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The prism manifold realization problem III

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Abstract

Every prism manifold can be parametrized by a pair of relatively prime integers $p > 1$ and q. In our earlier papers, we determined a complete list of prism manifolds $P(p, q)$ that can be realized by positive integral surgeries on knots in S^3 when $q < 0$ or $q > p$; in the present work, we solve the case when $0 < q < p$. This completes the solution of the realization problem for prism manifolds.

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1 | INTRODUCTION

Let $P(p, q)$ be an oriented prism manifold with Seifert invariants

$$(-1; (2, 1), (2, 1), (p, q)),$$

where q and $p > 1$ are relatively prime integers. See [1, Section 2] for the convention of Seifert invariants and basic topological properties of prism manifolds. In [1, 2], we solved the Dehn surgery realization problem of prism manifolds for $q < 0$ and for $q > p$. The theme of the present work is to settle the remaining case $0 < q < p$. In [1, Tables 1 and 2], the authors give a tabulation of prism manifolds that can be obtained by positive integral Dehn surgery on *Berge–Kang knots* [4]. The tables conjecturally account for all realizable prism manifolds; in particular, [1, Table 2] suggests that for a realizable $P(p, q)$ with $q > 0$, we must have $p \leqslant 2q + 1$. Indeed, this is the case:

TABLE 1 $\mathcal{P}$, table of $P(p, q)$ that are realizable

Type	$P(p,q)$ (let $P(-p,-q) = P(p,q)$)	Range of parameters (p and r are always odd, $\lvert p \rvert > 1$)
1A	$P(p, -\frac{1}{2}(p^2 - 3p + 4))$	$p \neq 3, 5$
1B	$P(p, -\frac{1}{22}(p^2 - 3p + 4))$	$p \equiv 17$ or $19 \pmod{22}$ $\lvert p \rvert > 22$
2	$P(p, -\frac{1}{\lvert 4r+2 \rvert}\lvert r^2 p + 1 \rvert)$	$p \equiv 2r - 3 \pmod{4r+2}$ $r \equiv -1 \pmod 4$ $r \neq -1, 3$
3A	$P(p, -\frac{1}{2r}(p+1)(p+4))$	$p \equiv -1 \pmod{2r}$ $r \geqslant 1$ and $p \geqslant 4r - 1$ if $p > 0$ $r \geqslant 5$ and $p \leqslant -4r - 1$ if $p < 0$
3B	$P(p, -\frac{1}{2r}(p+1)(p+4))$	$p \equiv r - 4 \pmod{2r}$ $r \geqslant 5$ and $p \geqslant 3r - 4$ if $p > 0$ $r \geqslant 1$ and $p \leqslant -3r - 4$ if $p < 0$
4	$P(p, -\frac{1}{2r^2}\lvert (2r+1)^2 p + 1 \rvert)$	$p \equiv 4r - 1 \pmod{2r^2}$ $r \neq -3, -1, 1$
5	$P(p, -\frac{1}{r^2 - 2r - 1}\lvert r^2 p + 1 \rvert)$	$p \equiv 2r - 5 \pmod{r^2 - 2r - 1}$ $r \neq 1$
Sporadic	$P(11, -30)$, $P(17, -31)$, $P(13, -47)$, $P(23, -64)$ $P(11, 19)$, $P(13, 34)$	

Theorem 1.1. *If $P(p, q)$ with $q > 0$ can be obtained by surgery on a knot $K \subset S^3$, then $p \leqslant 2q + 1$. If $p = 2q + 1$, then K is the torus knot $T(2q + 1, 2)$.*

Doig, in [7, Conjecture 12], conjectured that if $P(p, q)$ is realizable, then $p \leqslant 2|q| + 1$. The main result of [1] settles the conjecture for $q < 0$; Theorem 1.1 verifies it for $q > 0$.

Our second main result, Theorem 1.2, provides the solution of the realization problem for those $P(p, q)$ with $q < p < 2q$.

Theorem 1.2. *The prism manifold $P(p, q)$ with $q < p < 2q$ can be obtained by $4q$–surgery on a knot $K \subset S^3$ if and only if $q = \frac{1}{r^2 - 2r - 1}(r^2 p - 1)$, with $r \leqslant -3$ odd and $p \equiv -2r + 5 \pmod{r^2 - 2r - 1}$, $p \geqslant -2r + 5$. Moreover, in this case, there exists a Berge–Kang knot K_0 such that $P(p, q) \cong S^3_{4q}(K_0)$, and that K and K_0 have isomorphic knot Floer homology groups.*

Remark 1.3. If we allow $r = -1$ in Theorem 1.2, we get $p = 2q + 1$: see Theorem 1.1.

1.1 | The complete list of realizable prism manifolds

Theorems 1.1 and 1.2 and our earlier results [1, 2] give a complete classification of prism manifolds which can be obtained by Dehn surgery on knots in S^3. These prism manifolds are tabulated in Table 1.

Remark 1.4. Table 1 is essentially the union of [1, Table 1 and Table 2], with input of the range of parameters from [2, Table 2] and Theorem 1.2. There are two differences from [1, Table 2]. The first difference is that we adapt the convention that $P(-p,-q) = P(p,q)$ in the table, so that the cases $pq > 0$ and $pq < 0$ can be unified in one expression. The second difference is that we move the family $P(p, -\frac{25p+1}{18})$, $p < 0$, from Type 4 to Type 2, so that the ranges of r are the same despite of the sign of pq.

Remark 1.5. In the arXiv version of [1], for each prism manifold in Table 1, we listed a Berge–Kang knot realizing the corresponding surgery following the work of Berge–Kang [4]. However, since Berge–Kang's work is not publicly available, we did not include this list of Berge–Kang knots in the published version of [1]. An explicit list of primitive/Seifert-fibered knots admitting prism manifold surgeries was given in [21], independent of [4]. The existence of such Berge–Kang knots now follows from [21].

Remark 1.6. In Table 1, we divide the surgeries into six different types so that no surgery appears in more than one type. However, a prism manifold may appear in different types, in the sense that it arises from multiple changemaker vectors and it is obtained from surgeries on Berge–Kang knots belonging to different families. Such prism manifolds are the two infinite families

$$P(8s+3, -(16s+14)) \text{ and } P(8s+13, 16s+18), \quad s \geqslant 0,$$

and

$$P(11,-18), P(5,22), P(25,36), P(43,117).$$

(The two infinite families are essentially the same family, if we allow $s < 0$ and the surgery slope to be negative.) More information about these surgeries can be found in [1, Table 3] and [2, Table 3].

1.2 | The spherical manifold realization problem

The spherical manifold realization problem asks which spherical manifolds arise from positive integral surgery along a knot in S^3. Theorems 1.1 and 1.2 and our earlier results [1, 2], combined with Gu's work [11] and Greene's work [9], provide a complete classification of realizable spherical manifolds. The interest is in finding a complete classification of knots in S^3 on which Dehn surgery produce spherical manifolds. In [3], Berge proposed a complete list of knots in S^3 with lens space surgeries. Indeed, Berge's conjecture states that the *P/P knots* form a complete list of knots in S^3 that admit lens space surgeries. All the known examples of knots on which surgeries will result in non-lens space spherical manifolds are *P/SF knots*. We repeat the following conjecture from [1, Conjecture 1.7]: it is a generalization of Berge's conjecture.

Conjecture 1.7. *Let K be a knot in S^3 that admits an integral surgery to a spherical manifold. Then K is either a P/SF or a P/P knot.*

1.3 | Methodology

We first provide a brief overview of the methodology undertaken to solve the prism manifold realization problem in the cases $q < 0$ and $q > p$: the proof in both cases draws inspiration from that of Greene for lens spaces [9]. We then discuss how (and why) the methodology is modified for the case of the present work.

We first require a combinatorial definition.

Definition 1.8. A vector $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{n+1}) \in \mathbb{Z}^{n+2}$ that satisfies $0 \leqslant \sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_{n+1}$ is a *changemaker vector* if for every k, with $0 \leqslant k \leqslant \sigma_0 + \sigma_1 + \cdots + \sigma_{n+1}$, there exists a subset $S \subset \{0, 1, \dots, n+1\}$ such that $k = \sum_{i \in S} \sigma_i$.

The key idea is to use the *correction terms* in Heegaard Floer homology in tandem with Donaldson's Theorem A. The following is immediate from [9, Theorem 3.3].

Theorem 1.9. *Suppose that $P(p,q)$ bounds a sharp four-manifold $X(p,q)$. If $P(p,q)$ arises from positive integer surgery on a knot K in S^3, then the intersection lattice on $X(p,q)$ embeds as the orthogonal complement $\sigma^\perp$ of some changemaker vector $\sigma \in \mathbb{Z}^{n+2}$, with $n+1 = b_2(X)$.*

See Section 5 for the definition of a *sharp* four-manifold, and see Subsection 1.4 for the definition of the *intersection lattice*. When $q < 0$ or $q > p$, it turns out that $P(p,q)$ bounds a sharp four-manifold $X(p,q)$. We then solved a combinatorial problem: we classified all lattices isomorphic to the intersection lattice of $X(p,q)$, whose complements are changemakers in $\mathbb{Z}^{n+2}$. There is a heavy analysis of lattices involved that forms the main body of [1, 2]. Finally, we verified that for every (p,q) corresponding to such a lattice, $P(p,q)$ is indeed realized by surgery on a P/SF knot.

We now turn our attention to the case $0 < q < p$. In light of Theorem 1.1, it suffices to consider $q < p < 2q$. When $q < p < 2q$, $P(p,q)$ does not bound a sharp four-manifold. Thus, we cannot use the embedding restriction of Theorem 1.9 — an essential step to the classification of realizable prism manifolds in the previous two cases. Our strategy to prove Theorem 1.2 is to replace Theorem 1.9 with another lattice theoretic obstruction for $P(p,q)$ to being realizable, as follows. The prism manifold $P(2,1)$ bounds a rational homology four-ball Z_2 (the left two components of Figure 2 where the 0-framed unknot is replaced by a dotted circle and $a_{-1} = 2$); and that there exists a negative definite cobordism W from $P(2,1)$ to $P(p,q)$ (the right $n+1$ components of Figure 2). Suppose that $P(p,q)$ arises from surgery on a knot $K \subset S^3$, and let $W_{4q} = W_{4q}(K)$ be the corresponding two-handle cobordism obtained by attaching a two-handle to the four-ball along the knot K with framing $4q$. Form $Z := Z_2 \cup_{P(2,1)} W$; it will be a smooth four-manifold with boundary $P(p,q)$. The intersection lattice on Z is $\Lambda(q,-p)$, which is defined in Definition 3.1. Form $X := W \cup (-W_{4q})$. We prove that the intersection lattice on X is isomorphic to $D_4 \oplus \mathbb{Z}^{n-2}$, where D_4 is the sublattice of $\mathbb{Z}^4$ consisting of vectors the sum of whose coordinates is even. Finally, form $\widehat{X} := Z \cup (-W_{4q})$; see Figure 1. It follows that $\widehat{X}$ is a smooth, closed, simply connected, negative definite four-manifold with $b_2(Z) = n+2$ for some $n \geqslant 0$. Now, Donaldson's Theorem A [8] implies that the intersection lattice on $\widehat{X}$ is the Euclidean integer lattice $\mathbb{Z}^{n+2}$. This provides a necessary condition for $P(p,q)$ to be realizable: the lattice $\Lambda(q,-p)$ embeds as a codimension one sublattice of $\mathbb{Z}^{n+2}$. Our new obstruction now reads as follows:

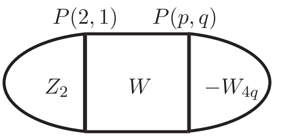

FIGURE 1 Schematic picture of the closed four-manifold $\widehat{X} = Z_2 \cup W \cup -W_{4q}$. We have $X = W \cup_{P(p,q)} -W_{4q}$, $Z = Z_2 \cup_{P(2,1)} W$

Theorem 1.10. *Suppose $P(p,q)$ with $q < p < 2q$ arises from positive integer surgery on a knot K in S^3.*

(a) *The linear lattice $\Lambda(q,-p)$ embeds as the orthogonal complement to a changemaker $\sigma \in \mathbb{Z}^{n+2}$, $n+1 = b_2(Z)$.*
(b) *There is an embedding of $D_4 \oplus \mathbb{Z}^{n-2}$ into $\mathbb{Z}^{n+2}$ such that there exists some short characteristic covector χ for $D_4 \oplus \mathbb{Z}^{n-2}$ with $\langle \chi, \sigma \rangle = i$ if and only if $-2q + g(K) \leqslant i \leqslant 2q - g(K)$.*

The strategy is now apparent: determine the list of all pairs (p,q) which pass the embedding restriction of Theorem 1.10. Finally, we verify that every manifold in our list is indeed realized by a knot surgery: we do so by comparing the list with the list of realizable manifolds tabulated in [1, Table 2]. The fact that the manifolds in [1, Table 2] are realizable is proved in [21]. It must be noted that Part (a) of Theorem 1.10 only provides a necessary condition for the prism manifold $P(p,q)$ to be realizable. Indeed, it is easy to find pairs (p,q) that satisfy Part (a) of Theorem 1.10, but the corresponding prism manifolds are not realizable; for example, $P(13,9)$ and $P(16,9)$. The 9-surgery on the torus knot $T(2,5)$ is $L(9,13) \cong L(9,16)$, then work of Greene [9] shows that the corresponding linear lattice satisfies Part (a) of Theorem 1.10. However, the manifold $P(16,9)$ is not realizable because of the parity of 16 (p is always odd for a realizable $P(p,q)$ [1]); and neither is $P(13,9)$ by Theorem 1.2.

In the previous cases $q < 0$ and $q > p$ as well as in the lens space realization problem [9], the first step was finding a sharp four-manifold bounded by $P(p,q)$ (respectively, the lens space $L(p,q)$): in each case, a negative definite four-manifold was found; then it was almost immediate from the previous works of Ozsváth and Szabó [16, 18] that the four-manifold is sharp. For the case at hand, however, $P(p,q)$ does not bound a sharp four-manifold. We need to carefully analyze the d-*invariants* of $P(p,q)$ in each Spin^c structure in terms of the d-invariants of certain Spin^c structures of $P(2,1)$ and the grading shift of the cobordism W. In particular, we generalize the notion of sharpness to cobordisms between rational homology spheres, and show that the cobordism W is sharp (Proposition 5.3): again, see Figure 1. Using that the intersection lattice on X is isomorphic to $D^4 \oplus \mathbb{Z}^{n-2}$, it will be immediate that X is a sharp four-manifold (Corollary 6.4). Using this finding, we are able to prove Theorem 1.10 and translate it into a more practical condition on the changemaker vector σ (Proposition 6.11).

1.4 | Notations

We use homology groups with integer coefficients throughout the paper. For a compact four-manifold X, regard $H_2(X)$ as equipped with the intersection pairing Q_X on X. Also, we refer to $(H_2(X), -Q_X)$ as the *intersection lattice* on X, where $-Q_X$ denotes the negation of the pairing of

Q_X. Finally, we call an oriented three-manifold Y a *realizable manifold* if it can be obtained by positive integral surgery on a knot in S^3 .

1.5 | Organization

This paper is organized as follows. In Section 2, we prove Theorem 1.1, thus solve the case of the realization problem when $2q < p$. In Section 3, we collect some basic results about linear lattices and changemaker lattices from [9]. In Section 4, we study the topology of a certain type of cobordism between rational homology 3-spheres. In Section 5, we define sharp cobordisms, and prove that the cobordism W between $P(2,1)$ and $P(p,q)$ is sharp. In Section 6, we use the result in Section 5 to prove a strengthened changemaker condition in the case $q < p < 2q$. In Sections 7 and 8, we use the strengthened changemaker condition to enumerate all the possible changemaker lattices we can have. In Section 9, we determine the pairs (p,q) corresponding to the changemaker lattices, thus finish the proof of Theorem 1.2.

2 | PROOF OF THEOREM 1.1

The goal of this section is to prove the following upper bound of p, and then to prove Theorem 1.1. Recall that we assume $q > 0$.

