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Propagation of symmetries for Ricci shrinkers

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Abstract: We will show that if a gradient shrinking Ricci soliton has an approximate symmetry on one scale, this symmetry propagates to larger scales. This is an example of the shrinker principle which roughly states that information radiates outwards for shrinking solitons.

Keywords: Ricci flow, solitons

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1 Introduction

A time-varying metric $\mathbf{g}$ on a manifold M is a Ricci flow if

$$\partial_t \mathbf{g} = -2\,\mathrm{Ric}_{\mathbf{g}}. \tag{1.1}$$

This is a nonlinear geometric evolution equation where singularities can form, and understanding them is the key for understanding the flow. For instance, one important ingredient in the rigidity of cylinders for the Ricci flow in [16] was a new estimate called propagation of almost splitting. This showed that if a gradient shrinking Ricci soliton almost splits off a line on one scale, then it also almost splits on a strictly larger set, though with a loss on the estimates.

tA manifold splits off a line if it is the metric product of a Euclidean factor $\mathbf{R}$ with a manifold of one dimension less. A splitting gives a linear function whose gradient is a parallel vector field. Similarly, an almost splitting gives an almost parallel vector field, which is a special type of almost Killing vector field. Thus, the propagation of almost splitting is equivalent to showing that a certain type of approximate symmetry extends to larger scales [16,17]. We will see here that this holds also for more general approximate symmetries.

A triple (M, g, f) of a manifold M, metric g, and function f is a gradient Ricci soliton if there is a constant κ so that

$$\mathrm{Ric} + \mathrm{Hess}_f = \kappa g. \tag{1.2}$$

Up to diffeomorphisms, the Ricci flow of a gradient Ricci soliton evolves by shrinking when $\kappa > 0$ is static (or steady) when $\kappa = 0$, and expanding when $\kappa < 0$. Gradient shrinking Ricci solitons model finite time singularities of the flow and describe the asymptotic structure at minus infinity for ancient flows.

Dedicated to our friend David Jerison.

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A Killing field V is a vector field that generates an isometry, i.e., the Lie derivative of the metric g vanishes

$$\mathbf{L}_V g = 0.$$

Equivalently, V is Killing when its covariant derivative ∇V is skew-symmetric. Many of the most important examples of solitons are highly symmetric; cf. [1,3–5,12,16,20–22].

On Euclidean space, there are two types of Killing fields: the parallel vector fields that generate translations and the linearly growing rotation vector fields. There are two very different types of symmetries on Ricci solitons depending on whether the symmetry preserves the function f. This dichotomy is already seen in the simplest examples of shrinkers: the Gaussian soliton $\left(\mathbf{R}^n, \delta_{ij}, \frac{|x|^2}{4}\right)$ and cylinders. Rotations around the origin on the Gaussian soliton preserve both the metric and $f = \frac{|x|^2}{4}$, while the translations along the axis of a cylinder preserve the metric but not the level sets of f. It is not hard to see that this is typical. To make this precise, use the metric g and function f, to define a weighted L^2 norm on functions, vector fields, and general tensors by

$$\|\cdot\|_{L^2}^2 = \int_M |\cdot|^2 \, \mathrm{e}^{-f}. \tag{1.3}$$

Proposition 1.1. *If Y is an L^2 Killing field on a gradient shrinking Ricci soliton, then either Y preserves f or M splits off a line.*

The metric and weight induce weighted divergence operators div_f on vector fields and symmetric two tensors and a drift Laplacian $\mathcal{L}$ on general tensors. The adjoint div_f^* of div_f takes a vector field V to the symmetric two tensor:

$$\mathrm{div}_f^* V = -\frac{1}{2}\mathbf{L}_V g. \tag{1.4}$$

As in [16], define a self-adjoint operator $\mathcal{P}$ on vector fields by

$$\mathcal{P}Y = \mathrm{div}_f \circ \mathrm{div}_f^* Y. \tag{1.5}$$

For a Killing field V, $\mathrm{div}_f^* V$ vanishes and, thus, $\mathcal{P}V = 0$.

For an *approximate* Killing field, the associated flow *almost* preserves the metric g. As mentioned earlier, approximate translations played an important role in [16] to show "propagation of almost splitting." Our interest here will be a corresponding propagation for more general symmetries.

