*	Mean (*SD*) of *ice*
≥ average	≥ average	Good students	67	0.065 (0.024)	4.4 (1.1)	61 (31)
< average	< average	Not-so-good students	20	0.023 (0.019)	4.4 (1.1)	28 (34)
≥ average	< average	Perfectionists	12	0.054 (0.025)	5.5 (0.8)	27 (32)
< average	≥ average	None	1	0.020 (0.009)	3.0 (0.0)	50 (27)

Note. *g* denotes decision determinism, *rt_mu* denotes mean retrieval threshold, and *ice* denotes initial counting experience.

Table 5. Children's Most Common Responses on Four Fraction Arithmetic Problems, and Frequencies of These Responses in UMA's Data.

		Percent of Responses	
Problem	Answer	Children (*N*=120)	UMA (*N*=1000)
2/3+3/5	**19/15**	**51**	**52**
	5/8	23	26
	5/15 or 1/3	8	6
3/5–1/4	**7/20**	**54**	**60**
	2 or 2/1	20	26
	2/20 or 1/10	5	5
	6/15 or 2/5	4	0
4/5×3/5	**12/25**	**40**	**54**
	12/5	37	25
	15/20 or 3/4	4	7
	20/15 or 4/3	3	8
3/5÷1/5	3/5	38	38
	3, 3/1, or 15/5	**36**	**43**
	5/15 or 1/3	6	5
	1/5	5	0

Note. Children's data are from Siegler and Pyke (2013). Answers generated by at least 3% of children are shown. Correct answers are **bolded**.

Table 6. Percentages of Children in Siegler and Pyke (2013) and Simulated Students Classified as Displaying Each Pattern of Fraction Arithmetic Strategy Use, and Mean Values of UMA's Parameters Within Each Pattern.

	Children (*N*=120)	UMA (*N*=1000)		
Pattern	%	%	Mean (*SD*) of *g*	Mean (*SD*) of *d*
Correct Strategies	31	15	0.070 (0.025)	0.69 (0.22)
Addition/Subtraction Perseveration	25	12	0.081 (0.018)	0.42 (0.26)
Multiplication Perseveration	12	8	0.079 (0.019)	0.18 (0.13)
Variable Strategies	22	61	0.042 (0.024)	0.50 (0.28)
None	10	4	0.071 (0.023)	0.57 (0.25)

Note. *g* denotes decision determinism, and *d* denotes error discount.

Table 7. Children's Most Common Responses on Six Decimal Arithmetic Problems, and Frequencies of These Responses in UMA's Data.

		Percent of Responses	
Problem	Answer	Children (*N*=92)	UMA (*N*=1000)
12.3+5.6	**17.9**	**93**	**89**
0.826+0.12	**0.946**	**70**	**86**
	0.838	12	5
	0.00838	4	6
0.415+52	**52.415**	**61**	**76**
	0.467	28	24
	0.935	5	0
2.4×1.2	**2.88**	**63**	**58**
	28.8	14	31
	7.2	8	4
	2.8	5	0
0.32×2.1	**0.672**	**42**	**59**
	6.72	18	28
	67.2	5	2
	0.96	5	2
	0.32	3	0
31×3.2	**99.2**	**61**	**78**
	9.92	10	7
	15.5	5	2
	992	4	4

Note. Children's data are from Braithwaite et al. (2021). Answers generated by at least 3% of children are shown. Correct answers are **bolded**.

Table 8. Percentages of Children in Braithwaite et al. (2021) and Simulated Students Classified as Displaying Each Pattern of Fraction Arithmetic Strategy Use, and Mean Values of UMA's Parameters Within Each Pattern.

