

# THE MOD $p$ RIEMANN–HILBERT CORRESPONDENCE AND THE PERFECT SITE

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ABSTRACT. The mod  $p$  Riemann–Hilbert correspondence (in covariant and contravariant forms) relates  $\mathbb{F}_p$ -étale sheaves on the spectrum of an  $\mathbb{F}_p$ -algebra  $R$  and Frobenius modules over  $R$ . We give an exposition of these correspondences using Breen’s vanishing results on the perfect site.

## 1. INTRODUCTION

Let  $R$  be a commutative  $\mathbb{F}_p$ -algebra. A *Frobenius module* over  $R$  is the datum of an  $R$ -module  $M$  equipped with a Frobenius-semilinear map  $\varphi_M : M \rightarrow M$ . Equivalently, a Frobenius module over  $R$  is a left module over the twisted polynomial ring  $R[F]$ , i.e., the free associative algebra over  $R$  on a generator  $F$  with the relations  $Fa = a^p F$ ; we will typically use the latter notation.

The mod  $p$  Riemann–Hilbert correspondence relates  $\mathbb{F}_p$ -étale sheaves on  $\mathrm{Spec}(R)$  with various types of Frobenius modules over  $R$ , and originates in the following result of Katz describing locally constant constructible sheaves on  $\mathrm{Spec}(R)$ , when  $R$  is smooth over a perfect field.

**Theorem 1.1** ([Kat73, Prop. 4.1.1]). *Suppose  $R$  is smooth over a perfect field  $k$  of characteristic  $p$ . Then there is an equivalence of categories between locally constant constructible étale  $\mathbb{F}_p$ -sheaves on  $\mathrm{Spec}(R)$  and finitely generated projective  $R$ -modules  $M$  equipped with an isomorphism  $\varphi^* M \simeq M$ , for  $\varphi : R \rightarrow R$  the Frobenius.*

The question of generalizing Theorem 1.1 to more general constructible sheaves, and to more general  $\mathbb{F}_p$ -schemes, has been studied by a number of authors, including [EK04] (who construct a contravariant Riemann–Hilbert correspondence) and [BP09, BL19] (who construct a covariant Riemann–Hilbert functor). The purpose of this note is to revisit these results via the vanishing results of Breen, [Bre81].

Let us first describe the contravariant Riemann–Hilbert correspondence of Emerton–Kisin, [EK04], cf. also [Onk18, Sch16] for extensions to singular schemes. We consider the additive group  $\mathbb{G}_a$  as a sheaf on  $\mathrm{Spec}(R)_{\mathrm{et}}$ , with a natural left  $R[F]$ -module structure.

**Theorem 1.2** ([EK04]). *Suppose  $R$  is smooth over a perfect field  $k$  of characteristic  $p$ . Then the functor*

$$\mathrm{RH}_{\mathrm{cont}} \stackrel{\mathrm{def}}{=} \mathrm{RHom}_{\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)}(-, \mathbb{G}_a) : \mathcal{D}_{\mathrm{cons}}^b(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)^{op} \rightarrow \mathcal{D}(R[F])$$

*is fully faithful, with essential image consisting of those objects  $X \in \mathcal{D}(R[F])$  which are  $t$ -bounded and such that, for each  $i \in \mathbb{Z}$ :*

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<sup>1</sup>In [EK04], the Riemann–Hilbert functor is normalized with an additional cohomological shift; this does not affect the conclusions of this statement.

- (1)  $H^i(X)$  is a unit  $R[F]$ -module: that is,  $F$  induces an isomorphism of  $R$ -modules  $\varphi^* H^i(X) \xrightarrow{\sim} H^i(X)$ .
- (2)  $H^i(X)$  is finitely generated as an  $R[F]$ -module.

The covariant mod  $p$  Riemann–Hilbert correspondence is developed in the following form by Bhatt–Lurie [BL19, 2] and earlier (in a slightly different form) when  $R$  is noetherian by Böckle–Pink [BP09].

**Theorem 1.3** ([BL19, Th. 12.1.5], [BP09]). *Let  $R$  be any  $\mathbb{F}_p$ -algebra. There is a fully faithful, cocontinuous,  $t$ -exact embedding*

$$\mathrm{RH}_{\mathrm{cov}} : \mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p) \rightarrow \mathcal{D}(R[F])$$

*of the derived  $\infty$ -category of étale sheaves of  $\mathbb{F}_p$ -modules on  $\mathrm{Spec} R$  into the derived  $\infty$ -category of  $R[F]$ -modules. The essential image consists of those objects of  $X \in \mathcal{D}(R[F])$  such that for each  $i \in \mathbb{Z}$ :*

- (1)  $F$  acts isomorphically on  $H^i(X)$  (i.e., induces an isomorphism of  $R$ -modules  $H^i(X) \xrightarrow{\sim} \varphi_* H^i(X)$ ).
- (2) Each  $x \in H^i(X)$  satisfies an equation of the form  $F^n(x) + a_1 F^{n-1}(x) + \cdots + a_n x = 0$  for some  $a_1, \dots, a_n \in R$ .

Unlike Theorem 1.2, Theorem 1.3 applies to all of  $\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$  (not only to the constructible objects), works for arbitrary  $\mathbb{F}_p$ -algebras, and is  $t$ -exact (hence induces an equivalence on abelian categories). However, the functor  $\mathrm{RH}_{\mathrm{cov}}$  of Theorem 1.3 is constructed indirectly, as the left adjoint to an explicit solutions functor from  $\mathcal{D}(R[F])$  to  $\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$ .

In this note, we give an alternative exposition of Theorem 1.3 (and its analog for  $p$ -power torsion sheaves, Theorem 5.5 below), based on the vanishing results of [Bre81]. The strategy is as follows: any sheaf on the étale site of  $\mathrm{Spec}(R)$  can be pulled back to the site of *perfect* schemes over  $\mathrm{Spec}(R)$ , denoted  $\mathrm{Sch}_R^{\mathrm{perf}}$ , equipped with the étale topology, and the pullback functor  $\pi^*$  is fully faithful on derived  $\infty$ -categories. The results of [Bre81] give a fully faithful embedding of those objects of  $\mathcal{D}(R[F])$  satisfying condition (1) into  $\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)$ , cf. Theorem 4.3. Via a henselian rigidity argument, one then identifies directly the image of  $\pi^*$  in  $\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)$  with the image of those objects of  $\mathcal{D}(R[F])$  satisfying (1) and (2), from which the result follows.

As a consequence, we obtain (Construction 5.3) the following description of the covariant Riemann–Hilbert functor:

**Corollary 1.4.** *The covariant Riemann–Hilbert functor  $\mathrm{RH}_{\mathrm{cov}}$  is given by*

$$(1) \quad \mathrm{RH}_{\mathrm{cov}}(\mathcal{F}) = \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)}(\mathbb{G}_a, \pi^* \mathcal{F})[1], \quad \mathcal{F} \in \mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p),$$

*for  $\mathbb{G}_a$  the additive group.*

This formula (1) for  $\mathrm{RH}_{\mathrm{cov}}$  is thus analogous to the formula in Theorem 1.2, except that we additionally pull back to the perfect site.

We also give an exposition of Theorem 1.2 using similar methods. In particular, we show that a variant of the solutions functor from [EK04] gives a fully faithful embedding from the subcategory  $\mathcal{D}_{\mathrm{unit}}(R[F])$  of  $\mathcal{D}(R[F])$  consisting of  $t$ -bounded unit objects (i.e., satisfying condition (1) of Theorem 1.2 but without any finite generation assumptions) into the *big* étale site (Theorem 6.16 which is stated for more general  $R$ ); this fact is closely related to the extension by Kato [Kat86].

<sup>2</sup>In [BL19], the result is stated with additional boundedness assumptions.

Prop. 2.1] of the results of [Bre81]. Once again, we can identify those objects in  $\mathcal{D}_{\text{unit}}(R[F])$  additionally satisfying (2) of Theorem 1.2 with those sheaves on the big étale site satisfying the necessary rigidity properties to be pulled back from constructible objects on the small étale site. Along the way, we also show that  $\mathcal{D}_{\text{unit}}(R[F])$  is identified with the derived  $\infty$ -category of its heart, which is Lyubeznik’s category of  $F$ -modules or unit  $R[F]$ -modules [Lyu97]; this sharpens results of Ma [Ma14].

It will be convenient to work with unbounded derived  $\infty$ -categories of various sorts. These unbounded derived  $\infty$ -categories (for example, of  $p$ -power torsion étale sheaves on a qcqs  $\mathbb{F}_p$ -scheme) behave quite well because of the presence of bounds on cohomological dimension; we have included some general results in this direction, using that the relevant abelian categories are locally regular coherent (Definition 2.6).

The arguments of Breen [Bre81] rely on some facts about the Steenrod algebra; purely algebraic proofs via functor homology have also appeared, cf. [FLS94, Lem. 0.3] and [Kuh95, Cor. 1.2]. In the appendix, we explain another argument, suggested by Scholze, which proves the result using instead the  $v$ -descent results of [BS17].

**Notation.** We will freely use the theory of Grothendieck prestable  $\infty$ -categories of [Lur18, App. C].

We will frequently use the Breen–Deligne resolution in the form of [Sch, Lec. 4]. Given a site  $\mathcal{T}$  with associated topos of sheaves  $\text{Sh}(\mathcal{T})$ , and an abelian group object  $A \in \text{Sh}(\mathcal{T})$ , there is a *functorial* resolution  $X_\bullet(A)$  of  $A$  where each term  $X_i(A)$  is a finite direct sum of terms of the form  $\mathbb{Z}[A^j]$ ,  $j \geq 0$ .

For an  $\mathbb{F}_p$ -algebra  $R$ , we let  $R_{\text{perf}} = \varinjlim_{\varphi} R$  to be the direct limit perfection of  $R$ .

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## 2. PRELIMINARIES ON THE DERIVED CATEGORIES

Let  $\mathcal{T}$  be a site (we use the conventions of [Sta22, Tag 00VG]); moreover, suppose  $\mathcal{T}$  is coherent, i.e., fiber products exist in  $\mathcal{T}$  and any cover has a finite refinement. This implies that abelian sheaf cohomology on  $\mathcal{T}$  commutes with filtered colimits [AGV72, Exp. VI, Cor. 5.2].