We will show that if a gradient shrinking Ricci soliton (*shrinker*) has an approximate symmetry on one scale, then this approximate symmetry propagates outwards to larger scales. This is an example of the shrinker principle [12,16,18], which roughly states that information radiates outwards for these types of equations.

Theorem 1.2. *Let (M, g, f) be a non-compact shrinker and C_1 be a constant with $|R| \le C_1$. There exist C_2 and R so that if Y is a vector field on $\{f < r^2/4\}$ with $r \ge R$ and*

(1) $\int_{f<r^2/4} |Y|^2 \mathrm{e}^{-f} = 1$ *and* $\int_{f<r^2/4} |\mathrm{div}_f^* Y|^2 \, \mathrm{e}^{-f} \le \bar{\mu} \le \frac{1}{4}$.

(2) $|Y| + |\nabla Y| \le C_1 r$ *on* $\{f < r^2/4\}$.

then there is a vector field Z on M with $\|Z\|_{L^2} = 1$, $\mathcal{P}Z = \mu Z$, and satisfying

(Z_1) $\|\mathrm{div}_f^* Z\|_{L^2}^2 = \mu \le C_2\left(\bar{\mu} + r^{4+n} \mathrm{e}^{-\frac{r^2}{4}}\right)$.

(Z_2) *For* $s \in (r^2/4, r^2)$, *we have* $\int_{f=s} |\mathrm{div}_f^* Z|^2 \le C_2 r^{C_2} \mu$.

The theorem shows that the approximate symmetry extends to the larger set, though with a loss in the estimates. This extension is powerful in situations where the loss can be recovered using some additional structure particular to the situation, thus leading to a global symmetry. In Theorem 0.2 in [16], the loss was recovered by solving a gauge problem and then using a rigidity property of cylinders that showed up at the quadratic level. This quadratic rigidity relied on the structure of the compact factor of the cylinder and not just on the existence of the approximate translation.

This kind of propagation of symmetry often plays an important role in understanding the structure of solutions, as well as the rate of convergence of a Ricci flow to a singularity.

2 Weighted manifolds and Killing fields

We will be most interested in gradient shrinking Ricci solitons, but many of the results hold more generally. Killing fields preserve the metric, so they also preserve volume and, thus, are always divergence free. However, they do not necessarily preserve the weighted volume, so the weighted divergence $\mathrm{div}_f Y$ of a Killing field Y need not vanish. We will see, however, that it does satisfy an eigenvalue equation on any gradient Ricci soliton (equation (2.3)).

To show this, we recall some general formulas from Lemma 2.8 in [16] on any gradient Ricci soliton (i.e., (M, g, f) satisfies (1.2) for some constant κ) for any vector field Y:

$$(\mathcal{L} + \kappa)\mathrm{div}_f Y = -\mathrm{div}_f(\mathcal{P} Y), \tag{2.1}$$

$$\mathcal{L}\nabla\mathrm{div}_f(Y) = -\nabla\mathrm{div}_f(\mathcal{P} Y). \tag{2.2}$$

If Y is a Killing field on a gradient Ricci soliton, then (2.1) gives that

$$(\mathcal{L} + \kappa)\,\mathrm{div}_f Y = 0. \tag{2.3}$$

As a consequence of this, $\mathrm{div}_f Y$ is harmonic:

Corollary 2.1. *If Y is a Killing field on a gradient Ricci soliton, then $\mathrm{div}_f Y$ is harmonic.*

Proof. By (2.3), $(\mathcal{L} + \kappa)\,\mathrm{div}_f Y = 0$. Since $\mathrm{div}\, Y = 0$ for any Killing field, it follows that

$$\begin{aligned}\Delta\mathrm{div}_f Y &= \mathcal{L}\mathrm{div}_f Y + \langle\nabla f, \nabla\mathrm{div}_f Y\rangle \\ &= -\kappa\mathrm{div}_f Y - \langle\nabla f, \nabla\langle Y, \nabla f\rangle\rangle \\ &= \kappa\langle\nabla f, Y\rangle - \langle\nabla_{\nabla f} Y, \nabla f\rangle - \mathrm{Hess}_f(Y, \nabla f).\end{aligned} \tag{2.4}$$

