T. Holck Colding and W. P. Minicozzi II (2022) "Parabolic Frequency on Manifolds,"
International Mathematics Research Notices, Vol. 2022, No. 15, pp. 11878–11890
Advance Access Publication April 19, 2021
https://doi.org/10.1093/imrn/rnab052

Parabolic Frequency on Manifolds

Tobias Holck Colding* and **William P. Minicozzi II**
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

**Correspondence to be sent to: e-mail: colding@math.mit.edu*

We prove monotonicity of a parabolic frequency on static and evolving manifolds without any curvature or other assumptions. These are parabolic analogs of Almgren's frequency function. When the static manifold is Euclidean space and the drift operator is the Ornstein–Uhlenbeck operator, this can been seen to imply Poon's frequency monotonicity for the ordinary heat equation. When the manifold is self-similarly evolving by the Ricci flow, we prove a parabolic frequency monotonicity for solutions of the heat equation. For the self-similarly evolving Gaussian soliton, this gives directly Poon's monotonicity. Monotonicity of frequency is a parabolic analog of the 19th century Hadamard three-circle theorem about log convexity of holomorphic functions on $\mathbf{C}$. From the monotonicity, we get parabolic unique continuation and backward uniqueness.

1 Introduction

Bounds on growth for functions satisfying a partial differential equation give crucial information and have many consequences. One of the oldest bounds of this type is Hadamard's three-circle theorem for holomorphic functions. For the Laplace equation, Almgren [1] proved the monotonicity of a frequency function that measures the rate of growth. Almgren's [1] frequency played a fundamental role in his regularity results and other areas; see, for example, [16, 22]. It was generalized to the heat equation by Poon [23]. The results of Almgren and Poon rely on the scaling structure of $\mathbf{R}^n$ (cf. [6]) and do not extend globally to general manifolds. Here we prove two very general

Received February 10, 2021; Revised February 16, 2021; Accepted February 17, 2021

monotonicities for heat equations on manifolds. Both recover Poon's monotonicity. One of these monotonicity is for drift heat equations on any static manifold. The other is a monotonicity for a parabolic frequency for self-similarly evolving solutions of the Ricci flow; when the shrinking flow is the Gaussian soliton, this recovers directly the monotonicity in [23]. Part of the strength is the simplicity of the arguments.

Let (M,g) be a Riemannian manifold, $\phi : M \to \mathbf{R}$ a smooth function, and define an operator $\mathcal{L}_\phi$ (drift Laplacian) on vector-valued functions $u : M \to \mathbf{R}^N$ by

$$\mathcal{L}_\phi\, u = \Delta\, u - \langle \nabla u, \nabla \phi \rangle = \mathrm{e}^{\phi}\,\mathrm{div}\left(\mathrm{e}^{-\phi}\,\nabla u\right) . \tag{1.1}$$

These operators play an important role in many parabolic problems; see, for example, [7–9, 12]. The prime example of $\mathcal{L}_\phi$ is on $\mathbf{R}^n$ with the flat metric, $\phi = \frac{|x|^2}{4}$, and $\mathcal{L}_{\frac{|x|^2}{4}}\, u = \Delta\, u - \frac{1}{2}\langle \nabla u, x\rangle$ is the Ornstein–Uhlenbeck operator. We let L^2_ϕ and $W^{1,2}_\phi$ be the spaces of square integrable $\mathbf{R}^N$-valued functions and Sobolev functions with respect to the weight $\mathrm{e}^{-\phi}$. It follows from (1.1) that $\mathcal{L}_\phi$ is self-adjoint on $W^{1,2}_\phi$ with

$$\int \langle u\,, \mathcal{L}_\phi\, v\rangle\, \mathrm{e}^{-\phi} = -\int \langle \nabla u, \nabla v\rangle\, \mathrm{e}^{-\phi} . \tag{1.2}$$

Suppose that $u : M \times [a,b] \to \mathbf{R}^N$ is smooth and (some growth assumption is necessary to rule out the classical Tychonoff example) $u, u_t \in W^{1,2}_\phi$ for each $t \in [a,b]$. Set

$$I(t) = \int |u|^2\, \mathrm{e}^{-\phi} , \tag{1.3}$$

$$D(t) = -\int |\nabla u|^2\, \mathrm{e}^{-\phi} = \int \langle u\,, \mathcal{L}_\phi\, u\rangle\, \mathrm{e}^{-\phi} , \tag{1.4}$$

$$U(t) = \frac{D}{I} . \tag{1.5}$$

Observe that with our convention U is always non-positive.

