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ON THE DIFFUSIVE LIMITS OF RADIATIVE HEAT TRANSFER SYSTEM I: WELL-PREPARED INITIAL AND BOUNDARY CONDITIONS*

MOHAMED GHATTASSI[†], XIAOKAI HUO[‡], AND NADER MASMOUDI[§]

Abstract. We study the diffusive limit approximation for a nonlinear radiative heat transfer system that arises in the modeling of glass cooling and greenhouse effects and in astrophysics. The model is considered with the reflective radiative boundary conditions for the radiative intensity and with periodic, Dirichlet and Robin boundary conditions for the temperature. The global existence of weak solutions for this system is given by using a Galerkin method with a careful treatment of the boundary conditions. Using the compactness method, averaging lemma and Young measure theory, we prove our main result that the weak solution converges to a nonlinear diffusion model in the diffusive limit. Moreover, under more regularity conditions on the limit system, the diffusive limit is also analyzed by using a relative entropy method. In particular, we get a rate of convergence. The initial and boundary conditions are assumed to be well-prepared in the sense that no initial or boundary layer exists.

Key words. radiative transfer system, compactness method, averaging lemma, Young measures, relative entropy

MSC codes. 35F50, 35D30, 35A10, 35Q79

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1. Introduction. The radiative transfer equation describes the physical phenomenon of energy transport in radiation. It has a variety of applications, such as cooling glass [50], radiation hydrodynamics [42], astrophysics [45] and greenhouse effect [7, 31]. In this paper we consider a model of glass cooling with the radiative heat transfer equation coupled with a heat equation. The model is given by

$$\begin{cases} c_m\rho_m\partial_t T = k_h\Delta T - \displaystyle\int_0^\infty\int_{\mathbb{S}^2}\kappa\left(\mathcal{B}-\psi\right)d\beta d\nu, \quad t>0, x\in\Omega, \\ \dfrac{1}{c}\partial_t\psi + \beta\cdot\nabla\psi = \kappa\left(\mathcal{B}-\psi\right), \quad t>0, (x,\beta,\nu)\in\Omega\times\mathbb{S}^2\times\mathbb{R}_+. \end{cases} \tag{1}$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded domain, and $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. The function $T=T(t,x)$ denotes the temperature of the medium and $\psi=\psi(t,x,\beta,\nu)$ is the specific radiation intensity at $x\in\Omega$ traveling in the direction $\beta\in\mathbb{S}^2$ with frequency $\nu>0$ at time $t>0$. The constants c_m, ρ_m, k_h, κ, and c are the specific heat, the density,

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[†]Department of Mathematics, New York University in Abu Dhabi, Saadiyat Island, Abu Dhabi, United Arab Emirates, and Department of Mathematics, Faculty of Science of Gabès, Gabès, Tunisia (mg6888@nyu.edu).
[‡]Institute for Analysis and Scientific Computing, Vienna University of Technology, 1040 Vienna, Austria (xiaokai.huo@tuwien.ac.at).
[§]Department of Mathematics, New York University in Abu Dhabi, Saadiyat Island, Abu Dhabi, United Arab Emirates, and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012 USA (masmoudi@cims.nyu.edu).

the thermal conductivity, the opacity coefficient, and the speed of light, respectively. Furthermore, $\mathcal{B} = \mathcal{B}(\nu, T)$ denotes the Planck's function

$$\mathcal{B}(\nu, T) := \frac{2h_p\nu^3}{c^2\left(e^{\frac{h_p\nu}{k_bT}} - 1\right)}$$

for black body radiation in glass. Here h_p is the Planck's constant and k_b is the Boltzmann's constant. We refer the reader to [22], [44] and references therein for more radiative heat transfer models.

In order to solve the glass cooling model (1), we need to provide initial and boundary conditions for T and ψ. The initial conditions are taken to be

$$\begin{aligned} T(t=0,x) = {}& T_0(x) \text{ for any } x \in \Omega, \\ \psi(t=0,x,\beta,\nu) = {}& \psi_0(x,\beta,\nu) \text{ for any } (x,\beta,\nu) \in \Omega \times \mathbb{S}^2 \times \mathbb{R}_+. \end{aligned}$$

The boundary condition for the temperature T is the following Robin boundary condition:

$$kn \cdot \nabla T(t,x) = h_c\left(T_b(t,x) - T(t,x)\right) \text{ for any } t > 0,\ x \in \partial\Omega. \tag{2}$$

Here $T_b = T_b(t,x) > 0$ is a nonnegative function, $n = n(x)$ is the outward unit normal vector to the boundary $\partial\Omega$, h_c is the convective heat transfer coefficient, and $k \geq 0$ is a constant. When $k = 0$, this corresponds to a nonhomogeneous Dirichlet boundary condition. To give the boundary condition for ψ, we define the boundary set $\Sigma = \partial\Omega \times \mathbb{S}^2$ and

$$\Sigma_- := \left\{(x,\beta) \in \partial\Omega \times \mathbb{S}^2,\ \beta \cdot n(x) < 0\right\}, \tag{3}$$

$$\Sigma_+ := \left\{(x,\beta) \in \partial\Omega \times \mathbb{S}^2,\ \beta \cdot n(x) > 0\right\}. \tag{4}$$

The boundary condition for the specific radiation intensity is taken to be the following reflecting absorbing mixed condition:

$$\psi(t,x,\beta,\nu) = \alpha\psi_b(t,x,\beta,\nu) + (1-\alpha)\psi(t,x,\beta',\nu), \quad t > 0,\ (x,\beta) \in \Sigma_-,\ \nu \in \mathbb{R}_+.$$

Here $\psi_b = \psi_b(t,x,\beta,\nu)$ is a given function defined in the in-flow direction, which is on the half surface Σ_-, and it describes the radiative intensity transmitted into the medium from outside. The coordinate $\beta^{'} \in \mathbb{S}^2$ is the exiting radius which specularly reflects into the incident radius β as $\beta' = \beta - 2(n(x)\cdot\beta)n(x)$, and $\alpha \in (0,1)$ is a constant.

Next, we give the dimensionless form of the system (1). We introduce the nondimensional parameter $\varepsilon = 1/\kappa_r x_r$, where x_r and κ_r are the length scale and reference absorption, respectively. Physically ε represents the ratio of a typical photon mean free path to a typical length scale of the problem. The rescaled system is given by

$$\begin{cases} \varepsilon^2\partial_t T = \varepsilon^2 k\Delta T - \displaystyle\int_0^\infty \int_{\mathbb{S}^2} \kappa\left(\mathcal{B} - \psi\right) d\beta d\nu, \quad t > 0,\ x \in \Omega, \\ \varepsilon^2 \dfrac{1}{c}\partial_t\psi + \varepsilon\beta\cdot\nabla\psi = \kappa\left(\mathcal{B} - \psi\right), \quad t > 0,\ (x,\beta,\nu) \in \Omega \times \mathbb{S}^2 \times (0,\infty). \end{cases}$$

See [22] for more details on the derivation. We consider the glass cooling in the gray medium, that is, $\mathcal{B}$ does not depend on the frequency ν. The specular black body intensity $\mathcal{B}$ is then given by $\mathcal{B} = \frac{\sigma}{\pi}T^4$, according to the Stefan–Boltzmann law. For simplicity, we take all the constants in (1) and (2) to be the same $k = \kappa = h_c = c = 1$, and take $\sigma = \pi$. Since the solutions of the above system depend on ε, we introduce new notations $T_\varepsilon = T_\varepsilon(t,x)$ and $\psi_\varepsilon = \psi_\varepsilon(t,x,\beta)$ to represent the temperature and the radiative intensity, respectively. We introduce the notation $\langle\psi_\varepsilon\rangle := \int_{\mathbb{S}^2}\psi_\varepsilon d\beta$, which is the radiative density; system (1) then can be written as

$$\partial_t T_\varepsilon = \Delta T_\varepsilon + \frac{1}{\varepsilon^2}\langle\psi_\varepsilon - T_\varepsilon^4\rangle, \tag{5}$$

$$\partial_t\psi_\varepsilon + \frac{1}{\varepsilon}\beta\cdot\nabla\psi_\varepsilon = -\frac{1}{\varepsilon^2}(\psi_\varepsilon - T_\varepsilon^4). \tag{6}$$

The initial conditions are taken to be

$$T_\varepsilon(t=0,x) = T_{\varepsilon 0}(x) \quad \text{for any } x\in\Omega, \tag{7}$$

$$\psi_\varepsilon(t=0,x,\beta) = \psi_{\varepsilon 0}(x,\beta) \quad \text{for any } x\in\Omega, \beta\in\mathbb{S}^2. \tag{8}$$

The boundary condition for ψ_ε is taken to be

$$\psi_\varepsilon(t,x,\beta) = \alpha\psi_b(t,x,\beta) + (1-\alpha)(L\psi_\varepsilon)(t,x,\beta),\ t>0, \quad (x,\beta)\in\Sigma_-, \tag{9}$$

where the reflection operator L is defined by

$$L(f(x,\beta)) := f(x,\beta') = f(x,\beta - 2(n(x)\cdot\beta)n(x)). \tag{10}$$

The boundary data for T_ε is taken to be one of the following three conditions:

(A) on the torus,

$$\Omega = \mathbb{T}^3, \tag{11}$$

(B) Dirichlet boundary condition,

$$T_\varepsilon(t,x) = T_b(t,x) \text{ for any } x\in\partial\Omega, \tag{12}$$

(C) Robin boundary condition,

$$\varepsilon^r n\cdot\nabla T_\varepsilon(t,x) = -T_\varepsilon(t,x) + T_b(t,x) \text{ for any } x\in\partial\Omega. \tag{13}$$

Here $r\geq 0$ is a nonnegative constant.

The parameter ε is usually small in applications and it plays an important role in the system (5)–(6). It is interesting and physically meaningful to study the behavior of its solutions as $\varepsilon\to 0$. We call such a limit the diffusive limit. The objective of this paper is to study the diffusive limit rigorously. First we derive the limit system formally.

1.1. Formal derivation of the limit system. By (6),

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\beta\cdot\nabla\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon. \tag{14}$$

Therefore, for small ε, we have

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\beta\cdot\nabla\left(T_\varepsilon^4 - \varepsilon\beta\cdot\nabla\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon\right) - \varepsilon^2\partial_t\left(T_\varepsilon^4 - \varepsilon\beta\cdot\nabla\psi_\varepsilon - \varepsilon^2\partial_t\psi_\varepsilon\right).$$

Combing the terms with the same order gives

$$\psi_\varepsilon = T_\varepsilon^4 - \varepsilon\beta\cdot\nabla T_\varepsilon^4 - \varepsilon^2\left(\partial_t T_\varepsilon^4 - \beta\cdot\nabla\left(\beta\cdot\nabla\psi_\varepsilon\right)\right) + \varepsilon^3(\beta\cdot\partial_t\psi_\varepsilon + \beta\cdot\nabla\partial_t\psi_\varepsilon) + \varepsilon^4\partial_t^2\psi_\varepsilon.$$

We can use (14) again in the fourth term on the right of the above equation and obtain

$$\begin{aligned}\psi_\varepsilon = {} & T_\varepsilon^4 - \varepsilon\beta\cdot\nabla T_\varepsilon^4 - \varepsilon^2\left(\partial_t T_\varepsilon^4 - \beta\cdot\nabla(\beta\cdot\nabla T_\varepsilon^4)\right)\\ & + \varepsilon^3(-\beta\cdot\nabla(\beta\cdot\nabla(\beta\cdot\nabla\psi_\varepsilon)) + \beta\cdot\partial_t\psi_\varepsilon + \beta\cdot\nabla\partial_t\psi_\varepsilon)\\ & + \varepsilon^4(-\beta\cdot\nabla(\beta\cdot\nabla\partial_t\psi_\varepsilon) + \partial_t^2\psi_\varepsilon).\end{aligned} \tag{15}$$

Assuming $T_\varepsilon \in C^2_{t,x}$ and $\psi_\varepsilon \in C^3_{t,x,\beta}$ are bounded, and assuming

$$T_\varepsilon \to \overline{T}, \quad \psi_\varepsilon \to \overline{\psi} \quad \text{as } \varepsilon \to 0,$$

we can pass to the limit $\varepsilon \to 0$ in (15) and get

$$\overline{\psi} = \overline{T}^4.$$

We can also use (15) to find that the radiative density $\langle\psi_\varepsilon\rangle$ satisfies

$$\begin{aligned}\langle\psi_\varepsilon\rangle = {} & 4\pi T_\varepsilon^4 - \varepsilon^2 4\pi\partial_t T_\varepsilon^4 + \varepsilon^2\frac{4\pi}{3}\Delta T_\varepsilon^4\\ & + \varepsilon^3\langle(-\beta\cdot\nabla(\beta\cdot\nabla(\beta\cdot\nabla\psi_\varepsilon)) + \beta\cdot\partial_t\psi_\varepsilon + \beta\cdot\nabla\partial_t\psi_\varepsilon)\rangle\\ & + \varepsilon^4\langle(-\beta\cdot\nabla(\beta\cdot\nabla\partial_t\psi_\varepsilon) + \partial_t^2\psi_\varepsilon)\rangle.\end{aligned}$$

This enables us to pass to the limit on the last term in (5):

$$\frac{1}{\varepsilon^2}(\langle\psi_\varepsilon - T_\varepsilon^4\rangle) \to -4\pi\partial_t\overline{T}^4 + \frac{4\pi}{3}\Delta\overline{T}^4.$$

We can then pass to the limit in (5) and derive the nonlinear limit system

$$\partial_t\left(\overline{T} + 4\pi\overline{T}^4\right) = \Delta\left(\overline{T} + \frac{4\pi}{3}\overline{T}^4\right), \tag{16}$$

$$\overline{T}(t=0,x) = \overline{T}_0(x) = \lim_{\varepsilon\to 0} T_{\varepsilon 0}(x), \quad x \in \Omega, \tag{17}$$

associated with suitable boundary conditions which will be given in section 3.

1.2. Main results of the paper. Before introducing our main results in this work, we start by giving some assumptions on the initial and boundary values.

- Well-prepared initial conditions,

$$\lim_{\varepsilon\to 0}(\psi_{\varepsilon 0}(x,\beta) - T_{\varepsilon 0}^4) = 0, \text{ for all } x \in \Omega,\ \beta \in \mathbb{S}^2, \tag{18}$$

- Well-prepared boundary conditions in the case of Dirichlet boundary condition (12), namely

$$\psi_b(t,x,\beta) = T_b^4(t,x), \text{ for all } t > 0, \text{ and } (x,\beta) \in \Sigma_-. \tag{19}$$

  Notice that for the case of Robin boundary condition (13), the well-prepared boundary condition assumption is not needed. The case of general boundary conditions will be discussed in [26, 24, 25].

We now state the main results in the following theorem.

Theorem 1. *Suppose the positive initial data satisfy $T_{\varepsilon 0} \in L^5(\Omega), \psi_{\varepsilon 0} \in L^2(\Omega \times \mathbb{S}^2)$ and assumption (18), and the positive boundary data satisfy $T_b \in L^5_{\mathrm{loc}}([0,\infty); L^5(\partial\Omega))$ and $\psi_b \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Sigma_-; |n\cdot\beta| d\beta d\sigma_x))$ and assumption (19). Then the following statements hold.*

(1) *Existence of weak solution: There exists a weak nonnegative solution* $(T_\varepsilon, \psi_\varepsilon) \in L^2_{\text{loc}}(0,\infty; L^5(\Omega)) \cap L^\infty_{\text{loc}}(0,\infty; L^2(\Omega \times \mathbb{S}^2))$ *for the system* (5)–(6) *with initial conditions* (7)–(8) *and boundary condition* (9) *for* ψ_ε *with boundary condition* (11)*,* (12)*, or* (13) *for* T_ε*.*

(2) *Diffusive limit: As* $\varepsilon \to 0$*, the weak solution* $(T_\varepsilon, \psi_\varepsilon)$ *to the system* (5)–(6) *converges to* $(\overline{T}, \overline{T}^4)$*, where* $\overline{T}$ *is the weak solution of the system* (16) *with boundary conditions that* $\Omega = \mathbb{T}^3$ *for the case* (A) *and* $\overline{T} = T_b$ *on the boundary* $\partial\Omega$ *for the case* (B) *and* (C)*.*

(3) *Rate of convergence: Assume* $\overline{T}$ *is a strong solution to the system* (16) *which has a positive lower bound. Then*

$$\|T_\varepsilon(t) - \overline{T}(t)\|^4_{L^4(\Omega)} + \|\psi_\varepsilon(t) - \overline{\psi}(t)\|^2_{L^2(\Omega)} \le C\|T_{\varepsilon 0} - \overline{T}_0\|^4_{L^4(\Omega)} + C\varepsilon^s,$$

where $s > 0$ *is a positive constant and takes the value* $s = 2, 1, \min(1, r)$ *for the case of boundary conditions* (11)*,* (12)*,* (13)*, respectively.*

The main contribution of the present work is to give a more rigorous study of the radiative heat transfer system and its diffusive limit. We prove the global existence of weak solutions for the system and the convergence of the weak solutions to a nonlinear diffusion model under the diffusive limit. Our work extends the analysis made by Klar and Schmeiser in [35], where the existence and diffusive limit were established for smooth solutions. In their work, some extra assumptions on the solutions (which are not known to hold) were needed. Here we do not need these assumptions. The major difficulties in our work lie in the nonlinearity and lack of compactness of the system (5)–(6). To overcome the difficulties, we use Young measure theory and averaging lemmas. The Young measure is applied to deal with the nonlinearity and the averaging lemma is applied to get the compactness. The diffusive limit can thus be rigorously justified. Assuming additional regularity on the limit system, the relative entropy method can be used to give the rate of convergence for the diffusive limit for the boundary conditions (11), (12), and (13) with $r > 0$. The case $r = 0$ can only be treated using the weak compactness method.

A lot of literature is devoted to the mathematical analysis and numerical computations of the radiative heat transfer system [36, 28, 46, 9, 23, 29]. Besides the work [35] on the same model considered here, there are some works on similar models [47, 32, 1, 2, 27]. For example, the existence and uniqueness of strong solutions for the nongray coupled convection-conduction radiation system were proved in [47] using accretive operators theory. In [32], the authors discussed the existence of weak solutions for a gray radiative transfer system without diffusion term in the temperature equation in a bounded domain with nonhomogeneous Dirichlet boundary conditions. They supposed that the radiative boundary data do not depend on the direction, in order to avoid the boundary layer. The main tools used to prove the existence of weak solutions are the compactness argument based on a maximum principle and the velocity averaging lemma. Furthermore, the existence and uniqueness of a weak solution for the stationary nonlinear heat equation and the integro-differential radiative transfer equation for semitransparent bodies were studied in [1], where the authors took into account the effects of reflection and refraction of radiation according to the Fresnel laws at the boundaries of bodies. More recently, in [27], the authors proved the local existence and uniqueness of strong solutions for the radiative heat transfer system under different types of boundary conditions by using the Banach fixed point theorem. The time derivative term in the radiative transfer equation (6) was also neglected therein. Besides analysis, there is a lot of literature on the applications on the radiative transfer models. For example, a radiative transfer system without thermal diffusion was used in [7] to study the greenhouse effect, where the authors found that the model was not capable of explaining the greenhouse effect due to the greenhouse gases. Note here our model (5)–(6) includes thermal diffusion. A more complicated model including thermal diffusion and coupled with fluid equations was studied in [31] to model greenhouse effects.

The diffusion limit in a radiative heat transfer system can be studied via Rosseland approximations [5, 6]. In [5], the authors derived the Rosseland approximation on a different radiative transfer equation where the solution also depends on the frequency variable ν. Using the so-called Hilbert's expansion method, they proved the strong convergence of the solution of the radiative transfer equation to the solution of the Rosseland equation for well-prepared boundary data. Then, in [6], under some weak hypotheses on the various parameters of the radiative transfer equation, the Rosseland approximation was justified in a weak sense. More recently, in [13, 14], the authors studied the diffusive limit of a stochastic kinetic radiative transfer equation, which is nonlinear and includes a smooth random term. They used a stochastic averaging lemma to show the convergence in distribution to a stochastic nonlinear fluid model. Moreover, there exists a wide literature on the diffusion limits for other kinds of kinetic systems, with various viewpoints and applications [16, 12, 11, 8, 41, 18, 48, 20, 37]. For example, in [41], the authors studied the diffusive limit of a semiconductor Boltzmann–Poisson system. The method of moments and a velocity averaging lemma were used to prove the convergence of its renormalized solution toward a global weak solution of a drift-diffusion-Poisson model. Similar methods have been used to study the hydrodynamic limit of the Boltzmann equation [48, 39, 40]. The hydrodynamic limit of the Boltzmann equation can also be studied using the relative entropy method, for example, to show the incompressible limit to Euler and Navier–Stokes equations [48]. The origins of the relative entropy method come from continuum mechanics; see [11] for more details. The principle of this method is to measure in a certain way the distance between two solutions in some given space. This method was also used in the stability and asymptotic limit for different types of PDEs; for instance, see [19, 15, 37, 51].

The paper is organized as follows. In the next section, the Galerkin approximation is used to show the existence of a global weak solution of the radiative heat transfer system. Then we prove the convergence of the weak solutions to the nonlinear parabolic equation (16) in the diffusive limit in section 3, by using the averaging lemma and the theory of Young measures. Moreover, we recover the boundary condition for the nonlinear parabolic limit equation by using trace theorems. In section 4, we give the convergence rate of the diffusive limit by using the relative entropy method.

Notations. In this paper, we use $\|\cdot\|_{L^p}$ to denote the natural norm on $L^p(\Omega)$, for $p \in [1,\infty]$, and $\|\cdot\|_{H^s}$ is the norm on the Sobolev space $H^s(\Omega)$, $s > 0$. We use $\langle\cdot\rangle$ to denote the integral over $\beta \in \mathbb{S}^2$. $C_{t,x}$ is the space of continuous functions in time and space. Let B be a Banach space. The space $C_w([0,\infty);B)$ is defined by time continuous functions in the weak topology, i.e., $f \in C_w([0,\infty);B)$ if and only if (f,ϕ) is time continuous for any $\phi \in B^*$ where B^* is the dual space of B and $(\cdot,\cdot)$ is the inner product.

2. Global existence of weak solutions.

In this section we prove the global existence of weak solutions for the radiative heat transfer system (5)–(6) under three different boundary conditions: torus, nonhomogeneous Dirichlet condition, and Robin condition. We first consider the case of torus, i.e., $\Omega = \mathbb{T}^3$.

2.1. The case of torus. We first prove the existence theorem for the case of torus. The idea of the proof can be modified to deal with bounded domain, which will be done later in this section. Before stating the existence theorem, we first introduce the definition of weak solutions.

DEFINITION 2. *Let $0 \le T_{\varepsilon 0} \in L^5(\mathbb{T}^3)$ and $0 \le \psi_{\varepsilon 0} \in L^2(\mathbb{T}^3 \times \mathbb{S}^2)$. We say that $(T_\varepsilon, \psi_\varepsilon)$ is a nonnegative weak solution of the system (5)–(6) with initial conditions (7)–(8) if*

$$T_\varepsilon \in L^\infty(0,\infty; L^5(\mathbb{T}^3)) \cap C_w([0,\infty); L^5(\mathbb{T}^3)),\ \nabla T_\varepsilon^{\frac{5}{2}} \in L^2([0,\infty); L^2(\mathbb{T}^3)), \tag{20}$$

$$\psi_\varepsilon \in L^\infty(0,\infty; L^2(\mathbb{T}^3 \times \mathbb{S}^2)) \cap C_w([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)), \tag{21}$$

and it solves (5)–(6) in the sense of distributions, i.e., for any test functions $\varphi \in C^\infty([0,\infty) \times \mathbb{T}^3)$ and $\rho \in C^\infty([0,\infty) \times \mathbb{T}^3 \times \mathbb{S}^2)$, the following equations hold:

$$-\iint_{[0,\infty)\times\mathbb{T}^3} \left(T_\varepsilon \partial_t \varphi + T_\varepsilon \Delta\varphi + \frac{1}{\varepsilon^2} \int_{\mathbb{S}^2} \varphi(\psi_\varepsilon - T_\varepsilon^4) d\beta \right) dxdt = \int_{\mathbb{T}^3} T_{\varepsilon 0} \varphi(0,\cdot) dx, \tag{22}$$

$$-\iiint_{[0,\infty)\times\mathbb{T}^3\times\mathbb{S}^2} \left(\psi_\varepsilon \partial_t \rho + \frac{1}{\varepsilon} \psi_\varepsilon \beta \cdot \nabla\rho - \frac{1}{\varepsilon^2} \rho(\psi_\varepsilon - T_\varepsilon^4) \right) d\beta dxdt$$
$$= \iint_{\mathbb{T}^3\times\mathbb{S}^2} \psi_{\varepsilon 0} \rho(0,\cdot,\cdot) d\beta dx. \tag{23}$$

Next we prove the following existence theorem.

