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Asymptotic stability for two-dimensional Boussinesq systems around the Couette flow in a finite channel

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ABSTRACT

In this paper, we study the asymptotic stability for the two-dimensional Navier-Stokes Boussinesq system around the Couette flow with small viscosity ν and small thermal diffusion μ in a finite channel. In particular, we prove that if the initial velocity and initial temperature (v_{in}, ρ_{in}) satisfies $\|v_{in} - (y, 0)\|_{H_{x,y}^2} \leq \varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}$ and $\|\rho_{in} - 1\|_{H_x^1 L_y^2} \leq \varepsilon_1 \min\{\nu, \mu\}^{\frac{1}{12}}$ for some small $\varepsilon_0, \varepsilon_1$ independent of ν, μ , then for the solution of the two-dimensional Navier-Stokes Boussinesq system, the velocity remains within $O(\min\{\nu, \mu\}^{\frac{1}{2}})$ of the Couette flow, and approaches to Couette flow as $t \rightarrow \infty$; the temperature remains within $O(\min\{\nu, \mu\}^{\frac{1}{12}})$ of the constant 1, and approaches to 1 as $t \rightarrow \infty$.

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1. Introduction

In this paper, we consider the two-dimensional Navier-Stokes Boussinesq system in a finite channel $\Omega = \{(x, y) : x \in \mathbb{T}, y \in (-1, 1)\}$:

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla P = -\rho g e_2 \\ \partial_t \rho + v \cdot \nabla \rho - \mu \Delta \rho = 0, \quad \nabla \cdot v = 0, \\ v(t, x, \pm 1) = (\pm 1, 0), \quad \rho(t, x, \pm 1) = c_0, \\ v(0, x, y) = v_{in}(x, y), \quad \rho(0, x, y) = \rho_{in}(x, y), \end{cases} \quad (1.1)$$

where ν is the viscosity coefficient and μ is the thermal diffusivity, $v(t, x, y) = (v^1, v^2)$ is the two-dimensional velocity field, $P(t, x, y)$ is the pressure, ρ is the temperature, $g = 1$ is the normalized gravitational constant and $e_2 = (0, 1)$ is the unit vector in the vertical direction. The boundary condition in (1.1) means that the fluid is moving together with the boundary and the temperature is fixed at the boundary. Let us also normalize $c_0 = 1$ for simplicity. The 2D Navier-Stokes Boussinesq system (1.1) is globally well-posed. One can refer to [1,2,15,16] and reference therein.

In this paper, we focus on the stability problem of the following steady-state

$$v_s = (y, 0), \quad \rho_s = 1, \quad p_s = y + c. \quad (1.2)$$

Now we introduce the perturbation: $v = u + (y, 0)$, $P = p + p_s$ and $\rho = \theta + \rho_s$, then (u, p, θ) satisfies

$$\begin{cases} \partial_t u + y \partial_x u + \begin{pmatrix} u^2 \\ 0 \end{pmatrix} + u \cdot \nabla u - \nu \Delta u + \nabla p = -\begin{pmatrix} 0 \\ \theta \end{pmatrix}, \\ \partial_t \theta + y \partial_x \theta + u \cdot \nabla \theta - \mu \Delta \theta = 0, \quad \nabla \cdot u = 0, \\ u(t, x, \pm 1) = 0, \quad \theta(t, x, \pm 1) = 0, \\ u(0, x, y) = u_{in}(x, y), \quad \theta(0, x, y) = \theta_{in}(x, y). \end{cases} \quad (1.3)$$

We also introduce the vorticity $\omega = \nabla \times u = \partial_y u^1 - \partial_x u^2$, which satisfies

$$\begin{cases} \partial_t \omega + y \partial_x \omega + u \cdot \nabla \omega - \nu \Delta \omega = -\partial_x \theta, \\ \partial_t \theta + y \partial_x \theta + u \cdot \nabla \theta - \mu \Delta \theta = 0, \\ u = \nabla^\perp \psi = (\partial_y \psi, -\partial_x \psi), \quad \Delta \psi = \omega. \end{cases} \quad (1.4)$$

Note that we can not impose the boundary condition on the vorticity, which is the main difficulty of this paper.

Before stating our main result, let us first recall previous works about the stability of flowing steady states. The linear inviscid two-dimensional Boussinesq system with shear flows has been extensively studied starting from the works of Taylor [24], Goldstein [14] and Synge [23]. We also refer to the book of Lin [19]. The system (1.3) is well studied in

the infinite channel case $\mathbb{T} \times \mathbb{R}$. We can refer to [7,11,28–30]. The best stability threshold results when $\nu = \mu$ is

$$\|\omega_{in}\|_{H^s} \leq \epsilon\nu^{\frac{1}{2}}, \quad \|\theta_{in}\|_{H^s} \leq \epsilon\nu, \quad \||D_x|^{\frac{1}{3}}\theta_{in}\|_{H^s} \leq \epsilon\nu^{\frac{5}{6}}, \tag{1.5}$$

with $s > 1$, which was proved by Deng, Wu and Zhang [11]. The mechanisms leading to stability are the so-called inviscid damping and enhanced dissipation, which are well studied for the Navier-Stokes system around Couette flow which we will introduce later. Without thermal diffusion, Masmoudi, Said-Houari and Zhao [22] considered the Navier-Stokes Boussinesq system with no heat diffusion in the thermal equation, and they studied the stability of Couette flow for the initial data perturbation in Gevrey- $\frac{1}{s}$ for $\frac{1}{3} < s \leq 1$ in the domain $\mathbb{T} \times \mathbb{R}$. For the Euler Boussinesq system $\nu = \mu = 0$, the global well-posedness for large data is an open problem. In [26], Yang and Lin proved the linear inviscid damping for the linearized two-dimensional Euler Boussinesq system which is generalized in [8]. The nonlinear inviscid damping for a large time is studied by Bedrossian, Bianchini, Coti Zelati and Dolce [6]. We also mention a very recent result [27], where the nonlinear asymptotic stability of Couette flow in a stratified fluid with the linear temperature was studied. We refer to [12,13] for some recent global stability results of other steady-states for the Boussinesq system, where the stabilization mechanism is different from that in this paper.

In this paper, we mainly study the boundary effect due to the non-slip boundary condition on the velocity. Our main result is stated as follows.

