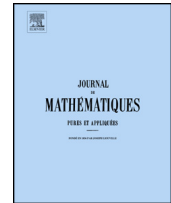




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Diffusive limits of the steady state radiative heat transfer system: Boundary layers

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ABSTRACT

In this paper, we study the diffusive limit of the steady state radiative heat transfer system for non-homogeneous Dirichlet boundary conditions in a bounded domain with flat boundaries. By taking account of the boundary layers, a composite approximate solution is constructed using asymptotic analysis. The convergence to the composite approximate solution in the diffusive limit is proved using a Banach fixed point theorem. The major difficulty lies in the nonlinear coupling between elliptic and kinetic transport equations. To overcome this difficulty, a spectral assumption is proposed to ensure the linear stability of boundary layers. Moreover, a combined L^2 - L^∞ estimate and the Banach fixed point theorem are used to obtain the convergence proof. This result extends our previous work [6] for the well-prepared boundary data case to the general boundary data.

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R É S U M É

Dans cet article, nous étudions la limite de diffusion du système de transfert de chaleur radiatif en régime stationnaire pour des conditions aux limites de type Dirichlet non homogènes dans un domaine borné avec frontière plate. Une solution approchée composite est construite à l'aide d'une analyse asymptotique prenant en compte les couches limites. La convergence vers la solution approchée dans la limite de diffusion est démontrée à l'aide d'un théorème de point fixe de Banach. La difficulté majeure réside dans le couplage non linéaire entre l'équation elliptique et l'équation de transport cinétique. Pour remédier à ce problème, une hypothèse spectrale assurant la stabilité linéaire des couches limites est proposée. De plus, une estimation combinée $L^2 - L^\infty$ et le théorème du point fixe de Banach sont utilisés pour obtenir la preuve de convergence. Ces résultats étendent nos travaux

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précédents [6] pour le cas des données aux limites bien préparées au cas mal préparé lorsque la couche limite existe.

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1. Introduction

1.1. Problem statements

We consider the following steady state radiative heat transfer system in the space $x \in \Omega = [0, 1] \times \mathbb{T}^2$ and $\beta \in \mathbb{S}^2$:

$$\varepsilon^2 \Delta T^\varepsilon + \langle \psi^\varepsilon - (T^\varepsilon)^4 \rangle = 0, \quad (1.1)$$

$$\varepsilon \beta \cdot \nabla \psi^\varepsilon + \psi^\varepsilon - (T^\varepsilon)^4 = 0, \quad (1.2)$$

with Dirichlet boundary conditions

$$T^\varepsilon(x) = T_b(x), \text{ for } x \in \partial\Omega = \{0, 1\} \times \mathbb{T}^2, \quad (1.3)$$

$$\psi^\varepsilon(x, \beta) = \psi_b(x, \beta), \text{ for } (x, \beta) \in \Gamma_-. \quad (1.4)$$

Here $T^\varepsilon = T^\varepsilon(x)$ is the temperature, $\psi^\varepsilon = \psi^\varepsilon(x, \beta)$ is the radiation intensity. The bracket $\langle \cdot \rangle$ denotes the momentum $\langle \psi(\beta) \rangle = \int_{\mathbb{S}^2} \psi(\beta) d\beta$. The boundary set Γ is given by

$$\Gamma = \{(x, \beta) : x \in \partial\Omega, \beta \in \mathbb{S}^2\},$$

and $\Gamma_+ = \Gamma \cap \{(x, \beta) : \beta \cdot n(x) > 0\}$ is the out-flow boundary, $\Gamma_- = \Gamma \cap \{(x, \beta) : \beta \cdot n(x) < 0\}$ is the in-flow boundary, where $n(x)$ is the exterior normal vector on the boundary. Note that the boundary conditions are imposed only on the in-flow boundary and the value of ψ on the out-flow boundary is determined by the system.

When the boundary data is well-prepared, i.e. $\psi_b(x, \beta) = T_b^4(x)$ for $(x, \beta) \in \Gamma_-$, the solution of (1.1)-(1.2) is shown in [6] to converge to the solution of the following nonlinear elliptic equation

$$\Delta T_0 + \frac{4\pi}{3} \Delta T_0^4 = 0, \quad (1.5)$$

$$\psi_0 = T_0^4, \quad (1.6)$$

subject to the boundary condition

$$T_0(x) = T_b(x), \quad \text{for any } x \in \partial\Omega. \quad (1.7)$$

The convergence is proved by using two methods: the weak convergence method and the relative entropy method. However, both methods rely on the assumption $\psi_b = T_b^4$ to obtain the estimates that are needed for the convergence estimates and fail for general boundary data, due to the presence of boundary layers.

The main objective of this paper is to study the diffusive limit ($\varepsilon \rightarrow 0$) for the general boundary data. By adding a boundary layer correction to (1.5)-(1.6), a composite approximation can be constructed. The convergence of the radiative transfer system (1.1)-(1.4) to this composite approximation in the diffusive limit will be established in this paper.

1.2. The boundary layer corrections

Here we recall the results of [7] on the boundary layer problem to system (1.1)-(1.2). For simplicity of notations, we assume that the boundary data at the top boundary $\{x_1 = 1\}$ is well-prepared, i.e. $\psi_b(x, \beta) = T_b^4(x)$ for $(x, \beta) \in \Gamma_- \cap \{x_1 = 1\}$, so that boundary layer only exists at the bottom $\{x_1 = 0\}$. Introducing $\eta = x_1/\varepsilon^2$, the corresponding boundary layer problem for system (1.1)-(1.2) reads as

$$\partial_\eta^2 \tilde{T}_0 + \langle \tilde{\psi}_0 - \tilde{T}_0^4 \rangle = 0, \quad (1.8)$$

$$\mu \partial_\eta \tilde{\psi}_0 + \tilde{\psi}_0 - \tilde{T}_0^4 = 0, \quad (1.9)$$

with boundary conditions

$$\tilde{T}_0(\eta = 0, x') = T_b(x'), \quad \text{for any } x' \in \mathbb{T}^2, \quad (1.10)$$

$$\tilde{\psi}_0(\eta = 0, x', \beta) = \psi_b(0, x', \beta), \quad \text{for any } x' \in \mathbb{T}^2, (0, x', \beta) \in \Gamma_-, \quad (1.11)$$

where $\mu = -n(x) \cdot \beta = \beta_1$ and $\tilde{T}_0 = \tilde{T}_0(\eta, x')$, $\tilde{\psi}_0 = \tilde{\psi}_0(\eta, x', \beta)$ with $x' = (x_2, x_3) \in \mathbb{T}^2$. The above problem is also called *nonlinear Milne problem* of the radiative heat transfer system.

Assuming T_b, ψ_b are bounded, the existence of weak solutions for the above problem is proved in [7]. Moreover, the weak solutions are shown to converge as $\eta \rightarrow \infty$ to some non-negative constants $\tilde{T}_{0,\infty}(x') := \lim_{\eta \rightarrow \infty} \tilde{T}_0(\eta, x')$, $\tilde{\psi}_{0,\infty}(x', \beta) := \lim_{\eta \rightarrow \infty} \tilde{\psi}_0(\eta, x', \beta)$. These constants give the boundary conditions for the nonlinear limit equation (1.5) by

$$T_0(x_1 = 0, x') = \tilde{T}_{0,\infty}(x'), \quad \text{for any } x' \in \mathbb{T}^2. \quad (1.12)$$

The linear stability for the nonlinear Milne problem is shown to be fulfilled under a spectral assumption, which reads as

(A) There exists a constant $\tau > 0$ such that the function $\tilde{T}_0 \in L_{\text{loc}}^2(\mathbb{R}_+ \times \mathbb{T}^2)$ satisfies the inequality

$$M \int_0^\infty e^{2\tau x} (2\tilde{T}_0^{\frac{3}{2}})^2 |\partial_x f|^2 dx \geq 4 \int_0^\infty e^{2\tau x} |\partial_x (2\tilde{T}_0^{\frac{3}{2}})|^2 f^2 dx \quad (1.13)$$

for any measurable function $f \in C^1(\mathbb{R}_+)$ with $f(0) = 0$, for some constant $M < 1$.

The spectral assumption holds when the boundary data is close to the well-prepared case ($\psi_b - T_b^4$ small) and uniqueness of the nonlinear Milne problem is proved in this situation [7].

By adding the boundary layer correction to the system (1.5)-(1.6), a composite approximate solution can be constructed. In details, we introduce a cut-off function $\chi = \chi(x_1) = \chi(\varepsilon\eta)$ as follows

$$\chi(x_1) = \begin{cases} 1, & \text{for } 0 \leq x_1 \leq \frac{1}{4}\delta, \\ 0, & \text{for } x_1 > \frac{3}{8}\delta, \\ \in (0, 1), & \text{otherwise,} \end{cases} \quad (1.14)$$

where $\delta > 0$ is a small constant and will be chosen later (see Theorem 9). The boundary layer corrections are given by

$$\bar{T}_0 = \chi(x_1)(\tilde{T}_0 - \tilde{T}_{0,\infty}), \quad \bar{\psi}_0 = \chi(x_1)(\tilde{\psi}_0 - \tilde{T}_{0,\infty}^4). \quad (1.15)$$

The composite approximate solution is obtained by adding the above terms to the solution of (1.5)-(1.6).

1.3. Main results

The main result of this paper is the following theorem.

Theorem 1. *Let $T_b \in C^2(\partial\Omega)$ and $\psi_b \in C^1(\Gamma_-)$ be non-negative functions. Let (T_0, ψ_0) be the smooth solution to system (1.5)-(1.6) with boundary condition (1.12) and $(\tilde{T}_0, \tilde{\psi}_0)$ be the smooth solution to the nonlinear Milne problem (1.8)-(1.11). Let the boundary layer correction $(\bar{T}_0, \bar{\psi}_0)$ be given by (1.15). Assume $\tilde{T}_0$ satisfies the spectral assumption (A) and has a lower bound $\tilde{T}_0 \geq a$ for some constant $a > 0$. Then for $\varepsilon > 0$ sufficiently small, there exists a unique solution $(T^\varepsilon, \psi^\varepsilon) \in L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2)$ to system (1.1)-(1.2) with boundary conditions (1.3)-(1.4), satisfying*

$$\|T^\varepsilon - T_0 - \bar{T}_0\|_{L^\infty(\Omega)} = O(\varepsilon), \quad \|\psi^\varepsilon - T_0^4 - \bar{\psi}_0\|_{L^\infty(\Omega \times \mathbb{S}^2)} = O(\varepsilon). \quad (1.16)$$

The above theorem shows the convergence of solutions of system (1.1)-(1.4) to the approximate solution $(T_0 + \bar{T}_0, T_0^4 + \bar{\psi}_0)$ as $\varepsilon \rightarrow 0$. Therefore, the diffusive limit of the radiative transfer system is rigorously justified.

In order to prove the above theorem, we need higher order approximate expansions beyond (1.5)-(1.6) and (1.8)-(1.9). Let $N \geq 1$ be an integer, we take the ansatz for the N -th order composite approximate solution (T^a, ψ^a) as

$$T^\varepsilon \sim T^a := \sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k), \quad \psi^\varepsilon \sim \psi^a := \sum_{k=0}^N \varepsilon^k (\psi_k + \bar{\psi}_k),$$

where (T_k, ψ_k) is the k -th order interior approximation (satisfying system (2.37)-(2.38)) and $(\bar{T}_k, \bar{\psi}_k)$ is the k -th order boundary layer correction (defined by (2.36) and system (2.32)-(2.33)), see Section 2 for details of the derivations. Note that due to the nonlinearity of $(T^\varepsilon)^4$, the interior expansions and boundary layer expansions are coupled. In particular, Taylor's expansions of the interior approximations are used for the construction of the boundary layer corrections. The composite approximate solution is shown to satisfy

$$\varepsilon^2 \Delta T^a + \langle \psi^a - (T^a)^4 \rangle = \mathcal{R}_1(T^a, \psi^a), \quad (1.17)$$

$$\varepsilon \beta \cdot \nabla \psi^a + \psi^a - (T^a)^4 = \mathcal{R}_2(T^a, \psi^a), \quad (1.18)$$

with boundary conditions

$$T^a(x) = T_b(x), \quad \text{for any } x \in \partial\Omega, \quad (1.19)$$

$$\psi^a(x, \beta) = \psi_b(x, \beta), \quad \text{for } (x, \beta) \in \Gamma_-, \quad (1.20)$$

where $\mathcal{R}_1(T^a, \psi^a)$ and $\mathcal{R}_2(T^a, \psi^a)$ are the approximation errors and are of order ε^{N+1} (see Theorem 9).

The main idea for the proof of Theorem 1 is to show the existence and uniqueness of system (1.1)-(1.2) in a small neighborhood of the approximate solution (T^a, ψ^a) . In order to achieve this, we construct a sequence of functions $\{T^k, \psi^k\}_{k=0}^\infty$ solving

$$\varepsilon^2 \Delta T^k + \langle \psi^k - 4(T^a)^3 T^k \rangle = \langle (T^{k-1})^4 - 4(T^a)^3 T^{k-1} \rangle, \quad (1.21)$$

$$\varepsilon \beta \cdot \nabla \psi^k + \psi^k - 4(T^a)^3 T^k = (T^{k-1})^4 - 4(T^a)^3 T^{k-1}, \quad (1.22)$$

with boundary conditions

$$\begin{aligned} T^k(x) &= T_b(x), \quad \text{for any } x \in \partial\Omega, \\ \psi^k(x, \beta) &= \psi_b(x, \beta), \quad \text{for } (x, \beta) \in \Gamma_-. \end{aligned}$$

This defines a mapping $\mathcal{T} : L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2) \rightarrow L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2)$ with $(T^k, \psi^k) = \mathcal{T}((T^{k-1}, \psi^{k-1}))$. For $\varepsilon > 0$ sufficiently small and $N \geq 5$, $s \geq 3$, the mapping $\mathcal{T}$ is shown in section 4 to be a contraction mapping in the set

$$\begin{aligned} \mathcal{O}_s &:= \{(T, \psi) \in L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2) : \\ &\|T - T^a\|_{L^\infty(\Omega)} \leq \varepsilon^s, \|\psi - \psi^a\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq \varepsilon^s\}, \end{aligned}$$

where $K > 0$ is a positive constant. Then by the Banach fixed point theorem, there exists a fixed point $(T^\varepsilon, \psi^\varepsilon)$ of $\mathcal{T}$ such that $\mathcal{T}((T^\varepsilon, \psi^\varepsilon)) = (T^\varepsilon, \psi^\varepsilon) \in \mathcal{O}_s$. By the definition of $\mathcal{T}$, $(T^\varepsilon, \psi^\varepsilon)$ solves system (1.1)-(1.2) with boundary conditions (1.3)-(1.4). For example, we can take $s = 3$ and $N = 5$ in the above arguments and conclude that there exists a unique solution $(T^\varepsilon, \psi^\varepsilon) \in \mathcal{O}_{s=3}$, i.e.

$$\left\| T^\varepsilon - \sum_{k=0}^5 \varepsilon^k T_k - \sum_{k=0}^5 \varepsilon^k \bar{T}_k \right\|_{L^\infty(\Omega)} \leq C\varepsilon^3, \quad (1.23)$$

$$\left\| \psi^\varepsilon - \sum_{k=0}^5 \varepsilon^k \psi_k - \sum_{k=0}^5 \varepsilon^k \bar{\psi}_k \right\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^3, \quad (1.24)$$

which leads to (1.16) and proves Theorem 1.

One of the most elusive and difficult issues to prove Theorem 1 is to show that $\mathcal{T}$ is a contraction mapping. This is solved by using an L^2 - L^∞ estimates on system (1.21)-(1.22). In order to get an L^2 estimate on this system, the following coercivity inequality

$$-\int_{\Omega} 4(T^a)^3 g \Delta g dx \geq -C\|g\|_{L^2(\Omega)}^2, \quad (1.25)$$

for any function $g \in H^1(\Omega)$, is used. This inequality can be shown to hold under the spectral assumption (A) (see Lemma 10). The L^∞ estimate is derived based on the elliptic regularity and the maximum principle for the radiative transport equation.

The spectral assumption (A) plays a key role in the proof of Theorem 1. First, it's required to show the well-posedness of the nonlinear Milne problem (1.8)-(1.11) and equations for higher order boundary corrections $(\bar{T}_k, \bar{\psi}_k)$, $k \geq 1$ (equations (2.32)-(2.33)). Second, under this assumption, the exponential decay of $\bar{T}_k, \bar{\psi}_k$, $k \geq 0$ can be shown, which is needed in order to find the boundary condition for equation (1.5). Third, the spectral assumption is used to show the inequality (1.25), which is crucial in order to get the suitable L^2 estimate on system (1.21)-(1.22) and prove $\mathcal{T}$ is a contraction mapping. Note since $\bar{T}_k = \bar{T}_k(\eta, x') = \bar{T}(x_1/\varepsilon, x')$, $\partial_\eta^2 \bar{T}_k$ may be of order $1/\varepsilon^2$ and thus the left term of (1.25) may be singular as $\varepsilon \rightarrow 0$. Thanks to the spectral assumption (A), inequality (1.25) holds and overcomes this singularity.

Our work also provides another approach to justify the diffusive limit in the well-prepared case, which was already done in [6] using different methods. Indeed, when the boundary data is well-prepared, $\psi_b = T_b^4$, no boundary layer exists and the boundary layer corrections $\bar{T}_k \equiv 0, \bar{\psi}_k \equiv 0$. We can thus take $T^a = \sum_{k=0}^N \varepsilon^k T_k$, $\psi^a = \sum_{k=0}^N \varepsilon^k \psi_k$. Since ∇T^a is bounded, inequality (1.25) holds obviously. Therefore, Theorem 1 holds and (1.16) implies the convergence of $(T^\varepsilon, \psi^\varepsilon)$ to (T_0, ψ_0) , which is the solution to (1.5)-(1.7).