THEOREM 3. *Let $0 \le T_{\varepsilon 0} \in L^5(\mathbb{T}^3)$ and $0 \le \psi_{\varepsilon 0} \in L^2(\mathbb{T}^3 \times \mathbb{S}^2)$. Then there exists a global nonnegative weak solution $(T_\varepsilon, \psi_\varepsilon)$ to the system (5)–(6) with initial data (7)–(8). Moreover the following energy inequality holds for all $t > 0$:*

$$\begin{aligned} &\frac{1}{5}\|T_\varepsilon(t,\cdot)\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\psi_\varepsilon(t,\cdot,\cdot)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 + \frac{16}{25}\int_0^t \|\nabla T_\varepsilon^{\frac{5}{2}}(\tau,\cdot)\|_{L^2}^2 d\tau \\ &\quad + \frac{1}{\varepsilon^2}\int_0^t \|\psi_\varepsilon(\tau,\cdot,\cdot) - T_\varepsilon^4(\tau,\cdot)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 d\tau \\ &\le \frac{1}{5}\|T_{\varepsilon 0}\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\psi_{\varepsilon 0}\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2. \end{aligned} \tag{24}$$

Proof. To prove Theorem 3, we construct an approximate system using Galerkin approximations in finite dimensions, and then show the system converges as the dimension goes to infinity with the limit satisfying (22)–(23).

Construction of a Galerkin approximate system. We first construct a finite dimensional approximation to the system (5)–(6) using Fourier series. We take the Fourier series of an L^s $(s \ge 1)$ function to be

$$f(x) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{ik\cdot x}$$

and define the operator $\mathbb{P}_m : L^s \mapsto L^s$ $(s \ge 1)$ as

$$\mathbb{P}_m f(x) = \sum_{|k|\le m} \hat{f}(k) e^{ik\cdot x}.$$

Notice that $\mathbb{P}_m$ commutes with derivatives and convolutions. For any function $g = g(x,\beta)$ defined on $\mathbb{T}^3 \times \mathbb{S}^2$,

$$\mathbb{P}_m g(x,\beta) = \sum_{|k|\le m} \hat{g}(k,\beta) e^{ik\cdot x}.$$

From Lemma 17 in Appendix C, there exists a complete basis $\{\phi_l(x)\chi_s(\beta)\}_{l,s=1}^{\infty}$ of $L^2(\mathbb{T}^3 \times \mathbb{S}^2)$. Define $\mathbb{Q}_m = \sum_{l=1}^{m_1}\sum_{s=1}^{m_2} \phi_l \chi_s$. Note that here for simplicity of notation, we take m as the superscripts. In the below $m \to \infty$ means $m \to \infty$ in $\mathbb{P}_m$ and m_1, $m_2 \to \infty$ in $\mathbb{Q}_m$. We take the mth Galerkin approximate system to be

$$\partial_t T_\varepsilon^m = \Delta T_\varepsilon^m + \frac{1}{\varepsilon^2}\int_{\mathbb{S}^2} \left(\psi_\varepsilon^m - \mathbb{P}_m\left((T_\varepsilon^m)^4\right)\right) d\beta, \tag{25}$$

$$\partial_t \psi_\varepsilon^m + \frac{1}{\varepsilon}\beta\cdot\nabla\psi_\varepsilon^m = -\frac{1}{\varepsilon^2}\left(\psi_\varepsilon^m - \mathbb{Q}_m\mathbb{P}_m\left((T_\varepsilon^m)^4\right)\right). \tag{26}$$

The initial data are taken to be

$$T_\varepsilon^m\big|_{t=0} = \mathbb{P}_m T_{\varepsilon 0}, \quad \psi_\varepsilon^m\big|_{t=0} = \mathbb{P}_m \psi_{\varepsilon 0}.$$

The above system then becomes an ODE in finite dimensional space. Therefore, system (25)–(26) has a unique solution $(T_\varepsilon^m, \psi_\varepsilon^m)$ on a maximal time interval $(0, t_m)$, according to the Cauchy–Lipschitz theorem. The maximal existence time t_m is characterized by

$$\lim_{t\to t_m^-} \sup\left(\|T_\varepsilon^m\|_{L^5(\mathbb{T}^3)}^5 + \|\psi_\varepsilon^m\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2\right) = \infty.$$

As will be seen next, the norms above are bounded uniformly in time and so the Galerkin approximate system (25)–(26) is globally well-posed. Note that the solutions to (25)–(26) are nonnegative, which is proved in Appendix A.

Uniform estimate of the Galerkin system. Next we derive the energy estimate for the system (25)–(26). Multiplying (25) by $(T_\varepsilon^m)^4$ and (26) by ψ_ε^m, adding the results together, integrating over $\mathbb{T}^3\times\mathbb{S}^2$, and using the fact that $\mathbb{P}_m$ is a self-adjoint operator, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{5}\|T_\varepsilon^m\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\psi_\varepsilon^m\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2\right) + \frac{16}{25}\|\nabla(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2(\mathbb{T}^3)}^2 \\ + \frac{1}{\varepsilon^2}\|\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 = \frac{1}{5}\|\mathbb{P}_m T_{\varepsilon 0}\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\mathbb{P}_m\psi_{\varepsilon 0}\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2.\end{aligned}$$

Integrating it over $[0,t]$ and using the inequality $\|\mathbb{P}_m f\|_{L^s} \le \|f\|_{L^s}$, we obtain the energy inequality

$$\begin{aligned}&\frac{1}{5}\|T_\varepsilon^m(t)\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\psi_\varepsilon^m(t)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 + \frac{16}{25}\int_0^t \|\nabla(T_\varepsilon^m)^{\frac{5}{2}}(\tau)\|_{L^2(\mathbb{T}^3)}^2 d\tau \\ &\quad + \frac{1}{\varepsilon^2}\int_0^t \|\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)(\tau)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 d\tau \\ &\quad \le \frac{1}{5}\|T_{\varepsilon 0}\|_{L^5(\mathbb{T}^3)}^5 + \frac{1}{2}\|\psi_{\varepsilon 0}\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2\end{aligned} \tag{27}$$

for all $t > 0$.

It follows from the above energy inequality that, up to a subsequence,

(28)
$$\{T_\varepsilon^m\}_{m>0} \text{ is uniformly bounded in } L^\infty([0,\infty);L^5(\mathbb{T}^3)),$$
(29)
$$\{\nabla(T_\varepsilon^m)^{\frac{5}{2}}\}_{m>0} \text{ is uniformly bounded in } L^2([0,\infty);L^2(\mathbb{T}^3)),$$
(30)
$$\{\psi_\varepsilon^m\}_{m>0} \text{ is uniformly bounded in } L^\infty([0,\infty);L^2(\mathbb{T}^3\times\mathbb{S}^2)),$$
(31)
$$\left\{\frac{1}{\varepsilon}\left(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\right)\right\}_{m>0} \text{ is uniformly bounded in } L^2([0,\infty);L^2(\mathbb{T}^3\times\mathbb{S}^2)).$$

Using (29) and the Sobolev inequality on the periodic domain

$$\|(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^6(\mathbb{T}^3)} \le C\|\nabla(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2},$$

we have

$$\begin{aligned}\int_0^t \|T_\varepsilon^m\|_{L^{15}(\mathbb{T}^3)}^5 d\tau &= \int_0^t \left(\int_{\mathbb{T}^3}(T_\varepsilon^m)^{15}dx\right)^{\frac{1}{3}} d\tau = \int_0^t \left(\int_{\mathbb{T}^3}((T_\varepsilon^m)^{\frac{5}{2}})^6 dx\right)^{\frac{2}{6}} d\tau \\ &= \int_0^t \|(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^6(\mathbb{T})}^2 d\tau \le C\int_0^t \|\nabla(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2}^2 d\tau,\end{aligned}$$

which is bounded. Moreover, from the embedding $L^6(\mathbb{T}^3)\subset L^5(\mathbb{T}^3)$ and the boundedness of $\|(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2([0,t];L^5(\mathbb{T}^3\times\mathbb{S}^2))}$, we can deduce using Littlewood's inequality $\|f\|_{L^8} \le \|f\|_{L^5}^{\frac{3}{8}}\|f\|_{L^{\frac{25}{2}}}^{\frac{5}{8}}$,

$$\begin{aligned}\int_0^t \|T_\varepsilon^m\|_{L^8(\mathbb{T}^3)}^8 d\tau &\le \int_0^t \|T_\varepsilon^m\|_{L^5(\mathbb{T}^3)}^3 \|T_\varepsilon^m\|_{L^{\frac{25}{2}}(\mathbb{T}^3)}^5 d\tau \\ &\le \|T_\varepsilon^m\|_{L^\infty([0,t];L^5(\mathbb{T}^3))}^3 \|T_\varepsilon^m\|_{L^5([0,t];L^{\frac{25}{2}}(\mathbb{T}^3))}^5 \\ &= \|T_\varepsilon^m\|_{L^\infty([0,t];L^5(\mathbb{T}^3))}^3 \|(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2([0,t];L^5(\mathbb{T}^3))}^2.\end{aligned}$$

Therefore,

(32)
$$\{(T_\varepsilon^m)^{\frac{5}{2}}\}_{m>0} \text{ is uniformly bounded in } L^2_{\mathrm{loc}}([0,\infty);L^6(\mathbb{T}^3)),$$
(33)
$$\{T_\varepsilon^m\}_{m>0} \text{ is uniformly bounded in } L^5_{\mathrm{loc}}([0,\infty);L^{15}(\mathbb{T}^3))\cap L^8_{\mathrm{loc}}([0,\infty);L^8(\mathbb{T}^3)).$$

Passing to the limit in the Galerkin system. From the energy estimate (27), we have

$$\begin{aligned}&\partial_t T_\varepsilon^m \text{ is uniformly bound in } L^2_{\mathrm{loc}}([0,\infty);H^{-2}(\mathbb{T}^3)),\\ &\partial_t \psi_\varepsilon^m \text{ is uniformly bound in } L^2_{\mathrm{loc}}([0,\infty);H^{-1}(\mathbb{T}^3));\end{aligned}$$

these, together with (28) and (29), by the Aubin–Lions lemma, imply that

$$T_\varepsilon^m \to T_\varepsilon, \text{ strongly in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3)), \tag{34}$$

$$\psi_\varepsilon^m \to \psi_\varepsilon, \text{ strongly in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)), \tag{35}$$

$$\partial_t T_\varepsilon^m \rightharpoonup \partial_t T_\varepsilon, \text{ weakly in } L^2_{\text{loc}}([0,\infty); H^{-2}(\mathbb{T}^3)), \tag{36}$$

$$\partial_t \psi_\varepsilon^m \rightharpoonup \partial_t \psi_\varepsilon, \text{ weakly in } L^2_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^3 \times \mathbb{S}^2)). \tag{37}$$

Therefore, the solution to the Galerkin approximate system converges. From the Sobolev embedding $C^{1/2}(\mathbb{R}) \subset H^1(\mathbb{R})$ and the above convergences, we have $\partial_t T_\varepsilon \in C^{1/2}_{\text{loc}}([0,\infty); H^{-2}(\mathbb{T}^3))$ and $\partial_t \psi_\varepsilon \in C^{1/2}_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^3 \times \mathbb{S}^2))$. This together with the fact that H^{-p}, $p \geq 1$, is dense in L^2 by using a standard density argument implies

$$T_\varepsilon \in C_w([0,\infty); L^2(\mathbb{T}^3)), \quad \psi_\varepsilon \in C_w([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)). \tag{38}$$

This means that T_ε and ψ_ε are weakly continuous with values in $L^5(\mathbb{T}^3)$ and $L^2(\mathbb{T}^3 \times \mathbb{S}^2)$, respectively.

Using (28)–(31) and (32)–(33), there exist subsequences $\{T_\varepsilon^{m_k}\}_{k>0}$ and $\{\psi_\varepsilon^{m_k}\}_{k>0}$ such that

$$\begin{aligned}
&T_\varepsilon^{m_k} \rightharpoonup T_\varepsilon, \text{ weakly in} \\
&\qquad L^2_{\text{loc}}([0,\infty); L^5(\mathbb{T}^5)) \cap L^5_{\text{loc}}([0,\infty); L^{15}(\mathbb{T}^3)) \cap L^8_{\text{loc}}([0,\infty); L^8(\mathbb{T}^3)),
\end{aligned} \tag{39}$$

$$T_\varepsilon^{m_k} \rightharpoonup^* T_\varepsilon, \text{ weakly star in } L^\infty([0,\infty); L^5(\mathbb{T}^5)), \tag{40}$$

$$(T_\varepsilon^{m_k})^{\frac{5}{2}} \rightharpoonup \overline{(T_\varepsilon^{m_k})^{\frac{5}{2}}}, \text{ weakly in } L^2_{\text{loc}}([0,\infty); H^1(\mathbb{T}^3)), \tag{41}$$

$$\psi_\varepsilon^{m_k} \rightharpoonup \psi_\varepsilon, \text{ weakly in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)), \tag{42}$$

$$\psi_\varepsilon^{m_k} \rightharpoonup^* \psi_\varepsilon, \text{ weakly star in } L^\infty([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)), \tag{43}$$

$$\frac{1}{\varepsilon}(\psi_\varepsilon^{m_k} - \mathbb{P}_m((T_\varepsilon^{m_k})^4)) \rightharpoonup A, \text{ weakly in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)), \tag{44}$$

as $k \to \infty$. Here, $\overline{(T_\varepsilon^{m_k})^{\frac{5}{2}}}$ denotes the weak limit of $(T_\varepsilon^{m_k})^{\frac{5}{2}}$. Note that here $\|A\|_{L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2))}$ is bounded. Due to the property of the operator $\mathbb{P}_m$,

$$\|(T_\varepsilon^m)^4 - \mathbb{P}_m((T_\varepsilon^m)^4)\|_{L^2(\mathbb{T}^3)} \to 0,$$

as $m \to \infty$ due to the property of the Fourier series, so by (44), we can conclude that

$$\frac{1}{\varepsilon}(\psi_\varepsilon^{m_k} - (T_\varepsilon^{m_k})^4) \rightharpoonup A, \text{ weakly in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^3 \times \mathbb{S}^2)). \tag{45}$$

The limit satisfies the system (5)–(6). To show the limit satisfies the system (5)–(6) in the sense of distributions, we apply test functions on (25)–(26). We take the convergence subsequence obtained in the previous step. Here we will drop the subscript k for simplicity. Fixing $t > 0$ and applying smooth function $\varphi \in C^\infty([0,t] \times \mathbb{T}^3)$ and $\rho \in C^\infty([0,t] \times \mathbb{T}^3 \times \mathbb{S}^2)$ to (25) and (26), respectively, we arrive at

$$
\begin{aligned}
&\int_{\mathbb{T}^3} T_\varepsilon^m(t)\cdot\varphi(t)dx - \int_0^t\int_{\mathbb{T}^3} T_\varepsilon^m \partial_t\varphi dx d\tau - \int_0^t\int_{\mathbb{T}^3} T_\varepsilon^m \Delta\varphi dx d\tau \\
&\quad - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \frac{1}{\varepsilon^2}\varphi(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4))d\beta dx d\tau = \int_{\mathbb{T}^3} \mathbb{P}_m T_{\varepsilon 0}\cdot\varphi(0)dx,
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&\iint_{\mathbb{T}^3\times\mathbb{S}^2} \psi_\varepsilon^m(t)\cdot\rho(t)d\beta dx - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \psi_\varepsilon^m \partial_t\rho d\beta dx d\tau \\
&\quad - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \frac{1}{\varepsilon}\psi_\varepsilon^m \beta\cdot\nabla\rho d\beta dx d\tau - \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \frac{1}{\varepsilon^2}\rho(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4))d\beta dx d\tau \\
&\quad = \iint_{\mathbb{T}^3\times\mathbb{S}^2} \mathbb{P}_m\psi_{\varepsilon 0}\cdot\rho(0)d\beta dx.
\end{aligned} \tag{47}
$$

From the property of the operator $\mathbb{P}_m$, $\|f-\mathbb{P}_m f\|_{L^2}\to 0$ as $m\to\infty$, we get

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \mathbb{P}_m T_{\varepsilon 0}\cdot\varphi(0)dx \to \int_{\mathbb{T}^3} T_{\varepsilon 0}\cdot\varphi(0)dx,\\
&\iint_{\mathbb{T}^3\times\mathbb{S}^2} \mathbb{P}_m\psi_{\varepsilon 0}\cdot\rho(0)d\beta dx \to \iint_{\mathbb{T}^3\times\mathbb{S}^2} \psi_{\varepsilon 0}\cdot\rho(0)d\beta dx.
\end{aligned}
$$

For the terms involving $(T_\varepsilon^m)^4$, we have

$$
\begin{aligned}
&\int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \varphi(\mathbb{P}_m((T_\varepsilon^m)^4) - T_\varepsilon^4)d\beta dx d\tau \\
&\quad = \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \varphi(\mathbb{P}_m((T_\varepsilon^m)^4) - \mathbb{P}_m T_\varepsilon^4 + \mathbb{P}_m T_\varepsilon^4 - T_\varepsilon^4)d\beta dx d\tau \\
&\quad = \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \mathbb{P}_m\varphi\cdot((T_\varepsilon^m)^4 - T_\varepsilon^4)d\beta dx d\tau + \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} T_\varepsilon^4(\mathbb{P}_m\varphi - \varphi)d\beta dx d\tau.
\end{aligned}
$$

Due to (39), $T_\varepsilon \in L^8_{\mathrm{loc}}([0,\infty);L^8(\mathbb{T}^3))$, hence

$$
\int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} T_\varepsilon^4(\mathbb{P}_m\varphi - \varphi)d\beta dx d\tau \le C\|T_\varepsilon\|^4_{L^8([0,t];L^8(\mathbb{T}^3))}\|\mathbb{P}_m\varphi-\varphi\|_{L^2([0,t];L^2(\mathbb{T}^3))},
$$

which goes to zero as $m\to\infty$. Moreover, the uniformly boundness of $T_\varepsilon^m \in L^8_{\mathrm{loc}}([0,\infty);L^8(\mathbb{T}^3))$ also implies $(T_\varepsilon^m)^4 \in L^2_{\mathrm{loc}}([0,\infty);L^2(\mathbb{T}^3))$ and

$$
(T_\varepsilon^m)^4 \rightharpoonup \overline{(T_\varepsilon^m)^4}, \quad \text{weakly in } L^2_{\mathrm{loc}}([0,\infty);L^2(\mathbb{T}^3)).
$$

Because of this and the strong convergence of (T_ε^m) in $L^2_{\mathrm{loc}}([0,\infty);L^2(\mathbb{T}^3))$,

$$
(T_\varepsilon^m)^4 T_\varepsilon^m \rightharpoonup \overline{(T_\varepsilon^m)^4}T_\varepsilon \quad \text{converges weakly in the sense of distributions.}
$$

Since T^4 is an increasing function of $T\in\mathbb{R}_+$, we can thus use the Minty's trick (see [43]) and conclude that $\overline{(T_\varepsilon^m)^4} = T_\varepsilon^4$. Hence

$$
(T_\varepsilon^m)^4 \rightharpoonup T_\varepsilon^4, \quad \text{weakly in } L^2_{\mathrm{loc}}([0,\infty);L^2(\mathbb{T}^3)),
$$

which implies

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \mathbb{P}_m\varphi\cdot((T_\varepsilon^m)^4 - T_\varepsilon^4)d\beta dx d\tau \to 0, \qquad \text{as } m\to\infty.$$

Therefore,

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \varphi(\mathbb{P}_m((T_\varepsilon^m)^4) - T_\varepsilon^4)d\beta dx d\tau \to 0, \qquad \text{as } m\to\infty. \tag{48}$$

We also have from the property of $\mathbb{P}_m$,

$$\int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \rho\mathbb{P}_m((T_\varepsilon^m)^4)d\beta dx d\tau \to \int_0^t \iint_{\mathbb{T}^3\times\mathbb{S}^2} \rho T_\varepsilon^4 d\beta dx d\tau.$$

Last, from (34) and (35), we have

$$\begin{aligned}&\int_{\mathbb{T}^d} T_\varepsilon^m(t)\varphi(t)dx \to \int_{\mathbb{T}^d} T_\varepsilon(t)\varphi(t)dx,\\ &\int_{\mathbb{T}^d} \psi_\varepsilon^m(t)\rho(t)dx \to \int_{\mathbb{T}^d} \psi_\varepsilon(t)\rho(t)dx.\end{aligned}$$

Notice that the weak continuity (38) gets rid of the possible bad zero measure set in time.

Using (34) and (35), we can pass to the limit in the other terms in (46) and (47). Finally we arrive at (22) and (23). Thus, for any $t > 0$, $(T_\varepsilon,\ \psi_\varepsilon)$ solves the system (5)–(6) in the sense of distributions and satisfies (20)–(21).

The energy inequality. To show the energy inequality, we consider the inequality (27). Note that since we have the strong convergence of $T_\varepsilon^m, \psi_\varepsilon^m$ according to (34) and (35), we can take a strong convergence subsequence to recover the energy estimate. The weak star convergences in (40) and (43) imply that

$$\|T_\varepsilon(t)\|_{L^5(\mathbb{T}^3)}^5 \le \limsup_{m\to\infty} \|T_\varepsilon^m(t)\|_{L^5(\mathbb{T}^3)}^5, \quad \|\psi_\varepsilon(t)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 \le \limsup_{m\to\infty} \|\psi_\varepsilon^m(t)\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2. \tag{49}$$

The weak convergence in (45) implies that

$$\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4) \rightharpoonup \psi_\varepsilon - T_\varepsilon^4 \quad \text{in} \quad L^2([0,t];L^2(\mathbb{T}^3\times\mathbb{S}^2)).$$

Therefore,

$$\int_0^t \|\psi_\varepsilon - T_\varepsilon^4\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 d\tau \le \liminf_{m\to\infty} \int_0^t \|\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m))^4\|_{L^2(\mathbb{T}^3\times\mathbb{S}^2)}^2 d\tau. \tag{50}$$

From (29),

$$\int_0^t \|\nabla(T_\varepsilon)^{\frac{5}{2}}\|_{L^2(\mathbb{T}^3)}^2 d\tau \le \liminf_{m\to\infty} \int_0^t \|\nabla(T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2(\mathbb{T}^3)}^2 d\tau. \tag{51}$$

Taking the $\limsup_{m\to\infty}$ in the energy inequality (27) and using the above estimates, we arrive at (24) and finish the proof. ∎

2.2. Case of nonhomogeneous Dirichlet boundary condition. We now consider the case of nonhomogeneous Dirichlet boundary condition (12) under the assumption of well-prepared initial and boundary conditions. For simplicity, here we assume the boundary data T_b and ψ_b are time independent such that $T_b = T_b(x)$ and $\psi_b = \psi_b(x)$. The case of time dependent boundary can be treated similarily. Before we give the definition of weak solutions, we introduce the trace operators that extend the functions in Sobolev spaces to the boundary.