Theorem 1.1. *Suppose that (u, θ) solves the system (1.3) with the initial data (u_{in}, θ_{in}) . Then there exist constants ν_0 and $\epsilon_0, \epsilon_1, C > 0$ independent of ν, μ so that if*

$$\begin{aligned} \|u_{in}\|_{H^2} &\leq \epsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}, \\ \|\theta_{in}\|_{H^1} + \||D_x|^{\frac{1}{6}}\theta_{in}\|_{H^1} &\leq \epsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}}, \end{aligned}$$

for some sufficiently small ϵ_0, ϵ_1 , $0 < \min\{\nu, \mu\} \leq \nu_0$, then the solution (u, θ) is global in time and satisfies the following stability estimates:

$$\begin{aligned} &\|(1 - |y|)^{\frac{1}{2}}\omega\|_{\tilde{L}_t^\infty \mathcal{F}L^1L_y^2} + \|\partial_x u\|_{\tilde{L}_t^2 \mathcal{F}L^1L_y^2} + \||D_x|^{\frac{1}{2}}u\|_{\tilde{L}_t^\infty \mathcal{F}L^1L_y^\infty} + \nu^{\frac{1}{4}}\||D_x|^{\frac{1}{2}}\omega\|_{\tilde{L}_t^2 \mathcal{F}L^1L_y^2} \\ &\leq C\epsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}, \end{aligned}$$

and

$$\|\theta\|_{\tilde{L}_t^\infty \mathcal{F}L^1L_y^2} + \||D_x|^{\frac{1}{6}}\theta\|_{\tilde{L}_t^\infty \mathcal{F}L^1L_y^2} + \mu^{\frac{1}{6}}\||D_x|^{\frac{2}{3}}\theta\|_{\tilde{L}_t^2 \mathcal{F}L^1L_y^2} \leq C\epsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}},$$

where $\|f\|_{\tilde{L}_t^p \mathcal{F}L^1L_y^q} = \sum_{k \in \mathbb{Z}} \|\hat{f}_k\|_{L_t^p L_y^q}$ and $\hat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)e^{-ikx} dx$ is the Fourier transform of f in the x direction and k is the wave number.

Remark 1.2. The function space $\widetilde{L}_t^p \mathcal{F}L^1 L_y^q$ is of the same spirit as the Chemin-Lerner's Besov space [9].

Remark 1.3. The asymptotic stability holds for the initial perturbation satisfying

$$\sum_{k \in \mathbb{Z}} \|\widehat{w}_{in,k}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-1} \|\partial_y \widehat{w}_{in,k}\|_{L^2} \leq C\varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}},$$

and

$$\|\widehat{\theta}_{in,0}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \| |k|^{\frac{1}{6}} \widehat{\theta}_{in,k} \|_{L^2} \leq C\varepsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}}.$$

Remark 1.4. The estimate $\|\partial_x u\|_{\widetilde{L}_t^2 \mathcal{F}L^1 L_y^2}$ is due to the inviscid damping and the estimates $\nu^{\frac{1}{4}} \| |D_x|^{\frac{1}{2}} \omega \|_{\widetilde{L}_t^2 \mathcal{F}L^1 L_y^2}$ and $\mu^{\frac{1}{6}} \| |D_x|^{\frac{2}{3}} \theta \|_{\widetilde{L}_t^2 \mathcal{F}L^1 L_y^2}$ are due to the enhanced dissipation.

Remark 1.5. Compared to [11], when $\nu = \mu$, the interpolation of Sobolev spaces gives that the stability threshold is more restrictive than the one in our paper. In [11], an extra smallness on lower frequencies is required, namely $\|\theta_{in}\|_{H^s} \leq \epsilon\nu$. The key point of improvement is that we can control the buoyancy term and nonlocal terms in the temperature equation by avoiding discussing the different sizes of θ in different frequencies. More precisely, there is one derivative loss of the buoyancy term $\partial_x \theta$. A half derivative can be absorbed by using the enhanced dissipation of the vorticity, namely $\nu^{\frac{1}{4}} \| |D_x|^{\frac{1}{2}} \omega \|_{\widetilde{L}_t^2 \mathcal{F}L^1 L_y^2}$. A $1/3$ derivative can be absorbed by using the enhanced dissipation of the temperature. Thus, there is only $\frac{1}{6}$ derivative loss in the buoyancy term that needs to be controlled, which requires the estimate of the temperature with an additional $\frac{1}{6}$ derivative, namely the term $\| |D_x|^{\frac{1}{6}} \theta \|_{\widetilde{L}_t^\infty \mathcal{F}L^1 L_y^2}$ in the energy.

Remark 1.6. If $\theta_{in} = 0$, $\nu = \mu$, Theorem 1.1 is consistent with the Navier-Stokes result in [10]. We also remark that the stability problem of two-dimensional Couette flow has previously been investigated. One may refer to [4,5,17,18,20,21] for infinite channel case, and to [3,10] for finite channel case.

In (1.4), the buoyancy term is a linear term, which makes it different from the linearized operator from the Navier-Stokes equation. Luckily, the linearized equation of the temperature is decoupled from the whole system, we can still use the linearized operator from the Navier-Stokes equation. In this paper, the linear estimates of the velocity and the vorticity can be obtained by the same method as [10], and to shorten this paper, we will use some linear estimates from [10] as a black box.

Remark 1.7. For the Navier-Stokes result, the restriction on the size of perturbations for the asymptotic stability is $\nu^{\frac{1}{2}}$ which was obtained in [10] due to the boundary effect. Without boundary, it is expected that the stability threshold is $\nu^{\frac{1}{3}}$ for perturbations in some higher regularity Sobolev spaces [21]. In the infinite channel setting $\Omega = \mathbb{T} \times \mathbb{R}$,

by modifying the time-dependent multiplier of [21] and treating the buoyancy term as in this paper, one can prove the nonlinear asymptotic stability holds for larger initial perturbations, namely,

$$\|\omega_{in}\|_{H^s} \leq \varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{3}}, \quad \|\theta_{in}\|_{H^s} \leq \varepsilon_1 \min\{\nu, \mu\}^\beta,$$

with some s large and $\beta \leq \frac{5}{6}$.

To control the buoyancy term, in section 2, we obtain the precise estimates of θ by decomposing the system of θ into the inhomogeneous problem and homogeneous problem. For the homogeneous part, we can obtain the sharp bound by using the Gearhart-Prüss lemma in [25]. And for the inhomogeneous part, we obtain Proposition 2.5 by some resolvent estimates which were obtained in section 3 of [10] with the Navier-slip boundary condition. Finally, in section 3, we will mainly give the proof of the nonlinear stability.

2. Space-time estimates of the linearized Boussinesq equations

In this section, we establish the space-time estimates of the linearized two-dimensional Boussinesq equation. By taking the Fourier transform in $x \in \mathbb{T}$, we have

$$\theta(t, x, y) = \sum_{k \in \mathbb{Z}} \widehat{\theta}_k(t, y)e^{ikx}, \quad \omega(t, x, y) = \sum_{k \in \mathbb{Z}} \widehat{\omega}_k(t, y)e^{ikx}, \quad u(t, x, y) = \sum_{k \in \mathbb{Z}} \widehat{u}_k(t, y)e^{ikx}.$$

And for convenience, we suppress the index k in $\widehat{\theta}_k, \widehat{\omega}_k, \widehat{u}_k$.