The next theorem is a parabolic analog of the classical Hadamard's three-circle theorem (the three-circle theorem was stated and proven by J. E. Littlewood in 1912, but he stated it as a known theorem; Harald Bohr and Edmund Landau, in 1896, attribute the theorem to Jacques Hadamard; Hadamard did not publish a proof) for holomorphic functions:

Theorem 1.6. When $(\partial_t - \mathcal{L}_\phi)\, u = 0$, then $(\log I)'(t) = 2\, U(t)$ and $\log I(t)$ is convex so $U' \geq 0$. Moreover, when U is constant, then $u(x,t) = e^{U\,t}\, u(x,0)$ and $u(\cdot,0)$ is an eigenfunction of $\mathcal{L}_\phi$ with eigenvalue $-U$.

Poon [23] proved a monotonicity that can be shown (see Section 2) to follow from the special case of Theorem 1.6 when $M = \mathbf{R}^n$, $N = 1$, and $\phi = \frac{|x|^2}{4}$. The monotonicity in [23] holds for non-negative sectional curvature and parallel Ricci curvature—exactly the assumptions needed to generalize Hamilton's work [17, 18] from $\mathbf{R}^n$ to manifolds. (See the discussion in [23] after Theorem 1.1′ on page 522 and the remark on page 530.) Theorem 1.6 holds on any manifold and no curvature assumption is needed.

We will also prove a frequency monotonicity for shrinking gradient Ricci flows with no curvature assumption (see Section 4 for the definitions of I and U in this case):

Theorem 1.7. If $u_t = \Delta\, u$ on a gradient shrinking Ricci flow, then $I'(t) = \frac{2\,D(t)}{-t}$ and $U(t)$ is monotone with $U' \geq 0$. Furthermore, if $U'(t) = 0$, then at t we have $\mathcal{L}_f\, u = \frac{U(t)}{-t}\, u$.

An immediate consequence of Theorems 1.6 and 1.7 is the following version of backwards uniqueness for these equations:

Corollary 1.8. Suppose that $u : M \times [a,b] \to \mathbf{R}^N$ and either the metric g is time independent and $(\partial_t - \mathcal{L}_\phi)\, u = 0$ or $(M, g(t))$ is a gradient shrinking Ricci flow and $u_t = \Delta\, u$. If $u(\cdot, b) = 0$, then $u \equiv 0$ for all $t \leq b$.

We are grateful to S. Brendle, R. Hamilton, and S. Klainerman for discussions and to Q. S. Zhang for useful comments.

2 Parabolic Frequency on Manifolds

Proof. (of Theorem 1.6). Calculating and integrating by parts give

$$I'(t) = 2 \int \langle u,\, u_t\rangle\, e^{-\phi} = 2 \int \langle u\,,\mathcal{L}_\phi\, u\rangle\, e^{-\phi} = -2 \int |\nabla u|^2\, e^{-\phi} = 2D(t)\,. \tag{2.1}$$

$$D'(t) = -2 \int \langle \nabla u, \nabla u_t\rangle\, e^{-\phi} = -2 \int \langle \nabla u, \nabla \mathcal{L}_\phi\, u\rangle\, e^{-\phi} = 2 \int |\mathcal{L}_\phi\, u|^2\, e^{-\phi}\,. \tag{2.2}$$

By (2.1) and the definition of U, we get

$$(\log I)'(t) = 2\,\frac{D(t)}{I(t)} = 2U(t)\,. \tag{2.3}$$

Therefore, using (2.1), (2.2), and (1.4),

$$I^2\,U' = D'\,I - I'\,D = \left(2\int |\mathcal{L}_\phi\, u|^2\,\mathrm{e}^{-\phi}\right)\left(\int |u|^2\,\mathrm{e}^{-\phi}\right) - 2D^2(t)$$
$$= \left(2\int |\mathcal{L}_\phi\, u|^2\,\mathrm{e}^{-\phi}\right)\left(\int |u|^2\,\mathrm{e}^{-\phi}\right) - 2\left(\int \langle u,\,\mathcal{L}_\phi\, u\rangle\,\mathrm{e}^{-\phi}\right)^2 \geq 0\,. \tag{2.4}$$

Here the inequality uses the Cauchy–Schwarz inequality (a similar estimate without the weight ϕ is used in [15] to prove backwards uniqueness when $u_t = \Delta u$ and the boundary values are specified).