	Children (*N*=92)	UMA (*N*=1000)		
Pattern	%	%	Mean (*SD*) of *g*	Mean (*SD*) of *d*
Correct Strategies	47	69	0.058 (0.028)	0.55 (0.27)
Addition/Subtraction Perseveration	25	13	0.071 (0.026)	0.29 (0.24)
Multiplication Perseveration	4	1	0.032 (0.024)	0.54 (0.28)
Variable Strategies	21	16	0.030 (0.020)	0.45 (0.28)
None	3	0	--	--

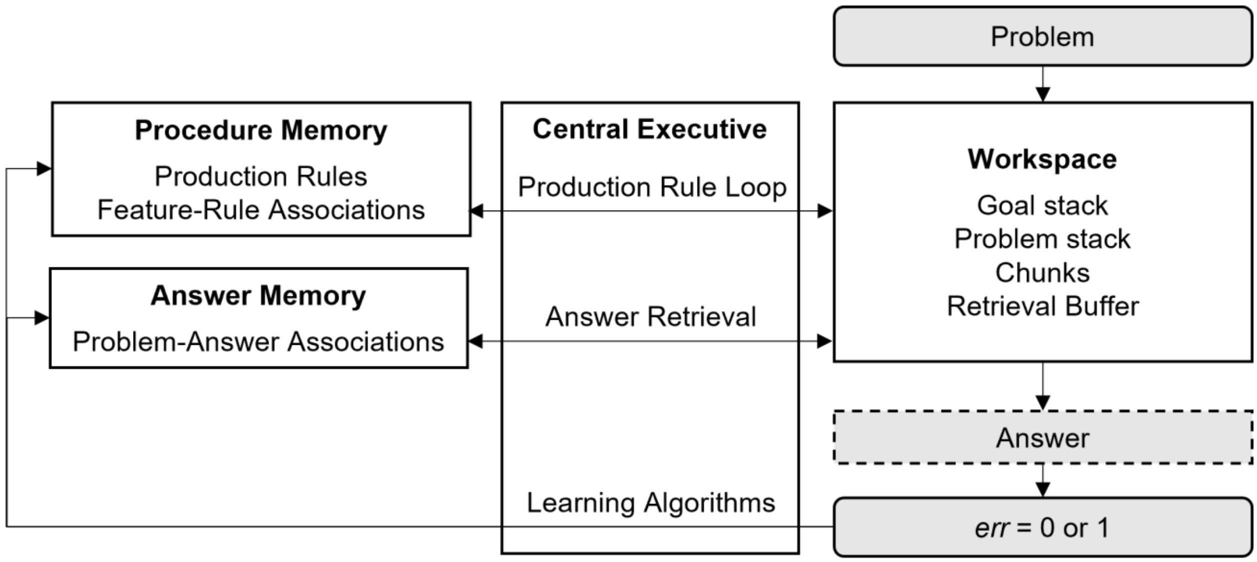
Note. *g* denotes decision determinism, and *d* denotes error discount.

Table 9. Results of Regressing UMA Accuracies on Advanced Arithmetic Skills on Basic Arithmetic Skills Without Control Variables and With Controls for UMA's Free Parameters.

		Without Control Variables				With Controls for UMA's Free Parameters				
Predictor	Outcome	*B*	*β*	*p*	R^2	*B*	*β*	*p*	R^2	ΔR^2
Grade 1 Addition	Grade 3 Multiplication	1.4	0.89	<.001	79%	1.2	0.78	<.001	83%	26%
Grade 1 Addition	Grade 6 Fraction Arithmetic	0.6	0.53	<.001	28%	0.3	0.30	<.001	46%	4%
Grade 1 Addition	Grade 6 Decimal Arithmetic	0.7	0.57	<.001	32%	0.4	0.36	<.001	52%	5%
Grade 3 Multiplication	Grade 6 Fraction Arithmetic	0.4	0.58	<.001	33%	0.3	0.40	<.001	49%	7%
Grade 3 Multiplication	Grade 6 Decimal Arithmetic	0.5	0.61	<.001	37%	0.3	0.39	<.001	53%	7%

