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The next-to-top term in knot Floer homology

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Abstract. Let K be a null-homologous knot in a generalized L-space Z with $b_1(Z) \leq 1$. Let F be a Seifert surface of K with genus g. We show that if $\widehat{\mathrm{HFK}}(Z, K, [F], g)$ is supported in a single $\mathbb{Z}/2\mathbb{Z}$-grading, then

$$\operatorname{rank} \widehat{\mathrm{HFK}}(Z, K, [F], g-1) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(Z, K, [F], g).$$

1. Introduction

Knot Floer homology is an invariant for null-homologous knots in 3-manifolds introduced by Ozsváth and Szabó [10] and Rasmussen [17]. Suppose that F is a Thurston norm minimizing Seifert surface for a null-homologous knot $K \subset Z$, then $\widehat{\mathrm{HFK}}(Z, K, [F], g(F))$, which is known as "the topmost term" in knot Floer homology, captures a lot of information about the knot complement. For example, $\widehat{\mathrm{HFK}}(Z, K, [F], g(F))$ always has positive rank [9]. Moreover, $\widehat{\mathrm{HFK}}(Z, K, [F], g(F))$ has rank 1 if and only if F is a fiber of a fibration of $Z \setminus K$ over S^1, see [2, 5].

It is natural to ask if one can say similar things for other terms in $\widehat{\mathrm{HFK}}(Z, K)$. Baldwin and Vela-Vick [1, Question 1.11] asked whether $\widehat{\mathrm{HFK}}(S^3, K, g(K)-1)$ is always nontrivial. More specifically, Sivek [1, Question 1.12] asked whether we always have

$$\operatorname{rank} \widehat{\mathrm{HFK}}(S^3, K, g(K)-1) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(S^3, K, g(K)). \tag{1}$$

This inequality has been known for knots with thin knot Floer homology [8], L-space knots [4], fibered knots in any closed oriented 3-manifolds [1]. In this paper, we will prove (1) when $\widehat{\mathrm{HFK}}(Z, K, [F], g)$ is supported in a single $\mathbb{Z}/2\mathbb{Z}$-grading.

Recall that a closed, oriented 3-manifold Z is a *generalized L-space* if

$$\mathrm{HF}_{\mathrm{red}}(Z) = 0.$$

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In [11], an absolute $\mathbb{Z}/2\mathbb{Z}$-grading was defined on Heegaard Floer homology. When the underlying Spin^c structure is torsion, one can define an absolute $\mathbb{Q}$-grading.

Theorem 1.1. *Let Z be a generalized L-space with $b_1(Z) \le 1$, and let $K \subset Z$ be a null-homologous knot with a Thurston norm minimizing Seifert surface F of genus $g > 0$. Suppose that $\widehat{\mathrm{HFK}}(Z, K, [F], g)$ is supported in a single $\mathbb{Z}/2\mathbb{Z}$-grading. Then for any $d \in \mathbb{Q}$, we have*

$$\operatorname{rank} \widehat{\mathrm{HFK}}_{d-1}(Z, K, [F], g-1) \ge \operatorname{rank} \widehat{\mathrm{HFK}}_{d}(Z, K, [F], g).$$

Theorem 1.1 contains some known cases of the conjectural inequality (1), including fibered knots and knots with thin knot Floer homology.

To prove Theorem 1.1, we need the following result about HF^+.

Theorem 1.2. *Let Y be a closed oriented 3-manifold. Suppose that $G \subset Y$ is a closed oriented surface of genus $g > 2$. If there exist two elements $\gamma_1, \gamma_2 \in H_1(G)$ with $\gamma_1 \cdot \gamma_2 \ne 0$, such that their images in $H_1(Y)$ are linearly dependent, then the map U is trivial on $\mathrm{HF}^+(Y, [G], g-2; \mathbb{Q})$.*

Remark 1.3. When $b_1(Y) \le 2$, a simple intersection number argument shows that the image of $H_1(G; \mathbb{Q}) \to H_1(Y; \mathbb{Q})$ is at most 1-dimensional for any $G \subset Y$ with $[G] \ne 0 \in H_2(Y)$. So, Theorem 1.2 can be applied to this case. Ozsváth and Szabó have computed $\mathrm{HF}^+(S^3_0(K))$ in the cases when K is an L-space knot [7, Proposition 8.1] and when K is an alternating knot [8, Theorem 1.4]. One can directly check Theorem 1.2 in these two cases.