We take $\gamma^1 : H^1(\Omega) \to L^2(\partial\Omega)$ to be the trace operator. From the trace theorem (see [21]) that if Ω is bounded and $\partial\Omega \in C^1$, there exists a trace operator γ^1 such that

$$\gamma^1 f = f|_{\partial\Omega}, \text{ if } f \in H^1(\Omega),$$

and

$$\|\gamma^1 f\|_{L^2(\partial\Omega)} \le C\|f\|_{H^1(\Omega)}.$$

Here the constant C only depends on Ω. For the weak formulation of (5), we can apply the trace operator on T_ε to get

$$\gamma^1 T_\varepsilon = T_b.$$

To consider the boundary condition (9), we define the trace operator following [48, Appendix (B.1)] as

$$\gamma^2 : \psi \in W^2(\mathbb{R}_+ \times \Omega \times \mathbb{S}^2) \mapsto \psi|_\Sigma \in L^2(\mathbb{R}_+ \times \partial\Omega \times \mathbb{S}^2, |n\cdot\beta|^2 d\beta d\sigma_x dt), \tag{52}$$

where the space $L^2(\mathbb{R}_+ \times \partial\Omega \times \mathbb{S}^2, |n\cdot\beta|^2 d\beta d\sigma_x dt)$ is endowed with the norm

$$\|\psi|_\Sigma\|^2_{L^2(\mathbb{R}_+\times\partial\Omega\times\mathbb{S}^2,|n\cdot\beta|^2 d\beta d\sigma_x dt)} = \int_{\mathbb{R}_+}\iint_{\partial\Omega\times\mathbb{S}^2} |n\cdot\beta|^2\psi^2 d\beta d\sigma_x dt.$$

Here W^2 is the space

$$W^2(\mathbb{R}\times\Omega\times\mathbb{S}^2) := \{\psi \in L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2) : (\varepsilon\partial_t + \beta\cdot\nabla)\psi \in L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2)\}. \tag{53}$$

The following lemma was proved in [48, Proposition B.1].

LEMMA 4. *The trace operator defined in* (52) *is continuous.*

Proof. For any bounded function $\rho \in C^1(\bar\Omega \times \mathbb{S}^2)$, we use Green's formula to get

$$\begin{aligned} 2\iiint \rho(x,\beta)\psi(\varepsilon\partial_t + \beta\cdot\nabla_x)\psi d\beta dx dt + \iiint (\beta\cdot\nabla_x\rho(x,\beta))\psi^2 d\beta dx dt \\ + \iiint \rho(x,\beta)\psi^2(0,x,\beta) d\beta d\sigma_x dt = \iiint \rho(x,\beta)\psi^2(t,x,\beta)(n\cdot\beta) d\beta d\sigma_x dt. \end{aligned} \tag{54}$$

Since $\partial\Omega$ is smooth, there exists some vector $n = n(x) \in W^{1,\infty}(\overline{\Omega})$ coinciding with the exterior normal vector on the boundary. Therefore, we can take $\rho(x,\beta) = n(x)\cdot\beta$ and get

$$\begin{aligned} &\|\psi|_\Sigma\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2,|n\cdot\beta|^2 d\beta d\sigma_x dt)} \\ &\quad \le C(\|\psi\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2)} + \|(\varepsilon\partial_t + \beta\cdot\nabla_x)\psi)\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2)} + \|\psi(0)\|_{L^2(\Omega\times\mathbb{S}^2)}). \qquad \square \end{aligned}$$

DEFINITION 5. *Assume* $\partial\Omega \in C^1$. *Let* $0 \le T_{\varepsilon 0} \in L^5(\Omega)$ *and* $0 \le \psi_{\varepsilon 0} \in L^2(\Omega\times\mathbb{S}^2)$. *Let* $0 \le T_b \in L^5_{\rm loc}([0,\infty); L^5(\partial\Omega))$ *and* $0 \le \psi_b \in L^2_{\rm loc}([0,\infty); L^2(\Sigma_-; |n\cdot\beta| d\beta d\sigma_x))$. *We say that* $(T_\varepsilon, \psi_\varepsilon)$ *is a weak solution of the system* (5)–(6) *with initial conditions* (7)–(8) *and boundary conditions* (9), (12) *if*

$$T_\varepsilon \in L^\infty_{\rm loc}(0,\infty; L^5(\Omega)), \quad T_\varepsilon^{\frac{5}{2}} \in L^2_{\rm loc}(0,\infty; H^1(\Omega)),$$
$$\psi_\varepsilon \in L^\infty_{\rm loc}(0,\infty; L^2(\Omega\times\mathbb{S}^2)) \cap W^2_{\rm loc}([0,\infty)\times\Omega\times\mathbb{S}^2),$$

and it solves (5)–(6) *in the sense of distributions, i.e., for any test functions* $\varphi \in C_0^\infty([0,\infty)\times\Omega)$ *and* $\rho \in C_0^\infty([0,\infty)\times\Omega\times\mathbb{S}^2)$, *the following equations hold:*

$$\begin{aligned} &-\iint_{[0,\infty)\times\Omega}\left(T_\varepsilon\partial_t\varphi + T_\varepsilon\Delta\varphi + \frac{1}{\varepsilon^2}\int_{\mathbb{S}^2}\varphi(\psi_\varepsilon - T_\varepsilon^4)d\beta\right)dxdt \\ &+\iint_{[0,\infty)\times\partial\Omega}(\gamma^1 T_\varepsilon) n\cdot\nabla\varphi|_{\partial\Omega} d\sigma_x dt = \int_\Omega T_{\varepsilon 0}\varphi(0,\cdot)dx, \end{aligned} \tag{55}$$

$$\begin{aligned} &-\iiint_{[0,\infty)\times\Omega\times\mathbb{S}^2}\left(\psi_\varepsilon\partial_t\rho + \frac{1}{\varepsilon}\psi_\varepsilon\beta\cdot\nabla\rho - \frac{1}{\varepsilon^2}\rho(\psi_\varepsilon - T_\varepsilon^4)\right)d\beta dxdt \\ &= \iint_{\Omega\times\mathbb{S}^2}\psi_{\varepsilon 0}\rho(0,\cdot,\cdot)d\beta dx, \end{aligned} \tag{56}$$

where

$$\gamma^1 T_\varepsilon\big|_{\partial\Omega} = T_b,$$
$$\gamma^2\psi_\varepsilon\big|_{\Sigma_-} = \alpha\psi_b + (1-\alpha)L\gamma^2\psi_\varepsilon\big|_{\Sigma_+}.$$

Next we prove the following existence theorem.

THEOREM 6. *Assume* $\partial\Omega \in C^1$. *Let* $0 \le T_{\varepsilon 0} \in L^5(\Omega)$ *and* $0 \le \psi_{\varepsilon 0} \in L^2(\Omega\times\mathbb{S}^2)$. *Let* $0 \le T_b \in L^5_{\rm loc}([0,\infty); L^5(\partial\Omega))$ *and* $0 \le \psi_b \in L^2_{\rm loc}([0,\infty); L^2(\Sigma_-; |n\cdot\beta| d\beta d\sigma_x))$. *Suppose the boundary data is well-prepared such that* $\psi_b = T_b^4$. *Then there exists a global nonnegative weak solution* $(T_\varepsilon, \psi_\varepsilon)$ *of the system* (5)–(6) *with initial conditions* (7)–(8) *and boundary conditions* (9), (12). *Furthermore, the following inequality holds:*

$$\begin{aligned} &\|T_\varepsilon(t)\|^5_{L^5(\Omega))} + \|\psi_\varepsilon(t)\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \int_0^t \|\nabla(T_\varepsilon)^{\frac{5}{2}}\|^2_{L^2(\Omega)}d\tau \\ &\quad + \frac{1}{\varepsilon^2}\int_0^t \|\psi_\varepsilon - T_\varepsilon^4)\|^2_{L^2(\Omega\times\mathbb{S}^2)}d\tau \\ &\quad + \frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t \|\psi_\varepsilon - \psi_b\|^2_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)}d\tau \\ &\le C(\|T_{\varepsilon 0}\|^5_{L^5(\Omega)} + \|\psi_{\varepsilon 0}\|^2_{L^2(\Omega\times\mathbb{S}^2)}) \end{aligned} \tag{57}$$

for any $t > 0$. *Here* C *depends on* t *and* Ω *but is independent of* ε. *The norm* $\|\cdot\|_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)}$ *in the above inequality is defined by*

$$\|f\|_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)} := \int_{\mathbb{S}^2\cap\beta\cdot n>0}\int_{\partial\Omega} n\cdot\beta f^2(x,\beta)d\sigma_x d\beta,$$

where σ_x *is the surface integral element.*

Proof. Since the boundary conditions (12) and (9) for T_ε and ψ_ε are not homogeneous, we need to lift up the boundary data and make the Galerkin approximations after subtracting the lifted data. For the boundary condition (5), we introduce $\widetilde{T}=\widetilde{T}(x)$ as the solution to the problem

$$\begin{aligned}\Delta\widetilde{T}&=0,\quad x\in\Omega,\\ \widetilde{T}(x)&=T_b(x),\quad x\in\partial\Omega.\end{aligned}\tag{58}$$

Owing to the well-prepared boundary assumption (19), we take $\widetilde{\psi}=\widetilde{T}^4$. After we introduce these variables, we can see that on the boundary,

$$\begin{aligned}&T_\varepsilon-\widetilde{T}=0 \text{ on } \partial\Omega,\\ &\psi_\varepsilon-\widetilde{\psi}=(1-\alpha)(L(\psi_\varepsilon-\widetilde{\psi}))(t,x,\beta) \text{ on } \in\Sigma_-.\end{aligned}$$

In order to find the Galerkin approximations of (5)–(6), we also need to define some truncation operators. We can take the complete set of the eigenvectors $\{w_k=w_k(x)\}_{k=1}^\infty$ of $H_0^1(\Omega)$ which is also an orthonormal basis in $L^2(\Omega)$. We take the operator $\overline{\mathbb{P}}_m: L^2(\Omega)\to L^2(\Omega)$ as

$$\overline{\mathbb{P}}_m f=\sum_{k\le m}(f,w_k)w_k(x),$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. Note that $\overline{\mathbb{P}}_m f=0$ on the boundary $\partial\Omega$. In order to deal with the boundary conditions, we take $\mathbb{P}_m f:=\overline{\mathbb{P}}_m(f-\tilde{T})+\tilde{T}$ so that $\mathbb{P}_m f(x)=T_b(x)$ for $x\in\partial\Omega$.

By Lemma 17 in Appendix C, we can also find an orthonormal basis $\{\phi_l(x)\chi_j(\beta)\}_{k,l=1}^\infty$ in $\{\varphi\in L^2(\Omega\times\mathbb{S}^2):\varphi|_{\Sigma_-}=(1-\alpha)\varphi|_{\Sigma_+}\}$. We define the operator $\mathbb{Q}_m$ as

$$\mathbb{Q}_m\psi(x,\beta)=\sum_{l=1}^{m_1}\sum_{j=1}^{m_2}((\psi,\phi_l\chi_j))\phi_l\chi_j.\tag{59}$$

Here $((\cdot,\cdot))$ denotes the inner product in the space $L^2(\Omega\times\mathbb{S}^2)$. After these preparations, now we proceed to prove Theorem 6.

As in the proof of Theorem 3, we take the Galerkin approximations to be

$$\partial_t T_\varepsilon^m= \Delta T_\varepsilon^m+\frac{1}{\varepsilon^2}\left(\int_{\mathbb{S}^2}\psi_\varepsilon^m-\mathbb{P}_m((T_\varepsilon^m)^4)d\beta\right),\tag{60}$$

$$\partial_t\psi_\varepsilon^m+\frac{1}{\varepsilon}\beta\cdot\nabla\psi_\varepsilon^m= -\frac{1}{\varepsilon^2}\left(\psi_\varepsilon^m-\mathbb{Q}_m\mathbb{P}_m((T_\varepsilon^m)^4)\right),\tag{61}$$

with initial conditions

$$\begin{aligned}T_\varepsilon^m(t=0,x)&= \mathbb{P}_m T_{\varepsilon 0}(x),\\ \psi_\varepsilon^m(t=0,x,\beta)&= \mathbb{Q}_m\psi_{\varepsilon 0}(\beta,x),\end{aligned}$$

and boundary conditions

$$T_\varepsilon^m(t,x)= T_b \text{ for } x\in\partial\Omega,\tag{62}$$

$$\psi_\varepsilon^m|_{\Sigma_-}= \alpha\psi_b+L\psi_\varepsilon^m|_{\Sigma_+}.\tag{63}$$

In the above we take $\mathbb{P}_m((T_\varepsilon^m)^4):=\bar{\mathbb{P}}((T_\varepsilon^m)^4-\tilde{T}^4)+\tilde{T}^4$.

We can take

$$T_\varepsilon^m - \widetilde{T} = \sum_{k=1}^{m} d_k(t) w_k(x),$$
$$\psi_\varepsilon^m - \widetilde{\psi} = \sum_{l=1}^{m_1}\sum_{j=1}^{m_2} \varphi_{lj}(t)\phi_l(x)\chi_j(\beta),$$

into (60) and (61), and get a system of ordinary differential equations for d_k and φ_{lj}. From the Cauchy–Lipschitz theorem, the ODE system has a global unique solution if T_ε^m and ψ_ε^m are bounded uniformly in time, which will be shown below. It follows that the system (60)–(61) has a global unique solutions. Note that the solutions $T_\varepsilon^m, \psi_\varepsilon^m \geq 0$, which is shown in Appendix A.

Next we derive the energy estimate for system (60)–(61). We multiply (60) by $(T_\varepsilon^m)^4 - \widetilde{T}^4$ and (61) by $\psi_\varepsilon^m - \widetilde{\psi}$, integrate over $[0,t]\times\Omega$ and $[0,t]\times\Omega\times\mathbb{S}^2$, respectively, and add the results together. We obtain considering $\mathbb{Q}_m(\psi_\varepsilon^m - \widetilde{\psi}) = \psi_\varepsilon^m - \widetilde{\psi}$,

$$\begin{aligned}
&\int_\Omega \left(\frac{(T_\varepsilon^m)^5}{5} - \widetilde{T}^4 T_\varepsilon^m\right)(t)dx + \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon^m - \widetilde{\psi})^2(t)d\beta dx \\
&= \int_\Omega \frac{(T_{\varepsilon 0}^m)^5}{5}dx + \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\psi_{\varepsilon 0}^m)^2 d\beta dx - \frac{1}{2\varepsilon}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta dx d\tau \\
&\quad + \int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla\widetilde{\psi}(\psi_\varepsilon^m - \widetilde{\psi})d\beta dx d\tau + \int_0^t\int_\Omega ((T_\varepsilon^m)^4 - \widetilde{T}^4)\Delta T_\varepsilon^m dx d\tau \\
&\quad - \frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \left(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\right)(\psi_\varepsilon^m - \widetilde{\psi} - \mathbb{P}_m((T_\varepsilon^m)^4) + \widetilde{T}^4) d\beta dx d\tau \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \qquad (64)
\end{aligned}$$

The first term on the left-hand side of the above equation can be estimated by

$$\begin{aligned}
\int_\Omega \left(\frac{(T_\varepsilon^m)^5}{5} - \widetilde{T}^4 T_\varepsilon^m\right)dx &\geq \int_\Omega \frac{(T_\varepsilon^m)^5}{5}dx - \int_\Omega \frac{1}{2}\frac{(T_\varepsilon^m)^5}{5}dx - 2\int_\Omega \frac{(\widetilde{T}^4)^{\frac{5}{4}}}{\frac{5}{4}}dx \\
&\geq \frac{1}{10}\|T_\varepsilon^m\|_{L^5(\Omega)}^5 - \frac{8}{5}\|\widetilde{T}\|_{L^5(\Omega)}^5. \qquad (65)
\end{aligned}$$

The second term on the left can be estimated by

$$\begin{aligned}
\frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta dx &\geq \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon^m)^2 d\beta dx - \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\widetilde{\psi})^2 d\beta dx \\
&= \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon^m)^2 d\beta dx - 2\pi\int_\Omega \widetilde{T}^8 dx. \qquad (66)
\end{aligned}$$

Next we estimate the right terms of (64). First, for I_1 and I_2, we can use the property of the operators $\mathbb{P}_m$ and $\mathbb{Q}_m$ to get

$$I_1 + I_2 = \int_\Omega \frac{(T_{\varepsilon 0}^m)^5}{5}dx + \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2}(\psi_{\varepsilon 0}^m)^2 d\beta dx \leq \frac{1}{5}\|T_{\varepsilon 0}\|_{L^5(\Omega)}^5 + \frac{1}{2}\|\psi_{\varepsilon 0}\|_{L^2(\Omega\times\mathbb{S}^2)}^2. \qquad (67)$$

Using the boundary condition (63), I_3 can be calculated as

$$
\begin{aligned}
I_3 =& -\frac{1}{2\varepsilon}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla(\psi_\varepsilon^m-\widetilde{\psi})^2 d\beta dx d\tau = -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma} \beta\cdot n(\psi_\varepsilon^m-\widetilde{\psi})^2 d\beta d\sigma_x d\tau\\
=& -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m-\widetilde{\psi})^2 d\beta d\sigma_x d\tau - \frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_-} \beta\cdot n(\psi_\varepsilon^m-\widetilde{\psi})^2 d\beta d\sigma_x d\tau\\
=& -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m-\psi_b)^2 d\beta d\sigma_x d\tau\\
& -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_-} \beta\cdot n(1-\alpha)^2(\psi_\varepsilon^m(\beta')-\psi_b)^2 d\beta d\sigma_x d\tau\\
=& -\frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m-\psi_b)^2 d\beta d\sigma_x d\tau.
\end{aligned}
\tag{68}
$$

The term I_4 can be estimated by using the fact that $\tilde{\psi}$ is independent of β as

$$
\begin{aligned}
I_4 =& \int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla\widetilde{\psi}(\psi_\varepsilon^m-\widetilde{\psi}) d\beta dx d\tau\\
=& \int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla\widetilde{\psi}(\psi_\varepsilon^m-\mathbb{P}_m((T_\varepsilon^m)^4)) d\beta dx d\tau\\
\le& 2\varepsilon^2\int_0^t \iint_{\Omega\times\mathbb{S}^2} |\beta\cdot\nabla\widetilde{\psi}|^2 d\beta dx d\tau + \frac{1}{2\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon^m-\mathbb{P}_m((T_\varepsilon^m)^4))^2 d\beta dx d\tau\\
\le& 8\pi\varepsilon^2\int_0^t \iint_{\Omega\times} |\nabla\widetilde{T}^4|^2 dx d\tau + \frac{1}{2\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon^m-\mathbb{P}_m((T_\varepsilon^m)^4))^2 d\beta dx d\tau.
\end{aligned}
\tag{69}
$$

We estimate the term I_5 by

$$
\begin{aligned}
I_5 =& \int_0^t\int_\Omega ((T_\varepsilon^m)^4-\widetilde{T}^4)\Delta T_\varepsilon^m dx d\tau\\
=& -\int_0^t\int_\Omega \nabla((T_\varepsilon^m)^4-\widetilde{T}^4)\nabla T_\varepsilon dx d\tau\\
=& -\int_0^t\int_\Omega \frac{16}{25}|\nabla(T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau - \int_0^t\int_\Omega T_\varepsilon^m\Delta\widetilde{T}^4 dx d\tau + \int_0^t\int_{\partial\Omega} T_b n\cdot\nabla\widetilde{T}^4 d\sigma_x d\tau\\
\le& -\int_0^t\int_\Omega \frac{16}{25}|\nabla(T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau + \int_0^t\int_\Omega \frac{(T_\varepsilon^m)^5}{5} dx d\tau + \int_0^t\int_\Omega \frac{(\Delta\widetilde{T}^4)^{\frac{5}{4}}}{\frac{5}{4}} dx d\tau\\
&+\int_0^t\int_{\partial\Omega} 4T_b^4 n\cdot\nabla\widetilde{T} d\sigma_x d\tau.
\end{aligned}
$$

Multiplying (58) by $\widetilde{T}^4$ and integrating over $[0,t]\times\Omega$ leads to

$$
\int_\Omega \frac{\widetilde{T}^5}{5}dx + \frac{16}{25}\int_0^t\int_\Omega |\nabla\widetilde{T}^{\frac{5}{2}}|^2 dx d\tau - \int_0^t\int_{\partial\Omega} T_b^4 n\cdot\nabla\widetilde{T} d\sigma_x d\tau = 0,
$$

thus

$$I_5 \leq -\int_0^t \int_\Omega \frac{16}{25}|\nabla (T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau + \int_0^t \int_\Omega \frac{(T_\varepsilon^m)^5}{5} dx d\tau + \int_0^t \int_\Omega \frac{(\Delta \widetilde{T}^4)^{\frac{5}{4}}}{\frac{5}{4}} dx d\tau$$

$$+ \frac{4}{5}\int_\Omega \widetilde{T}^5 dx + \frac{64}{25}\int_0^t \int_\Omega |\nabla \widetilde{T}^{\frac{5}{2}}|^2 dx d\tau. \tag{70}$$

The term I_6 can be treated by

$$I_6 = -\frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \left(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\right)\left(\psi_\varepsilon^m - \widetilde{\psi} - \mathbb{P}_m((T_\varepsilon^m)^4) + \widetilde{T}^4\right) d\beta dx d\tau$$

$$= -\frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4))^2 d\beta dx d\tau. \tag{71}$$

Here we use $\widetilde{\psi} = \widetilde{T}^4$ in the above equality. Taking the estimates (65)–(71) into (64) leads to the estimate

$$\begin{aligned}
&\frac{1}{10}\|T_\varepsilon^m(t)\|_{L^5(\Omega)}^5 + \frac{1}{2}\|\psi_\varepsilon^m(t)\|_{L^2(\Omega\times\mathbb{S}^2)}^2 + \frac{16}{25}\int_0^t \|\nabla (T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2(\Omega)}^2 d\tau \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4))^2 d\beta dx d\tau \\
&\quad + \frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m - \psi_b)^2 d\beta d\sigma_x d\tau \\
&\qquad \leq \frac{1}{5}\|T_{\varepsilon 0}\|_{L^5}^5 + \frac{1}{2}\|\psi_{\varepsilon 0}\|_{L^2(\Omega\times\mathbb{S}^2)}^2 + \frac{8}{5}\|\widetilde{T}\|_{L^5(\Omega)}^5 + 2\pi\|\widetilde{T}\|_{L^8(\Omega)}^8 \\
&\qquad + 8\pi\varepsilon^2\int_0^t \iint_\Omega |\nabla \widetilde{T}^4|^2 dx d\tau + \frac{1}{5}\int_0^t \int_\Omega \|T_\varepsilon^m\|_{L^5(\Omega)}^5 + \frac{4}{5}\int_0^t \|\Delta\widetilde{T}^4\|_{L^{\frac{5}{4}}(\Omega)}^{\frac{5}{4}} d\tau \\
&\qquad + \frac{4}{5}\|\widetilde{T}\|_{L^5(\Omega)}^5 + \frac{64}{25}\int_0^t \|\nabla\widetilde{T}^{\frac{5}{2}}\|_{L^2(\Omega)}^2 d\tau.
\end{aligned}$$

Since $\widetilde{T}$ is the solution of the Laplace's equation (58), it is smooth and thus the terms including $\widetilde{T}$ of the above inequality are bounded. We can apply Gronwall's inequality to obtain

$$\begin{aligned}
&\|T_\varepsilon^m(t)\|_{L^5(\Omega))}^5 + \|\psi_\varepsilon^m(t)\|_{L^2(\Omega\times\mathbb{S}^2)}^2 + \int_0^t \|\nabla (T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2(\Omega)}^2 d\tau \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \|\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\|_{L^2(\Omega\times\mathbb{S}^2)}^2 d\tau \\
&\quad + \frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t \|\psi_\varepsilon^m - \psi_b\|_{L^2(\Sigma_+;|n\cdot\beta| d\beta d\sigma_x)}^2 d\tau \\
&\leq Ce^{Ct}(\|T_{\varepsilon 0}\|_{L^5}^5 + \|\psi_{\varepsilon 0}\|_{L^2(\Omega\times\mathbb{S}^2)}^2),
\end{aligned} \tag{72}$$

where C depends on $\widetilde{T}$.

From this estimate, we can get the same bounds on T_ε^m and ψ_ε^m inside the domain, i.e., (28)–(31) still hold. We can follow the same proof of Theorem 3 to get

$$T_\varepsilon^m \to T_\varepsilon \text{ almost everywhere}, \tag{73}$$

$$\psi_\varepsilon^m \rightharpoonup \psi_\varepsilon \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega\times\mathbb{S}^2)), \tag{74}$$

and that $(T_\varepsilon, \psi_\varepsilon)$ satisfies (55)–(56).