Proposition 2.1. *If $P(p,q)$ is realizable, then $p \leqslant 2q+1$.*

Remark 2.2. If $P(p,q)$ is realizable with $p = 2|q| \pm 1$, then K must be a torus knot [14, Theorem 1.6]. Recall that for a realizable $P(p,q)$, p is odd [1]. In particular, if we restrict attention to hyperbolic knots on which surgeries will result in $P(p,q)$, then $p \leqslant 2|q| - 3$.

2.1 | The Casson–Walker invariant of $P(p,q)$

Let

$$\Delta_K(T) = \alpha_0 + \sum_{i>0} \alpha_i (T^i + T^{-i}) \tag{1}$$

be the normalized Alexander polynomial of K. If K admits an L-space surgery, then $|\alpha_i| \leqslant 1$, $\alpha_{g(K)} = 1$, and $+1$ and -1 appear alternatingly among the non-zero α_i [17, Theorem 1.2].

Given a real number x, let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Given a pair of coprime integers n, m with $n > 0$, let $\mathbf{s}(m,n)$ be the Dedekind sum

$$\mathbf{s}(m,n) = \sum_{i=1}^{n-1} \left(\left(\frac{i}{n}\right)\right)\left(\left(\frac{im}{n}\right)\right),$$

where

$$((x)) = \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let $\lambda(\cdot)$ be the Casson–Walker invariant [22], normalized so that

$$\lambda(S^3_1(T(3,2))) = 2.$$

By [12, Proposition 6.1.1], the Casson–Walker invariant of $P(p,q)$ can be computed by the formula

$$\lambda(P(p,q)) = \frac{1}{12}\left(-\frac{p}{q}\left(\frac{1}{p^2}-\frac{1}{2}\right)-\frac{q}{p}+3+12\mathbf{s}(q,p)\right).$$

Since the Dedekind sum satisfies the reciprocity law

$$\mathbf{s}(q,p)+\mathbf{s}(p,q) = \frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}\right)-\frac{1}{4},$$

we get

$$\lambda(P(p,q)) = \frac{p}{8q} - \mathbf{s}(p,q). \tag{2}$$

On the other hand, the surgery formula for the Casson–Walker invariant [5, Theorem 2.8] implies

$$\begin{aligned}\lambda(S^3_{4q}(K)) &= -\mathbf{s}(1,4q) + \frac{1}{4q}\Delta''_K(1)\\ &= -\frac{(2q-1)(4q-1)}{24q} + \frac{1}{4q}\Delta''_K(1).\end{aligned} \tag{3}$$

Lemma 2.3. *For realizable $P(p,q)$ with q odd, $p \equiv -1 \pmod 4$.*

Proof. By combining (2) and (3), we have

$$\begin{aligned}&-\frac{(2q-1)(4q-1)}{24q} + \frac{1}{4q}\Delta''_K(1)\\ &= \lambda(P(p,q))\\ &\equiv \frac{p}{8q} - \sum_{i=1}^{q-1}\left(\frac{i}{q}-\frac{1}{2}\right)\left(\frac{pi}{q}-\frac{1}{2}\right) \pmod 1\\ &= \frac{p}{8q} - \frac{p(q-1)(2q-1)}{6q} + \frac{p(q-1)}{4}.\end{aligned}$$

Multiplying both sides by $24q$, we get

$$1-6q+8q^2+p(-1+6q-2q^2) \equiv 6\Delta''_K(1) \pmod{24q}.$$

Since $\Delta''_K(1)$ is even and p, q are odd, we get

$$2q+1+p(2q+1) \equiv 0 \pmod 4.$$

So $p \equiv -1 \pmod 4$. □

2.2 | The Spin^c structures

The ith *torsion coefficient* of a knot K is defined to be

$$t_i(K) = \sum_{j \geqslant 1} j\alpha_{i+j},$$

for $i \geqslant 0$, where the α_i are as in (1). Let

$$\varepsilon_i = t_i - t_{i+1}.$$

When K admits an L-space surgery, it is proved in [20, Proposition 7.6] that

$$\varepsilon_i \in \{0, 1\}.$$

Suppose $4q$–surgery on K is $P(p, q)$, then $4q \geqslant 2g(K) - 1$ [19]. So

$$g(K) \leqslant 2q. \tag{4}$$

Since $\alpha_{g(K)} = 1$ and $\alpha_i = 0$ when $i > g(K)$, it follows from the definition of t_i that

$$t_i = 0 \quad \text{if and only if } i \geqslant g(K). \tag{5}$$

In particular, by (4), we get

$$t_{2q} = 0. \tag{6}$$

For $i > 0$,

$$\begin{aligned} \alpha_i &= t_{i-1} - 2t_i + t_{i+1} \\ &= \varepsilon_{i-1} - \varepsilon_i. \end{aligned}$$

Since $1 = \Delta_K(1) = \alpha_0 + 2\sum_{i>0} \alpha_i$, we can also get

$$\alpha_0 = 1 - 2\sum_{i>0} \alpha_i.$$

Thus

$$\begin{aligned} \Delta_K(-1) &= \alpha_0 + 2\textstyle\sum_{i>0}(-1)^i\alpha_i \\ &= 1 - 4\textstyle\sum_{i\geqslant 0}(-1)^i\varepsilon_i. \end{aligned} \tag{7}$$

Given a knot $K \subset S^3$ and an integer $n > 0$, there is an affine isomorphism [15]

$$\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Spin}^c(S^3_n(K)).$$

For simplicity, let $d(S^3_n(K), i) = d(S^3_n(K), \varphi(i))$.

From [15], we have

$$d(L(n,1),i) = -\frac{1}{4} + \frac{(2i-n)^2}{4n}. \tag{8}$$

Using [19, Theorem 1.2], we get

$$d(S^3_n(K),i) = d(L(n,1),i) - 2t_{\min\{i,n-i\}}. \tag{9}$$

Lemma 2.4. *Suppose that $P(p,q)$ is obtained by the $4q$-surgery on K. Let i be an integer with $0 \leqslant i \leqslant q$. If i is even, we have*

$$d(S^3_{4q}(K), q-i) = d(S^3_{4q}(K), q+i),$$

and

$$t_{q-i} - t_{q+i} = \frac{i}{2}.$$

If i is odd, we have

$$d(S^3_{4q}(K), q-i) = d(S^3_{4q}(K), q+i) \pm 1,$$

and

$$t_{q-i} - t_{q+i} = \frac{i \mp 1}{2}.$$

Proof. Since $S^3_{4q}(K)$ is a prism manifold, it contains a Klein bottle. So the order-2 element in $H_1(S^3_{4q}(K))$ is represented by a curve in the Klein bottle, such that the complement of the curve in the Klein bottle is an annulus. By [13, Theorem 1.1], for any $j \in \mathbb{Z}/4q\mathbb{Z}$, we have

$$|d(S^3_{4q}(K), j) - d(S^3_{4q}(K), j+2q)| \leqslant 1. \tag{10}$$

Since the conjugate of $\varphi(j+2q)$ is $\varphi(2q-j)$, we have

$$d(S^3_{4q}(K), j+2q) = d(S^3_{4q}(K), 2q-j). \tag{11}$$

Let $j = q-i$. Using (8) and (9), we get

$$\begin{aligned} &d(S^3_{4q}(K), q-i) - d(S^3_{4q}(K), q+i) \\ &= -\frac{1}{4} + \frac{(2q-2i-4q)^2}{16q} - 2t_{q-i} - \left(-\frac{1}{4} + \frac{(2q+2i-4q)^2}{16q} - 2t_{q+i}\right) \\ &= i - 2t_{q-i} + 2t_{q+i} \in \mathbb{Z}, \end{aligned} \tag{12}$$

so $d(S^3_{4q}(K), q-i) - d(S^3_{4q}(K), q+i)$ has the same parity as i. Using (10) and (11), we get

$$|d(S^3_{4q}(K), q-i) - d(S^3_{4q}(K), q+i)| \leqslant 1.$$

So $d(S^3_{4q}(K), q-i) - d(S^3_{4q}(K), q+i) = 0$ when i is even, and $d(S^3_{4q}(K), q-i) - d(S^3_{4q}(K), q+i) = \pm 1$ when i is odd. Now $t_{q-i} - t_{q+i}$ can be computed from (12). □

2.3 | The proof of Proposition 2.1

Proof of Proposition 2.1. By Lemma 2.4 and (6),

$$t_0 = t_0 - t_{2q} \leqslant \left\lfloor \frac{q+1}{2} \right\rfloor.$$

By [14, Lemma 6.1], $p = |\Delta_K(-1)|$. Using (7), we get

$$\begin{aligned} p &\leqslant 1 + 4\sum_{i \geqslant 0} \varepsilon_i \\ &= 1 + 4t_0 \\ &\leqslant 1 + 4\left\lfloor \frac{q+1}{2} \right\rfloor. \end{aligned}$$

When q is even, $p \leqslant 2q+1$. When q is odd, $p \leqslant 2q+3$. By Lemma 2.3, $p \neq 2q+3$, so we must have $p \leqslant 2q+1$. □

Proof of Theorem 1.1. The first statement is Proposition 2.1. The second statement follows from combining [14, Theorem 1.6] and [1, Lemma 2.1]. □

3 | INPUT FROM LATTICE THEORY

This section assembles facts about lattices that will be used in the paper. We mainly follow the treatment of [1, 2, 9, 10].

Recall that an *integral lattice* is a finitely generated free abelian group L endowed with a positive definite symmetric bilinear form $\langle , \rangle : L \times L \to \mathbb{Z}$. Given $v \in L$, let $|v| = \langle v, v \rangle$ be the *norm* of v. We can extend $\langle , \rangle$ to a $\mathbb{Q}$–valued pairing on $L \otimes \mathbb{Q}$; using it we define

$$L^* = \{x \in L \otimes \mathbb{Q} | \langle x, y \rangle \in \mathbb{Z}, \forall y \in L\}.$$

The pairing on L descends to a non-degenerate, symmetric bilinear form on the *discriminant group* $\overline{L} = L^*/L$

$$\begin{aligned} b &: \overline{L} \times \overline{L} \to \mathbb{Q}/\mathbb{Z} \\ b(\overline{x}, \overline{y}) &\equiv \langle x, y \rangle \pmod 1, \end{aligned}$$

the *linking form*, where $\overline{x}$ denotes the class of $x \in L$ in $\overline{L}$. The *discriminant* of L is the order of the finite group $\overline{L}$. Let

$$\mathrm{Char}(L) = \{x \in L^* | \langle x, y \rangle \equiv \langle y, y \rangle \pmod 2, \forall y \in L\}$$

denote the set of *characteristic covectors* for L. The set $C(L) = \mathrm{Char}(L)/2L$ forms a torsor over the discriminant group $\overline{L}$. Given $\chi \in C(L)$, define

$$d_L([\chi]) = \min\left\{ \frac{|\chi'| - \mathrm{rk}(L)}{4} \middle| \chi' \in [\chi] \right\}, \tag{13}$$

and call an element $\chi \in \mathrm{Char}(L)$ *short* if its norm is minimal in $[\chi]$. We call the pair $(C(L), d_L)$ the *d-invariant* of the lattice L; in particular it is an invariant of the stable isomorphism type of the lattice L [18, Theorem 4.7]. We drop L from the notation when the lattice L is understood from the context.

3.1 | Linear lattices

Given a pair of relatively prime positive integers p, q, write $\frac{p}{q}$ in a Hirzebruch–Jung continued fraction

$$\frac{p}{q} = a_{-1} - \cfrac{1}{a_0 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}} = [a_{-1}, a_0, \dots, a_n]^-, \tag{14}$$

with $a_i \geqslant 2$ when $i \geqslant 0$ in equation (14).

Definition 3.1. The *linear lattice* $\Lambda(q, -p)$ has a basis

$$\{x_0, \dots, x_n\}, \tag{15}$$

and inner product given by

$$\langle x_i, x_j \rangle = \begin{cases} a_i, & i = j \\ -1, & |i-j| = 1 \\ 0, & |i-j| > 1, \end{cases} \tag{16}$$

where the coefficients a_i, for $i \in \{0, \dots, n\}$, are defined by the continued fraction (14). We call (15) the *vertex basis* of $\Lambda(q, -p)$.

Remark 3.2. The reason that we use $\Lambda(q, -p)$ instead of $\Lambda(q, p)$ is that our convention for lens spaces is different from that of [9]. In our paper, the lens space $L(q, p)$ is oriented as the $\frac{q}{p}$–surgery on the unknot, and $P(p, q)$ is the $\frac{q}{p}$-surgery on $\mathbb{R}P^1 \# \mathbb{R}P^1 \subset \mathbb{R}P^3 \# \mathbb{R}P^3$, so they both bound 4-manifolds with intersection lattice $\Lambda(q, -p)$.

An element $\ell \in L$ is *reducible* if $\ell = x + y$ for some non-zero $x, y \in L$ with $\langle x, y \rangle \geqslant 0$, and *irreducible* otherwise. An element $\ell \in L$ is *breakable* if $\ell = x + y$ with $|x|, |y| \geqslant 3$ and $\langle x, y \rangle = -1$, and *unbreakable* otherwise.

Definition 3.3. In a linear lattice, if I is any subset of $\{x_0, x_1, \dots, x_n\}$ then write $[I] = \sum_{x \in I} x$. An *interval* is an element of the form $[I]$ with $I = \{x_a, x_{a+1}, \dots, x_b\}$ for $0 \leqslant a \leqslant b \leqslant n$. We say that a is the left endpoint of the interval, and b is the right endpoint of the interval. Say that $[I]$ contains x_i if I does: we often write $x_i \in [I]$ in this case.

When $[I]$ is an interval, it is easy to compute

$$|[I]| = 2 + \sum_{x_i \in [I]} (|x_i| - 2). \tag{17}$$

Proposition 3.4 [9, Proposition 3.3]. *If $v \in \Lambda(q, -p)$ is irreducible, $v = \epsilon[I]$ for some $\epsilon = \pm 1$ and $[I]$ an interval.*

From now on, let $[v]$ be the interval corresponding to v when v is irreducible.

Definition 3.5. A vertex x_i has *high weight* if $|x_i| = a_i > 2$.

Proposition 3.6 [9, Corollary 3.5(4)]. *An element $\epsilon[I] \in \Lambda(q, -p)$ with $\epsilon \in \{\pm 1\}$ is unbreakable if and only if $[I]$ contains at most one vertex with high weight.*

Definition 3.7. For two intervals $[I]$ and $[J]$ with left endpoints i_0, j_0 and right endpoints i_1, j_1, say that $[I]$ and $[J]$ are *distant* if either $i_1 + 1 < j_0$ or $j_1 + 1 < i_0$, that $[I]$ and $[J]$ *share a common end* if $i_0 = j_0$ or $i_1 = j_1$, and that $[I]$ and $[J]$ are *consecutive* if $i_1 + 1 = j_0$ or $j_1 + 1 = i_0$. Write $[I] \prec [J]$ if $I \subset J$ and $[I]$ and $[J]$ share a common end, and $[I] \dagger [J]$ if they are consecutive. If $[I]$ and $[J]$ are either consecutive or share a common end, say that they *abut*. If $I \cap J$ is non-empty and $[I]$ and $[J]$ do not share a common end, write $[I] \pitchfork [J]$.

Direct computations show the following lemma.

Lemma 3.8. *Let $[I], [J]$ be two intervals. Then*

$$\langle [I], [J] \rangle = \begin{cases} 0, & [I] \text{ and } [J] \text{ are distant,} \\ |[I \cap J]| - 1, & [I] \text{ and } [J] \text{ share a common end,} \\ -1, & [I] \text{ and } [J] \text{ are consecutive,} \\ |[I \cap J]| - 2, & [I] \pitchfork [J]. \end{cases}$$

Proposition 3.9 [9, Corollary 3.5(2)]. *The lattice $\Lambda(q, -p)$ is indecomposable; that is, $\Lambda(q, -p)$ is not the direct sum of two non-trivial lattices.*

Proposition 3.10 (Proposition 3.6 of [9]). *If $\Lambda(q, p) \cong \Lambda(q', p')$, then $q = q'$ and either $p \equiv p'$ or $pp' \equiv 1$ (mod q).*

3.2 | Changemaker lattices

When a lattice L is isomorphic to $\sigma^{\perp}$, the orthogonal complement of a changemaker vector $\sigma \in \mathbb{Z}^{n+2}$, L is called a *changemaker lattice*.