If U is constant, then we have equality in the Cauchy–Schwarz inequality (2.4) and, thus,

$$\mathcal{L}_\phi\, u = c(t)\, u\,. \tag{2.5}$$

To evaluate $c(t)$, use the 2nd equality in (1.4) to get

$$D(t) = \int \langle u, \mathcal{L}_\phi\, u\rangle\,\mathrm{e}^{-\phi} = c(t)\int |u|^2\,\mathrm{e}^{-\phi} = c(t)\,I(t)\,. \tag{2.6}$$

It follows that $c(t) = U$ and $\mathcal{L}_\phi\, u = U\,u$. If we set $v(x,t) = \mathrm{e}^{-U\,t}\,u(x,t)$, then we have that

$$\partial_t v = \mathrm{e}^{-U\,t}\,(-U\,u + \partial_t u) = \mathrm{e}^{-U\,t}\,(-U\,u + \mathcal{L}_\phi\, u) = 0\,. \tag{2.7}$$

From this, the 2nd claim follows. ■

Proof of Corollary 1.8. The proof is essentially the same in both cases. We will give the details when g is time independent and u satisfies the drift heat equation. By Theorem 1.6, we get for $a < b$ that

$$\log I(b) - \log I(a) = \int_a^b (\log I)'(s)\,\mathrm{d}s = 2\int_a^b U(s)\,\mathrm{d}s \geq 2\,U(a)\,(b-a)\,. \tag{2.8}$$

Integrating this gives that

$$I(b) \geq I(a)\,\mathrm{e}^{2\,U(a)\,(b-a)}\,. \tag{2.9}$$

Backwards uniqueness follows immediately. ■

Equation (2.9) can be thought of as a bound for the vanishing order at ∞. The bound implies strong unique continuation at ∞. That is, if u vanishes to infinite order at ∞, that is, $\lim_{t\to\infty} e^{ct} I(t) = 0$ for all constants c, then it vanishes identically.

There is a natural correspondence on $\mathbf{R}^n$ between solutions of the ordinary heat equation and solutions of the drift heat equation: given $u : \mathbf{R}^n \times (-\infty, 0) \to \mathbf{R}$, define $v(x,t) = u(\sqrt{-t}\,x, t)$, $w(x,s) = v(x, -e^{-s})$, and $t = -e^{-s}$. We have the following:

Lemma 2.10. The function $w : \mathbf{R}^n \times (-\infty, 0) \to \mathbf{R}$ defined as above satisfies

$$(\partial_s - \mathcal{L}_{\frac{|x|^2}{4}})\, w(x,s) = e^{-s} \left(u_t - \Delta u\right)(e^{-\frac{s}{2}} x, -e^{-s})\,. \tag{2.11}$$

Proof. To prove (2.11), we use the chain rule to get

$$\partial_t v = -\frac{1}{2\sqrt{-t}} \langle \nabla u, x\rangle + u_t\,, \tag{2.12}$$

$$\partial_s w = -t\, \partial_t\, v = -\frac{\sqrt{-t}}{2} \langle \nabla u, x\rangle - t\, u_t\,, \tag{2.13}$$

$$\mathcal{L}_{\frac{|x|^2}{4}}\, w = \Delta\, w - \frac{1}{2} \langle x, \nabla w\rangle = -t\, \Delta\, u - \frac{1}{2} \langle x, \sqrt{-t}\, \nabla u\rangle\,. \tag{2.14}$$

Combining (2.13) and (2.14) gives (2.11). ∎

Poon [23] considered solutions the ordinary heat equation on $\mathbf{R}^n$. He showed a monotonicity that is equivalent to $s \to \log H(e^{\frac{s}{2}})$ being convex, where

$$H(R) = R^{-n} \int u^2(y, -R^2)\, e^{-\frac{|y|^2}{4R^2}}\,. \tag{2.15}$$

The convexity of $\log H(e^{\frac{s}{2}})$ follows from Theorem 1.6 when $M = \mathbf{R}^n$ and $\phi = \frac{|x|^2}{4}$. To see this, suppose $u_t = \Delta u$, so that $(\partial_s - \mathcal{L}_{\frac{|x|^2}{4}})\, w = 0$ by Lemma 2.10. Using the definition of $I_w(s)$ and making the change of variables $y = e^{-\frac{s}{2}}\, x$ and $R = e^{-\frac{s}{2}}$ give

$$I_w(s) = \int u^2(e^{-\frac{s}{2}}\, x, -e^{-s})\, e^{-\frac{|x|^2}{4}}\, dx = R^{-n} \int u^2(y, -R^2)\, e^{-\frac{|y|^2}{4R^2}}\, dy = H(e^{\frac{s}{2}})\,. \tag{2.16}$$

The convexity of $\log H(e^{\frac{s}{2}})$ now follows from Theorem 1.6.

3 More General Operators

Our results hold also for more general operators (cf. [2, 4, 13, 24]) where

$$|(\partial_t - \mathcal{L}_\phi)\, u| \leq C(t)\, (|u| + |\nabla u|)\,, \tag{3.1}$$

and $C(t)$ is allowed to depend on t.