We consider the unbounded derived  $\infty$ -category  $\mathcal{D}(\mathcal{T})$  of the Grothendieck abelian category of sheaves of abelian groups on  $\mathcal{T}$ , or equivalently [Lur18, Cor. 2.1.2.3] the  $\infty$ -category of hypercomplete sheaves on  $\mathcal{T}$  with values in  $\mathcal{D}(\mathbb{Z})$ . The presentable, stable  $\infty$ -category  $\mathcal{D}(\mathcal{T})$  is equipped with a natural  $t$ -structure, whose connective objects  $\mathcal{D}(\mathcal{T})^{\leq 0}$  form a Grothendieck prestable  $\infty$ -category in the sense of [Lur18, App. C]. We let  $\mathcal{D}(\mathcal{T})_{p\text{-tors}} \subset \mathcal{D}(\mathcal{T})$  denote the subcategory of  $p$ -power torsion objects: i.e., those  $X$  with  $X[1/p] = 0$ . Then  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  inherits a natural  $t$ -structure from  $\mathcal{D}(\mathcal{T})$ , and  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}^{\leq 0}$  is Grothendieck prestable by [Lur18, Prop. C.5.2.1]. We say that an object  $t \in \mathcal{T}$  has mod  $p$  cohomological dimension  $\leq d$  if for every sheaf  $\mathcal{F}$  of  $\mathbb{F}_p$ -vector spaces (and thus more generally every  $p$ -power torsion sheaf) on  $\mathcal{T}$ , the groups  $H^i(t, \mathcal{F}) = 0$  for  $i > d$ .

**Proposition 2.1.** *Suppose moreover that  $\mathcal{T}$  has a basis  $\mathcal{B}$  consisting of objects of mod  $p$  cohomological dimension  $\leq d$  for some fixed  $d$ . Then:*

- (1)  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is left complete.
- (2)  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is the derived  $\infty$ -category of the heart (of  $p$ -power torsion abelian sheaves on  $\mathcal{T}$ ).
- (3) The  $\infty$ -category  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}^{\leq 0}$  of connective objects in  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is compactly generated by the objects  $\mathbb{Z}/p^n[h_t], t \in \mathcal{B}, n \geq 0$ , for  $h_t$  the sheafification of the representable presheaf (of sets) defined by  $t$ . Moreover,  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is generated as a localizing subcategory by the  $\mathbb{Z}/p^n[h_t], t \in \mathcal{B}, n \geq 0$ .

*Proof.* Fix  $X \in \mathcal{D}(\mathcal{T})_{p\text{-tors}}$ . In  $\mathcal{D}(\mathcal{T})$ , the object  $X$  is the inverse limit of its Postnikov tower thanks to [CM21, Prop. 2.10], since by our assumptions the objects  $\mathbb{Z}[h_t], t \in \mathcal{B}$  have cohomological dimension  $\leq d$  with  $p$ -power torsion coefficients. Therefore, we find that for any  $t \in \mathcal{B}$ , the sections  $R\Gamma(t, X) = \mathrm{RHom}_{\mathcal{D}(\mathcal{T})}(\mathbb{Z}[h_t], X) \in \mathcal{D}(\mathbb{Z})$  is  $p$ -power torsion: this follows by passage up the Postnikov tower of  $X$ , using the cohomological dimension bound on  $t$  to see that the limit stabilizes in any range of degrees. Moreover, the same argument shows that the construction  $X \mapsto R\Gamma(t, X)$  therefore commutes with filtered colimits, from  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  to  $\mathcal{D}(\mathbb{Z})$ . This implies that  $\{\mathbb{Z}/p^n[h_t], t \in \mathcal{B}, n \geq 0\}$  are compact in  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$ .

We now show that  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is 0-complicial [Lur18, Def. C.5.3.1], i.e., for any object  $X \in \mathcal{D}(\mathcal{T})_{p\text{-tors}}^{\leq 0}$ , there exists a discrete object  $X'$  (which will in fact be a sum of  $\mathbb{Z}/p^n[h_t]$ 's) and a map  $X' \rightarrow X$  inducing a surjection on  $H^0$ . For every  $t \in \mathcal{B}$ , any map  $\mathbb{Z}[h_t] \rightarrow X$  necessarily factors over  $\mathbb{Z}/p^n[h_t] \rightarrow X$  for some  $n \gg 0$ , by the above. It follows that the natural map

$$\bigoplus_{n \geq 0, t \in \mathcal{B}, f: \mathbb{Z}/p^n[h_t] \rightarrow X} \mathbb{Z}/p^n[h_t] \rightarrow X$$

induces a surjection on  $H^0$ , and the former is discrete. This gives the desired 0-complicial statement. Since  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is clearly left separated, we find by [Lur18, Prop. C.5.4.5] (see also [Lur18, Rem. C.5.4.11]) that  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is the derived  $\infty$ -category of its heart. Moreover, [Lur18, Cor. C.2.1.7] now shows that the  $\mathbb{Z}/p^n[h_t], t \in \mathcal{B}, n \geq 0$  generate  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}^{\leq 0}$  under colimits (and the full  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  as a localizing subcategory).

Since the abelian category of  $p$ -power torsion sheaves on  $\mathcal{T}$  is generated by the objects  $\mathbb{Z}/p^n[h_t], t \in \mathcal{B}, n \geq 0$ , we find from [CM21, Ex. 2.21] that  $\mathcal{D}(\mathcal{T})_{p\text{-tors}}$  is Postnikov complete.  $\square$

**Example 2.2** (The small étale site). Let  $X$  be any qcqs  $\mathbb{F}_p$ -scheme. The étale site  $X_{\text{et}}$  has a basis of objects of mod  $p$  cohomological dimension  $\leq 1$ . In fact, for any affine  $\mathbb{F}_p$ -scheme  $U$ , and any sheaf  $\mathcal{F}$  of  $\mathbb{F}_p$ -vector spaces on  $U_{\text{et}}$ , we have  $H^i(U, \mathcal{F}) = 0$  for  $i > 1$ ; we may therefore take the affines as the desired basis.

To see this claim, we may assume that  $\mathcal{F}$  is constructible by writing  $\mathcal{F}$  as a filtered colimit of constructible sheaves [Sta22, Tag 03SA] and using [Sta22, Tag 073E]. By [Sta22, Tag 09YU], we may then assume that  $\mathcal{F}$  is pulled back from the spectrum of a finitely generated  $\mathbb{F}_p$ -algebra. Using [AGV72, Cor. 5.10, Exp. VII], we are thus reduced to proving the cohomological dimension bound in the noetherian case, where it is [AGV72, Th. 5.1, Exp. X].

Thus, we obtain the above conclusions for  $\mathcal{D}(X_{\text{et}})_{p\text{-tors}}$ . Namely,  $\mathcal{D}(X_{\text{et}})_{p\text{-tors}}$  is the derived  $\infty$ -category of the abelian category of  $p$ -power torsion sheaves of abelian groups on  $X$ , and it is left complete and compactly generated.

**Example 2.3** (The perfect site). Let  $X$  be a qcqs perfect  $\mathbb{F}_p$ -scheme. Let  $\mathrm{Sch}_X^{\mathrm{perf}}$  (also written  $\mathrm{Sch}_R^{\mathrm{perf}}$  if  $X = \mathrm{Spec}(R)$  is affine) be the site of all perfect qcqs  $X$ -schemes, equipped with the étale topology. Again, as in Example 2.2, the basis of affines shows that the above conclusions hold:  $\mathcal{D}(\mathrm{Sch}_X^{\mathrm{perf}})_{p\text{-tors}}$  is the derived  $\infty$ -category of the abelian category of  $p$ -power torsion sheaves of abelian groups on  $\mathrm{Sch}_X^{\mathrm{perf}}$ , and it is left complete and compactly generated.

We let  $(\pi^*, \pi_*)$  denote the natural adjunction

$$(\pi^*, \pi_*) : \mathcal{D}(X_{\mathrm{et}}) \rightleftarrows \mathcal{D}(\mathrm{Sch}_X^{\mathrm{perf}}).$$

from the derived  $\infty$ -category of the étale site to the derived  $\infty$ -category of  $\mathrm{Sch}_R^{\mathrm{perf}}$ , arising from the inclusion of sites  $X_{\mathrm{et}} \subset \mathrm{Sch}_X^{\mathrm{perf}}$ .

**Proposition 2.4.** *For any  $\mathcal{F} \in \mathcal{D}(X_{\mathrm{et}})$ , the adjunction map  $\mathcal{F} \rightarrow \pi_* \pi^* \mathcal{F}$  is an equivalence. Moreover,  $\pi_*$  is cocontinuous.*

*Proof.* For discrete objects, the result follows classically from the comparisons between big and small sites of [AGV72, Exp. VII, Sec. 4] or [Sta22, Tag 00XU]. The adjoint functors on hearts pass to adjoint functors on derived  $\infty$ -categories (note that both  $\pi^*, \pi_*$  are cocontinuous and exact on abelian categories), whence the claim, cf. [Lur18, Th. C.5.4.9]. This also follows from [CM21, Prop. 7.1].  $\square$

**Remark 2.5.** We can identify the image of the fully faithful embedding  $\pi^*$  of Proposition 2.4, at least on  $p$ -power torsion objects. An object  $\mathcal{F} \in \mathcal{D}(\mathrm{Sch}_X^{\mathrm{perf}})_{p\text{-tors}}$  belongs to the image of  $\pi^*$  if and only if:

- (1) As a functor on rings,  $\mathcal{F}$  commutes with filtered colimits.
- (2) For any local homomorphism of perfect, strictly henselian local rings  $R_1 \rightarrow R_2$  over  $X$ , the map  $\mathcal{F}(R_1) \rightarrow \mathcal{F}(R_2)$  is an equivalence.

In fact, the pullback of any object of  $\mathcal{D}(X_{\mathrm{et}})_{p\text{-tors}}$  clearly has these properties, since étale cohomology commutes with filtered colimits of rings [AGV72, Cor. 5.8, Exp. VII] and because of mod  $p$  cohomological dimension  $\leq 1$  for affines. Conversely, if  $\mathcal{F}$  has these properties, then the adjunction map  $\pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F}$  is an equivalence on strictly henselian local perfect rings over  $X$  (already by (2) alone), and hence in general by étale descent of both  $\pi^* \pi_* \mathcal{F}, \mathcal{F}$  and by (1) to identify the stalks of both sides.