Definition 3.11. The *standard basis* of $\sigma^\perp$ is the collection $S = \{v_1, \dots, v_{n+1}\}$, where

$$v_j = \left(2e_0 + \sum_{i=1}^{j-1} e_i\right) - e_j,$$

whenever $\sigma_j = 1 + \sigma_0 + \cdots + \sigma_{j-1}$, and

$$v_j = \left(\sum_{i\in A} e_i\right) - e_j$$

whenever $\sigma_j = \sum_{i\in A} \sigma_i$, with $A \subset \{0, \dots, j-1\}$ chosen to maximize the quantity $\sum_{i\in A} 2^i$. A vector $v_j \in S$ is called *tight* in the first case, *just right* in the second case as long as $i < j-1$ and $i \in A$ implies that $i+1 \in A$, and *gappy* if there is some index i with $i \in A$, $i < j-1$, and $i+1 \notin A$. Such an index, i, is a *gappy index* for v_j.

Definition 3.12. For $v \in \mathbb{Z}^{n+2}$, $\operatorname{supp} v = \{i | \langle e_i, v\rangle \neq 0\}$.

Lemma 3.13 (Lemma 3.12 (3) in [9]). *If $|v_{k+1}| = 2$, then k is not a gappy index for any v_j with $j \in \{1, \dots, n+1\}$.*

Lemma 3.14 (Lemma 3.13 in [9]). *Each $v_j \in S$ is irreducible. Furthermore, for any $A \subset \{0, 1, \dots, j-1\}$, if the vector*

$$-e_j + \sum_{i\in A} e_i$$

is in $\sigma^\perp$, then it is irreducible.

Lemma 3.15. *Let $v = \sum_{i\in A} b_i e_i \in \sigma^\perp$, with $A \subset \{0, 1, \dots, n+1\}$ and each $b_i \in \{-1, 1\}$. If $v = x + y$ with $\langle x, y\rangle \geqslant 0$, then there exists a subset $B \subset A$ such that*

$$x = \sum_{i\in B} b_i e_i, y = \sum_{i\in A\setminus B} b_i e_i.$$

Proof. Let $x = \sum x_i e_i$, $y = \sum y_i e_i$. Since $x_i + y_i \in \{-1, 0, 1\}$, $x_i y_i \leqslant 0$. If $\langle x, y\rangle \geqslant 0$, then each $x_i y_i = 0$, namely, one of x_i, y_i is 0. So our conclusion holds. □

Lemma 3.16 (Lemma 3.15 in [9]). *If $v_j \in S$ is breakable, then it is tight.*

Lemma 3.17 (Lemma 4.2(1) in [9]). *If $\Lambda(q, -p)$ is isomorphic to a changemaker lattice, then it contains at most one tight standard basis vector.*

Lemma 3.18 (Lemma 3.12(1) in [9]). *For any $v_j \in S$, we have $j - 1 \in \operatorname{supp}(v_j)$.*

Definition 3.19. If T is a set of irreducible vectors in a linear lattice $\Lambda(q, -p)$, the *intersection graph* $G(T)$ has vertex set T, and an edge between v and w if the intervals corresponding to v and w abut. We write $v \sim w$ if v and w are connected in $G(T)$.

Lemma 3.20. *If v, w are irreducible elements of a linear lattice and the intervals corresponding $[v]$ and $[w]$ abut, then $\langle v, w\rangle \neq 0$.*

Lemma 3.21 (Lemma 4.4 in [9]). *If v_i and v_j are distinct unbreakable standard basis vectors with $|v_i|, |v_j| \geqslant 3$, then $|\langle v_i, v_j\rangle| \leqslant 1$, with equality if and only if $[v_i] \dagger [v_j]$.*

Lemma 3.22 (Corollary 4.5 in [9]). *If v_i and v_j are distinct unbreakable standard basis vectors with $|v_i|, |v_j| \geqslant 3$, then the high weight vertices contained in $[v_i], [v_j]$ are different.*

Definition 3.23. A *claw* in a graph G is a quadruple $(v; w_1, w_2, w_3)$ of vertices such that v neighbors all the w_i, but no two of the w_i neighbor each other.

Lemma 3.24 (Lemma 4.8 of [9]). *For any set T of irreducible elements in a linear lattice, the intersection graph $G(T)$ has no claws.*

Definition 3.25. Given a set T of unbreakable elements in a linear lattice and $v_1, v_2, v_3 \in T$, (v_1, v_2, v_3) is a *heavy triple* if $|v_i| \geqslant 3$, and if each pair among the v_i is connected by a path in $G(T)$ disjoint from the third.

Lemma 3.26 (Based on Lemma 4.10 of [9]). *For any set T of unbreakable elements in a linear lattice, $G(T)$ has no heavy triples.*

4 | THE TOPOLOGY OF CERTAIN COBORDISMS

In this section, we will consider the topology of a certain cobordism $W : Y_0 \to Y_1$. We assume that W is obtained by adding $n+1$ two-handles along a link $L \subset Y_0$, such that one component L_0 of L represents a 2-torsion in $H_1(Y_0)$, and all other components of L are null-homologous in Y_0. Moreover, we assume that $|H_1(Y_0)| = 4$ and W is negative definite. Under these assumptions, Y_1 is a rational homology sphere. Let $\iota_i : Y_i \to W$ be the inclusion map, $\iota_i^* : H^2(W) \to H^2(Y_i)$ be the induced maps on cohomology, and $\iota_i^s : \mathrm{Spin}^c(W) \to \mathrm{Spin}^c(Y_i)$ be the induced maps on Spin^c, $i = 0, 1$.

We make the further assumption that Y_0 is the boundary of a compact 4-manifold Z_0 with $H_1(Z_0) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(Z_0) = 0$, and L_0 is null-homologous in Z_0. Let $Z = Z_0 \cup_{Y_0} W$.

From the handle structure of W, we can compute

$$H_1(W) \cong \mathbb{Z}/2\mathbb{Z}, H_2(W) \cong \mathbb{Z}^{n+1}, H_1(W, Y_i) = 0, H_2(W, Y_i) \cong \mathbb{Z}^{n+1}, i = 0, 1.$$

By the Universal Coefficient Theorem,

$$H^2(W) \cong \mathbb{Z}^{n+1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In particular, there exists a unique torsion class $\alpha \in H^2(W)$. Let $\alpha_i = \iota_i^*(\alpha), i = 0, 1$.

Since Z is obtained by adding two-handles to Z_0, such that all attaching curves are null-homologous in Z_0, we have

$$H_1(Z) \cong H_1(Z_0) \cong \mathbb{Z}/2\mathbb{Z},$$

and the map $H_2(Z) \to H_2(Z, Z_0)$ is an isomorphism.

Lemma 4.1. *The map* $\iota^*_{W,Z} : H^2(Z) \to H^2(W)$ *is injective with image containing* α. *The map* $\iota^*_{Y_0,Z_0} : H^2(Z_0) \to H^2(Y_0)$ *is injective with image generated by* α_0. *Moreover,* $[L_0] \in H_1(Y_0)$ *is the Poincaré dual of* α_0.

Proof. Using the long exact sequences

$$H^2(Z,W) \to H^2(Z) \to H^2(W), \quad H^2(Z_0,Y_0) \to H^2(Z_0) \to H^2(Y_0),$$

and the fact that $0 = H^2(Z_0,Y_0) \cong H^2(Z,W)$, we get that $\iota^*_{W,Z}$ and $\iota^*_{Y_0,Z_0}$ are injective.

By the Universal Coefficient Theorem, $H^2(Z) \cong Hom(H_2(Z),\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$, so it has a unique 2-torsion $\overline{\alpha}$. Since $\iota^*_{W,Z}$ is injective, $\iota^*_{W,Z}(\overline{\alpha})$ is a 2-torsion in $H^2(W)$, which must be α. Let $\overline{\alpha}_0$ be the restriction of $\overline{\alpha}$ to $H^2(Z_0)$. Using the commutative diagram

$$\begin{array}{ccc} H^2(Z) & \longrightarrow & H^2(Z_0), \\ \downarrow & & \downarrow \\ H^2(W) & \longrightarrow & H^2(Y_0) \end{array}$$

we see that $\iota^*_{Y_0,Z_0}(\overline{\alpha}_0) = \alpha_0$. Since $H^2(Z_0) \cong \mathbb{Z}/2\mathbb{Z}$, the image of $\iota^*_{Y_0,Z_0}$ is generated by α_0.

Since L_0 is null-homologous in Z_0, there exists a properly embedded oriented surface $F_0 \subset Z_0$ such that $\partial F_0 = L_0$. Thus the image of the Poincaré dual of $[F_0]$ under $\iota^*_{Y_0,Z_0}$ is the Poincaré dual of $[L_0]$. Since both $[L_0]$ and $[\alpha_0]$ have order 2, and $\iota^*_{Y_0,Z_0}(\overline{\alpha}_0) = \alpha_0$, we get that $[L_0]$ is the Poincaré dual of α_0. □

Lemma 4.2.

(1) *For* $i = 0,1$, *we have* $\ker \iota_i^* \cong H^2(W,Y_i)$, *and* ι_i^* *is surjective. In particular,* $\alpha_i \neq 0$ *in* $H^2(Y_i)$.
(2) *The kernel of the restriction map* $(\iota_0')^* : \ker \iota_1^* \to H^2(Y_0)$ *is isomorphic to* $H^2(W,\partial W)$, *and its image is generated by* α_0.

Proof.

(1) The first statement follows from the long exact sequence

$$0 = H^1(Y_i) \to H^2(W,Y_i) \to H^2(W) \xrightarrow{\iota_i^*} H^2(Y_i) \to H^3(W,Y_i) = 0.$$

It follows that $\ker \iota_i^*$ is torision-free, so $\alpha \notin \ker \iota_i^*$. Thus $\alpha_i \neq 0$.
(2) By (1), the map $(\iota_0')^*$ can be identified with $H^2(W,Y_1) \to H^2(Y_0)$, which is part of the long exact sequence

$$0 = H^1(\partial W,Y_1) \to H^2(W,\partial W) \to H^2(W,Y_1) \to H^2(\partial W,Y_1) = H^2(Y_0).$$

Thus $\ker(\iota_0')^*$ is $H^2(W,\partial W)$.

By Poincaré duality, $(\iota_0')^*$ can be identified with the boundary map $\partial_0' : H_2(W,Y_0) \to H_1(Y_0)$. By the handle decomposition of W, we see that the image of ∂_0' is generated by $[L_0]$. By Lemma 4.1, $\mathrm{im}(\iota_0')^*$ is generated by α_0. □

Corollary 4.3. *For each* $\mathfrak{t} \in \mathrm{Spin}^c(Y_1)$*, there exists a subset*

$$\mathfrak{R}(\mathfrak{t}) = \{\mathfrak{r}_0, \mathfrak{r}_1 = \mathfrak{r}_0 + \alpha_0\} \subset \mathrm{Spin}^c(Y_0)$$

such that for each $\mathfrak{r} \in \mathrm{Spin}^c(Y_0)$*, the set*

$$(\iota_0^s, \iota_1^s)^{-1}(\mathfrak{r}, \mathfrak{t}) := (\iota_0^s)^{-1}(\mathfrak{r}) \cap (\iota_1^s)^{-1}(\mathfrak{t}) \tag{18}$$

is non-empty if and only if $\mathfrak{r} \in \mathfrak{R}(\mathfrak{t})$*. Moreover, the set (18) is an* $H^2(W, \partial W)$*–torsor when it is non-empty.*

Proof. This follows from Lemma 4.2 and the fact that Spin^c is an H^2-torsor. □

By the long exact sequence

$$0 = H_2(Y_0) \to H_2(W) \to H_2(W, Y_0) \to H_1(Y_0),$$

$H_2(W)$ embeds as an index-2 subgroup of $H_2(W, Y_0) \cong \mathbb{Z}^{n+1}$. Thus we can extend the intersection form on $H_2(W)$ to $H_2(W, Y_0)$, with value in $\frac{1}{4}\mathbb{Z}$. Let

$$\mathcal{L} \cong H_2(W, Y_0) \cong H_2(Z, Z_0) \cong H_2(Z)$$

be the intersection lattice on the pair (W, Y_0). Suppose that the generators corresponding to the two-handles are $x_0, \ldots, x_n$, where x_0 corresponds to the two-handle attached along L_0. Let

$$\mathcal{L}_0 = \langle 2x_0, x_1, \ldots, x_n \rangle$$

be the sublattice of $\mathcal{L}$ generated by $2x_0, x_1, \ldots, x_n$; then $\mathcal{L}_0$ can be identified with the intersection lattice $H_2(W)$. Let

$$\mathcal{L}^* = Hom(\mathcal{L}, \mathbb{Z}), \mathcal{L}_0^* = Hom(\mathcal{L}_0, \mathbb{Z}) \supset \mathcal{L}^*.$$

Using the inner product on $\mathcal{L}$, we can embed $\mathcal{L}^*$ and $\mathcal{L}_0^*$ as sublattices of $\mathcal{L} \otimes \mathbb{Q}$.

Let

$$\widetilde{C} = \{y \in \mathcal{L}_0^* | \langle y, 2x_0 \rangle \equiv \langle 2x_0, 2x_0 \rangle, \langle y, x_j \rangle \equiv \langle x_j, x_j \rangle \pmod 2, \quad j > 0\}.$$

Let $\overline{H}^2(W) = H^2(W)/\mathrm{Tors} = \mathcal{L}_0^*$, and let $\bar{c}_1 : \mathrm{Spin}^c(W) \to \overline{H}^2(W)$ be the composition of the map $c_1 : \mathrm{Spin}^c(W) \to H^2(W)$ and the quotient map $H^2(W) \to \overline{H}^2(W)$. Then $\widetilde{C}$ is the image of $\bar{c}_1$.

Proposition 4.4.

(1) *The quotient* $\mathrm{Spin}^c(Y_1)/\langle \alpha_1 \rangle$ *can be identified with* $\widetilde{C}/2\mathcal{L}$*.*
(2) *Under the previous identification, suppose that the* $\langle \alpha_1 \rangle$*–orbit* $\{\mathfrak{t}, \mathfrak{t} + \alpha_1\}$ *is identified with* $y + 2\mathcal{L}$ *for some* $y \in \widetilde{C}$*. Let* $\mathfrak{R}(\mathfrak{t}) = \{\mathfrak{r}_0, \mathfrak{r}_1\}$*. Then there exist* $y_0, y_1 \in y + 2\mathcal{L}$*, such that*

$$\bar{c}_1((\iota_0^s, \iota_1^s)^{-1}(\mathfrak{r}_0, \mathfrak{t})) = y_0 + 2\mathcal{L}_0, \quad \bar{c}_1((\iota_0^s, \iota_1^s)^{-1}(\mathfrak{r}_1, \mathfrak{t})) = y_1 + 2\mathcal{L}_0,$$

and

$$\bar{c}_1((\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_0,\mathfrak{t}+\alpha_1)) = y_1 + 2\mathcal{L}_0, \quad \bar{c}_1((\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_1,\mathfrak{t}+\alpha_1)) = y_0 + 2\mathcal{L}_0.$$

Proof.

(1) By Lemma 4.2, every $\mathfrak{t} \in \mathrm{Spin}^c(Y_1)$ is in the image of ι_1^s, and $\mathfrak{s}_1,\mathfrak{s}_2 \in \mathrm{Spin}^c(W)$ restrict to the same $\mathfrak{t} \in \mathrm{Spin}^c(Y_1)$ if and only if $\mathfrak{s}_1 - \mathfrak{s}_2 \in H^2(W,Y_1) \cong H_2(W,Y_0) = \mathcal{L}$. So $\mathrm{Spin}^c(Y_1) \cong \mathrm{Spin}^c(W)/\mathcal{L}$. Consider the map $\bar{c}_1 : \mathrm{Spin}^c(W) \to \widetilde{C}$. It is surjective, and $\bar{c}_1(\mathfrak{s}_1) = \bar{c}_1(\mathfrak{s}_2)$ if and only if $\mathfrak{s}_1 - \mathfrak{s}_2 \in \langle\alpha\rangle$. Using the formula

$$c_1(\mathfrak{s}_1) - c_1(\mathfrak{s}_2) = 2(\mathfrak{s}_1 - \mathfrak{s}_2)$$

we get that $\mathrm{Spin}^c(Y_1)/\langle\alpha_1\rangle \cong \mathrm{Spin}^c(W)/(\mathcal{L}+\langle\alpha\rangle) \cong \widetilde{C}/2\mathcal{L}$.