Remark 1.4. If $G \subset Y$ is a closed oriented surface of genus $g > 1$, the map U on $\mathrm{HF}^+(Y, [G], g-1)$ is trivial. The author first learned this result from Peter Ozsváth, and learned a sketch of a proof of it from Yankı Lekili using a similar argument as in [13, Theorem 3.1]. A proof of a more general result using the same idea as Lekili's was given by Wu [18]. The proof of Theorem 1.2 uses the same argument. Our proof justifies the use of the Künneth formula for HF^+ in [18].

This paper is organized as follows. In Section 2, we will collect some results about Heegaard Floer homology we will use. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.1.

We will use the following notations, If N is a submanifold of another manifold M, let $\nu(N)$ be a closed tubular neighborhood of N in M, and let $\nu^\circ(N)$ be the interior of $\nu(N)$. If K is a null-homologous knot in a 3-manifold Z, let $Z_{p/q}(K)$ be the manifold obtained by $\frac{p}{q}$-surgery on K.

2. Preliminaries on Heegaard Floer homology

Heegaard Floer homology [12], in its most fundamental form, assigns a package of invariants

$$\widehat{\mathrm{HF}}, \quad \mathrm{HF}^+, \quad \mathrm{HF}^-, \quad \mathrm{HF}^\infty$$

to a closed, connected, oriented 3-manifold Y equipped with a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

As described in [16, Section 2], let $\mathbf{HF}^-$ and $\mathbf{HF}^\infty$ denote the completions of HF^- and HF^∞ with respect to the maximal ideal (U) in the ring $\mathbb{Z}[U]$. By [16, (5)], when $c_1(\mathfrak{s})$ is non-torsion, $\mathbf{HF}^\infty(Y,\mathfrak{s}) = 0$. By [16, (4)], which is an exact sequence relating $\mathbf{HF}^-(Y,\mathfrak{s}), \mathbf{HF}^\infty(Y,\mathfrak{s}), \mathrm{HF}^+(Y,\mathfrak{s})$, one gets [16, (6)], which is

$$\mathrm{HF}^+(Y,\mathfrak{s}) \cong \mathbf{HF}^-(Y,\mathfrak{s}), \tag{2}$$

if $c_1(\mathfrak{s})$ is non-torsion.

Let $\mathrm{CF}^{\leq 0}(Y,\mathfrak{s})$ be the subcomplex of $\mathrm{CF}^\infty(Y,\mathfrak{s})$ which consists of $[\mathbf{x}, i]$, $i \leq 0$. This chain complex is clearly isomorphic to $\mathrm{CF}^-(Y,\mathfrak{s})$ via the U-action. We have a similar completion $\mathbf{HF}^{\leq 0}$.

We often use HF° to denote one of the above invariants.

When W is a cobordism from Y_1 to Y_2, and $\mathfrak{S} \in \mathrm{Spin}^c(W)$, there is an induced homomorphism

$$F^\circ_{W,\mathfrak{S}} \colon \mathrm{HF}^\circ(Y_1, \mathfrak{S}|_{Y_1}) \to \mathrm{HF}^\circ(Y_2, \mathfrak{S}|_{Y_2}).$$

In [12, Section 4.2.5], Ozsváth and Szabó defined an action of $H_1(Y)/\operatorname{Tors}$ on $\mathrm{HF}^\circ(Y)$. Given $\gamma \in H_1(Y)/\operatorname{Tors}$, there is a homomorphism

$$A_\gamma \colon \mathrm{HF}^\circ(Y) \to \mathrm{HF}^\circ(Y)$$

satisfying $A_\gamma^2 = 0$. The following theorem is the $\mathbf{HF}^{\leq 0}$ version of [3, Theorem 3.6]. See the remark following the proof.

Theorem 2.1. *Suppose Y_1, Y_2 are two closed, oriented, connected 3-manifolds, and W is a cobordism from Y_1 to Y_2. Let*

$$\mathbf{F}^{\leq 0}_W \colon \mathbf{HF}^{\leq 0}(Y_1) \to \mathbf{HF}^{\leq 0}(Y_2)$$

be the homomorphism induced by W. Suppose $\zeta_1 \subset Y_1$, $\zeta_2 \subset Y_2$ are two closed curves which are homologous in W. Then

$$\mathbf{F}^{\leq 0}_W \circ A_{[\zeta_1]} = A_{[\zeta_2]} \circ \mathbf{F}^{\leq 0}_W.$$

3. The next-to-top term in $\mathbf{HF}^+$

We will use $\mathbb{Q}$-coefficients for Heegaard Floer homology in the rest of this paper.