2.1. Space-time estimates for the vorticity

Let us first study the following system for $k \neq 0$:

$$\begin{cases} \partial_t \widehat{w} + \nu(\partial_y^2 - k^2)\widehat{w} +iky\widehat{w} = -ikf^1 - \partial_y f^2, & w|_{t=0} = \widehat{w}_{in}(k, y), \\ \widehat{w} = \partial_y \widehat{u}^1 - ik\widehat{u}^2, & \widehat{u}(t, k, \pm 1) = 0. \end{cases} \tag{2.1}$$

We also introduce the space-time norm:

$$\|f\|_{L^p L^q} = \left\| \|f(t)\|_{L^q(-1,1)} \right\|_{L^p(\mathbb{R}^+)}.$$

Let us introduce the following estimate for (2.1).

Proposition 2.1. (Proposition 6.1 in [10].) *Let $0 < \nu \leq \nu_0$ and \widehat{w} be a solution of (2.1) with $\widehat{w}_{in} \in H^1(-1, 1)$ and $f^1, f^2 \in L^2 L^2$, where \widehat{w}_{in} satisfies $\langle \widehat{w}_{in}, e^{\pm ky} \rangle = 0$. Then there exists a constant $C > 0$ independent of ν, k so that*

$$\begin{aligned} & |k| \|\widehat{u}\|_{L^\infty L^\infty}^2 + k^2 \|\widehat{u}\|_{L^2 L^2}^2 + (\nu k^2)^{\frac{1}{2}} \|\widehat{w}\|_{L^2 L^2}^2 + \|(1 - |y|)^{\frac{1}{2}} \widehat{w}\|_{L^\infty L^2}^2 \\ & \leq C (\|\widehat{w}_{in}\|_{L^2}^2 + k^{-2} \|\partial_y \widehat{w}_{in}\|_{L^2}^2) + C (\nu^{-\frac{1}{2}} |k| \|f^1\|_{L^2 L^2}^2 + \nu^{-1} \|f^2\|_{L^2 L^2}^2). \end{aligned}$$

2.2. Space-time estimates for θ

First of all, we consider the linearized equation:

$$\partial_t \widehat{\theta} - \mu(\partial_y^2 - k^2)\widehat{\theta} +iky\widehat{\theta} = -ikg^1 - \partial_y g^2, \quad \widehat{\theta}|_{t=0} = \widehat{\theta}_{in}, \quad \widehat{\theta}|_{y=\pm 1} = 0. \quad (2.2)$$

By the standard energy estimates for $\widehat{\theta}$, we can easily get the following proposition, which is important for the estimates of the high frequency of $\widehat{\theta}$.

Proposition 2.2. *Let θ be a solution of (2.2) with $\widehat{\theta}_{in} \in L^2(-1, 1)$ and $g^1, g^2 \in L^2L^2$. Then there exists a constant $C > 0$ independent in μ, k so that*

$$\|\widehat{\theta}\|_{L^\infty L^2}^2 + \mu k^2 \|\widehat{\theta}\|_{L^2 L^2}^2 + \mu \|\partial_y \widehat{\theta}\|_{L^2 L^2}^2 \leq C\mu^{-1} (\|g^1\|_{L^2 L^2}^2 + \|g^2\|_{L^2 L^2}^2) + \|\widehat{\theta}_{in}\|_{L^2}^2.$$

Proof. Taking L^2 inner product between (2.2) and $\widehat{\theta}$, we get

$$\langle \partial_t \widehat{\theta}, \widehat{\theta} \rangle - \mu \langle (\partial_y^2 - k^2)\widehat{\theta}, \widehat{\theta} \rangle + \langle iky\widehat{\theta}, \widehat{\theta} \rangle = \langle -ikg^1 - \partial_y g^2, \widehat{\theta} \rangle.$$

By taking the real part and integration by parts in the above equality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{\theta}\|_{L^2}^2 + \mu \|\partial_y \widehat{\theta}\|_{L^2}^2 + \mu k^2 \|\widehat{\theta}\|_{L^2}^2 &\leq C \|g^1\|_{L^2} \|k\widehat{\theta}\|_{L^2} + C \|g^2\|_{L^2} \|\partial_y \widehat{\theta}\|_{L^2} \\ &\leq \frac{1}{4} \mu \|k\widehat{\theta}\|_{L^2}^2 + C\mu^{-1} \|g^1\|_{L^2}^2 + \frac{1}{4} \mu \|\partial_y \widehat{\theta}\|_{L^2}^2 \\ &\quad + C\mu^{-1} \|g^2\|_{L^2}^2. \end{aligned}$$

Thus by integrating in time, we have

$$\|\widehat{\theta}\|_{L^\infty L^2}^2 + \mu k^2 \|\widehat{\theta}\|_{L^2 L^2}^2 + \mu \|\partial_y \widehat{\theta}\|_{L^2 L^2}^2 \leq C\mu^{-1} (\|g^1\|_{L^2 L^2}^2 + \|g^2\|_{L^2 L^2}^2) + \|\widehat{\theta}_{in}\|_{L^2}^2. \quad \square$$

To deal with the buoyancy term $\partial_x \theta$ in the vorticity equation, we also need to give the following estimates about $\widehat{\theta}$.

First, we decompose $\widehat{\theta} = \widehat{\theta}_I + \widehat{\theta}_H$, where $\widehat{\theta}_I$ solves

$$\partial_t \widehat{\theta}_I - \mu(\partial_y^2 - k^2)\widehat{\theta}_I +iky\widehat{\theta}_I = -ikg^1 - \partial_y g^2, \quad \widehat{\theta}_I|_{t=0} = 0, \quad \widehat{\theta}_I|_{y=\pm 1} = 0, \quad (2.3)$$

and $\widehat{\theta}_H$ solves

$$\partial_t \widehat{\theta}_H - \mu(\partial_y^2 - k^2)\widehat{\theta}_H + ik\widehat{\theta}_H = 0, \quad \widehat{\theta}_H|_{t=0} = \widehat{\theta}_{in}, \quad \widehat{\theta}_H|_{y=\pm 1} = 0. \quad (2.4)$$

For the homogeneous part, $\widehat{\theta}_H$, by using transport diffusion structure and the Gearhart-Prüss type lemma with sharp bound [25], we use the following estimates.