We include for completeness some additional results on the structure of  $\mathcal{D}(X_{\mathrm{et}})_{p\text{-tors}}$  which will not be used in the construction of the Riemann–Hilbert correspondence. In particular, we identify the compact objects precisely, using the following definition.

**Definition 2.6.** We say that a Grothendieck abelian category  $\mathcal{A}$  is *locally regular coherent* if:

- (1)  $\mathcal{A}$  is compactly generated.
- (2) The subcategory  $\mathcal{A}_0 \subset \mathcal{A}$  of compact objects is closed under finite limits (so it is an abelian subcategory).
- (3) Given  $X \in \mathcal{A}_0$ , there exists an integer  $d_X$  such that one has  $\mathrm{Ext}_{\mathcal{A}}^i(X, Y) = 0$  for  $i > d_X$  and for all  $Y \in \mathcal{A}_0$ .
- (4) There exists an integer  $d$  and a subcategory  $\mathcal{A}'_0 \subset \mathcal{A}_0$  such that if  $X \in \mathcal{A}'_0, Y \in \mathcal{A}_0$ , then  $\mathrm{Ext}_{\mathcal{A}}^i(X, Y) = 0$  for  $i > d$ . The subcategory  $\mathcal{A}'_0$  generates  $\mathcal{A}$  under colimits.

The category of modules over any coherent ring such that any finitely presented module is of finite projective dimension (e.g., the infinite-dimensional polynomial algebra  $\mathbb{F}_p[x_1, x_2, x_3, \dots]$ ) is

an example of a locally regular coherent Grothendieck abelian category (take  $\mathcal{A}'_0$  to be the category of finitely generated free modules).

**Proposition 2.7.** *Suppose  $\mathcal{A}$  is a locally regular coherent Grothendieck abelian category. Then:*

- (1)  $\mathcal{D}(\mathcal{A})$  is Postnikov complete.
- (2) The compact objects of  $\mathcal{D}(\mathcal{A})$  consist of those objects which are  $t$ -bounded and all of whose cohomology groups belong to  $\mathcal{A}_0$ .

*Proof.* Let us first show that the objects of  $\mathcal{A}_0 \subset \mathcal{A}$  are pseudo-coherent (also called almost compact), i.e., they are compact in each truncation  $\mathcal{D}(\mathcal{A})^{\geq -n}$ . In fact, by [Lur18, Cor. C.6.5.9], the Grothendieck prestable  $\infty$ -category  $\mathcal{D}(\mathcal{A})^{\leq 0}$  is coherent in the sense of [Lur18, Def. C.6.5.1], and in particular any object of  $\mathcal{A}$  can be written as a filtered colimit of objects in  $\mathcal{A}$  which are pseudo-coherent objects in  $\mathcal{D}(\mathcal{A})^{\leq 0}$ , cf. [Lur18, Prop. C.6.5.6]; by compactness in the heart, this means that any object of  $\mathcal{A}_0$  is necessarily pseudo-coherent. Since the subcategory of pseudo-coherent objects of  $\mathcal{D}(\mathcal{A})^{\leq 0}$  is closed under finite limits (again by coherence, [Lur18, Def. C.6.5.1]) and therefore under extensions<sup>3</sup>, we conclude that  $\mathcal{A}_0$  is closed under extensions.

Now by pseudo-coherence, it follows that if  $X \in \mathcal{A}_0, Y \in \mathcal{A}$ , then  $\mathrm{Ext}_{\mathcal{A}}^i(X, Y) = 0$  for  $i > d_X$ . Moreover, if  $X \in \mathcal{A}'_0$ , then the vanishing holds for  $i > d$ . This also follows by pseudo-coherence of  $X$ , since we have assumed the vanishing for  $Y \in \mathcal{A}_0$ .

The first claim follows from [CM21, Ex. 2.21] since we have the generating objects  $\mathcal{A}'_0$ , which we have just seen to be of cohomological dimension  $\leq d$ .

Given  $X \in \mathcal{A}_0$ , we claim that  $X$  is actually compact in  $\mathcal{D}(\mathcal{A})$ . In fact, we have seen that  $X$  is pseudo-coherent. Since  $X$  has finite cohomological dimension by assumption, and since (as just proved) Postnikov towers converge in  $\mathcal{D}(\mathcal{A})$ , we conclude that  $X$  is compact in  $\mathcal{D}(\mathcal{A})$ . Consequently, any  $t$ -bounded object in  $\mathcal{D}(\mathcal{A})$  all of whose cohomology groups belong to  $\mathcal{A}_0$  is compact in  $\mathcal{D}(\mathcal{A})$ ; the converse claim follows now since  $\mathcal{A}_0$  yields a set of compact generators of  $\mathcal{D}(\mathcal{A})$  as a localizing subcategory, whence any compact object belongs to the thick subcategory generated by  $\mathcal{A}_0$ .  $\square$

**Proposition 2.8.** *For any qcqs  $\mathbb{F}_p$ -scheme  $X$ , the abelian category of  $p$ -power torsion étale sheaves on  $X$  is locally regular coherent. Thus, the compact objects of  $\mathcal{D}(X_{\mathrm{et}})_{p\text{-tors}}$  are precisely the objects which are bounded in the  $t$ -structure and such that each cohomology group is constructible.*

*Proof.* The abelian category  $\mathcal{A}$  of  $p$ -power torsion sheaves on  $X_{\mathrm{et}}$  is compactly generated by the constructible  $p$ -power torsion sheaves, which form an abelian subcategory  $\mathcal{A}_0 \subset \mathcal{A}$ , cf. [Sta22, Tags 05BE, 095M] or [AGV72, Exp. IX, sec. 2]. For each affine scheme  $U$  and étale map  $j : U \rightarrow X$ , the object  $j_!(\mathbb{Z}/p^n)$  is of cohomological dimension  $\leq 2$  and these objects generate  $\mathcal{A}$  under colimits.

In light of Proposition 2.7 (and Proposition 2.1) with  $\mathcal{A}'_0$  the collection  $\{j_!(\mathbb{Z}/p^n), j : U \rightarrow X \text{ étale}\}$ , it suffices to show that if  $\mathcal{F}$  is a constructible  $p$ -power torsion sheaf on  $X_{\mathrm{et}}$ , then  $\mathcal{F}$  has finite cohomological dimension in  $\mathcal{A}$ , i.e., there exists  $n$  such that  $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$  for any constructible  $p$ -power torsion sheaf  $\mathcal{G}$  on  $X_{\mathrm{et}}$  and  $i > n$ . By [AGV72, Exp. IX, Prop. 2.5], we can assume that  $\mathcal{F} = f_!\mathcal{F}_0$  for  $f : U \rightarrow X$  a locally closed constructible embedding and  $\mathcal{F}_0$  a locally constant constructible sheaf on  $U$ . Now  $Rf^!$  has finite cohomological dimension: in fact, this follows from factoring  $f$  into an open embedding and a closed embedding and using [AGV72, Exp. X, Th. 5.1] (and Example 2.2 in the non-noetherian case), which shows that any qcqs  $\mathbb{F}_p$ -scheme has bounded mod  $p$  cohomological dimension.

By duality, the result now follows because (writing  $\mathcal{F}_0^\vee = \mathrm{RHom}(\mathcal{F}_0, \mathbb{Z})$ )  $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = H^i(U, \mathcal{F}_0^\vee \otimes^{\mathbb{L}} Rf^!\mathcal{G})$  and  $U$  has bounded  $\mathbb{F}_p$ -cohomological dimension by [AGV72, Exp. X, Th. 5.1] again.  $\square$

<sup>3</sup>An extension is obtained as a fiber of a map to the suspension.



3.  $W(R)[F^{\pm 1}]$ -MODULES

Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra. The purpose of this section is to introduce the ring  $W(R)[F^{\pm 1}]$  and review the notion of an *algebraic*  $W(R)[F^{\pm 1}]$ -module, following [BL19]. Let  $W(R)$  denote the ring of Witt vectors of  $R$  with Witt vector Frobenius  $\varphi : W(R) \rightarrow W(R)$ .

**Construction 3.1** (The ring  $W(R)[F^{\pm 1}]$ ). Let  $W(R)[F^{\pm 1}]$  denote the *noncommutative* ring over  $W(R)$  defined as follows:  $W(R)[F^{\pm 1}]$  is spanned freely as a left module over  $W(R)$  by the powers  $F^i, i \in \mathbb{Z}$ , with the commutation rule  $F^i a = \varphi^i(a) F^i$  for  $a \in W(R), i \in \mathbb{Z}$ . We will also consider the quotient  $R[F^{\pm 1}] = W(R)[F^{\pm 1}]/p$ ; a module over  $W(R)[F^{\pm 1}]/p$  is a perfect Frobenius module over  $R$  in the sense of [BL19].

We will consider the derived  $\infty$ -category  $\mathcal{D}(W(R)[F^{\pm 1}])$  of left  $W(R)[F^{\pm 1}]$ -modules, and the full subcategory  $\mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}} \subset \mathcal{D}(W(R)[F^{\pm 1}])$  consisting of objects  $M$  with  $M[1/p] = 0$ .

**Remark 3.2.** Given  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  and a right  $W(R)[F^{\pm 1}]$ -module  $M'$  (more generally,  $M'$  could belong to the appropriate derived  $\infty$ -category), we can form the relative derived tensor product  $M' \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \in \mathcal{D}(\mathbb{Z})$ , cf. [Lur17] Sec. 4.4 for a very general account.