Next we consider the boundary conditions. According to (72), we have

$$(T_\varepsilon^m)^{\frac{5}{2}} \text{ is uniformly bounded in } L^2_{\rm loc}([0,\infty); H^1(\Omega)]),$$

so by the Rellich–Kondrachov emdedding theorem, as $m \to \infty$,

$$(T_\varepsilon^m)^{\frac{5}{2}} \to T_\varepsilon^{\frac{5}{2}} \text{ strongly in } L^2_{\rm loc}([0,\infty); H^{1-\delta}(\Omega)])$$

for $\delta > 0$ small. We can thus use the continuity of the trace operator to get

$$\gamma^1 (T_\varepsilon^m)^{\frac{5}{2}} = T_b^{\frac{5}{2}} \to \gamma^1 T_\varepsilon^{\frac{5}{2}} = T_b^{\frac{5}{2}} \text{ strongly in } L^2_{\rm loc}([0,\infty); H^{\frac{1}{2}-\delta}(\Omega)]). \tag{75}$$

Therefore, we get

$$\gamma^1 T_\varepsilon = T_b.$$

To pass to the limit on the boundary of ψ_ε^m, we learn from Lemma 4 that

$$\|\psi_\varepsilon^m|_\Sigma\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{S}^2, |n\cdot\beta|^2 d\beta d\sigma_x dt)}$$

is bounded.

Therefore, there exists a subsequence $\{\psi_\varepsilon^{m_k}\}_{k>0}$ such that

$$\gamma^2 \psi_\varepsilon^{m_k} \rightharpoonup \overline{\gamma^2\psi_\varepsilon^{m_k}} \text{ weakly in } L^2_{\rm loc}([0,\infty); L^2(\Sigma; |n\cdot\beta|^2 d\beta d\sigma_x d\tau)).$$

We can thus take the weak limit in (63) to obtain

$$\overline{\gamma^2\psi_\varepsilon^{m_k}}|_{\Sigma_-} = \alpha\psi_b + L\overline{\gamma^2\psi_\varepsilon^{m_k}}|_{\Sigma_+}. \tag{76}$$

To show $\overline{\gamma^2\psi_\varepsilon^{m_k}} = \gamma^2\psi_\varepsilon$, we use the fact that $(T_\varepsilon, \psi_\varepsilon)$ solves

$$\varepsilon\partial_t\psi_\varepsilon + \beta\cdot\nabla\psi_\varepsilon = -\frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4).$$

We apply test function $\rho \in C^\infty([0,\infty)\times\Omega\times\mathbb{S}^2)$ on the above equation and deduce

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon(t)\rho(t) d\beta dx - \iint_\Omega \psi_{\varepsilon 0}\rho(0) d\beta dx - \int_0^t \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon \partial_t\rho d\beta dx d\tau \\
&\quad - \frac{1}{\varepsilon}\int_0^t \int_{\Omega\times\mathbb{S}^2} \psi_\varepsilon \beta\cdot\nabla\rho d\beta dx d\tau + \frac{1}{\varepsilon}\int_0^t \iint_\Sigma (n\cdot\beta)\gamma^2\psi_\varepsilon \rho d\beta dx d\tau \\
&\quad = -\frac{1}{\varepsilon^2}\int_0^t \int_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon)^4))\rho d\beta dx d\tau.
\end{aligned}$$

We can also apply the same test function on (61) and get

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon^m(t)\rho(t) d\beta dx - \iint_\Omega \psi_{\varepsilon 0}^m\rho(0) d\beta dx - \int_0^t \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon^m \partial_t\rho d\beta dx d\tau \\
&\quad - \frac{1}{\varepsilon}\int_0^t \int_{\Omega\times\mathbb{S}^2} \psi_\varepsilon^m \beta\cdot\nabla\rho d\beta dx d\tau + \frac{1}{\varepsilon}\int_0^t \iint_\Sigma (n\cdot\beta)\gamma^2\psi_\varepsilon^m \rho d\beta dx d\tau \\
&\quad = -\frac{1}{\varepsilon^2}\int_0^t \int_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4)\rho d\beta dx d\tau.
\end{aligned}$$

Passing $m \to \infty$ in the above equation and comparing with the previous one lead to

$$\overline{\gamma^2 \psi_\varepsilon^m} = \gamma^2 \psi_\varepsilon.$$

This combined with (76) implies that ψ_ε satisfies

$$\gamma^2 \psi_\varepsilon|_{\Sigma_-} = \alpha \psi_b + L\gamma^2 \psi_\varepsilon|_{\Sigma_+} \tag{77}$$

on the boundary. The energy inequality can be shown as in the proof of Theorem 3, except that here we need to use

$$\int_0^t \|\psi_\varepsilon - \psi_b\|_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)} d\tau \le \liminf_{m\to\infty} \int_0^t \|\psi_\varepsilon^m - \psi_b\|_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)} d\tau,$$

which is due to the weak convergence of ψ_ε^m on the boundary. ∎

Remark 1. In the above proof we assume the boundary condition to be well-prepared $\psi_b = T_b^4$. When the boundary condition is not well-prepared, a similar estimate on the term (68) gives

$$\begin{aligned}
I_3 &= -\frac{1}{2\varepsilon}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta dx d\tau = -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma} \beta\cdot n(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta d\sigma_x d\tau \\
&= -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta d\sigma_x d\tau - \frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_-} \beta\cdot n(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta d\sigma_x d\tau \\
&= -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m - T_b^4)^2 d\beta d\sigma_x d\tau \\
&\quad -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_-} \beta\cdot n\left(\alpha(\psi_b - T_b^4) + (1-\alpha)(\psi_\varepsilon^m(\beta') - T_b^4)\right)^2 d\beta d\sigma_x d\tau \\
&\le -\frac{(2\alpha-\alpha^2)}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m - T_b^4)^2 d\beta d\sigma_x d\tau \\
&\quad + \frac{\alpha^2}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_b - T_b^4)^2 d\beta d\sigma_x d\tau \\
&\quad + \frac{2\alpha(1-\alpha)}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_\varepsilon^m - T_b^4)(\psi_b - T_b^4) d\beta d\sigma_x d\tau \\
&= -\frac{(2\alpha-\alpha^2)}{2\varepsilon}\int_0^t \iint_{\Sigma_+} \beta\cdot n\left(\psi_\varepsilon^m - T_b^4 - \frac{1-\alpha}{2-\alpha}(\psi_b - T_b^4)\right)^2 d\beta d\sigma_x d\tau \\
&\quad + \frac{\alpha}{2(2-\alpha)}\int_0^t \iint_{\Sigma_+} \beta\cdot n(\psi_b - T_b^4)^2 d\beta d\sigma_x d\tau.
\end{aligned}$$

So the estimate (57) holds with the constant C depending on ε. We then get the existence of the weak solutions for fixed ε. However, the well-prepared assumption on the boundary conditions is still required to study the diffusive limit in section 3.

2.3. Case of Robin boundary condition. We now proceed to consider the case of Robin boundary condition (13). We first give the definition of the weak solution.

DEFINITION 7. *Assume* $\partial\Omega \in C^1$. *Let* $0 \le T_{\varepsilon 0} \in L^5(\Omega)$ *and* $0 \le \psi_{\varepsilon 0} \in L^2(\Omega\times\mathbb{S}^2)$. *Let* $0 \le T_b \in L^5_{\text{loc}}([0,\infty); L^5(\partial\Omega))$ *and* $0 \le \psi_b \in L^2_{\text{loc}}([0,\infty); L^2(\Sigma_-; |n\cdot\beta| d\beta d\sigma_x))$.

We say that $(T_\varepsilon,\psi_\varepsilon)$ *is a weak solution of the system* (5)–(6) *with initial conditions* (7)–(8) *and boundary conditions* (9), (13) *if*

$$T_\varepsilon\in L^\infty_{\rm loc}(0,\infty;L^5(\Omega)),\quad T_\varepsilon^{\frac52}\in L^2_{\rm loc}(0,\infty;H^1(\Omega)),$$
$$\psi_\varepsilon\in L^\infty_{\rm loc}(0,\infty;L^2(\Omega\times\mathbb{S}^2))\cap W^2_{\rm loc}([0,\infty)\times\Omega\times\mathbb{S}^2),$$

and it solves (5)–(6) *in the sense of distributions, i.e., for any test functions* $\varphi\in C^\infty([0,\infty),\Omega)$ *and* $\rho\in C^\infty([0,\infty);\Omega\times\mathbb{S}^2)$, *the following equations hold:*

$$\begin{aligned}&-\iint_{[0,\infty)\times\Omega}\left(T_\varepsilon\partial_t\varphi+T_\varepsilon\Delta\varphi+\frac{1}{\varepsilon^2}\int_{\mathbb{S}^2}\varphi(\psi_\varepsilon-T_\varepsilon^4)d\beta\right)dxdt\\&-\iint_{[0,\infty)\times\partial\Omega}\varphi\cdot\frac{T_b-(\gamma^1T_\varepsilon)}{\varepsilon^r}d\sigma_xdt+\iint_{[0,\infty)\times\partial\Omega}(\gamma^1T_\varepsilon)n\cdot\nabla\varphi d\sigma_xdt\\&=\int_\Omega T_{\varepsilon0}\varphi(0,\cdot)dx,\end{aligned}\tag{78}$$

$$\begin{aligned}&-\iiint_{[0,\infty)\times\Omega\times\mathbb{S}^2}\left(\psi_\varepsilon\partial_t\rho+\frac1\varepsilon\psi_\varepsilon\beta\cdot\nabla\rho-\frac{1}{\varepsilon^2}\rho(\psi_\varepsilon-T_\varepsilon^4)\right)d\beta dxdt\\&+\iiint_{[0,\infty)\times\Sigma}(n\cdot\beta)\rho\cdot(\gamma^2\psi_\varepsilon)d\beta d\sigma_xdt=\iint_{\Omega\times\mathbb{S}^2}\psi_{\varepsilon0}\rho(0,\cdot,\cdot)d\beta dx,\end{aligned}\tag{79}$$

where

$$\gamma^2\psi_\varepsilon\big|_{\Sigma_-}=\alpha\psi_b+(1-\alpha)\gamma^2L\psi_\varepsilon\big|_{\Sigma_+},\tag{80}$$

with the reflection operator L *defined in* (10).

Next, we prove the following existence theorem.

THEOREM 8. *Assume* $\partial\Omega\in C^1$. *Let* $0\le T_{\varepsilon0}\in L^5(\Omega)$ *and* $0\le\psi_{\varepsilon0}\in L^2(\Omega\times\mathbb{S}^2)$. *Let* $0\le T_b\in L^5_{\rm loc}([0,\infty);L^5(\partial\Omega))$ *and* $0\le\psi_b\in L^2_{\rm loc}([0,\infty);L^2(\Sigma_-;|n\cdot\beta|d\beta d\sigma_x))$. *Then there exists a global nonnegative weak solution* $(T_\varepsilon,\psi_\varepsilon)$ *of the system* (5) *and* (6) *with initial conditions* (7)–(8) *and boundary conditions* (9), (13). *Moreover, the following energy inequality holds for all* $t>0$*:*

$$\begin{aligned}&\|T_\varepsilon(t)\|^5_{L^5(\Omega))}+\|\psi_\varepsilon(t)\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\int_0^t\|\nabla T_\varepsilon^{\frac52}\|^2_{L^2(\Omega)}d\tau\\&\quad+\frac{1}{\varepsilon^2}\int_0^t\|\psi_\varepsilon-T_\varepsilon^4\|^2_{L^2(\Omega\times\mathbb{S}^2)}d\tau+\frac{1}{\varepsilon^r}\int_0^t\|\gamma^1T_\varepsilon-T_b\|^5_{L^5(\partial\Omega)}d\tau\\&\quad+\frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t\|\gamma^2\psi_\varepsilon-\psi_b\|^2_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)}d\tau\\&\le C(\|T_{\varepsilon0}\|^5_{L^5}+\|\psi_{\varepsilon0}\|^2_{L^2(\Omega\times\mathbb{S}^2)}).\end{aligned}\tag{81}$$

Here C *is a positive constant independent on* ε.

Proof. The positivity of the solution is given in Appendix A. To deal with the case of Robin boundary condition (13), for $s\ge-\frac12$, we define the Robin map $\mathcal{R}:H^s(\partial\Omega)\to H^{s+\frac32}(\Omega)$ (for example, see [33]) with $f=\mathcal{R}g$ as the weak solution for the equation

$$\Delta f=0\text{ in }\Omega,\tag{82}$$
$$\varepsilon^rn\cdot\nabla f+f=g\text{ on }\partial\Omega.\tag{83}$$

We define the operator Δ_r in $L^2(\Omega)$ by

$$\begin{aligned}&\Delta_r : \mathcal{D}(\Delta_r) \subset L^2(\Omega) \to L^2(\Omega),\\&\Delta_r = -\Delta,\ \mathcal{D}(\Delta_r) = \Big\{ f \in H^1(\Omega) : \Delta f \in L^2(\Omega),\ \varepsilon^r n \cdot \nabla f + f = 0 \text{ on } \partial\Omega \Big\}.\end{aligned}$$

The space $\mathcal{D}(\Delta_r)$ is equipped with the norm

$$\|f\|_{\mathcal{D}(\Delta_r)} = (\|\nabla f\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^r}\|\gamma f\|^2_{L^2(\partial\Omega)})^{\frac{1}{2}}$$

for all $f, h \in H^1(\Omega)$. With the above definitions, we can see that $T_\varepsilon - \mathcal{R}T_b \in \mathcal{D}(\Delta_r)$ satisfies the following condition on the boundary:

$$\varepsilon^r n \cdot \nabla(T_\varepsilon - \mathcal{R}T_b) + T_\varepsilon - \mathcal{R}T_b = 0 \text{ on } \partial\Omega.$$

We take $\{w_m(x)\}_{m=1}^\infty$ to be an orthogonal basis in $\mathcal{D}(\Delta_r)$; for example, we can take the complete set of the eigenvectors of $-\Delta_r$ as the basis. It is also orthonormal in $L^2(\Omega)$. We take the operator $\mathbb{P}_m$ to be

$$\mathbb{P}_m f = \sum_{k=1}^m (f, w_k) w_k(x).$$

Here we take $\mathbb{Q}_m$ as the same with (59).

We consider the Galerkin approximate system

$$\partial_t T^m_\varepsilon = \Delta_r(T^m_\varepsilon - \mathcal{R}T_b) + \frac{1}{\varepsilon^2}\int_{\mathbb{S}^2} (\mathbb{P}_m \psi^m_\varepsilon - \mathbb{P}_m((T^m_\varepsilon)^4)) d\beta, \tag{84}$$

$$\partial_t \psi^m_\varepsilon + \frac{1}{\varepsilon}\beta \cdot \nabla \psi^m_\varepsilon = -\frac{1}{\varepsilon^2}(\psi^m_\varepsilon - \mathbb{Q}_m \mathbb{P}_m (T^m_\varepsilon)^4)). \tag{85}$$

We take $\widetilde{T}$ as defined in (58) but with boundary data

$$\widetilde{T} = \psi_b^{\frac{1}{4}} \text{ on } \partial\Omega$$

and take $\widetilde{\psi} = \widetilde{T}^4$. We take

$$\begin{aligned}T^m_\varepsilon - \mathcal{R}T_b &= \sum_{k=1}^m d_k(t) w_k(x),\\ \psi^m_\varepsilon - \widetilde{\psi} &= \sum_{k=1}^m \phi_k(t) \varphi_k(x, \beta)\end{aligned}$$

into the above system to get an ODE system of $d_k(t)$ and $\phi_k(t)$ with $k = 1, \dots, m$. The existence of the ODE system is guaranteed by the Cauchy–Lipschitz theorem. Note the solutions $T^m_\varepsilon, \psi^m_\varepsilon \geq 0$, which are proved in Appendix A.

We next derive the energy estimate. We multiply (84) by $(T_\varepsilon^m)^4 - \widetilde{T}^4$ and (85) by $\psi_\varepsilon^m - \widetilde{\psi}$ and integrate over time and space to get (same as (64))

$$\begin{aligned}
&\int_\Omega \left(\frac{(T_\varepsilon^m)^5}{5} - \widetilde{T}^4 T_\varepsilon^m \right)(t)dx + \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon^m - \widetilde{\psi})^2(t) d\beta dx \\
&= \int_\Omega \frac{(T_{\varepsilon 0}^m)^5}{5} dx + \frac{1}{2}\iint_{\Omega\times\mathbb{S}^2} (\psi_{\varepsilon 0}^m)^2 d\beta dx - \frac{1}{2\varepsilon}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla(\psi_\varepsilon^m - \widetilde{\psi})^2 d\beta dx d\tau \\
&\quad + \int_0^t \iint_{\Omega\times\mathbb{S}^2} \beta\cdot\nabla\widetilde{\psi}(\psi_\varepsilon^m - \widetilde{\psi}) d\beta dx d\tau + \int_0^t\int_\Omega ((T_\varepsilon^m)^4 - \widetilde{T}^4)\Delta T_\varepsilon^m dx d\tau \\
&\quad - \frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} \left(\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\right)(\psi_\varepsilon^m - \widetilde{\psi} - \mathbb{P}_m((T_\varepsilon^m)^4) + \widetilde{T}^4) d\beta dx d\tau
\end{aligned}$$

$$=I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{86}$$

The terms can be treated as in the proof of Theorem 6 except I_5, which can be estimated by

$$\begin{aligned}
I_5 &= \int_0^t \int_\Omega ((T_\varepsilon^m)^4 - \widetilde{T}^4)\Delta T_\varepsilon^m dx d\tau \\
&= -\frac{16}{25}\int_0^t\int_\Omega |\nabla (T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau + \int_0^t\int_{\partial\Omega} ((T_\varepsilon^m)^4 - \widetilde{T}^4) n\cdot\nabla T_\varepsilon^m d\sigma_x d\tau \\
&\quad - \int_0^T \int_\Omega T_\varepsilon^m \Delta\widetilde{T} dx d\tau + \int_0^t\int_{\partial\Omega} T_b n\cdot\nabla\widetilde{T}^4 d\sigma_x d\tau \\
&= -\frac{16}{25}\int_0^t\int_\Omega |\nabla (T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau + \int_0^t\int_{\partial\Omega} ((T_\varepsilon^m)^4 - T_b^4)\frac{1}{\varepsilon^r}(-T_\varepsilon^m + T_b) d\sigma_x d\tau \\
&\quad + \int_0^t\int_\Omega \frac{(T_\varepsilon^m)^5}{5} dx d\tau + \int_0^t\int_\Omega \frac{(\Delta\widetilde{T}^4)^{\frac{5}{4}}}{\frac{5}{4}} dx d\tau \\
&\quad + \frac{4}{5}\int_\Omega \widetilde{T}^5(t)dx + \frac{64}{25}\int_0^t\int_\Omega |\nabla\widetilde{T}^{\frac{5}{2}}|^2 dx d\tau \\
&= -\frac{16}{25}\int_0^t\int_\Omega |\nabla (T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau + \int_0^t\int_\Omega \frac{(T_\varepsilon^m)^5}{5} dx d\tau \\
&\quad + \int_0^t\int_\Omega \frac{(\Delta\widetilde{T}^4)^{\frac{5}{4}}}{\frac{5}{4}} dx d\tau + \frac{4}{5}\int_\Omega \widetilde{T}^5(t)dx + \frac{64}{25}\int_0^t\int_\Omega |\nabla\widetilde{T}^{\frac{5}{2}}|^2 dx d\tau \\
&\quad - \frac{1}{\varepsilon^r}\int_0^t\int_{\partial\Omega} ((T_\varepsilon^m)^2 + T_b^2)(T_\varepsilon^m + T_b)(T_\varepsilon^m - T_b)^2 d\sigma_x d\tau.
\end{aligned}$$

The last term appears additional to the estimate (70) of I_5 for the Dirichlet case. Using the positivity of T_ε^m and T_b, we get

$$\begin{aligned}
&((T_\varepsilon^m)^2 + T_b^2)(T_\varepsilon^m + T_b)(T_\varepsilon^m - T_b)^2 - (T_\varepsilon^m - T_b)^5 \\
&\quad = (T_\varepsilon^m - T_b)^2 2T_b(2(T_\varepsilon^m)^2 - T_\varepsilon^m T_b + T_b^2) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&((T_\varepsilon^m)^2 + T_b^2)(T_\varepsilon^m + T_b)(T_\varepsilon^m - T_b)^2 + (T_\varepsilon^m - T_b)^5 \\
&\quad = (T_\varepsilon^m - T_b)^2 2T_\varepsilon^m((T_\varepsilon^m)^2 - T_\varepsilon^m T_b + 2T_b^2) \geq 0,
\end{aligned}$$

so that

$$((T_\varepsilon^m)^2 + T_b^2)(T_\varepsilon^m + T_b)(T_\varepsilon^m - T_b)^2 \geq |T_\varepsilon^m - T_b|^5. \tag{87}$$

Consequently, we obtain

$$I_5 \leq -\frac{16}{25}\int_0^t \int_\Omega |\nabla (T_\varepsilon^m)^{\frac{5}{2}}|^2 dx d\tau - \frac{1}{\varepsilon^r}\int_0^t \|\gamma^1 T_\varepsilon - T_b\|_{L^5(\partial\Omega)}^5 d\tau.$$

The energy inequality (72) then becomes

$$\begin{aligned}
&\|T_\varepsilon^m(t)\|_{L^5(\Omega))}^5 + \|\psi_\varepsilon^m(t)\|_{L^2(\Omega\times\mathbb{S}^2)}^2 + \int_0^t \|\nabla (T_\varepsilon^m)^{\frac{5}{2}}\|_{L^2(\Omega)}^2 d\tau \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \|\psi_\varepsilon^m - \mathbb{P}_m((T_\varepsilon^m)^4)\|_{L^2(\Omega\times\mathbb{S}^2)}^2 d\tau + \frac{1}{\varepsilon^r}\int_0^t \|\gamma^1 T_\varepsilon - T_b\|_{L^5(\partial\Omega)}^5 d\tau \\
&\quad + \frac{2\alpha-\alpha^2}{2\varepsilon}\int_0^t \|\gamma^2\psi_\varepsilon^m - \psi_b\|_{L^2(\Sigma_+;|n\cdot\beta|d\beta d\sigma_x)}^2 d\tau \\
&\leq Ce^{Ct}\left(\|T_{\varepsilon 0}\|_{L^5}^5 + \|\psi_{\varepsilon 0}\|_{L^2(\Omega\times\mathbb{S}^2)}^2\right).
\end{aligned} \tag{88}$$

We can pass to the limit $m \to \infty$ and use the trace theorem like in the proof of Theorem 6 to get

$$\begin{aligned}
&\gamma^1 T_\varepsilon^m \to \gamma^1 T_\varepsilon, \text{ strongly in } L^5_{\mathrm{loc}}([0,\infty); L^5(\partial\Omega)),\\
&\gamma^2\psi_\varepsilon^m \rightharpoonup \gamma^2\psi_\varepsilon, \text{ weakly in } L^2_{\mathrm{loc}}([0,\infty); L^2(\Sigma; |n\cdot\beta|d\beta d\sigma_x d\tau)).
\end{aligned}$$

With this we can apply test functions on the Galerkin system (84)–(85) and pass to the limit $m \to \infty$ to show (78)–(79) holds.

To show the energy inequality (81), we can pass to the limit $m \to \infty$ in the above energy estimate and follow the proof of Theorems 3 and 6, except here we additionally need

$$\int_0^t \|\gamma^1 T_\varepsilon - T_b\|_{L^5(\partial\Omega)}^5 d\tau \leq \liminf_{m\to\infty}\int_0^t \|\gamma^1 T_\varepsilon^m - T_b\|_{L^5(\partial\Omega)}^5 d\tau,$$

which comes from the convergence of $\gamma^1 T_\varepsilon$. ∎

2.4. Uniform boundness of the solutions. We finish this section by showing the nonnegative weak solutions are uniformly bounded. We have the following lemma.

LEMMA 9. *Assume the intial and boundary data satisfy* $0 \leq T_{\varepsilon 0} \leq \gamma, 0 \leq \psi_{\varepsilon 0} \leq \gamma^4$ *and* $0 \leq T_b \leq \gamma, 0 \leq \psi_b \leq \gamma^4$ *for some constant* $\gamma > 0$. *Then the weak solutions* $(T_\varepsilon, \psi_\varepsilon)$ *to system* (5)–(6) *subject to boundary condition* (11), (12), *or* (13) *are uniformly bounded, i.e.,* $0 \leq T_\varepsilon \leq \gamma, 0 \leq \psi_\varepsilon \leq \gamma^4$.

The proof of the above lemma is given in Appendix A.