Let G be a closed oriented surface of genus $g > 2$. Let

$$V: S^3 \to G \times S^1$$

be the cobordism which consists of $2g$ one-handles and 1 two-handle with attaching curve being the Borromean knot B_g. Let $\mathfrak{S}_{g-2} \in \mathrm{Spin}^c(V)$ be the Spin^c structure with $\langle c_1(\mathfrak{S}_{g-2}), [G]\rangle = 2g-4$, and let $\mathfrak{s}_{g-2} \in \mathrm{Spin}^c(G \times S^1)$ be the restriction of $\mathfrak{S}_{g-2}$ to $G \times S^1$.

Let

$$\mathbf{F}^{\leq 0}_{V,\mathfrak{S}_{g-2}}: \mathbf{HF}^{\leq 0}(S^3) \to \mathbf{HF}^{\leq 0}(G \times S^1, \mathfrak{s}_{g-2})$$

be the map induced by the cobordism $(V, \mathfrak{S}_{g-2})$, and let

$$y = \mathbf{F}^{\leq 0}_{V,\mathfrak{S}_{g-2}}(\mathbf{1}). \tag{3}$$

In [10, Theorem 9.3], it is shown that

$$\mathrm{HF}^+(G \times S^1, \mathfrak{s}_{g-2}) \cong X(g,1) = H^0(G) \otimes \mathbb{Q}[U]/(U^2) \oplus H^1(G) \otimes \mathbb{Q}[U]/(U), \tag{4}$$

with the homological action given by

$$A_\gamma(\theta \otimes 1) = \mathrm{PD}(\gamma) \otimes \mathbf{1}, \quad A_\gamma(\eta \otimes 1) = \langle \eta, \gamma \rangle \otimes U. \tag{5}$$

Here θ is a generator of $H^0(G)$, and $\eta \in H^1(G)$. We will fix an identification as in (4). By abuse of notation, we often use θ to denote $\theta \otimes 1 \in X(g,1)$.

We will prove the following proposition.

Proposition 3.1. *The element y defined in (3) has the form $a\theta + bU\theta$ for some $a, b \in \mathbb{Q}$, $a \neq 0$.*

Let Y be a closed, oriented 3-manifold and suppose that G embeds into Y as a homologically essential surface. Consider the trivial cobordism

$$Y \times [0,1]: Y \to Y.$$

Let p be a point in G, and let W_1 be a tubular neighborhood of

$$(Y \times \{0\}) \cup \left(\mathsf{p} \times \left[0, \frac{1}{2}\right]\right) \cup \left(G \times \left\{\frac{1}{2}\right\}\right).$$

Then W_1 is a cobordism from Y to $Y\#(G \times S^1)$. Let $W_2 = \overline{Y \times [0,1] \setminus W_1}$.

Let $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ be a Spin^c structure satisfying $\langle c_1(\mathfrak{t}), [G]\rangle = 2(g-2)$, and let $\mathfrak{T} \in \mathrm{Spin}^c(Y \times [0,1])$ be the corresponding Spin^c structure. If we think of $G \times S^1$ as the boundary of a regular neighborhood of $G \times \{\frac{1}{2}\}$, then we clearly have $\mathfrak{T}|_{G\times S^1} = \mathfrak{s}_{g-2}$. By [6, Lemma 2.1],

$$F^{\circ}_{W_2,\mathfrak{T}|_{W_2}} \circ F^{\circ}_{W_1,\mathfrak{T}|_{W_1}} = \mathrm{id}\colon \mathrm{HF}^{\circ}(Y,\mathfrak{t}) \to \mathrm{HF}^{\circ}(Y,\mathfrak{t}). \tag{6}$$

Lemma 3.2. *Suppose that* $x \in \mathbf{HF}^{\le 0}(Y,\mathfrak{t})$, *then* $\mathbf{F}^{\le 0}_{W_1,\mathfrak{T}|_{W_1}}(x) = x \otimes y$. *Here* y *is defined in* (3), *and*

$$x \otimes y \in \mathbf{HF}^{\le 0}(Y,\mathfrak{t}) \otimes_{\mathbb{Q}[U]} \mathbf{HF}^{\le 0}(G\times S^1,\mathfrak{s}_{g-2}) \subset \mathbf{HF}^{\le 0}(Y\#(G\times S^1),\mathfrak{t}\#\mathfrak{s}_{g-2})$$

by the Künneth formula.