Lemma 2.3. (Lemma 6.3 in [10].) Let $\widehat{\theta}_{in} \in L^2(-1, 1)$. Then for any $k \in \mathbb{Z}$, there exist constants $C, c > 0$ independent of μ, k such that

$$\|\widehat{\theta}_H\|_{L^2} \leq C e^{-c\mu^{\frac{1}{3}}|k|^{\frac{2}{3}}t - \mu t} \|\widehat{\theta}_{in}\|_{L^2}.$$

Moreover, for any $|k| \geq 1$,

$$(\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_H\|_{L^2 L^2}^2 \leq C \|\widehat{\theta}_{in}\|_{L^2}^2.$$

For the inhomogeneous part, considering the system

$$-\mu(\partial_y^2 - k^2)\widehat{\Theta} + ik(y - \lambda)\widehat{\Theta} = F, \quad \widehat{\Theta}(\pm 1) = 0, \tag{2.5}$$

we have the following sharp resolvent estimates for the linearized operator, which is very important for the space-time estimates of $\widehat{\theta}_I$.

Lemma 2.4. (Proposition 3.1 and Proposition 3.3 in [10].) Let $\widehat{\Theta} \in H^2(-1, 1)$ be a solution of (2.5) with $\lambda \in \mathbb{R}$. Then it holds for $F \in L^2(-1, 1)$,

$$\mu^{\frac{2}{3}}|k|^{\frac{1}{3}}\|\partial_y \widehat{\Theta}\|_{L^2} + (\mu k^2)^{\frac{1}{3}}\|\widehat{\Theta}\|_{L^2} + |k|\|(y - \lambda)\widehat{\Theta}\|_{L^2} \leq C\|F\|_{L^2},$$

and for $F \in H^{-1}(-1, 1)$,

$$\mu\|\partial_y \widehat{\Theta}\|_{L^2} + \mu^{\frac{2}{3}}|k|^{\frac{1}{3}}\|\widehat{\Theta}\|_{L^2} \leq C\|F\|_{H^{-1}}.$$

Proposition 2.5. Let $\widehat{\theta}_I$ be a solution of (2.3). Then there exists a constant $C > 0$ independent of μ, k such that

$$(\mu k^2)^{\frac{1}{3}}\|\widehat{\theta}_I\|_{L^2 L^2}^2 + \|\widehat{\theta}_I\|_{L^\infty L^2}^2 \leq C(\mu^{-\frac{1}{3}}|k|^{\frac{4}{3}}\|g^1\|_{L^2 L^2}^2 + \mu^{-1}\|g^2\|_{L^2 L^2}^2).$$

Proof. Now we use the resolvent estimates in Lemma 2.4 to obtain the semigroup estimates. We first decompose $\widehat{\theta}_I = \widehat{\theta}_I^{(1)} + \widehat{\theta}_I^{(2)}$, where $\widehat{\theta}_I^{(1)}$ and $\widehat{\theta}_I^{(2)}$ solve

$$\begin{aligned} \partial_t \widehat{\theta}_I^{(1)} - \mu(\partial_y^2 - k^2)\widehat{\theta}_I^{(1)} + ik y \widehat{\theta}_I^{(1)} &= -ik g^1, \quad \widehat{\theta}_I^{(1)}|_{t=0} = 0, \quad \widehat{\theta}_I^{(1)}|_{y=\pm 1} = 0, \\ \partial_t \widehat{\theta}_I^{(2)} - \mu(\partial_y^2 - k^2)\widehat{\theta}_I^{(2)} + ik y \widehat{\theta}_I^{(2)} &= -\partial_y g^2, \quad \widehat{\theta}_I^{(2)}|_{t=0} = 0, \quad \widehat{\theta}_I^{(2)}|_{y=\pm 1} = 0 \end{aligned}$$

By taking the Fourier transform in t :

$$\begin{aligned} \widehat{\Theta}_I(\lambda, k, y) &= \int_0^{+\infty} \widehat{\theta}_I(t, k, y) e^{-it\lambda} dt, \\ \widehat{\Theta}_I^{(j)}(\lambda, k, y) &= \int_0^{+\infty} \widehat{\theta}_I^{(j)}(t, k, y) e^{-it\lambda} dt, \quad j = 1, 2, \end{aligned}$$

$$G^j(\lambda, k, y) = \int_0^{+\infty} g^j(t, k, y) e^{-it\lambda} dt, \quad j = 1, 2.$$

Then $\widehat{\theta}_I = \widehat{\theta}_I^{(1)} + \widehat{\theta}_I^{(2)}$, and $\widehat{\theta}_I^{(1)}$ and $\widehat{\theta}_I^{(2)}$ satisfy

$$\begin{aligned} (i\lambda - \mu(\partial_y^2 - k^2) +iky)\widehat{\theta}_I^{(1)}(\lambda, k, y) &= -ikG^1(\lambda, k, y), \quad \widehat{\theta}_I^{(1)}|_{y=\pm 1} = 0, \\ (i\lambda - \mu(\partial_y^2 - k^2) +iky)\widehat{\theta}_I^{(2)}(\lambda, k, y) &= -\partial_y G^2(\lambda, k, y), \quad \widehat{\theta}_I^{(2)}|_{y=\pm 1} = 0. \end{aligned}$$

By Lemma 2.4, we get

$$(\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_I^{(1)}(\lambda)\|_{L^2} \leq C \|kG^1(\lambda)\|_{L^2},$$

and

$$\mu^{\frac{2}{3}} |k|^{\frac{1}{3}} \|\widehat{\theta}_I^{(2)}(\lambda)\|_{L^2} \leq C \|G^2(\lambda)\|_{L^2}.$$

Then, by Plancherel's theorem, we have

$$\begin{aligned} (\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_I\|_{L^2 L^2}^2 &\sim (\mu k^2)^{\frac{1}{3}} \left\| \|\widehat{\theta}_I(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 \\ &\leq 2(\mu k^2)^{\frac{1}{3}} \left(\left\| \|\widehat{\theta}_I^{(1)}(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 + \left\| \|\widehat{\theta}_I^{(2)}(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 \right) \\ &\leq C(\mu k^2)^{\frac{1}{3}} \left(\left\| (\mu k^2)^{-\frac{1}{3}} \|kG^1(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 + \left\| \mu^{-\frac{2}{3}} |k|^{-\frac{1}{3}} \|G^2(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 \right) \\ &= C\mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \left\| \|G^1(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 + C\mu^{-1} \left\| \|G^2(\lambda)\|_{L^2} \right\|_{L^2(\mathbb{R})}^2 \\ &\sim \mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2 L^2}^2 + \mu^{-1} \|g^2\|_{L^2 L^2}^2. \end{aligned}$$