1.4. Related work

When the diffusion operator is not considered, system (1.1)-(1.2) reduces to the linear transport equation $\varepsilon\beta \cdot \nabla U^\varepsilon + U^\varepsilon - \langle U^\varepsilon \rangle / (4\pi) = 0$. As $\varepsilon \rightarrow 0$, its solution converges to $U_0 + \bar{U}_0$, where $U_0 = U_0(x)$ is the solution of the Laplacian equation $\Delta U_0 = 0$ and $\bar{U}_0 = \bar{U}_0(x, \beta)$ is a boundary layer correction defined by $\bar{U}_0 = \chi(x_1)(\tilde{U}_0 - \tilde{U}_{0,\infty})$ where $\tilde{U}_0 = \tilde{U}(\eta, x', \beta)$ is the solution to $\mu\partial_\eta \tilde{U}_0 + \tilde{U}_0 - \langle \tilde{U}_0 \rangle / (4\pi) = 0$. This convergence was first rigorously proved in [4] whereas the study of this linear Milne problem was done in [3]. However, when the boundary is not flat, $\|U^\varepsilon - U_0 - \bar{U}_0\|_{L^\infty}$ does not converge to zero as $\varepsilon \rightarrow 0$. In [16], a geometric correction $U_0^\varepsilon, \bar{U}_0^\varepsilon$ is constructed with considerations of the curvature effects and $\|U^\varepsilon - U_0^\varepsilon - \bar{U}_0^\varepsilon\|_{L^\infty}$ is shown to converge to zero as $\varepsilon \rightarrow 0$ in the 2D unit disk [16], in the annulus [17], in the 2D convex domain [9,14] and in the 3D convex domain [15]. For more references on the diffusive limit of the linear transport equation, we refer the reader to [13,14] and references therein.

When the term $\varepsilon^2 \Delta T^\varepsilon$ is replaced by $\varepsilon^2 T^\varepsilon$, the diffusive limit for system (1.1)-(1.2) has been studied in many works [8,1,2,5,12]. In particular the problem has motivated the introduction of compactness techniques specific to kinetic equations, the average lemma being at the basis of the results in [2]. Moreover, the boundary layer problem for system (1.1)-(1.2) without the Laplacian term is constructed in [12] and the boundary condition for the limiting system (1.5) is shown to satisfy a mixed Robin boundary condition. Such a method is extended in [11] to construct the boundary layer approximations for system (1.1)-(1.2). However, the method only provides the zeroth and first order approximations near the boundary and could not extend to get higher order approximations. Since our estimates (1.23)-(1.24) need higher order expansions, here we provide a different way to obtain the approximation boundary layer solutions up to any order.

1.5. Plan of the paper

The paper is organized as follows. In the next section, we construct the composite approximate solution and show that it satisfies the radiative heat transfer system in the perturbative sense, in Lemma 2. In section 3, the properties of the approximate solutions are studied and the approximation errors are shown in Theorem 9, and inequality (1.25) is proved in Lemma 10. Finally, the proof of Theorem 1 is given, which consists of showing the linearized stability and nonlinear stability of (1.1)-(1.2) in the neighborhood of the approximate solution, in section 4.1 and 4.2, respectively.

Notations. Throughout the paper, some standard notations are used. The norm $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega \times \mathbb{S}^2)}$ are defined by $\|f\|_{L^2(\Omega)}^2 = \int_\Omega f^2 dx$, $\forall f \in L^2(\Omega)$ and $\|g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 = \int_{\Omega \times \mathbb{S}^2} g^2 d\beta dx$, $\forall g \in L^2(\Omega \times \mathbb{S}^2)$. The norm $\|\cdot\|_{L^2(\Gamma_+)}$ and $\|\cdot\|_{L^2(\Gamma_-)}$ are defined respectively by $\|g\|_{L^2(\Gamma_+)}^2 = \int_{\Gamma_+} \beta \cdot n(x) g^2 d\beta d\sigma_x$, and $\|g\|_{L^2(\Gamma_-)}^2 = \int_{\Gamma_-} |\beta \cdot n(x)| g^2 d\beta d\sigma_x$, where σ_x is the surface element.

2. Asymptotic analysis

In this section, an approximate solution to system (1.1)-(1.2) is constructed via asymptotic analysis. An interior expansion is first constructed which is valid in the interior of the domain and then a boundary layer correction is constructed accounting for the boundary layer effects. Finally, we combine the results to get a composite approximate solution to system (1.1)-(1.2). We here recall the simplifying assumption that the boundary layer only occurs near the bottom $\{x \in \partial\Omega, x_1 = 0\}$.

2.1. Interior expansion

We take the interior expansion to be

$$T^\varepsilon(x) \sim \sum_{k=0}^N \varepsilon^k T_k(x), \quad \psi^\varepsilon(x, \beta) \sim \sum_{k=0}^N \varepsilon^k \psi_k(x, \beta). \quad (2.1)$$

Define

$$\mathcal{R}_1(T, \psi) := \varepsilon^2 \Delta T + \langle \psi - T^4 \rangle, \quad \mathcal{R}_2(T, \psi) := \varepsilon \beta \cdot \nabla \psi + (\psi - T^4). \quad (2.2)$$

Plugging (2.1) into the above formulas gives

$$\begin{aligned} \mathcal{R}_1 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) \\ = \sum_{k=0}^N \varepsilon^k (\Delta T_{k-2} + \langle \psi_k - \mathcal{C}(T, k) \rangle) + \varepsilon^{N+1} \Delta T_{N-1} + \varepsilon^{N+2} \Delta T_N - \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T, k) \rangle, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \mathcal{R}_2 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) \\ = \sum_{k=0}^N \varepsilon^k (\beta \cdot \nabla \psi_{k-1} + \psi_k - \mathcal{C}(T, k)) + \varepsilon^{N+1} \beta \cdot \nabla \psi_N - \sum_{k=N+1}^{4N} \varepsilon^k \mathcal{C}(T, k), \end{aligned} \quad (2.4)$$

where

$$\mathcal{C}(T, k) := \sum_{\substack{i+j+l+m=k \\ i,j,l,m \geq 0}} T_i T_j T_l T_m, \quad (2.5)$$

and $(T_k, \psi_k), k < 0$ are taken to be zero.

Collecting terms with the same order, we take

$$\Delta T_{k-2} + \langle \psi_k - \mathcal{C}(T, k) \rangle = 0, \quad (2.6)$$

$$\beta \cdot \nabla \psi_{k-1} + \psi_k - \mathcal{C}(T, k) = 0, \quad (2.7)$$

for any $k = 0, \dots, N$. From the above two equations we obtain

$$\Delta T_{k-2} = \langle \beta \cdot \nabla \psi_{k-1} \rangle.$$

By (2.7), an iterative process on the above equation leads to

$$\begin{aligned} \Delta T_{k-2} &= \langle \beta \cdot \nabla (-\beta \cdot \nabla \psi_{k-2} + \mathcal{C}(T, k-1)) \rangle = -\langle \beta \cdot \nabla (\beta \cdot \nabla \psi_{k-2}) \rangle \\ &= -\langle (\beta \cdot \nabla)^2 (-\beta \cdot \nabla \psi_{k-3} + \mathcal{C}(T, k-2)) \rangle = -\frac{4\pi}{3} \Delta \mathcal{C}(T, k-2) + \langle (\beta \cdot \nabla)^3 \psi_{k-3} \rangle. \end{aligned}$$

Consequently, equations (2.6)-(2.7) can be rewritten as

$$\Delta T_k + \frac{4\pi}{3} \Delta \mathcal{C}(T, k) = \langle (\beta \cdot \nabla)^3 \psi_{k-1} \rangle, \quad (2.8)$$

$$\psi_k = -\beta \cdot \nabla \psi_{k-1} + \mathcal{C}(T, k), \quad (2.9)$$

for any $k = 0, \dots, N$.

Note that here T_k is solved by (2.8) and the solution is plugged into (2.9) to get ψ_k . In addition, for $k = 0$, equation (2.8) becomes

$$\Delta T_0 + \frac{4\pi}{3} \Delta T_0^4 = 0,$$

which gives the nonlinear elliptic equation (1.5). For $k \geq 1$, equation (2.8) can be rewritten as the following linear elliptic equation

$$\Delta T_k + \frac{4\pi}{3} \Delta(4T_0^3 T_k) = -\frac{4\pi}{3} \Delta \mathcal{E}(T, k-1) + \langle (\beta \cdot \nabla)^3 \psi_{k-1} \rangle,$$

where

$$\mathcal{E}(T, k-1) := \sum_{\substack{i+j+l+m=k \\ i,j,l,m \geq 1}} T_i T_j T_l T_m.$$

From the above equations, residuals (2.3) and (2.4) are given by

$$\begin{aligned} \mathcal{R}_1 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) &= \varepsilon^{N+1} \Delta T_{N-1} + \varepsilon^{N+2} \Delta T_N - \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T, k) \rangle, \\ \mathcal{R}_2 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) &= \varepsilon^{N+1} \beta \cdot \nabla \psi_N - \sum_{k=N+1}^{4N} \varepsilon^k \mathcal{C}(T, k), \end{aligned}$$

where the right hand side is both formally of order ε^{N+1} .

2.2. Boundary layer corrections

We next find the approximation of system (1.1)–(1.2) near the boundary. Let $\eta = x_1/\varepsilon$, we take the ansatz

$$T^\varepsilon(x) \sim \sum_{k=0}^N \varepsilon^k (\bar{T}_k(\eta, x') + T_k(x)), \quad \psi^\varepsilon(x, \beta) \sim \sum_{k=0}^N \varepsilon^k (\bar{\psi}_k(\eta, x', \beta) + \psi_k(x, \beta)),$$

where $\bar{T}_k, \bar{\psi}_k$ are the correction terms near the boundary and (T_k, ψ_k) are the interior expansions derived in the previous subsection. Taking the above ansatz into (2.2) gives

$$\begin{aligned} & \mathcal{R}_1 \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k), \sum_{k=0}^N \varepsilon^k (\psi_k + \bar{\psi}_k) \right) \\ &= \mathcal{R}_1 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) - \sum_{k=0}^N \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) \rangle \\ & \quad - \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) \rangle \\ & \quad + \varepsilon^{N+1} \Delta_{x'} \bar{T}_{N-1} + \varepsilon^{N+2} \Delta_{x'} \bar{T}_N + \sum_{k=0}^N \varepsilon^k \langle \mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k) \rangle \\ & \quad + \sum_{k=0}^N \varepsilon^k (\partial_\eta^2 \bar{T}_k + \Delta_{x'} \bar{T}_{k-2} + \langle \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
& \mathcal{R}_2 \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k), \sum_{k=0}^N \varepsilon^k (\psi_k + \bar{\psi}_k) \right) \\
&= \mathcal{R}_2 \left(\sum_{k=0}^N \varepsilon^k T_k, \sum_{k=0}^N \varepsilon^k \psi_k \right) - \sum_{k=0}^N \varepsilon^k (\mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k)) \\
&\quad - \sum_{k=N+1}^{4N} \varepsilon^k (\mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k)) \\
&\quad + \varepsilon^{N+1} \beta' \cdot \nabla_{x'} \bar{\psi}_N + \sum_{k=0}^N \varepsilon^k (\mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k)) \\
&\quad + \sum_{k=0}^N \varepsilon^k (\mu \partial_\eta \bar{\psi}_k + \beta' \cdot \nabla_{x'} \bar{\psi}_{k-1} + \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k)). \tag{2.11}
\end{aligned}$$

Here $\mu = \beta_1$, $x' = (x_2, x_3)$, $\beta' = (\beta_2, \beta_3)$, $T_k(0) = T_k(x_1 = 0)$ and $P_k = P_k(\eta, x')$, $k = 0, \dots, N$ are the Taylor's expansions of $T_k(\varepsilon\eta, x')$, $k = 0, \dots, N$ around $\eta = 0$, given by

$$P_k(\eta, x') = \sum_{l=0}^k \frac{\eta^l}{l!} \frac{\partial^l}{\partial x_1^l} T_{k-l}(0, x'). \tag{2.12}$$

Collecting terms of the same order in (2.10)-(2.11), we take

$$\partial_\eta^2 \bar{T}_k + \Delta_{x'} \bar{T}_{k-2} + \langle \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle = 0, \tag{2.13}$$

$$\mu \partial_\eta \bar{\psi}_k + \beta' \cdot \nabla_{x'} \bar{\psi}_{k-1} + \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) = 0, \tag{2.14}$$

for $k = 0, \dots, N$. Let $\tilde{T}_k = \bar{T}_k + P_k(0) = \bar{T}_k + T_k(0)$ for any $k = 0, \dots, N$, and $\tilde{\psi}_0 = \bar{\psi}_0 + T_0^4(0)$ and $\tilde{\psi}_k = \bar{\psi}_k + 4T_0^3(0)T_k(0)$, then

$$-(\bar{T}_0 + P_0)^4 + P_0^4 = -\tilde{T}_0^4 + T^4(0),$$

and for $k \geq 1$,

$$-4(\bar{T}_0 + P_0)^3(\bar{T}_k + P_k) + 4P_0^3 P_k = -4\tilde{T}_0^3 \tilde{T}_k + 4T_0^3(0)T_k(0) - 4(\tilde{T}_0^3 - T_0^3(0))(P_k - P_k(0)).$$

Therefore, $\tilde{T}_k, \tilde{\psi}_k$ satisfies the equations

$$\partial_\eta^2 \tilde{T}_0 + \langle \tilde{\psi}_0 - \tilde{T}_0^4 \rangle = 0, \tag{2.15}$$

$$\mu \partial_\eta \tilde{\psi}_0 + \tilde{\psi}_0 - \tilde{T}_0^4 = 0, \tag{2.16}$$

and for $k = 1, \dots, N$,

$$\begin{aligned}
& \partial_\eta^2 \tilde{T}_k + \Delta_{x'} \tilde{T}_{k-2} + \langle \tilde{\psi}_k - 4\tilde{T}_0^3 \tilde{T}_k \rangle \\
& + \langle -4(\tilde{T}_0^3 - T_0^3(0))(P_k - P_k(0)) - \mathcal{E}(\bar{T} + P(0), k-1) + \mathcal{E}(P(0), k-1) \rangle = 0, \tag{2.17}
\end{aligned}$$

$$\begin{aligned}
& \mu \partial_\eta \tilde{\psi}_k + \beta' \cdot \nabla_{x'} \tilde{\psi}_{k-1} + \tilde{\psi}_k \\
& - 4\tilde{T}_0^3 \tilde{T}_k - 4(\tilde{T}_0^3 - T_0^3(0))(P_k - P_k(0)) - \mathcal{E}(\bar{T} + P(0), k-1) + \mathcal{E}(P(0), k-1) = 0. \tag{2.18}
\end{aligned}$$

The boundary conditions for the above equations are

$$\tilde{T}_0(\eta = 0, x') = T_b(0, x'), \quad \text{for any } x' \in \mathbb{T}^2, \quad (2.19)$$

$$\tilde{\psi}_0(\eta = 0, x', \beta) = \psi_b(0, x', \beta), \quad \text{for any } (0, x', \beta) \in \Gamma_-, \quad (2.20)$$

and for $k = 1, \dots, N$,

$$\tilde{T}_k(\eta = 0, x') = 0, \quad \text{for any } x' \in \mathbb{T}^2, \quad (2.21)$$

$$\tilde{\psi}_k(\eta = 0, x', \beta) = \psi_k(0) - 4T_0^3(0)T_k(0), \quad \text{for any } (0, x', \beta) \in \Gamma_-. \quad (2.22)$$

The boundary conditions above are taken to be consistency with boundary conditions (1.3)-(1.4) such that

$$\begin{aligned} \sum_{k=0}^N (\bar{T}_k(0, x') + T_k(0, x')) &= T_b(0, x'), \quad \text{for } x' \in \mathbb{T}^2, \\ \sum_{k=0}^N (\bar{\psi}_k(0, x', \beta) + \psi_k(0, x', \beta)) &= \psi_b(0, x', \beta), \quad \text{for } (0, x', \beta) \in \Gamma_-. \end{aligned} \quad (2.23)$$

2.3. Construction of the composition approximate solution

In order to combine the interior expansions and boundary corrections, we use the cut-off function $\chi(x_1)$ defined in (1.14) and we also introduce another cut-off function $\chi_0(x_1)$ by

$$\chi_0(x_1) = \begin{cases} 1, & \text{for } 0 \leq x_1 \leq \frac{1}{2}\delta, \\ 0, & \text{for } x \geq \frac{3}{4}\delta, \\ \in (0, 1), & \text{otherwise.} \end{cases} \quad (2.24)$$

The construction of the composite approximate solution is done via the following procedure.