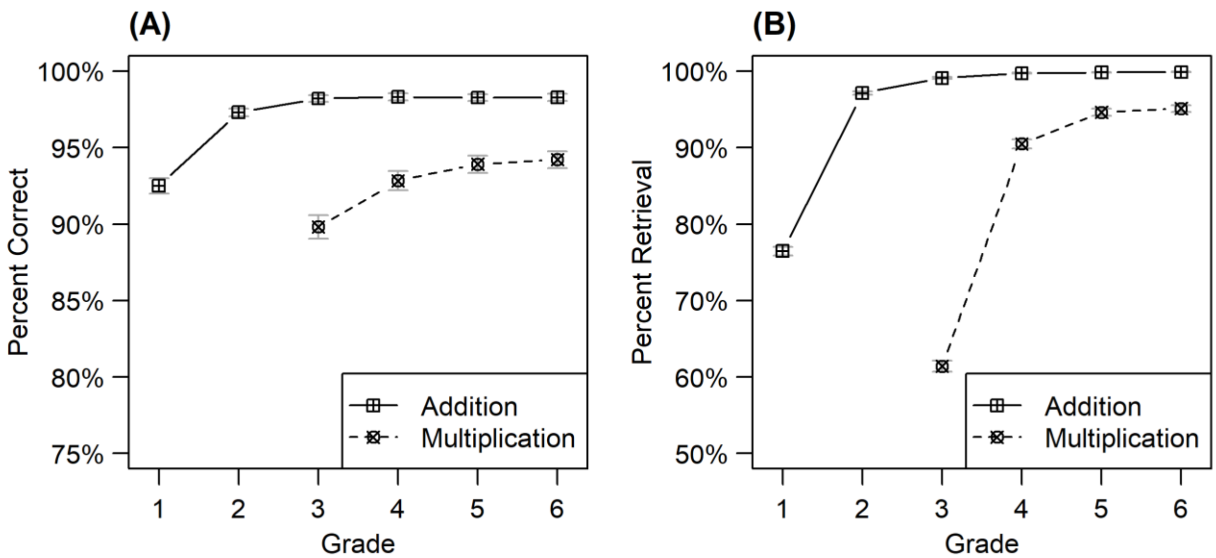
Note. Details regarding analyses are presented in the text. Each predictor and outcome denotes percent correct on the indicated assessment. *B* and *β* denote nonstandardized and standardized effects of the predictors in column 1 on the outcomes in column 2, *p* denotes the *p* values for these effects, and R^2 denotes percent variance in the outcome explained by the model. For the models with controls for UMA's free parameters, ΔR^2 denotes increase in percent variance explained by each model relative to a model that includes only the control variables and not the predictor in column 1—in other words, percent variance uniquely explained by the predictor.

Figure 1. Diagram of UMA.