The second to last term vanishes because of the Killing equation. For the last term, we bring in the soliton equation $\mathrm{Ric} + \mathrm{Hess}_f = \kappa g$ to obtain

$$\Delta\mathrm{div}_f Y = \mathrm{Ric}(\nabla f, Y). \tag{2.5}$$

Finally, this last term vanishes since $\mathrm{Ric}(\nabla f, Y) = \frac{1}{2}\langle\nabla S, Y\rangle$ (see, e.g., (1.14) in [16]) and the scalar curvature must be constant along a Killing field. □

Later, we will need a second-order self-adjoint operator L defined on symmetric two-tensors by

$$Lh = \mathcal{L}h + 2R(h), \tag{2.6}$$

where R is the Riemann curvature acting on h in an orthonormal frame by

$$[R(h)]_{ij} = \sum_{m,n} R_{imjn} h_{mn}. \tag{2.7}$$

Theorem 1.32 in [16] gives the following relation between L and div_f^*

$$L\mathrm{div}_f^*(Y) = \mathrm{div}_f^*(\mathcal{L} + \kappa)Y. \tag{2.8}$$

It will also be useful that the operator $\mathcal{P}$ is related to $\mathcal{L}$ by

$$-2\mathcal{P} = \nabla \mathrm{div}_f + \mathcal{L} + \kappa. \tag{2.9}$$

Finally, an easy integration by parts argument shows that if Y and $\mathcal{P}Y$ are in L^2, then $Y \in W^{1,2}$ and $\mathrm{div}_f Y \in L^2$; this is given by the next lemma:

Lemma 2.2. (Lemma 2.15, [16]) *For any gradient Ricci soliton if* Y, $\mathcal{P}Y \in L^2$, *then* $\mathrm{div}_f(Y)$, $\nabla Y \in L^2$ *and*

$$\|\nabla Y\|_{L^2}^2 + \|\mathrm{div}_f Y\|_{L^2}^2 \leq 2\|Y\|_{L^2}\|(2\mathcal{P} + \kappa)Y\|_{L^2}. \tag{2.10}$$

2.1 Shrinkers

We now specialize to shrinkers, where $\kappa = \frac{1}{2}$. We will use the spectral theory of cL in the next proof; see, e.g., [9,14,17,21].

Proof of Proposition 1.1. We consider two cases. Suppose first that Y preserves f, so that $\langle \nabla f, Y\rangle = 0$. Since Y is Killing, $\mathrm{div}\, Y = 0$, and, thus, $\mathrm{div}_f Y$ vanishes.

Suppose now that Y does not preserve f and, thus, $\mathrm{div}_f Y$ does not vanish identically. In this case, (2.3) gives that $\mathrm{div}_f Y$ is a non-trivial solution to

$$\mathcal{L}\mathrm{div}_f Y = -\frac{1}{2}\mathrm{div}_f Y. \tag{2.11}$$

Moreover, Lemma 2.2 implies that $\mathrm{div}_f Y \in L^2$. If $\mathcal{L}v = -\mu v$, then the drift Bochner formula gives that

$$\frac{1}{2}\mathcal{L}|\nabla v|^2 = |\mathrm{Hess}_v|^2 + \left(\frac{1}{2} - \mu\right)|\nabla v|^2. \tag{2.12}$$

Applying this with $v = \mathrm{div}_f Y$ and $\mu = \frac{1}{2}$ and integrating over M, we conclude that

$$\|\mathrm{Hess}_{\mathrm{div}_f Y}\|_{L^2} = 0. \tag{2.13}$$

It follows that $\nabla \mathrm{div}_f Y$ is a non-trivial parallel vector field, giving the desired splitting. □

There is an interesting distinction between the Killing fields that preserve f and those that do not. The Killing fields that preserve f turn out to be orthogonal to all gradient vector fields. The translations, which do not preserve f, are generated by gradient vector fields coming from least eigenfunctions on the shrinking soliton. Both types of vector fields satisfy eigenvalue equations for the drift Laplacian $\mathcal{L}$, but at different eigenvalues.