Next we estimate $\|\widehat{\theta}_I\|_{L^\infty L^2}$. Notice that

$$\begin{aligned} &\frac{1}{2} \partial_t \|\widehat{\theta}_I\|_{L^2}^2 + \mu \|\partial_y \widehat{\theta}_I\|_{L^2}^2 + \mu k^2 \|\theta_I\|_{L^2}^2 \\ &= \operatorname{Re} \langle (\partial_t - \mu(\partial_y^2 - k^2) +iky)\widehat{\theta}_I, \widehat{\theta}_I \rangle \\ &= \operatorname{Re} \langle -ikg^1 - \partial_y g^2, \widehat{\theta}_I \rangle = \operatorname{Re} \left(-ik \langle g^1, \widehat{\theta}_I \rangle + \langle g^2, \partial_y \widehat{\theta}_I \rangle \right) \\ &\leq |k| \|g^1\|_{L^2} \|\widehat{\theta}_I\|_{L^2} + \|g^2\|_{L^2} \|\partial_y \widehat{\theta}_I\|_{L^2}, \end{aligned}$$

which gives

$$\partial_t \|\widehat{\theta}_I\|_{L^2}^2 + \mu \|\partial_y \widehat{\theta}_I\|_{L^2}^2 + 2\mu k^2 \|\theta_I\|_{L^2}^2 \leq \mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2}^2 + (\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_I\|_{L^2}^2 + \mu^{-1} \|g^2\|_{L^2}^2.$$

As $\widehat{\theta}_I|_{t=0} = 0$, this shows that

$$\begin{aligned} \|\widehat{\theta}_I(t)\|_{L^2}^2 &\leq \int_0^t \left(\mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1(s)\|_{L^2}^2 + (\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_I(s)\|_{L^2}^2 + \mu^{-1} \|g^2(s)\|_{L^2}^2 \right) ds \\ &\leq \mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2 L^2}^2 + (\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}_I\|_{L^2 L^2}^2 + \mu^{-1} \|g^2\|_{L^2 L^2}^2 \\ &\leq C(\mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2 L^2}^2 + \mu^{-1} \|g^2\|_{L^2 L^2}^2). \end{aligned}$$

Thus, we get

$$\|\widehat{\theta}_I\|_{L^\infty L^2}^2 \leq C(\mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2 L^2}^2 + \mu^{-1} \|g^2\|_{L^2 L^2}^2).$$

This completes the proof of Proposition 2.5. \square

Thus, combining Lemma 2.3 and Proposition 2.5, we immediately obtain the following space-time estimates of $\widehat{\theta}$.

Proposition 2.6. *Let $\widehat{\theta}$ be a solution of (2.2) with $\widehat{\theta}_{in} \in L^2(-1, 1)$ and $g^1, g^2 \in L^2 L^2$. Then there exists a constant $C > 0$ independent in μ, k such that*

$$\|\widehat{\theta}\|_{L^\infty L^2}^2 + (\mu k^2)^{\frac{1}{3}} \|\widehat{\theta}\|_{L^2 L^2}^2 \leq \|\widehat{\theta}_{in}\|_{L^2}^2 + C\left(\mu^{-\frac{1}{3}} |k|^{\frac{4}{3}} \|g^1\|_{L^2 L^2}^2 + \mu^{-1} \|g^2\|_{L^2 L^2}^2\right).$$

3. Nonlinear stability

In this section, we prove Theorem 1.1. Due to the buoyancy term $\partial_x \theta$ in the equation of the vorticity, we need to estimate $\| |D_x|^{\frac{1}{6}} \theta(t) \|_{L^2}$ in order to control the buoyancy term. In fact, for the two-dimensional Boussinesq equation, the global existence of smooth solution is well-known for the data $u_{in} \in H^2(\Omega), \theta_{in} \in H^1(\Omega)$ and $|D_x|^{\frac{1}{6}} \theta_{in} \in H^1(\Omega)$. The main interest of Theorem 1.1 is the stability estimates

$$\sum_{k \in \mathbb{Z}} E_k \leq C \varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}, \quad \sum_{k \in \mathbb{Z}} H_k \leq C \varepsilon_1 \min\{\nu, \mu\}^{\frac{1}{12}}. \tag{3.1}$$

Here $E_0 = \|\widehat{w}_0\|_{L^\infty L^2}$ and $H_0 = \|\widehat{\theta}_0\|_{L^\infty L^2}$, and for $k \neq 0$,

$$E_k = \|(1 - |y|)^{\frac{1}{2}} \widehat{w}_k\|_{L^\infty L^2} + |k| \|\widehat{u}_k\|_{L^2 L^2} + |k|^{\frac{1}{2}} \|\widehat{u}_k\|_{L^\infty L^\infty} + (\nu k^2)^{\frac{1}{4}} \|\widehat{w}_k\|_{L^2 L^2},$$

and

$$H_k = |k|^{\frac{1}{6}} \|\widehat{\theta}_k\|_{L^\infty L^2} + \mu^{\frac{1}{6}} |k|^{\frac{1}{2}} \|\widehat{\theta}_k\|_{L^2 L^2}.$$

And we can get the following estimates, which along with bootstrap arguments, then we can easily deduce the estimates (3.1).

Proposition 3.1. *There hold that, for $k \neq 0$,*

$$E_k \leq \|\widehat{w}_{in,k}\|_{L^2} + |k|^{-1} \|\partial_y \widehat{w}_{in,k}\|_{L^2} + C\nu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l E_{k-l} + C\nu^{-\frac{1}{4}} \mu^{-\frac{1}{6}} H_k, \quad (3.2)$$

and

$$E_0 \leq \|\widehat{w}_{in,0}\|_{L^2} + C\nu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0\}} E_l E_{-l}. \quad (3.3)$$

For H_0 , there holds that

$$H_0 \lesssim \|\widehat{\theta}_{in,0}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0\}} |l|^{-\frac{2}{3}} E_l H_{-l}. \quad (3.4)$$

For $k \neq 0$, there hold that

1. for $\mu k^2 \leq 1$,

$$H_k \lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l}; \quad (3.5)$$

2. for $\mu k^2 > 1$,

$$H_k \lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} E_k H_0. \quad (3.6)$$

Proof. *Proof of (3.2).* Denoting $\widehat{w}_k(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} \omega(t, x, y) e^{-ikx} dx$ and

$$f_k^1(t, y) = \sum_{l \in \mathbb{Z}} \widehat{u}_l^1(t, y) \widehat{w}_{k-l}(t, y), \quad f_k^2(t, y) = \sum_{l \in \mathbb{Z}} \widehat{u}_l^2(t, y) \widehat{w}_{k-l}(t, y),$$

we have

$$(\partial_t - \nu(\partial_y^2 - k^2) + ik y) \widehat{w}_k(t, y) = -ik \widehat{\theta}_k(t, y) - ik f_k^1(t, y) - \partial_y f_k^2(t, y). \quad (3.7)$$

It follows from Proposition 2.1 that

$$E_k \leq C \left(\nu^{-\frac{1}{4}} |k|^{\frac{1}{2}} \|\widehat{\theta}_k\|_{L^2 L^2} + \nu^{-\frac{1}{4}} |k|^{\frac{1}{2}} \|f_k^1\|_{L^2 L^2} + \nu^{-\frac{1}{2}} \|f_k^2\|_{L^2 L^2} \right) + \|\widehat{w}_{in,k}\|_{L^2} + |k|^{-1} \|\partial_y \widehat{w}_{in,k}\|_{L^2}. \quad (3.8)$$