Proof. By [7, Proposition 4.4], there is a commutative diagram (note that we switch the order of the tensor product)

$$\begin{array}{ccc}
\mathbf{HF}^{\le 0}(Y,\mathfrak{t}) \otimes \mathbf{HF}^{\le 0}(S^3) & \xrightarrow{\mathbf{F}^{\le 0}_{Y\#S^3,\mathfrak{t}}} & \mathbf{HF}^{\le 0}(Y,\mathfrak{t}) \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes \mathbf{F}^{\le 0}_{V,\mathfrak{S}_{g-2}}} & & \Big\downarrow {\scriptstyle \mathbf{F}^{\le 0}_{W_1,\mathfrak{T}|_{W_1}}} \\
\mathbf{HF}^{\le 0}(Y,\mathfrak{t}) \otimes \mathbf{HF}^{\le 0}(G\times S^1,\mathfrak{s}_{g-2}) & \xrightarrow{\mathbf{F}^{\le 0}_{Y\#(G\times S^1),\mathfrak{t}\#\mathfrak{s}_{g-2}}} & \mathbf{HF}^{\le 0}(Y\#(G\times S^1),\mathfrak{t}\#\mathfrak{s}_{g-2})
\end{array}$$

Our conclusion follows from this commutative diagram. ∎

Proof of Proposition 3.1. We choose $Y = G \times S^1$ and $x = U\theta$. By (6) and Lemma 3.2,

$$U\theta = \mathbf{F}^{\le 0}_{W_2} \circ \mathbf{F}^{\le 0}_{W_1}(U\theta) = \mathbf{F}^{\le 0}_{W_2}(U\theta \otimes y) = \mathbf{F}^{\le 0}_{W_2}(\theta \otimes Uy).$$

Since $U\theta \ne 0$, $Uy \ne 0$. From the structure of $X(g,1)$ in (4), we see that any homogeneous element y (with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading) satisfying $Uy \ne 0$ must be of the form $a\theta + bU\theta$, $a \ne 0$. ∎

Lemma 3.3. *For any* $\gamma_1, \gamma_2 \in H_1(G) \subset H_1(G\times S^1)$, *we have*

$$A_{\gamma_2} \circ A_{\gamma_1}(y) = (\gamma_1 \cdot \gamma_2)Uy.$$

Proof. By Proposition 3.1, $y = a\theta + bU\theta$. By the module structure of $X(g,1)$ in (4) and (5), $Uy = aU\theta$, and

$$A_{\gamma_2} \circ A_{\gamma_1}(y) = \langle \mathrm{PD}(\gamma_1), \gamma_2\rangle aU\theta = (\gamma_1 \cdot \gamma_2)aU\theta. \qquad ∎$$

Proof of Theorem 1.2. Let $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ be as above. Assume further that $U \ne 0$ on $\mathrm{HF}^+(Y,\mathfrak{t})$. By (2), $Ux \ne 0$ for some $x \in \mathbf{HF}^{\le 0}(Y,\mathfrak{t})$. By (6) and Lemma 3.2,

$$x = \mathbf{F}^{\le 0}_{W_2} \circ \mathbf{F}^{\le 0}_{W_1}(x) = \mathbf{F}^{\le 0}_{W_2}(x \otimes y). \tag{7}$$

Let $c_i \subset G$ be a closed curve representing γ_i, $i = 1, 2$. Let $\gamma_i' \in H_1(Y\#(G \times S^1))$ be represented by $c_i \times \text{point} \subset G \times S^1$, and let $\gamma_i'' \in H_1(Y)$ be represented by $c_i \subset G \subset Y$. Then $(c_i \times [\frac{1}{2}, 1]) \cap W_2$ defines a homology between γ_i' and γ_i''. By Lemma 3.3 and (7) we have

$$\begin{aligned}(\gamma_1 \cdot \gamma_2)Ux &= \mathbf{F}^{\leq 0}_{W_2}(x \otimes (\gamma_1 \cdot \gamma_2)Uy) \\ &= \mathbf{F}^{\leq 0}_{W_2}(x \otimes A_{\gamma_2} \circ A_{\gamma_1}(y)) \\ &= \mathbf{F}^{\leq 0}_{W_2}(A_{\gamma_2'} \circ A_{\gamma_1'}(x \otimes y)),\end{aligned}$$

where the last equality follows from the fact that the actions of $A_{\gamma_1'}$ and $A_{\gamma_2'}$ on the $\mathbf{HF}^{\leq 0}(Y, \mathfrak{t})$ factor are trivial.