2.2 Geometric estimates for shrinkers

We will next recall several useful formulas for shrinkers and some geometric estimates. First, taking the trace of the shrinker equation gives that

$$\Delta f + S = \frac{n}{2}, \tag{2.14}$$

where S is the scalar curvature. On a complete shrinker, it is well known ([19], cf. [6,10]) that f can be normalized so that

$$|\nabla f|^2 + S = f. \tag{2.15}$$

Since $S \geq 0$ by [8] (see [11] for an improved bound on non-compact shrinkers), the function $b = 2\sqrt{f}$ is nonnegative and satisfies

$$|\nabla b| \leq 1. \tag{2.16}$$

We will also need some geometric estimates by Cao-Zhou for shrinkers. First, Theorem 1.1 in [7] gives c_1 and c_2, depending only on $B_1(x_0) \subset M$ so that

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2, \tag{2.17}$$

where $r(x)$ is the distance to a fixed point x_0 (the constants can be made universal if x_0 is chosen at the minimum of f). Second, Theorem 1.2 in [7] gives that shrinkers have at most Euclidean volume growth: There exists c_3 so that

$$\mathrm{Vol}(B_r(x_0)) \leq c_3 r^n. \tag{2.18}$$

3 Small eigenvalues and almost Killing fields

Throughout this section, (M, g, f) is a complete non-compact gradient shrinking Ricci soliton (i.e., $\kappa = \frac{1}{2}$).

The operator $\mathcal{P}$ was constructed to vanish on Killing fields. The next lemma uses the variational characterizations of eigenvalues and the exponential decay of the weight to find low eigenvalues for $\mathcal{P}$ when there is an approximate symmetry on a large ball.

Lemma 3.1. *Suppose that there is a (non-trivial) compactly supported vector field V with*

$$\|\mathrm{div}_f^* V\|^2 \leq \bar{\mu}\|V\|_{L^2}^2, \tag{3.1}$$

where $\bar{\mu} < 1$. Then there is a $W^{1,2}$ vector field Z with $\|Z\|_{L^2} = 1$, so that
(A) $\mathcal{P}Z = \mu Z$, with $\mu \in [0, \bar{\mu}]$, and $\|\mathrm{div}_f^* Z\|_{L^2}^2 = \mu$.
(B) $\mathrm{div}_f Z \in W^{1,2}$ satisfies $\mathcal{L}\mathrm{div}_f Z = -\left(\frac{1}{2} + \mu\right)\mathrm{div}_f Z$ and $\|\mathrm{div}_f Z\|_{L^2}^2 \leq 4\mu + 1$.

Proof. The operator $\mathcal{P}$ is self-adjoint by construction. Lemma 4.20 in [16] gives that $\mathcal{P}$ has a complete basis of smooth $W^{1,2}$ eigen-vector fields Y_i with eigenvalues

$$\mu_i \to \infty, \tag{3.2}$$

and with $\|Y_i\|_{L^2} = 1$. Moreover, Lemma 4.20 in [16] also gives that

$$\int |\mathrm{div}_f^* Y_i|^2 \, \mathrm{e}^{-f} = \int \langle Y_i, \mathcal{P}Y_i\rangle \, \mathrm{e}^{-f} = \mu_i. \tag{3.3}$$

Expanding V by projecting onto the Y_i's, we write V as follows:

$$V = \sum_i a_i Y_i, \tag{3.4}$$

where each $a_i \in \mathbf{R}$ and

$$\sum_i a_i^2 = \|V\|_{L^2}^2. \tag{3.5}$$

Since the Y_i's are L^2-orthonormal, (3.1) gives that

$$\bar{\mu}\|V\|_{L^2}^2 \geq \|\mathrm{div}_f^* V\|_{L^2} = \int \langle V, \mathcal{P}V\rangle \mathrm{e}^{-f} = \sum_i a_i^2 \mu_i \geq \mu_1 \sum_i a_i^2. \tag{3.6}$$

Comparing this with (3.5), we see that μ_1 (the smallest μ_i) is at most $\bar{\mu}$. Set $Z = Y_1$.