3. Passage to the limit with weak compactness method. Now we state the main theorem of this paper.

THEOREM 10. *Consider a family of nonnegative weak solution* $(T_\varepsilon, \psi_\varepsilon)$ *of* (5)–(6) *with initial conditions* (7)–(8) *and boundary condition* (9) *for* ψ_ε *and boundary condition* (11) *or* (12) *or* (13) *for* T_ε, *defined in Definitions* 2, 5, *and* 7, *respectively. Assume the nonnegative initial data satisfy*

$$\|T_{\varepsilon 0} - \overline{T}_0\|_{L^5(\Omega)} \to 0 \quad \textit{and} \quad \|\psi_{\varepsilon 0} - \overline{T}_0^4\|_{L^2(\Omega\times\mathbb{S}^2)} \to 0 \quad \textit{as } \varepsilon \to 0,$$

where $\overline{T}_0 \in L^8(\Omega)$. *We also suppose that the well-prepared data condition* (19) *is satisfied and the boundary data satisfy* $T_b, \psi_b \geq 0$. *Then, when* $\varepsilon \to 0$, *we can extract a subsequence of* $(T_\varepsilon, \psi_\varepsilon)$ *such that for* $t > 0$,

$$T_\varepsilon \to \overline{T} \quad \textit{almost everywhere}, \tag{89}$$

$$\psi_\varepsilon \to \overline{T}^4 \quad \textit{strongly in } L^2([0,t] \times \Omega \times \mathbb{S}^2). \tag{90}$$

Moreover, $\overline{T} = \overline{T}(t,x)$ *is the weak solution of the limit equation*

$$\partial_t \left(\overline{T} + 4\pi \overline{T}^4\right) = \Delta \left(\overline{T} + \frac{4\pi}{3}\overline{T}^4\right) \tag{91}$$

with initial condition

$$\overline{T}(0,x) = \overline{T}_0(x), x \in \Omega, \tag{92}$$

and boundary condition

$$\overline{T}(t,x) = T_b(t,x),\ t > 0 \textit{ and } x \in \partial\Omega. \tag{93}$$

Proof. The proof can be divided into two steps. First we show the convergence of the solutions of the system (5)–(6), i.e., (89) and (90) hold. Then we show that the limit $\overline{T}$ satisfies (91) as well as the initial and boundary conditions.

Convergence of the solutions for system (5)–(6). From Theorem 3, or Theorem 6 or Theorem 8, we get, under any of the three types of boundary conditions considered in the above theorems,

$$\begin{aligned} &\|T_\varepsilon(t)\|^5_{L^5(\Omega)} + \|\psi_\varepsilon(t)\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \int_0^t \|\nabla T_\varepsilon^{\frac{5}{2}}\|^2_{L^2} d\tau \\ &\quad + \int_0^t \left\| \frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4) \right\|^2_{L^2(\Omega\times\mathbb{S}^2)} d\tau \leq C. \end{aligned} \tag{94}$$

Here C does not depend on ε.

Therefore, it follows that up to a subsequence,

$$\psi_\varepsilon \rightharpoonup \overline{\psi},\ \text{weakly in } L^2([0,t]; L^2(\Omega\times\mathbb{S}^2)), \tag{95}$$

$$T_\varepsilon \rightharpoonup \overline{T},\ \text{weakly in } L^5([0,t]; L^5(\Omega)), \tag{96}$$

$$T_\varepsilon^{\frac{5}{2}} \rightharpoonup \overline{T_\varepsilon^{\frac{5}{2}}},\ \text{weakly in } L^2([0,t]; H^1(\Omega)), \tag{97}$$

$$\psi_\varepsilon - T_\varepsilon^4 \to 0,\ \text{strongly in } L^2([0,t]; L^2(\Omega\times\mathbb{S}^2)), \tag{98}$$

$$\frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon) \rightharpoonup A,\ \text{weakly in } L^2([0,t]; L^2(\Omega\times\mathbb{S}^2)). \tag{99}$$

Here and below, $\overline{f_\varepsilon}$ denotes the weak limit of $\{f_\varepsilon\}_{\varepsilon>0}$ while $\varepsilon \to 0$.

Since

$$4\pi\|T_\varepsilon^4\|_{L^2(\Omega)} = \|T_\varepsilon^4\|_{L^2(\Omega\times\mathbb{S}^2)} \leq \|\psi_\varepsilon - T_\varepsilon^4\|_{L^2(\Omega\times\mathbb{S}^2)} + \|\psi_\varepsilon\|_{L^2(\Omega\times\mathbb{S}^2)},$$

we have

$$T_\varepsilon \text{ is uniformly bounded in } L^8([0,t]; L^8(\Omega)).$$

It follows that

$$T_\varepsilon^p \rightharpoonup \overline{T_\varepsilon^p}, \text{ weakly in } L^{q_1}([0,t];L^{q_2}(\Omega)), \tag{100}$$

for any $1 \le p \le 8$ and $q_1 \le \frac{8}{p}, q_2 \le \frac{8}{p}$.

Taking the integral of (6) over $\beta \in \mathbb{S}^2$ and adding (5), we get

$$\partial_t \left(T_\varepsilon + \langle \psi_\varepsilon \rangle\right) + \frac{1}{\varepsilon}\nabla \cdot \langle \psi_\varepsilon \beta\rangle = \Delta T_\varepsilon. \tag{101}$$

Since for all $t > 0$,

$$\|T_\varepsilon\|_{L^1([0,t];L^1(\Omega))} \le C\|T_\varepsilon\|_{L^5([0,t];L^5(\Omega))}$$

is uniform bounded in ε, we get that ΔT_ε is bounded in $L^1([0,t];W^{-2,1}(\Omega))$. Moreover, using (94), we have

$$\begin{aligned}
&\int_0^t \int_\Omega \int_{\mathbb{S}^2} \frac{1}{\varepsilon}\psi_\varepsilon \beta d\beta dx d\tau \\
&= \int_0^t \int_\Omega \int_{\mathbb{S}^2} \frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4)\beta d\beta dx d\tau \\
&\le \left(\frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4)^2 d\beta dx d\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega \int_{\mathbb{S}^2} |\beta|^2 d\beta dx d\tau\right)^{\frac{1}{2}} \\
&\le C\int_0^t \left\|\frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4)^2 d\beta dx d\tau\right\|_{L^2(\Omega\times\mathbb{S}^2)} \le C.
\end{aligned}$$

Therefore, $\frac{1}{\varepsilon}\nabla \cdot \langle \psi_\varepsilon \beta \rangle$ is bounded in $L^1([0,t];W^{-1,1}(\Omega))$. Consequently, we have

$$\partial_t (T_\varepsilon + \langle \psi_\varepsilon \rangle) \in L^1([0,t];W^{-2,1}(\Omega)). \tag{102}$$

In addition, from (94), we can get

$$T_\varepsilon + \langle \psi_\varepsilon \rangle \in L^2([0,t]\times\Omega). \tag{103}$$

From (95) and (96), we deduce

$$T_\varepsilon + \langle \psi_\varepsilon \rangle \rightharpoonup \overline{T} + \langle \overline{\psi} \rangle, \quad \text{weakly in } L^2([0,t]\times\Omega). \tag{104}$$

On the other hand, from (97), we have

$$T_\varepsilon^{\frac{5}{2}} \rightharpoonup \overline{T_\varepsilon^{\frac{5}{2}}}, \quad \text{weakly in } L^2([0,t];H^1(\Omega)). \tag{105}$$

By the embedding $L^6(\Omega) \subset H^1(\Omega)$, $\|(T_\varepsilon)^{\frac{5}{2}}\|_{L^2([0,t];L^6(\Omega))}$ is uniformly bounded with respect to ε. Thus by the continuity of translation on L^p space, $\|(T_\varepsilon)^{\frac{5}{2}}(\cdot,\cdot,\cdot) - (T_\varepsilon)^{\frac{5}{2}}(\cdot,\cdot+\xi,\cdot)\|_{L^2([0,t];L^6(\Omega))} \to 0$ as $|\xi| \to 0$. Then Lemma 15, with its assumptions verified by (102)–(105), implies that

$$\left(T_\varepsilon + \langle \psi_\varepsilon \rangle\right) T_\varepsilon^{\frac{5}{2}} \rightharpoonup \left(\overline{T} + \langle \overline{\psi}\rangle\right) \overline{T_\varepsilon^{\frac{5}{2}}} \quad \text{in the sense of distributions.} \tag{106}$$

Moreover, due to (98), we have $\overline{\psi} - \overline{T_\varepsilon^4} = 0$. Taking this into (106), we conclude that

$$\left(T_\varepsilon + \langle \psi_\varepsilon \rangle\right) T_\varepsilon^{\frac{5}{2}} \rightharpoonup (\overline{T} + 4\pi\overline{T_\varepsilon^4})\overline{T_\varepsilon^{\frac{5}{2}}} \quad \text{in the sense of distributions.} \tag{107}$$

On the other hand, using the weak convergence (100) with $p = \frac{7}{2}, \frac{13}{2}$ and the strong convergence (98), we get

$$
\begin{aligned}
&T_\varepsilon^{\frac{7}{2}} \rightharpoonup \overline{T_\varepsilon^{\frac{7}{2}}}, \quad \text{weakly in } L^2([0,t]\times\Omega),\\
&\langle\psi_\varepsilon\rangle T_\varepsilon^{\frac{5}{2}} = \langle\psi_\varepsilon - T_\varepsilon^4\rangle T_\varepsilon^{\frac{5}{2}} + 4\pi T_\varepsilon^{\frac{13}{2}} \rightharpoonup 4\pi\overline{T_\varepsilon^{\frac{13}{2}}}, \quad \text{weakly in } L^{\frac{16}{13}}([0,t]\times\Omega).
\end{aligned}
$$

Therefore,

$$
\left(T_\varepsilon + \int_{\mathbb{S}^2}\psi_\varepsilon\right)T_\varepsilon^{\frac{5}{2}} \rightharpoonup \overline{T_\varepsilon^{\frac{7}{2}}} + 4\pi\overline{T_\varepsilon^{\frac{13}{2}}} \text{ in the sense of distributions.}
$$

Comparing (107) and using the uniqueness of weak limits, we arrive at

$$
\overline{T_\varepsilon^{\frac{7}{2}}} + 4\pi\overline{T_\varepsilon^{\frac{13}{2}}} = \overline{T}\,\overline{T_\varepsilon^{\frac{5}{2}}} + 4\pi\overline{T_\varepsilon^4}\,\overline{T_\varepsilon^{\frac{5}{2}}}. \tag{108}
$$

Next we use the family of Young measures $\{\nu_x\}_{x\in\Omega}$ (see [10, Theorems 2.2, 2.3] and [49, 3, 4]) associated with the $\{T_{\varepsilon_n} : n \in \mathbb{N}\}$ to prove that (108) implies the strong convergence of $\overline{T_\varepsilon}$ to $\overline{T}$. Indeed, we have

$$
(T_{\varepsilon_n})^p \rightharpoonup \int_{\mathbb{R}} \lambda^p d\nu_x(\lambda) \tag{109}
$$

for any $p \geq 1$. Hence,

$$
\begin{aligned}
\overline{T_\varepsilon^{\frac{7}{2}}} + 4\pi\overline{T_\varepsilon^{\frac{13}{2}}} &= \int_{\mathbb{R}} \lambda^{\frac{7}{2}} d\nu_x(\lambda) + 4\pi\int_{\mathbb{R}} \lambda^{\frac{13}{2}} d\nu_x(\lambda),\\
\overline{T}\,\overline{T_\varepsilon^{\frac{5}{2}}} + 4\pi\overline{T_\varepsilon^4}\,\overline{T_\varepsilon^{\frac{5}{2}}} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mu\lambda^{\frac{5}{2}} d\nu_x(\lambda)d\nu_x(\mu) + 4\pi\int_{\mathbb{R}}\int_{\mathbb{R}} \lambda^4\mu^{\frac{5}{2}} d\nu_x(\lambda)d\nu_x(\mu).
\end{aligned}
$$

From (108), the above two equations equal, that is,

$$
\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\lambda^{\frac{5}{2}}(\lambda-\mu) + 4\pi\lambda^4(\lambda^{\frac{5}{2}} - \mu^{\frac{5}{2}})\right) d\nu_x(\lambda)d\nu_x(\mu) = 0.
$$

Using the symmetric property of the above formula leads to

$$
0 \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \left((\lambda^{\frac{5}{2}} - \mu^{\frac{5}{2}})(\lambda-\mu) + 4\pi(\lambda^4-\mu^4)(\lambda^{\frac{5}{2}} - \mu^{\frac{5}{2}})\right) d\nu_x(\lambda)d\nu_x(\mu) = 0.
$$

Since the function inside the integral is strictly positive unless $\lambda = \mu$, we can conclude that $\nu_x(\lambda)$ reduces almost all points of x to a family of Dirac masses concentrated at $\nu_x = \delta_{\overline{T}(x)}$. With this and the uniform boundness of $T_\varepsilon, \psi_\varepsilon$ according to Lemma 9, we can apply [10, Theorem 2.3] and conclude that

$$
T_\varepsilon \to \overline{T} \text{ almost everywhere and strongly in } L^p([0,t]\times\Omega), 1 < p \leq \infty. \tag{110}
$$

From this, we get $T_\varepsilon^4 \to \overline{T}^4$ almost everywhere. This combined with (98) implies that

$$
\psi_\varepsilon \to \overline{T}^4, \quad \text{strongly in } L^2([0,t]\times\Omega\times\mathbb{S}^2). \tag{111}
$$

Therefore, we have proved (89) and (90).

The limiting system. To show the limit function $\overline{T}$ satisfies (91), we define

$$\rho_\varepsilon = \int_{\mathbb{S}^2} \psi_\varepsilon d\beta,\ j_\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{S}^2} \psi_\varepsilon \beta d\beta\cdot$$

We have

$$\partial_t \rho_\varepsilon + \nabla \cdot j_\varepsilon = -\frac{1}{\varepsilon^2}\int_{\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4) d\beta\cdot \tag{112}$$

Comparing (5) and (112), we get

$$\partial_t T_\varepsilon - \Delta T_\varepsilon = -(\partial_t \rho_\varepsilon + \nabla \cdot j_\varepsilon).$$

Using (110)–(111) and $\rho_\varepsilon \to 4\pi \overline{T}^4$ we can pass to the limit in the above equation to get

$$\partial_t \overline{T} - \Delta \overline{T} = -4\pi \partial_t \overline{T}^4 + \nabla \cdot \overline{j_\varepsilon} \tag{113}$$

in the sense of distributions. Here we use $\overline{j_\varepsilon}$ to denote the weak limit of j_ε.

Next we find the weak limit of j_ε. Using (6), we can get

$$j_\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{S}^2} \psi_\varepsilon \beta d\beta = \frac{1}{\varepsilon}\int_{\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4)\beta d\beta = -\varepsilon \partial_t \int_{\mathbb{S}^2} \psi_\varepsilon \beta d\beta - \nabla \cdot \int_{\mathbb{S}^2} (\psi_\varepsilon \beta \otimes \beta) d\beta.$$

From the convergence of ψ_ε, we can see that for test function $\varphi \in C_c^1([0,t]\times\Omega)$,

$$\begin{aligned}&\int_0^\infty \iint_{\Omega\times\mathbb{S}^2} \partial_t \psi_\varepsilon \beta \varphi d\beta dx dt\\ &\quad = -\int_0^\infty \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon \beta \partial_t \varphi dx dt - \iint_{\Omega\times\mathbb{S}^2} \psi_{\varepsilon 0} \beta \varphi(0,\cdot) d\beta dx dt,\end{aligned}$$

which is bounded. Hence

$$\varepsilon \partial_t \int_{\mathbb{S}^2} \psi_\varepsilon \beta d\beta \rightharpoonup 0,\ \text{weakly in } L^2([0,t];L^2(\Omega))$$

as $\varepsilon \to 0$, and

$$\int_{\mathbb{S}^2} \psi_\varepsilon \beta \otimes \beta d\beta \to \overline{T}^4 \int_{\mathbb{S}^2} \beta \otimes \beta d\beta = \frac{4\pi}{3}\overline{T}^4 I,$$

where I is the identity matrix in $\mathbb{R}^3$. Therefore, we get

$$\nabla \cdot j_\varepsilon \rightharpoonup -\Delta(4\pi T_\varepsilon^4) \text{ in the sense of distributions.}$$

It follows from this and (113) that

$$\partial_t \overline{T} - \Delta \overline{T} = -4\pi \partial_t \overline{T}^4 + \frac{4\pi}{3}\Delta \overline{T}^4$$

holds in the sense of distributions, i.e., (91) holds.

The initial condition. Next we show the initial condition (92) of the limit system (91) holds in a weak sense. We consider

$$\int_0^t \int_\Omega \left(\frac{1}{\varepsilon}\int_{\mathbb{S}^2} \psi_\varepsilon \beta d\beta\right)^2 dx d\tau = \int_0^t \int_\Omega \left(\frac{1}{\varepsilon}\int_{\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4)\beta d\beta\right)^2 dx d\tau$$
$$\leq \frac{4\pi}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} \left(\psi_\varepsilon - T_\varepsilon^4\right)^2 d\beta dx d\tau,$$

which is uniformly bounded due to (94). This leads to

$$\partial_t(T_\varepsilon + \langle\psi_\varepsilon\rangle) \in L^2([0,t]; H^{-2}(\Omega)),$$

which combined with (110)–(111) implies that

$$\overline{T} + 4\pi\overline{T}^4 \in C_w([0,t]; L^2(\Omega)).$$

Consequently, we get

$$\overline{T}(t=0) + 4\pi\overline{T}^4(t=0) = \lim_{\varepsilon\to 0}(T_{\varepsilon 0} + \langle\psi_{\varepsilon 0}\rangle) = \overline{T}_0 + 4\pi\overline{T}_0^4.$$

Hence, from (18) it follows that

$$\overline{T}(t=0) = \overline{T}_0 = \lim_{\varepsilon\to 0} T_{\varepsilon 0},$$

in a weak sense.

Now let us deal with the boundary conditions.

Dirichlet boundary condition. For the boundary condition of $\overline{T}$, due to (97) and (110), there exists a subsequence $\{T_{\varepsilon_k}\}_{k=1}^\infty$ satisfying

$$T_{\varepsilon_k}^{\frac{5}{2}} \to \overline{T}^{\frac{5}{2}}, \text{ strongly in } L^2([0,t]; H^{1-\delta}(\Omega)),$$

for $0 < \delta < \frac{1}{2}$ small. Then we can use the continuity of the trace operator to get

$$\gamma^1 T_{\varepsilon_k}^{\frac{5}{2}} \to \gamma^1\overline{T}^{\frac{5}{2}}, \text{ strongly in } L^2([0,t]; L^2(\partial\Omega)). \tag{114}$$

For the Dirichlet boundary condition (12), we have $\gamma^1 T_\varepsilon = T_b$, hence

$$\gamma^1\overline{T} = T_b.$$

This verifies (93) in the Dirichlet case.

Robin boundary condition with $r > 0$. We can use the inequality from the energy inequality (81),

$$\frac{1}{\varepsilon^r}\int_0^t \|\gamma^1 T_\varepsilon - T_b\|_{L^5(\partial\Omega)}^5 d\tau \leq C,$$

to deduce that

$$\gamma^1 T_\varepsilon \to T_b, \text{ strongly in } L^5([0,t]; L^5(\partial\Omega)).$$

This combined with(114) leads to

$$\gamma^1\overline{T} = T_b.$$

The case of Robin boundary condition (13) with $r = 0$ will be shown later.

Boundary condition for $\overline{\psi}$. We show the boundary condition for $\overline{\psi}$. From (94), we get that

$$\psi_\varepsilon \in L^2([0,t]; L^2(\Omega \times \mathbb{S}^2)),$$
$$(\varepsilon\partial_t + \beta \cdot \nabla)\psi_\varepsilon = \frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4) \in L^2([0,t]; L^2(\Omega \times \mathbb{S}^2)),$$

and we can use the definition of the trace operator γ^2 to get

$$\gamma^2\psi_\varepsilon \rightharpoonup \overline{\gamma^2\psi_\varepsilon}, \text{ weakly in } L^2([0,t]; L^2(\Sigma; |n \cdot \beta| d\beta d\sigma_x)).$$

To show $\overline{\gamma^2\psi_\varepsilon} = \gamma^2\overline{T}^4$, we multiply (6) by $\varepsilon\rho$ with $\rho \in C^\infty([0,t] \times \Omega)$ and integrate over time and space; we get

$$\begin{aligned}
&\varepsilon \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon(t)\rho(t) d\beta dx d\tau - \varepsilon \iint_{\Omega\times\mathbb{S}^2} \psi_{\varepsilon 0}\rho(0) d\beta dx d\tau - \varepsilon \int_0^t \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon \partial_t \rho d\beta dx \\
&- \int_0^t \iint_{\Omega\times\mathbb{S}^2} \psi_\varepsilon \beta \cdot \nabla\rho d\beta dx d\tau + \int_0^t \iint_\Sigma (\beta \cdot n)\gamma^2\psi_\varepsilon \cdot \rho d\beta d\sigma_x d\tau \\
&= -\frac{1}{\varepsilon}\int_0^t \int_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4)\rho d\beta dx d\tau.
\end{aligned}$$

We pass to the limit $\varepsilon \to 0$ in the above equation and use (111) and (99) to get

$$\begin{aligned}
&- \int_0^t \iint_{\Omega\times\mathbb{S}^2} \overline{\psi}\beta \cdot \nabla\rho d\beta dx d\tau + \int_0^t \iint_\Sigma (\beta \cdot n)\overline{\gamma^2\psi_\varepsilon} \cdot \rho d\beta d\sigma_x d\tau \\
&= -\int_0^t \int_{\Omega\times\mathbb{S}^2} A \cdot \rho d\beta dx d\tau.
\end{aligned} \tag{115}$$

On the other hand, taking the weak limit in

$$\varepsilon\partial_t\psi_\varepsilon + \beta \cdot \nabla\psi_\varepsilon = -\frac{1}{\varepsilon}(\psi_\varepsilon - T_\varepsilon^4),$$

we get

$$\beta \cdot \nabla\overline{\psi} = A.$$

Applying the test function ρ on this equation leads to

$$\begin{aligned}
&- \int_0^t \iint_{\Omega\times\mathbb{S}^2} \overline{\psi}\beta \cdot \nabla\rho d\beta dx d\tau + \int_0^t \iint_\Sigma (\beta \cdot n)\gamma^2\overline{\psi} \cdot \rho d\beta d\sigma_x d\tau \\
&= -\int_0^t \int_{\Omega\times\mathbb{S}^2} A \cdot \rho d\beta dx d\tau.
\end{aligned}$$

Comparing the above equation with (115) leads to

$$\gamma^2\overline{\psi} = \overline{\gamma^2\psi_\varepsilon}.$$

On the other hand, we can use the boundary term in the energy inequality (57) or (81),

$$\frac{2\alpha - \alpha^2}{2\varepsilon} \int_0^t \|\gamma^2\psi_\varepsilon - \psi_b\|^2_{L^2(\Sigma_+; |n\cdot\beta| d\beta d\sigma_x)} d\tau \le C,$$

to deduce that

$$\gamma^2\psi_\varepsilon \to \psi_b, \text{ strongly in } L^2(\Sigma_+; |n\cdot\beta| d\beta d\sigma_x).$$

It follows that

$$\gamma^2\overline{\psi}|_{\Sigma_+} = \psi_b.$$

In addition, we can pass to the limit $\varepsilon \to 0$ in the boundary condition (9) to get

$$\gamma^2\overline{\psi}|_{\Sigma_-} = \alpha\psi_b + (1-\alpha)L\gamma^2\psi_\varepsilon|_{\Sigma_+} = \psi_b.$$

Therefore,

$$\gamma^2\overline{\psi} = \psi_b = T_b^4. \tag{116}$$

Robin boundary condition with $r = 0$. We can use the above formula to get

$$\gamma^2\overline{T}^4 = T_b^4,$$

from which we can deduce that

$$\gamma^2\overline{T} = T_b,$$

i.e., (93) also holds in the case of the Robin boundary condition with $r = 0$. Hence we finish the proof. ∎