As in [10], we get that for $k \neq 0$,

$$\left\| \frac{\widehat{u}_k^2(t, y)}{(1 - |y|)^{\frac{1}{2}}} \right\|_{L^2 L^\infty}^2 = \left\| \sup_{y \in [-1, 1]} \frac{|\widehat{u}_k^2(t, y)|^2}{1 - |y|} \right\|_{L^1}$$

$$\begin{aligned}
 &= \left\| \max \left\{ \sup_{y \in [0,1]} \frac{|\int_1^y \partial_z \widehat{u}_k^2(t, z) dz|^2}{1 - |y|}, \sup_{y \in [-1,0]} \frac{|\int_{-1}^y \partial_z \widehat{u}_k^2(t, z) dz|^2}{1 - |y|} \right\} \right\|_{L^1} \\
 &\leq 4 \|\partial_y \widehat{u}_k^2\|_{L^2 L^2}^2 = 4|k|^2 \|\widehat{u}_k^1\|_{L^2 L^2}^2 \leq 4E_k^2.
 \end{aligned}$$

From which, we infer that, for $k \in \mathbb{Z}$,

$$\|f_k^2\|_{L^2 L^2} \leq \sum_{l \in \mathbb{Z}} \left\| \frac{\widehat{u}_l^2(t, y)}{(1 - |y|)^{\frac{1}{2}}} \right\|_{L^2 L^\infty} \|(1 - |y|)^{\frac{1}{2}} \widehat{w}_{k-l}\|_{L^\infty L^2} \leq 2 \sum_{l \in \mathbb{Z}} E_l E_{k-l}, \tag{3.9}$$

and

$$\begin{aligned}
 \|f_k^1\|_{L^2 L^2} &\leq \|\widehat{u}_0^1\|_{L^\infty L^\infty} \|\widehat{w}_k\|_{L^2 L^2} + \|\widehat{u}_k^1\|_{L^2 L^\infty} \|\widehat{w}_0\|_{L^\infty L^2} \\
 &\quad + \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \|\widehat{u}_l^1\|_{L^\infty L^\infty} \|\widehat{w}_{k-l}\|_{L^2 L^2}.
 \end{aligned}$$

Thanks to $|l||k - l| \gtrsim |k|(l \neq 0, k)$, we have

$$\begin{aligned}
 \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \|\widehat{u}_l^1\|_{L^\infty L^\infty} \|\widehat{w}_{k-l}\|_{L^2 L^2} &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0, k\}} |l|^{-\frac{1}{2}} E_l \nu^{-\frac{1}{4}} |k - l|^{-\frac{1}{2}} E_{k-l} \\
 &\lesssim |k|^{-\frac{1}{2}} \nu^{-\frac{1}{4}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} E_l E_{k-l},
 \end{aligned}$$

and

$$\|\widehat{u}_0^1\|_{L^\infty L^\infty} \|\widehat{w}_k\|_{L^2 L^2} + \|\widehat{u}_k^1\|_{L^2 L^\infty} \|\widehat{w}_0\|_{L^\infty L^2} \lesssim \|\widehat{w}_0\|_{L^\infty L^2} \|\widehat{w}_k\|_{L^2 L^2} \lesssim (\nu k^2)^{-\frac{1}{4}} E_k E_0.$$

This shows that

$$\|f_k^1\|_{L^2 L^2} \lesssim (\nu k^2)^{-\frac{1}{4}} \sum_{l \in \mathbb{Z}} E_l E_{k-l}. \tag{3.10}$$

Thus, by (3.8), (3.9) and (3.10), we obtain that

$$E_k \leq \|\widehat{w}_{in, k}\|_{L^2} + |k|^{-1} \|\partial_y \widehat{w}_{in, k}\|_{L^2} + C\nu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l E_{k-l} + C\nu^{-\frac{1}{4}} \mu^{-\frac{1}{6}} H_k.$$

Proof of (3.3). Due to $\operatorname{div} u = 0$, we have $\widehat{u}_0^2(t, y) = 0$. By $P_0(\widehat{u}^1 \partial_x \widehat{u}^1) = 0$, we have

$$\begin{aligned}
 \partial_t \widehat{u}_0^1(t, y) - \nu \partial_y^2 \widehat{u}_0^1(t, y) &= - \sum_{l \in \mathbb{Z} \setminus \{0\}} \widehat{u}_l^2(t, y) \partial_y \widehat{u}_{-l}^1(t, y) \\
 &= - \sum_{l \in \mathbb{Z} \setminus \{0\}} \widehat{u}_l^2(t, y) \widehat{w}_{-l}(t, y) = -f_0^2(t, y). \tag{3.11}
 \end{aligned}$$

By integration by parts in (3.11), we get

$$\langle (\partial_t - \nu \partial_y^2) \widehat{u}_0^1, -\partial_y^2 \widehat{u}_0^1 \rangle = \frac{1}{2} \partial_t \|\partial_y \widehat{u}_0^1(t)\|_{L^2}^2 + \nu \|\partial_y^2 \widehat{u}_0^1(t)\|_{L^2}^2 = \langle f_0^2, \partial_y^2 \widehat{u}_0^1 \rangle,$$

which gives

$$\partial_t \|\partial_y \widehat{u}_0^1(t)\|_{L^2}^2 + \nu \|\partial_y^2 \widehat{u}_0^1(t)\|_{L^2}^2 \leq C \nu^{-1} \|f_0^2(t, y)\|_{L^2}^2,$$

from which, along with $\partial_y \widehat{u}_0^1(t, y) = \widehat{w}_0(t, y)$, we infer that

$$E_0^2 = \|\widehat{w}_0\|_{L^\infty L^2}^2 \leq C \nu^{-1} \|f_0^2(t, y)\|_{L^2 L^2}^2 + \|\widehat{w}_{in,0}\|_{L^2}^2. \tag{3.12}$$

Thus, by using (3.9), we obtain

$$E_0 \leq \|\widehat{w}_{in,0}\|_{L^2} + C \nu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0\}} E_l E_{-l}.$$

Proof of (3.4). Similarly, we can derive the evolution equation of $\widehat{\theta}_0$,

$$\partial_t \widehat{\theta}_0 - \mu \partial_y^2 \widehat{\theta}_0 = - \sum_{l \in \mathbb{Z} \setminus \{0\}} \partial_y (\widehat{u}_l^2 \widehat{\theta}_{-l})(t, y) = -\partial_y g_0^2(t, y). \tag{3.13}$$