Theorem 3.2. If $u : M \times [a,b] \to \mathbf{R}^N$ satisfies (3.1), then

$$(\log I)' \geq \left(2 + \frac{C}{2}\right) U - \frac{3C}{2}\,, \tag{3.3}$$

$$U' \geq C^2\, (U-1)\,, \tag{3.4}$$

$$C^2 \geq \left[\log(1-U)\right]'\,. \tag{3.5}$$

Proof. First, we rewrite D as follows:

$$D = \int \langle u, \mathcal{L}_\phi\, u\rangle\, \mathrm{e}^{-\phi} = \int \left\langle u\,, \left[u_t - \frac{1}{2}\,(u_t - \mathcal{L}_\phi\, u)\right]\right\rangle \mathrm{e}^{-\phi} - \frac{1}{2}\int \langle u,\, (u_t - \mathcal{L}_\phi\, u)\rangle\, \mathrm{e}^{-\phi}\,. \tag{3.6}$$

Differentiating $I(t)$ and rewriting give

$$\begin{aligned} I'(t) &= 2\int \langle u,\, u_t\rangle\, \mathrm{e}^{-\phi} = 2\int \langle u\,, \mathcal{L}_\phi\, u\rangle\, \mathrm{e}^{-\phi} + 2\int \langle u,\, (u_t - \mathcal{L}_\phi\, u)\rangle\, \mathrm{e}^{-\phi} \\ &= 2\int \left\langle u, \left[u_t - \frac{1}{2}\,(u_t - \mathcal{L}_\phi u)\right]\right\rangle \mathrm{e}^{-\phi} + \int \langle u,\, (u_t - \mathcal{L}_\phi\, u)\rangle\, \mathrm{e}^{-\phi}\,. \end{aligned} \tag{3.7}$$

It follows from the 1st line in (3.7), the Cauchy–Schwarz inequality, and the elementary inequality $a \leq \frac{1}{2}\,\left(a^2+1\right)$ applied to $a = \sqrt{-U}$ that

$$(\log I)' \geq 2U - \frac{C}{I}\int |u|\, (|u| + |\nabla u|)\, \mathrm{e}^{-\phi} \geq 2U - C\,(1+\sqrt{-U}) \geq \left(2 + \frac{C}{2}\right) U - \frac{3C}{2}\,.$$

This gives the 1st claim. Next, (3.7) gives that

$$I'(t)\, D(t) = 2\left(\int \left\langle u, \left[u_t - \frac{1}{2}\,(u_t - \mathcal{L}_\phi\, u)\right]\right\rangle \mathrm{e}^{-\phi}\right)^2 - \frac{1}{2}\left(\int \langle u,\, (u_t - \mathcal{L}_\phi\, u)\rangle\, \mathrm{e}^{-\phi}\right)^2. \tag{3.8}$$

Differentiating $D(t)$ and integrating by parts give

$$D'(t) = -2 \int \langle \nabla u, \nabla u_t \rangle \, e^{-\phi} = 2 \int \langle u_t, \mathcal{L}_\phi \, u \rangle \, e^{-\phi} = 2 \int \langle u_t, (u_t - [u_t - \mathcal{L}_\phi \, u]) \rangle \, e^{-\phi}$$
$$= 2 \int \left\{ \left| u_t - \frac{1}{2} [u_t - \mathcal{L}_\phi \, u] \right|^2 - \frac{1}{4} |u_t - \mathcal{L}_\phi \, u|^2 \right\} e^{-\phi} . \quad (3.9)$$

We conclude that

$$D'(t) \, I(t) = 2 \, I(t) \int \left| u_t - \frac{1}{2} [u_t - \mathcal{L}_\phi \, u] \right|^2 e^{-\phi} - \frac{I(t)}{2} \int |u_t - \mathcal{L}_\phi \, u|^2 \, e^{-\phi} . \quad (3.10)$$

Combining (3.8) and (3.10) and using the Cauchy-Schwarz inequality, (3.1) and the elementary inequality $(a+b)^2 \leq 2\,(a^2+b^2)$ give

$$\begin{aligned} I^2 \, U' &= 2 \left[\int |u|^2 \, e^{-\phi} \int \left| u_t - \frac{1}{2} [u_t - \mathcal{L}_\phi \, u] \right|^2 e^{-\phi} - \left(\int \left\langle u, \left[u_t - \frac{1}{2} (u_t - \mathcal{L}_\phi \, u) \right] \right\rangle e^{-\phi} \right)^2 \right] \\ &\quad - \frac{I}{2} \int |u_t - \mathcal{L}_\phi \, u|^2 \, e^{-\phi} + \frac{1}{2} \left(\int \langle u, (u_t - \mathcal{L}_\phi \, u) \rangle \, e^{-\phi} \right)^2 \\ &\geq -\frac{I}{2} \int |u_t - \mathcal{L}_\phi \, u|^2 \, e^{-\phi} \geq -\frac{C^2 I}{2} \int (|u| + |\nabla u|)^2 \, e^{-\phi} \geq -C^2 \, I \, (I - D) . \end{aligned} \quad (3.11)$$