Step 1. Construction of $(\bar{T}_0, \bar{\psi}_0)$ and (T_0, ψ_0) . We first solve (2.17)-(2.18) when $k = 0$:

$$\partial_\eta^2 \tilde{T}_0 + \langle \tilde{\psi}_0 - \tilde{T}_0^4 \rangle = 0, \quad (2.25)$$

$$\mu \partial_\eta \tilde{\psi}_0 + \tilde{\psi}_0 - \tilde{T}_0^4 = 0, \quad (2.26)$$

with boundary conditions

$$\tilde{T}_0(\eta = 0, x') = T_b(0, x'), \quad \text{for any } x' \in \mathbb{T}^2, \quad (2.27)$$

$$\tilde{\psi}_0(\eta = 0, x', \beta) = \psi_b(0, x', \beta), \quad \text{for any } x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2 \text{ and } \mu = \beta_1 > 0. \quad (2.28)$$

This is the nonlinear Milne problem (1.8)-(1.11). It has been shown in [7] that the above problem has a global weak solution $(\tilde{T}_0, \tilde{\psi}_0) \in L_{\text{loc}}^2(\mathbb{R}_+ \times \mathbb{T}^2) \cap L_{\text{loc}}^2(\mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{S}^2)$ and as $\eta \rightarrow \infty$, the solution converges to some bounded functions $(\tilde{T}_{0,\infty}, \tilde{\psi}_{0,\infty})$ independent of η . We define the lowest order boundary layer correction $(\bar{T}_0, \bar{\psi}_0)$ by

$$\bar{T}_0(\eta, x') = \chi(\varepsilon\eta)(\tilde{T}_0(\eta, x') - \tilde{T}_{0,\infty}(x')), \quad \bar{\psi}_0(\eta, x', \beta) = \chi(\varepsilon\eta)(\tilde{\psi}_0(\eta, x', \beta) - \tilde{\psi}_{0,\infty}(x', \beta)). \quad (2.29)$$

Note due to the property of the nonlinear Milne problem, we have $\tilde{\psi}_{0,\infty} = \tilde{T}_{0,\infty}^4$. Here and below, $\bar{T}_k, \bar{\psi}_k$ are redefined by using a cut-off and is different from the same notation in section 2.2.

We next give the leading order of the interior expansion (T_0, ψ_0) , which is obtained by solving (2.8)-(2.9) for $k = 0$:

$$\Delta T_0 + \frac{4\pi}{3} \Delta T_0^4 = 0, \quad \psi_0 = T_0^4 \quad (2.30)$$

with boundary conditions

$$T_0(0, x') = \tilde{T}_{0,\infty}(x'), \quad T_0(1, x') = T_b(1, x'), \quad \text{for any } x' \in \mathbb{T}^2. \quad (2.31)$$

Step 2. Construction of $(\bar{T}_k, \bar{\psi}_k)$ and (T_k, ψ_k) for $k = 1, \dots, N$. We solve (2.17)-(2.18) for $1 \leq k \leq N$:

$$\begin{aligned} \partial_\eta^2 \tilde{T}_k + \langle \tilde{\psi}_k - 4\tilde{T}_0^3 \tilde{T}_k \rangle &= -\chi_0 \Delta_{x'} \tilde{T}_{k-2} + \langle 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \rangle \\ &+ \langle \mathcal{E}(\tilde{T} + P, k-1) - \mathcal{E}(P, k-1) \rangle, \end{aligned} \quad (2.32)$$

$$\begin{aligned} \mu \partial_\eta \tilde{\psi}_k + \tilde{\psi}_k - 4\tilde{T}_0^3 \tilde{T}_k &= -\chi_0 \beta' \cdot \nabla_{x'} \bar{\psi}_{k-1} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \\ &+ (\mathcal{E}(\tilde{T} + P, k-1) - \mathcal{E}(P, k-1)), \end{aligned} \quad (2.33)$$

with boundary conditions

$$\tilde{T}_k(\eta = 0, x') = 0, \quad \text{for any } x' \in \mathbb{T}^2, \quad (2.34)$$

$$\tilde{\psi}_k(\eta = 0, x', \beta) = \beta \cdot \nabla \psi_{k-1}(0) - \mathcal{E}(T(0), k-1), \quad \text{for any } x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2 \text{ and } \mu > 0. \quad (2.35)$$

The above problem was studied in [7] where existence and uniqueness of solutions are proved and the solutions are also shown to converge to some bounded functions as $\eta \rightarrow \infty$. We thus obtain $(\bar{T}_k, \bar{\psi}_k)$ by

$$\bar{T}_k(\eta, x') = \chi(\varepsilon\eta)(\tilde{T}_k(\eta, x') - \tilde{T}_{k,\infty}(x')), \quad \bar{\psi}_k(\eta, x', \beta) = \chi(\varepsilon\eta)(\tilde{\psi}_k(\eta, x', \beta) - \tilde{\psi}_{k,\infty}(x', \beta)) \quad (2.36)$$

where

$$\tilde{T}_{k,\infty}(x') = \lim_{\eta \rightarrow \infty} \tilde{T}_k(\eta, x'), \quad \tilde{\psi}_{k,\infty}(x', \beta) = \lim_{\eta \rightarrow \infty} \tilde{\psi}_k(\eta, x', \beta).$$

The interior expansions (T_k, ψ_k) are given, according to (2.8)-(2.9), by solving the system

$$\Delta T_k + \frac{4\pi}{3} \Delta(4T_0^3 T_k) = \langle (\beta \cdot \nabla)^3 \psi_{k-3} \rangle - \frac{4\pi}{3} \Delta \mathcal{E}(T, k-1), \quad (2.37)$$

$$\psi_k = -\beta \cdot \nabla \psi_{k-1} + 4T_0^3 T_k + \mathcal{E}(T, k-1) \quad (2.38)$$

with boundary conditions

$$T_k(0, x') = \tilde{T}_{k,\infty}(x'), \quad T_k(1, x') = T_b(1, x'), \quad \text{for any } x' \in \mathbb{T}^2. \quad (2.39)$$

Step 3: The composite approximate solution. With the above results, the composite approximate solution is given by

$$T^a = \sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k), \quad \psi^a = \sum_{k=0}^N \varepsilon^k (\psi_k + \bar{\psi}_k). \quad (2.40)$$

From (2.23), (T^a, ψ^a) also satisfies the boundary conditions (1.3)-(1.4).

2.4. Error of the composite expansion

In this section, we give the approximation errors. By the definition of $(\bar{T}_0, \bar{\psi}_0)$ of (2.29) and equations (2.25)-(2.26), as well as the relation (2.31) and $\tilde{\psi}_{0,\infty} = \tilde{T}_{0,\infty}^4$, a direct computation gives

$$\begin{aligned} E_0^0 &:= \partial_\eta^2 \bar{T}_0 + \langle \bar{\psi}_0 - (\bar{T}_0 + T_0(0))^4 + T_0^4(0) \rangle \\ &= (\bar{T}_0 - \tilde{T}_{0,\infty}) \partial_\eta^2 \chi + 2 \partial_\eta \chi \partial_\eta \tilde{T}_0 + \chi \langle \tilde{T}_0^4 - \tilde{T}_{0,\infty}^4 \rangle - \langle (\chi(\bar{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4 \rangle, \end{aligned} \quad (2.41)$$

and

$$\begin{aligned} E_0^1 &:= \mu \partial_\eta \bar{\psi}_0 + \bar{\psi}_0 - (\bar{T}_0 + T_0(0))^4 + T_0^4(0) \\ &= \mu(\tilde{\psi}_0 - \tilde{\psi}_{0,\infty}) \partial_\eta \chi + \chi(\tilde{T}_0^4 - \tilde{T}_{0,\infty}^4) - ((\chi(\bar{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4). \end{aligned} \quad (2.42)$$

Using the definition of $(\bar{T}_k, \bar{\psi}_k)$ in (2.36) and equations (2.32)-(2.33) as well as the relation (2.34)-(2.35) and $\tilde{\psi}_{k,\infty} = 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty}$, we get

$$\begin{aligned} E_k^0 &:= \partial_\eta^2 \bar{T}_k + \Delta_{x'} \bar{T}_{k-2} + \langle \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle \\ &= (\tilde{T}_k - \tilde{T}_{k,\infty}) \partial_\eta^2 \chi + 2 \partial_\eta \chi \partial_\eta \tilde{T}_k \\ &\quad + \chi \langle (4\tilde{T}_0^3 \tilde{T}_k - 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0))) \rangle \\ &\quad + \langle \mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1) \rangle - \langle \mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k) \rangle \\ &\quad + (1 - \chi \chi_0) \Delta_{x'} \bar{T}_{k-2}, \end{aligned} \quad (2.43)$$

and

$$\begin{aligned} E_k^1 &:= \mu \partial_\eta \bar{\psi}_k + \beta' \cdot \nabla_{x'} \bar{\psi}_{k-1} + \bar{\psi}_k - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \\ &= \mu(\tilde{\psi}_k - \tilde{\psi}_{k,\infty}) \partial_\eta \chi + \chi(4\tilde{T}_0^3 \tilde{T}_k - 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0))) \\ &\quad + \mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1) - \langle \mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k) \rangle \\ &\quad + (1 - \chi \chi_0) \beta' \cdot \nabla_{x'} \bar{\psi}_{k-1}. \end{aligned} \quad (2.44)$$

Combining formulas (2.10) and (2.11), we get from the above equations

$$\begin{aligned} \mathcal{R}_1(T^a, \psi^a) &= \varepsilon^{N+1} \Delta T_{N-1} + \varepsilon^{N+2} \Delta T_N - \sum_{k=0}^N \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) \\ &\quad - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle - \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) \rangle \\ &\quad + \varepsilon^{N+1} \Delta_{x'} \bar{T}_{N-1} + \varepsilon^{N+2} \Delta_{x'} \bar{T}_N + \sum_{k=0}^N \varepsilon^k E_k^0, \end{aligned} \quad (2.45)$$

and

$$\begin{aligned} \mathcal{R}_2(T^a, \psi^a) &= \varepsilon^{N+1} \beta \cdot \nabla \psi_N - \sum_{k=0}^N \varepsilon^k (\mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k)) \\ &\quad - \sum_{k=N+1}^{4N} \varepsilon^k (\mathcal{C}(T + \bar{T}, k)) + \varepsilon^{N+1} \beta' \cdot \nabla_{x'} \bar{\psi}_N + \sum_{k=0}^N \varepsilon^k E_k^1. \end{aligned} \quad (2.46)$$

Therefore, we summarize the above results in the following lemma.

Lemma 2. *Let (T^a, ψ^a) be given by (2.40). Then (T^a, ψ^a) satisfies system (1.17)-(1.18) with boundary conditions (1.19)-(1.20), where $\mathcal{R}_1(T^a, \psi^a)$, $\mathcal{R}_2(T^a, \psi^a)$ are given by (2.45) and (2.46), respectively.*

3. Properties of the approximate solution

In this section, we study the properties of the approximate solution (T^a, ψ^a) constructed in the previous section. The properties of the interior approximations are shown first, followed by the properties of the boundary layer approximations. Then, the approximate errors $\mathcal{R}_1(T^a, \psi^a)$ and $\mathcal{R}_2(T^a, \psi^a)$, obtained in the previous section, are estimated in Theorem 9. Finally, the coercivity inequality (1.25) is shown in Lemma 10.

3.1. Interior approximations

The interior approximation (T_0, ψ_0) is obtained by solving (2.30) and (T_k, ψ_k) , $k = 1, \dots, N$ are obtained by solving (2.37). For equation (2.30), the following lemma holds.

Lemma 3. *Assume $\tilde{T}_{0,\infty} \in L^2(\mathbb{T}^2)$ and $T_b(1, x') \in L^2(\mathbb{T}^2)$ satisfy $\tilde{T}_{0,\infty} \geq 0$, $T_b(1, x') \geq 0$ for any $x' \in \mathbb{T}^2$. Then, there exists a unique solution $T_0 \in C^\infty(\Omega)$ to equation (2.30) with boundary condition (2.31) and the solution satisfies $T_0(x) \geq 0$ for any $x \in \Omega$.*

Proof. The proof of the above lemma follows directly from elliptic theory. Actually, we may take $u = T_0 + 4\pi T_0^4/3$ and then $\Delta u = 0$ in Ω and $u(0, x') = \tilde{T}_{0,\infty}(x') + 4\pi \tilde{T}_{0,\infty}^4(x')/3$, $u(1, x') = T_b(1, x') + 4\pi T_b^4(1, x')/3$ on the boundary. We have $u \in C^\infty(\Omega)$ and so $T \in C^\infty(\Omega)$. $\square$

We now provide in the following lemma a well-posedness result for the elliptic equation (2.37).

Lemma 4. *Assume $\tilde{T}_{k,\infty} \in L^2(\mathbb{T}^2)$ satisfy $\tilde{T}_{k,\infty} \geq 0$. Given T_s , $1 \leq s \leq k-1$ and ψ_s , $1 \leq s \leq k-1$ satisfy $T_s \in L^2(\Omega)$, $\psi_s \in L^2(\Omega \times \mathbb{S}^2)$. Then there exists a unique solution $T_k \in C^\infty(\Omega)$ to equation (2.37) with boundary conditions (2.39).*

Proof. The proof of the above lemma also follows directly from elliptic theory by taking $u = (1 + 16\pi T_0^3/3)T_k$. $\square$

We finish this part by giving an L^∞ -estimate of the Taylor expansion P_k defined by (2.12).

Lemma 5. *Let T_s , $s = 0, \dots, N$ be the solution to equation (2.30) and (2.37), and P_s , $s = 0, \dots, N$ be given by (2.12). Then*

$$\|P_s(\eta)\|_{L^\infty(\mathbb{T}^2)} \leq C(1 + \eta^s), \quad \text{for any } s = 0, \dots, N,$$

for some constant $C > 0$.

Proof. Since $T_s \in C^\infty(\Omega)$, $\partial_{x_1}^l T_s(0)$ are bounded for any $l \geq 0$. Therefore,

$$|P_s| = \left| \sum_{l=0}^s \frac{\eta^l}{l!} \frac{\partial^l}{\partial x_1^l} T_{s-l}(0, x') \right| \leq C_1 \sum_{l=0}^s \frac{\eta^l}{l!} \leq C(1 + \eta^s),$$

and thus the lemma holds. $\square$

3.2. Boundary layer approximation

The analyses of the half-space nonlinear and linear Milne problems (2.25)-(2.26) and (2.32)-(2.33) have been carried out in [7]. Here we summarize the results in Lemma 6 and 7, for the proof we refer the reader to [7]. First, we have the following lemma for system (2.25)-(2.26).

Lemma 6 ([7]). *Given $(T_b(0, x'), \psi_b(0, x', \beta)) \in L^\infty(\mathbb{T}^2) \times L^\infty(\Gamma_-)$ satisfying $T_b(0, x') \geq 0$, $\psi_b(0, x', \beta) \geq 0$ for any $x' \in \mathbb{T}^2$ and $\beta \in \mathbb{S}^2$ with $\beta_1 \geq 0$. There exists a bounded solution $(\tilde{T}_0, \tilde{\psi}_0) \in L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2) \times L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{S}^2)$ to system (2.25)-(2.26) with boundary conditions (2.27)-(2.28). Moreover, there exists a constant function $\tilde{T}_{0,\infty} \in L^\infty(\mathbb{T}^2)$ such that*

$$|\tilde{T}_0(\eta, x') - \tilde{T}_{0,\infty}(x')| \leq Ce^{-\lambda_0 \eta}, \quad |\tilde{\psi}(\eta, x', \beta) - \tilde{T}_{0,\infty}^4(x')| \leq Ce^{-\lambda_0 \eta}, \quad (3.1)$$

for any $\eta \in [0, \infty)$ and $\beta \in \mathbb{S}^2$, $x' \in \mathbb{T}^2$. Here $C > 0$ are constants depending linearly on the constant $\left(\int_0^1 \mu(\psi_b - T_b^4)^2 d\mu\right)^{\frac{1}{2}}$ and λ_0 is any fixed constant in $[0, 1)$. Furthermore, for sufficiently small value of $\left(\int_0^1 \mu(\psi_b - T_b^4)^2 d\mu\right)^{\frac{1}{2}}$, the solution $(\tilde{T}_0, \tilde{\psi}_0)$ is unique.

Note that by the above lemma, $\tilde{\psi}_{0,\infty} := \lim_{\eta \rightarrow 0} \tilde{\psi}_0(\eta) = \tilde{T}_{0,\infty}^4$.

For system (2.32)-(2.33), the spectral assumption (A) on $\tilde{T}_0$ is needed in order to show the existence, uniqueness and exponential decay of solutions. The following lemma holds.

Lemma 7 ([7]). *Given $\bar{T}_s \in L^2(\mathbb{R}_+; C^2(\mathbb{T}^2))$, $\bar{\psi}_s \in L^2(\mathbb{R}_+; C^2(\mathbb{T}^2 \times \mathbb{S}^2))$, $0 \leq s \leq k-1$ such that*

$$|\bar{T}_s(\eta, x')| \leq Ce^{-\lambda_{k-1} \eta}, \quad |\bar{\psi}_s(\eta, x', \beta)| \leq Ce^{-\lambda_{k-1} \eta} \quad (3.2)$$

for some constants $C > 0$, $\lambda_{k-1} > 0$ and for any $x' \in \mathbb{T}^2$, $\beta \in \mathbb{S}^2$. Given $P_s \in C([0, \infty) \times \mathbb{T}^2)$ satisfying $|P_s| \leq C(1 + \eta^{k-1})$ for some constant $C > 0$ for any $0 \leq s \leq k-1$. Assume $\tilde{T}_0$ satisfies the spectral assumption (A). Then there exists a unique bounded solution $(\tilde{T}_k, \tilde{\psi}_k) \in L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2) \times L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{S}^2)$ to system (2.32)-(2.33) with boundary conditions (2.34)-(2.35). Moreover, there exist constants $(\tilde{T}_{k,\infty}, \tilde{\psi}_{k,\infty}) \in L^\infty(\mathbb{T}^2) \times L^\infty(\mathbb{T}^2 \times \mathbb{S}^2)$ such that

$$|\tilde{T}_k(\eta, x') - \tilde{T}_{k,\infty}(x')| \leq Ce^{-\lambda' \eta}, \quad |\tilde{\psi}(\eta, x', \beta) - \tilde{\psi}_{k,\infty}(x', \beta)| \leq Ce^{-\lambda' \eta}, \quad (3.3)$$

for any constant $0 < \lambda' < \lambda_{k-1}$, where $C > 0$ is a positive constant independent of k and the relation $\tilde{\psi}_{k,\infty} = 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty}$ holds.