Similarly, as the estimate of E_0 , we get that by integration by parts in (3.13),

$$H_0^2 = \|\widehat{\theta}_0\|_{L^\infty L^2}^2 + \mu \|\partial_y \widehat{\theta}_0\|_{L^2 L^2}^2 \leq C \mu^{-1} \|g_0^2(t, y)\|_{L^2 L^2}^2 + \|\widehat{\theta}_{in,0}\|_{L^2}^2. \tag{3.14}$$

By using the Gagliardo-Nirenberg inequality and $\partial_y \widehat{u}_k^2 = -ik \widehat{u}_k^1$, we have

$$\|\widehat{u}_k^2\|_{L^2 L^\infty} \leq C |k|^{\frac{1}{2}} \|\widehat{u}_k^2\|_{L^2 L^2}^{\frac{1}{2}} \|\widehat{u}_k^1\|_{L^2 L^2}^{\frac{1}{2}} \leq C |k|^{-\frac{1}{2}} E_k. \tag{3.15}$$

And then, we obtain

$$\begin{aligned} \|g_0^2\|_{L^2 L^2} &\leq \sum_{l \in \mathbb{Z} \setminus \{0\}} \|\widehat{u}_l^2\|_{L^2 L^\infty} \|\widehat{\theta}_{-l}\|_{L^\infty L^2} \lesssim \sum_{l \in \mathbb{Z} \setminus \{0\}} |l|^{-\frac{1}{2}} |l|^{-\frac{1}{6}} E_l H_{-l} \\ &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0\}} |l|^{-\frac{2}{3}} E_l H_{-l}. \end{aligned} \tag{3.16}$$

Thus, from (3.14) and (3.16), we have

$$H_0 \lesssim \|\widehat{\theta}_{in,0}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0\}} |l|^{-\frac{2}{3}} E_l H_{-l}.$$

To control the nonlinear term $\|g_k^1\|_{L^2 L^2}$, during the estimates of H_k , we need to divide them into the low-frequency part $\mu k^2 \leq 1$ and the high-frequency part $\mu k^2 > 1$.

Proof of (3.5). First, we can derive the evolution equations of $\widehat{\theta}_k(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} \theta(t, x, y) \times e^{-ikx} dx$. Denoting

$$g_k^1(t, y) = \sum_{l \in \mathbb{Z}} \widehat{u}_l^1(t, y) \widehat{\theta}_{k-l}(t, y), \quad g_k^2(t, y) = \sum_{l \in \mathbb{Z}} \widehat{u}_l^2(t, y) \widehat{\theta}_{k-l}(t, y),$$

we have that $\widehat{\theta}_k(t, y)$ satisfies,

$$(\partial_t - \mu(\partial_y^2 - k^2) +iky)\widehat{\theta}_k(t, y) = -ikg_k^1(t, y) - \partial_y g_k^2(t, y). \tag{3.17}$$

For $\mu k^2 \leq 1$, it follows from Proposition 2.6 that

$$H_k \leq |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + C \left(\mu^{-\frac{1}{6}} |k|^{\frac{5}{6}} \|g_k^1\|_{L^2 L^2} + \mu^{-\frac{1}{2}} |k|^{\frac{1}{6}} \|g_k^2\|_{L^2 L^2} \right). \tag{3.18}$$

On the one hand, by using $\widehat{u}_0^2 = 0$ and (3.15), we have that for $k \neq 0$,

$$\begin{aligned} \|g_k^2\|_{L^2 L^2} &\leq \|\widehat{u}_k^2\|_{L^2 L^\infty} \|\widehat{\theta}_0\|_{L^\infty L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^2\|_{L^2 L^\infty} \|\widehat{\theta}_{k-l}\|_{L^\infty L^2} \\ &\leq |k|^{-\frac{1}{2}} E_k H_0 + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} |l|^{-\frac{1}{2}} |k-l|^{-\frac{1}{6}} E_l H_{k-l} \\ &\leq |k|^{-\frac{1}{2}} E_k H_0 + |k|^{-\frac{1}{6}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l H_{k-l}. \end{aligned} \tag{3.19}$$

On the other hand, for g_k^1 and $k \neq 0$, by using Gagliardo-Nirenberg inequality, we have

$$\|\widehat{u}_k^1\|_{L^2 L^\infty} \leq C \|\widehat{u}_k^1\|_{L^2 L^2}^{\frac{1}{2}} \|\partial_y \widehat{u}_k^1\|_{L^2 L^2}^{\frac{1}{2}} \leq C \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k, \tag{3.20}$$

and then we obtain that

$$\begin{aligned} \|g_k^1\|_{L^2 L^2} &\leq \|\widehat{u}_0^1\|_{L^\infty L^\infty} \|\widehat{\theta}_k\|_{L^2 L^2} + \|\widehat{u}_k^1\|_{L^2 L^\infty} \|\widehat{\theta}_0\|_{L^\infty L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} \\ &\leq \|\widehat{w}_0\|_{L^\infty L^2} \|\widehat{\theta}_k\|_{L^2 L^2} + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k \|\widehat{\theta}_0\|_{L^\infty L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} \\ &\leq \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} E_0 H_k + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k H_0 + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2}. \end{aligned} \tag{3.21}$$

To estimate $\sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2}$, we divide it into two parts and get that

$$\begin{aligned} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} &\leq \sum_{l \in \mathbb{Z} \setminus \{0,k\}, |k-l| \leq \frac{|k|}{2}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}, |k-l| > \frac{|k|}{2}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} \\ &\stackrel{\text{def}}{=} \text{HL} + \text{LH}, \end{aligned}$$

whereas by (3.20),

$$\begin{aligned}
 \text{HL} &\leq \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} \|\widehat{u}_l^1\|_{L^2 L^\infty} \|\widehat{\theta}_{k-l}\|_{L^\infty L^2} \\
 &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} \nu^{-\frac{1}{8}} |l|^{-\frac{3}{4}} |k-l|^{-\frac{1}{6}} E_l H_{k-l} \\
 &\lesssim \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l},
 \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
 \text{LH} &\leq \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} \|\widehat{u}_l^1\|_{L^\infty L^\infty} \|\widehat{\theta}_{k-l}\|_{L^2 L^2} \\
 &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} |l|^{-\frac{1}{2}} |k-l|^{-\frac{1}{2}} \mu^{-\frac{1}{6}} E_l H_{k-l} \\
 &\lesssim \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} E_l H_{k-l}.
 \end{aligned} \tag{3.23}$$

And then, substituting (3.22) and (3.23) into (3.21), we get

$$\begin{aligned}
 \|g_k^1\|_{L^2 L^2} &\lesssim \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} E_0 H_k + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k H_0 \\
 &\quad + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l} + \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} E_l H_{k-l}.
 \end{aligned} \tag{3.24}$$