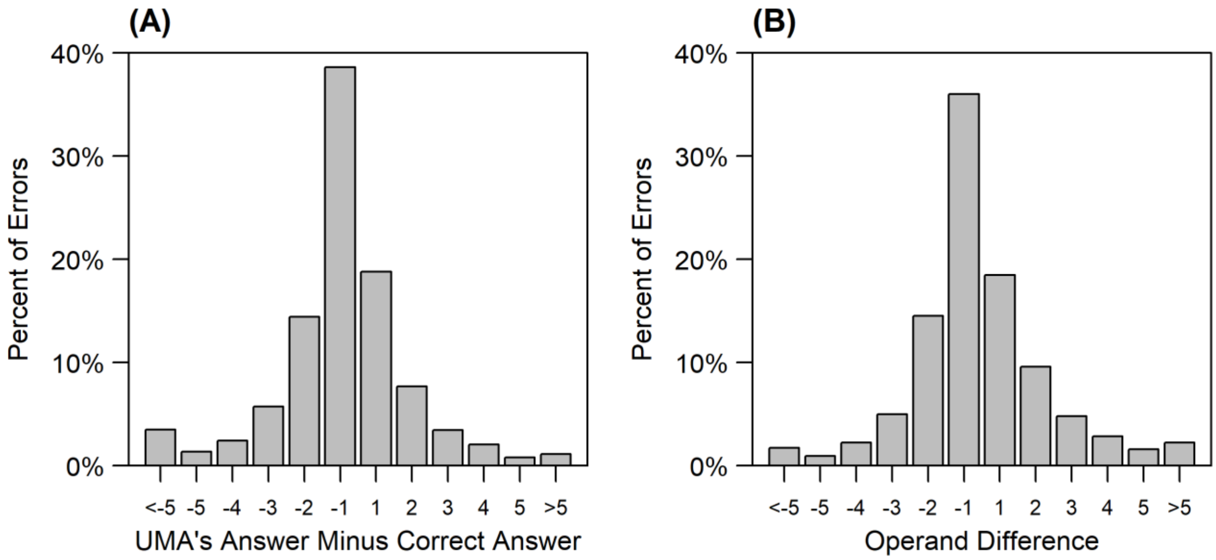
Note. White boxes indicate components of UMA's architecture, grey boxes with solid borders indicate input to the model, and the grey box with dashed border indicates model output.

Figure 2. UMA's Performance on Whole Number Addition and Multiplication by Grade.



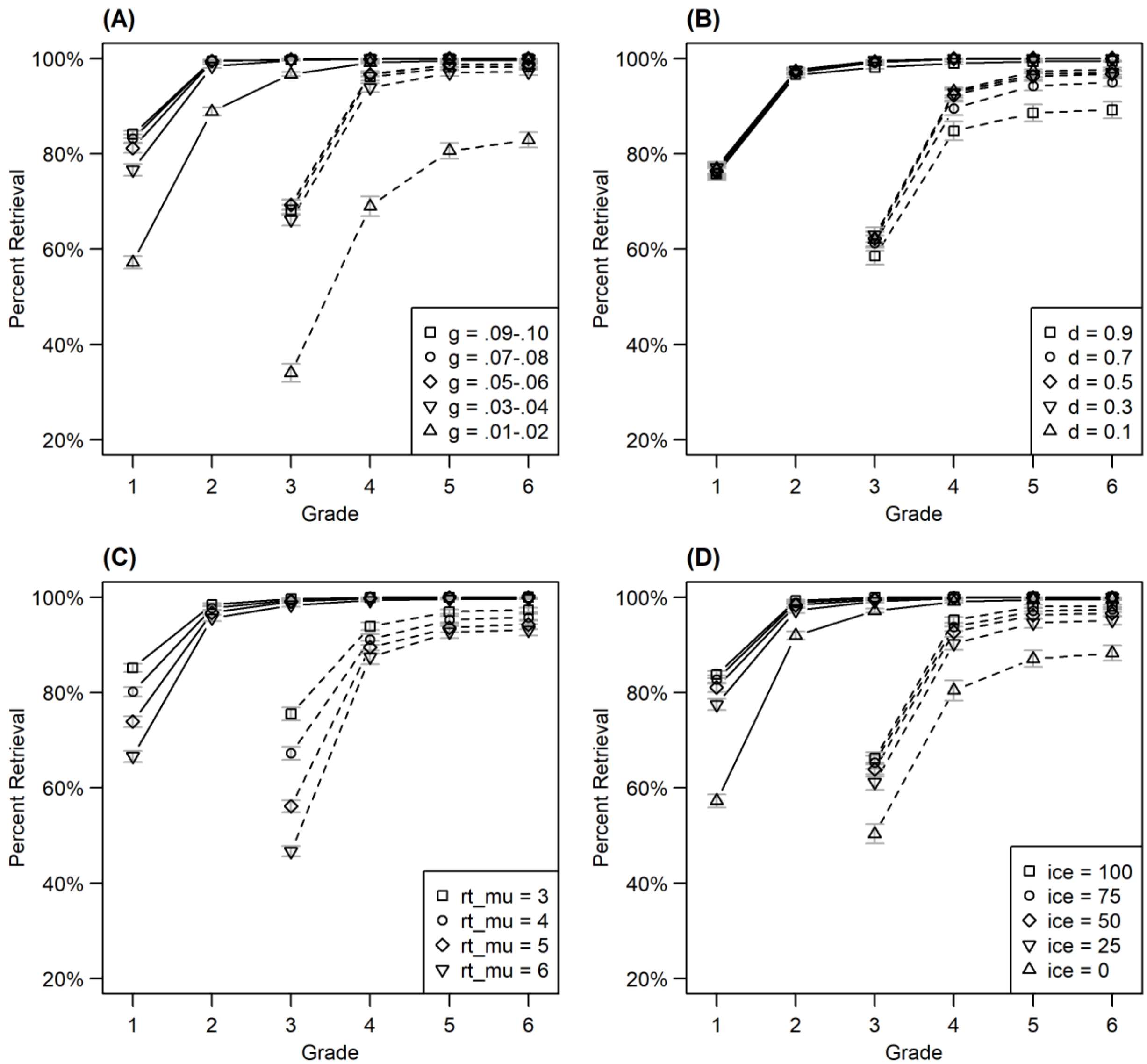
Note. (A) Percent correct; (B) Percent retrieval of answers from answer memory. Error bars indicate standard errors.