Proof. It has been proved in [7] that given functions $S_1 = S_1(\eta, x')$, $S_2 = S_2(\eta, x', \beta)$ such that

$$\int_0^\infty e^{2\lambda' \eta} S_1^2 d\eta, \int_0^\infty \int_{\mathbb{S}^2} e^{2\lambda' \eta} S_2^2 d\eta d\beta < \infty \quad (3.4)$$

are bounded, the following equations

$$\partial_\eta^2 g + \langle \phi - 4\tilde{T}_0^3 g \rangle = S_1, \quad (3.5)$$

$$\mu \partial_\eta \phi + \phi - 4\tilde{T}_0^3 g = S_2, \quad (3.6)$$

with boundary conditions

$$\begin{aligned} g(\eta = 0, x') &= 0, \quad \text{for any } x' \in \mathbb{T}^2, \\ \phi(\eta = 0, x', \beta) &= \phi_b, \quad \text{for any } (x, \beta) \in \Gamma_-, \end{aligned}$$

have a unique bounded solution $(g, \phi) \in L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2) \times L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{S}^2)$. Moreover, there exists a constant g_∞ such that $|g - g_\infty| \leq Ce^{-\lambda'\eta}$, $|\phi - 4\tilde{T}_{0,\infty}^3 g_\infty| \leq Ce^{-\lambda'\eta}$, for the proof of this result we refer to the [7, Theorem 2]. Therefore, we deduce that $\tilde{\psi}_{k,\infty} = 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty}$.

Taking $g = \tilde{T}_k$, $\phi = \tilde{\psi}_k$ and $S_1 = -\chi_0 \Delta_{x'} \tilde{T}_{k-2} + \langle 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \rangle + \langle \mathcal{E}(\bar{T} + P, k-1) - \mathcal{E}(P, k-1) \rangle$, $S_2 = -\chi_0 \beta' \cdot \nabla_{x'} \tilde{\psi}_{k-1} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) + (\mathcal{E}(\bar{T} + P, k-1) - \mathcal{E}(P, k-1))$, $\phi_b = \beta \cdot \nabla \psi_{k-1}(0) - \mathcal{E}(T(0), k-1)$, system (3.5)-(3.6) becomes system (2.32)-(2.33) with boundary conditions (2.34)-(2.35). Therefore, Lemma 7 holds if S_1, S_2 satisfy (3.4).

To show (3.4), from the assumption (3.2),

$$\int_0^\infty e^{2\lambda'\eta} |\chi_0 \Delta_{x'} \tilde{T}_{k-2}|^2 d\eta \leq C \int_0^\infty e^{2\lambda'\eta} e^{-2\lambda_{k-1}\eta} d\eta = \frac{C}{2(\lambda_{k-1} - \lambda')}, \quad (3.7)$$

$$\int_0^\infty e^{2\lambda'\eta} |-\chi_0 \beta' \cdot \nabla_{x'} \tilde{\psi}_{k-1}|^2 d\eta \leq C \int_0^\infty e^{2\lambda'\eta} e^{-2\lambda_{k-1}\eta} d\eta = \frac{C}{2(\lambda_{k-1} - \lambda')}. \quad (3.8)$$

Moreover, due to the assumption (3.2) and $|P_s| \leq C(1 + \eta^k)$, we have

$$|4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0))| = |4(\tilde{T}_0 - \tilde{T}_{0,\infty})(P_k - P_k(0))| |\tilde{T}_0^2 + \tilde{T}_{0,\infty}^2| \leq Ce^{-\lambda_0\eta}(1 + \eta^k),$$

and

$$\begin{aligned} & |\mathcal{E}(\bar{T} + P, k-1) - \mathcal{E}(P, k-1)| \\ &= \left| \sum_{\substack{i+j+l+m=k \\ i,j,l,m \geq 1}} (\bar{T}_i + P_i)(\bar{T}_j + P_j)(\bar{T}_l + P_l)(\bar{T}_m + P_m) - \sum_{\substack{i+j+l+m=k \\ i,j,l,m \geq 1}} P_i P_j P_l P_m \right| \\ &= \left| \sum_{\substack{i+j+l+m=k \\ i,j,l,m \geq 1}} (\bar{T}_i \bar{T}_j \bar{T}_l \bar{T}_m + 3\bar{T}_i P_j P_l P_m + 6\bar{T}_i \bar{T}_j P_l P_m + 3\bar{T}_i \bar{T}_j \bar{T}_l P_m) \right| \\ &\leq C(e^{-4\lambda_{k-1}\eta} + e^{-\lambda_{k-1}\eta}(1 + \eta^{k-1})^3 + e^{-2\lambda_{k-1}\eta}(1 + \eta^{k-1})^2 + e^{-3\lambda_{k-1}\eta}(1 + \eta^{k-1})) \\ &\leq Ce^{-\lambda_{k-1}\eta}(1 + \eta^{3k-3}). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty e^{2\lambda'\eta} |4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0))|^2 d\eta &\leq C \int_0^\infty e^{2\lambda'\eta} e^{-2\lambda_0\eta} (1 + \eta^k) d\eta \\ &= \frac{C}{2(\lambda_0 - \lambda')} + \frac{C}{2^{k+1}(\lambda_0 - \lambda')^{k+1}} \Gamma(k+1), \end{aligned}$$

and

$$\int_0^\infty e^{2\lambda'\eta} |\mathcal{E}(\bar{T} + P, k-1) - \mathcal{E}(P, k-1)|^2 d\eta$$

$$\leq C \int_0^\infty e^{2\lambda'\eta} e^{-2\lambda_{k-1}\eta} (1 + \eta^{6k-6}) d\eta + \frac{C}{2(\lambda_{k-1} - \lambda')} + \frac{C}{2^{6k-5}(\lambda_{k-1} - \lambda')^{6k-5}} \Gamma(6k-5),$$

where $\Gamma(n) := (n-1)!$ is the Gamma function. Combining the above inequalities with (3.7)-(3.8) implies (3.4) and finishes the proof. $\square$

In the above lemma, the constant λ' may vary for different k . In order to get a uniform constant, we apply the above lemma iteratively. First, according to (3.1) from Lemma 6, $\bar{T}_0 = \chi(\tilde{T}_0 - \tilde{T}_{0,\infty})$ and $\bar{\psi}_0 = \chi(\tilde{\psi}_0 - \tilde{\psi}_{0,\infty})$ satisfy

$$|\bar{T}_0(\eta, x')| \leq C e^{-\lambda_0 \eta}, \quad |\bar{\psi}_0(\eta, x', \beta)| \leq C e^{-\lambda_0 \eta}, \quad \text{for any } x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2, \quad (3.9)$$

for some $C > 0$ and $\lambda > 0$. Therefore, with the definition of P_0 in (2.12), we have that $P_0 = T_0(0)$ is bounded, i.e. $|P_0| \leq C$ for some constant $C > 0$. Therefore, the assumptions of Lemma 7 with $k = 1$ hold. We take $\lambda' = \lambda_0 - \varepsilon_0$, with $\varepsilon_0 > 0$ being a sufficiently small number, in (3.3) and get that for any $x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2$,

$$|\tilde{T}_1(\eta, x') - \tilde{T}_{1,\infty}(x')| \leq C e^{-(\lambda_0 - \varepsilon_0)\eta}, \quad |\tilde{\psi}_1(\eta, x', \beta) - \tilde{\psi}_{1,\infty}(x', \beta)| \leq C e^{-(\lambda_0 - \varepsilon_0)\eta}. \quad (3.10)$$

With these, $\bar{T}_1 = \chi(\tilde{T}_1 - \tilde{T}_{1,\infty})$ and $\bar{\psi}_1 = \chi(\tilde{\psi}_1 - \tilde{\psi}_{1,\infty})$ satisfy

$$|\bar{T}_1(\eta, x')| \leq C e^{-(\lambda_0 - \varepsilon_0)\eta}, \quad |\bar{\psi}_k(\eta, x', \beta)| \leq C e^{-(\lambda_0 - \varepsilon_0)\eta}, \quad \forall (x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2). \quad (3.11)$$

Moreover, from Lemma 5, $|P_1| \leq C(1 + \eta)$. Therefore, the assumption of Lemma 7 with $k = 2$ hold with $\lambda_1 = \lambda_0 - \varepsilon_0$. We take $\lambda' = \lambda_1 - \varepsilon_0/4$ in (3.3) and get that for any $x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2$,

$$|\tilde{T}_2(\eta, x') - \tilde{T}_{2,\infty}(x')| \leq C e^{-(\lambda_1 - \varepsilon_0/4)\eta}, \quad |\tilde{\psi}_2(\eta, x', \beta) - \tilde{\psi}_{2,\infty}(x', \beta)| \leq C e^{-(\lambda_1 - \varepsilon_0/4)\eta}. \quad (3.12)$$

These again implies $|\bar{T}_2(\eta, x')|, |\bar{\psi}_2(\eta, x', \beta)| \leq C e^{-(\lambda_2 - \varepsilon_0/8)\eta}$ with $\lambda_2 = \lambda_1 - \varepsilon_0/4$ if we take $\lambda' = \lambda_2 - \varepsilon_0/9$. We can thus apply Lemma 7 iteratively with $\lambda' = \lambda_{s-1} - \varepsilon_0/s^2$ and $\lambda_s = \lambda_{s-1} - \varepsilon_0/(s-1)^2$ in the s -th step and get that

$$|\tilde{T}_s(\eta, x') - \tilde{T}_{s,\infty}(x')| \leq C e^{-(\lambda_{s-1} - \varepsilon_0/s^2)\eta}, \quad |\tilde{\psi}_k(\eta, x', \beta) - \tilde{\psi}_{2,\infty}(x', \beta)| \leq C e^{-(\lambda_s - \varepsilon_0/s^2)\eta}, \quad (3.13)$$

hold for any $0 \leq s \leq N$ for all $x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2$. By $\bar{T}_s = \chi(\tilde{T}_s - \tilde{T}_{s,\infty})$ and $\bar{\psi}_s = \chi(\tilde{\psi}_s - \tilde{\psi}_{s,\infty})$, the above inequalities imply for any $s = 0, \dots, N$,

$$|\bar{T}_s(\eta, x')| \leq C e^{-(\lambda_{s-1} - \varepsilon_0/s^2)\eta}, \quad |\bar{\psi}_s(\eta, x')| \leq C e^{-(\lambda_{s-1} - \varepsilon_0/s^2)\eta}, \quad \forall (x' \in \mathbb{T}^2, \beta \in \mathbb{S}^2). \quad (3.14)$$

Since $\lambda_s = \lambda_{s-1} - \varepsilon_0/(s-1)^2$,

$$\lambda_{N-1} - \varepsilon_0/N^2 = \lambda_{N-2} - \varepsilon_0/(N-1)^2 - \varepsilon_0/N^2 = \dots = \lambda_0 - \varepsilon_0 \left(1 + \frac{1}{2^2} + \dots + \frac{1}{N^2}\right). \quad (3.15)$$

From the formula $\sum_{n=1}^\infty \frac{1}{n^2} = \pi^2/6$, the above equation implies

$$\lambda_{N-1} - \varepsilon_0/N^2 = \lambda_0 - \varepsilon_0 \sum_{n=1}^N \frac{1}{N^2} \leq \lambda_0 - \varepsilon_0 \sum_{n=1}^\infty \frac{1}{n^2} \leq \lambda_0 - \frac{\pi^2}{6} \varepsilon_0. \quad (3.16)$$

Therefore, taking $\varepsilon_0 = 3\lambda_0/\pi^2$ and setting $\lambda = \lambda_0/2$, the following lemma holds.

Lemma 8. Let $\left\{(\tilde{T}_s, \tilde{\psi}_s)\right\}_{0 \leq s \leq N}$ be solutions to system (2.25)-(2.26) for $s = 0$ and (2.32)-(2.33) for $s \geq 1$. Then there exist constants $\bar{\lambda} > 0$ and $C > 0$ such that for any $s = 0, \dots, N$,

$$\begin{aligned} |\tilde{T}_s(\eta, x') - \tilde{T}_{s,\infty}(x')|, |\tilde{\psi}_s(\eta, x', \beta) - \tilde{\psi}_{s,\infty}(x', \beta)| &\leq Ce^{-\lambda\eta}, \\ |\bar{T}_s(\eta, x')|, |\bar{\psi}_s(\eta, x', \beta)| &\leq Ce^{-\lambda\eta}, \quad \text{for any } (x', \beta) \in \mathbb{T}^2 \times \mathbb{S}^2, \end{aligned} \quad (3.17)$$

where $C > 0$ is a positive constant independent of s .

3.3. Residual estimates

Next we estimate $\mathcal{R}_1(T^a, \psi^a)$ and $\mathcal{R}_2(T^a, \psi^a)$ (given by (2.45) and (2.46), respectively). We prove the following theorem.

Theorem 9. Assume $\tilde{T}_0$ satisfies the spectral assumption (A). Then the composite approximate solution (T^a, ψ^a) constructed in section 2, satisfies (1.17)-(1.18) with boundary conditions (1.19)-(1.20). Moreover, the error terms $\mathcal{R}_1 = \mathcal{R}_1(T^a, \psi^a)$, $\mathcal{R}_2 = \mathcal{R}_2(T^a, \psi^a)$ satisfy

$$\|\mathcal{R}_1(T^a, \psi^a)\|_{L^\infty(\Omega)}, \quad \|\mathcal{R}_2(T^a, \psi^a)\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\gamma_N \varepsilon^{N+1} + (N+2)Ce^{-\frac{\lambda\delta}{4\varepsilon}}, \quad (3.18)$$

for some constant $\gamma_N > 1$. Furthermore, for $\delta > -\frac{4}{\lambda}(N+1)\varepsilon \log \varepsilon$, the above estimate implies

$$\|\mathcal{R}_1(T^a, \psi^a)\|_{L^\infty(\Omega)}, \quad \|\mathcal{R}_2(T^a, \psi^a)\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^{N+1}, \quad (3.19)$$

where $C > 0$ is a positive constant independent of ε .

Proof. We first consider $\mathcal{R}_1(T^a, \psi^a)$. From (2.45), we have

$$\begin{aligned} \mathcal{R}_1(T^a, \psi^a) &= (\varepsilon^{N+1}\Delta T_{N-1} + \varepsilon^{N+2}\Delta T_N + \varepsilon^{N+1}\Delta_{x'}\bar{T}_{N-1} + \varepsilon^{N+2}\Delta_{x'}\bar{T}_N) \\ &\quad - \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) \rangle \\ &\quad - \sum_{k=0}^N \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle \\ &\quad + \sum_{k=0}^N \varepsilon^k E_k^0 \\ &=: R_{11} + R_{12} + R_{13} + R_{14}. \end{aligned} \quad (3.20)$$

Estimate of R_{11} and R_{12} . By Lemma 4, $\|\Delta T_{N-1}\|_{L^2(\Omega)}$, $\|\Delta T_N\|_{L^2(\Omega)}$, $\|\Delta T_{N-1}\|_{L^\infty(\Omega)}$ and $\|\Delta T_N\|_{L^\infty(\Omega)}$ are bounded. Moreover, from Lemma 8,

$$|\bar{T}_{N-1}| = |\chi(\varepsilon\eta)(\bar{T}_{N-1}(\eta) - \tilde{T}_{N-1,\infty})| \leq Ce^{-\lambda\eta} \leq C,$$

and thus is bounded. Similarly, $\Delta_{x'}\bar{T}_{N-1}$ and $\Delta_{x'}\bar{T}_N$ are bounded. Therefore,

$$\|R_{11}\|_{L^\infty(\Omega)} \leq C\varepsilon^{N+1}. \quad (3.21)$$

Due to (3.17), $\|\bar{T}_s\|_{L^\infty(\Omega)}$ are bounded for any $s = 0, \dots, N$. Therefore,

$$\|R_{12}\|_{L^\infty(\Omega)} \leq C\varepsilon^{N+1}. \quad (3.22)$$

Estimate of R_{13} . Since $\chi(x_1) = 0$ for $x_1 \geq \frac{3}{8}\delta$, $R_{13}(x_1, x') = 0$ for $x_1 \geq \frac{3}{8}\delta$. For the region $x_1 \leq \frac{3}{8}\delta$, we rewrite it as

$$\begin{aligned} R_{13}1_{\{x_1 \leq \frac{3}{8}\delta\}} &= - \sum_{k=0}^{4N} \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle 1_{\{x_1 \leq \frac{3}{8}\delta\}} \\ &\quad + \sum_{k=N+1}^{4N} \varepsilon^k \langle \mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k) \rangle 1_{\{x_1 \leq \frac{3}{8}\delta\}} \\ &=: R_{131} + R_{132}. \end{aligned} \quad (3.23)$$

Similarly as R_{12} , R_{132} satisfies

$$\|R_{132}\|_{L^\infty(\Omega)} \leq C\varepsilon^{N+1}. \quad (3.24)$$

For R_{131} , from the definition (2.5) of $\mathcal{C}$,

$$R_{131} = - \left\langle \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k) \right)^4 - \left(\sum_{k=0}^N \varepsilon^k T_k \right)^4 - \left(\sum_{k=0}^N \varepsilon^k (P_k + \bar{T}_k) \right)^4 + \left(\sum_{k=0}^N \varepsilon^k P_k \right)^4 \right\rangle. \quad (3.25)$$