Dividing both sides by I^2 gives the 2nd claim, which implies the remaining claim. ■

This leads to the following generalization of Corollary 1.8:

Corollary 3.12. If $u : M \times [a,b] \to \mathbf{R}^N$ satisfies (3.1), then

$$I(b) \geq I(a) \, \exp \left((b-a) \, (2 + \sup_{[a,b]} C) \left[\exp \left(\int_a^b C^2(s) \, ds \right) [U(a) - 1] + 1 - \frac{3}{2} \sup_{[a,b]} C \right] \right) .$$

In particular, if $u(\cdot, b) = 0$, then $u \equiv 0$.

Proof. Integrating the 1st claim in Theorem 3.2, we get that

$$\log I(b) - \log I(a) \geq \frac{1}{2} \left(4 + \sup_{[a,b]} C \right) \int_a^b U(s) \, ds - \frac{3}{2} \sup_{[a,b]} C \, (b-a) . \quad (3.13)$$

From (3.5), we get that for $s \in [a, b]$

$$\log(1 - U(s)) \leq \log(1 - U(a)) + \int_a^s C^2(r)\,\mathrm{d}r \leq \log(1 - U(a)) + \int_a^b C^2(s)\,\mathrm{d}s\,. \tag{3.14}$$

Therefore,

$$U(s) \geq \exp\left(\int_a^s C^2(s)\,\mathrm{d}s\right)(U(a) - 1) + 1\,. \tag{3.15}$$

Inserting this lower bound in (3.13) and integrating give

$$\log I(b) - \log I(a) \geq (b-a)\left[\exp\left((2 + \sup_{[a,b]} C)\int_a^b C^2(s)\,ds\right)[U(a) - 1] + 1 - \frac{3}{2}\sup_{[a,b]} C\right].$$

■

Recall that $u : M \times (a, \infty) \to \mathbf{R}^N$ vanishes to infinite order at ∞ if $\lim_{t\to\infty} \mathrm{e}^{ct} I(t) = 0$ for all constants c. Theorem 3.12 implies the following strong unique continuation at ∞:

Corollary 3.16. Suppose that $\sup C + \int_a^\infty C^2(s)\,\mathrm{d}s < \infty$ and $u : M \times [a, \infty) \to \mathbf{R}^N$ is a solution of (3.1) that vanishes to infinite order at ∞, then u vanishes.

This corollary implies the unique continuation of Poon [23] for $u : \mathbf{R}^n \to \mathbf{R}$ with

$$u_t - \Delta u = \langle b(x,t), \nabla u\rangle + c(x,t)\,u\,, \tag{3.17}$$

where $|b| + |c| \leq C$ is uniformly bounded (cf. [4, 19, 21]). See also [14] for a Carleman estimate approach where it was observed that parabolic unique continuation is related to backward uniqueness for the Ornstein–Uhlenbeck operator. The methods here apply even more generally than to u satisfying (3.17). In particular, suppose that u satisfies (note that functions satisfying (3.18) do not necessarily satisfy (3.1) because the drift term may give an unbounded first order coefficient)

$$\left|u_t - \Delta u\right| \leq C\,(|u| + |\nabla u|)\,. \tag{3.18}$$

Applying the transformation in Lemma 2.10 to u, we get a function $w(y,s)$ with

$$\left|\left(\partial_s - \mathcal{L}_{\frac{|y|^2}{4}}\right) w\right| = \mathrm{e}^{-s} \left|(\partial_t - \Delta)\, u\right| \leq C\, \mathrm{e}^{-s} \left(|u| + |\nabla u|\right)$$
$$\leq C\, \mathrm{e}^{-s} |w| + C\, \mathrm{e}^{-\frac{s}{2}} \, |\nabla w|\,. \tag{3.19}$$

Since $\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s < \infty$, Corollary 3.16 applies. Exponential decay of order c, that is, decay like e^{-cs}, corresponds to polynomial decay t^c in the transformed variable $t = -\mathrm{e}^{-s}$.