We will need the following example of this later. Given a *perfect*  $R$ -algebra  $R'$ , we can also make  $W(R')$  into a *right*  $W(R)[F^{\pm 1}]$ -module, where the action of  $F$  is by  $x \mapsto \varphi^{-1}(x)$ . If  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$ , then we can describe  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  as a cofiber

$$(2) \quad W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M = \text{cofib}(F - 1 : W(R') \otimes_{W(R)}^{\mathbb{L}} M \rightarrow W(R') \otimes_{W(R)}^{\mathbb{L}} M),$$

where  $W(R') \otimes_{W(R)}^{\mathbb{L}} M = W(R')[F^{\pm 1}] \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \in \mathcal{D}(W(R')[F^{\pm 1}])$  is the extension of scalars. In fact, this follows from the short exact sequence of right  $W(R)[F^{\pm 1}]$ -modules,

$$0 \rightarrow W(R')[F^{\pm 1}] \xrightarrow{(F-1)\cdot} W(R')[F^{\pm 1}] \rightarrow W(R') \rightarrow 0.$$

**Definition 3.3** (Cf. [BL19] Def. 2.4.1). A discrete  $W(R)[F^{\pm 1}]$ -module  $M$  is *algebraic* if  $M[1/p] = 0$  and if any  $x \in M$  satisfies an equation of the form  $F^n x + a_1 F^{n-1} x + \cdots + a_n x = 0$  for some  $a_1, \dots, a_n \in W(R)$  (equivalently, if the  $W(R)$ -submodule spanned by  $\{F^i x\}_{i \geq 0}$  is finitely generated).

An object of  $\mathcal{D}(W(R)[F^{\pm 1}])$  is *algebraic* if the cohomology modules are algebraic. We let  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}]) \subset \mathcal{D}(W(R)[F^{\pm 1}])$  be the full subcategory spanned by the algebraic objects.

**Proposition 3.4.** (1) *If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $W(R)[F^{\pm 1}]$ -modules, then  $M$  is algebraic if and only if  $M', M''$  are algebraic.*  
 (2) *The collection of algebraic  $W(R)[F^{\pm 1}]$ -modules is closed under arbitrary colimits.*

*Proof.* This follows from [BL19] Prop. 9.5.5; note that an  $W(R)[F^{\pm 1}]$ -module  $N$  with  $N[1/p] = 0$  is algebraic if and only if for each  $n$ , the  $p^n$ -torsion  $N[p^n]$  is an algebraic Frobenius  $W_n(R)$ -module in the sense of [BL19] Def. 9.5.2.  $\square$

**Corollary 3.5.** *The subcategory  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}]) \subset \mathcal{D}(W(R)[F^{\pm 1}])$  is a localizing subcategory.*

*Proof.* The subcategory  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}]) \subset \mathcal{D}(W(R)[F^{\pm 1}])$  is a localizing subcategory thanks to Proposition 3.4.  $\square$

Moreover, the  $t$ -structure on  $\mathcal{D}(W(R)[F^{\pm 1}])$  restricts to a  $t$ -structure on  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])$ , which is clearly left and right complete, and the subcategory  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])^{\leq 0}$  of connective objects is Grothendieck prestable, [Lur18] Prop. C.5.2.1].

**Example 3.6.** For any monic element  $T = F^n + a_1 F^{n-1} + \cdots + a_n \in W(R)[F^{\pm 1}]$  and  $j \geq 0$ , the quotient  $W(R)[F^{\pm 1}]/(p^j, W(R)[F^{\pm 1}]T)$  is algebraic. This can be checked directly or follows from [BL19, Prop. 4.2.1] (first reducing to  $j = 1$ ), since (in the terminology of *loc. cit.*)  $R[F^{\pm 1}]/R[F^{\pm 1}]T$  is the perfection of the Frobenius module  $R[F]/R[F]T$ , which is finite free over  $R$ , whence the former is holonomic in the sense of [BL19, Def. 4.1.1].

**Proposition 3.7.** *The  $\infty$ -category  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])^{\leq 0} \subset \mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])$  is compactly generated by the objects of Example 3.6. Consequently,  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])$  is generated as a localizing subcategory by the objects of Example 3.6. Moreover,  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])$  is the derived  $\infty$ -category of the abelian category of algebraic Frobenius modules.*

*Proof.* The objects of Example 3.6 are evidently compact, since the sequence  $(p^j, T) \in W(R)[F^{\pm 1}]$  is regular. Let  $M \in \mathcal{D}(W(R)[F^{\pm 1}])^{\leq 0}$  be algebraic. We claim that for any  $x \in H^0(M)$ , there exists a monic polynomial  $T = F^n + a_1 F^{n-1} + \cdots + a_n \in W(R)[F^{\pm 1}]$  and a map  $W(R)[F^{\pm 1}]/(p^n, T) \rightarrow M$  carrying the unit to  $x$ . In fact, since  $x$  is annihilated by some  $T$ , we obtain a map  $W(R)[F^{\pm 1}]/T \rightarrow M$  in  $\mathcal{D}(W(R)[F^{\pm 1}])$ ; this map is annihilated by some power of  $p$  by compactness of  $W(R)[F^{\pm 1}]/T \in \mathcal{D}(W(R)[F^{\pm 1}])^{\leq 0}$ , and thus factors over  $W(R)[F^{\pm 1}]/(p^n, T)$ . This proves the claim, whence the result follows in light of [Lur18, Cor. C.2.1.7] and [Lur18, Rem. C.5.4.11].  $\square$

**Proposition 3.8.** *Let  $R \rightarrow R'$  be any map of perfect  $\mathbb{F}_p$ -algebras. The extension of scalars functor  $W(R')[F^{\pm 1}] \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-) : \mathcal{D}(W(R)[F^{\pm 1}]) \rightarrow \mathcal{D}(W(R')[F^{\pm 1}])$  carries algebraic objects to algebraic objects.*

*Proof.* This follows from Proposition 3.7, since the generators of Proposition 3.7 are preserved under extension of scalars and extension of scalars is cocontinuous.  $\square$

We next observe that restriction of scalars along a closed embedding is fully faithful on  $p$ -power torsion objects of  $\mathcal{D}(W(R)[F^{\pm 1}])$ ; this is an analog of Kashiwara’s theorem, cf. [BL19, Th. 5.3.1].

**Proposition 3.9.** *Let  $R \rightarrow R'$  be any surjection of perfect  $\mathbb{F}_p$ -algebras. The restriction of scalars functor  $\mathcal{D}(W(R')[F^{\pm 1}])_{p\text{-tors}} \rightarrow \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  is fully faithful. The restriction of scalars functor carries algebraic objects into algebraic objects.*

*Proof.* The last assertion is evident, so it suffices to show that if  $M' \in \mathcal{D}(W(R')[F^{\pm 1}])_{p\text{-tors}}$ , the adjunction map  $W(R')[F^{\pm 1}] \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M' \rightarrow M'$  is an equivalence. This reduces by taking colimits and Postnikov towers to the case where  $M'$  is discrete and annihilated by  $p$ . As objects of  $\mathcal{D}(R')$ , the adjunction map is  $R' \otimes_R^{\mathbb{L}} M' \rightarrow M'$ , which is an equivalence since  $R' \otimes_R^{\mathbb{L}} R' \rightarrow R'$  is an equivalence, cf. [BS17, Lem. 11.10].  $\square$

**Proposition 3.10.** *Let  $C$  be a perfect  $R$ -algebra. Suppose  $C$  is integral over  $R$ . Consider  $C$  as a left  $W(R)[F^{\pm 1}]$ -module with  $F$  acting by Frobenius; then  $C$  is algebraic.*

*Proof.* Fix an element  $x \in C$ ; we need to show that it is annihilated by a monic polynomial in  $F$  in  $R[F^{\pm 1}]$ . Since  $C$  is a filtered colimit of perfections of finite, finitely presented  $R$ -algebras, we may reduce to the case when  $C$  is of this form. Now, we may descend  $C$  as follows: there exists a finitely generated subalgebra  $R_0 \subset R$ , a finite  $R_0$ -algebra  $C_0$ , and an isomorphism  $C = (C_0 \otimes_{R_0} R)_{\text{perf}}$ . Moreover, we can arrange that  $x$  descends to an element  $x_0 \in C_0$ . Since  $C_0$  is a finite  $R_0$ -module, the  $R_0$ -submodule of  $C_0$  generated by the Frobenius iterates of  $x_0$  is finitely generated, whence we obtain the algebraicity claim. See also [BL19, Prop. 4.2.1] for a more general result: any holonomic  $R[F^{\pm 1}]$ -module is algebraic.  $\square$



In the remainder of this section, we observe that the category of algebraic  $W(R)[F^{\pm 1}]$ -modules is locally regular coherent in the sense of Definition 2.6. This will not be needed for the proof of Theorem 5.5 (and in fact also follows from Theorem 5.5 together with Proposition 2.8), but we include an independent argument (relying on the results on holonomic  $R[F^{\pm 1}]$ -modules from [BL19]) for completeness.

An  $R[F^{\pm 1}]$ -module is defined to be *holonomic* in [BL19, Def. 4.1.1] if it is obtained via extension of scalars from an  $R[F]$ -module which is finitely presented as an  $R$ -module. The subcategory of holonomic modules is an abelian subcategory of the category of  $R[F^{\pm 1}]$ -modules and is stable under extensions, [BL19, Cor. 4.3.3]. The category of algebraic  $R[F^{\pm 1}]$ -modules (an  $R[F^{\pm 1}]$ -module is algebraic if it is algebraic as a  $W(R)[F^{\pm 1}]$ -module) is compactly generated, with the holonomic modules as the subcategory of compact objects [BL19, Th. 4.2.9] (where we use also that the subcategory of holonomic modules is idempotent-complete since it is abelian). We extend the definition to  $W(R)[F^{\pm 1}]$ -modules as follows.

**Definition 3.11.** A (discrete)  $W(R)[F^{\pm 1}]$ -module  $M$  is said to be *holonomic* if it is algebraic and if it is compact in the category of algebraic  $W(R)[F^{\pm 1}]$ -modules.

**Remark 3.12.** Since  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])^{\leq 0}$  is compactly generated by Proposition 3.7, it follows that the category of algebraic  $W(R)[F^{\pm 1}]$ -modules (the heart) is compactly generated, with compact objects the cokernels of maps between finite direct sums of the objects of Example 3.6. It also follows that an algebraic  $W(R)[F^{\pm 1}]$ -module is holonomic if and only if it is finitely presented as a module over  $W(R)[F^{\pm 1}]$ .