Figure 3. Distributions of UMA's Errors in Whole Number Addition and Multiplication.



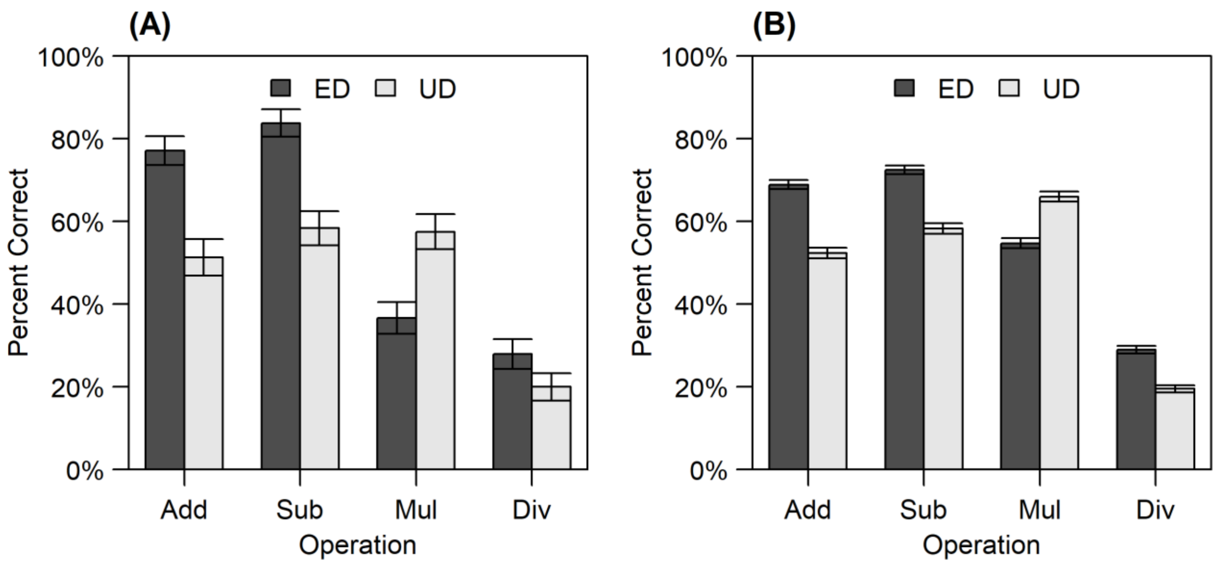
Note. (A) shows addition errors. (B) shows multiplication errors that were classified as operand errors (see text for explanation). "Operand difference" denotes the difference between the operand for which UMA's answer would have been correct and the actual operand. For example, if UMA claimed 3×4 = 6, operand difference would be -2 because 3×2 = 6 and 2–4 = -2.