Using the formula $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$ and for $a - b = c - d = f$

$$\begin{aligned} (a^4 - b^4) - (c^4 - d^4) &= (a - b)(a + b)(a^2 + b^2) - (c - d)(c + d)(c^2 + d^2) \\ &= f(a - c) (2a^2 + 2b^2 + (c + d)(a + b + c + d)), \end{aligned}$$

with $a = \sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k)$, $b = \sum_{k=0}^N \varepsilon^k T_k$, $c = \sum_{k=0}^N \varepsilon^k (P_k + \bar{T}_k)$, $d = \sum_{k=0}^N \varepsilon^k P_k$, $f = \sum_{k=0}^N \varepsilon^k \bar{T}_k$, we obtain

$$\begin{aligned} &\left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k) \right)^4 - \left(\sum_{k=0}^N \varepsilon^k T_k \right)^4 - \left(\sum_{k=0}^N \varepsilon^k (P_k + \bar{T}_k) \right)^4 + \left(\sum_{k=0}^N \varepsilon^k P_k \right)^4 \\ &= \left(\sum_{k=0}^N \varepsilon^k \bar{T}_k \right) \left(\sum_{k=0}^N \varepsilon^k (T_k - P_k) \right) (2a^2 + 2b^2 + (c + d)(a + b + c + d)). \end{aligned} \quad (3.26)$$

Due to (3.17),

$$\left| \sum_{k=0}^N \varepsilon^k \bar{T}_k \right| \leq C e^{-\lambda \eta} = C e^{-\frac{\lambda x_1}{\varepsilon}}. \quad (3.27)$$

Taylor's formula yields

$$T_k(x_1, x') = \sum_{l=0}^{N-k} \frac{x_1^l}{l!} \partial_{x_1}^l T_l(0, x') + \frac{\partial_{x_1}^{N-k+1} T_k(\xi_k, x')}{(N-k+1)!} x_1^{N-k+1}.$$

Using the above formula, we get

$$\begin{aligned}
\sum_{k=0}^N \varepsilon^k (T_k - P_k) &= \sum_{k=0}^N \varepsilon^k T_k - \sum_{k=0}^N \varepsilon^k P_k \\
&= \sum_{k=0}^N \varepsilon^k \left(\sum_{l=0}^{N-k} \frac{x_1^l}{l!} \partial_{x_1}^l T_k(0, x') + \frac{\partial_{x_1}^{N-k+1} T_k(\xi_k, x')}{(N-k+1)!} x_1^{N-k+1} \right) \\
&\quad - \sum_{k=0}^N \varepsilon^k \sum_{l=0}^k \frac{\eta^l}{l!} \frac{\partial^l}{\partial x_1^l} T_{k-l}(0, x'),
\end{aligned} \tag{3.28}$$

with $\xi_k \in [0, x_1]$. Using the formula

$$\sum_{k=0}^N \sum_{l=0}^k f(l, k) = \sum_{l=0}^N \sum_{k=l}^N f(l, k) \stackrel{s=k-l}{=} \sum_{l=0}^N \sum_{s=0}^{N-l} f(l, l+s) \stackrel{l \rightarrow k, s \rightarrow l}{=} \sum_{k=0}^N \sum_{l=0}^{N-k} f(k, k+l)$$

and taking $f(k, k+l) = \varepsilon^k \frac{x_1^l}{l!} \partial_{x_1}^l T_k(0, x')$, we get

$$f(l, k) = \varepsilon^l \frac{x_1^{k-l}}{(k-l)!} \partial_{x_1}^{k-l} T_l(0, x') = \varepsilon^k \frac{\eta^{k-l}}{(k-l)!} \partial_{x_1}^{k-l} T_l(0, x')$$

and so

$$\begin{aligned}
\sum_{k=0}^N \sum_{l=0}^{N-k} \varepsilon^k \frac{x_1^l}{l!} \partial_{x_1}^l T_k(0, x') &= \sum_{k=0}^N \sum_{l=0}^k \varepsilon^k \frac{\eta^{k-l}}{(k-l)!} \partial_{x_1}^{k-l} T_l(0, x') \\
&\stackrel{k \rightarrow k, k-l \rightarrow l}{=} \sum_{k=0}^N \sum_{l=0}^N \varepsilon^k \frac{\eta^l}{l!} \partial_{x_1}^l T_{k-l}(0, x').
\end{aligned}$$

Taking this relation into (3.28) leads to

$$\sum_{k=0}^N \varepsilon^k (T_k - P_k) = \sum_{k=0}^N \varepsilon^k \frac{\partial_{x_1}^{N-k+1} T_k(\xi_k, x')}{(N-k+1)!} x_1^{N-k+1}.$$

Combining this with (3.27), (3.26) satisfies

$$\begin{aligned}
&\left| \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k) \right)^4 - \left(\sum_{k=0}^N \varepsilon^k T_k \right)^4 - \left(\sum_{k=0}^N \varepsilon^k (P_k + \bar{T}_k) \right)^4 + \left(\sum_{k=0}^N \varepsilon^k P_k \right)^4 \right| \\
&\leq C e^{-\frac{\lambda x_1}{\varepsilon}} \sum_{k=0}^N \varepsilon^k \frac{|\partial_{x_1}^{N-k+1} T_k(\xi_k, x')|}{(N-k+1)!} x_1^{N-k+1} |2a^2 + 2b^2 + (c+d)(a+b+c+d)| \\
&\leq C \sum_{k=0}^N \varepsilon^k \frac{1}{(N-k+1)!} x_1^{N-k+1} e^{-\frac{\lambda x_1}{\varepsilon}},
\end{aligned}$$

where we use the fact that $\|a, b, c, d\|_{L^\infty(\Omega)} \leq C$ are bounded and $\|T_k\|_{C^s(\Omega)} \leq C$ is bounded for any $s \geq 0$. Note that the function $h(x_1) := x_1^{N-k+1} e^{-\lambda x_1/\varepsilon}$ attains its maximum at $x_1^* = (N-k+1)\varepsilon/\lambda$ with the maximum value $h(x^*) = (N-k+1)\varepsilon^{N-k+1}/\lambda^{N-k+1} \cdot e^{-(N-k+1)}$. Therefore,

$$\left| \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k) \right)^4 - \left(\sum_{k=0}^N \varepsilon^k T_k \right)^4 - \left(\sum_{k=0}^N \varepsilon^k (P_k + \bar{T}_k) \right)^4 + \left(\sum_{k=0}^N \varepsilon^k P_k \right)^4 \right|$$

$$\leq C\varepsilon^{N+1} \sum_{k=0}^N \frac{(N-k+1)^{N-k+1}}{\lambda^{N-k+1}(N-k+1)!} e^{-(N-k+1)} = C\varepsilon^{N+1} \sum_{n=1}^{N+1} \frac{n^n}{\lambda^n n!} e^{-n} \leq C(\gamma_N - 1)\varepsilon^{N+1},$$

where $\gamma_N := \sum_{n=0}^{N+1} n^n / (\lambda^n n!) > 1$ is a constant depending on N . Therefore,

$$\|R_{131}\|_{L^\infty(\Omega)} \leq C(\gamma_N - 1)\varepsilon^{N+1}.$$

Combining this with (3.24), we get

$$\|R_{13}\|_{L^\infty(\Omega)} \leq C\gamma_N \varepsilon^{N+1}. \quad (3.29)$$

Estimate of R_{14} . Recalling (2.43), we have

$$E_0^0 = (\tilde{T}_0 - \tilde{T}_{0,\infty})\partial_\eta^2 \chi + 2\partial_\eta \chi \partial_\eta \tilde{T}_0 + \chi \langle \tilde{T}_0^4 - \tilde{T}_{0,\infty}^4 \rangle - \langle (\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4 \rangle. \quad (3.30)$$

Since $\tilde{T}_0(\eta, x') - \tilde{T}_{0,\infty}(x')$ and $\partial_\eta \tilde{T}_0(\eta, x')$ exponentially decay to 0 as $\eta \rightarrow \infty$ and $\partial_\eta \chi(\varepsilon\eta) = \varepsilon \partial_{x_1} \chi(x_1)$, $\partial_\eta^2 \chi(\varepsilon\eta) = \varepsilon^2 \partial_{x_1}^2 \chi(x_1)$ are supported on the interval $\varepsilon\eta \in (\frac{3}{8}\delta, \infty)$, hence

$$|(\tilde{T}_0 - \tilde{T}_{0,\infty})\partial_\eta^2 \chi + 2\partial_\eta \chi \partial_\eta \tilde{T}_0| \leq C e^{-\lambda\eta} (\varepsilon |\partial_{x_1} \chi| + \varepsilon^2 |\partial_{x_1}^2 \chi|) 1_{\eta \geq \frac{3\delta}{8\varepsilon}} \leq C \varepsilon e^{-\frac{3\lambda\delta}{8\varepsilon}}. \quad (3.31)$$

Since $\tilde{T}_0(\eta, x') - \tilde{T}_{0,\infty}(x')$ decays exponentially,

$$|\tilde{T}_0^4 - \tilde{T}_{0,\infty}^4| = |(\tilde{T}_0 - \tilde{T}_{0,\infty}) \cdot (\tilde{T}_0 + \tilde{T}_{0,\infty})(\tilde{T}_0^2 + \tilde{T}_{0,\infty}^2)| \leq C e^{-\lambda\eta},$$

and

$$\begin{aligned} & |(\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4| \\ &= \chi |(\tilde{T}_0 - \tilde{T}_{0,\infty})| \cdot |\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty}| \cdot |(\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}))^2 + \tilde{T}_{0,\infty}^2| \\ &\leq C e^{-\lambda\eta}. \end{aligned}$$

Note that for $\varepsilon\eta < \frac{1}{4}$, $\chi(\varepsilon\eta) = 1$ and

$$\langle \chi(\tilde{T}_0^4 - \tilde{T}_{0,\infty}^4) \rangle - \langle (\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4 \rangle = \langle \tilde{T}_0^4 - \tilde{T}_{0,\infty}^4 \rangle - \langle \tilde{T}_0^4 - \tilde{T}_{0,\infty}^4 \rangle = 0.$$

Hence the left term of the above equation is supported on the interval $\varepsilon\eta \in [\frac{1}{4}\delta, \infty)$. Therefore,

$$|\langle \chi(\tilde{T}_0^4 - \tilde{T}_{0,\infty}^4) \rangle - \langle (\chi(\tilde{T}_0 - \tilde{T}_{0,\infty}) + \tilde{T}_{0,\infty})^4 - \tilde{T}_{0,\infty}^4 \rangle| \leq C e^{-\lambda\eta} 1_{\eta \geq \frac{\delta}{4\varepsilon}} \leq C e^{-\frac{\lambda\delta}{4\varepsilon}}. \quad (3.32)$$

Combining the above inequality with (3.31), (3.30) satisfies

$$|E_0^0| \leq C \varepsilon e^{-\frac{3\delta}{8\varepsilon}} + C e^{-\frac{\lambda\delta}{4\varepsilon}} \leq C e^{-\frac{\lambda\delta}{4\varepsilon}}. \quad (3.33)$$

For $k \geq 1$, we recall (2.43):

$$\begin{aligned} E_k^0 &= ((\tilde{T}_k - \tilde{T}_{k,\infty})\partial_\eta^2 \chi + 2\partial_\eta \chi \partial_\eta \tilde{T}_k) + \left(\chi \langle 4\tilde{T}_0^3 \tilde{T}_k - 4\tilde{T}_{0,\infty}^3 \tilde{T}_{k,\infty} \right. \\ &\quad \left. + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \rangle + \langle \mathcal{E}(\bar{T} + P, k-1) - \mathcal{E}(P, k-1) \rangle \right) \\ &\quad \left. - \langle \mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k) \rangle \right) + (1 - \chi\chi_0)\Delta_{x'} \bar{T}_{k-2} \\ &=: E_{k1}^0 + E_{k2}^0 + E_{k3}^0. \end{aligned}$$

The term $(\tilde{T}_k - \tilde{T}_{k,\infty})\partial_\eta^2\chi + 2\partial_\eta\chi\partial_\eta\tilde{T}_k$ can be estimated in the same way as (3.31) as

$$|E_{k1}^0| = |(\tilde{T}_k - \tilde{T}_{k,\infty})\partial_\eta^2\chi + 2\partial_\eta\chi\partial_\eta\tilde{T}_k| \leq C\varepsilon e^{-\frac{3\lambda\delta}{8\varepsilon}}. \quad (3.34)$$

For E_{k3}^0 , $(1 - \chi(\varepsilon\eta)\chi_0(\varepsilon\eta))$ is supported in $\varepsilon\eta \in [\frac{1}{4}\delta, \infty)$. By (3.17),

$$|E_{k3}^0| \leq C e^{-\lambda\eta} 1_{\eta > \frac{\delta}{4\varepsilon}} \leq C e^{-\frac{\lambda\delta}{4\varepsilon}}. \quad (3.35)$$

To estimate E_{k2}^2 , due to (3.17),

$$|4\tilde{T}_0^3\tilde{T}_k - 4\tilde{T}_{0,\infty}^3\tilde{T}_{k,\infty} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0))| \leq C e^{-\lambda\eta} + C e^{-\lambda\eta}(1 + \eta^k) \leq C(1 + \eta^k)e^{-\lambda\eta}.$$

Since $(\mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1))/\bar{T}$ is a polynomial of $\bar{T}$, P of order no bigger than η^{k-1} and thus is bounded, we have

$$\begin{aligned} |\mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1)| &= |\bar{T}| \left| \frac{\mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1)}{\bar{T}} \right| \\ &\leq C(1 + \eta^{k-1})e^{-\lambda\eta}. \end{aligned}$$

Similarly,

$$|\mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k)| = |\bar{T}| \left| \frac{\mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k)}{\bar{T}} \right| \leq C(1 + \eta^k)e^{-\lambda\eta}.$$

Therefore,

$$E_{k2}^0 \leq C(1 + \eta^k)e^{-\lambda\eta}.$$

Moreover, when $\varepsilon\eta < \frac{1}{4}$, $\chi(\varepsilon\eta) = 1$ and due to the formula $\mathcal{C}(\bar{T} + P, k) - \mathcal{C}(P, k) = 4(\bar{T}_0 + P_0)^3(\bar{T}_k + P_k) - 4P_0^3P_k + \mathcal{E}(\bar{T} + P, k - 1) - \mathcal{E}(P, k - 1)$ as well as

$$\begin{aligned} &4\tilde{T}_0^3\tilde{T}_k - 4\tilde{T}_{0,\infty}^3\tilde{T}_{k,\infty} + 4(\tilde{T}_0^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \\ &= 4(\bar{T}_0 + P_0)^3(\bar{T}_k + P_k(0)) - 4\tilde{T}_{0,\infty}^3P_k(0) + 4((\bar{T}_0 + P_0)^3 - \tilde{T}_{0,\infty}^3)(P_k - P_k(0)) \\ &= 4(\bar{T}_0 + P_0)^3(\bar{T}_k + P_k) - 4P_0^3P_k, \end{aligned}$$

where we use the fact that $P_0 = T_0(0) = \tilde{T}_{0,\infty}$, we have $E_{k2}^0 = 0$ on the interval $\varepsilon\eta \in [0, \frac{1}{4}\delta)$. Thus

$$E_{k2}^0 \leq C(1 + \eta^k)e^{-\lambda\eta} 1_{\eta > \frac{\delta}{4\varepsilon}} \leq C(1 + \frac{1}{\varepsilon^k})e^{-\frac{\lambda\delta}{4\varepsilon}}.$$

Combining this with (3.34) and (3.35), we obtain

$$|E_k^0| \leq C\varepsilon e^{-\frac{3\delta}{8\varepsilon}} + C e^{-\frac{\lambda\delta}{4\varepsilon}} + C(1 + \frac{1}{\varepsilon^k})e^{-\frac{\lambda\delta}{4\varepsilon}} \leq C e^{-\frac{\lambda\delta}{4\varepsilon}} + C \frac{1}{\varepsilon^k} e^{-\frac{\lambda\delta}{4\varepsilon}}.$$

Combining the above estimate with (3.33), we arrive at

$$\left| \sum_{k=0}^N E_k^0 \right| \leq \sum_{k=0}^N |E_k^0| \leq C e^{-\frac{\lambda\delta}{4\varepsilon}} + \sum_{k=1}^N \varepsilon^k \left(1 + \frac{1}{\varepsilon^k} \right) e^{-\frac{\lambda\delta}{4\varepsilon}} \leq (N + 1) C e^{-\frac{\lambda\delta}{4\varepsilon}} + C e^{-\frac{\lambda\delta}{4\varepsilon}} \sum_{k=1}^N \varepsilon^k.$$

We can take $\varepsilon < 1/2$ so that $\sum_{k=1}^N \varepsilon^k \leq \sum_{k=1}^N 1/2^k \leq 1$ and get

$$|R_{14}| = \left| \sum_{k=0}^N E_k^0 \right| \leq (N+2)Ce^{-\frac{\lambda\delta}{4\varepsilon}}.$$

By the regularity of solutions, we have

$$\|R_{14}\|_{L^\infty(\Omega)} \leq (N+2)Ce^{-\frac{\lambda\delta}{4\varepsilon}}.$$

Taking the above inequality and (3.21), (3.22) and (3.29), we obtain

$$\|\mathcal{R}_1(T^a, \psi^a)\|_{L^\infty(\Omega)} \leq C\gamma_N \varepsilon^{N+1} + (N+2)Ce^{-\frac{\lambda\delta}{4\varepsilon}}. \quad (3.36)$$

One can estimate $\mathcal{R}_2(T^a, \psi^a)$ given by (2.46) in the same way. Recalling (2.46),

$$\begin{aligned} \mathcal{R}_2(T^a, \psi^a) &= (\varepsilon^{N+1}\beta \cdot \nabla \psi_N + \varepsilon^{N+1}\beta' \cdot \nabla_{x'} \bar{\psi}_N) - \sum_{k=N+1}^{4N} \varepsilon^k (\mathcal{C}(T + \bar{T}, k)) \\ &\quad - \sum_{k=0}^N \varepsilon^k (\mathcal{C}(T + \bar{T}, k) - \mathcal{C}(T, k) - \mathcal{C}(\bar{T} + P, k) + \mathcal{C}(P, k)) + \sum_{k=0}^N \varepsilon^k E_k^1 \\ &=: R_{21} + R_{22} + R_{23} + R_{24}. \end{aligned}$$

By the boundness of solutions to the interior expansion and boundary layer corrections, $|R_{21}| = O(\varepsilon^{N+1})$, $|R_{22}| = O(\varepsilon^{N+1})$. The term R_{23} is the same as that of R_{13} without integration over $\beta \in \mathbb{S}^2$ and thus can be estimated in the same way, $|R_{13}| \leq C\gamma_N \varepsilon^{N+1}$. Finally, R_{24} can be estimated in the same way as R_{14} , whereas the only difference is the term $\mu(\tilde{\psi}_k - \tilde{\psi}_{k,\infty})\partial_\eta \chi$, which is supported on $\varepsilon\eta \in [\frac{1}{4}\delta, \infty)$ and thus can be estimate by

$$|\mu(\tilde{\psi}_k - \tilde{\psi}_{k,\infty})\partial_\eta \chi| \leq Ce^{-\lambda\eta} 1_{\eta \geq \frac{\delta}{4\varepsilon}} \leq Ce^{-\lambda\delta/4\varepsilon},$$

and $|R_{24}| \leq (N+2)Ce^{-\frac{\lambda}{4\varepsilon}}$. Thus (3.36) also holds for $\mathcal{R}_2(T^a, \psi^a)$, i.e.

$$\|\mathcal{R}_2(T^a, \psi^a)\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\gamma_N \varepsilon^{N+1} + (N+2)Ce^{-\frac{\lambda\delta}{4\varepsilon}}. \quad \square$$

3.4. Proof of inequality (1.25)

Next we prove inequality (1.25) under the spectral assumption (A).