3.1 Without u term

In this subsection, we assume that $u : M \times [a,b] \to \mathbf{R}^N$ satisfies

$$|(\partial_t - \mathcal{L}_\phi)\, u| \leq C(t) |\nabla u|\,. \tag{3.20}$$

In this case, we get better estimates when $U(a)$ is small. It follows from (3.11), with obvious simplifications in the 2nd to last inequality from using (3.20) in place of (3.1), that $U' \geq \frac{C^2}{2}\, U$ or, equivalently, $[\log(-U)]' \leq \frac{C^2}{2}$. We therefore get that

$$U(s) \geq U(a)\, \exp\left(\frac{1}{2}\int_a^s C^2(\tau)\,\mathrm{d}\tau\right). \tag{3.21}$$

With similar simplifications in (3.3), we get that for $s \in [a,b]$

$$(\log I)' \geq 2U - C\sqrt{-U}$$
$$\geq 2U(a)\, \exp\left(\frac{1}{2}\int_a^b C^2(\tau)\,\mathrm{d}\tau\right) - C\sqrt{-U(a)}\, \exp\left(\frac{1}{4}\int_a^b C^2(\tau)\,\mathrm{d}\tau\right). \tag{3.22}$$

Integrating gives

$$I(b) \geq I(a)\, \exp\left[(b-a)\left\{2U(a)\, \exp\left(\frac{1}{2}\int_a^b C^2(\tau)\,\mathrm{d}\tau\right) - C\sqrt{-U(a)}\, \exp\left(\frac{1}{4}\int_a^b C^2(\tau)\,\mathrm{d}\tau\right)\right\}\right].$$

4 Monotonicity for Shrinking Ricci Flows

A Ricci flow $(M, g(t))$ for $t < 0$ is a gradient shrinking soliton flow if there is a function $f(x,t)$ satisfying

$$\mathrm{Hess}_f + \mathrm{Ric} = \frac{1}{-2t}\, g\,. \tag{4.1}$$

Shrinkers model Ricci flow singularities. The function f can be normalized to satisfy (see, e.g., [3, 5])

$$f_t = -\Delta f - S + |\nabla f|^2 - \frac{n}{2t}\,, \tag{4.2}$$

where S is the scalar curvature. Given a function $u(x,t)$, define $I(t)$ and $D(t)$ by

$$I(t) = (-t)^{-\frac{n}{2}} \int u^2\, \mathrm{e}^{-f}\,, \tag{4.3}$$

$$D(t) = t\,(-t)^{-\frac{n}{2}} \int |\nabla u|^2\, \mathrm{e}^{-f}\,, \tag{4.4}$$

and then define the frequency $U(t) = \frac{D(t)}{I(t)}$. The scaling here differs from (1.3) and (1.4), but both are scale invariant (e.g., I is constant if $u \equiv 1$). In the simplest case of the shrinking Gaussian soliton, the two Is are logarithmically related via rescaling in space as in (2.16).

Lemma 4.5. If u is a function on a gradient shrinking soliton flow, then

$$2\int (\mathcal{L}_f\, u)^2\, \mathrm{e}^{-f} = \int \left\{ 2\, \left|\mathrm{Hess}_u\right|^2 - \frac{|\nabla u|^2}{t} \right\} \mathrm{e}^{-f}\,. \tag{4.6}$$

Proof. The drift Bochner formula gives

$$\frac{1}{2}\,\mathcal{L}_f\, |\nabla u|^2 = \left|\mathrm{Hess}_u\right|^2 + \langle \nabla u, \nabla\, \mathcal{L}_f\, u\rangle + \mathrm{Hess}_f(\nabla u, \nabla u) + \mathrm{Ric}(\nabla u, \nabla u)\,. \tag{4.7}$$

The lemma follows from integrating this, then integrating by parts on the 2nd term on the right and using the shrinker equation (4.1) on the last two terms. ∎

We will use some standard evolution formulas for Ricci flow:

Lemma 4.8. Suppose that $g_t = -2\,\mathrm{Ric}$. The derivative of the volume element is $\partial_t\, dv = -S\, dv$ where S is the scalar curvature. If $u(x,t)$ is a function, then

$$\partial_t\, |\nabla u|^2 = 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2\,\langle \nabla u, \nabla u_t\rangle\,. \tag{4.9}$$