(2) By Corollary 4.3, there exist $\mathfrak{s}_0,\mathfrak{s}_1 \in \mathrm{Spin}^c(W)$, such that

$$(\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_0,\mathfrak{t}) = \mathfrak{s}_0 + \mathcal{L}_0, \quad (\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_1,\mathfrak{t}) = \mathfrak{s}_1 + \mathcal{L}_0.$$

Since

$$\iota_0^s(\mathfrak{s}_1+\alpha) = \iota_0^s(\mathfrak{s}_1) + \alpha_0 = \mathfrak{r}_1 + \alpha_0 = \mathfrak{r}_0, \iota_0^s(\mathfrak{s}_0+\alpha) = \mathfrak{r}_1,$$

we also have

$$(\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_0,\mathfrak{t}+\alpha_1) = \mathfrak{s}_1 + \alpha + \mathcal{L}_0, \quad (\iota_0^s,\iota_1^s)^{-1}(\mathfrak{r}_1,\mathfrak{t}+\alpha_1) = \mathfrak{s}_0 + \alpha + \mathcal{L}_0.$$

Applying $\bar{c}_1$ to the above equalities, we get our conclusion. □

For any $\mathfrak{s} \in \mathrm{Spin}^c(W)$, let

$$gr(W,\mathfrak{s}) = \frac{c_1^2(\mathfrak{s}) + b_2(W)}{4}. \tag{19}$$

For any $\mathfrak{t} \in \mathrm{Spin}^c(Y_1)$, let

$$D_W(Y_1,\mathfrak{t}) = \max_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(W)\\ \mathfrak{s}|Y_1=\mathfrak{t}}} (d(Y_0,\mathfrak{s}|Y_0) + gr(W,\mathfrak{s})). \tag{20}$$

Lemma 4.5. *There are exactly two Spin^c structures $\mathfrak{e}_0,\mathfrak{e}_1 \in \mathrm{Spin}^c(Y_0)$ which can be extended over Z_0. Moreover,*

$$\mathfrak{e}_1 = \mathfrak{e}_0 + \alpha_0, \quad d(Y_0,\mathfrak{e}_i) = 0, \quad i = 0,1.$$

Proof. By Lemma 4.1, α_0 is the restriction of a cohomology class in $H^2(Z_0)$. Let $\mathfrak{e}_0 \in \mathrm{Spin}^c(Y_0)$ be a Spin^c structure which is the restriction of a Spin^c structure on Z_0, then $\mathfrak{e}_1 := \mathfrak{e}_0 + \alpha_0$ also extends over Z_0. Since $H^2(Z_0) \cong \mathbb{Z}/2\mathbb{Z}$, $\mathfrak{e}_0,\mathfrak{e}_1$ are the only two Spin^c structures which can be extended over Z_0. It follows from [15, Proposition 9.9] that $d(Y_0,\mathfrak{e}_i) = 0$. □

Lemma 4.6. *The image of*

$$\bar{c}_1 : (\iota_0^s)^{-1}(\{\mathfrak{e}_0, \mathfrak{e}_1\}) \to \overline{H}^2(W)$$

is $C := \mathrm{Char}(\mathcal{L})$.

Proof. Let $\mathfrak{s}_0$ be the restriction of a Spin^c structure on Z to W, then $\mathfrak{s}_0 \in (\iota_0^s)^{-1}(\{\mathfrak{e}_0, \mathfrak{e}_1\})$. Clearly, $\bar{c}_1(\mathfrak{s}_0) \in C$. By Lemma 4.1, $\iota_{W,Z}^*$ is injective, so the image of $H^2(Z)$ in $\overline{H}^2(W)$ can be identified with $Hom(H_2(Z), \mathbb{Z}) = Hom(H_2(W, Y_0), \mathbb{Z}) = \mathcal{L}^*$. Thus $\bar{c}_1((\iota_0^s)^{-1}(\{\mathfrak{e}_0, \mathfrak{e}_1\}))$ is a $2\mathcal{L}^*$-torsor. Since C is the unique $2\mathcal{L}^*$-torsor containing $\bar{c}_1(\mathfrak{s}_0)$, our conclusion holds. □

Corollary 4.7. *The sum*

$$\sum_{\mathfrak{t} \in \mathrm{Spin}^c(Y_1)} D_W(Y_1, \mathfrak{t}) \tag{21}$$

only depends on the lattice $\mathcal{L}$ and the correction terms of Y_0. In fact, if we write (21) as a function

$$\mathcal{D}(\mathcal{L}, \{d_0, d_1\})$$

of $\mathcal{L}$ and the multiset $\{d_0, d_1\}$ of the correction terms of the two Spin^c structures other than $\mathfrak{e}_0, \mathfrak{e}_1$, then

$$\mathcal{D}(\mathcal{L}, \{d_0 + c, d_1 + c\}) = \mathcal{D}(\mathcal{L}, \{d_0, d_1\}) + c|\mathcal{L}_0^*/\mathcal{L}| \tag{22}$$

for any $c \in \mathbb{Q}$. Note that, by Proposition 4.4, $|H_1(Y_1)| = 2|\mathcal{L}_0^*/\mathcal{L}|$.

Proof. We will give the procedure of computing (21) from $\mathcal{L}$ and the correction terms of Y_0. Let $\mathfrak{o}_0, \mathfrak{o}_1$ be the two Spin^c structures other than $\mathfrak{e}_0, \mathfrak{e}_1$ on Y_0. We choose $[z] \in \widetilde{C}/2\mathcal{L}$. By Proposition 4.4, $[z]$ corresponds to a pair of Spin^c structures $\mathfrak{t}_0, \mathfrak{t}_1 = \mathfrak{t}_0 + \alpha_1 \in \mathrm{Spin}^c(Y_1)$. There are exactly two $2\mathcal{L}_0$-torsors contained in $z + 2\mathcal{L}$, denoted by $\mathcal{T}_0, \mathcal{T}_1$.

Next we check whether $z + 2\mathcal{L}$ is contained in C. If it is contained in C, it follows from Lemma 4.6 that each $\mathfrak{t}_i$ is cobordant to $\mathfrak{e}_0$ and $\mathfrak{e}_1$, $i = 0, 1$. Since $d(Y_0, \mathfrak{e}_0) = d(Y_0, \mathfrak{e}_1) = 0$, by Proposition 4.4,

$$D_W(Y_1, \mathfrak{t}_0) = D_W(Y_1, \mathfrak{t}_1) = 0 + \max_{y \in z + 2\mathcal{L}} \frac{-\langle y, y\rangle + b_2(W)}{4}.$$

If $z + 2\mathcal{L}$ is not contained in C, then each $\mathfrak{t}_i$ is cobordant to $\mathfrak{o}_0$ and $\mathfrak{o}_1$. By Proposition 4.4, the multiset $\{D_W(Y_1, \mathfrak{t}_0), D_W(Y_1, \mathfrak{t}_1)\}$ is equal to

$$\left\{ \max\left\{ d(Y_0, \mathfrak{o}_0) + \max_{y \in \mathcal{T}_0} \frac{-\langle y, y\rangle + b_2(W)}{4}, d(Y_0, \mathfrak{o}_1) + \max_{y \in \mathcal{T}_1} \frac{-\langle y, y\rangle + b_2(W)}{4} \right\}, \right.$$
$$\left. \max\left\{ d(Y_0, \mathfrak{o}_0) + \max_{y \in \mathcal{T}_1} \frac{-\langle y, y\rangle + b_2(W)}{4}, d(Y_0, \mathfrak{o}_1) + \max_{y \in \mathcal{T}_0} \frac{-\langle y, y\rangle + b_2(W)}{4} \right\} \right\}.$$

Finally, to get (21), we add all the $D_W(Y_1, \mathfrak{t}_0) + D_W(Y_1, \mathfrak{t}_1)$ together, for all $[z] \in \widetilde{C}/2\mathcal{L}$.

The equality (22) follows from the above procedure, since exactly $\frac{1}{2}|H_1(Y_1)|$ values of $D_W(Y_1, \mathfrak{t})$ are increased by c after increasing $d(Y_0, \mathfrak{o}_i)$ by c, $i = 0, 1$. □

5 | SHARP COBORDISMS

In this section, we will generalize the notion of sharp 4-manifolds defined by Greene [10] to 4-dimensional cobordisms, and prove that certain cobordisms between prism manifolds are sharp. Recall that a smooth, compact, negative definite 4-manifold X with $\partial X = Y$ is *sharp* if for every $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, there exists some $\mathfrak{s} \in \mathrm{Spin}^c(X)$ extending $\mathfrak{t}$ such that

$$c_1(\mathfrak{s})^2 + b_2(X) = 4d(Y, \mathfrak{t})$$

Definition 5.1. Let $W : Y_0 \to Y_1$ be a smooth, connected, negative definite cobordism between two rational homology spheres Y_0 and Y_1. We say W is *sharp*, if for any $\mathfrak{t} \in \mathrm{Spin}^c(Y_1)$, we have

$$d(Y_1, \mathfrak{t}) = D_W(Y_1, \mathfrak{t}).$$

Here D_W is defined using the formula (20).

Lemma 5.2. *Let Y_1, Y_2, Y_3 be rational homology spheres, $W_1 : Y_1 \to Y_2$ and $W_2 : Y_2 \to Y_3$ be two negative definite cobordisms. If $W = W_1 \cup_{Y_2} W_2$ is sharp, then W_2 is sharp.*

Proof. Let $\mathfrak{s} \in \mathrm{Spin}^c(W)$ and let $\mathfrak{s}_i = \mathfrak{s}|W_i$, $i = 1, 2$, then

$$c_1^2(\mathfrak{s}) = c_1^2(\mathfrak{s}_1) + c_1^2(\mathfrak{s}_2).$$

Our conclusion follows from the above equality. □

5.1 | A Kirby diagram of $P(p, q)$

Suppose that

$$\frac{p}{q} = [a_{-1}, a_0, \dots, a_n]^-$$

as in (14), where each a_i is $\geqslant 2$ when $i \geqslant 0$.

Figure 2 is a surgery diagram of $P(p, q)$. The leftmost two components give rise to a surgery diagram of $P(a_{-1}, 1)$, and other components give rise to a negative definite cobordism

$$W(p, q) : P(a_{-1}, 1) \to P(p, q).$$

If we replace the leftmost component, which is unknotted with slope 0, with a dotted circle representing a one-handle, we get a negative definite 4-manifold $Z(p, q)$ bounded by $P(p, q)$, and the two leftmost components give rise to a rational homology ball $Z_{a_{-1}}$ bounded by $P(a_{-1}, 1)$, with $H_1(Z_{a_{-1}}) = \mathbb{Z}/2\mathbb{Z}$.

The main result of this section is the following proposition.

Proposition 5.3. *The cobordism $W(p, q)$ is sharp.*

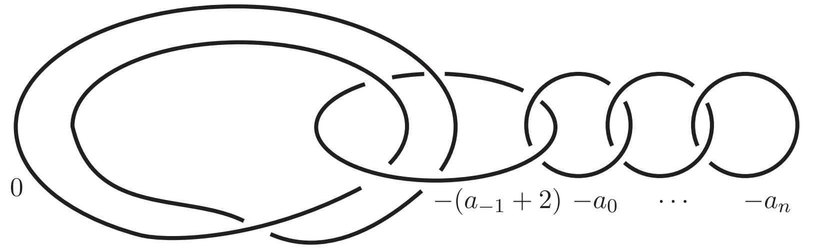

FIGURE 2 A manifold bounded by $P(p,q)$. If we replace the leftmost component with a dotted circle, we get a negative definite 4-manifold $Z(p,q)$

For simplicity, we only prove the case $q < p < 2q$. The proof of the general case is similar. From now on, let $W = W(p,q)$.

5.2 | More Kirby diagrams

We will consider three other cobordisms.

When $q < p < 2q$, $a_{-1} = 2$. We have

$$\frac{2q-(p-q)}{q-(p-q)} = 1 + \frac{q}{2q-p} = [a_0+1, a_1, \ldots, a_n]^-.$$

Consider the following surgery diagram of $P(p-q,q)$. By [2], this diagram gives rise to a sharp 4-manifold bounded by $P(p-q,q)$. The component with label -4 gives rise to $P(1,1) = L(4,-1)$, and the other two-handles give rise to a cobordism

$$W_1 : P(1,1) \to P(p-q,q).$$

Let

$$\frac{p+q}{p} = [a'_0, a'_1, \ldots, a'_m]^-.$$

By [1], $P(p,-q)$ has a surgery diagram as in Figure 4, which gives rise to a sharp 4–manifold bounded by $P(p,-q)$. The two components with label -2 give rise to $P(0,1) = \mathbb{R}P^3 \# \mathbb{R}P^3$, and the other two-handles give rise to a cobordism

$$W' : P(0,1) \to P(p,-q).$$

Using the continued fraction

$$\frac{-2q-(p-q)}{-q-(p-q)} = \frac{p+q}{p} = [a'_0, a'_1, \ldots, a'_m]^-,$$

by [2], we get a surgery diagram of $P(p-q,-q)$ as in Figure 5, which gives rise to a sharp 4-manifold bounded by $P(p-q,-q)$. The component with label -4 gives rise to $P(1,1) = L(4,-1)$, and the other two-handles give rise to a cobordism

$$W'_1 : P(1,1) \to P(p-q,-q).$$

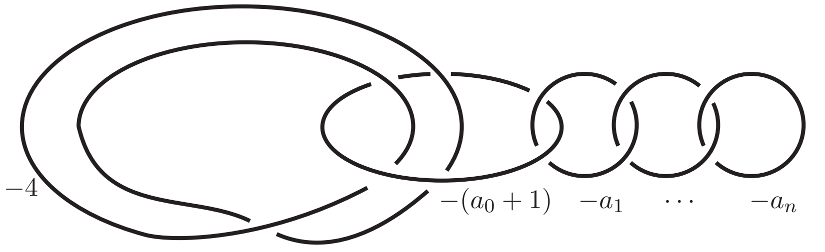

FIGURE 3 A sharp 4-manifold $X(p-q,q)$ bounded by $P(p-q,q)$

By Lemma 5.2, W_1, W', W'_1 are all sharp cobordisms.

Lemma 5.4. *The intersection lattices on* $(W, P(2,1))$ *and* $(W_1, P(1,1))$ *are isomorphic; also, the intersection lattices on* $(W', P(0,1))$ *and* $(W'_1, P(1,1))$ *are isomorphic.*

Proof. In Figure 2, consider the knot L_0 with label $-a_0$. The canonical longitude on L_0 is clearly rationally null-homologous in $P(2,1)\setminus L_0$. As a result, the square of the generator of $H_2(W, P(2,1))$ corresponding to the two-handle attached along L_0 is $-a_0$. In Figure 3, consider the knot K_0 with label $-(a_0+1)$. If the framing on K_0 is -1, the manifold we get by doing surgery on the two leftmost components is $P(1,0)$ which has $b_1 > 0$. Thus the slope -1 on K_0 is rationally null-homologous in $P(1,1)\setminus K_0$. As a result, the square of the generator of $H_2(W_1, P(1,1))$ corresponding to the two-handle attached along K_0 is $-a_0$. So the intersection lattices on $(W, P(2,1))$ and $(W_1, P(1,1))$ are isomorphic.

Similarly, we see that the square of the generator of $H_2(W', P(0,1))$ and $H_2(W'_1, P(1,1))$ corresponding to the two-handle attached along the knot with label $-a'_0$ is $-(a'_0-1)$. So the intersection lattices are isomorphic. □

Lemma 5.5. *All four cobordisms* W, W_1, W', W'_1 *satisfy the assumptions in the beginning of Section 4.*

Proof. The cobordism W satisfies the assumptions by its construction.