Remark 2. Here in the proof we use the continuity of $T_\varepsilon^{\frac{5}{2}}$ and Lemma 15 to show the strong convergence of T_ε. If we drop the Laplacian term in (5), we can no longer show this by the above proof. However, thanks to the averaging lemma, i.e., Lemma 16, we have that for any $\eta \in C^\infty(\mathbb{S}^2)$,

$$\left\| \int_{\mathbb{S}^2} (\psi_\varepsilon(\cdot, \cdot + y, \beta) - \psi_\varepsilon(\cdot, \cdot, \beta))\eta(\beta)d\beta \right\|_{L^2([0,t];L^2(\mathbb{T}^3))} \to 0, \tag{117}$$

as $y \to 0$ uniformly in ε. Thus we can take h to be

$$\langle\psi_\varepsilon\rangle = \int_{\mathbb{S}^2} \psi_\varepsilon d\beta$$

instead of $(T_\varepsilon)^{\frac{5}{2}}$ in Lemma 15 and (106) becomes

$$(T_\varepsilon + \langle\psi_\varepsilon\rangle)\langle\psi_\varepsilon\rangle \rightharpoonup \left(\overline{T} + \langle\overline{\psi}\rangle\right)\langle\overline{\psi}\rangle. \tag{118}$$

Due to the strong convergence in (98),

$$\langle\psi_\varepsilon - T_\varepsilon^4\rangle \to 0,$$

which leads to

$$\begin{aligned}
&T_\varepsilon\langle\psi_\varepsilon\rangle = T_\varepsilon\langle\psi_\varepsilon - T_\varepsilon^4\rangle + 4\pi T_\varepsilon^5 \rightharpoonup \langle A\rangle\overline{T} + 4\pi\overline{T_\varepsilon^5},\\
&\langle\psi_\varepsilon\rangle\langle\psi_\varepsilon\rangle = \langle\psi_\varepsilon - T_\varepsilon^4\rangle\left(\langle\psi_\varepsilon - T_\varepsilon^4\rangle + 4\pi T_\varepsilon^4\right) + 4\pi T_\varepsilon^4\langle\psi_\varepsilon - T_\varepsilon^4\rangle + 16\pi^2 T_\varepsilon^8\\
&\qquad \rightharpoonup \langle A\rangle(\langle A\rangle + 4\pi\overline{T_\varepsilon^4}) + 4\pi\overline{T_\varepsilon^4}\langle A\rangle + 16\pi^2\overline{T_\varepsilon^8},
\end{aligned}$$

weakly in $L^2_{\mathrm{loc}}([0,\infty);L^2(\mathbb{T}^3))$ as $\varepsilon\to 0$. Thus we can also pass to the limit $\varepsilon\to 0$ in (118) and get the same limit with

$$\begin{aligned}&\langle A\rangle\overline{T}+4\pi\overline{T_\varepsilon^5}+\langle A\rangle(\langle A\rangle+4\pi\overline{T_\varepsilon^4})+4\pi\overline{T_\varepsilon^4}\langle A\rangle+16\pi^2\overline{T_\varepsilon^8}\\&=(\overline{T}+(\langle A\rangle+4\pi\overline{T_\varepsilon^4}))(\langle A\rangle+4\pi\overline{T_\varepsilon^4}),\end{aligned}$$

which implies that

$$4\pi\overline{T_\varepsilon^5}+16\pi^2\overline{T_\varepsilon^8}=\overline{T}4\pi\overline{T_\varepsilon^4}+16\pi^2\overline{T_\varepsilon^4}\cdot\overline{T_\varepsilon^4}.$$

We can also apply the Young measure theory to get

$$\begin{aligned}&4\pi\int_{\mathbb{R}}\lambda^5 d\nu_x(\lambda)+16\pi^2\int_{\mathbb{R}}\lambda^8 d\nu_x(\lambda)\\&=4\pi\int_{\mathbb{R}}\int_{\mathbb{R}}\lambda\mu^4 d\nu_x(\lambda)d\nu_x(\mu)+16\pi^2\int_{\mathbb{R}}\int_{\mathbb{R}}\lambda^4\mu^4 d\nu_x(\lambda)d\nu_x(\mu).\end{aligned}$$

It implies that

$$\begin{aligned}&4\pi\int_{\mathbb{R}}\int_{\mathbb{R}}\lambda(\lambda^4-\mu^4)d\nu_x(\lambda)d\nu_x(\mu)+16\pi^2\int_{\mathbb{R}}\int_{\mathbb{R}}\lambda^4(\lambda^4-\mu^4)d\nu_x(\lambda)d\nu_x(\mu)\\&=4\pi\int_{\mathbb{R}}\int_{\mathbb{R}}(\lambda-\mu)(\lambda^4-\mu^4)d\nu_x(\lambda)d\nu_x(\mu)+16\pi^2\int_{\mathbb{R}}\int_{\mathbb{R}}(\lambda^4-\mu^4)^2 d\nu_x(\lambda)d\nu_x(\mu)\\&=0.\end{aligned}$$

Therefore, ν_x is concentrated at $\delta_{T_\varepsilon(x)}$. We can thus conclude that (110) holds. Therefore, the above theorem also holds for the system

$$\begin{aligned}\partial_t T_\varepsilon=&\ \frac{1}{\varepsilon^2}(\langle\psi_\varepsilon\rangle-4\pi T_\varepsilon^4),\\ \partial_t\psi_\varepsilon+\frac{1}{\varepsilon}\beta\cdot\nabla\psi_\varepsilon=&\ -\frac{1}{\varepsilon^2}(\psi_\varepsilon-T_\varepsilon^4).\end{aligned}$$

4. The relative entropy method. The compactness method gives a clear justification of the diffusive limit of system (5)–(6). In this section, we give the rate of convergence of the diffusive limit under some regularity assumption of the limit system (16).

We will introduce a relative entropy functional to compare the solutions between (5)–(6) and (16). The difference of their solutions are estimated using this relative entropy functional.

To compare solutions of (5)–(6) and (16), we note that the limit system (16) does not include the equation for ψ_ε. To use the relative entropy method, we define $\overline{\psi}$ as

$$\overline{\psi}=\overline{T}^4-\varepsilon\beta\cdot\nabla\overline{T}^4-\varepsilon^2\partial_t\overline{T}^4+\varepsilon^2\beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4),$$

so that $\overline{T}$ and $\overline{\psi}$ satisfy

$$\partial_t\overline{T}=\ \Delta\overline{T}+\frac{1}{\varepsilon^2}\int_{\mathbb{S}^2}(\overline{\psi}-\overline{T}^4)d\beta,\tag{119}$$

$$\partial_t\overline{\psi}+\frac{1}{\varepsilon}\beta\cdot\nabla\overline{\psi}=\ -\frac{1}{\varepsilon^2}(\overline{\psi}-\overline{T}^4)+\overline{R},\tag{120}$$

where

$$\begin{aligned}\overline{R} =&\partial_t\overline{\psi} + \frac{1}{\varepsilon}\beta\cdot\nabla\overline{\psi} + \frac{1}{\varepsilon^2}(\overline{\psi}-\overline{T}^4)\\ =&\partial_t\overline{T}^4 + \frac{1}{\varepsilon}\beta\cdot\nabla\overline{T}^4 - \beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4) - \frac{1}{\varepsilon}\beta\cdot\nabla\overline{T}^4 - \partial_t\overline{T}^4 + \beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4)\\ &-\varepsilon\beta\cdot\nabla\partial_t\overline{T}^4 + \varepsilon\beta\cdot\nabla(-\partial_t\overline{T}^4 + \beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4)) - \varepsilon^2\partial_t^2\overline{T}^4 + \varepsilon^2\beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4)\\ =&\varepsilon\beta\cdot\nabla(-2\partial_t\overline{T}^4 + \beta\cdot\nabla(\beta\cdot\nabla\partial_t\overline{T}^4)) - \varepsilon^2(\partial_t\overline{T}^4 - \beta\cdot\nabla(\beta\cdot\nabla\partial_t\overline{T}^4)).\end{aligned}$$

We define the energy functional

$$H(T_\varepsilon,\psi_\varepsilon) := \int_\Omega \frac{T_\varepsilon^5}{5}dx + \iint_{\Omega\times\mathbb{S}^2}\frac{\psi_\varepsilon^2}{2}d\beta dx.$$

The relative energy functional is defined to be

$$\begin{aligned}&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\\ &:=E(T_\varepsilon,\psi_\varepsilon) - E(\overline{T},\overline{\psi}) - \left\langle\frac{\delta E}{\delta T}(\overline{T},\overline{\psi}), T_\varepsilon-\overline{T}\right\rangle - \left\langle\frac{\delta E}{\delta\psi}(\overline{T},\overline{\psi}),\psi_\varepsilon-\overline{\psi}\right\rangle\\ &=\int_\Omega\frac{T_\varepsilon^5-\overline{T}^5-5\overline{T}^4(T_\varepsilon-\overline{T})}{5}dx + \int_\Omega\int_{\mathbb{S}^2}\frac{(\psi_\varepsilon-\overline{\psi})^2}{2}d\beta dx.\end{aligned}$$

We will apply the relative entropy method to compare the solutions $(T_\varepsilon,\psi_\varepsilon)$ and $(\overline{T},\overline{\psi})$ under three different boundary conditions: in the torus (11), Dirichlet boundary condition (12), and Robin boundary condition (13). The main result of this section is the following theorem.

THEOREM 11. *Assume $(T_\varepsilon,\psi_\varepsilon)$ is a weak solution of the system* (5)–(6) *and $\overline{T}$ is a strong solution of* (5) *with $\overline{T}\in C^1([0,t];C^2(\Omega))$ for any $t>0$. Assume the well-prepared boundary condition* (19) *holds. Suppose $\overline{T}\geq c>0$. Then the following inequality holds:*

$$\begin{aligned}&\int_\Omega(T_\varepsilon-\overline{T})^2+(T_\varepsilon-\overline{T})^4\Big|_t dx + \iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})^2\Big|_t d\beta dx\\ &\leq\int_\Omega(T_{\varepsilon0}-\overline{T}_0)^2+(T_{\varepsilon0}-\overline{T}_0)^4dx + \iint_{\Omega\times\mathbb{S}^2}(\psi_{\varepsilon0}-\overline{\psi}_0)^2d\beta dx + C\varepsilon^s.\end{aligned}\tag{121}$$

Here $s=2$ for the case of torus, $s=\min\{1,r\}$ for the case of Robin boundary condition (13) *with $r>0$ and $s=1$ for the case of nonhomogeneous Dirichlet condition* (12).

Furthermore, if the initial data is well-prepared so that (18) *holds, and $T_{\varepsilon0}-\overline{T}_0\to 0$ as $\varepsilon\to0$, then $T_\varepsilon\to\overline{T}$ and $\psi_\varepsilon\to\overline{\psi}$ strongly in $L^2(\Omega)$ and $L^2(\Omega\times\mathbb{S}^2)$, respectively, for any $t>0$.*

4.1. The case of torus. We next derive the relative entropy inequality for the case of torus $\Omega=\mathbb{T}^3$.

LEMMA 12. *Assume T_ε is a weak solution of the system* (5)–(6), *and $\overline{T}$ is a smooth solution of* (16). *The following inequality holds:*

$$
\begin{aligned}
&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t + \frac{16}{25}\int_0^t\int_\Omega (\nabla T_\varepsilon^{\frac{5}{2}} - \nabla \overline{T}^{\frac{5}{2}})^2 dx d\tau \\
\le& H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0 + \frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac{5}{2}} - \overline{T}^{\frac{5}{2}} - \frac{5}{2}\overline{T}^{\frac{3}{2}}(T_\varepsilon - \overline{T}))\Delta\overline{T}^{\frac{5}{2}} dx d\tau \\
&+ \frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})\right)\left(\overline{\psi} - \overline{T}^4\right) d\beta dx d\tau \\
&- \int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon - \overline{\psi})\overline{R} d\beta dx d\tau - \frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon - T_\varepsilon^4 - (\overline{\psi} - \overline{T}^4))^2 d\beta dx d\tau.
\end{aligned} \tag{122}
$$

Proof. First, we recall that from (24), the energy functional for (5)–(6) satisfies

$$
H(T_\varepsilon,\psi_\varepsilon)\Big|_0^t + \frac{16}{25}\int_0^t\int_\Omega \left|\nabla (T_\varepsilon)^{\frac{5}{2}}\right|^2 dx d\tau + \frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2}(\psi_\varepsilon - (T_\varepsilon)^4)^2 d\beta dx d\tau \le 0. \tag{123}
$$

The function $(\overline{T},\overline{\psi})$ also satisfies a similar equality:

$$
\begin{aligned}
&H(\overline{T},\overline{\psi})\Big|_0^t + \frac{16}{25}\int_0^t\int_\Omega \left|\nabla (\overline{T})^{\frac{5}{2}}\right|^2 dx d\tau + \frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2}(\overline{\psi} - \overline{T}^4)^2 d\beta dx d\tau \\
&= \int_0^t\iint_{\Omega\times\mathbb{S}^2}\overline{\psi}\cdot\overline{R} d\beta dx d\tau.
\end{aligned} \tag{124}
$$

Next we consider the equations of the difference $(T_\varepsilon - \overline{T},\ \psi_\varepsilon - \overline{\psi})$ using the definition of weak solutions (22)–(23):

$$
\begin{aligned}
&-\int_0^\infty\int_\Omega \varphi_t (T_\varepsilon - \overline{T}) dx dt - \int_\Omega \varphi (T_\varepsilon - \overline{T})\Big|_{t=0} dx \\
&= \int_0^\infty\int_\Omega \Delta\varphi (T_\varepsilon - \overline{T}) dx dt + \frac{1}{\varepsilon^2}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4)\varphi d\beta dx dt,
\end{aligned} \tag{125}
$$

$$
\begin{aligned}
&-\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}\rho_t(\psi_\varepsilon - \overline{\psi}) d\beta dx dt - \int_\Omega\int_{\mathbb{S}^2}\rho(\psi_\varepsilon - \overline{\psi})\Big|_{t=0} d\beta dx \\
&\qquad - \frac{1}{\varepsilon}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon - \overline{\psi})\beta\cdot\nabla\rho d\beta dx dt \\
&= -\frac{1}{\varepsilon^2}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4)\rho d\beta dx dt - \int_0^\infty\iint_{\Omega\times\mathbb{S}^2}\rho\overline{R} d\beta dx dt.
\end{aligned} \tag{126}
$$

We introduce the following test function:

$$
\varphi = \theta(\tau)\overline{T}^4, \quad \rho = \theta(\tau)\overline{\psi}, \tag{127}
$$

where

$$
\theta(\tau) := \begin{cases} 1 & \text{for } 0 \le \tau < t, \\ \frac{t-\tau}{\delta} + 1 & \text{for } t \le \tau < t+\delta, \\ 0 & \text{for } \tau \ge t+\delta. \end{cases}
$$

Taking these test functions into (125)–(126) and letting $\delta \to 0$, we obtain

$$
\begin{aligned}
&\int_\Omega \overline{T}^4(T_\varepsilon - \overline{T})\Big|_{\tau=0}^{t} dx - \int_0^t \int_\Omega \partial_\tau(\overline{T}^4)(T_\varepsilon - \overline{T}) dx d\tau \\
&\quad = \int_0^t \int_\Omega \Delta\overline{T}^4 (T_\varepsilon - \overline{T}) dx d\tau + \frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{T}^4(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau. \\
&\int_\Omega \int_{\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi})\Big|_{\tau=0}^{t} d\beta dx - \int_0^t \int_\Omega \int_{\mathbb{S}^2} (\partial_\tau \overline{\psi})(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau \\
&\qquad - \frac{1}{\varepsilon}\int_0^t \int_\Omega \int_{\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})\beta \cdot \nabla\overline{\psi} d\beta dx d\tau \\
&\quad = -\frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau - \int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{\psi} \cdot \overline{R} d\beta dx d\tau.
\end{aligned}
$$

Using (119) and (120), the above equation becomes

$$
\begin{aligned}
&\int_\Omega \overline{T}^4(T_\varepsilon - \overline{T})\Big|_{\tau=0}^{t} dx \\
&= \int_0^t \int_\Omega 4\overline{T}^3 \Delta\overline{T}(T_\varepsilon - \overline{T}) dx d\tau + \int_0^t \int_\Omega \Delta\overline{T}^4 (T_\varepsilon - \overline{T}) dx d\tau \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} 4\overline{T}^3(\overline{\psi} - \overline{T}^4)(T_\varepsilon - \overline{T}) d\beta dx d\tau \\
&\quad + \frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{T}^4(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau, \\
&\int_\Omega \int_{\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi})\Big|_{\tau=0}^{t} d\beta dx \\
&= \int_0^t \int_\Omega \int_{\mathbb{S}^2} (-\frac{1}{\varepsilon}\beta \cdot \nabla\overline{\psi})(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau + \frac{1}{\varepsilon}\int_0^t \int_\Omega \int_{\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})\beta \cdot \nabla\overline{\psi} d\beta dx d\tau \\
&\quad - \frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} (\overline{\psi} - \overline{T}^4)(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau + \int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{R}(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau \\
&\quad - \frac{1}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau - \int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{\psi} \cdot \overline{R} d\beta dx d\tau.
\end{aligned}
$$

Adding them together gives

$$
\begin{aligned}
&\int_\Omega \overline{T}^4(T_\varepsilon - \overline{T})\Big|_{\tau=0}^{t} dx + \iint_{\Omega\times\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi})\Big|_{\tau=0}^{t} d\beta dx \\
&= \int_0^t \int_\Omega 4\overline{T}^3 \Delta\overline{T}(T_\varepsilon - \overline{T}) + \Delta\overline{T}^4 (T_\varepsilon - \overline{T}) dx d\tau \\
&\quad - \frac{1}{\varepsilon^2}\int_0^t \iint_{\Omega\times\mathbb{S}^2} (\overline{\psi} - \overline{T}^4)(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})) d\beta dx d\tau \\
&\quad - \frac{2}{\varepsilon^2}\int_0^t \int_\Omega \int_{\mathbb{S}^2} (\overline{\psi} - \overline{T}^4)(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau \\
&\quad + \int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{R}(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau - \int_0^t \int_\Omega \int_{\mathbb{S}^2} \overline{\psi} \cdot \overline{R} d\beta dx d\tau.
\end{aligned}
$$

We substract the above equation from the difference between (123) and (124) and arrive at the following inequality:

$$
\begin{aligned}
&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t \\
&\quad\le H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0-\frac{16}{25}\int_0^t\int_\Omega\left(\left|\nabla(T_\varepsilon)^{\frac52}\right|^2-\left|\nabla(\overline{T})^{\frac52}\right|^2\right)dxd\tau\\
&\qquad-\int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon-\overline{T})+\Delta\overline{T}^4(T_\varepsilon-\overline{T})dxd\tau\\
&\qquad-\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})\overline{R}d\beta dxd\tau\\
&\qquad-\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-T_\varepsilon^4-(\overline{\psi}-\overline{T}^4))^2d\beta dxd\tau\\
&\qquad+\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T})\right)\left(\overline{\psi}-\overline{T}^4\right)d\beta dxd\tau.
\end{aligned}\tag{128}
$$

To simplify the inequality, we rewrite the third term on the right-hand side as

$$
\begin{aligned}
&-\int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon-\overline{T})+\Delta\overline{T}^4(T_\varepsilon-\overline{T})dxd\tau\\
&\quad=\int_0^t\int_\Omega(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T}))\Delta\overline{T}dxd\tau\\
&\qquad-\int_0^t\int_\Omega(T_\varepsilon^4-\overline{T}^4)\Delta\overline{T}+\Delta\overline{T}^4(T_\varepsilon-\overline{T})dxd\tau\\
&\quad=\int_0^t\int_\Omega(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T}))\Delta\overline{T}dxd\tau-\frac{32}{25}\int_0^t\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dxd\tau\\
&\qquad-\int_0^t\int_\Omega T_\varepsilon^4\Delta\overline{T}+T_\varepsilon\Delta\overline{T}^4dxd\tau.
\end{aligned}\tag{129}
$$

Here we use the fact that

$$
\int_\Omega\overline{T}^4\Delta\overline{T}dx=\int_\Omega\overline{T}\Delta\overline{T}^4dx=-\frac{16}{25}\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dx.\tag{130}
$$

We calculate the last term in (129) as

$$
-\int_0^t\int_\Omega(T_\varepsilon^4\Delta\overline{T}+T_\varepsilon\Delta\overline{T}^4)dxd\tau=-\int_0^t\int_\Omega T_\varepsilon^4\Delta\overline{T}+4T_\varepsilon\overline{T}^3\Delta\overline{T}+12T_\varepsilon\overline{T}^2|\nabla\overline{T}|^2dxd\tau.
$$

Using

$$
\Delta\overline{T}^{\frac52}=\nabla\cdot\left(\frac52\overline{T}^{\frac32}\nabla\overline{T}\right)=\frac52\overline{T}^{\frac32}\Delta\overline{T}+\frac{15}{4}\overline{T}^{\frac12}|\nabla\overline{T}|^2,
$$

we obtain

$$
\begin{aligned}
&-\int_0^t\int_\Omega (T_\varepsilon^4\Delta\overline{T}+T_\varepsilon\Delta\overline{T}^4)dxd\tau\\
&\quad=-\int_0^t\int_\Omega T_\varepsilon^4\Delta\overline{T}+4T_\varepsilon\overline{T}^3\Delta\overline{T}+12T_\varepsilon\overline{T}^{\frac32}\cdot\frac{4}{15}(\Delta\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}\Delta\overline{T})dxd\tau\\
&\quad=-\int_0^t\int_\Omega T_\varepsilon^4\overline{T}+4T_\varepsilon\overline{T}^3\Delta\overline{T}+\frac{16}{5}T_\varepsilon\overline{T}^{\frac32}\Delta\overline{T}^{\frac52}-8T_\varepsilon\overline{T}^3\Delta\overline{T}dxd\tau\\
&\quad=-\int_0^t\int_\Omega T_\varepsilon^4\overline{T}-4T_\varepsilon\overline{T}^3\Delta\overline{T}+\frac{16}{5}T_\varepsilon\overline{T}^{\frac32}\Delta\overline{T}^{\frac52}dxd\tau\\
&\quad=-\int_0^t\int_\Omega (T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T}))\Delta\overline{T}dxd\tau+3\int_0^t\int_\Omega\overline{T}^4\Delta\overline{T}dxd\tau\\
&\qquad-\frac{16}{5}\int_0^t\int_\Omega T_\varepsilon\overline{T}^{\frac32}\Delta\overline{T}^{\frac52}dxd\tau.
\end{aligned}\tag{131}
$$

The last term in the above equation can be calculated as

$$
\begin{aligned}
-\frac{16}{5}\int_0^t\int_\Omega T_\varepsilon\overline{T}^{\frac32}\Delta\overline{T}^{\frac52}dxd\tau=&\frac{16}{5}\int_0^t\int_\Omega\frac25(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau\\
&-\frac{32}{25}\int_0^t\int_\Omega T_\varepsilon^{\frac52}\Delta\overline{T}^{\frac52}dxd\tau-\frac{16}{5}\frac35\int_0^t\int_\Omega\overline{T}^{\frac52}\Delta\overline{T}^{\frac52}dxd\tau.
\end{aligned}
$$

Taking this equation into (131) and using (130), we obtain

$$
\begin{aligned}
&-\int_0^t\int_\Omega (T_\varepsilon^4\Delta\overline{T}+T_\varepsilon\Delta\overline{T})dxd\tau\\
&\quad=-\int_0^t\int_\Omega (T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T}))\Delta\overline{T}dxd\tau\\
&\qquad+\frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau-\frac{32}{25}\int_0^t\int_\Omega T_\varepsilon^{\frac52}\Delta\overline{T}^{\frac52}dxd\tau.
\end{aligned}
$$

Taking it into (129) and using

$$
\begin{aligned}
&-\frac{16}{25}\int_0^t\left|\nabla T_\varepsilon^{\frac52}\right|^2-\frac{16}{25}\int_0^t\left|\nabla\overline{T}^{\frac52}\right|^2dxd\tau+\frac{32}{25}\int_0^t\nabla T_\varepsilon^{\frac52}\cdot\nabla\overline{T}^{\frac52}\\
&\quad=-\frac{16}{25}\int_0^t\int_\Omega(\nabla T_\varepsilon^{\frac52}-\nabla\overline{T}^{\frac52})^2dxd\tau,
\end{aligned}
$$

inequality (128) becomes (122) and finishes the proof. ∎

We now prove Theorem 11.