Thus, combining (3.18), (3.19) and (3.24), we get that for $k \neq 0$ and $\mu k^2 \leq 1$,

$$\begin{aligned}
 H_k &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0\}} E_l H_{k-l} + \mu^{-\frac{1}{3}} |k|^{\frac{1}{3}} E_0 H_k + \nu^{-\frac{1}{8}} \mu^{-\frac{1}{6}} |k|^{\frac{1}{12}} E_k H_0 \\
 &\quad + \nu^{-\frac{1}{8}} \mu^{-\frac{1}{6}} |k|^{\frac{1}{12}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l} + \mu^{-\frac{1}{3}} |k|^{\frac{1}{3}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} E_l H_{k-l} \\
 &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} E_k H_0 \\
 &\quad + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| > \frac{|k|}{2}} E_l H_{k-l} \\
 &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}, |k-l| \leq \frac{|k|}{2}} E_l H_{k-l}.
 \end{aligned}$$

Proof of (3.6). For $\mu k^2 > 1$, it follows from Proposition 2.2 that

$$\begin{aligned}
 H_k &\leq |k|^{\frac{1}{6}} \|\widehat{\theta}_k\|_{L^\infty L^2} + |k|^{\frac{1}{6}} (\mu k^2)^{\frac{1}{2}} \|\widehat{\theta}_k\|_{L^2 L^2} \\
 &\leq |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + C \mu^{-\frac{1}{2}} |k|^{\frac{1}{6}} (\|g_k^1\|_{L^2 L^2} + \|g_k^2\|_{L^2 L^2}).
 \end{aligned}
 \tag{3.25}$$

For g_k^1 and $k \neq 0$, by (3.20), we obtain

$$\begin{aligned}
 \|g_k^1\|_{L^2 L^2} &\leq \|\widehat{u}_0^1\|_{L^\infty L^\infty} \|\widehat{\theta}_k\|_{L^2 L^2} + \|\widehat{u}_k^1\|_{L^2 L^\infty} \|\widehat{\theta}_0\|_{L^\infty L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} \\
 &\lesssim \|\widehat{w}_0\|_{L^\infty L^2} \|\widehat{\theta}_k\|_{L^2 L^2} + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k \|\widehat{\theta}_0\|_{L^\infty L^2} + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} \\
 &\lesssim \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} E_0 H_k + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k H_0 + \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2}.
 \end{aligned}
 \tag{3.26}$$

Whereas for the term $\sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2}$, we can obtain that by using $|l||k-l| \gtrsim |k|(l \neq 0, k)$,

$$\begin{aligned}
 \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1 \widehat{\theta}_{k-l}\|_{L^2 L^2} &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|\widehat{u}_l^1\|_{L^\infty L^\infty} \|\widehat{\theta}_{k-l}\|_{L^2 L^2} \\
 &\lesssim \sum_{l \in \mathbb{Z} \setminus \{0,k\}} |l|^{-\frac{1}{2}} E_l \mu^{-\frac{1}{6}} |k-l|^{-\frac{1}{2}} H_{k-l} \\
 &\lesssim \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l H_{k-l}.
 \end{aligned}$$

And then, we obtain

$$\|g_k^1\|_{L^2 L^2} \lesssim \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} E_0 H_k + \nu^{-\frac{1}{8}} |k|^{-\frac{3}{4}} E_k H_0 + \mu^{-\frac{1}{6}} |k|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l H_{k-l}.
 \tag{3.27}$$

Thus, combining (3.25), (3.19) and (3.27), we get that for $k \neq 0$ and $\mu k^2 > 1$,

$$\begin{aligned}
 H_k &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \mu^{-\frac{2}{3}} |k|^{-\frac{1}{3}} E_0 H_k + \nu^{-\frac{1}{8}} \mu^{-\frac{1}{2}} |k|^{-\frac{7}{12}} E_k H_0 \\
 &\quad + \mu^{-\frac{2}{3}} |k|^{-\frac{1}{3}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l H_{k-l} \\
 &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \mu^{-\frac{1}{2}} E_0 H_k + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} E_k H_0 \\
 &\quad + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l H_{k-l} \\
 &\lesssim |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} E_k H_0.
 \end{aligned}$$

This completes the proof of Proposition 3.1. \square

Now we prove Theorem 1.1. From (3.3) and (3.2), we deduce

$$\begin{aligned} \sum_{k \in \mathbb{Z}} E_k &\leq \sum_{k \in \mathbb{Z}} \|\widehat{w}_{in,k}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-1} \|\partial_y \widehat{w}_{in,k}\|_{L^2} \\ &+ C\nu^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} E_l E_{k-l} + C\nu^{-\frac{1}{4}} \mu^{-\frac{1}{6}} \sum_{k \in \mathbb{Z} \setminus \{0\}} H_k. \end{aligned} \quad (3.28)$$

And by the fact that

$$\sum_{k \in \mathbb{Z}} H_k = H_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}, \mu k^2 \leq 1} H_k + \sum_{k \in \mathbb{Z} \setminus \{0\}, \mu k^2 > 1} H_k,$$

combining (3.4), (3.5) and (3.6), we can deduce that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} H_k &\lesssim \|\widehat{\theta}_{in,0}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} + \mu^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} E_l H_{k-l} \\ &+ \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} \sum_{k \in \mathbb{Z} \setminus \{0\}, \mu k^2 \leq 1} \sum_{l \in \mathbb{Z}} E_l H_{k-l} + \nu^{-\frac{1}{8}} \mu^{-\frac{5}{24}} \sum_{k \in \mathbb{Z} \setminus \{0\}, \mu k^2 > 1} E_k H_0. \end{aligned} \quad (3.29)$$

On the other hand, it is easy to verify that from $\|u_{in}\|_{H^2} \leq \varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}$ and $\|\theta_{in}\|_{H^1} + \| |D_x|^{\frac{1}{6}} \theta_{in} \|_{H^1} \leq \varepsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}}$,

$$\sum_{k \in \mathbb{Z}} \|\widehat{w}_{in,k}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-1} \|\partial_y \widehat{w}_{in,k}\|_{L^2} \leq C\varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}},$$

and

$$\|\widehat{\theta}_{in,0}\|_{L^2} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{\frac{1}{6}} \|\widehat{\theta}_{in,k}\|_{L^2} \leq C\varepsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}}.$$

Thus, for $\varepsilon_0, \varepsilon_1$ suitably small, by bootstrap arguments, we can deduce from (3.28) and (3.29) that

$$\sum_{k \in \mathbb{Z}} E_k \leq C\varepsilon_0 \min\{\nu, \mu\}^{\frac{1}{2}}, \quad \sum_{k \in \mathbb{Z}} H_k \leq C\varepsilon_1 \min\{\nu, \mu\}^{\frac{11}{12}}.$$

This completes the proof of Theorem 1.1. \square

Data availability

No data was used for the research described in the article.

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