Since γ_1'' and γ_2'' in $H_1(Y)$ are linearly dependent, we get

$$\mathbf{F}^{\leq 0}_{W_2}(A_{\gamma_2'} \circ A_{\gamma_1'}(x \otimes y)) = A_{\gamma_2''} \circ A_{\gamma_1''}\mathbf{F}^{\leq 0}_{W_2}(x \otimes y) = 0$$

by Theorem 2.1 and the fact that $A_\gamma^2 = 0$ for any $\gamma \in H_1(Y)$. This contradicts the assumption that $\gamma_1 \cdot \gamma_2 \neq 0$ and $Ux \neq 0$. ∎

4. Proof of the main theorem

Let K be a null-homologous knot in a generalized L-space Z. Let F be a Thurston norm minimizing Seifert surface of K with genus $g > 2$. By the proof of [10, Theorem 5.1], we can choose a Heegaard diagram for (Z, K) such that

$$\widehat{\mathrm{CFK}}(Z, K, [F], i) = 0 \quad \text{if } |i| > g.$$

Given $\mathfrak{s} \in \mathrm{Spin}^c(Z)$, let

$$C = \mathrm{CFK}^\infty(Z, K, \mathfrak{s}, [F]),$$

then

$$C(i, j) = 0, \quad \text{if } |i - j| > g. \tag{8}$$

Let

$$A_k^+ = C\{i \geq 0 \text{ or } j \geq k\}, \quad B^+ = C\{i \geq 0\}$$

and define maps

$$v_k^+, h_k^+ \colon A_k^+ \to B^+$$

as in [15]. More precisely, v_k^+ is the natural quotient map (or the vertical projection) onto B^+, and h_k^+ is essentially a horizontal projection. By [15, Theorem 2.3],

v_k^+ and h_k^+ can be identified with certain chain maps induced by a two-handle cobordism $W_n'(K)\colon Z_n(K) \to Z$.

When $\mathfrak{s}$ is a torsion Spin^c structure, by [14], there is an absolute Q-grading on $\mathrm{HF}^+(Z, \mathfrak{s})$, so there is an absolute Q-grading on C. The shift of the absolute grading of maps induced by cobordisms is computed as in [14, Theorem 7.1]. In particular, if we identify v_k^+ and h_k^+ with maps induced by the cobordism $W_n'(K)$, the difference between the grading shifts of v_k^+ and h_k^+ is

$$-\frac{(2k-n)^2 - (2k+n)^2}{4n} = 2k. \tag{9}$$

Proposition 4.1. *Let $\widehat{F}$ be the closed surface in $Z_0(K)$ obtained by capping off ∂F with a disk. Let $\mathfrak{s}_{g-2} \in \mathrm{Spin}^c(Z_0(K))$ be the Spin^c structure satisfying that*

$$\mathfrak{s}_{g-2}|_{Z\setminus\nu^\circ(K)} = \mathfrak{s}|_{Z\setminus\nu^\circ(K)}, \quad \langle c_1(\mathfrak{s}_{g-2}), [\widehat{F}]\rangle = 2(g-2).$$

If there exists an element $a \in H_*(C\{i < 0, j \geq g-2\})$ such that $Ua \neq 0$, then there also exists an element $a' \in \mathrm{HF}^+(Z_0(K), \mathfrak{s}_{g-2})$ such that $Ua' \neq 0$.

Proof. Consider the short exact sequence of chain complexes

$$0 \to C\{i < 0, j \geq g-2\} \to A_{g-2}^+ \xrightarrow{v_{g-2}^+} B^+ \to 0, \tag{10}$$

which induces an exact triangle.

By [15, Section 4.8], $\mathrm{CF}^+(Z_0(K), \mathfrak{s}_{g-2})$ is quasi-isomorphic to the mapping cone of

$$v_{g-2}^+ + h_{g-2}^+\colon A_{g-2}^+ \to B^+.$$

So, there is also an exact triangle. We will use a standard argument to compare these two exact triangles.

Case 1: $\mathfrak{s}$ is a torsion Spin^c structure. Since Z is a generalized L-space,

$$v = (v_{g-2}^+)_*\colon H_*(A_{g-2}^+) \to H_*(B^+)$$

is surjective. So

$$H_*(C\{i < 0, j \geq g-2\}) \cong \ker v$$

as a $\mathbb{Q}[U]$-module.