Proof of Theorem 11. From Lemma 12, we have

$$
\begin{aligned}
&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t+\frac{16}{25}\int_0^t\int_\Omega(\nabla T_\varepsilon^{\frac52}-\nabla\overline{T}^{\frac52})^2dxd\tau\\
&\le H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0+\frac{32}{25}\int_0^t\int_\Omega(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau\\
&\quad+\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T})\right)\left(\overline{\psi}-\overline{T}^4\right)d\beta dxd\tau\\
&\quad-\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})\overline{R}d\beta dxd\tau-\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-T_\varepsilon^4-(\overline{\psi}-\overline{T}^4))^2d\beta dxd\tau\\
&=I_1+I_2+I_3+I_4+I_5. \qquad (132)
\end{aligned}
$$

To control the relative entropy, we need the following lemma.

LEMMA 13. *Let $c>0$. Suppose $A\ge c$, $A+g\ge 0$; then*

$$(A+g)^5-A^5-5A^4g\ge(c^3|g|^2+c|g|^4).$$

Proof. We can prove this lemma by direct calculations:

$$
\begin{aligned}
&(A+g)^5-A^5-5A^4g\\
&\qquad=A^5+5A^4g+10A^3g^2+10A^2g^3+5Ag^4+g^5-A^5-5A^4g\\
&\qquad=10A^3g^2+10A^2g^3+5Ag^4+g^5\\
&\qquad\ge10A^3g^2+10A^2g^3+5Ag^4-Ag^4\\
&\qquad=10A^3g^2+10A^2g^3+4Ag^4\\
&\qquad=A^3g^2+\left(9A^3g^2+10A^2g^3+\frac{25}{9}Ag^4\right)+\frac{11}{9}Ag^4\\
&\qquad\ge A^3g^2+A^5\left(3\frac{g}{A}+\frac53\frac{g^2}{A^2}\right)^2+\frac{11}{9}Ag^4\\
&\qquad\ge c^3g^2+cg^4.
\end{aligned}
$$
∎

By applying Lemma 13 with $g:=T_\varepsilon-\overline{T}$ and $A=\overline{T}\ge c$, we have

$$H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\ge\int_\Omega\frac{T_\varepsilon^5}{5}-\frac{\overline{T}^5}{5}-\overline{T}^4(T_\varepsilon-\overline{T})dx\ge C\int_\Omega(T_\varepsilon-\overline{T})^2+(T_\varepsilon-\overline{T})^4dx. \qquad (133)$$

Now we estimate the right-hand side of inequality (132).

We first consider I_2. Using the mean value theorem, we obtain

$$
\begin{aligned}
T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T})&=\frac{15}{4}\int_0^1\int_0^r\left(s(T_\varepsilon-\overline{T})+\overline{T}\right)^{\frac12}dsdr\cdot(T_\varepsilon-\overline{T})^2\\
&\le\frac{15}{4}\int_0^1\int_0^r((s|T_\varepsilon-\overline{T}|)^{\frac12}+\overline{T}^{\frac12})dsdr\cdot(T_\varepsilon-\overline{T})^2\\
&\le C|T_\varepsilon-\overline{T}|^2+C|T_\varepsilon-\overline{T}|^4.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
I_2 =& \frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac{5}{2}} - \overline{T}^{\frac{5}{2}} - \frac{5}{2}\overline{T}^{\frac{3}{2}}(T_\varepsilon - \overline{T}))\Delta\overline{T}^{\frac{5}{2}} dxd\tau \\
\le& C\int_0^t\int_\Omega (T_\varepsilon - \overline{T})^2 + (T_\varepsilon - \overline{T})^4 dxd\tau. \qquad (134)
\end{aligned}
$$

Next we consider I_3. From the property that

$$
\begin{aligned}
T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T}) =& 6\overline{T}^2(T_\varepsilon - \overline{T})^2 + 4\overline{T}(T_\varepsilon - \overline{T})^3 + (T_\varepsilon - \overline{T})^4 \\
\le& C(T_\varepsilon - \overline{T})^2 + C(T_\varepsilon - \overline{T})^4,
\end{aligned}
$$

we have

$$
\int_0^t\int_\Omega (T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T}))\Delta\overline{T} dxd\tau \le C\int_0^t\int_\Omega (T_\varepsilon - \overline{T})^2 + (T_\varepsilon - \overline{T})^4 dxd\tau.
$$

So I_3 can be estimated as

$$
\begin{aligned}
I_3 =& \frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2} \left(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})\right)\left(\overline{\psi} - \overline{T}^4\right) d\beta dxd\tau \\
=& \frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2} (T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T}) \\
& \cdot (\overline{T}^4 - \varepsilon\beta\cdot\nabla\overline{T}^4 - \varepsilon^2\partial_t\overline{T}^4 + \varepsilon^2\beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4) - \overline{T}^4) d\beta dxd\tau \\
=& \int_0^t\int_\Omega \left(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})\right)(-4\pi\partial_t\overline{T}^4 + \frac{4}{3}\pi\Delta\overline{T}^4) dxd\tau \\
\le& C(\|\partial_t\overline{T}\|_{L^\infty} + \|\Delta\overline{T}\|_{L^\infty})\int_0^t\int_\Omega \left(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})\right) dxd\tau \\
\le& C\int_0^t\int_\Omega (6\overline{T}^2(T_\varepsilon - \overline{T})^2 + 4\overline{T}(T_\varepsilon - \overline{T})^3 + (T_\varepsilon - \overline{T})^4) dxd\tau \\
\le& C\int_0^t\int_\Omega (T_\varepsilon - \overline{T})^2 + (T_\varepsilon - \overline{T})^4 dxd\tau. \qquad (135)
\end{aligned}
$$

For I_4, we have

$$
\begin{aligned}
I_4 =& \int_0^t\int_\Omega\int_{\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})\overline{R} d\beta dxd\tau \\
\le& \int_0^t\int_\Omega\int_{\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})^2 d\beta dxd\tau + \int_0^t\int_\Omega\int_{\mathbb{S}^2} \overline{R}^2 d\beta dxd\tau \\
\le& \int_0^t\int_\Omega\int_{\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})^2 d\beta dxd\tau + C\varepsilon^2. \qquad (136)
\end{aligned}
$$

Taking the above estimate and (134)–(136) into (132), we get the estimate

$$\begin{aligned}&\int_\Omega (T_\varepsilon-\overline{T})^2+(T_\varepsilon-\overline{T})^4\Big|_t dx+\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})^2\Big|_t d\beta dx\\&\quad+\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-T_\varepsilon^4-(\overline{\psi}-\overline{T}^4))^2 d\beta dx d\tau\\&\quad+\frac{16}{25}\int_0^t\int_\Omega\left(\nabla(T_\varepsilon)^{\frac52}-\nabla(\overline{T})^{\frac52}\right)^2 dxd\tau\\&\le C\int_0^t\int_\Omega(T_\varepsilon-\overline{T})^2+(T_\varepsilon-\overline{T})^4dxd\tau+\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})^2d\beta dxd\tau+C\varepsilon^2.\end{aligned}$$

Applying Gronwall's lemma to the above inequality leads to (121) and finishes the proof. ∎

4.2. Dirichlet boundary conditions. In this case, we can do similar calculations as before and use the boundary condition

$$T_\varepsilon=\overline{T}=T_b\quad\text{for } x\in\partial\Omega$$

to get the relative entropy inequality.

LEMMA 14. *Assume T_ε is the weak solution of the system* (5)–(6) *with boundary conditions* (9) *and* (12)*, and $\overline{T}$ is a smooth solution of* (16) *with boundary condition $\overline{T}(t,x)=T_b$ for $x\in\partial\Omega$. We have the following inequality:*

(137)

$$\begin{aligned}&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t+\frac{16}{25}\int_0^t\int_\Omega(\nabla T_\varepsilon^{\frac52}-\nabla\overline{T}^{\frac52})^2dxd\tau\\&\le H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0+\frac{32}{25}\int_0^t\int_\Omega(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau\\&\quad+\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T})\right)\left(\overline{\psi}-\overline{T}^4\right)d\beta dxd\tau\\&\quad-\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})\overline{R}d\beta dxd\tau-\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-T_\varepsilon^4-(\overline{\psi}-\overline{T}^4))^2d\beta dxd\tau\\&\quad-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-\overline{\psi})^2d\sigma_xdxd\tau\\&\quad-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon'-\overline{\psi})^2d\sigma_xdxd\tau.\end{aligned}$$

Here, the above inequality does not include boundary terms of T_ε since $T_\varepsilon-\overline{T}$ vanishes on the boundary.

Proof. We can slightly modify the proof of Theorem 6 to show that the following energy inequality holds:

$$\begin{aligned}&H(T_\varepsilon,\psi_\varepsilon)\Big|_0^t+\frac{16}{25}\int_0^t\int_\Omega\left|\nabla(T_\varepsilon)^{\frac52}\right|^2dxd\tau+\frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2}(\psi_\varepsilon-(T_\varepsilon)^4)^2d\beta dxd\tau\\&\quad+\int_0^t\int_{\partial\Omega}T_b^4n\cdot\nabla T_\varepsilon d\sigma_xd\tau+\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)\psi_\varepsilon^2d\sigma_xdxd\tau\\&\quad+\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon')^2d\sigma_xdxd\tau\le 0.\end{aligned}\tag{138}$$

Similarily (119) and (120) also satisfy

$$
\begin{aligned}
&H(\overline{T},\overline{\psi})\Big|_0^t + \frac{16}{25}\int_0^t\int_\Omega \left|\nabla(\overline{T})^{\frac{5}{2}}\right|^2 dxd\tau + \frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2}(\overline{\psi}-\overline{T}^4)^2 d\beta dxd\tau\\
&\quad -\int_0^t\int_{\partial\Omega} T_b^4 n\cdot\nabla\overline{T}d\sigma_x d\tau + \frac{1}{2\varepsilon}\int_0^t\iint_\Sigma(\beta\cdot n)\overline{\psi}^2 d\sigma_x dxd\tau\\
&= \int_0^t\iint_{\Omega\times\mathbb{S}^2}\overline{\psi}\cdot\overline{R}d\beta dxd\tau. \qquad (139)
\end{aligned}
$$

We recall that from the definition of weak solutions (55)–(56), the difference $T_\varepsilon-\overline{T},\psi_\varepsilon-\overline{\psi}$ satisfy

$$
\begin{aligned}
&-\int_0^\infty\int_\Omega\varphi_t(T_\varepsilon-\overline{T})dxdt - \int_\Omega\varphi(T_\varepsilon-\overline{T})\Big|_{t=0}dx\\
&= \int_0^\infty\int_\Omega\Delta\varphi(T_\varepsilon-\overline{T})dxdt + \int_0^\infty\int_{\partial\Omega} n\cdot\nabla T_\varepsilon\varphi d\sigma_x dt\\
&\quad -\int_0^\infty\int_{\partial\Omega}\varphi n\cdot\nabla\overline{T}d\sigma_x dt - \int_0^\infty\int_{\partial\Omega}(T_\varepsilon-\overline{T})n\cdot\nabla\varphi dxdt\\
&\quad +\frac{1}{\varepsilon^2}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi}-T_\varepsilon^4+\overline{T}^4)\varphi d\beta dxdt, \qquad (140)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}\rho_t(\psi_\varepsilon-\overline{\psi})d\beta dxdt - \int_\Omega\int_{\mathbb{S}^2}\rho(\psi_\varepsilon-\overline{\psi})\Big|_{t=0}d\beta dx\\
&\quad -\frac{1}{\varepsilon}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})\beta\cdot\nabla\rho d\beta dxdt + \frac{1}{\varepsilon}\int_0^\infty\iint_{\Sigma_+}(\beta\cdot n)\psi_\varepsilon\rho d\sigma_x dxdt\\
&\quad +\frac{1}{\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon)\rho d\sigma_x dxdt - \frac{1}{\varepsilon}\int_0^t\iint_\Sigma(\beta\cdot n)\overline{\psi}\rho d\sigma_x dxdt\\
&= -\frac{1}{\varepsilon^2}\int_0^\infty\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi}-T_\varepsilon^4+\overline{T}^4)\rho d\beta dxdt - \int_0^\infty\iint_{\Omega\times\mathbb{S}^2}\rho\overline{R}d\beta dxdt. \qquad (141)
\end{aligned}
$$

We choose the test function the same as (127) and let $\delta\to 0$. We will get

$$
\begin{aligned}
&\int_\Omega\overline{T}^4(T_\varepsilon-\overline{T})\Big|_{\tau=0}^t dx\\
&= \int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon-\overline{T})dxd\tau + \int_0^t\int_\Omega\Delta\overline{T}^4(T_\varepsilon-\overline{T})dxd\tau\\
&\quad +\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}4\overline{T}^3(\overline{\psi}-\overline{T}^4)(T_\varepsilon-\overline{T})d\beta dxd\tau\\
&\quad +\frac{1}{\varepsilon^2}\int_0^t\int_\Omega\int_{\mathbb{S}^2}\overline{T}^4(\psi_\varepsilon-\overline{\psi}-T_\varepsilon^4+\overline{T}^4)d\beta dxd\tau\\
&\quad +\int_0^t\int_{\partial\Omega} n\cdot\nabla T_\varepsilon\overline{T}^4 d\sigma_x d\tau - \int_0^t\int_{\partial\Omega}T_b^4 n\cdot\nabla\overline{T}d\sigma_x d\tau\\
&\quad -\int_0^t\int_{\partial\Omega}(T_\varepsilon-\overline{T})n\cdot\nabla\overline{T}^4 dxd\tau,
\end{aligned}
$$

$$\iint_{\Omega\times\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi})\Big|_{\tau=0}^{t} d\beta dx$$
$$= -\frac{1}{\varepsilon}\int_0^t \iint_{\Sigma_+} (\beta\cdot n)\psi_\varepsilon \overline{\psi} d\sigma_x dx d\tau$$
$$-\frac{1}{\varepsilon}\int_0^t \iint_{\Sigma_-} (\beta\cdot n)(\alpha T_b^4 + (1-\alpha)\psi_\varepsilon)\overline{\psi} d\sigma_x dx d\tau$$
$$+\frac{1}{\varepsilon}\int_0^t \iint_{\Sigma} (\beta\cdot n)\overline{\psi}\cdot\overline{\psi} d\sigma_x dx d\tau - \frac{1}{\varepsilon^2}\int_0^t \int_\Omega\int_{\mathbb{S}^2} (\overline{\psi} - \overline{T}^4)(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau$$
$$+\int_0^t \int_\Omega\int_{\mathbb{S}^2} \overline{R}(\psi_\varepsilon - \overline{\psi}) d\beta dx d\tau - \frac{1}{\varepsilon^2}\int_0^t \int_\Omega\int_{\mathbb{S}^2} \overline{\psi}(\psi_\varepsilon - \overline{\psi} - T_\varepsilon^4 + \overline{T}^4) d\beta dx d\tau$$
$$-\int_0^t \int_\Omega\int_{\mathbb{S}^2} \overline{\psi}\cdot\overline{R} d\beta dx d\tau.$$

We substract the summation of the above two equations from the difference of (138) and (139) to get

(142)
$$H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t$$
$$\le H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0 - \frac{16}{25}\int_0^t\int_\Omega \left(\left|\nabla(T_\varepsilon)^{\frac{5}{2}}\right|^2 - \left|\nabla(\overline{T})^{\frac{5}{2}}\right|^2\right) dx d\tau$$
$$-\int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon - \overline{T}) + \Delta\overline{T}^4(T_\varepsilon - \overline{T}) dx d\tau$$
$$-\int_0^t\iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})\overline{R} d\beta dx d\tau - \frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4 - (\overline{\psi} - \overline{T}^4))^2 d\beta dx d\tau$$
$$+\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2} \left(T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T})\right)\left(\overline{\psi} - \overline{T}^4\right) d\beta dx d\tau$$
$$+\int_0^t\int_{\partial\Omega} (T_\varepsilon - \overline{T}) n\cdot\nabla\overline{T}^4 d\sigma_x d\tau$$
$$-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+} (\beta\cdot n)(\psi_\varepsilon - \overline{\psi})^2 d\sigma_x dx d\tau$$
(143)
$$-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-} (\beta\cdot n)(\alpha T_b^4 + (1-\alpha)\psi_\varepsilon' - \overline{\psi})^2 d\sigma_x dx d\tau.$$

By considering the boundary conditions, equation (129) becomes

$$-\int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon - \overline{T}) + \Delta\overline{T}^4(T_\varepsilon - \overline{T}) dx d\tau$$
$$= \int_0^t\int_\Omega (T_\varepsilon^4 - \overline{T}^4 - 4\overline{T}^3(T_\varepsilon - \overline{T}))\Delta\overline{T} dx d\tau + \int_0^t\int_\Omega (\overline{T}^4\Delta\overline{T} + \overline{T}\Delta\overline{T}^4) dx d\tau$$
$$-\int_0^t\int_\Omega (T_\varepsilon^4\Delta\overline{T} + T_\varepsilon\Delta\overline{T}^4) dx d\tau.$$

The last term is

$$
\begin{aligned}
&-\int_0^t\int_\Omega (T_\varepsilon^4\Delta\overline{T}+T_\varepsilon\Delta\overline{T}^4)dxd\tau\\
&\quad=-\int_0^t\int_\Omega (T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T}))\Delta\overline{T}dxd\tau\\
&\qquad+\frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau-\frac{32}{25}\int_0^t\int_\Omega T_\varepsilon^{\frac52}\Delta\overline{T}^{\frac52}dxd\tau\\
&\qquad+3\int_0^t\int_\Omega \overline{T}^4\Delta\overline{T}dxd\tau-\frac{48}{25}\int_0^t\int_\Omega \overline{T}^{\frac52}\Delta\overline{T}^{\frac52}dxd\tau.
\end{aligned}
$$

Adding the above two equations and using the integration-by-parts formulas

$$
\begin{aligned}
\int_\Omega \overline{T}^4\Delta\overline{T}dx&=-\frac{16}{25}\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dx+\int_{\partial\Omega}\overline{T}^4 n\cdot\nabla\overline{T}d\sigma_x,\\
\int_\Omega \overline{T}\Delta\overline{T}^4&=-\frac{16}{25}\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dx+\int_{\partial\Omega}\overline{T}n\cdot\nabla\overline{T}^4d\sigma_x,\\
\int_\Omega \overline{T}^{\frac52}\Delta\overline{T}^{\frac52}dx&=-\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dx+\frac52\int_{\partial\Omega}\overline{T}^4 n\cdot\nabla\overline{T}dx,\\
\int_\Omega T_\varepsilon^{\frac52}\Delta\overline{T}^{\frac52}dx&=-\int_\Omega \nabla T_\varepsilon^{\frac52}\cdot\nabla\overline{T}^{\frac52}dx+\frac52\int_{\partial\Omega}T_\varepsilon^{\frac52}\overline{T}^{\frac32}n\cdot\nabla\overline{T}dx,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&-\int_0^t\int_\Omega 4\overline{T}^3\Delta\overline{T}(T_\varepsilon-\overline{T})+\Delta\overline{T}^4(T_\varepsilon-\overline{T})dxd\tau\\
&=\frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau \qquad (144)\\
&\quad-\frac{32}{25}\int_0^t\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dxd\tau+\frac{32}{25}\int_0^t\int_\Omega \nabla T_\varepsilon^{\frac52}\cdot\nabla\overline{T}^{\frac52}dxd\tau\\
&\quad+\frac{16}{5}\int_0^t\int_{\partial\Omega}\overline{T}^4 n\cdot\nabla\overline{T}d\sigma_xd\tau-\frac{16}{5}\int_0^t\int_{\partial\Omega}T_\varepsilon^{\frac52}\overline{T}^{\frac32}n\cdot\nabla\overline{T}d\sigma_xd\tau\\
&=\frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau-\frac{32}{25}\int_0^t\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dxd\tau\\
&\quad+\frac{32}{25}\int_0^t\int_\Omega \nabla T_\varepsilon^{\frac52}\cdot\nabla\overline{T}^{\frac52}dxd\tau\\
&\quad-\frac{16}{5}\int_0^t\int_{\partial\Omega}\overline{T}^{\frac32}(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))n\cdot\nabla\overline{T}d\sigma_xd\tau\\
&\quad-2\int_0^t\int_{\partial\Omega}(T_\varepsilon-\overline{T})n\cdot\nabla\overline{T}^4d\sigma_xd\tau\\
&=\frac{32}{25}\int_0^t\int_\Omega (T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52}dxd\tau-\frac{32}{25}\int_0^t\int_\Omega\left|\nabla\overline{T}^{\frac52}\right|^2dxd\tau\\
&\quad+\frac{32}{25}\int_0^t\int_\Omega \nabla T_\varepsilon^{\frac52}\cdot\nabla\overline{T}^{\frac52}dxd\tau. \qquad (145)
\end{aligned}
$$

Taking the above equation into (128) leads to the inequality and finishes the proof. ∎

We now proceed to prove Theorem 11.

Proof of Theorem 11. Note that the relative entropy formula (137) only differs from (122) of the torus case by the last two boundary terms on the right-hand side of (137). To control these two terms, we recall

$$\overline{\psi}|_{\partial\Omega} = T_b^4 - \varepsilon\beta\cdot\nabla\overline{T}^4 - \varepsilon^2\partial_t\overline{T}^4 + \varepsilon^2\beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4) = T_b^4 + \varepsilon\overline{R}_b$$

with $\overline{R}_b = -\beta\cdot\nabla\overline{T}^4 - \varepsilon\partial_t\overline{T}^4 + \varepsilon\beta\cdot\nabla(\beta\cdot\nabla\overline{T}^4)$ bounded. So we have

$$\begin{aligned}
&-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-\overline{\psi})^2 d\sigma_x dx d\tau\\
&\quad=-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4-\varepsilon\overline{R}_b)^2 d\sigma_x dx d\tau\\
&\quad=-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau+\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)\overline{R}_b d\sigma_x dx d\tau\\
&\qquad-\frac{\varepsilon}{2}\int_0^t\iint_{\Sigma_+}\overline{R}_b^2 d\sigma_x dx d\tau.
\end{aligned}$$

From the coordinate transform $\beta'=\beta-2n(n\cdot\beta)$, we get $n\cdot\beta=-n\cdot\beta'$. We can also get $\beta=\beta'+2n(n\cdot\beta')$ and $\overline{R}_b$ can be also expressed using β'. We denote $\overline{R}_b'[\beta']=\overline{R}_b[\beta]$. We have

$$\begin{aligned}
&-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon'-\overline{\psi})^2 d\sigma_x dx d\tau\\
&\quad=-\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon'-T_b^4-\varepsilon\overline{R}_b)^2 d\sigma_x dx d\tau\\
&\quad=\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta'\cdot n)((1-\alpha)(\psi_\varepsilon'-T_b^4)-\varepsilon\overline{R}_b')^2 dS_{\beta',x} d\tau\\
&\quad=\frac{(1-\alpha)^2}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau\\
&\qquad-\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)\overline{R}_b' d\sigma_x dx d\tau+\frac{\varepsilon}{2}\int_0^t\iint_{\Sigma_+}(\overline{R}_b')^2 d\sigma_x dx d\tau.
\end{aligned}$$

Adding the above two equations together, we have

$$
\begin{aligned}
&-\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-\overline{\psi})^2 d\sigma_x dx d\tau \\
&\qquad -\frac{1}{2\varepsilon}\int_0^t \iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon'-\overline{\psi})^2 d\sigma_x dx d\tau \\
&\le -\frac{(2\alpha-\alpha^2)}{2\varepsilon}\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau \\
&\quad +\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)(\overline{R}_b-\overline{R}_b')d\sigma_x dx d\tau + C\varepsilon \\
&\le -\frac{(2\alpha-\alpha^2)}{2\varepsilon}\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau \\
&\quad +\frac{(2\alpha-\alpha^2)}{4\varepsilon}\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau \\
&\quad +\frac{C\varepsilon}{2\alpha-\alpha^2}\int_0^t \iint_{\Sigma_+}(\overline{R}_b-\overline{R}_b')^2 d\sigma_x dx d\tau + C\varepsilon \\
&\le -\frac{(2\alpha-\alpha^2)}{4\varepsilon}\int_0^t \iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-T_b^4)^2 d\sigma_x dx d\tau + C\varepsilon. \qquad (146)
\end{aligned}
$$

Combining the above estimates with the estimates of other terms in proof of the torus case and applying Gronwall's inequality lead to (121) with $s=1$ and finishes the proof. ∎

4.3. Robin boundary condition with $r>0$. For the case of Robin boundary condition (13), a similar relative entropy inequality can also be derived like for the Dirichlet case. The result is

$$
\begin{aligned}
&H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_t+\frac{16}{25}\int_0^t\int_\Omega(\nabla T_\varepsilon^{\frac52}-\nabla\overline{T}^{\frac52})^2 dx d\tau \\
&\le H(T_\varepsilon,\psi_\varepsilon|\overline{T},\overline{\psi})\Big|_0+\frac{32}{25}\int_0^t\int_\Omega(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))\Delta\overline{T}^{\frac52} dx d\tau \\
&\quad +\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left(T_\varepsilon^4-\overline{T}^4-4\overline{T}^3(T_\varepsilon-\overline{T})\right)\left(\overline{\psi}-\overline{T}^4\right)d\beta dx d\tau \\
&\quad -\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-\overline{\psi})\overline{R}d\beta dx d\tau-\frac{1}{\varepsilon^2}\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\psi_\varepsilon-T_\varepsilon^4-(\overline{\psi}-\overline{T}^4))^2 d\beta dx d\tau \\
&\quad +\int_0^t\int_{\partial\Omega}(T_\varepsilon^4-\overline{T}^4)\frac{T_b-T_\varepsilon}{\varepsilon^r}d\sigma_x d\tau-\int_0^t\int_{\partial\Omega}(T_\varepsilon-\overline{T})n\cdot\nabla\overline{T}^4 d\sigma_x d\tau \\
&\quad -\frac{16}{5}\int_0^t\int_{\partial\Omega}\overline{T}^{\frac32}(T_\varepsilon^{\frac52}-\overline{T}^{\frac52}-\frac52\overline{T}^{\frac32}(T_\varepsilon-\overline{T}))n\cdot\nabla\overline{T}d\sigma_x d\tau \\
&\quad -\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_+}(\beta\cdot n)(\psi_\varepsilon-\overline{\psi})^2 d\sigma_x dx d\tau \\
&\quad -\frac{1}{2\varepsilon}\int_0^t\iint_{\Sigma_-}(\beta\cdot n)(\alpha T_b^4+(1-\alpha)\psi_\varepsilon'-\overline{\psi})^2 d\sigma_x dx d\tau. \qquad (147)
\end{aligned}
$$

With the above relative entropy inequality we now proceed to prove Theorem 11.