**Proposition 3.13.** *If  $R \rightarrow R'$  is any map of perfect  $\mathbb{F}_p$ -algebras, the extension of scalars functor  $W(R')[F^{\pm 1}] \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-) : \mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}]) \rightarrow \mathcal{D}_{\text{alg}}(W(R')[F^{\pm 1}])$  is  $t$ -exact.*

*Proof.* Since the functor is clearly right  $t$ -exact, it suffices to show that if  $M$  is a discrete algebraic  $W(R)[F^{\pm 1}]$ -module, then  $W(R')[F^{\pm 1}] \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  is discrete. This follows from [BL19, Th. 3.5.1] (and reduction by dévissage to the case  $pM = 0$ ).  $\square$

**Lemma 3.14.** *Suppose that  $R$  is the perfection of a finitely generated algebra over  $\mathbb{F}_p$ . Then:*

- (1) *The category of algebraic  $W(R)[F^{\pm 1}]$ -modules is locally noetherian (cf. [Lur18, Sec. C.6.8] for a treatment) with the holonomic modules as noetherian objects; the holonomic modules are also precisely the algebraic  $W(R)[F^{\pm 1}]$ -modules which are finitely generated as  $W(R)[F^{\pm 1}]$ -modules.*
- (2) *Any  $W(R)[F^{\pm 1}]$ -module annihilated by a power of  $p$  has finite projective dimension.*

*Proof.* To prove local noetherianity, we show that if  $M$  is an algebraic  $W(R)[F^{\pm 1}]$ -module such that  $M$  is finitely generated as a module over  $W(R)[F^{\pm 1}]$ , then any  $W(R)[F^{\pm 1}]$ -submodule  $M' \subset M$  is also finitely generated. In fact,  $M/pM$  is a finitely generated algebraic  $R[F^{\pm 1}]$ -module, and the category of algebraic  $R[F^{\pm 1}]$ -modules is locally noetherian [BL19, Prop. 4.3.1 and 3.4.3], with the holonomic (or in this case finitely generated algebraic) modules as noetherian objects. Intersecting the (finite)  $p$ -adic filtration on  $M$  with  $M'$ , we see that  $M'$  admits a finite filtration by finitely generated  $R[F^{\pm 1}]$ -modules, and hence is finitely generated over  $W(R)[F^{\pm 1}]$ . It follows that the category of finitely generated algebraic  $W(R)[F^{\pm 1}]$ -modules is an abelian category, and is precisely the subcategory of holonomic  $W(R)[F^{\pm 1}]$ -modules. This gives the local noetherianity and proves (1).

Finally, we claim that any  $W(R)[F^{\pm 1}]$ -module  $N$  which is annihilated by a power of  $p$  has finite projective dimension as a  $W(R)[F^{\pm 1}]$ -module. It suffices to show any  $R[F^{\pm 1}]$ -module has finite

projective dimension as an  $R[F^{\pm 1}]$ -module; this follows from [BL19, Rem. 3.1.8 and 3.2.8] together with [BS17, Prop. 11.31].  $\square$

**Proposition 3.15.** *For any perfect ring  $R$ , the category of algebraic  $W(R)[F^{\pm 1}]$ -modules is locally regular coherent. Moreover, any holonomic  $W(R)[F^{\pm 1}]$ -module  $M$  has finite projective dimension over  $W(R)[F^{\pm 1}]$ . Thus, the subcategory of compact objects in  $\mathcal{D}_{\text{alg}}(W(R)[F^{\pm 1}])$  is precisely the subcategory of  $t$ -bounded objects whose cohomology  $W(R)[F^{\pm 1}]$ -modules are holonomic.*

*Proof.* Given a holonomic  $W(R)[F^{\pm 1}]$ -module  $M$ , there exists a perfectly finitely generated subring  $R_0 \subset R$ , a holonomic  $W(R_0)[F^{\pm 1}]$ -module  $M_0$ , and an isomorphism  $M \simeq W(R)[F^{\pm 1}] \otimes_{W(R_0)[F^{\pm 1}]} M_0$ , thanks to Remark 3.12. By Proposition 3.13, the isomorphism also holds with the derived tensor product. It follows from Lemma 3.14 that  $M$  has finite projective dimension as a  $W(R)[F^{\pm 1}]$ -module.

Similarly, given any map  $f : M \rightarrow N$  between holonomic  $W(R)[F^{\pm 1}]$ -modules, we can descend the map to some  $R'_0$  and deduce (using Lemma 3.14 and  $t$ -exactness of base-change again) that the kernel of  $f$  is holonomic.

The objects of Proposition 3.7, which generate the category of algebraic  $W(R)[F^{\pm 1}]$ -modules, have projective dimension  $\leq 2$  as  $W(R)[F^{\pm 1}]$ -modules.

Combining all these assertions, we conclude that algebraic  $W(R)[F^{\pm 1}]$ -modules form a locally regular coherent Grothendieck abelian category, and the last claim follows from Proposition 2.7.  $\square$

#### 4. $W(R)[F^{\pm 1}]$ -MODULES AND SHEAVES

Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra. The purpose of this section is to embed  $\mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  as a full subcategory of  $\mathcal{D}(\text{Sch}_R^{\text{perf}})$ ; this embedding is a consequence of the results of [Bre81].

Let us first recall some aspects of Morita theory. Let  $\mathcal{C}$  be a presentable, stable  $\infty$ -category. Given an object  $X \in \mathcal{C}$ , we have an associative ring spectrum  $\text{End}_{\mathcal{C}}(X)$  and an adjunction

$$(X \otimes_{\text{End}_{\mathcal{C}}(X)} (-), \text{Hom}_{\mathcal{C}}(X, -)) : \text{RMod}(\text{End}_{\mathcal{C}}(X)) \rightleftarrows \mathcal{C}.$$

This can be constructed as follows. The Ind-completion of the thick subcategory  $\mathcal{C}_0$  of  $\mathcal{C}$  generated by  $X$  is equivalent to  $\text{RMod}(\text{End}_{\mathcal{C}}(X))$  by the Schwede–Shipley theorem [Lur17, Th. 7.1.2.1]; now the inclusion  $\mathcal{C}_0 \subset \mathcal{C}$  extends to a cocontinuous functor  $\text{Ind}(\mathcal{C}_0) \rightarrow \mathcal{C}$  and gives the desired adjunction. Compare also [Lur18, Ex. C.1.5.11] for an account of a very similar construction.

**Construction 4.1** (From Frobenius modules to sheaves). We define a cocontinuous functor

$$W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-) : \mathcal{D}(W(R)[F^{\pm 1}]) \rightarrow \mathcal{D}(\text{Sch}_R^{\text{perf}})$$

as follows. The Witt vector functor defines an object  $W$  of  $\mathcal{D}(\text{Sch}_R^{\text{perf}})$  with a right action of  $W(R)[F^{\pm 1}]$  (where  $F$  acts as the *inverse* of the Witt vector Frobenius, cf. Remark 3.2), whence Morita theory produces a cocontinuous functor  $\mathcal{D}(W(R)[F^{\pm 1}]) \rightarrow \mathcal{D}(\text{Sch}_R^{\text{perf}})$  whose the right adjoint is given by  $\text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W, -)$ .

**Remark 4.2.** A priori, the tensor product of Construction 4.1 involves étale sheafification, but in fact the construction can be carried out purely at the *presheaf* level for  $p$ -power torsion objects: that is, for any  $M \in \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$ , the construction carrying a perfect  $R$ -algebra  $R'$  to  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \in \mathcal{D}(\mathbb{Z})$  is already a hypercomplete étale sheaf.

In fact, let  $R'^{-1}$  be a perfect  $R$ -algebra, and let  $R'^{\bullet}$  be an étale hypercover of  $R'^{-1}$ . We need to show that the map  $W(R'^{-1}) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \rightarrow \varprojlim W(R'^{\bullet}) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  is an equivalence. Without loss of generality, we may base change and thus assume  $R'^{-1} = R$ .

By working up the Postnikov tower of  $M$ , we may assume that  $M$  is truncated and then (by flatness of each term in  $R'^\bullet$  over  $R$ ) that  $M$  is discrete, and even annihilated by  $p$ . In this case,  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  is expressed as the cofiber of the self-map  $F - 1$  on  $R' \otimes_R^{\mathbb{L}} M$ , cf. Remark 3.2. Since  $R' \mapsto R' \otimes_R^{\mathbb{L}} M$  satisfies flat hyperdescent [Lur18, Cor. D.6.3.3], we conclude.

**Theorem 4.3.** *The functor  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-) : \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}} \rightarrow \mathcal{D}(\text{Sch}_R^{\text{perf}})$  of Construction 4.1 is fully faithful, with right adjoint (and left inverse)  $\text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W, -)$ . The essential image is a localizing subcategory.<sup>4</sup>*

*Proof.* We need to show that if  $M, M' \in \mathcal{D}(W(R)[F^{\pm 1}])$  and if  $M'$  is  $p$ -power torsion, then the comparison map

$$(3) \quad \text{RHom}_{W(R)[F^{\pm 1}]}(M, M') \rightarrow \text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M, W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M')$$

is an equivalence. We may assume for the rest of the proof that  $M = W(R)[F^{\pm 1}]$ ; indeed, given  $M'$ , the collection of  $M$  such that the comparison map is an equivalence is a localizing subcategory of  $\mathcal{D}(W(R)[F^{\pm 1}])$  and  $\mathcal{D}(W(R)[F^{\pm 1}])$  is generated by  $W(R)[F^{\pm 1}]$  as a localizing subcategory. Thus, we want the map

$$(4) \quad M' \rightarrow \text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W, W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M')$$

to be an equivalence for any  $M' \in \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$ .

Let  $(f^*, f_*)$  be the geometric morphism between the perfect site of  $\mathbb{F}_p$  and the perfect site of  $R$ . Then  $W \in \mathcal{D}(\text{Sch}_R^{\text{perf}})$  is  $f^*W$  by representability, and  $f_*(W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M') = W \otimes_{W(\mathbb{F}_p)[F^{\pm 1}]}^{\mathbb{L}} M'$  (thanks to Remark 4.2) whence we reduce showing (4) is an equivalence to the absolute case  $R = \mathbb{F}_p$ , which we do for the remainder of the proof.