For W_1, W'_1, note that $P(1,1)$ bounds a rational homology ball Z_1 with $H_1(Z_1)\cong \mathbb{Z}/2\mathbb{Z}$. Since $H_1(P(1,1))$ is cyclic, the kernel of the surjective map $H_1(P(1,1))\to H_1(Z_1)$ is $2H_1(P(1,1))$. From Figures 3 and 5, we see that the knot with label $-(a_0+1)$ or $-a'_0$ represents an element in $2H_1(P(1,1))$. So W_1, W'_1 satisfy the assumptions.

For W', the rational ball bounded by $\mathbb{R}P^3\#\mathbb{R}P^3$ is $Z_0 = (\mathbb{R}P^3\setminus B^3)\times I$. Clearly, the knot labeled with $-a'_0$ in Figure 4 is null-homologous in Z_0. □

5.3 | The proof of Proposition 5.3

Recall from Section 5.1 that $P(a,1)$ bounds a rational homology ball Z_a with $H_1(Z_a)\cong\mathbb{Z}/2\mathbb{Z}$. There are exactly two Spin^c structures $\mathfrak{e}_0, \mathfrak{e}_1 \in \mathrm{Spin}^c(P(a,1))$ which extend over Z_a. Let $\mathfrak{o}_0, \mathfrak{o}_1 \in \mathrm{Spin}^c(P(a,1))$ be two other Spin^c structures, such that $d(P(a,1),\mathfrak{o}_1) \geqslant d(P(a,1),\mathfrak{o}_0)$.

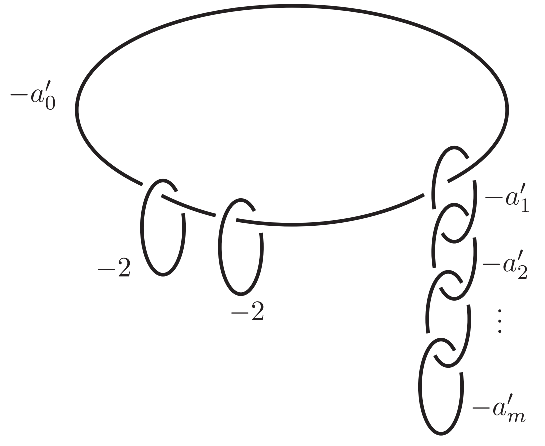

FIGURE 4 A sharp 4-manifold bounded by $P(p, -q)$

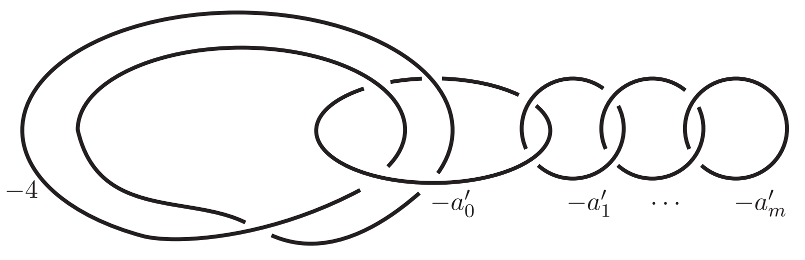

FIGURE 5 A sharp 4-manifold bounded by $P(p - q, -q)$

Lemma 5.6. *The correction terms of* $P(a, 1)$ *are*

$$d(P(a,1), \mathfrak{e}_0) = d(P(a,1), \mathfrak{e}_1) = 0,$$

$$d(P(a,1), \mathfrak{o}_0) = -\frac{a+2}{4}, \quad d(P(a,1), \mathfrak{o}_1) = -\frac{a-2}{4}.$$

Proof. The correction terms of $P(a, 1)$ are computed in [6, Example 15], and they are $\{0, 0, -\frac{a+2}{4}, \frac{a-2}{4}\}$. It is a standard fact that $d(P(a,1), \mathfrak{e}_i) = 0, i = 0, 1$ [15, Proposition 9.9]. So we must have $d(P(a,1), \mathfrak{o}_i) = -\frac{a+2}{4} + i, i = 0, 1$, by our choice of $\mathfrak{o}_0, \mathfrak{o}_1$. □

Proof of Proposition 5.3 in the case $q < p < 2q$. By [15, Theorem 9.6],

$$d(P(p,q), \mathfrak{t}) \geqslant D_W(P(p,q), \mathfrak{t}). \tag{23}$$

Also, since W_1, W', W_1' are sharp, we have

$$d(P(p-q,q), \mathfrak{t}_1) = D_{W_1}(P(p-q,q), \mathfrak{t}_1),$$
$$d(P(p,-q), \mathfrak{t}) = D_{W'}(P(p,-q), \mathfrak{t})$$
$$d(P(p-q,-q), \mathfrak{t}_1) = D_{W_1'}(P(p-q,-q), \mathfrak{t}_1).$$

By Corollary 4.7, Lemma 5.4 and Lemma 5.6,

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,q))} D_W(P(p,q),\mathfrak{t}) = -\frac{2q}{4} + \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,q))} D_{W_1}(P(p-q,q),\mathfrak{t}_1),$$

$$-\frac{2q}{4} + \sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,-q))} D_{W'}(P(p,-q),\mathfrak{t}) = \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,-q))} D_{W_1'}(P(p-q,-q),\mathfrak{t}_1).$$

Adding the above two equalities together, and using (23) and the three equalities after it, we get

$$\begin{aligned}
0 &= \sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,q))} d(P(p,q),\mathfrak{t}) + \sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,-q))} d(P(p,-q),\mathfrak{t}) \quad (\text{since } P(p,q) = -P(p,-q))\\
&\geqslant \sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,q))} D_W(P(p,q),\mathfrak{t}) + \sum_{\mathfrak{t}\in \mathrm{Spin}^c(P(p,-q))} D_{W'}(P(p,-q),\mathfrak{t})\\
&= \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,q))} D_{W_1}(P(p-q,q),\mathfrak{t}_1) + \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,-q))} D_{W_1'}(P(p-q,-q),\mathfrak{t}_1)\\
&= \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,q))} d(P(p-q,q),\mathfrak{t}_1) + \sum_{\mathfrak{t}_1\in \mathrm{Spin}^c(P(p-q,-q))} d(P(p-q,-q),\mathfrak{t}_1)\\
&= 0.
\end{aligned}$$

So the equality in (23) must hold. □

6 | THE CHANGEMAKER CONDITION WHEN $q < p < 2q$

6.1 | Positive definite manifold with boundary $P(2,1)$

The goal of this subsection is to prove the following proposition.

Proposition 6.1. *If X is a positive definite, simply connected four-manifold with $\partial X \cong P(2,1)$, then the intersection form of X is isomorphic to $D_4 \oplus \mathbb{Z}^{n-4}$ for some n.*

Here, D_k is the sublattice of $\mathbb{Z}^k$ consisting of vectors for which the sum of the coefficients is even.

Lemma 6.2. *If $L \subset \mathbb{Z}^n$ is an index-two sublattice, then $L \cong D_k \oplus \mathbb{Z}^{n-k}$ for some $k \geqslant 1$. In fact, there are indices $i_1, \ldots, i_k$ such that L contains exactly the elements of $\mathbb{Z}^n$ that have even pairing with $e_{i_1} + \cdots + e_{i_k}$. There are always two elements $x \in \overline{L}$ with $b(x,x) = 0 \pmod 1$, and the other two elements satisfy $b(x,x) = k/4 \pmod 1$.*

Proof. Let $L \subset \mathbb{Z}^n$ have index two, and let $i_1, \ldots, i_k$ be an enumeration of the indices i for which $e_i \notin L$. Since L has index two, the elements $\pm e_{i_j} \pm e_{i_{j'}}$ are all in L. Since these elements generate D_k, we have $L \cong D_k \oplus \mathbb{Z}^{n-k}$.

The dual lattice L^* is the set of elements of $\mathbb{Q}^n$ with integral inner product with each element of L, and in this representation we have that L^* is the set of vectors with integer components in all entries other than $i_1, \dots, i_k$, and with the components in entries $i_1, \dots, i_k$ either all integers or all half integers. Therefore, the discriminant group $\overline{L}$ can be represented by the four vectors 0, $z = e_{i_1}$, and

$$a = \frac{1}{2}\left(e_{i_1} + e_{i_2} + \cdots + e_{i_k}\right),$$
$$b = \frac{1}{2}\left(-e_{i_1} + e_{i_2} + \cdots + e_{i_k}\right).$$

We have $\langle z, z\rangle = 1 \equiv 0 \pmod 1$, and $\langle a, a\rangle = \langle b, b\rangle = k/4$. □

Lemma 6.3. *The d-invariant of $L = D_k \oplus \mathbb{Z}^{n-k}$ takes on the values* $0, 0, -k/4, 1 - k/4$.

Proof. The d-invariant is invariant under stable isomorphisms, so we can assume $L = D_k$. Then a set of short representatives of the classes of characteristic covectors is $(1, \dots, 1)$, $(-1, 1, \dots, 1)$, $(0, \dots, 0)$, and $(2, 0, \dots, 0)$. These have norms k, k, 0, and 4. The result now follows: see equation (13). □

Proof of Proposition 6.1. As in Section 5.1, $P(2,1)$ bounds a rational homology ball Z_2 with

$$H_1(Z_2) \cong \mathbb{Z}/2\mathbb{Z}, H_2(Z_2) = 0.$$

If X is any simply connected positive definite 4-manifold with boundary $P(2,1)$, then $\widehat{X} := X \cup_{P(2,1)} (-Z_2)$ is a closed, positive definite 4-manifold. Since $\widehat{X}$ can be obtained from X by attaching a two-handle, a three-handle and a four-handle, $\widehat{X}$ is also simply connected. By [8], $\widehat{X}$ has intersection form $\mathbb{Z}^n$.

In the long exact sequence for the pair $(\widehat{X}, X)$, we have

$$H_3(\widehat{X}, X) \to H_2(X) \to H_2(\widehat{X}) \to H_2(\widehat{X}, X) \to H_1(X).$$

We have

$$H_3(\widehat{X}, X) \cong H_3(Z_2, \partial Z_2) \cong H^1(Z_2) = 0, \quad H_2(\widehat{X}, X) \cong H^2(Z_2) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(X) = 0,$$

and both $H_2(X)$ and $H_2(\widehat{X})$ are torsionfree. Therefore, we have a short exact sequence

$$0 \to H_2(X) \to H_2(\widehat{X}) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

so $H_2(X)$ is an index-two subgroup of $H_2(\widehat{X})$ under the natural inclusion map. Since $\widehat{X}$ has intersection lattice $\mathbb{Z}^n$, the intersection lattice of X is an index-two sublattice of $\mathbb{Z}^n$, so, by Lemma 6.2, is isomorphic to $D_k \oplus \mathbb{Z}^{n-k}$.

Let X_0 be the positive definite plumbing 4-manifold with intersection form D_4, then $P(2,1) = \partial X_0$. Since the discriminant group and linking pairing of the intersection form of a 4-manifold are invariants of its boundary, Lemma 6.2 implies that k must be divisible by 4. Since the d-invariant of the intersection form of a positive definite 4-manifold gives an upper bound on the d-invariant of its boundary [15] and $-X_0$ is sharp [16], Lemma 6.3 implies $k \leqslant 4$. Therefore, $k = 4$, and the result follows. □

Corollary 6.4. *Any negative definite, simply connected* 4*-manifold with boundary* $-P(2,1)$ *is sharp.*

Proof. The 4-manifold $-X_0$ is sharp. By Proposition 6.1, any negative definite, simply connected 4-manifold with boundary $-P(2,1)$ has the same intersection form as that of $-X_0\#(n-4)\overline{\mathbb{C}P^2}$. □

6.2 | The changemaker condition

Whenever $q < p < 2q$, using Proposition 5.3, there is a sharp cobordism W from $P(2,1)$ to $P(p,q)$. Suppose $P(p,q)$ is positive surgery on some knot $K \subset S^3$. Let $X = W \cup_{P(p,q)} (-W_{4q}(K))$, then X is a negative definite manifold with boundary $-P(2,1)$. Since X is obtained from W_{4q} (which is simply connected) by adding two-handles, X is simply connected. By combining Corollary 6.4 and Proposition 6.1, X is sharp and has intersection lattice $-(D_4 \oplus \mathbb{Z}^{n-2})$. Also, for Z_2 the rational homology ball with boundary $P(2,1)$, the manifold $\widehat{X} = X \cup_{P(2,1)} (-Z_2)$ is closed, simply connected and negative definite, so has intersection lattice $-\mathbb{Z}^{n+2}$. From Kirby diagrams for W and $Z = W \cup_{P(2,1)} (-Z_2)$ (see Figure 2), we can also see that the intersection lattice of Z is the linear lattice $\Lambda(q,-p)$ with vertex basis $x_0, \dots, x_n$, and the intersection lattice of W is (as a sublattice of $\Lambda(q,-p)$) spanned by $2x_0, x_1, \dots, x_n$. Therefore, the following diagram of homology groups

$$\begin{array}{ccc} H_2(W) & \longrightarrow & H_2(Z) \\ \downarrow & & \downarrow \\ H_2(X) & \longrightarrow & H_2(\widehat{X}) \end{array}$$

with maps induced by inclusions is isomorphic to the diagram

$$\begin{array}{ccc} \langle 2x_0, x_1, \dots, x_n\rangle & \longrightarrow & \langle x_0, x_1, \dots, x_n\rangle = -\Lambda(q,-p) \\ \downarrow & & \downarrow \\ -(D_4 \oplus \mathbb{Z}^{n-2}) & \longrightarrow & -\mathbb{Z}^{n+2}. \end{array}$$

Lemma 6.5. *As subgroups of $H_2(\widehat{X})$,*

$$H_2(W) = H_2(Z) \cap H_2(X).$$

Proof. By the exact sequence $H_2(Z) \to H_2(\widehat{X}) \to H_2(\widehat{X}, Z)$, an element $\beta \in H_2(\widehat{X})$ is contained in the image of $H_2(Z)$ if and only if the image of β in $H_2(\widehat{X}, Z) \cong H_2(W_{4q}(K), \partial W_{4q}(K))$ is zero. Similarly, β is contained in the image of $H_2(X)$ if and only if the image of β in $H_2(\widehat{X}, X) \cong H_2(Z_2, \partial Z_2)$ is zero, and β is contained in the image of $H_2(W)$ if and only if the image of β in $H_2(\widehat{X}, W) \cong H_2(Z_2, \partial Z_2) \oplus H_2(W_{4q}(K), \partial W_{4q}(K))$ is zero. Our conclusion follows easily. □

The last piece of data we need is the class $[\widehat{F}] \in H_2(-W_{4q}(K)) \subset H_2(X)$, where $\widehat{F}$ is obtained by smoothly gluing the core of the handle attachment to a copy of a minimal genus Seifert surface F for K; its homology class generates the second homology. Note that $H_2(-W_{4q}(K))$ is orthogonal to all of $H_2(W)$ and satisfies $\langle [\widehat{F}], [\widehat{F}]\rangle = -4q$ since $-W_{4q}(K)$ is negative definite. Let

$$\varphi : \mathbb{Z}/4q\mathbb{Z} \to \mathrm{Spin}^c(P(p,q))$$

be the correspondence with $\varphi(i)$ equal $\mathfrak{s}_0|_{P(p,q)}$ for $\mathfrak{s}_0$ any Spin^c structure on $-W_{4q}(K)$ satisfying

$$\langle c_1(\mathfrak{s}_0), [\widehat{F}]\rangle \equiv -4q + 2i \pmod{8q}.$$

Proposition 6.6. *There is an extension $\mathfrak{r} \in \mathrm{Spin}^c(X)$ of $\varphi(i)$ over X with $c_1(\mathfrak{r})$ a short characteristic covector of $D_4 \oplus \mathbb{Z}^{n-2}$ if any only if $g(K) \leqslant i \leqslant 4q - g(K)$.*

Proof. Since X has boundary $-P(2,1)$ and $b_2(X) = n+2$, we have that for any $\mathfrak{r} \in \mathrm{Spin}^c(X)$,

$$d(-P(2,1), \mathfrak{r}|_{P(2,1)}) \geqslant \frac{(c_1(\mathfrak{r}))^2 + (n+2)}{4}, \tag{24}$$

and since X is sharp this is an equality if and only if $c_1(\mathfrak{r})$ is a short characteristic covector of $-H_2(X) = D_4 \oplus \mathbb{Z}^{n-2}$. Similarly, for any $\mathfrak{s}_1 \in \mathrm{Spin}^c(W)$,

$$d(P(p,q), \mathfrak{s}_1|_{P(p,q)}) \geqslant d(P(2,1), \mathfrak{s}_1|_{P(2,1)}) + \frac{(c_1(\mathfrak{s}_1))^2 + (n+1)}{4} \tag{25}$$

and since W is sharp as a cobordism, for each $\mathfrak{t} \in \mathrm{Spin}^c(P(p,q))$ there is some $\mathfrak{s}_1 \in \mathrm{Spin}^c(W)$ such that this is an equality and $\mathfrak{s}_1|_{P(p,q)} = \mathfrak{t}$.