Proof. Let x_i be local coordinates and ∂_{x_i} coordinate vector fields, so that $g(\partial_{x_i}, \partial_{x_j}) = g_{ij}$. The 1st claim follows since the volume element is given by $dv = \sqrt{\det g_{ij}}\, dx$. Next, if

g^{ij} denotes the inverse of g_{ij}, then $\partial_t g^{km} = -(\partial_t g_{ij}) g^{ki} g^{mj} = 2\,\mathrm{Ric}_{ij} g^{ki} g^{mj}$. Since $\nabla u = g^{ik} u_k \partial_{x_i}$, we have $|\nabla u|^2 = g_{ij}(g^{ik} u_k)(g^{jm} u_m) = g^{km} u_k u_m$ and, thus,

$$\begin{aligned}\partial_t |\nabla u|^2 &= \partial_t \left(g^{km} u_k u_m\right) = (\partial_t g^{km}) u_k u_m + g^{km} (u_t)_k u_m + g^{km} u_k (u_t)_m \\ &= 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2 \langle \nabla u, \nabla u_t \rangle . \end{aligned} \tag{4.10}$$

■

We will now prove the monotonicity of the frequency U for the heat equation on shrinking Ricci solitons. This has many analytic implications (cf. [10, 11, 20] for similar situations).

Proof of Theorem 1.7. Note that

$$\Delta\, \mathrm{e}^{-f} = \mathrm{div}\left(- \mathrm{e}^{-f} \nabla f\right) = \mathrm{e}^{-f} \left(|\nabla f|^2 - \Delta f\right). \tag{4.11}$$

Observe that $\partial_t\, dv = -S\, dv$ and $\partial_t (\mathrm{e}^{-f}) = -f_t\, \mathrm{e}^{-f}$, so that

$$\begin{aligned}(-t)^{\frac{n}{2}} I'(t) &= \int \left\{2u\, u_t - u^2 \left(S + f_t + \frac{n}{2t}\right)\right\} \mathrm{e}^{-f} = \int \left\{2u \,\Delta\, u - u^2 \left(|\nabla f|^2 - \Delta f\right)\right\} \mathrm{e}^{-f} \\ &= \int \left\{2u \,\Delta\, u\, \mathrm{e}^{-f} - u^2 \,\Delta\, \mathrm{e}^{-f}\right\} = -2 \int |\nabla u|^2 \mathrm{e}^{-f} = 2 \int (u\, \mathcal{L}_f\, u)\, \mathrm{e}^{-f} . \end{aligned} \tag{4.12}$$

In particular, $I'(t) = \frac{2\,D(t)}{-t}$. Differentiating $D(t)$ and using Lemma 4.8 and (4.2) give

$$\begin{aligned}(-t)^{\frac{n}{2}} D'(t) &= t \int \left\{2\,\mathrm{Ric}(\nabla u, \nabla u) + 2 \langle \nabla u, \nabla u_t\rangle - |\nabla u|^2 \left(S + f_t + \frac{n-2}{2t}\right)\right\} \mathrm{e}^{-f} \\ &= t \int \left\{2\,\mathrm{Ric}(\nabla u, \nabla u) + \frac{|\nabla u|^2}{t} + 2 \langle \nabla u, \nabla \,\Delta\, u\rangle - |\nabla u|^2 \left(|\nabla f|^2 - \Delta f\right)\right\} \mathrm{e}^{-f} . \end{aligned} \tag{4.13}$$

Using (4.11), the divergence theorem, and the Bochner formula, we see that

$$\begin{aligned}\int |\nabla u|^2 \left(|\nabla f|^2 - \Delta f\right) \mathrm{e}^{-f} &= \int |\nabla u|^2 \,\Delta\, \mathrm{e}^{-f} = \int \left(\Delta\, |\nabla u|^2\right) \mathrm{e}^{-f} \\ &= 2 \int \left\{|\mathrm{Hess}_u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla\, \Delta\, u\rangle\right\} \mathrm{e}^{-f} . \end{aligned} \tag{4.14}$$

Substituting this into (4.13) and then applying Lemma 4.5 give

$$(-t)^{\frac{n}{2}}\, D'(t) = t \int \left\{ \frac{|\nabla u|^2}{t} - 2 \left|\mathrm{Hess}_u\right|^2 \right\} \mathrm{e}^{-f} = -2\, t \int (\mathcal{L}_f\, u)^2\, \mathrm{e}^{-f}\,. \tag{4.15}$$

Therefore, since $I^2\, U' = I\, D' - D\, I'$ and $t\, I' = -2\, D$, we have

$$\begin{aligned} t\, I^2\, U' &= 2\, D^2 + t\, I\, D' \\ &= 2\,(-t)^{-n} \left(t \int (u\, \mathcal{L}_f\, u)\, \mathrm{e}^{-f} \right)^2 - 2\,(-t)^{-n} \left(\int u^2\, \mathrm{e}^{-f} \right) \left(t^2 \int (\mathcal{L}_f\, u)^2\, \mathrm{e}^{-f} \right). \end{aligned} \tag{4.16}$$

The Cauchy–Schwarz inequality then implies that $U' \geq 0$ (recall that $t < 0$). Furthermore, if $U'(t) = 0$, then the equality in this Cauchy–Schwarz inequality implies that (at time t) $\mathcal{L}_f\, u = c\, u$ for some constant c; the definition of $U(t)$ then gives that $c = -\frac{U(t)}{t}$. ∎

Funding

This work was supported by National Science Foundation Grants [DMS 1812142 and DMS 1707270].