Since $Z, \mathcal{P}Z \in L^2$, Lemma 2.2 and Proposition 3.3 give that

$$\mathrm{div}_f Z \in W^{1,2} \quad \text{and} \quad \nabla Z \in L^2. \tag{3.7}$$

The L^2 bound on $\mathrm{div}_f Z$ in (B) follows from Lemma 2.2. Finally, (2.1) gives that $\mathrm{div}_f Z$ satisfies the eigenfunction equation $\mathcal{L}\mathrm{div}_f Z = -\left(\frac{1}{2} + \mu\right)\mathrm{div}_f Z$. □

The weighted L^2 bound on $\mathrm{div}_f^* Z$ forces it to be small in the region where f is small, but it says almost nothing where f is large and the weight e^{-f} is very small. To obtain better bounds when f is large, we instead rely on polynomial growth estimates developed in [16]. To explain this, given a tensor w define the "weighted spherical average" $I_w(r)$ by

$$I_w(r) = r^{1-n} \int_{b=r} |w|^2 |\nabla b|. \tag{3.8}$$

A priori, this is well defined at regular values of b, but Lemma 3.27 in [16] shows that $I_w(r)$ can be extended to all r, this extension is differentiable almost everywhere and is absolutely continuous as a function of r; cf. [2,13,15].

We have the following polynomial growth bounds:

Proposition 3.2. [16] *Given $\bar{\lambda}$, there exists r_0 so that if w is an L^2 tensor with*

$$\langle \mathcal{L}w, w\rangle \geq -\bar{\lambda}\,|w|^2, \tag{3.9}$$

then for any $r_2 > r_1 \geq r_0$, we have that

$$I_w(r_2) \leq 2\left(\frac{r_2}{r_1}\right)^{5\bar{\lambda}} I_w(r_1). \tag{3.10}$$

Proposition 3.2 is a special case of Theorem 3.4 in [16]. This proposition requires the lower bound (3.9) for $\langle \mathcal{L}w, w\rangle$, which would hold if w satisfied an eigenvalue equation for $\mathcal{L}$. However, we will want to apply it to a vector field that satisfies an eigenvalue equation for $\mathcal{P}$. The next result gives a decomposition for that equation that makes this possible (this is a special case of Proposition 4.6 in [16]):

Proposition 3.3. [16] *If $\mathcal{P}Y = \mu Y$, and we set $Z = Y + \frac{2}{2\mu+1}\nabla \mathrm{div}_f(Y)$, then $\mathrm{div}_f(Z) = 0$ and*

$$(\mathcal{L} + \mu)\nabla \mathrm{div}_f(Y) = 0, \tag{3.11}$$

$$\left(\mathcal{L} + 2\mu + \frac{1}{2}\right)Z = 0. \tag{3.12}$$

Moreover, if $Y \in L^2$, then $\|Y\|^2 = \|Z\|^2 + \left(\mu + \frac{1}{2}\right)^{-2}\|\nabla \mathrm{div}_f(Y)\|^2$.

We are now prepared to prove that $\mathrm{div}_f^* Y$ grows at most polynomially when Y is an eigenvector field for $\mathcal{P}$.

Theorem 3.4. *Suppose that (M, g) has bounded curvature $|R| \leq C_1$ and Y is a $W^{1,2}$ vector field with $\|Y\|_{L^2} = 1$ that satisfies (A) and (B). There exist C_2, R so that for all $r \geq R$*

$$I_{\mathrm{div}_f^* Y}(r) \leq C_1 r^{C_2 \mu}. \tag{3.13}$$

Proof. Since $\mathcal{P}Y = \mu Y$, Proposition 3.3 gives that $Z = Y + \frac{2}{1+2\mu}\nabla \mathrm{div}_f Y$ satisfies

$$\mathcal{L}Z = -\left(2\mu + \frac{1}{2}\right)Z, \tag{3.14}$$

$$\mathcal{L}\nabla \mathrm{div}_f Y = -\mu \nabla \mathrm{div}_f Y. \tag{3.15}$$

Applying (2.8) to (3.14) and (3.15) gives

$$L\mathrm{div}_f^* Z = \mathrm{div}_f^*\left(\mathcal{L} + \frac{1}{2}\right)Z = -2\mu \mathrm{div}_f^* Z, \tag{3.16}$$

$$L\mathrm{div}_f^* \nabla \mathrm{div}_f Y = \mathrm{div}_f^*\left(\mathcal{L} + \frac{1}{2}\right)\nabla \mathrm{div}_f Y = \left(\frac{1}{2} - \mu\right)\mathrm{div}_f^* \nabla \mathrm{div}_f Y. \tag{3.17}$$