By (9), v_{g-2}^+ and h_{g-2}^+ have different grading shifts. Since Z is a generalized L-space,

$$v + h = (v_{g-2}^+)_* + (h_{g-2}^+)_*\colon H_*(A_{g-2}^+) \to H_*(B^+)$$

is surjective. So

$$\mathrm{HF}^+(Z_0(K), \mathfrak{s}_{g-2}) \cong \ker(v+h)$$

as a $\mathbb{Q}[U]$-module.

Since v is homogeneous and surjective, there exists a homogeneous homomorphism ρ: $H_*(B^+) \to H_*(A^+_{g-2})$ satisfying

$$v \circ \rho = \mathrm{id}\,.$$

By (9) and the assumption that $g(F) > 2$, the grading shift of h is strictly less than the grading shift of v, so the grading shift of ρh is negative. As the grading of $H_*(A^+_{g-2})$ is bounded from below, for any $x \in H_*(A^+_{g-2})$, $(\rho h)^m(x) = 0$ when m is sufficiently large. So, the map

$$\mathrm{id} - \rho h + (\rho h)^2 - (\rho h)^3 + \cdots : H_*(A^+_{g-2}) \to H_*(A^+_{g-2})$$

is well defined, and it maps $\ker v$ to $\ker(v + h)$.

Assume that $a \in \ker v$ is a homogeneous element with $Ua \neq 0$. Then

$$a' = (\mathrm{id} - \rho h + (\rho h)^2 - (\rho h)^3 + \cdots)(a) = a + \text{lower grading terms} \in \ker(v + h)$$

so

$$Ua' = Ua + \text{lower grading terms}$$

which is nonzero since $Ua \neq 0$.

Case 2. $\mathfrak{s}$ is non-torsion. Since Z is a generalized L-space, $\mathrm{HF}^+(Z, \mathfrak{s}) = 0$. Namely, $H_*(B^+) = 0$. By the two exact triangles at the beginning of this proof, we have

$$H_*(C\{i < 0, j \geq g - 2\}) \cong \mathrm{HF}^+(Z_0(K), \mathfrak{s}_{g-2})$$

as $\mathbb{Q}[U]$-modules. So, our conclusion holds. ■

We will use the following elementary lemma in linear algebra.

Lemma 4.2. *Let* V, W *be two linear spaces over a field* $\mathbb{F}$*, and let* $\mathsf{V}_1, \mathsf{W}_1$ *be their subspaces, respectively. If* $v \in \mathsf{V} \setminus \mathsf{V}_1$*,* $w \in \mathsf{W} \setminus \mathsf{W}_1$*, then*

$$v \otimes w \notin \mathsf{V}_1 \otimes \mathsf{W} + \mathsf{V} \otimes \mathsf{W}_1.$$

Proof. Suppose that $\dim \mathsf{V} = m$, $\dim \mathsf{V}_1 = m_1$, $\dim \mathsf{W} = n$, $\dim \mathsf{W}_1 = n_1$. We can choose a basis

$$v_1, \quad \ldots, \quad v_m$$

of V, such that $v_1, \ldots, v_{m_1}$ is a basis of V_1, and $v = v_{m_1+1}$. Similarly, we choose a basis

$$w_1, \quad \ldots, \quad w_n$$

of W, such that $w_1, \ldots, w_{n_1}$ is a basis of W_1, and $w = w_{n_1+1}$. Then

$$v_i \otimes w_j, \quad 1 \leq i \leq m, 1 \leq j \leq n,$$

is a basis for $\mathsf{V} \otimes \mathsf{W}$. Now, $\mathsf{V}_1 \otimes \mathsf{W} + \mathsf{V} \otimes \mathsf{W}_1$ is spanned by

$$v_i \otimes w_j, \quad 1 \le i \le m_1 \text{ or } 1 \le j \le n_1.$$

So, $v \otimes w = v_{m_1+1} \otimes w_{n_1+1}$ is not in this subspace. ■

Let ∂ be the differential in $C = \mathrm{CFK}^\infty$, ∂_0 be the component of ∂ which preserves the (i, j)-grading, ∂_z be the component of ∂ which decreases the (i, j)-grading by $(0, 1)$, and ∂_w be the component which decreases the (i, j)-grading by $(1, 0)$. Since $\partial^2 = 0$, each homogeneous summand of ∂^2 is zero. If we consider the summand of ∂^2 which preserves the (i, j)-grading, we get