References

[1] Almgren, Jr., F. "Q-valued functions minimizing Dirichlets integral and the regularity of area minimizing rectifiable currents up to codimension two." *Bull. Amer. Math. Soc. (N.S.)* 8, no. 2 (1983): 327–328.

[2] Bernstein, J. "Asymptotic structure of almost eigenfunctions of drift Laplacians on conical ends." *Amer. J. Math.* 142, no. 6 (2020): 1897–929.

[3] Cao, H. D. "Recent Progress on Ricci Solitons." In *Recent Advances in Geometric Analysis*, 1–38. Advanced Lectures in Mathematics 11. Somerville, MA: Int. Press, 2010.

[4] Chen, X.-Y. "A strong unique continuation theorem for parabolic equations." *Math. Ann.* 311, no. 4 (1998): 603–30.

[5] Chow, B., P. Lu, and L. Ni. *Hamiltons Ricci Flow*. GSM 77. Providence, RI: AMS, 2006.

[6] Colding, T. H. and W. P. Minicozzi II. "Harmonic functions with polynomial growth." *J. Differential Geom.* 46, no. 1 (1997): 1–77.

[7] Colding, T. H. and W. P. Minicozzi, II. "Heat equations in analysis, geometry and probability." (forthcoming).

[8] Colding, T. H. and W. P. Minicozzi, II. "Generic mean curvature flow I; generic singularities." *Ann. of Math. (2)* 175 (2012): 755–833.

[9] Colding, T. H. and W. P. Minicozzi, II. "Sharp frequency bounds for eigenfunctions of the Ornstein–Uhlenbeck operator." *Calc. Var. Partial Differential Equations (138)* 57, no. 5 (2018): 16.

[10] Colding, T. H. and W. P. Minicozzi, II. "Optimal bounds for ancient caloric functions." *Duke Math. J.* (forthcoming).

[11] Colding, T. H. and W. P. Minicozzi, II. "Complexity of parabolic systems." *Publ. Math. Inst. Hautes Etudes Sci.* 132 (2020): 83–135.

[12] Colding, T. H. and W. P. Minicozzi, II. "Singularities of Ricci flow and diffeomorphisms." (forthcoming).

[13] Escauriaza, L., G. Seregin, and V. Sverak. "Backward uniqueness for parabolic equations." *Arch. Rational Mech. Anal.* 169 (2003): 147–57.

[14] Escauriaza, L. "Carleman inequalities and the heat operator." *Duke Math. J.* 104, no. 1 (2000): 113–27.

[15] Evans, L. C. *Partial Differential Equations*, 2nd ed. Graduate Studies in Mathematics 19. Providence, RI: American Mathematical Society, 2010.

[16] Garofalo, N. and F. H. Lin. "Monotonicity properties of variational integrals, A_p weights and unique continuation." *Indiana Univ. Math. J.* 35, no. 2 (1986): 245–68.

[17] Hamilton, R. S. "A matrix Harnack estimate for the heat equation." *Comm. Anal. Geom.* 1 (1993): 88–99.

[18] Hamilton, R. S. "Monotonicity formulas for parabolic flows on manifolds." *Comm. Anal. Geom.* 1 (1993): 100–8.

[19] Han, Q. and F. H. Lin. "Nodal sets of solutions of parabolic equations. II." *Comm. Pure Appl. Math.* 47, no. 9 (1994): 1219–38.

[20] Kotschwar, B. and L. Wang. "Rigidity of asymptotically conical shrinking gradient Ricci solitons." *J. Differential Geom.* 100, no. 1 (2015): 55–108.

[21] Lin, F. H. "A uniqueness theorem for parabolic equations." *Comm. Pure Appl. Math.* 42 (1988): 125–36.

[22] Logunov, A. "Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure." *Ann. of Math. (2)* 187, no. 2, 1 (2018): 221–39.

[23] Poon, C. C. "Unique continuation for parabolic equations." *Comm. Partial Differential Equations* 21 (1996): 521–39.

[24] Wang, L. "Uniqueness of self-similar shrinkers with asymptotically conical ends." *J. Amer. Math. Soc.* 27, no. 3 (2014): 613–38.