For $\mathfrak{s}_0 \in \mathrm{Spin}^c(-W_{4q}(K))$ with

$$\langle c_1(\mathfrak{s}_0), [\widehat{F}]\rangle = -4q + 2i$$

(so that in particular $\varphi(i) = \mathfrak{s}_0|_{P(p,q)}$), we have

$$(c_1(\mathfrak{s}_0))^2 = -\frac{(-4q+2i)^2}{4q}.$$

Using (8) and (9), we have

$$d(P(p,q), \mathfrak{s}_0|_{P(p,q)}) = \frac{-(c_1(\mathfrak{s}_0))^2 - 1}{4} - 2t_{\min\{i,4q-i\}}(K).$$

Since $t_i(K) \geqslant 0$ and (5),

$$d(P(p,q), \mathfrak{s}_0|_{P(p,q)}) \leqslant \frac{-(c_1(\mathfrak{s}_0))^2 - 1}{4} \tag{26}$$

with equality if and only if $\langle c_1(\mathfrak{s}_0), [\widehat{F}]\rangle = -4q + 2i$ for some i with $g(K) \leqslant i \leqslant 4q - g(K)$. Note that inequality (24) is the difference of inequalities (26) and (25) if $\mathfrak{s}_0|_{P(p,q)} = \mathfrak{s}_1|_{P(p,q)}$. If $g(K) \leqslant i \leqslant 4q - g(K)$, then there is some extension $\mathfrak{s}_0$ of $\varphi(i)$ over $-W_{4q}(K)$ that achieves equality in (26), and there is always some extension s_1 of $\varphi(i)$ over W achieving equality in (25). These two Spin^c structures glue to a Spin^c structure $\mathfrak{r}$ on $X = W \cup (-W_{4q}(K))$ that will achieve equality in (24), so $c_1(\mathfrak{r})$ is short and $\mathfrak{r}|_{P(p,q)} = \varphi(i)$.

Conversely, if $\mathfrak{r} \in \mathrm{Spin}^c(X)$ has $c_1(\mathfrak{r})$ short, then $\mathfrak{r}$ achieves equality in (24), so $\mathfrak{s}_0 = \mathfrak{r}|_{-W_{4q}(K)}$ and $\mathfrak{s}_1 = \mathfrak{r}|_W$ will achieve equality in (25) and (26), respectively. Therefore, $\mathfrak{s}_0|_{P(p,q)} = \mathfrak{r}|_{P(p,q)}$ will equal $\varphi(i)$ for some $g(K) \leqslant i \leqslant 4q - g(K)$. □

Putting all of these together, we have a Euclidean lattice $\mathbb{Z}^{n+2} = -H_2(\widehat{X})$, with a corank–1, linear sublattice

$$-H_2(W) \cong \Lambda(q, -p) = \langle x_0, \ldots, x_n \rangle$$

and a sublattice $D_4 \oplus \mathbb{Z}^{n-2} = -H_2(X)$ such that

$$\langle 2x_0, \ldots, x_n \rangle = \langle x_0, \ldots, x_n \rangle \cap (D_4 \oplus \mathbb{Z}^{n-2}). \tag{27}$$

Since $\Lambda(q, -p)$ has discriminant q and corank 1 and is embedded primitively in $\mathbb{Z}^{n+2}$ (this follows from the long exact sequence of the pair $(X \cup Z_0, W \cup Z_0)$), the orthogonal complement of $\Lambda(q, -p)$ has discriminant q and rank 1, so is generated by a vector σ with $\langle \sigma, \sigma \rangle = q$. Since $|\langle [\widehat{F}], [\widehat{F}] \rangle| = 4q$ and $[\widehat{F}]$ is contained in the orthogonal complement of $\Lambda(q, -p)$, we must have $[\widehat{F}] = 2\sigma$. Therefore, Proposition 6.6 gives the following:

Proposition 6.7. *If $P(p, q)$ is the result of $4q$ surgery on some knot $K \subset S^3$ and $q < p < 2q$, then there is an embedding of $\Lambda(q, -p)$ into $\mathbb{Z}^{n+2}$ as the orthogonal complement of a vector σ and an embedding $D_4 \oplus \mathbb{Z}^{n-2} \hookrightarrow \mathbb{Z}^{n+2}$ such that there exists some short characteristic covector χ for $D_4 \oplus \mathbb{Z}^{n-2}$ with $\langle \chi, \sigma \rangle = i$ if and only if $-2q + g(K) \leqslant i \leqslant 2q - g(K)$.*

Pushing the logic of Proposition 6.6 a little further, the Alexander polynomial of K can be recovered from σ:

Proposition 6.8. *For $0 \leqslant i \leqslant 2q$, the torsion coefficient $t_i(K)$ satisfies*

$$t_i(K) = \min_{\substack{\chi \in \mathrm{Char}(D_4 \oplus \mathbb{Z}^{n-2}) \\ \langle \chi, \sigma \rangle = 2q - i}} \left\lceil \frac{\langle \chi, \chi \rangle - n - 2}{8} \right\rceil .$$

Proof. Since $[\widehat{F}] = 2\sigma$ and the intersection lattice on X is $D_4 \oplus \mathbb{Z}^{n-2}$, any characteristic covector χ for $D_4 \oplus \mathbb{Z}^{n-2}$ with $\langle \chi, \sigma \rangle = 2q - i$ is the first Chern class of a Spin^c structure $\mathfrak{r}$ on X with

$$\langle c_1(\mathfrak{r}), [\widehat{F}] \rangle = -4q + 2i. \tag{28}$$

(Note that we need to change the sign of the inner product.) Then, exactly as in the proof of Proposition 6.6, the restriction of $\mathfrak{r}$ to $-W_{4q} = -W_{4q}(K)$ satisfies

$$d(P(p, q), \mathfrak{r}|_{P(p,q)}) = \frac{-(c_1(\mathfrak{r}|_{-W_{4q}}))^2 - 1}{4} - 2t_i(K). \tag{29}$$

Let $\mathfrak{s}_1$ be the restriction of $\mathfrak{r}$ to W, then $\mathfrak{s}_1$ satisfies

$$d(P(p, q), \mathfrak{s}_1|_{P(p,q)}) \geqslant d(P(2, 1), \mathfrak{s}_1|_{P(2,1)}) + \frac{(c_1(\mathfrak{s}_1))^2 + (n + 1)}{4}. \tag{30}$$

Combining (29) and (30) together,

$$t_i(K) \leqslant \frac{-(c_1(\mathfrak{r}))^2 - (n + 2)}{8} - \frac{d(P(2, 1), \mathfrak{r}|_{P(2,1)})}{2}. \tag{31}$$

Using Proposition 5.3, some $\mathfrak{s}_1 \in \mathrm{Spin}^c(W)$ achieves equality in (30) with $\mathfrak{s}_1|_{P(p,q)} = \varphi(i)$. Let $\mathfrak{r} \in \mathrm{Spin}^c(X)$ be the extension of $\mathfrak{s}_1$ with (28), then $\mathfrak{r}$ achieves equality in (31). Therefore,

$$t_i(K) = \min_{\substack{\mathfrak{r}\in\mathrm{Spin}^c(X)\\ \langle c_1(\mathfrak{r}),[\widehat{F}]\rangle = -4q+2i}} \frac{-(c_1(\mathfrak{r}))^2 - (n+2)}{8} - \frac{d(P(2,1), \mathfrak{r}|_{P(2,1)})}{2} \tag{32}$$

Since $t_i(K)$ is an integer and $d(P(2,1), \mathfrak{r}|_{P(2,1)})$ will always be either 0 or -1, we get

$$t_i(K) = \min_{\substack{\mathfrak{r}\in\mathrm{Spin}^c(X)\\ \langle c_1(\mathfrak{r}),[\widehat{F}]\rangle = -4q+2i}} \left\lceil \frac{-(c_1(\mathfrak{r}))^2 - (n+2)}{8} \right\rceil. \tag{33}$$

Finally, Spin^c structures $\mathfrak{r}$ on X with (28) correspond (under the first Chern class and a change in the sign of the inner product) with characteristic covectors χ of $D_4 \oplus \mathbb{Z}^{n-2}$ with $\langle \chi, \sigma\rangle = 2q - i$, and $-(c_1(\mathfrak{r}))^2 = \langle \chi, \chi\rangle$, so the desired formula follows. □

By Proposition 6.2, specifying a sublattice $D_4 \oplus \mathbb{Z}^{n-2} \subset \mathbb{Z}^{n+2}$ is equivalent to choosing four indices $a > b > c > d$ such that for $v \in \mathbb{Z}^{n+2}$, $v \in D_4 \oplus \mathbb{Z}^{n-2}$ if and only if $\langle v, e_a + e_b + e_c + e_d\rangle$ is even. The characteristic covectors for $D_4 \oplus \mathbb{Z}^{n-2}$ come in two types: those that are the restrictions of characteristic covectors of $\mathbb{Z}^{n+2}$, which can be represented by elements of $\mathbb{Z}^{n+2}$ with all entries odd, and those that are not, which can be represented by elements of $\mathbb{Z}^{n+2}$ with the entries in positions a, b, c, and d even and all other entries odd. Call these two types of covectors *even* and *odd*, respectively. The short characteristic covectors are exactly the ones with all odd entries equal to ± 1, and the even entries (if any) equal to $\pm 2, 0, 0$, and 0 in some order.

As in [9], we will assume $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{n+1})$ with

$$0 \leqslant \sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_{n+1}.$$

Moreover, we can assume that for any two indices $i, j \in \{0, 1, \dots, n+1\}$, we always have

$$i > j, \qquad \text{if } \sigma_i = \sigma_j, i \in \{a, b, c, d\}, \text{and } j \notin \{a, b, c, d\}. \tag{34}$$

Definition 6.9. Let $\mathrm{Short}(D_4 \oplus \mathbb{Z}^{n-2}) = \mathrm{Short}_0 \cup \mathrm{Short}_1$, with $\mathrm{Short}_0 = \mathrm{Short}(\mathbb{Z}^n)$ the set of even short characteristic covectors and $\mathrm{Short}_1 = \mathrm{Short}(D_4 \oplus \mathbb{Z}^{n-2}) - \mathrm{Short}_0$ the set of odd characteristic covectors. Let

$$\chi^0 = -\sum_{i=0}^{n+3} e_i$$

and

$$\chi^1 = -2e_a - \sum_{i\notin\{a,b,c,d\}} e_i$$

be the elements of Short_0 and Short_1, respectively, minimizing $\langle \chi, \sigma\rangle$. Let

$$\mathcal{T}_0 = \left\{ \frac{1}{2}(\chi - \chi^0) \,\middle|\, \chi \in \mathrm{Short}_0 \right\}$$

and

$$\mathcal{T}_1 = \left\{ \frac{1}{2}(\chi - \chi^1) \middle| \chi \in \text{Short}_1 \right\}$$

be called the sets of even and odd test vectors, respectively.

For $\chi \in \mathbb{Z}^{n+2}$, let χ_i denote the component of χ corresponding to the index i. The following result is easy to see.

Proposition 6.10. *For $\chi \in \mathcal{T}_1$, $(\chi_d, \chi_c, \chi_b, \chi_a)$ is one of $(\pm 1, 0, 0, 1)$, or $(0, \pm 1, 0, 1)$, or $(0, 0, \pm 1, 1)$, or $(0,0,0,2)$, or $(0,0,0,0)$.*

Proposition 6.11. *The sets $\{\langle \chi, \sigma \rangle \mid \chi \in \mathcal{T}_0\}$ and $\{\langle \chi, \sigma \rangle \mid \chi \in \mathcal{T}_1\}$ are both intervals of integers beginning at 0. Also,*

$$\sum_{i=0}^{n+1} \sigma_i = \max\{\langle \chi, \sigma \rangle \mid \chi \in \mathcal{T}_0\} = \max\{\langle \chi, \sigma \rangle \mid \chi \in \mathcal{T}_1\} \pm 1. \tag{35}$$

Proof. By Proposition 6.7, the set $\{\langle \chi, \sigma \rangle \mid \chi \in \text{Short}(D_4 \oplus \mathbb{Z}^{n-2})\}$ is an interval of integers. For each $i \in \{0, 1\}$, the set $\{\langle \chi, \sigma \rangle \mid \chi \in \text{Short}_i\}$ contains the elements of this interval with the same parity. So the parities are different for $i = 0$ and $i = 1$. In particular, both sets are arithmetic progressions of step size 2, so subtracting off the smallest element and dividing by 2 gives intervals beginning at 0. □

Corollary 6.12. *σ is a changemaker.*

Proof. The set $\mathcal{T}_0$ consists of just vectors with all entries 0 or 1. □

Proof of Theorem 1.10. This follows from the combination of Corollary 6.12 and Proposition 6.7. □

Corollary 6.13. $\sigma_a = \sigma_b + \sigma_c + \sigma_d + \theta$, *where $\theta \in \{-1, 1\}$.*

Proof. Using (35), we see that

$$\sum_{i=0}^{n+1} \sigma_i = 2\sigma_a + \left(\sum_{j \notin \{a,b,c,d\}} \sigma_j \right) \pm 1.$$

The result is now immediate. □

Lemma 6.14. *An irreducible vector $v \in \sigma^\perp$ has an odd pairing with the vector $e_a + e_b + e_c + e_d$ if and only if $[v]$ contains x_0.*

Proof. Suppose $v \in \sigma^\perp$ is irreducible. The pairing $\langle v, e_a + e_b + e_c + e_d \rangle$ is even if and only if $v \in D_4 \oplus \mathbb{Z}^{n-2}$, which is equivalent to $v \in \langle 2x_0, \dots, x_n \rangle$ by (27). Since v is irreducible, $v \notin \langle 2x_0, \dots, x_n \rangle$ if and only if $[v]$ contains x_0. □

Let

$$G = 1 + \sigma_0 + \sigma_1 + \cdots + \sigma_{d-1}. \tag{36}$$

Lemma 6.15. *There exists $\chi \in \mathcal{T}_1$ with $\langle \chi, \sigma \rangle = G$. Let f be the minimal index such that $f > d$ and $f \notin \{a, b, c\}$. If $\chi_a = 0$, then*

$$G \geqslant \sigma_f.$$

If $\chi_a \neq 0$, then

$$G \geqslant \sigma_a - \sigma_b = \sigma_c + \sigma_d + \theta.$$

Proof. Using Proposition 6.11, there exists $\chi \in \mathcal{T}_1$ with $\langle \chi, \sigma \rangle = G$. If $\chi_a = 0$, by Proposition 6.10 we have $\chi_b = \chi_c = \chi_d = 0$, then there must be an index $i > d$, $i \notin \{a, b, c\}$, with $\chi_i \neq 0$ as otherwise $\langle \chi, \sigma \rangle < G$. So

$$G = \langle \chi, \sigma \rangle \geqslant \sigma_i \geqslant \sigma_f.$$

If $\chi_a \neq 0$, by Proposition 6.10, we have

$$G = \langle \chi, \sigma \rangle \geqslant \sigma_a - \sigma_b = \sigma_c + \sigma_d + \theta. \qquad \square$$

7 | BOUNDING THE INDEX d

In Sections 7 and 8, we will classify the linear changemaker lattices restricted by Theorem 1.10. As in our previous papers [1, 2], we will use the techniques introduced by Greene [9]. The basic strategy is, for such a lattice $\sigma^\perp$, we will analyze the standard basis vectors, which are irreducible by Lemma 3.14, and other irreducible vectors of interest. By Proposition 3.4, irreducible vectors are intervals up to sign reversal, hence the pairings between them can be computed (up to sign reversal) by (17) and Lemma 3.8, which only involve the weights of high weight vertices and the relative positions of intervals.