$$\partial_0^2 = 0.$$

Similarly, considering the summands of ∂^2 which decrease the (i, j)-grading by $(0, 1)$, $(1, 0)$, and $(1, 1)$, respectively, we get

$$\partial_z \circ \partial_0 + \partial_0 \circ \partial_z = 0, \quad \partial_w \circ \partial_0 + \partial_0 \circ \partial_w = 0, \tag{11}$$

and

$$\partial_w \circ \partial_z + \partial_{zw} \circ \partial_0 + \partial_0 \circ \partial_{zw} = 0 \quad \text{on } C(0, g), \tag{12}$$

where in the last equation we use the fact that $C(-1, g) = 0$ (see (8)).

It follows from (11) that ∂_z and ∂_w induces homomorphisms on the homology with respect to the differential ∂_0, denoted by $(\partial_z)_*$ and $(\partial_w)_*$. By (12),

$$(\partial_w)_* \circ (\partial_z)_* = 0 \tag{13}$$

on $H_*(C(0, g))$.

Theorem 4.3. *Let Z be a generalized L-space, $K \subset Z$ be a null-homologous knot. Let F be a Seifert surface of K with genus $g > 2$. Let $d \in \mathbb{Q}$ satisfy*

$$\widehat{\mathrm{HFK}}_{d\pm1}(Z, K, [F], g) = 0. \tag{14}$$

If there exist two elements $\gamma_1, \gamma_2 \in H_1(F)$ with $\gamma_1 \cdot \gamma_2 \neq 0$, such that the images of γ_1, γ_2 in $H_1(Z)$ are linearly dependent, then

$$\operatorname{rank} \widehat{\mathrm{HFK}}_d(Z, K, [F], g) \le \operatorname{rank} \widehat{\mathrm{HFK}}_{d-1}(Z, K, [F], g-1).$$

Proof. By (8), the chain complex $C\{i < 0, j \ge g-2\}$ has the form

$$\begin{array}{ccc} & & C(-1, g-1) \\ & \swarrow^{\partial_{zw}} & \downarrow \partial_z \\ C(-2, g-2) & \xleftarrow[\partial_w]{} & C(-1, g-2) \end{array} \tag{15}$$

where

$$C_{*-2}(-1, g-1) \cong C_{*-4}(-2, g-2) \cong \widehat{\mathrm{CFK}}_*(Z, K, [F], g),$$

and

$$C_{*-2}(-1, g-2) \cong \widehat{\mathrm{CFK}}_*(Z, K, [F], g-1).$$

By abuse of notation, we will use ∂_z and ∂_w to denote their restrictions

$$\partial_z \colon \widehat{\mathrm{CFK}}_d(Z, K, [F], g) \to \widehat{\mathrm{CFK}}_{d-1}(Z, K, [F], g-1)$$

and

$$\partial_w \colon \widehat{\mathrm{CFK}}_{d-1}(Z, K, [F], g-1) \to \widehat{\mathrm{CFK}}_d(Z, K, [F], g).$$

Using (13), we have

$$\begin{aligned} &\operatorname{rank}\ker(\partial_z)_* \\ &\quad = \operatorname{rank}\widehat{\mathrm{HFK}}_d(Z, K, [F], g) - \operatorname{rank}\operatorname{im}(\partial_z)_* \\ &\quad \geq \operatorname{rank}\widehat{\mathrm{HFK}}_d(Z, K, [F], g) - \operatorname{rank}\ker(\partial_w)_* \\ &\quad = \operatorname{rank}\widehat{\mathrm{HFK}}_d(Z, K, [F], g) - \operatorname{rank}\widehat{\mathrm{HFK}}_{d-1}(Z, K, [F], g-1) + \operatorname{rank}\operatorname{im}(\partial_w)_*. \end{aligned}$$

If

$$\operatorname{rank}\widehat{\mathrm{HFK}}_d(Z, K, [F], g) > \operatorname{rank}\widehat{\mathrm{HFK}}_{d-1}(Z, K, [F], g-1), \tag{16}$$

then

$$\operatorname{rank}\ker(\partial_z)_* > \operatorname{rank}\operatorname{im}(\partial_w)_*,$$

so there exists an element $x \in \ker(\partial_z)_*$, such that $Ux \notin \operatorname{im}(\partial_w)_*$. Let $\xi \in C_{d-2}(-1, g-1)$ be a closed chain representing x, then $\partial_z(\xi)$ is an exact chain in $C_{d-3}(-1, g-2)$. So, there exists an element $\eta \in C_{d-2}(-1, g-2)$ with $\partial_0\eta = \partial_z(\xi)$. By (11) and (12),

$$\partial_0\partial_w\eta = -\partial_w\partial_0\eta = -\partial_w\partial_z(\xi) = \partial_0\partial_{zw}(\xi).$$