Lemma 10. *Let (T^a, ψ^a) be the composite approximate solution constructed in the previous section. Assume the spectral assumption (A) holds for the solution $\tilde{T}_0$ of the nonlinear Milne problem (1.8)-(1.11) where $\tilde{T}_0 \geq a$ for some constant $a > 0$. Then, for $\varepsilon > 0$ sufficiently small, the following inequality holds*

$$\begin{aligned} - \int_{\Omega} 4(T^a)^3 g \Delta g dx &= \int_{\Omega} 4(T^a)^3 |\nabla g|^2 dx - \int_{\Omega} \nabla(4(T^a)^3) \cdot g \nabla g dx \\ &\geq \kappa \int_{\Omega} |\nabla g|^2 dx - C \|g\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.37)$$

for any function g satisfying $g(0) = 0$ and for some constants $\kappa > 0$, $C > 0$ depending on M , where $M < 1$ is the constant in (1.13) of the spectral assumption.

Proof. Note that $T^a = \sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k)$ where $\bar{T}_k = \chi(x_1)(\tilde{T}_k - \tilde{T}_{k,\infty})$. We split the integration over domain inside and outside the boundary layers.

Outside the boundary layer. In the domain $x_1 > \frac{3}{8}\delta$, $\chi(x_1) = 0$ and $T^a = \sum_{k=0}^N \varepsilon^k T_k$, which only contains the interior approximations. Since $\|T_k\|_{C^s(\Omega)}$ is bounded for any $s > 0$ and $k = 1, \dots, N$,

$$\begin{aligned}
 & \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla g|^2 dx - \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} \nabla(4(T^a)^3) \cdot g \nabla g dx \\
 &= \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla g|^2 dx - \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 2(T^a)^{3/2} \nabla g \cdot 6(T^a)^{1/2} \nabla T^a g dx \\
 &\geq \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla g|^2 dx - \frac{1}{2} \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla g|^2 dx \\
 &\quad - \frac{1}{2} \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 36 T^a |\nabla T^a|^2 g^2 dx \\
 &\geq \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 2(T^a)^3 |\nabla g|^2 dx - C \|g\|_{L^2(\Omega)}^2. \tag{3.38}
 \end{aligned}$$

Inside the boundary layer. In the domain $x_1 \leq \frac{3}{8}\delta$, boundary layer effects play a role. First we split the integral as

$$\begin{aligned}
 & \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla g|^2 dx - \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} \nabla(4(T^a)^3) \cdot g \nabla g dx \\
 &= \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} 4(T^a)^3 |\nabla_{x'} g|^2 dx - \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} \nabla_{x'}(4(T^a)^3) \cdot g \nabla_{x'} g dx \\
 &\quad + \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} 4(T^a)^3 |\partial_{x_1} g|^2 dx - \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} \partial_{x_1}(4(T^a)^3) \cdot g \partial_{x_1} g dx =: I_1 + I_2.
 \end{aligned}$$

Since $\|\nabla_{x'}(T^a)\|_{L^2(\Omega)}$ is bounded, we can estimate I_1 the same as (3.38):

$$I_1 \geq \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} 2(T^a)^3 |\nabla_{x'} g|^2 dx - C \|g\|_{L^2(\Omega)}^2. \tag{3.39}$$

To estimate I_2 , we use the spectral assumption **(A)**. Near the boundary, the composite approximate solution (T^a, ψ^a) is close to the solution $(\tilde{T}_0, \tilde{\psi}_0)$ of the nonlinear Milne problem (1.8)-(1.11). Using the equation

$$(T^a)^3 = \left(\sum_{k=0}^N \varepsilon^k (T_k + \bar{T}_k) \right)^3 = (T_0 + \bar{T}_0)^3 + \varepsilon G,$$

where $G = 3(T_0 + \bar{T}_0)^2 \sum_{k=1}^N \varepsilon^{k-1} (T_k + \bar{T}_k) + 6(T_0 + \bar{T}_0)^2 (\sum_{k=1}^N \varepsilon^{k-1} (T_k + \bar{T}_k))^2 + 3(T_0 + \bar{T}_0) (\sum_{k=1}^N \varepsilon^{k-1} (T_k + \bar{T}_k))^2$, we can rewrite I_2 as

$$\begin{aligned}
I_2 &= \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} \int_0^{\frac{3}{8\varepsilon}} 4(T_0 + \bar{T}_0)^3 + 4\varepsilon G |\partial_\eta g|^2 d\eta dx' \\
&\quad - \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} \int_0^{\frac{3}{8\varepsilon}} \partial_\eta (4(T_0 + \bar{T}_0)^3 + 4\varepsilon G) g \partial_\eta g d\eta dx' \\
&= \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4\tilde{T}_0^3 |\partial_\eta g|^2 - \partial_\eta (4\tilde{T}_0^3) g \partial_\eta g) d\eta \\
&\quad + \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4(T_0 + \bar{T}_0)^3 - 4\tilde{T}_0^3) |\partial_\eta g|^2 d\eta \\
&\quad - \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} \partial_\eta (4(T_0 + \bar{T}_0)^3 - 4\tilde{T}_0^3) g \partial_\eta g d\eta dx' \\
&\quad + \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} \varepsilon 4(G |\partial_\eta g|^2 - \partial_\eta G g \partial_\eta g) d\eta \\
&=: I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}$$

The spectral assumption **(A)** implies

$$\begin{aligned}
I_{21} &= \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4\tilde{T}_0^3 |\partial_\eta g|^2 - \partial_\eta (4\tilde{T}_0^3) g \partial_\eta g) d\eta \\
&\geq \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4\tilde{T}_0^3 |\partial_\eta g|^2 - \frac{1}{2} (4\tilde{T}_0^3 |\partial_\eta g|^2 + 36\tilde{T}_0 |\partial_\eta \tilde{T}_0|^2 g^2)) d\eta \\
&\geq \frac{1}{2\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4\tilde{T}_0^3 |\partial_\eta g|^2 - 36\tilde{T}_0 |\partial_\eta \tilde{T}_0|^2 g^2) d\eta \\
&\geq \frac{1-M}{2\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} 4\tilde{T}_0^3 |\partial_\eta g|^2 d\eta.
\end{aligned}$$

For I_{22} , since $\bar{T}_0 = T_0 + \chi(\varepsilon\eta)(\tilde{T}_0 - T_0(0))$, it holds that

$$\begin{aligned}
(T_0 + \bar{T}_0)^3 - \tilde{T}_0^3 &= (T_0 + \chi(\varepsilon\eta)(\tilde{T}_0 - T_0(0)) - \tilde{T}_0)(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\
&= ((T_0 - T_0(0)) - (1 - \chi(\varepsilon\eta))(\tilde{T}_0 - \bar{T}_0))(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\
&= (\partial_{x_1} T_0(\xi) \varepsilon\eta - (1 - \chi(\varepsilon\eta))(\tilde{T}_0 - \bar{T}_0))(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2).
\end{aligned}$$

Since we are considering the integration over $x_1 = \varepsilon\eta \in [0, \frac{3}{8}\delta]$ and $(1 - \chi(\varepsilon\eta))$ is supported on $[\frac{1}{4}\delta, \frac{3}{8}\delta]$,

$$I_{22} = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3}{8\varepsilon}} (4(T_0 + \bar{T}_0)^3 - 4\tilde{T}_0^3) |\partial_\eta g|^2 d\eta \leq \frac{3\delta C}{8\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{\delta}{2\varepsilon}} |\partial_\eta g|^2 d\eta.$$

For I_{23} , due to

$$\begin{aligned} & \partial_\eta ((T_0 + \bar{T}_0)^3 - \tilde{T}_0^3) \\ &= \partial_\eta (T_0 + \chi(\varepsilon\eta)(\tilde{T}_0 - T_0(0)) - \tilde{T}_0)(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\ & \quad + (T_0 + \chi(\varepsilon\eta)(\tilde{T}_0 - T_0(0)) - \tilde{T}_0) \partial_\eta (\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\ &= \varepsilon \chi'(\varepsilon\eta)(\tilde{T}_0 - T_0(0))(\tilde{T}_0 - T_0(0)) - \tilde{T}_0(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\ & \quad + (\chi(\varepsilon\eta) - 1) \partial_\eta \tilde{T}_0(\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2) \\ & \quad + (\partial_{x_1} T_0(\xi) \varepsilon\eta - (1 - \chi(\varepsilon\eta))(\tilde{T}_0 - \bar{T}_0)) \partial_\eta (\tilde{T}_0^2 + \tilde{T}_0(T_0 + \bar{T}_0) + (T_0 + \bar{T}_0)^2), \end{aligned}$$

with consideration of $\varepsilon\eta \in [0, \frac{3}{8}\delta]$ and $(1 - \chi(\varepsilon\eta))$ being supported on $[\frac{1}{4}\delta, \frac{3}{8}\delta]$, it holds that

$$\begin{aligned} I_{23} &= -\frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} \partial_\eta (4(T_0 + \bar{T}_0)^3 - 4\tilde{T}_0^3) g \partial_\eta g d\eta \\ &\leq \frac{C}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} (\varepsilon |g| |\partial_\eta g| + \frac{3\delta}{8} |g| |\partial_\eta g|) d\eta \\ &\leq \frac{3\delta C}{8\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} (g^2 + |\partial_\eta g|^2) d\eta. \end{aligned}$$

For I_{24} , we have

$$I_{24} = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} \varepsilon 4(G |\partial_\eta g|^2 - \partial_\eta G g \partial_\eta g) d\eta \leq \frac{C}{\varepsilon} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} (|g|^2 + |\partial_\eta g|^2) d\eta.$$

Combining the above estimates gives

$$\begin{aligned} I_2 &\geq \frac{1-M}{2\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} 4\tilde{T}_0^3 |\partial_\eta g|^2 d\eta - \frac{3\delta C}{8\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} (|g|^2 + |\partial_\eta g|^2) d\eta \\ &\quad - \frac{C}{\varepsilon} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} (|g|^2 + |\partial_\eta g|^2) d\eta. \end{aligned}$$

By the assumption of the lemma, $\tilde{T}_0 \geq a$, hence $4\tilde{T}_0^3 \geq 4a^3$ for some constant $a > 0$. We can take sufficiently small ε and δ such that $\varepsilon < (1-M)a^3/C$ and $3\delta C/8 \leq (1-M)/8$, and we get from the above inequality

$$I_2 \geq \frac{1-M}{4\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} 4\tilde{T}_0^3 |\partial_\eta g|^2 d\eta.$$

Combining this with (3.38) and (3.39) implies

$$\begin{aligned}
 - \int_{\Omega} 4(T^a)^3 g \Delta g dx &= \int_{\Omega} 4(T^a)^3 |\nabla g|^2 dx - \int_{\Omega} \nabla(4(T^a)^3) \cdot g \nabla g dx \\
 &\geq \int_{\Omega \cap \{x_1 > \frac{3}{8}\delta\}} 2(T^a)^3 |\nabla g|^2 dx + \int_{\Omega \cap \{x_1 \leq \frac{3}{8}\delta\}} 2(T^a)^3 |\nabla_{x'} g|^2 dx \\
 &\quad + \frac{1-M}{4\varepsilon^2} \int_{\mathbb{T}^2} dx' \int_0^{\frac{3\delta}{8\varepsilon}} 4\tilde{T}_0^3 |\partial_\eta g|^2 d\eta - C \|g\|_{L^2(\Omega)}^2 \\
 &\geq \kappa \|\nabla g\|_{L^2(\Omega)}^2 - C \|g\|_{L^2(\Omega)}^2,
 \end{aligned}$$

where $\kappa = \min\{2a^3, (1-M)a^3\}$, which finishes the proof of Lemma 10. $\square$

4. Diffusive limit

In this section, we prove Theorem 1 by estimating the difference between the solution $(T^\varepsilon, \psi^\varepsilon)$ to system (1.1)-(1.4) and the constructed approximate solution (T^a, ψ^a) , which satisfies (1.17)-(1.20). Setting $g := T^\varepsilon - T^a$, $\phi := \psi^\varepsilon - \psi^a$, functions (g, ϕ) then satisfy

$$\varepsilon^2 \Delta g + \langle \phi - (T^a + g)^4 + (T^a)^4 \rangle = -\mathcal{R}_1(T^a, \psi^a), \quad (4.1)$$

$$\varepsilon \beta \cdot \nabla \phi + \phi - (T^a + g)^4 + (T^a)^4 = -\mathcal{R}_2(T^a, \psi^a), \quad (4.2)$$

with boundary conditions

$$\begin{aligned}
 g(x) &= 0, \quad \text{for } x \in \partial\Omega, \\
 \phi(x, \beta) &= 0, \quad \text{for } (x, \beta) \in \Gamma_-.
 \end{aligned}$$

In order to prove Theorem 1, we first derive suitable estimates on a linearized system and then use the Banach fixed point theorem to show the existence of the above problem near zero solutions, leading to the convergence of $(T^\varepsilon, \psi^\varepsilon)$ to (T^a, ψ^a) as $\varepsilon \rightarrow 0$. Note that the right hand sides of equations (4.1) and (4.2) reduces to zero as $\varepsilon \rightarrow 0$.

4.1. Linearized system

We first linearize system (4.1)-(4.2) around zero and consider the following linear system:

$$\varepsilon^2 \Delta g + \langle \phi - 4(T^a)^3 g \rangle = r_1 + \langle r \rangle, \quad (4.3)$$

$$\varepsilon \beta \cdot \nabla \phi + \phi - 4(T^a)^3 g = r_2 + r, \quad (4.4)$$

where $r_1 = r_1(x)$, $r = r(x, \beta)$, and $r_2 = r_2(x, \beta)$ are given functions and the boundary conditions are taken to be

$$g(x) = 0, \quad \text{for any } x \in \partial\Omega, \quad (4.5)$$

$$\phi(x, \beta) = 0, \quad \text{for any } (x, \beta) \in \Gamma_-. \quad (4.6)$$

The existence of solutions to the above problem and suitable estimates on the solutions are stated in the following lemma.

Lemma 11. *Let $\varepsilon > 0$ and (T^a, ψ^a) be the composite approximate solution constructed in section 2. Assume $r_1 \in L^\infty(\Omega)$, and $r, r_2 \in L^\infty(\Omega \times \mathbb{S}^2)$. Then, there exists a unique solution $(g, \phi) \in L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2)$ to system (4.3)-(4.4) with boundary conditions (4.5)-(4.6). Moreover, the solution (g, ϕ) satisfies the following estimates*

$$\begin{aligned} & \varepsilon \|\phi\|_{L^2(\Omega \times \mathbb{S})} + \varepsilon \|g\|_{H^1(\Omega)} + \sqrt{\varepsilon} \|\phi\|_{L^2(\Gamma_+)} + \|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)} \\ & \leq C \|r\|_{L^2(\Omega \times \mathbb{S}^2)} + \frac{C}{\varepsilon} (\|r_1\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}), \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} & \|\phi\|_{L^\infty(\Omega \times \mathbb{S}^2)} + \|g\|_{L^\infty(\Omega)} \\ & \leq \frac{C}{\varepsilon^2} \|r\|_{L^2(\Omega \times \mathbb{S}^2)} + \frac{C}{\varepsilon^3} (\|r_1\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}) + C \|r_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} + C \|r\|_{L^\infty(\Omega)}, \end{aligned} \quad (4.8)$$

where $C > 0$ is a constant depending on $\kappa, |\Omega|$ not depending on ε . Here κ is the constant in inequality (3.37) of Lemma 10.