Using Remark 4.2, we can further reduce to the case where  $M'$  is truncated by passage up the Postnikov tower of  $M'$ . Since  $W$  is pseudo-coherent (i.e., compact in any truncation) in  $\mathcal{D}(\text{Sch}_{\mathbb{F}_p}^{\text{perf}})$  by the Breen–Deligne resolution (since  $W$  is representable, cf. [Sch, Lec. 4]), we can reduce further to the case where  $M'$  is discrete and  $pM' = 0$ .

Now when  $M' = \mathbb{F}_p[F^{\pm 1}]$  as well, the equivalence (4) is precisely the vanishing result of Breen, [Bre81] (also recalled below as Theorem A.1), which gives  $\text{RHom}_{\mathcal{D}(\text{Sch}_{\mathbb{F}_p}^{\text{perf}}, \mathbb{F}_p)}(\mathbb{G}_a, \mathbb{G}_a) = \mathbb{F}_p[F^{\pm 1}]$ . By pseudo-coherence again, the equivalence follows for any free (discrete)  $W(R)[F^{\pm 1}]/p = \mathbb{F}_p[F^{\pm 1}]$ -module. Since  $W(R)[F^{\pm 1}]/p = \mathbb{F}_p[F^{\pm 1}]$  has global dimension 1, we deduce (4) for any discrete  $W(R)[F^{\pm 1}]/p$ -module, and conclude.  $\square$

## 5. THE COVARIANT RIEMANN–HILBERT CORRESPONDENCE

Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra. In this section, we construct the  $\text{RH}_{\text{cov}}$  functor from  $\mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}}$  to  $\mathcal{D}(W(R)[F^{\pm 1}])$  and prove Theorem 5.5 below. We start by reviewing a version of the Artin–Schreier sequence.

**Remark 5.1** (The Artin–Schreier sequence for animated  $\mathbb{F}_p$ -algebras). Let  $A$  be any animated  $\mathbb{F}_p$ -algebra. Then there is a natural fiber sequence

$$R\Gamma_{\text{et}}(\text{Spec}(A), \mathbb{F}_p) \rightarrow A \xrightarrow{F-1} A$$

<sup>4</sup>In the next section, we will modify this by a shift, but this does not affect the conclusions.

where the first term is also  $R\Gamma_{\text{et}}(\text{Spec}(\pi_0 A), \mathbb{F}_p)$ . This follows from the usual Artin–Schreier sequence for classical  $\mathbb{F}_p$ -algebras and the fact that Frobenius acts by zero on the higher homotopy groups of an animated  $\mathbb{F}_p$ -algebra, [BS17] Prop. 11.6 and proof].

**Proposition 5.2.** *Let  $\mathcal{F} \in \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}}$ . Then there exists a (unique)  $M \in \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  and an equivalence  $\pi^* \mathcal{F} \simeq W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1]$ . Moreover,  $M$  is algebraic.*

*Proof.* By Theorem 4.3, the functor  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-)[-1]$  is fully faithful with image a localizing subcategory. It suffices to prove the proposition when  $\mathcal{F} = j_! \mathbb{F}_p$ , for  $j : U \rightarrow \text{Spec}(R)$  an affine, étale map, since these generate  $\mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}}$  as a localizing subcategory. Using Zariski’s main theorem [Sta22] Tag 05K0], we can factor  $j$  as the composite of an open immersion  $U \rightarrow Z$  and a finite morphism  $g : Z \rightarrow \text{Spec}(R)$ . Thus,  $j_!(\mathbb{F}_p)$  is the kernel (or fiber) of the (surjective) map from  $g_*(\mathbb{F}_p)$  to its restriction to a constructible closed subset. It thus suffices to prove the result for  $\mathcal{F}$  the pushforward of the constant sheaf along any finite map to  $\text{Spec}(R)$ .

Let  $f : \text{Spec } B \rightarrow \text{Spec } R$  be a finite map. Then the object  $\pi^*(f_* \mathbb{F}_p)$  carries a perfect  $R$ -algebra  $R'$  to the mod  $p$  étale cohomology of  $R' \otimes_R B$ , or equivalently  $R' \otimes_R^{\mathbb{L}} B_{\text{perf}}$ . Thanks to the Artin–Schreier sequence (in the form of Remark 5.1),  $\pi^*(f_* \mathbb{F}_p)$  can be described as the functor which carries a perfect  $R$ -algebra  $R'$  to the Frobenius fixed points of  $R' \otimes_R^{\mathbb{L}} B_{\text{perf}}$ . In particular, it corresponds (cf. (2)) to the  $W(R)[F^{\pm 1}]$  module  $B_{\text{perf}}$ , which is also algebraic by Proposition 3.10.  $\square$

**Construction 5.3** (The functor  $\text{RH}_{\text{cov}}$ ). By Proposition 5.2 the fully faithful left adjoint functor  $\pi^* : \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}} \rightarrow \mathcal{D}(\text{Sch}_R^{\text{perf}})$  has essential image inside the image of the fully faithful left adjoint functor  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-)[-1]$  (cf. Theorem 4.3), so  $\pi^*$  uniquely factors through a fully faithful left adjoint functor  $\text{RH}_{\text{cov}} : \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}} \rightarrow \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$ , i.e., we have a commutative diagram of fully faithful, left adjoint functors,

$$\begin{array}{ccc} & & \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}} \\ & \nearrow \text{RH}_{\text{cov}} & \downarrow W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} (-)[-1] \\ \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}} & \xrightarrow{\pi^*} & \mathcal{D}(\text{Sch}_R^{\text{perf}}). \end{array}$$

Explicitly, for any  $\mathcal{F} \in \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}}$ , we have by Theorem 4.3,

$$(5) \quad \text{RH}_{\text{cov}}(\mathcal{F}) = \text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W, \pi^* \mathcal{F}[1]).$$

**Remark 5.4** (Comparison with the  $\text{RH}_{\text{cov}}$  of [BL19]). Note that  $\text{RH}_{\text{cov}}$  is a left adjoint, and by the above diagram of left adjoints (with the right vertical arrow fully faithful), we find that  $\text{RH}_{\text{cov}}$  is left adjoint to the functor (referred to as the *solution* functor in [BL19]) which carries  $M \in \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  to  $\pi_*(W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M)[-1] \in \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}}$ . In particular, the construction of  $\text{RH}_{\text{cov}}$  here (in light also of the expression (2) for the tensor product, and Remark 4.2) agrees with that of [BL19] Th. 6.1.1].

Let us now formulate the main result, i.e., the  $p$ -power torsion version of the Riemann–Hilbert correspondence for  $\mathbb{F}_p$ -schemes, cf. [BL19] Th. 9.6.1] for the mod  $p^n$ -version (and on abelian categories); the present formulation of the result is not difficult to deduce from there.

**Theorem 5.5.** *The functor  $\text{RH}_{\text{cov}} : \mathcal{D}(\text{Spec}(R)_{\text{et}})_{p\text{-tors}} \rightarrow \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  is fully faithful,  $t$ -exact, and cocontinuous. The image consists of those  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  which are algebraic.*

By the full faithfulness of  $\pi^*$ , the essential image of  $\mathrm{RH}_{\mathrm{cov}}$  consists of precisely those objects  $M \in \mathcal{D}(W(R)[F^{\pm 1}])_{p\text{-tors}}$  such that  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  belongs to the essential image of  $\pi^*$ . We have already seen (in Proposition 5.2) that any such  $M$  is algebraic, so it remains to prove the converse.

**Proposition 5.6.** *Let  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  be algebraic. Let  $(R', I)$  be a henselian pair of perfect  $R$ -algebras (so  $R'/I$  is also assumed perfect). Then the map  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \rightarrow W(R'/I) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  is an equivalence.*

*Proof.* In view of the generators of the localizing subcategory of algebraic objects of  $\mathcal{D}(W(R)[F^{\pm 1}])$  of Proposition 3.7, we see that it suffices to show that any monic element  $F^n + a_1 F^{n-1} + \cdots + a_n$  acts invertibly on  $I = \ker(W(R') \rightarrow W(R'/I))/p$  by right multiplication. In other words, it suffices to show that the self-map of  $I$  given by

$$(6) \quad x \mapsto x^{1/p^n} + a_1^{1/p^{n-1}} x^{1/p^{n-1}} + \cdots + a_n x$$

is an isomorphism. Since  $I$  is perfect as a nonunital ring, it suffices to precompose by the  $n$ th iterate of Frobenius and to show that the map

$$(7) \quad x \mapsto x + a_1^{1/p^{n-1}} x^p + \cdots + a_n x^{p^n}, \quad I \rightarrow I$$

is an isomorphism. However, this follows from the definition of a henselian pair. In fact, for any  $t \in I$ , the polynomial  $x + a_1^{1/p^{n-1}} x^p + \cdots + a_n x^{p^n} - t \in R'[x]$  has nonvanishing derivative, so any solution in  $R'/I$  (e.g., zero) lifts uniquely to a solution in  $I$ , whence the claim.  $\square$

**Proposition 5.7.** *Let  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  be algebraic. Then  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1] \in \mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}})$  belongs to the essential image of  $\pi^* : \mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}})_{p\text{-tors}} \rightarrow \mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}})$ .*

*Proof.* Thanks to Remark 4.2, the construction  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1]$  as a functor on perfect  $R$ -algebras commutes with filtered colimits. Let  $R_1 \rightarrow R_2$  be a local homomorphism of strictly henselian, perfect  $R$ -algebras. We show that the map

$$W(R_1) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M \rightarrow W(R_2) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$$

is an equivalence, which will suffice, cf. Remark 2.5. Thanks to Proposition 5.6, it suffices to show this when  $R_1, R_2$  are algebraically closed fields. Since the algebraic subcategory of  $\mathcal{D}(W(R)[F^{\pm 1}])$  is generated as a localizing subcategory (cf. Proposition 3.7) by objects of the form  $W(R)[F^{\pm 1}]/(W(R)[F^{\pm 1}]T, p)$  for  $T \in W(R)[F^{\pm 1}]$  of the form  $F^n + a_1 F^{n-1} + \cdots + a_n$ , it suffices to assume  $M$  to be of this form.