In this section, we will prove that $d = 0$, where d is the index defined after the proof of Proposition 6.8. We assume that $d > 0$ for contradiction.

Recall that we write $(e_0, e_1, \ldots, e_{n+1})$ for the orthonormal basis of $\mathbb{Z}^{n+2}$, and $\sigma = \sum_i \sigma_i e_i$. Since $\Lambda(q, -p)$ is indecomposable (Proposition 3.9), $\sigma_0 \neq 0$, otherwise $\sigma^\perp$ would have a direct summand $\mathbb{Z}$. So $\sigma_0 = 1$. By Lemma 6.14, we have that $[v_d]$ contains x_0. Set

$$w = \theta e_0 + e_d + e_c + e_b - e_a, \tag{37}$$

where $\theta \in \{-1, 1\}$ is as in Corollary 6.13. The strategy in this section is to analyze v_d and w.

Lemma 7.1. *w is an irreducible vector of $\sigma^\perp$. Also, $x_0 \notin [w]$.*

Proof. Corollary 6.13 shows that w is in $\sigma^\perp$. Suppose $w = x + y$ with $x, y \in \sigma^\perp$ and $\langle x, y \rangle \geqslant 0$. If both x, y are non-zero, by Lemma 3.15 we may assume that one of the vectors is $e_d - e_0$ and the other is $-e_a + e_b + e_c$. Both vectors will then be irreducible and $x_0 \in [x], [y]$ by Lemma 6.14.

That implies $\langle x, y\rangle \neq 0$, which is a contradiction. The second statement is immediate from Lemma 6.14. ◻

Corollary 7.2. *If one of the following two conditions holds, then* $\theta = 1$*:*

(1) $\sigma_d = 1$;
(2) *there exists a vector* v *with* $\langle v, e_0\rangle = -\langle v, e_d\rangle = 1$, $\max \operatorname{supp}(v) = d$ *and* $|\langle v, w\rangle| \leqslant 1$.

Proof. If $\sigma_d = 1$ and $\theta = -1$, then $w = (-e_0 + e_d) + (e_c + e_b - e_a)$ is reducible, a contradiction to Lemma 7.1.

If there exists a vector v as in the statement, then since $\langle v, e_0\rangle = -\langle v, e_d\rangle = 1$ and $\max \operatorname{supp}(v) = d$, we have $\langle v, w\rangle = \theta - 1$. Using $|\langle v, w\rangle| \leqslant 1$, we have $\theta = 1$. ◻

Remark 7.3. When $d > 0$, we have $[v_d]$ contains x_0. For any $0 < i < d$, $[v_i]$ does not contain x_0. Also, $\operatorname{supp}(v_i) \cap \operatorname{supp}(w) = \emptyset$ or $\{0\}$, so $|\langle w, v_i\rangle| \leqslant 2$.

Lemma 7.4. *Suppose that* $0 \notin \operatorname{supp}(v_d)$*, then* $[v_d] \dagger [w]$.

Proof. Since $0 \notin \operatorname{supp}(v_d)$, we can compute

$$\langle w, v_d\rangle = -1. \tag{38}$$

Note that $x_0 \in [v_d]$ and $x_0 \notin [w]$. Assume that $[v_d] \dagger [w]$ does not happen, then either $[w] \prec [v_d]$ or $[v_d] \pitchfork [w]$.

If $[w] \prec [v_d]$, then by Lemma 3.8 we have $|\langle w, v_d\rangle| = |w| - 1 = 4$, contradicting (38).

If $[v_d] \pitchfork [w]$, by Lemma 3.8 and (38), we have $|[v_d] \cap [w]| = 3$, and there exists $\epsilon \in \{-1, 1\}$ such that $w = \epsilon[w]$ and $v_d = -\epsilon[v_d]$. So $w + v_d = \epsilon([w] - [v_d]) = x + y$ with $[x]$ and $[y]$ being distant, and we may assume $x_0 \in [x]$. Since v_d is not tight, v_d is unbreakable (Lemma 3.16). By Proposition 3.6 and (17), $|v_d| = |[w] \cap [v_d]| = 3$, and $|x| = 2$. We get $v_d = e_i + e_{d-1} - e_d$ for some $0 < i < d - 1$, and

$$w + v_d = \theta e_0 + e_i + e_{d-1} + e_c + e_b - e_a.$$

Using Lemmas 3.15 and 6.14, and the fact that $x_0 \in [x]$, we have either $x = e_j - e_a$ for some $j \in \{0, i, d-1\}$ or $x = -e_0 + e_k$ for some $k \in \{c, b\}$. If $x = e_j - e_a$, then $\sigma_j = \sigma_a$, which forces $\sigma_a = \sigma_b$, contradicting Corollary 6.13. If $x = -e_0 + e_k$, then $\theta = -1$ and $\sigma_k = 1$, which forces $\sigma_d = 1$, contradicting Corollary 7.2. ◻

Lemma 7.5. *Suppose that* $0 \notin \operatorname{supp}(v_d)$ *and* $|\langle v_i, v_d\rangle| = 1$ *for some* i *with* $0 < i < d$*. Then* $i = 1$.

Proof. Since $i < d$, $x_0 \notin [v_i]$ by Lemma 6.14. We have $[v_d] \dagger [w]$ by Lemma 7.4.

If $[v_i] \dagger [v_d]$, then $[v_i]$ and $[w]$ share their left end. If $|v_i| > 2$, by Lemma 3.8 we have $2 \leqslant |\langle v_i, w\rangle|$, and the equality holds only when $|v_i| = 3$. Since $\langle v_i, w\rangle = \langle v_i, e_0\rangle$, we have $\langle v_i, e_0\rangle = 2$ and $|v_i| = 3$, which is not possible. So $|v_i| = 2$ in this case.

If $[v_i]$ and $[v_d]$ share their right end, then we must have $|v_i| = 2$ by Lemma 3.8.

In the above two cases, we have $|v_i| = 2$ and $[v_i]$ abuts the right end of $[v_d]$, so $|\langle v_i, w\rangle| = 1$, which implies $i = 1$ since $v_i = e_{i-1} - e_i$ by Lemma 3.18.

If $[v_i] \pitchfork [v_d]$, then $|[v_i] \cap [v_d]| = |v_d| = 3$. By Lemma 3.22, v_i is tight. If $i > 1$, $|v_i| \geqslant 6 = |w| + |v_d| - 2 = |[v_d] \cup [w]|$. Since $[v_d] \dagger [w]$, the interval $[v_i]$ must contain all high weight vertices of $[w]$ by (17). Thus $|\langle w, v_i \rangle| \geqslant |w| - 2 = 3$, a contradiction (Remark 7.3). □

Lemma 7.6. *v_d is not gappy.*

Proof. Suppose for contradiction that v_d is gappy. Take the index i to be the smallest gappy index of v_d. First suppose that $i = 0$. Then, using Lemma 3.13, v_1 will be tight with $|v_1| = 5$. Note that $\langle w, v_1 \rangle = 2\theta$, $|v_1| = |w| = 5$, so $[w] \pitchfork [v_1]$ with $|[v_1] \cap [w]| = 4$, and there exists $\epsilon \in \{-1, 1\}$ such that $w = \epsilon[w]$ and $v_1 = \theta\epsilon[v_1]$. It follows that $w - \theta v_1 = x + y$ with $[x]$ and $[y]$ being distant, $|x| = |y| = 3$. Now

$$w - \theta v_1 = -\theta e_0 + \theta e_1 + e_d + e_c + e_b - e_a.$$

Since $x_0 \notin [w], [v_1]$, we have $x_0 \notin [x], [y]$. Using Lemma 3.15, one of x, y has the form $\pm e_j + e_k + e_l$, where $j \in \{0, 1\}, \{k, l\} \subset \{d, c, b\}$, but this vector is not in $\sigma^\perp$, a contradiction.

Suppose $i > 0$. Then $i = \min \operatorname{supp}(v_d)$ by [9, Paragraph 2 in Section 6, and Propositions 8.6, 8.7, 8.8]. Since $\langle v_{i+1}, v_d \rangle = 1$ (Lemma 3.18), by Lemma 7.5 we have $i + 1 = 1$, a contradiction. □

Proposition 7.7. $\min \operatorname{supp}(v_d) \leqslant 1$.

Proof. Set $i = \min \operatorname{supp}(v_d)$. If $i > 0$, since $\langle v_i, v_d \rangle = -1$, by Lemma 7.5 we have $i = 1$. □

Let G be defined as in (36). Our strategy is to first find a bound for G, and then find a bound for the integer d. Next, we do a case-by-case analysis to find that indeed $d = 0$.

Lemma 7.8. *v_d is not tight.*

Proof. Suppose for contradiction that v_d is tight. Using Lemma 6.15, we get

$$\sigma_d = G \geqslant \min\{\sigma_f, \sigma_d + \sigma_c + \theta\} \geqslant \min\{\sigma_f, 2\sigma_d - 1\},$$

which is not possible by (34) and Corollary 7.2. □

Combining Proposition 7.7 and Lemmas 7.6 and 7.8, we have:

Corollary 7.9. *$v_d = v_{d,0}e_0 + e_1 + \cdots - e_d$ with $v_{d,0} \in \{0, 1\}$.*

With the notation of Corollary 7.9 in place, we start the analysis to deduce $d = 0$. The following identity will be useful to keep in mind:

$$\sigma_d = G - 2 + v_{d,0}. \tag{39}$$

Lemma 7.10. *If either $|v_d| > 2$ or $d = 1$, then*

$$G \geqslant \sigma_d + \sigma_c + \theta.$$

Proof. Let χ be the vector as in Lemma 6.15. By that lemma, it will suffice to show $\chi_a \neq 0$. Assume that $\chi_a = 0$, then Lemma 6.15 implies that $G \geqslant \sigma_f > \sigma_d$. Using (39), we have that $G \leqslant \sigma_d + 2$, so $\sigma_f \in \{\sigma_d + 1, \sigma_d + 2\}$. Moreover, if $\sigma_f = \sigma_d + 2$, then $v_{d,0} = 0$, hence $d > 1$ by Corollary 7.9.

If $\sigma_f = \sigma_d + 1$, set $v'_f = -e_f + e_d + e_0$. If $\sigma_f = \sigma_d + 2$, set

$$v'_f = \begin{cases} -e_f + e_d + e_1 + e_0, & \text{if } \sigma_1 = 1, \\ -e_f + e_d + e_1, & \text{if } \sigma_1 = 2. \end{cases}$$

In either case, v'_f is irreducible and also in $\sigma^\perp$. Since $\langle v'_f, e_a + e_b + e_c + e_d\rangle = 1$, we get that $x_0 \in [v'_f]$. So $[v_d]$ and $[v'_f]$ share their left endpoint. If $|v_d| > 2$, then $|\langle v_d, v'_f\rangle| \geqslant 2$ by Lemma 3.8, which contradicts the direct computation $|\langle v_d, v'_f\rangle| \leqslant 1$. If $d = 1$, then $v_d = e_0 - e_1$ by Corollary 7.9, and $v'_f = -e_f + e_1 + e_0$. We get $\langle v_d, v'_f\rangle = 0$: this is still giving a contradiction since the intervals $[v_d]$ and $[v'_f]$ share their left endpoints, and so $\langle v_d, v'_f\rangle \neq 0$. □

Proposition 7.11. *If $|v_d| = 2$, then either $d = 1, G = 2$, or else $d = 2$, $G \in \{3, 4\}$.*
If $|v_d| > 2$, then $d \in \{3, 4\}$, $\theta = -1$, $v_{d,0} = 0$, and $1 + d \leqslant G \leqslant 5$.

Proof. If $|v_d| = 2$, our conclusion follows from Corollary 7.9 and (39).

Now we assume $|v_d| > 2$. Using Lemma 7.10 and (39), we have

$$G \geqslant \sigma_d + \sigma_c + \theta \geqslant 2\sigma_d + \theta = 2(G - 2 + v_{d,0}) + \theta,$$

thus

$$G \leqslant 4 - \theta - 2v_{d,0}. \tag{40}$$

If $d \leqslant 2$, by Corollary 7.9 and the assumption that $|v_d| > 2$ we have $v_{d,0} = 1$ and $d = 2$. We have $x_0 \in [v_2]$ while $x_0 \notin [w]$. Since $|v_2| = 3 < |w|$, $[v_2]$ and $[w]$ do not share their right endpoint, so we must have $|\langle v_2, w\rangle| \leqslant 1$ by Lemma 3.8. Then $\theta = 1$ by Corollary 7.2. So $G \leqslant 1$ by (40), which is not possible.

If $d \geqslant 3$, it follows from (40) and (36) that

$$4 - \theta - 2v_{d,0} \geqslant G \geqslant d + 1 \geqslant 4,$$

so $\theta = -1$, $v_{d,0} = 0$, $d \leqslant 4$ and $G \leqslant 5$. □

Proposition 7.11 implies $d \in \{0, 1, 2, 3, 4\}$. We now argue that $d = 0$.

Proposition 7.12. $d = 0$.

Proof. Suppose that $d = 1$. Using Lemma 7.8, we get that $v_1 = -e_1 + e_0$. By (40), $G = 2$ and $\sigma_1 = 1$. By Corollary 7.2 and Lemma 7.10, we get that

$$2 = G \geqslant \sigma_c + \sigma_1 + 1 \geqslant 3,$$

which is a contradiction.

Suppose $d = 2$. It follows from Proposition 7.11 that $|v_2| = 2$. We separate the cases to whether $\sigma_1 (= \sigma_2)$ is 1 or 2.

First assume that $\sigma_1 = \sigma_2 = 1$. If $c \neq 3$, then $x_0 \in [v_3]$ by Lemma 6.14, thus $[v_2]$ and $[v_3]$ share their left endpoint. So $\langle v_3, v_2\rangle \neq 0$. In particular, $1 \notin \mathrm{supp}(v_3)$. Since $\sigma_0 = \sigma_1 = 1$, $0 \notin \mathrm{supp}(v_3)$, so $|v_3| = 2$, which is impossible as $\sigma_3 > 1$ by (34). If $c = 3$, note that $\theta = 1$ by Corollary 7.2, by Lemma 6.15 we have

$$3 = G \geqslant \min\{\sigma_f, \sigma_3 + 2\}.$$

By (34), $\sigma_f > \sigma_3$, so we have $\sigma_3 \leqslant 2$. If $\sigma_3 = 1$, then $|v_3| = 2$, $\langle v_3, w\rangle = 0$ and $\langle v_3, v_2\rangle = -1$. Since $x_0 \notin [v_3]$, $[v_3]$ abuts the right endpoint of $[v_2]$. Since $[v_2] \dagger [w]$ by Lemma 7.4, we get $\langle v_3, w\rangle \neq 0$, a contradiction. If $\sigma_3 = 2$, then $v_3 = -e_3 + e_2 + e_1$. We have $v_3 \sim v_1 \sim v_2$, $|v_1| = |v_2| = 2$, $[v_2] \dagger [w]$, so $[v_3]$ contains the leftmost high weight vertex of $[w]$, which contradicts the fact that $\langle v_3, w\rangle = 0$.

Next we suppose ($d = 2$ and) $\sigma_1 = \sigma_2 = 2$. Then $v_1 = 2e_0 - e_1$, $v_2 = e_1 - e_2$. We have $x_0 \in [v_2]$, $x_0 \notin [v_1]$, $[w]$, $[v_2]$ abuts both $[v_1]$ and $[w]$, and $2 = |v_2| < |v_1| = |w|$. So $[v_1]$ and $[w]$ share their left endpoint. It follows from Lemma 3.8 that $|\langle v_1, w\rangle| = 4$, which is not possible by Remark 7.3.