Proof. Existence of the linear system (4.3)-(4.4) with homogeneous boundary conditions (4.5)-(4.6) follows from standard theory of elliptic and transport equations. To derive estimates (4.7) and (4.8), we first derive the energy estimate. Then the L^2 type estimate is derived. Finally, the L^∞ type estimate is shown.

Step 1: The energy estimate. We multiply (4.3) by $4(T^a)^3 g$ and (4.4) by ϕ , and integrate over $x \in \Omega$ and $\beta \in \mathbb{S}^2$ to get

$$\begin{aligned} & - \int_{\Omega} \varepsilon^2 4(T^a)^3 g \Delta g dx + \iint_{\Omega \times \mathbb{S}^2} \varepsilon \beta \cdot \nabla \frac{\phi^2}{2} d\beta dx + \iint_{\Omega \times \mathbb{S}^2} (\phi - 4(T^a)^3 g)^2 d\beta dx \\ & = \iint_{\Omega \times \mathbb{S}^2} \phi r_2 d\beta dx - \int_{\Omega} 4(T^a)^3 g r_1 dx + \iint_{\Omega \times \mathbb{S}^2} (\phi - 4(T^a)^3 g) r d\beta dx. \end{aligned} \quad (4.9)$$

The boundary condition (4.6) implies

$$\iint_{\Omega \times \mathbb{S}^2} \varepsilon \beta \cdot \nabla \frac{\phi^2}{2} d\beta dx = \varepsilon \iint_{\Gamma} \beta \cdot n \phi^2 d\beta d\sigma_x = \varepsilon \iint_{\Gamma_+} \beta \cdot n \phi^2 d\beta d\sigma_x = \varepsilon \|\phi\|_{L^2(\Gamma_+)}^2. \quad (4.10)$$

By Lemma 10, inequality (3.37) implies

$$- \int_{\Omega} 4(T^a)^3 g \Delta g dx \geq \kappa \|\nabla g\|_{L^2(\Omega)}^2 - C \|g\|_{L^2(\Omega)}^2. \quad (4.11)$$

Applying Young's inequality on the last term of (4.9) gives

$$\iint_{\Omega \times \mathbb{S}^2} (\phi - 4(T^a)^3 g) r d\beta dx \leq \frac{1}{2} \iint_{\Omega \times \mathbb{S}^2} (\phi - 4(T^a)^3 g)^2 d\beta dx + \frac{1}{2} \iint_{\Omega \times \mathbb{S}^2} r^2 d\beta dx.$$

Taking the above inequality and (4.10), (4.11) into (4.9), we obtain the following energy estimate:

$$\begin{aligned}
& \varepsilon^2 \kappa \|\nabla g\|_{L^2(\Omega)}^2 + \varepsilon \|\phi\|_{L^2(\Gamma_+)}^2 + \frac{1}{2} \|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 \\
& \leq C \varepsilon^2 \|g\|_{L^2(\Omega)}^2 + \iint_{\Omega \times \mathbb{S}^2} \phi r_2 d\beta dx - \int_{\Omega} 4(T^a)^3 g r_1 dx + \frac{1}{2} \|r\|_{L^2(\Omega \times \mathbb{S}^2)}^2.
\end{aligned} \tag{4.12}$$

Step 2: The L^2 estimate. First we estimate the L^2 norm of g . Given $\rho = \rho(x, \beta) \in L^2(\Omega \times \mathbb{S}^2)$, define the operator $\mathcal{A} : L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{A}h = \langle \varphi - h \rangle, \quad \text{where } \varphi \text{ solves } \begin{cases} \varepsilon \beta \cdot \nabla \varphi + \varphi - h = \rho, & \text{in } \Omega \times \mathbb{S}^2, \\ \varphi(x, \beta) = 0, & \text{for } (x, \beta) \in \Gamma_-, \end{cases} \tag{4.13}$$

for $h = h(x) \in L^2(\Omega)$. Then for any function $\ell = \ell(x) \in C^2(\Omega)$ satisfying $\ell = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_{\Omega} \mathcal{A}h \cdot \ell dx &= \int_{\Omega} \int_{\mathbb{S}^2} (\varphi - h) \ell d\beta dx \\
&= - \int_{\Omega} \int_{\mathbb{S}^2} \varepsilon \beta \cdot \nabla \varphi \ell d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= \int_{\Omega} \int_{\mathbb{S}^2} \varepsilon \varphi \beta \cdot \nabla \ell d\beta dx - \varepsilon \iint_{\Gamma} \beta \cdot n \varphi \ell d\beta d\sigma_x + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= \varepsilon \int_{\Omega} \int_{\mathbb{S}^2} (\varphi - h) \beta \cdot \nabla \ell d\beta dx - 0 + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= -\varepsilon \int_{\Omega} \int_{\mathbb{S}^2} \varepsilon \beta \cdot \nabla \varphi \beta \cdot \nabla \ell d\beta dx + \varepsilon \int_{\Omega} \int_{\mathbb{S}^2} \rho \beta \cdot \nabla \ell d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= \varepsilon^2 \int_{\Omega} \int_{\mathbb{S}^2} (\varphi - h) (\beta \cdot \nabla)^2 \ell d\beta dx + \varepsilon^2 \int_{\Omega} \int_{\mathbb{S}^2} h (\beta \cdot \nabla)^2 \ell d\beta dx \\
&\quad - \varepsilon^2 \int_{\Gamma_+} \beta \cdot n \varphi \beta \cdot \nabla \ell d\beta d\sigma_x + \varepsilon \int_{\Omega} \int_{\mathbb{S}^2} \rho \beta \cdot \nabla \ell d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= -\varepsilon^3 \int_{\Omega} \int_{\mathbb{S}^2} (\beta \cdot \nabla)^2 \ell \beta \cdot \nabla \varphi d\beta dx + \varepsilon^2 \int_{\Omega} \int_{\mathbb{S}^2} \rho (\beta \cdot \nabla)^2 \ell d\beta dx + \varepsilon^2 \frac{4\pi}{3} \int_{\Omega} h \Delta \ell dx \\
&\quad - \varepsilon^2 \int_{\Gamma_+} \beta \cdot n \varphi \beta \cdot \nabla \ell d\beta d\sigma_x + \varepsilon \int_{\Omega} \int_{\mathbb{S}^2} \rho \beta \cdot \nabla \ell d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&= \varepsilon^3 \int_{\Omega} \int_{\mathbb{S}^2} (\varphi - \langle \varphi \rangle / 4\pi) (\beta \cdot \nabla)^3 \ell d\beta dx - \varepsilon^3 \int_{\Gamma_+} (\beta \cdot \nabla)^2 \ell \beta \cdot n \varphi d\beta d\sigma_x \\
&\quad + \varepsilon^2 \int_{\Omega} \int_{\mathbb{S}^2} \rho (\beta \cdot \nabla)^2 \ell d\beta dx + \varepsilon^2 \frac{4\pi}{3} \int_{\Omega} h \Delta \ell dx - \varepsilon^2 \int_{\Gamma_+} \beta \cdot n \varphi \beta \cdot \nabla \ell d\beta d\sigma_x \\
&\quad + \varepsilon \int_{\Omega} \int_{\mathbb{S}^2} \rho \beta \cdot \nabla \ell d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} \rho \ell d\beta dx \\
&\geq -C \varepsilon^3 \|\varphi - \langle \varphi \rangle / (4\pi)\|_{L^2(\Omega \times \mathbb{S}^2)} \|\ell\|_{H^3(\Omega)} - C \varepsilon^3 \|\nabla^2 \ell\|_{L^2(\partial\Omega)} \|\varphi\|_{L^2(\Gamma_+)}
\end{aligned}$$

$$\begin{aligned}
& -C\varepsilon^2\|\rho\|_{L^2(\Omega\times\mathbb{S}^2)}\|\ell\|_{H^2(\Omega)}+\varepsilon^2\frac{4\pi}{3}\int_{\Omega}h\Delta\ell dx-C\varepsilon^2\|\nabla\ell\|_{L^2(\partial\Omega)}\|\varphi\|_{L^2(\Gamma_+)} \\
& -C\varepsilon\|\rho\|_{L^2(\Omega\times\mathbb{S}^2)}\|\ell\|_{H^1(\Omega)}+\int_{\Omega}\int_{\mathbb{S}^2}\rho\ell d\beta dx,
\end{aligned}$$

where $\langle\beta\cdot\nabla\ell\rangle=0$ and $\langle(\beta\cdot\nabla)^3\ell\rangle=0$ are used. These two equalities are due to $\ell=\ell(x)$ not depending on β . By the trace theorem and Sobolev embeddings, the above inequality implies

$$\begin{aligned}
& \int_{\Omega}\mathcal{A}h\cdot\ell dx \\
& \geq\frac{4\pi}{3}\varepsilon^2\int_{\Omega}h\Delta\ell dx-C\varepsilon^3\|\varphi-\langle\varphi\rangle/(4\pi)\|_{L^2(\Omega\times\mathbb{S}^2)}\|\ell\|_{H^3(\Omega)}-C\varepsilon^3\|\ell\|_{H^3(\Omega)}\|\varphi\|_{L^2(\Gamma_+)} \\
& -C\varepsilon^2\|\ell\|_{H^2(\Omega)}\|\varphi\|_{L^2(\Gamma_+)}-C\varepsilon^2\|\rho\|_{L^2(\Omega\times\mathbb{S}^2)}\|\ell\|_{H^2(\Omega)}-C\varepsilon\|\rho\|_{L^2(\Omega\times\mathbb{S}^2)}\|\ell\|_{H^1(\Omega)} \\
& +\int_{\Omega}\int_{\mathbb{S}^2}\rho\ell d\beta dx.
\end{aligned} \tag{4.14}$$

Let $\rho=r_2+r$, $\varphi=\phi$, $h=4(T^a)^3g$ in (4.13), equation (4.3) can be written as

$$\varepsilon^2\Delta g+\mathcal{A}(4(T^a)^3g)=r_1+\langle r\rangle.$$

Let ℓ be the solution to

$$\begin{aligned}
\Delta\ell &= g, \text{ in } \Omega, \\
\ell &= 0, \text{ on } \partial\Omega.
\end{aligned}$$

We multiply the previous equation by ℓ and integration by parts to get

$$\varepsilon^2\int_{\Omega}g^2dx+\int_{\Omega}\mathcal{A}(4(T^a)^3g)\cdot\ell dx=\int_{\Omega}(r_1+\langle r\rangle)\ell dx.$$

By (4.14), and the properties of the elliptic equations $\|\ell\|_{H^2(\Omega)}\leq C\|g\|_{L^2(\Omega)}$, $\|\ell\|_{H^3(\Omega)}\leq C\|g\|_{H^1(\Omega)}$, we have

$$\begin{aligned}
& \int_{\Omega}\mathcal{A}(4(T^a)^3g)\ell dx \\
& \geq\varepsilon^2\frac{4\pi}{3}\int_{\Omega}4(T^a)^3g^2dx-C\varepsilon^3\|\phi-\langle\phi\rangle/(4\pi)\|_{L^2(\Omega\times\mathbb{S}^2)}\|g\|_{H^1(\Omega)}-C\varepsilon^3\|g\|_{H^1(\Omega)}\|\phi\|_{L^2(\Gamma_+)} \\
& -C\varepsilon^2\|g\|_{L^2(\Omega)}\|\phi\|_{L^2(\Gamma_+)}-C\varepsilon(\|r_2\|_{L^2(\Omega\times\mathbb{S}^2)}+\|r\|_{L^2(\Omega\times\mathbb{S}^2)})\|g\|_{L^2(\Omega)} \\
& +\int_{\Omega}\int_{\mathbb{S}^2}(r_2+r)\ell d\beta dx \\
& \geq\varepsilon^2\frac{4\pi}{3}\int_{\Omega}4(T^a)^3g^2dx-C\varepsilon^3\|\phi-4(T^a)^3g\|_{L^2(\Omega\times\mathbb{S}^2)}^2-C\varepsilon^3\|g\|_{H^1(\Omega)}^2-C\varepsilon^3\|g\|_{H^1(\Omega)}^2 \\
& -C\varepsilon^3\|\phi\|_{L^2(\Gamma_+)}^2-\frac{1}{4}\varepsilon^2\|g\|_{L^2(\Omega)}^2-C\varepsilon^2\|\phi\|_{L^2(\Gamma_+)}^2-C(\|r\|_{L^2(\Omega\times\mathbb{S}^2)}^2+\|r_2\|_{L^2(\Omega\times\mathbb{S}^2)}^2)
\end{aligned}$$

$$-\frac{1}{4}\varepsilon^2\|g\|_{L^2(\Omega)}^2 + \int_{\Omega} \int_{\mathbb{S}^2} (r_2 + r)\ell d\beta dx,$$

where

$$\begin{aligned}\|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)} &= \|\phi - \langle \phi \rangle / 4\pi + \langle \phi \rangle / 4\pi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)} \\ &= \|\phi - \langle \phi \rangle / 4\pi\|_{L^2(\Omega \times \mathbb{S}^2)} + \|\langle \phi \rangle / 4\pi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)} \\ &\geq \|\phi - \langle \phi \rangle / 4\pi\|_{L^2(\Omega \times \mathbb{S}^2)}\end{aligned}$$

is used in the last inequality. Taking it into the previous equation gives

$$\begin{aligned}&\varepsilon^2\|g\|_{L^2(\Omega)}^2 + \varepsilon^2\|2(T^a)^{3/2}g\|_{L^2(\Omega)}^2 \\ &\leq C\varepsilon^3\|\phi - (4T^a)^3g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + C\varepsilon^3\|g\|_{H^1(\Omega)}^2 + C\varepsilon^2\|\phi\|_{L^2(\Gamma_+)}^2 \\ &\quad + C(\|r\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2) + \int_{\Omega} r_1 \ell dx - \int_{\Omega} \int_{\mathbb{S}^2} r_2 \ell d\beta dx \\ &\leq C\varepsilon^3\|\phi - (4T^a)^3g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + C\varepsilon^3\|g\|_{H^1(\Omega)}^2 + C\varepsilon^2\|\phi\|_{L^2(\Gamma_+)}^2 \\ &\quad + C(\|r\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2) + \frac{C}{\varepsilon^2}(\|r_1\|_{L^2(\Omega)}^2 + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2) + \frac{1}{4}\varepsilon^2\|g\|_{L^2(\Omega)}^2.\end{aligned}$$

Combining this inequality with (4.12), we obtain for ε sufficiently small ($\varepsilon \leq \min\{1, \kappa\}/C$),

$$\begin{aligned}&\varepsilon^2\kappa\|\nabla g\|_{L^2(\Omega)}^2 + \varepsilon^2\|2(T^a)^{3/2}g\|_{L^2(\Omega)}^2 + \varepsilon\|\phi\|_{L^2(\Gamma_+)}^2 + \|\phi - 4(T^a)^3g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 \\ &\leq C\|r\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \frac{C}{\varepsilon^2}(\|r_1\|_{L^2(\Omega)}^2 + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2) + \iint_{\Omega \times \mathbb{S}^2} \phi r_2 d\beta dx \\ &\quad - \int_{\Omega} 4(T^a)^3 g r_1 dx.\end{aligned}$$

Using the inequalities

$$\begin{aligned}\int_{\Omega} \int_{\mathbb{S}^2} \phi r_2 d\beta dx &= \int_{\Omega} \int_{\mathbb{S}^2} (\phi - 4(T^a)^3 g) r_2 d\beta dx + \int_{\Omega} \int_{\mathbb{S}^2} 4(T^a)^3 g r_2 d\beta dx \\ &\leq \frac{1}{2}\|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \frac{1}{2}\|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2 \\ &\quad + \frac{1}{4}\varepsilon^2\|2(T^a)^{3/2}g\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon^2}\|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2\end{aligned}$$

and

$$-\int_{\Omega} 4(T^a)^3 g r_1 dx \leq \frac{1}{4}\varepsilon^2\|2(T^a)^{3/2}g\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon^2}\|r_1\|_{L^2(\Omega)}^2,$$

we get

$$\begin{aligned}&\varepsilon^2\kappa\|g\|_{H^1(\Omega)}^2 + \varepsilon^2\|2(T^a)^{3/2}g\|_{L^2(\Omega)}^2 + \varepsilon\|\phi\|_{L^2(\Gamma_+)}^2 + \|\phi - 4(T^a)^3g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 \\ &\leq C\|r\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \frac{C}{\varepsilon^2}(\|r_1\|_{L^2(\Omega)}^2 + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}^2).\end{aligned}$$

Note that

$$\begin{aligned}\varepsilon^2 \|\phi\|_{L^2(\Omega \times \mathbb{S}^2)}^2 &\leq \varepsilon^2 \|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + \varepsilon^2 \|4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 \\ &\leq \varepsilon^2 \|\phi - 4(T^a)^3 g\|_{L^2(\Omega \times \mathbb{S}^2)}^2 + C\varepsilon^2 \|g\|_{L^2(\Omega)}^2,\end{aligned}$$

due to T^a being bounded. Therefore, we combine the previous two inequalities and arrive at the estimate (4.7).