In this case,  $W(R_1) \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M$  is given by the cofiber of the map (6) on  $R_1$ . This is clearly zero in  $H^0$  and a finite set of cardinality  $\leq p^n$  in  $H^{-1}$  (the set of solutions of the equation  $x^{1/p^n} + a_1^{1/p^{n-1}} x^{1/p^{n-1}} + \cdots + a_n x = 0$ ), which is unchanged under extensions of algebraically closed fields.  $\square$

*Proof of Theorem 5.5.* We have already seen the fully faithful, cocontinuous functor  $\mathrm{RH}_{\mathrm{cov}}$  in Construction 5.3. The essential image was seen in Proposition 5.2 to be contained inside the algebraic subcategory of  $\mathcal{D}(W(R)[F^{\pm 1}])$ , but Proposition 5.7 now shows that the essential image is precisely the algebraic subcategory.

It only remains to check  $t$ -exactness, i.e., that if an algebraic object  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  is connective (resp. coconnective), then  $W \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1] \in \mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}})$  is connective (resp. coconnective). Here we also use that  $\pi^*$  reflects connectivity (resp. coconnectivity). We carry this out in the following two paragraphs, and implicitly use Remark 4.2

If  $M$  is coconnective and  $M[1/p] = 0$ , then for any *flat*, perfect  $R$ -algebra  $R'$ , we have that  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1]$  is coconnective. In fact, this reduces by taking colimits and extensions to the case where  $M$  is discrete and annihilated by  $p$ , whence the result follows from the expression (2). We thus obtain the same coconnectivity claim for any perfect  $R$ -algebra  $R'$ , since we can replace (thanks to Proposition 5.6)  $R'$  with the henselization (along the kernel) of the perfection of a polynomial  $R$ -algebra surjecting onto  $R'$ , which is flat over  $R$ .

If  $M \in \mathcal{D}(W(R)[F^{\pm 1}])$  is connective and algebraic, then for any strictly henselian, perfect  $R$ -algebra  $R'$ , we claim that  $W(R') \otimes_{W(R)[F^{\pm 1}]}^{\mathbb{L}} M[-1]$  is connective. By Proposition 5.6, we may assume that  $R' = k$  is actually an algebraically closed field  $k$ . In light of Proposition 3.7 (and closure under colimits and extensions), we may assume that  $M = W(R)[F^{\pm 1}]/(p, T)$  for  $T = F^n + a_1 F^{n-1} + \cdots + a_n$ ,  $a_1, \dots, a_n \in W(R)$ . But right multiplication by  $T$ , i.e. the map (6) is clearly surjective on  $k$ ; in fact, this reduces to showing that (7) is surjective on  $k$ , which follows because  $k$  is algebraically closed.  $\square$

*Proof of Theorem 1.3.* Note that Theorem 1.3 is stated for an arbitrary  $\mathbb{F}_p$ -algebra  $R$ , not necessarily assumed to be perfect; however, the statement reduces to the perfect case since passage to the perfection does not change the étale site. This reduction follows because the multiplicative subset  $\{F^i\}_{i \geq 0} \subset R[F]$  satisfies the left Ore condition, and the localization is given by  $R_{\text{perf}}[F^{\pm 1}]$ . The restriction functor on derived  $\infty$ -categories  $\mathcal{D}(R_{\text{perf}}[F^{\pm 1}]) \rightarrow \mathcal{D}(R[F])$  is fully faithful, cf. [Lur17, Sec. 7.2.3]. Hence, we assume that  $R$  is perfect, whence the result follows by passing to  $\mathbb{F}_p$ -modules in Theorem 5.5.  $\square$

**Remark 5.8.** The treatment in [BL19, BP09] first proves the correspondence at the level of abelian categories. Here, it is essential to work at the level of derived  $\infty$ -categories throughout, for the use of Morita theory. Of course, in the contravariant approach [EK04], the use of derived  $\infty$ -categories is essential as the functors involved are not  $t$ -exact.

**Remark 5.9.** The equivalence of Theorem 5.5 yields an equivalence on compact objects: one obtains a  $t$ -exact equivalence between the bounded constructible  $p$ -power torsion derived  $\infty$ -category of  $\text{Spec}(R)_{\text{et}}$  and the bounded derived  $\infty$ -category of  $W(R)[F^{\pm 1}]$  consisting of objects with holonomic cohomologies, by Proposition 2.8 and Proposition 3.15.

**Remark 5.10** (Compatibilities). As proved in [BL19], Theorem 1.3 is compatible with pullback, proper pushforward, and tensor products. The first two claims can also be proved analogously using the perfect site: the functor from Frobenius modules to sheaves on the perfect site is compatible with pullback and (when globalized as in [BL19, Sec. 10]) with proper pushforward. However, the compatibility with the symmetric monoidal structure appears less clear from this perspective; for example, the description of  $\text{RH}_{\text{cov}}(\mathcal{F}) = \text{RHom}_{\mathcal{D}(\text{Sch}_R^{\text{perf}})}(W, \pi^* \mathcal{F}[1])$  does not have an evident symmetric monoidal structure.

## 6. THE CONTRAVARIANT RIEMANN–HILBERT CORRESPONDENCE

Let  $R$  be a regular noetherian  $\mathbb{F}_p$ -algebra. In this case, the work of Emerton–Kisin [EK04] gives a contravariant description of the bounded derived  $\infty$ -category of constructible  $\mathbb{F}_p$ -sheaves,  $\mathcal{D}_{\text{cons}}^b(\text{Spec}(R)_{\text{et}}, \mathbb{F}_p)$ , in terms of finitely generated unit Frobenius modules.<sup>5</sup>

<sup>5</sup>The work [EK04] also treats the case of  $\mathbb{Z}/p^n$ -sheaves when a lift of  $R$  to  $\mathbb{Z}/p^n$  is specified, and the correspondence has been extended to certain singular cases in [Ohk18, Sch16]. An analog of this result for arbitrary  $\mathbb{F}_p$ -algebras can



In this section, we give another proof of the contravariant Riemann–Hilbert correspondence of [EK04] (Theorem 1.2, reproduced below as Theorem 6.20). The essential observation is that the subcategory  $\mathcal{D}_{\text{unit}}^{\text{fproj}}(R[F])^{op} \subset \mathcal{D}(R[F])$  consisting of objects of finite projective dimension whose cohomology  $R[F]$ -modules are unit embeds fully faithfully into the derived  $\infty$ -category of sheaves on the *big* étale site of  $\text{Spec}(R)$  (Theorem 6.16).

**6.1. Unit  $R[F]$ -modules.** Let  $R$  be an  $\mathbb{F}_p$ -algebra whose Frobenius  $\varphi : R \rightarrow R$  is flat. In this subsection, we review some basic facts about unit  $R[F]$ -modules.

We recall that an  $R[F]$ -module can be equivalently regarded as an  $R$ -module  $M$  equipped with an  $R$ -linear map  $\varphi^*M \rightarrow M$  (adjoint to  $F : M \rightarrow \varphi_*M$ ); we will abuse notation and often write this map as  $F$  too.

**Definition 6.1** (Unit  $R[F]$ -modules). An  $R[F]$ -module  $M$  is *unit* if the map  $F : \varphi^*M \rightarrow M$  of  $R$ -modules is an isomorphism.

Since  $\varphi^*$  is exact by assumption, the collection of unit  $R[F]$ -modules is closed under all colimits, finite limits, and extensions inside the category of all  $R[F]$ -modules; in particular, it is an abelian subcategory.

When  $R$  is regular noetherian,<sup>6</sup> the abelian category of unit  $R[F]$ -modules is extensively studied by Lyubeznik, [Lyu97], under the name *F-modules*.

**Remark 6.2** (Unitality via the perfection). An  $R[F]$ -module  $M$  is unit if and only if the base-change  $R_{\text{perf}} \otimes_R M$ , considered as an  $R_{\text{perf}}[F]$ -module, has  $F$  acting invertibly. This follows because the condition that  $\varphi^*M \rightarrow M$  is an isomorphism can be checked after faithfully flat base-change, and for a perfect  $\mathbb{F}_p$ -algebra is equivalent to  $F$  acting invertibly.

**Remark 6.3** (Unit  $R[F]$ -modules as modules over a ring). Suppose  $R$  is  $F$ -finite regular noetherian. The abelian category of unit  $R[F]$ -modules admits a compact projective generator. In fact, this follows because the forgetful functor from unit  $R[F]$ -modules to  $R$ -modules preserves limits and colimits (because  $\varphi^*$  preserves limits by the  $F$ -finiteness assumption; preservation of colimits always holds), and so admits a left adjoint; the image of  $R$  under this left adjoint is a compact projective generator. Consequently, the category of unit  $R[F]$ -modules is identified with the category of modules over a certain large ring (containing  $R[F]$ ). A description of this ring (at least when  $R$  is smooth over a perfect field) in terms of differential operators on  $R$  appears in [EK04, Prop. 15.1.4] (see also [EK04, Sec. 15.2]).

**Construction 6.4** (Unitalization, cf. [Lyu97, Def. 1.9] or [BL19, Cons. 11.2.2]). Let  $M$  be an  $R$ -module together with a map  $f : M \rightarrow \varphi^*M$  of  $R$ -modules. The colimit

$$M \xrightarrow{f} \varphi^*M \xrightarrow{\varphi^*f} (\varphi^*)^2M \rightarrow \dots$$

naturally has the structure of a unit  $R[F]$ -module  $M^{\text{unit}}$ : in fact, the construction provides an identification of  $M^{\text{unit}}$  and  $\varphi^*M^{\text{unit}}$ . More generally, one can carry out this construction when  $M \in \mathcal{D}(R)$ .

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also be deduced from Theorem 1.3 using a duality argument at the level of Frobenius modules, cf. [BL19, Sec. 12]. We do not treat these extensions here.

<sup>6</sup>Recall that a noetherian  $\mathbb{F}_p$ -algebra is regular if and only if its Frobenius is flat, [Kun69].

**Proposition 6.5.** *Let  $M$  be an  $R$ -module equipped with a map  $f : M \rightarrow \varphi^* M$ . Then the unitalization  $M^{\text{unit}}$  of  $(M, f)$  has the following universal mapping property: given  $N \in \mathcal{D}(R[F])$ , there is an equivalence in  $\mathcal{D}(\mathbb{Z})$ ,*

$$\text{RHom}_{R[F]}(M^{\text{unit}}, N) = \text{eq}(\text{RHom}_R(M, N) \rightrightarrows \text{RHom}_R(M, N))$$

where the first map is the identity and the second map carries  $v : M \rightarrow N$  to the composite  $M \xrightarrow{f} \varphi^* M \xrightarrow{\varphi^* v} \varphi^* N \xrightarrow{F} N$ .