Figure 4. UMA's Use of Retrieval on Single-digit Whole Number Addition and Multiplication Problems by Grade and Parameter Value.



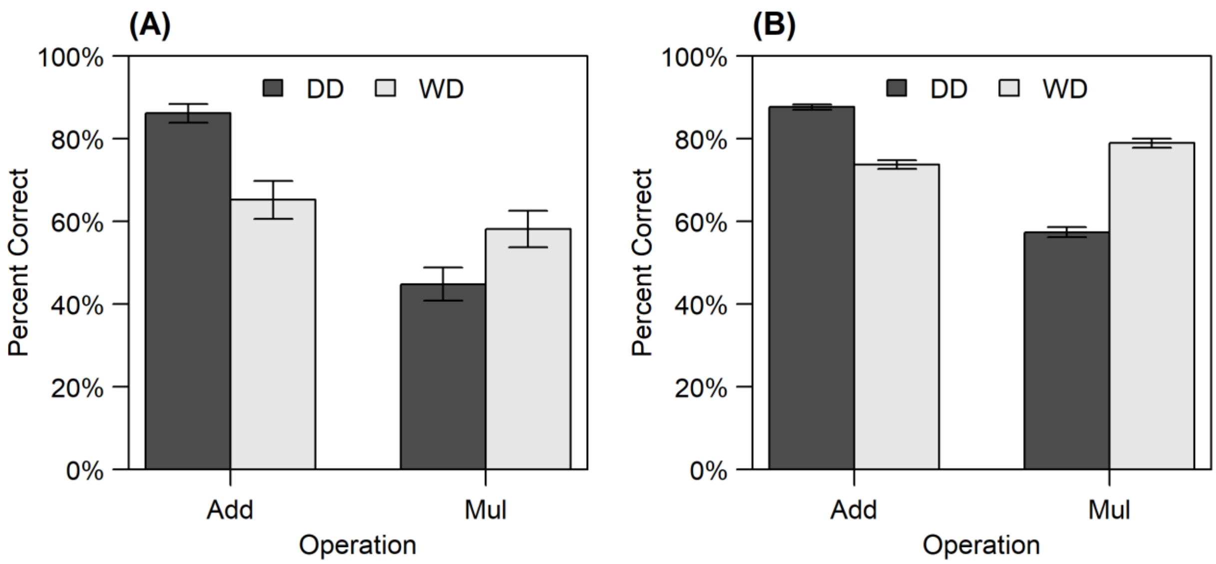
Note. Solid and dashed lines represent addition and multiplication, respectively. Panels (A), (B), (C), and (D) show results for varying values of decision determinism (*g*), error discount (*d*), mean retrieval threshold (*rt_mu*), and initial counting experience (*ice*), respectively. Error bars indicate standard errors.

Figure 5. Fraction Arithmetic Accuracies by Arithmetic Operation and Operand Pair Type.



Note. (A) Children from Siegler and Pyke (2013). (B) UMA's performance on the sixth grade test. Add, Sub, Mul, and Div denote addition, subtraction, multiplication, and division. ED and UD denote "equal denominators" and "unequal denominators." Error bars indicate standard errors.

Figure 6. Decimal Arithmetic Accuracies by Arithmetic Operation and Operand Pair Type.



Note. (A) Children from Braithwaite et al. (2021). (B) UMA's performance on the sixth grade test. Add and Mul denote addition and multiplication. DD and WD denote "two decimal operands" and "one whole number and one decimal operand." Error bars indicate standard errors.