Suppose $d \in \{3, 4\}$. Proposition 7.11 implies $v_d = -e_d + e_{d-1} + \cdots + e_1$, $\theta = -1$. Also, by Proposition 7.11 and (36), we have $5 \geqslant G \geqslant 2 + \sigma_1 + \sigma_2$, so $\sigma_1 = 1$. Let $v'_d = v_d - e_1 + e_0$, then v'_d is irreducible by Lemma 3.15, $|v'_d| = d < |w| = 5$, and $\langle v'_d, w\rangle = -2$. Since $x_0 \in [v'_d]$ and $x_0 \notin [w]$, by Lemma 3.8 we must have $[v'_d] \pitchfork [w]$ with $|[v'_d] \cap [w]| = 4$, and there exists $\epsilon = \pm 1$, such that $v'_d = \epsilon[v'_d]$ and $w = -\epsilon[w]$. Hence

$$v'_d + w = \epsilon([v'_d] - [w]) = -e_a + e_b + e_c + e_{d-1} + \cdots + e_2$$

is reducible, a contradiction to Lemma 3.15. □

8 | THE CASE $d = 0$

We now turn our attention to the classification in the case $d = 0$: in what follows, we classify all changemaker linear lattices of this sort.

Lemma 8.1. *$c = 1$, $\sigma_c = 1$, and $\sigma_a = \sigma_b + 1$.*

Proof. By Lemma 6.15, we have

$$1 = G \geqslant \min\{\sigma_f, \sigma_a - \sigma_b\} \geqslant \min\{\sigma_f, \sigma_c + \sigma_0 - 1\} = \min\{\sigma_f, \sigma_c\}.$$

If $f = 1$, then $1 \geqslant \sigma_f$, which contradicts (34). So $f > 1$, thus $c = 1$ and $\sigma_c = 1$. Hence the above inequality becomes an inequality, which means $\sigma_a = \sigma_b + 1$. □

For the rest of the section, we will replace w in (37) with

$$w' = -e_a + e_b + e_c. \tag{41}$$

The following is an immediate corollary of Lemma 8.1.

Corollary 8.2. *The vector w' is an irreducible, unbreakable vector in $\sigma^\perp$, and $x_0 \in [w']$.*

Proof. It follows from Lemma 8.1 that $w' \in \sigma^\perp$. Since $|w'| = 3$, it is irreducible and unbreakable. The fact that $x_0 \in [w']$ follows from Lemma 6.14. □

Lemma 8.3. *$b = 2$, $\sigma_b = 1$, and $\sigma_a = 2$. Hence $(\sigma_0, \dots, \sigma_a) = (1, 1, 1, 2^{[s]}, 2)$ for some $s \geqslant 0$.*

Proof. Suppose toward a contradiction that $b > 2$. Since $\sigma_0 = \sigma_1 = 1$ and $b > 2$, $\sigma_2 \in \{2, 3\}$.

If $\sigma_2 = 2$, then $\langle v_2, v_1\rangle = 0$, $\langle v_2, w'\rangle = 1$ and $\langle v_1, w'\rangle = -1$. Since $|v_1| = 2$ and $x_0 \notin [v_1]$, $[v_1]$ abuts the right end of $[w']$. If $[v_2]$ also abuts $[w']$, noting that $x_0 \notin [v_2]$, it abuts the right end of $[w']$, so $[v_2]$ abuts $[v_1]$, contradicting the fact that $\langle v_2, v_1\rangle = 0$. Thus by Lemma 3.8 we must have $[v_2] \pitchfork [w']$, $|[v_2] \cap [w']| = 3$, $v_2 = \epsilon[v_2]$ and $w' = \epsilon[w']$ for some $\epsilon \in \{1, -1\}$. It follows that $w' - v_2$ is reducible. However, $w' - v_2 = -e_a + e_b + e_2 - e_0$ is irreducible by Lemma 3.15 and the fact that $\sigma_a = \sigma_b + 1$, a contradiction.

If $\sigma_2 = 3$, then $[v_2]$ contains x_0, so $[w'] \prec [v_2]$. However, since $|w'| = 3$, Lemma 3.8 implies that $|\langle v_2, w'\rangle| = 2$, contradicting the direct computation $\langle v_2, w'\rangle = 1$.

Having proved $b = 2$, we must have $\sigma_2 \in \{1, 2, 3\}$. If $\sigma_2 = 2$, the interval $[v_2]$ contains x_0, so $[v_2]$ and $[w']$ share their left end, a contradiction to the direct computation $\langle v_2, w'\rangle = 0$. If $\sigma_2 = 3$, using Proposition 6.11, there must be some $\chi \in \mathcal{T}_1$ with $\langle \chi, \sigma\rangle = 2$. Moreover, since $\{0, 1, 2\} = \{d, c, b\}$, $\sigma_f > \sigma_2 = 3$. Therefore, $\chi_a \neq 0$ by Proposition 6.10. Using Lemma 8.1, $\sigma_a = 4$. So $\sigma_k \geqslant 4$ if $k > b = 2$ by (34). To get $\langle \chi, \sigma\rangle = 2$, it must be the case that for some $i \in \{b, c, d\}$, $\chi_i = -1$ and $\chi_j = 0$ for $j \neq i, a$. Then $\langle \chi, \sigma\rangle$ is either 1 or 3, a contradiction.

Therefore, $b = 2$, $\sigma_2 = 1$, and $\sigma_a = \sigma_b + 1 = 2$. □

Lemma 8.4. *$\sigma_i = 2s + 3$ for $i > a$. That is, $\sigma = (1, 1, 1, 2^{[s]}, 2, 2s + 3^{[t]})$ with $s, t \geqslant 0$.*

Proof. First, consider v_{a+1}. By (34), $\sigma_{a+1} > 2$, so $m := \min \operatorname{supp}(v_{a+1}) < a$. If $m \geqslant 3$, then $s := a - 3 > 0$. Let $j = \min \operatorname{supp}(v_m)$, by Lemma 8.3, $j = m - 1$ if $m > 3$, and $j = m - 2$ if $m = 3$. There would be a claw $(v_m; v_{a+1}, v_{m+1}, v_j)$, a contradiction to Lemma 3.24. Therefore, $\operatorname{supp}(v_{a+1}) \cap \{0, 1, 2\}$ is non-empty, thus is one of $\{0, 1, 2\}$, $\{1, 2\}$, or $\{2\}$ by Lemma 3.13.

We note that $x_0 \in [v_a]$ no matter $s = 0$ or $s > 0$.

We claim that there is no index j such that v_j is tight. Otherwise, we have $j > a$ and $[v_j]$ contains x_0, so $[v_a] \prec [v_j]$. If $s > 0$, $\langle v_a, v_j\rangle = 0$, a contradiction to $[v_a] \prec [v_j]$. If $s = 0$, then $|v_a| = 3$, hence $|\langle v_a, v_j\rangle| = 2$ by Lemma 3.8, contradicting the direct computation $\langle v_a, v_j\rangle = 1$.

If $m \in \{0, 1\}$, then $3 \in \operatorname{supp}(v_{a+1})$ since otherwise $\langle v_3, v_{a+1}\rangle = 2$, a contradiction to Lemma 3.21. Then since $|v_i| = 2$ for $3 < i \leqslant a$, v_{a+1} is just right by Lemma 3.13 and the claim in the last paragraph.

If $m = 0$, we have $\langle v_3, v_{a+1}\rangle = 1$, and $x_0 \notin [v_{a+1}]$. We also have $\langle v_3, v_1\rangle = -1$, and $x_0 \notin [v_1]$. If $s > 0$, then $(v_3; v_4, v_1, v_{a+1})$ will give a claw, a contradiction (Lemma 3.24). If $s = 0$ then $[v_3]$ contains x_0, so $[v_1]$ and $[v_{a+1}]$ must both abut the right end of $[v_3]$, contradicting the fact that they are orthogonal.

If $m = 1$, since $|\{a, b, c, d\} \cap \operatorname{supp}(v_{a+1})| = 3$, $x_0 \in [v_{a+1}]$. So $[v_a] \prec [v_{a+1}]$ and $|\langle v_{a+1}, v_a\rangle| = |v_a| - 1$. This contradicts the direct computation of $\langle v_a, v_{a+1}\rangle$ no matter $s = 0$ or $s > 0$.

If $m = 2$, then $v_{a+1} = e_2 + e_k + \cdots + e_a - e_{a+1}$ for some $3 \leqslant k \leqslant a$. If $3 < k < a$, there is a claw $(v_k; v_{k-1}, v_{k+1}, v_{a+1})$ (Lemma 3.24). If $k = a$ and $a > 3$, then $x_0 \in [v_a]$ but $x_0 \notin [v_{a+1}]$. Since $\langle v_a, v_{a+1}\rangle = -1$ and $|v_a| = 2 < |v_{a+1}|$, we have $[v_a] \dagger [v_{a+1}]$. If $s = 1$, then since $|v_a| = 2$, $\langle v_3, v_a\rangle = -1$, $[v_3]$ and $[v_{a+1}]$ will share a hight weight vertex, which is not possible by Lemma 3.22. If $s > 1$, then $\langle v_a, v_{a-1}\rangle = -1$ and $x_0 \notin [v_{a-1}]$, so $[v_{a-1}]$ abuts the right endpoint of $[v_a]$. Recall that $[v_{a+1}]$ also abuts the right endpoint of $[v_a]$, hence $\langle v_{a+1}, v_{a-1}\rangle = \pm 1$, a contradiction to the direct computation $\langle v_{a+1}, v_{a-1}\rangle = 0$. Therefore, $k = 3$, so v_{a+1} is just right and $\sigma_{a+1} = 2s + 3$.

Finally, suppose that for some $j > a+1$, $|v_j| > 2$. Take j to be the smallest such index. Then v_j is unbreakable by our earlier claim. Let $\ell = \min \operatorname{supp}(v_j)$. If either $\ell \geqslant a+1$ or $3 \leqslant \ell < a$, there will be a claw $(v_\ell; v_{\ell-1}, v_{\ell+1}, v_j)$ (when $\ell > 3$) or $(v_\ell; v_{\ell-2}, v_{\ell+1}, v_j)$ (when $\ell = 3$), contradicting Lemma 3.24.

If $\ell = a$, then $[v_j]$ contains x_0, so $[v_a]$ and $[v_j]$ share their left endpoint. No matter $s = 0$ or $s > 0$, $[v_3]$ is connected to $[v_a]$ via a (possibly empty) sequence of norm 2 vectors, so the intervals $[v_3]$ and $[v_j]$ will share a high weight vertex, a contradiction to Lemma 3.22.

If $\ell < 3$, then v_j is connected to the path $v_3 \sim v_1 \sim v_2 \sim v_{a+1}$ in the intersection graph $G(S)$. If $\ell \in \{1, 2\}$, $v_j \sim v_\ell$, so there is a heavy triple (v_3, v_{a+1}, v_j), contradicting Lemma 3.26. If $\ell = 0$, it follows from Lemma 3.13 that $1, 2 \in \operatorname{supp}(v_j)$. By Lemma 3.21, $|\langle v_3, v_j\rangle| \leqslant 1$, so $3 \in \operatorname{supp}(v_j)$ and $v_j \sim v_3$. Now $4, \dots, a \in \operatorname{supp}(v_j)$ by Lemma 3.13, so $\langle v_{a+1}, v_j\rangle \geqslant s + 2 - 1 \geqslant 1$, so $v_j \sim v_{a+1}$. We again have a heavy triple (v_3, v_{a+1}, v_j), contradicting Lemma 3.26. □

9 | PROOF OF THEOREM 1.2

Lemma 8.4 specifies a changemaker vector in $\mathbb{Z}^{n+2}$ whose orthogonal complement is the linear changemaker lattice $\Lambda(q, -p)$. From the integers $a_0, a_1, \dots a_n$ in (16), we can recover p and q using (14). Since $q < p < 2q$, we have

$$\frac{p}{q} = [2, a_0, a_1, \dots, a_n]^-.$$

We use the following facts:

Lemma 9.1 [9, Lemma 9.5 (2) and (3)]. *For integers s, t, b with $b \geqslant 2$ and $s, t \geqslant 0$:*

(1) $[\cdots, b, 2^{[t-1]}]^- = [\cdots, b-1, -t]^-$;
(2) *If* $[2^{[s+1]}, b, \dots]^- = \frac{p}{q}$, *then* $[-(s+2), b-1, \dots]^- = \frac{p}{q-p}$.

We have

$$\sigma = (1, 1, 1, 2^{[s]}, 2, 2s+3^{[t]}),$$

with $s, t \geqslant 0$. One can check that the standard basis of the linear changemaker lattice

$$S = \{v_{s+3}, \dots, v_3, v_1, v_2, v_{s+4}, \dots, v_{s+t+3}\}$$

coincides with its vertex basis with norms given by

$$\{2^{[s]}, 3, 2, 2, s+3, 2^{[t-1]}\}.$$

By Lemma 6.14, $[v_{s+3}]$ contains x_0, so $v_{s+3} = x_0$. Hence, we have

$$\frac{p}{q} = [2^{[s+1]}, 3, 2, 2, s+3, 2^{[t-1]}]^-.$$

Using Lemma 9.1, we see that

$$q = 7 + 4s + 9t + 12st + 4s^2t, \text{ and}$$

$$p = 11 + 4s + 14t + 16st + 4s^2t.$$

It is straightforward to check that

$$q = \frac{1}{r^2 - 2r - 1}(r^2 p - 1),$$

with $r = -2s - 3$ and $p = t(r^2 - 2r - 1) - 2r + 5$.

Proof of Theorem 1.2. Suppose $P(p, q) \cong S^3_{4q}(K)$, the above computation shows that (p, q) must be as in the statement. On the other hand, if (p, q) is as in the statement, it follows from [1, Table 2] and [21] that there exists a Berge–Kang knot K_0 such that $P(p, q) \cong S^3_{4q}(K_0)$. For the second statement, we note that K and K_0 correspond to the same changemaker vector. Using Proposition 6.8, we know that $\Delta_K = \Delta_{K_0}$, so $\widehat{HFK}(K) \cong \widehat{HFK}(K_0)$ by [17, Theorem 1.2]. □

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REFERENCES

1. W. Ballinger, C. Ching-Yun Hsu, W. Mackey, Y. Ni, T. Ochse, and F. Vafaee, *The prism manifold realization problem*, Algebr. Geom. Topol. **20** (2020), no. 2, 757–815.
2. W. Ballinger, Y. Ni, T. Ochse, and F. Vafaee, *The prism manifold realization problem II*, Comm. Anal. Geom. **29** (2021), no. 6, 1279–1334.
3. J. Berge, *Some knots with surgeries yielding lens spaces*, Preprint, arXiv:1802.09722, 2018.
4. J. Berge and S. Kang, *The hyperbolic P/P, P/SF_d, and P/SF_m knots in S^3*, Preprint.
5. S. Boyer and D. Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. reine angew. Math. **405** (1990), 181–220.
6. M. I. Doig, *Finite knot surgeries and Heegaard Floer homology*, Algebr. Geom. Topol. **15** (2015), no. 2, 667–690.
7. M. I. Doig, *On the number of finite p/q-surgeries*, Proc. Amer. Math. Soc. **144** (2016), no. 5, 2205–2215.
8. S. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), no. 2, 279–315.
9. J. E. Greene, *The lens space realization problem*, Ann. of Math. (2) **177** (2013), no. 2, 449–511.
10. J. E. Greene, *L-space surgeries, genus bounds, and the cabling conjecture*, J. Differential Geom. **100** (2015), no. 3, 491–506.
11. L. Gu, *Integral finite surgeries on knots in S^3*, Preprint, arXiv:1401.6708, 2014.
12. C. Lescop, *Global surgery formula for the Casson-Walker invariant*, Ann. Math. Stud., vol. 140, Princeton Univ. Press, Princeton, N.J., 1996.
13. Y. Ni and Z. Wu, *Heegaard Floer correction terms and rational genus bounds*, Adv. Math. **267** (2014), 360–380.
14. Y. Ni and X. Zhang, *Finite Dehn surgeries on knots in S^3*, Algebr. Geom. Topol. **18** (2018), no. 1, 441–492.
15. P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.
16. P. Ozsváth and Z. Szabó, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. **7** (2003), 185–224.

17. P. Ozsváth and Z. Szabó, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.
18. P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.
19. P. Ozsváth and Z. Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68.
20. J. Rasmussen, *Floer homology and knot complements*, ProQuest LLC, Ann Arbor, MI, Ph.D. thesis, Harvard University, 2003.
21. Z. Shang, *Knots with prism manifold surgeries*, Preprint, arXiv:1909.02252, 2019.
22. K. Walker, *An extension of Casson's invariant to rational homology spheres*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), no. 2, 261–267.