Step 3: L^∞ estimate. We now derive the L^∞ estimate of g, ϕ . First, by the maximum principle for linear transport equation (see for example Lemma 3.1 in [16]), the following estimate holds for (4.4):

$$\|\phi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq \|4(T^a)^3 g\|_{L^\infty(\Omega)} + \|r_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} + \|r\|_{L^\infty(\Omega \times \mathbb{S}^2)}.$$

Since $\|T^a\|_{L^\infty(\Omega)}$ is bounded,

$$\|\phi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\|g\|_{L^\infty(\Omega)} + \|r_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} + \|r\|_{L^\infty(\Omega \times \mathbb{S}^2)}. \quad (4.15)$$

We now give the L^∞ estimate of g . Equation (4.3) can be written as

$$\Delta g = f, \quad (4.16)$$

with $f = (-\langle \phi - 4(T^a)^3 g \rangle + r_1 + \langle r \rangle)/\varepsilon^2$. According to the elliptic regularity, we have

$$\|g\|_{L^\infty(\Omega)} \leq C(\|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Combining the above inequality with (4.7), we obtain

$$\begin{aligned}\|g\|_{L^\infty(\Omega)} &\leq C(\|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \\ &\leq \frac{C}{\varepsilon} \|r\|_{L^2(\Omega \times \mathbb{S}^2)} + \frac{C}{\varepsilon^2} (\|r_1\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega \times \mathbb{S}^2)}) \\ &\quad + \frac{C}{\varepsilon^2} (\|\phi - 4(T^a)^3 g\|_{L^2(\Omega)} + \|r_1\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega \times \mathbb{S}^2)}).\end{aligned}$$

Adding the above inequality with (4.15) and using (4.7) on the last term of the above inequality, we arrive at estimate (4.8), which finishes the proof. $\square$

4.2. Nonlinear system

We now show the existence and uniqueness of solutions to system (1.1)-(1.2) around the constructed composite approximate solution (T^a, ψ^a) and finish the proof of Theorem 1. Due to the equivalence between system (4.1)-(4.2) and (1.1)-(1.2), we only need to show the existence and uniqueness of solutions for (4.1)-(4.2) in the neighborhood of zero.

Proof of Theorem 1. The proof of existence and uniqueness is obtained using the Banach fixed point theorem. We first construct a sequence of functions and then show the sequence is a contraction sequence. Finally we show the convergence of (g, ϕ) to zero as $\varepsilon \rightarrow 0$.

Construction of the sequence of functions. Let $\{g^0, \phi^0\}$ be zero functions

$$g^0(x) = 0, \quad \phi^0(x, \beta) = 0,$$

and for $n \geq 1$, $\{g^n, \phi^n\}$ are defined recursively by

$$\varepsilon^2 \Delta g^n + \langle \phi^n - 4(T^a)^3 g^n \rangle = -\mathcal{R}_1 + 4\pi(6(T^a)^2(g^{n-1})^2 + 4(T^a)(g^{n-1})^3 + (g^{n-1})^4), \quad (4.17)$$

$$\varepsilon \beta \cdot \nabla \phi^n + (\phi^n - 4(T^a)^3 g^n) = -\mathcal{R}_2 + 6(T^a)^2(g^{n-1})^2 + 4(T^a)(g^{n-1})^3 + (g^{n-1})^4, \quad (4.18)$$

with boundary conditions

$$g^n(x) = 0, \text{ for } x \in \partial\Omega,$$

$$\phi^n(x, \beta) = \phi_b(x, \beta), \text{ for } (x, \beta) \in \Gamma_-.$$

The above system defines a mapping $\mathcal{T}$ with $(g^n, \phi^n) = \mathcal{T}((g^{n-1}, \phi^{n-1}))$.

Note that the above system is the same with (1.21)-(1.22) in the introduction. Let $T^n = g^n + T^a$, $\psi^n = \phi^n + \psi^a$, then a direct computation using (4.17)-(4.18) gives

$$\begin{aligned} \varepsilon^2 \Delta T^n + \langle \psi^n - 4(T^a)^3 T^n \rangle &= \langle (T^{n-1})^4 - 4(T^a)^3 T^{n-1} \rangle + \varepsilon^2 \Delta T^a + \langle \psi^a - (T^a)^4 \rangle - \mathcal{R}_1, \\ \varepsilon \beta \cdot \nabla \psi^n + \psi^n - 4(T^a)^3 T^n &= (T^{n-1})^4 - 4(T^a)^3 T^{n-1} + \varepsilon \beta \cdot \nabla \psi^a + \psi^a - (T^a)^4 - \mathcal{R}_2, \end{aligned}$$

which implies (1.21)-(1.22) after using (1.17)-(1.18).

The contraction mapping. Let $Y = L^\infty(\Omega)$ and $W = L^\infty(\Omega \times \mathbb{S}^2)$. We consider the solution in the function space

$$\mathcal{O}_s := \{(g, \phi) \in Y \times W : \|g\|_{L^\infty(\Omega)} \leq \varepsilon^s, \|\phi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq \varepsilon^s\},$$

with $s > 0$ is a constant to be chosen later.

First we show $\mathcal{T}$ maps the space $\mathcal{O}_s$ onto itself. Assume the residuals satisfy $\|\mathcal{R}_1\|_{L^\infty(\Omega)}, \|\mathcal{R}_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq \varepsilon^p$ for some constant $p > 0$. Next, we show if $(g^{n-1}, \psi^{n-1}) \in \mathcal{O}_s$, then $(g^n, \psi^n) \in \mathcal{O}_s$. By (4.7) with $r_1 = -\mathcal{R}_1, r_2 = -\mathcal{R}_2$ and $r = 6(T^a)^2(g^{n-1})^2 + 4T^a(g^{n-1})^3 + (g^{n-1})^4$, the following estimate holds (assuming $s \geq 1$):

$$\begin{aligned} &\varepsilon \|\phi^n\|_{L^2(\Omega \times \mathbb{S}^2)} + \varepsilon \|g^n\|_{H^1(\Omega)} + \sqrt{\varepsilon} \|\phi^n\|_{L^2(\Gamma_+)} + \|\phi^n - 4(T^a)^3 g^n\|_{L^2(\Omega \times \mathbb{S}^2)} \\ &\leq \frac{C}{\varepsilon} (\|\mathcal{R}_1\|_{L^2(\Omega)} + \|\mathcal{R}_2\|_{L^2(\Omega \times \mathbb{S}^2)}) + C (\|6(T^a)^2(g^{n-1})^2 + 4T^a(g^{n-1})^3 + (g^{n-1})^4\|_{L^2(\Omega)}) \\ &\leq C\varepsilon^{p-1} + C(\|g^{n-1}\|_{L^4}^2 + \|g^{n-1}\|_{L^s}^4) \\ &\leq C\varepsilon^{p-1} + C(\|g^{n-1}\|_{L^\infty(\Omega)} \|g^{n-1}\|_{L^2(\Omega)} + \|g^{n-1}\|_{L^\infty}^3 \|g^{n-1}\|_{L^2(\Omega)}) \\ &\leq C\varepsilon^{p-1} + C\|g^{n-1}\|_{L^\infty}^2 + C\|g^{n-1}\|_{L^\infty}^4 \\ &\leq C\varepsilon^{p-1} + C\varepsilon^{2s}. \end{aligned}$$

Assuming $p - 2 > s$ and $2s - 1 > s$, i.e. $p > s + 2$ and $s > 1$, the above inequality implies

$$\|g^n\|_{H^1(\Omega)}, \|\phi^n\|_{L^2(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^{p-2} + C\varepsilon^{2s-1} \leq \varepsilon^s,$$

for ε sufficiently small ($\varepsilon \leq (1/C)^{1/(s-1)}$). Moreover, by (4.8),

$$\begin{aligned} \|\phi\|_{L^\infty(\Omega \times \mathbb{S}^2)} + \|g\|_{L^\infty(\Omega)} &\leq \frac{C}{\varepsilon^2} (\|6(T^a)^2(g^{n-1})^2 + 4T^a(g^{n-1})^3 + (g^{n-1})^4\|_{L^2(\Omega)}) \\ &\quad + \frac{1}{\varepsilon^3} (\|\mathcal{R}_1\|_{L^2(\Omega)} + \|\mathcal{R}_2\|_{L^2(\Omega \times \mathbb{S}^2)}) + \|\mathcal{R}_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} \end{aligned}$$

$$\begin{aligned}
& + \|6(T^a)^2(g^{n-1})^2 + 4T^a(g^{n-1})^3 + (g^{n-1})^4\|_{L^\infty(\Omega)} \\
& \leq C\varepsilon^{2s-2} + \varepsilon^{p-3} + \varepsilon^p + C\varepsilon^{2s} \\
& \leq C\varepsilon^{2s-2} + \varepsilon^{p-3}.
\end{aligned}$$

Assuming $p - 3 > s$ and $2s - 2 > s$, i.e. $p > s + 3$ and $s > 2$, the above inequality implies

$$\|g^n\|_{L^\infty(\Omega)}, \|\phi^n\|_{L^\infty(\Omega)} \leq C\varepsilon^{2s-2} + \varepsilon^{p-3} \leq \varepsilon^s,$$

for ε sufficiently small ($\varepsilon \leq (1/C)^{1/(s-2)}$). Thus we obtain that $(g^n, \phi^n) \in \mathcal{O}_s$ and therefore $\mathcal{T}$ maps $\mathcal{O}_s$ onto itself.

Next we show the map $\mathcal{T}$ is a contraction mapping. Let $h^n = g^n - g^{n-1}$, $\varphi^n = \phi^n - \phi^{n-1}$, then they satisfy

$$\begin{aligned}
\varepsilon^2 \Delta h^n + \langle \varphi^n - 4(T^a)^3 h^n \rangle &= 4\pi f_n, \\
\varepsilon \beta \cdot \nabla \varphi^n + \varphi^n - 4(T^a)^3 h^n &= f_n,
\end{aligned}$$

where

$$\begin{aligned}
f_n &= 6(T^a)^2(g^{n-1})^2 + 4(T^a)(g^{n-1})^3 + (g^{n-1})^4 - (6(T^a)^2(g^{n-2})^2 + 4(T^a)(g^{n-2})^3 + (g^{n-2})^4) \\
&= 6(T^a)^2(g^{n-1} + g^{n-2})h^{n-1} + 4T^a h^{n-1}((g^{n-1})^2 - g^{n-1}g^{n-2} + (g^{n-2})^2) \\
&\quad + h^{n-1}(g^{n-1} + g^{n-2})((g^{n-1})^2 + (g^{n-2})^2).
\end{aligned}$$

Using (4.7) with $r_1 = r_2 = 0$, $r = f_n$, we obtain

$$\begin{aligned}
& \varepsilon \|\varphi^n\|_{L^2(\Omega \times \mathbb{S})} + \varepsilon \|h^n\|_{H^1(\Omega)} + \sqrt{\varepsilon} \|\varphi^n\|_{L^2(\Gamma_+)} + \|\varphi^n - 4(T^a)^3 h^n\|_{L^2(\Omega \times \mathbb{S}^2)} \\
& \leq \|f_n\|_{L^2(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^s \|h^{n-1}\|_{L^2(\Omega)},
\end{aligned}$$

hence

$$\|h^n\|_{H^1(\Omega)} + \|\varphi^n\|_{L^2(\Omega \times \mathbb{S})} \leq C\varepsilon^{s-1} \|h^{n-1}\|_{L^2(\Omega)}.$$

Using (4.8) with $r_1 = r_2 = 0$, $r = f_n$, we obtain

$$\begin{aligned}
& \|\varphi^n\|_{L^\infty(\Omega \times \mathbb{S}^2)} + \|h^n\|_{L^\infty(\Omega)} \\
& \leq \frac{C}{\varepsilon^2} \|f_n\|_{L^2(\Omega \times \mathbb{S}^2)} + C\|f_n\|_{L^\infty(\Omega)} \leq C\varepsilon^{s-2} \|h^{n-1}\|_{L^2(\Omega)} + C\varepsilon^s \|h^{n-1}\|_{L^\infty(\Omega)}.
\end{aligned}$$

Assume $s \geq 3$, then $C\varepsilon^{s-2}, C\varepsilon^s < 1$ for ε sufficiently small, the above inequality implies

$$\|h^n\|_{L^\infty(\Omega)} + \|\varphi^n\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C_1(\|h^{n-1}\|_{L^\infty(\Omega)} + \|\varphi^{n-1}\|_{L^\infty(\Omega \times \mathbb{S}^2)})$$

for some constant $0 < C_1 < 1$. Therefore, for $p \geq s + 3$ and $s \geq 3$, $\mathcal{T}$ is a contraction mapping. By the Banach fixed point theorem, there exists a unique fixed point (g, ϕ) such that $(g, \phi) = \mathcal{T}((g, \phi))$. Therefore, there exists a unique solution to (4.1)-(4.2) in $\mathcal{O}_s$.

Taking $s = 3$ and $p = 6$, we can conclude that

$$\|g\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^3.$$

Note that in order to obtain $\mathcal{R}_1, \mathcal{R}_2 = O(\varepsilon^6)$, we need to take $n = 5$ in the expansion and by Lemma 9, $\|\mathcal{R}_1\|_{L^\infty(\Omega)}, \|\mathcal{R}_2\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leq C\varepsilon^6$. We have

$$\begin{aligned} \left\| T^\varepsilon - \sum_{k=0}^5 \varepsilon^k T_k - \sum_{k=0}^5 \varepsilon^k \bar{T}_k \right\|_{L^\infty(\Omega)} &\leq C\varepsilon^3, \\ \left\| \psi^\varepsilon - \sum_{k=0}^5 \varepsilon^k \psi_k - \sum_{k=0}^5 \varepsilon^k \bar{\psi}_k \right\|_{L^\infty(\Omega \times \mathbb{S}^2)} &\leq C\varepsilon^3. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \|T^\varepsilon - T_0 - \bar{T}_0\|_{L^\infty(\Omega)} &\leq C\varepsilon, \\ \|\psi^\varepsilon - T_0^4 - \bar{\psi}_0\|_{L^\infty(\Omega \times \mathbb{S}^2)} &\leq C\varepsilon, \end{aligned}$$

which is (1.16) and finishes the proof. $\square$

Remark 12. The existence for system (1.1)-(1.4) can also be obtained using the maximum principle (see Appendix A for the proof).

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Appendix A. Existence of the steady state radiative transfer system

Next we prove the existence for the steady state radiative transfer system (1.1)-(1.2) with boundary conditions (1.3)-(1.4). The existence for the corresponding time dependent case is given in [10].

Theorem 13. Assume $\gamma_1 \leq T_b \leq \gamma_2$ and $\gamma_1^4 \leq \psi_b \leq \gamma_2^4$ for some constants $0 \leq \gamma_1 \leq \gamma_2$. There exists a weak solution $(T^\varepsilon, \psi^\varepsilon) \in L^\infty(\Omega) \times L^\infty(\Omega \times \mathbb{S}^2)$ to (1.1)-(1.2).

Proof. We show the existence by using the fixed-point theorem. Let $\mathcal{A} := \{T^\varepsilon \in L^\infty(\Omega) : \gamma_1 \leq T^\varepsilon \leq \gamma_2\}$. We define the operator $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ with $\theta = \mathcal{F}T$ by solving

$$\begin{aligned} \varepsilon \beta \cdot \nabla \psi + \psi &= T^4, \\ \psi^\varepsilon(x, \beta) &= \psi_b(x, \beta), \quad \text{for } (x, \beta) \in \Gamma_-, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \varepsilon^2 \Delta \theta - 4\pi^2 \theta^4 &= -\langle \psi \rangle, \\ \theta(x) &= T_b(x), \quad \text{for } x \in \partial\Omega. \end{aligned} \tag{A.2}$$

Next we show if $\gamma_1 \leq T \leq \gamma_2$, then $\gamma_1 \leq \theta \leq \gamma_2$. First the maximum principle for the transport equation implies

$$\gamma_1^4 \leq \psi \leq \gamma_2^4. \tag{A.3}$$

The maximum principle for equation (A.2) also implies $\gamma_1 \leq \theta \leq \gamma_2$. Suppose θ reaches its maximum at $x_M \in \Omega$, then if $x_M \in \partial\Omega$, $\theta(x_M) = T_b \leq \gamma_2$ and thus $\theta(x) \leq \theta(x_M) \leq \gamma_2$ for any $x \in \Omega$. Otherwise if x_M is an interior point, then $\Delta\theta(x_M) \leq 0$, and so

$$4\pi^2\theta^4(x_M) \leq \langle \psi(x_M, \cdot) \rangle \leq 4\pi^2\gamma_2^4, \quad (\text{A.4})$$

hence $\theta(x) \leq \theta(x_M) \leq \gamma_2$. Using a similar contradiction argument, $\theta(x) \geq \gamma_1$ can be shown.

Since $\mathcal{F}$ maps $\mathcal{A}$ to itself and $\mathcal{A}$ is a convex compact subset of the Banach space L^∞ . Hence by Schauder's fixed point theorem, there exists a fixed point $(T^\varepsilon, \psi^\varepsilon)$ of $\mathcal{F}$. Since this fixed point satisfies (A.1) and (A.2), hence $(T^\varepsilon, \psi^\varepsilon)$ is a solution to (1.1)-(1.2) with boundary conditions (1.3)-(1.4). $\square$

Remark 14. Unlike the time dependent case, where the uniqueness can be shown by showing $\mathcal{F}$ is a contraction mapping (with time step small), we are not able to show $\mathcal{F}$ in the above proof is a contraction mapping and thus uniqueness is not guaranteed by the above theorem. However, the uniqueness can be obtained from Theorem 1.

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