The last equation can be rewritten as follows:

$$L\mathrm{Hess}_{\mathrm{div}_f Y} = \left(\frac{1}{2} - \mu\right)\mathrm{Hess}_{\mathrm{div}_f Y}. \tag{3.18}$$

Since M, g has bounded curvature and μ is also bounded, (3.16) and (3.18) give C so that

$$\langle \mathcal{L}w, w\rangle \geq -C|w|^2 \quad \text{for } w = \mathrm{Hess}_{\mathrm{div}_f Y} \quad \text{or} \quad w = \mathrm{div}_f^* Z. \tag{3.19}$$

Thus, Proposition 3.2 applies, giving R and C_1 so that

$$w = \mathrm{Hess}_{\mathrm{div}_f Y} \quad \text{or} \quad \mathrm{div}_f^* Z$$

both satisfy

$$I_w(r_2) \leq \left(\frac{r_2}{r_1}\right)^{C_1} I_w(r_1) \quad \text{for any } r_2 > r_1 \geq R. \tag{3.20}$$

Next, observe that (B) and the drift Bochner formula give that

$$\|\mathrm{Hess}_{\mathrm{div}_f Y}\|_{L^2}^2 \leq \mu. \tag{3.21}$$

On the other hand, (A) gives that $\|\mathrm{div}_f^* Y\|_{L^2}^2 \leq \mu$ as well. It follows that

$$\|\mathrm{div}_f^* Z\|_{L^2}^2 \leq C\mu. \tag{3.22}$$

Since $|\nabla b| \leq 1$, the co area formula gives some $r_1 \geq R$ and a constant C_2 so that

$$I_w(r_1) \leq C_2\mu \quad \text{for } w = \mathrm{Hess}_{\mathrm{div}_f Y} \quad \text{or} \quad w = \mathrm{div}_f^* Z. \tag{3.23}$$

Using this in (3.20) and then writing $\mathrm{div}_f^* Y$ as a combination gives the theorem. □

We are now prepared to prove the main theorem.

Proof of Theorem 1.2. The first step is to cutoff the vector field to obtain a compactly supported vector field V that we can use in Lemma 3.1. To do this, define a cutoff function η with $0 \leq \eta \leq 1$ that has support in $f < \frac{r^2}{4}$, cuts off in distance r^{-1}, has $|\nabla \eta| \leq 2r$ and so that

$$e^{-f} \leq e^{-1} e^{-\frac{r^2}{4}} \text{ on the support of } |\nabla \eta|. \tag{3.24}$$

The last bound uses that $|\nabla b| \leq 1$ by (2.16). Now set $V = \eta Y$. Since $|Y| \leq C_1 r$ on the support of η, it follows that

$$\|V\|_{L^2}^2 \geq \int_{\eta=1} |Y|^2 e^{-f} = 1 - \int_{0\leq\eta<1} |Y|^2 e^{-f} \geq 1 - Cr^{2+n} e^{-\frac{r^2}{4}}, \tag{3.25}$$

where the last bound also used the volume bound (2.18). Similarly, we have that

$$\|\mathrm{div}_f^* V\|_{L^2}^2 \leq 2\int_{f<r^2/4} |\mathrm{div}_f^* Y|^2 e^{-f} + Cr^{4+n} e^{-\frac{r^2}{4}} \leq 2\bar{\mu} + Cr^{4+n} e^{-\frac{r^2}{4}}. \tag{3.26}$$

After multiplying V by a constant so that the L^2 norm is one, we have that

$$\|\mathrm{div}_f^* V\|_{L^2}^2 \leq 3\bar{\mu} + Cr^{4+n} e^{-\frac{r^2}{4}} < 1. \tag{3.27}$$

Lemma 3.1 gives a $W^{1,2}$ vector field Z with $\|Z\|_{L^2} = 1$ and satisfying (A) and (B). In particular, $\mathcal{P}Z = \mu Z$ with

$$\mu \leq 3\bar{\mu} + Cr^{4+n} e^{-\frac{r^2}{4}} < 1. \tag{3.28}$$

This gives (Z1). Theorem 3.4 now gives a polynomially growing bound on $I_{\mathrm{div}_f^* V}$. Since the scalar curvature is bounded in this range, there is a lower bound for $|\nabla b|$ here. Therefore, the bound on $I_{\mathrm{div}_f^* V}$ implies the fine growth bound (Z2). □

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