So, $\partial_w\eta - \partial_{zw}(\xi)$ is a closed chain in $C_{d-3}(-2, g-2) \cong \widehat{\mathrm{CFK}}_{d+1}(Z, K, [F], g)$. By (14), $\partial_w\eta - \partial_{zw}(\xi)$ is exact, so there exists an element $\zeta \in C_{d-2}(-2, g-2)$ with $\partial_0\zeta = \partial_w\eta - \partial_{zw}(\xi)$. This means that $\xi - \eta + \zeta$ is a cycle in the mapping cone (15).

Now, we want to prove $U(\xi - \eta + \zeta) = U\xi$ is not exact in (15). Otherwise, assume

$$U\xi = \partial(\xi' + \eta' + \zeta'), \tag{17}$$

where

$$\xi' \in C_{d-3}(-1, g-1), \quad \eta' \in C_{d-3}(-1, g-2), \quad \zeta' \in C_{d-3}(-2, g-2).$$

Considering the components of (17), we get

$$0 = \partial_0 \xi', \tag{18}$$

$$0 = \partial_z \xi' + \partial_0 \eta', \tag{19}$$

$$U\xi = \partial_{zw}\xi' + \partial_w \eta' + \partial_0 \zeta'. \tag{20}$$

By (18), ξ' is a cycle in $C_{d-3}(-1, g-1) \cong \widehat{\mathrm{CFK}}_{d-1}(Z, K, [F], g)$. By (14), ξ' is exact, so there exists $\omega \in C_{d-2}(-1, g-1)$ with $\partial_0\omega = \xi'$. Using (11) and (19), we get

$$\partial_0(\eta' - \partial_z\omega) = 0.$$

Using (12) and (20), we get

$$U\xi = -\partial_0\partial_{zw}\omega + \partial_w(\eta' - \partial_z\omega) + \partial_0\zeta',$$

which means that $U\xi$ is homologous to an element in $\partial_w(\ker \partial_0)$. Since $[U\xi] = Ux \notin \mathrm{im}(\partial_w)_*$, we get a contradiction.

Now, we have proved that $U \neq 0$ in the mapping cone (15). By Proposition 4.1, we have $U \neq 0$ in $\mathrm{HF}^+(Z_0(K), [\widehat{F}], g-2)$, a contradiction to Theorem 1.2. ∎

Remark 4.4. The above proof can be greatly simplified if we use the "reduction lemma" [4, 17] in homological algebra. In fact, the author's original approach was using the Reduction Lemma. The reason that we choose the current argument is that we want to understand the diagonal map

$$H_*(C(-1, g-1)) \to H_*(C(-2, g-2))$$

after reduction, which may be important if we try to generalize our result to other knots.

Proof of Theorem 1.1. When $g > 2$, this follows from Theorem 4.3.

If $g = 2$, we assume (16) holds. As in the proof of Theorem 4.3, there exists an element $x \in \ker(\partial_z)_*$, such that $Ux \notin \mathrm{im}(\partial_w)_*$. Consider the element

$$x \otimes x \in \widehat{\mathrm{HFK}}_d(Z, K, [F], g) \otimes \widehat{\mathrm{HFK}}_d(Z, K, g) \cong \widehat{\mathrm{HFK}}_{2d}(Z\#Z, K\#K, [F\natural F], 2g).$$

In the complex $\mathrm{CFK}^\infty(Z\#Z, K\#K)$, we can check $x \otimes x \in \ker(\partial_z)_*$, while, by Lemma 4.2, $U(x \otimes x) \notin \mathrm{im}(\partial_w)_*$. Let γ_1, γ_2 be a pair of elements in $H_1(F)$ with $\gamma_1 \cdot \gamma_2 \neq 0$. We can think of γ_1, γ_2 as elements in the first summand of $H_1(F\natural F) \cong H_1(F) \oplus H_1(F)$. Then the images of γ_1, γ_2 in $H_1(Z\#Z)$ are linearly dependent. So, we can apply Theorem 1.2 to get a contradiction as in the proof of Theorem 4.3.

The case $g = 1$ can be proved similarly by considering a three-fold connected sum. ∎

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