Proof of Theorem 11. Compare the relative entropy inequality (147) with (137) of the Dirichlet case; there are two additional terms with the boundary of T_ε that we need to control. First, by (87) we have

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} (T_\varepsilon^4 - \overline{T}^4) \frac{T_b - T_\varepsilon}{\varepsilon^r} d\sigma_x d\tau \\
&= -\int_0^t \int_{\partial\Omega} (T_\varepsilon^4 - T_b^4) \frac{T_b - T_\varepsilon}{\varepsilon^r} d\sigma_x d\tau \\
&= -\frac{1}{\varepsilon^r} \int_0^t \int_{\partial\Omega} (T_\varepsilon - T_b)^2 (T_\varepsilon + T_b)(T_\varepsilon^2 + T_b^2) d\sigma_x d\tau \\
&\le -\frac{1}{\varepsilon^r} \int_0^t \int_{\partial\Omega} |T_\varepsilon - T_b|^5 d\sigma_x d\tau.
\end{aligned} \tag{148}$$

The second bounary term is

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} (T_\varepsilon - \overline{T}) n \cdot \nabla \overline{T}^4 d\sigma_x d\tau \\
&= \int_0^t \int_{\partial\Omega} (T_\varepsilon - T_b) n \cdot \nabla \overline{T}^4 d\sigma_x d\tau \\
&\le \frac{1}{2\varepsilon^r} \int_0^t \int_{\partial\Omega} |T_\varepsilon - T_b|^5 d\sigma_x d\tau + C\varepsilon^r \int_0^t \int_{\partial\Omega} |n \cdot \nabla \overline{T}^4|^2 d\sigma_x d\tau \\
&\le \frac{1}{2\varepsilon^r} \int_0^t \int_{\partial\Omega} |T_\varepsilon - T_b|^5 d\sigma_x d\tau + C\varepsilon^r.
\end{aligned} \tag{149}$$

Combining (148), (149), and the result of the Dirichlet case, we can conclude that the following inequality holds:

$$\begin{aligned}
&\int_\Omega (T_\varepsilon - \overline{T})^2 + (T_\varepsilon - \overline{T})^4 \Big|_t dx + \iint_{\Omega \times \mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})^2 \Big|_t d\beta dx \\
&\quad + \frac{1}{\varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{S}^2} (\psi_\varepsilon - T_\varepsilon^4 - (\overline{\psi} - \overline{T}^4))^2 d\beta dx d\tau \\
&\quad + \int_0^t \int_\Omega \left(\nabla (T_\varepsilon)^{\frac{5}{2}} - \nabla (\overline{T})^{\frac{5}{2}} \right)^2 dx d\tau + \frac{1}{2\varepsilon^r} \int_0^t \int_{\partial\Omega} |T_\varepsilon - T_b|^5 d\sigma_x d\tau \\
&\quad + \frac{(2\alpha - \alpha^2)}{\varepsilon} \int_0^t \iint_{\Sigma_+} (\beta \cdot n)(\psi_\varepsilon - T_b^4)^2 d\sigma_x dx d\tau \\
&\le C \int_0^t \int_\Omega (T_\varepsilon - \overline{T})^2 + (T_\varepsilon - \overline{T})^4 dx d\tau + \int_0^t \iint_{\Omega \times \mathbb{S}^2} (\psi_\varepsilon - \overline{\psi})^2 d\beta dx d\tau \\
&\quad + C\varepsilon + C\varepsilon^r.
\end{aligned}$$

Applying Gronwall's inequality leads to (121) and finishes the proof. ∎

For the case of the Robin boundary condition with $r = 0$, a boundary layer exists for T_ε; thus we cannot apply the above relative entropy method directly to show the convergence of T_ε, although with the compactness method this is done in Theorem 10. This boundary layer problem will be investigated in our future work.

Appendix A. Positivity of the solution of system (5)–(6). This section is devoted to the proof of Lemma 9 such that the nonnegativity of the solution for

(5)–(6), associated to the radiative boundary condition (9) and for a general boundary condition

$$a\varepsilon^r n\cdot\nabla T_\varepsilon(t,x) = -bT_\varepsilon(t,x) + bT_b(t,x) \text{ for any } x\in\partial\Omega, \tag{150}$$

where a and b are real numbers satisfying the condition of $|a|+|b|>0$. We add the reals number a and b to (13) in order to cover different boundary conditions.

The nonnegativity of the solutions to the approximate Galerkin approximate systems constructed in section 2 can be proved in the same way.

Proof of Lemma 9. For nonnegative initial and boundary radiative data $\psi_{\varepsilon 0}\geq 0, \psi_b\geq 0$, and due to $T_\varepsilon^4\geq 0$, following from the maximum principle for linear transport equations, the solution ψ_ε of the radiative transfer radiative (6) is nonnegative.

Next we show $T_\varepsilon\geq 0$. We define F on $\mathbb{R}_+\times\Omega\times\mathbb{R}$ by

$$F(t,x,y) = \frac{1}{\varepsilon^2}\langle\psi_\varepsilon(t,x,\beta) - y^4(t,x)\rangle. \tag{151}$$

The system (5), (150) can be rewritten as

$$\partial_t T_\varepsilon = \Delta T_\varepsilon + F(t,x,T_\varepsilon) \text{ for any } t>0\;\; x\in\Omega, \tag{152}$$

$$a\varepsilon^r n\cdot\nabla T_\varepsilon(t,x) = -bT_\varepsilon(t,x) + bT_b(t,x) \text{ for any } x\in\partial\Omega, \tag{153}$$

$$T_\varepsilon(t=0,x) = T_{\varepsilon 0}(x) \quad \text{for any } x\in\Omega. \tag{154}$$

Now, we define $\bar F$ in $\mathbb{R}_+\times\Omega\times\mathbb{R}$ by

$$\bar F(t,x,y) = \begin{cases} \frac{1}{\varepsilon^2}\langle\psi_\varepsilon(t,x,\beta) - y^4(t,x)\rangle & \text{if} \quad y\geq 0,\\ \frac{1}{\varepsilon^2}\langle\psi_\varepsilon(t,x,\beta)\rangle & \text{if} \quad y<0.\end{cases} \tag{155}$$

Let us consider $\bar T_\varepsilon$ the solution of the following system:

$$\begin{aligned} &\partial_t\bar T_\varepsilon = \;\Delta\bar T_\varepsilon + F(t,x,\bar T_\varepsilon) \text{ for any } t>0\;\; x\in\Omega,\\ &a\varepsilon^r n\cdot\nabla\bar T_\varepsilon(t,x) = -b\bar T_\varepsilon(t,x) + bT_b(t,x) \text{ for any } x\in\partial\Omega,\\ &\bar T_\varepsilon(t=0,x) = \;\bar T_{\varepsilon 0}(x) \quad \text{for any } x\in\Omega.\end{aligned} \tag{156}$$

The objective is to show that the solution $\bar T_\varepsilon$ of this equation remains nonnegative over time. Indeed, in this case $\bar F$ and F coincide; therefore we have by the uniqueness of the solution $T_\varepsilon = \bar T_\varepsilon$ which is nonnegative.

We set $\bar T_\varepsilon^+ = \max(T_\varepsilon,0)$ and $\bar T_\varepsilon^- = \max(-T_\varepsilon,0)$ such that $\bar T_\varepsilon = \bar T_\varepsilon^+ - \bar T_\varepsilon^-$. Multiplying (156) by $(-\bar T_\varepsilon^-)$ and integrating over Ω, we obtain

$$-\int_\Omega \partial_t\bar T_\varepsilon(t,x)\bar T_\varepsilon^-(t,x)dx + \int_\Omega \Delta\bar T_\varepsilon(t,x)\bar T_\varepsilon^-(t,x)dx = -\int_\Omega \bar F(t,x,\bar T_\varepsilon)\bar T_\varepsilon^-(t,x)dx.$$

Now, we have

$$-\int_\Omega \partial_t\bar T_\varepsilon(t,x)\bar T_\varepsilon^-(t,x)dx = \frac{1}{2}\partial_t\int_\Omega(\bar T_\varepsilon^-(t,x))^2dx, \tag{157}$$

$$\begin{aligned} -\int_\Omega \bar F(t,x,\bar T_\varepsilon)\bar T_\varepsilon^-(t,x)dx &= -\int_{\{\bar T_\varepsilon<0\}}\bar F(t,x,\bar T_\varepsilon)\bar T_\varepsilon^-(t,x)dx\\ &= -\int_{\{\bar T_\varepsilon<0\}}\frac{1}{\varepsilon^2}\langle\psi_\varepsilon(t,x,\beta)\rangle\bar T_\varepsilon^-(t,x)dx\leq 0,\end{aligned} \tag{158}$$

and

$$\int_\Omega \Delta \bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)dx = \int_\Omega (\nabla \bar{T}_\varepsilon^-(t,x))^2 dx + \int_{\partial\Omega} n\cdot\nabla\bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x.$$

If $a \neq 0$ (Robin or Neumann boundary conditions), then

$$\begin{aligned}\int_{\partial\Omega} n\cdot\nabla\bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x =& -\frac{b}{a\varepsilon^r}\int_{\partial\Omega}\bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x \\ &+\frac{b}{a\varepsilon^r}\int_{\partial\Omega} T_b(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x \\ =&\frac{b}{a\varepsilon^r}\int_{\partial\Omega}\left(\bar{T}_\varepsilon^-(t,x)\right)^2 d\sigma_x \\ &+\frac{b}{a\varepsilon^r}\int_{\partial\Omega} T_b(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x \geqslant 0.\end{aligned} \tag{159}$$

Now, if we have $a \neq 0$ (thus $b > 0$), since $\bar{T}_\varepsilon^- = 0$ on $\partial\Omega$ then

$$\int_{\partial\Omega} n\cdot\nabla\bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)d\sigma_x = 0. \tag{160}$$

In both cases, we have

$$\int_\Omega \Delta\bar{T}_\varepsilon(t,x)\bar{T}_\varepsilon^-(t,x)dx \geqslant 0. \tag{161}$$

Consequently, (157), (158), and (161) imply

$$\frac{1}{2}\partial_t\int_\Omega(\bar{T}_\varepsilon^-(t,x))^2dx \leq 0. \tag{162}$$

As $T_{\varepsilon 0}$ is nonnegative, we deduce from (162) that $\bar{T}_\varepsilon^- \equiv 0$. It follows that $\bar{T}_\varepsilon$ and consequently T_ε are nonnegative in $\mathbb{R}_+\times\Omega$.

Finally, we show if $T_{\varepsilon 0}, T_b \leq \gamma$ and $\psi_{\varepsilon 0}, \psi_b \leq \gamma^4$, then the solution $(T_\varepsilon, \psi_\varepsilon)$ to system (5)–(6) satisfies $T_\varepsilon \leq \gamma, \psi_\varepsilon \leq \gamma$ in Ω for any $t > 0$. Let $\gamma - T_\varepsilon = g, \gamma^4 - \psi_\varepsilon = \phi$; then (g,ϕ) satisfies

$$\partial_t g = \Delta g + \frac{1}{\varepsilon^2}\langle\phi - \gamma^4 + (\gamma - g)^4\rangle, \tag{163}$$

$$\partial_t\phi + \frac{1}{\varepsilon}\beta\cdot\nabla\phi = -\frac{1}{\varepsilon^2}(\phi - \gamma^4 + (g+\gamma)^4), \tag{164}$$

subject to the initial condition $g(t=0) = \gamma - T_{\varepsilon 0} \geq 0, \psi(t=0) = \gamma^4 - \psi_{\varepsilon 0} \geq 0$ and boundary condition

$$a\varepsilon^r n\cdot\nabla g = -b(g - (T_b - \gamma)).$$

We can take define $\bar{G}$ (155) to be

$$\bar{G} = \begin{cases} \frac{1}{\varepsilon^2}(\phi - (\gamma^4 - (\gamma - y)^4)) & \text{if} \quad y \geq 0, \\ \frac{1}{\varepsilon^2}\phi & \text{if} \quad y < 0, \end{cases}$$

and due to $\gamma - (\gamma - y)^4 \geq 0$ for $y \geq 0$, the same argument as before leads to the nonnegativity of the solutions to

$$\partial_t g = \Delta g + \langle G(\bar{g})\rangle,$$

$$\partial_t \phi + \frac{1}{\varepsilon}\beta \cdot \nabla\phi = -\bar{G}(g).$$

Due to $g \geq 0, \bar{G}(g) = (\phi - (\gamma^4 - (\gamma - g)^4))$ and so the solutions to (163)–(164) are nonnegative $g = \gamma - T_\varepsilon \geq 0, \phi = \gamma^4 - \psi_\varepsilon \geq 0$ and hence $T_\varepsilon \leq \gamma, \psi_\varepsilon \leq \gamma^4$ holds. ∎

Appendix B. Lemmas used in the compactness method. We recall now the compactness method to prove the weak convergence.

LEMMA 15 ([38, Lemma 5.1]). *Let g^n,h^n converge weakly to g, h, respectively, in $L^{p_1}(0,\tau;L^{p_2}(\Omega))$, $L^{q_1}(0,\tau;L^{q_2}(\Omega))$, where $1 \leq p_1, p_2 \leq +\infty$,*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

We assume in addition that

$$\frac{\partial g^n}{\partial t} \text{ is bounded in } L^1(0,\tau;W^{-m,1}(\Omega)) \text{ for some } m \geqslant 0 \text{ independent of } n,$$

$$\|h^n(.,t) - h^n(.+\xi,t)\|_{L^{q_1}(0,\tau;L^{q_2}(\Omega))} \to 0, \text{ as } |\xi| \to 0 \text{ uniformly in } n.$$

Then $g^n h^n$ converges to gh in the sense of distributions on $\Omega \times (0,\tau)$.

Let us recall the averaging lemma (see [30, 17, 41]).

LEMMA 16. *Let $\tau > 0$. Assume that ψ_ε is bounded in $L^2((0,\tau)\times\Omega\times\mathbb{S}^2)$, that g_ε is bounded in $L^2((0,\tau)\times\Omega\times\mathbb{S}^2)$, and that*

$$\varepsilon\partial_t\psi_\varepsilon + \beta\cdot\nabla_x\psi_\varepsilon = g_\varepsilon.$$

Then, for all $\rho \in C_0^\infty(\mathbb{S}^2)$,

$$\left\|\int_{\mathbb{S}^2}(\psi_\varepsilon(t,x+y,\beta) - h(t,x,v))\,\rho(\beta)d\beta\right\|_{L^2_{t,x}} \to 0, \text{ as } y \to 0 \text{ uniformly in } \varepsilon, \tag{165}$$

where ψ_ε has been prolonged by 0 for $x \notin \Omega$.

We consider only an average on the sphere and for the L^2 regularity of the solution. However, Lemma 16 can be obtained from the proof of [17, Theorems 3 and 6]. In our proof we follow [17, Theorem 3] with $q = p = 2$, $m = 1$, and $\tau = 0$ and we prove that $\int_{\mathbb{R}^3}\psi_\varepsilon(t,x,\beta)\rho(\beta)d\beta \in L^{r,\infty}(0,\tau;B^{s,r}_{\infty,\infty})$ where $r = 2$ and $s = \frac{1}{4}$, which implies the compactness result given in (165).

Proof. We start by rewriting the problem in the whole domain. Let us introduce the following cut-off functions χ_1 and χ_2 such that $\chi_1(t) = 1$ on $(\delta,\tau-\delta)$ for δ small enough, and $\chi_2(x) = 1$ on $\{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > \delta\}$. Denoting by $\chi(t,x) = \chi_1(t)\chi_1(x)$ and $\tilde{\psi}_\varepsilon(t,x) = \psi_\varepsilon(t,x)\chi(t,x)$, then

$$\varepsilon\partial_t\tilde{\psi}_\varepsilon + \beta\cdot\nabla_x\tilde{\psi}_\varepsilon = \chi g_\varepsilon + (\varepsilon\partial_t + \beta\cdot\nabla_x)\,\chi(t,x)\psi_\varepsilon(t,x,\beta). \tag{166}$$

From the uniform bound of ψ_ε and g_ε in L^2 space

$$\left\|\int_{\mathbb{S}^2}\left(\psi_\varepsilon(t,x,\beta) - \tilde{\psi}_\varepsilon(t,x,\beta)\right)\rho(\beta)d\beta\right\|_{L^2((0,\tau)\times\Omega)} \to 0,$$

when $\delta \to 0$ uniformly in ε.

Now, we have g_ε in L^2 and it is enough to prove the lemma for $\mathbb{R}_t \times \mathbb{R}^3_x$ instead of $(0,\tau)\times\Omega$. As the proof of [41, Lemma 4.2] is enough to prove

$$\int_{\mathbb{R}^3} \psi_\varepsilon(t,x,\beta)\rho(\beta)d\beta \in L^{r,\infty}(0,\tau; B^{s,r}_{\infty,\infty}),$$

where $r=2$ and $s=\frac{1}{4}$; for more details about the definition of Besov space built on the Lorentz space $L^{r,\infty}$ see [34]. As we said in the beginning we follow the proof of [41, Lemma 4.2]. So, we add $\lambda\psi_\varepsilon$ in both sides of (166), and we obtain

$$\lambda\psi_\varepsilon + \varepsilon\partial_t\psi_\varepsilon + \beta\cdot\nabla_x\psi_\varepsilon = g_\varepsilon + \lambda\psi_\varepsilon.$$

Then

$$\int_{\mathbb{S}^2} \psi_\varepsilon(t,x,\beta)\rho(\beta)d\beta = T_\lambda\left(g_\varepsilon + \lambda\psi_\varepsilon\right),$$

where

$$T_\lambda(g) = \int_0^\infty \int_{\mathbb{S}^2} g(t-s, x-s\beta, \beta)e^{\lambda s}\rho(\beta)d\beta ds.$$

Consequently, from [34, Proposition 3.1] it follows that

$$\|T_\lambda(g)\|_{L^2_t H^{1/2}_x} \le \lambda^{-1/2}\|g\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)}$$

and

$$\|T_\lambda(g)\|_{\lambda^{-3/2}L^2_t H^{-1/2}_x + \lambda^{-1/2}L^2_t H^{1/2}_x} \le C\|g\|_{L^2(L^2(\mathbb{R}\times\mathbb{R}^3;H^{-1}_x(\mathbb{R}^3))}.$$

Moreover, we have

$$\int_{\mathbb{S}^2} \psi_\varepsilon(t,x,\beta)\rho(\beta)d\beta = \eta = \eta^1 + \eta^2,$$

where

$$\|\eta^1\|_{L^2_t H^{1/2}_x} \le C\lambda^{1/2}\|\psi_\varepsilon\|_{L^2},$$

and

$$\|\eta^2\|_{\lambda^{-3/2}L^2_t H^{-1/2}_x + \lambda^{-1/2}L^2_t H^{1/2}_x} \le C\|g_\varepsilon\|_{L^2}.$$

Then by rewriting $\eta^2 = \eta^2_1 + \eta^2_2$ as

$$\|\eta^2_1\|_{L^2_t H^{-1/2}_x} \le C\lambda^{-3/2}\|g_\varepsilon\|_{L^2},$$

$$\|\eta^2_2\|_{L^2_t H^{1/2}_x} \le C\lambda^{-1/2}\|g_\varepsilon\|_{L^2},$$

we deduce that $\eta \in (L^2_t H^{1/2}_x, L^2_t H^{-1/2}_x)$ for the real interpolation of order $(\frac{1}{4},\infty)$; see [34, Proposition 3.1]. Then, for all $t\in\mathbb{R}_+$, we have

$$K(t) = \inf_{a_1+a_2=\eta} \|a_1\|_{L^2_t H^{1/2}_x} + t\|a_2\|_{L^2_t H^{-1/2}_x}.$$

Thus from [34], we need to show that $K(t) \le Ct^{1/4}$. We choose λ such that $t=\lambda^2$ for $t>0$.

- If $0<t<1$, we have $\|\eta^2_2\|_{L^2_t H^{-1/2}_x} \le C\lambda^{-3/2}$. Then η^2_2 and $a_1 = \eta^1$ and $a_2 = \eta^2$ we conclude that $K(t) \le Ct^{1/4}$.

- For $t > 1$, we can rewrite η^2 as $\eta^2 = \eta_3^2 + \eta_4^2$ such that $\eta_3^2 \in \lambda^{-3/2} L_t^2 H_x^{-1/2}$ and $\eta_4^2 \in \lambda^{1/2} L_t^2 H_x^{3/2}$; then we define

$$K_1(t) = \inf_{a_1+a_2=\eta} \|a_1\|_{L_t^2 H_x^{1/2} + L_t^2 H_x^{3/2}} + t\|a_2\|_{L_t^2 H_x^{-1/2}}$$

and we obtain $K_1(t) \leq Ct^{1/4}$ for $a_1 = \eta_1 + \eta_4^2$ and $a_2 = \eta_1^2 + \eta_3^2$, which implies that

$$\eta \in \left(L_t^2 H^{1/2} + L_t^2 H^{3/2}, L_t^2 H_x^{-1/2} \right). \qquad \square$$

Finally, we deduce the compactness result. This finishes the proof.

Appendix C. Basis in $L^2(\Omega \times \mathbb{S}^2)$.

LEMMA 17. *There exists an orthonormal basis* $\{\varphi_k\}_{k=1}^{\infty}$ *of* $L^2(\Omega \times \mathbb{S}^2)$ *with* $\varphi_k \in H^1(\Omega) \otimes L^2(\mathbb{S}^2)$.

Proof. We take $\{\phi_i\}_{i=1}^{\infty}$ to be an orthogonal basis in $H^1(\Omega)$ which is also an orthonormal basis in $L^2(\Omega)$ and $\{\chi_j\}_{j=1}^{\infty}$ to be a orthonormal basis in $L^2(\mathbb{S}^2)$. Then

$$\varphi_{ij} = \phi_i \chi_j$$

is an orthonormal basis in $L^2(\Omega \times \mathbb{S}^2)$. $\square$

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