*Proof.* This follows from the presentation of the unitalization given in [BL19, Prop. 11.2.5] upon taking maps in the derived  $\infty$ -category.  $\square$

**Definition 6.6** (Unit objects of  $\mathcal{D}(R[F])$ ). An object of  $\mathcal{D}(R[F])$  is said to be *unit* if all the cohomology  $R[F]$ -modules are unit. We let  $\mathcal{D}_{\text{unit}}(R[F]) \subset \mathcal{D}(R[F])$  denote the subcategory of unit objects.

The subcategory  $\mathcal{D}_{\text{unit}}(R[F]) \subset \mathcal{D}(R[F])$  is closed under colimits and finite limits, since unit  $R[F]$ -modules are closed under all colimits, finite limits, and extensions inside all  $R[F]$ -modules. Moreover,  $\mathcal{D}_{\text{unit}}(R[F])$  is accessible by general accessibility results (e.g., the stability of accessible  $\infty$ -categories under limits, [Lur09, Prop. 5.4.7.3]) and is therefore presentable stable.

Note that  $\mathcal{D}_{\text{unit}}(R[F])$  inherits a natural  $t$ -structure from the inclusion into  $\mathcal{D}(R[F])$ .

**Proposition 6.7.**  *$\mathcal{D}_{\text{unit}}(R[F])^{\leq 0}$  is Grothendieck prestable. Moreover,  $\mathcal{D}_{\text{unit}}(R[F])$  is identified with the derived  $\infty$ -category of its heart (i.e., the abelian category of unit  $R[F]$ -modules).*

*Proof.* Since the  $t$ -structure on  $\mathcal{D}_{\text{unit}}(R[F])$  induced from  $\mathcal{D}(R[F])$  is evidently right-complete and compatible with filtered colimits, we find that  $\mathcal{D}_{\text{unit}}(R[F])^{\leq 0}$  is Grothendieck prestable.

It is clear that  $\mathcal{D}_{\text{unit}}(R[F])^{\leq 0}$  is left separated, so it suffices to show that it is 0-complicial in light of [Lur18, Prop. C.5.4.5]. Let  $M \in \mathcal{D}_{\text{unit}}(R[F])^{\leq 0}$ . We can choose a free  $R$ -module  $P$  together with a map  $g : P \rightarrow M$  in  $\mathcal{D}(R)$  which induces a surjection on  $H^0$ . Moreover, since  $g$  is surjective on  $H^0$  and since  $P$  is free, we can choose a map of  $R$ -modules  $f : P \rightarrow \varphi^* P$  such that the composite  $P \xrightarrow{f} \varphi^* P \xrightarrow{\varphi^* g} \varphi^* M \xrightarrow{F} M$  agrees with  $g$ . Using the universal property of the unitalization, Proposition 6.5, we obtain a map in  $\mathcal{D}(R[F])$  from the unitalization of  $(P, f : P \rightarrow \varphi^* P)$  to  $M$  which necessarily induces a surjection on  $H^0$ . This proves 0-compliciality and thus the result.  $\square$

**Remark 6.8.** If  $R$  is  $F$ -finite regular noetherian, it follows that  $\mathcal{D}_{\text{unit}}(R[F])$  is simply the derived  $\infty$ -category of the ring of Remark 6.3.

As a consequence, we obtain the following result on the category of unit  $R[F]$ -modules; for regular noetherian  $\mathbb{F}_p$ -algebras satisfying  $F$ -finiteness assumptions, this result is due to Ma [Ma14, Th. 1.3] (who also proves the lower bound).

**Corollary 6.9.** *Let  $R$  be an  $\mathbb{F}_p$ -algebra whose Frobenius is flat and which has global dimension  $d$  (e.g., a regular noetherian  $\mathbb{F}_p$ -algebra of Krull dimension  $d$ ). Then the abelian category of unit  $R[F]$ -modules has global dimension  $\leq d + 1$ .*

*Proof.* In light of [BL19, Rem. 3.1.8], the abelian category of  $R[F]$ -modules has global dimension  $\leq d + 1$ . Since Ext-groups in unit  $R[F]$ -modules are computed in all  $R[F]$ -modules thanks to Proposition 6.7, the result follows.  $\square$

In the remainder of the subsection, we assume that  $R$  is a regular noetherian  $\mathbb{F}_p$ -algebra.

**Definition 6.10** (Finitely generated unit  $R[F]$ -modules). An  $R[F]$ -module  $M$  is *finitely generated unit* if it is unit and if it is finitely generated as an  $R[F]$ -module. We let  $\mathcal{D}_{\text{fgu}}^b(R[F]) \subset \mathcal{D}(R[F])$  denote the subcategory of objects which are  $t$ -bounded and whose cohomology  $R[F]$ -modules are finitely generated unit.

**Proposition 6.11.** (1) (Cf. [EK04] Th. 6.1.3] or [BL19] Cor. 11.2.6, 11.2.12]) An  $R[F]$ -module  $M$  is finitely generated unit if and only if there exists a finitely generated  $R$ -module  $N$  with a map  $f : N \rightarrow \varphi^* N$  such that  $M$  arises as the unitalization of  $(N, f)$ .  
 (2) (Cf. [Lyu97] Th. 2.8]) The collection of finitely generated unit  $R[F]$ -modules is closed (inside unit  $R[F]$ -modules) under subobjects, quotients, and extensions.

It follows that  $\mathcal{D}_{\text{fgu}}^b(R[F]) \subset \mathcal{D}(R[F])$  is a thick subcategory.

**Corollary 6.12** (Cf. [BL19] Lem. 11.3.12]). Any object of  $\mathcal{D}_{\text{fgu}}^b(R[F])$  has finite projective dimension over  $R[F]$ .

**6.2. The Emerton–Kisin correspondence.** Let  $R$  be an  $\mathbb{F}_p$ -algebra. We consider the solutions functor, as in [EK04]; however, we do so on the *big* étale site rather than the small étale site. The basic observation is that doing so leads to full faithfulness on a larger subcategory (Theorem 6.16).

**Definition 6.13** (The big étale site). We let  $\text{Sch}_R$  denote the site of all qcqs  $R$ -schemes, equipped with the étale topology, and let  $\mathcal{D}(\text{Sch}_R, \mathbb{F}_p)$  denote the derived  $\infty$ -category of sheaves of  $\mathbb{F}_p$ -vector spaces on  $\text{Sch}_R$ .

Let  $(\lambda^*, \lambda_*)$  denote the pullback and pushforward from the small étale site to the big étale site; note that  $\lambda^*$  is fully faithful, and the essential image of  $\lambda^*$  consists of those objects of  $\mathcal{D}(\text{Sch}_R, \mathbb{F}_p)$  such that on  $R$ -algebras, they commute with filtered colimits and carry local homomorphisms of strictly henselian local  $R$ -algebras to equivalences (analogously to Proposition 2.4 and Remark 2.5).

**Construction 6.14** (The solutions functor). We have a functor

$$\text{Sol} = \text{RHom}_{R[F]}(-, \mathbb{G}_a) : \mathcal{D}(R[F])^{op} \rightarrow \mathcal{D}(\text{Sch}_R, \mathbb{F}_p)$$

which carries  $M \in \mathcal{D}(R[F])$  to the functor that sends an  $R$ -algebra  $R'$  to  $\text{RHom}_{R[F]}(M, R') \in \mathcal{D}(\mathbb{F}_p)$ . This defines a hypercomplete étale sheaf with values in  $\mathcal{D}(\mathbb{F}_p)$  on the category of  $R$ -algebras (since  $R' \mapsto R'$  does), whence an object of  $\mathcal{D}(\text{Sch}_R, \mathbb{F}_p)$ .

**Remark 6.15** (Compatibility of Sol and base-change). The solutions functor Sol carries a free  $R[F]$ -module  $\bigoplus_I R[F]$  to the direct product  $\prod_I \mathbb{G}_a$ . Using this, one concludes that Sol is compatible with base-change *on the subcategory of objects with bounded projective amplitude*: if  $M \in \mathcal{D}(R[F])$  has bounded projective amplitude, and if  $f : \text{Spec}(R') \rightarrow \text{Spec}(R)$  is any map, then  $f^*(\text{Sol}(M)) \simeq \text{Sol}(R' \otimes_R M)$  for  $f^*$  the pullback on sheaves on the big étale site; indeed, this follows because  $f^*$  carries the representable  $\prod_I \mathbb{G}_a$  to  $\prod_I \mathbb{G}_a$ .

In the following, we let  $\mathcal{D}_{\text{unit}}^{\text{fproj}}(R[F]) \subset \mathcal{D}(R[F])$  denote the subcategory spanned by objects which are unit and of bounded projective amplitude. In the case when  $R$  has finite global dimension, this is equivalent to simply being  $t$ -bounded, thanks to [BL19] Rem. 3.1.8].

**Theorem 6.16.** Suppose  $R$  is any  $\mathbb{F}_p$ -algebra whose Frobenius is flat. The restriction of Sol induces a fully faithful embedding  $\mathcal{D}_{\text{unit}}^{\text{fproj}}(R[F])^{op} \rightarrow \mathcal{D}(\text{Sch}_R, \mathbb{F}_p)$  with left inverse given by  $\text{RHom}_{\mathcal{D}(\text{Sch}_R, \mathbb{F}_p)}(-, \mathbb{G}_a)$ . More generally, for any  $M \in \mathcal{D}_{\text{unit}}^{\text{fproj}}(R[F])$ ,  $N \in \mathcal{D}(R[F])$ , the natural map

$$(8) \quad \text{RHom}_{R[F]}(N, M) \rightarrow \text{RHom}_{\mathcal{D}(\text{Sch}_R, \mathbb{F}_p)}(\text{Sol}(M), \text{Sol}(N))$$

is an equivalence.

*Proof.* It suffices to treat the case where  $N = R[F]$  itself, so that  $\mathrm{Sol}(N) = \mathbb{G}_a$ , since both sides carry colimits in  $N$  to limits.

We let  $R_{\mathrm{perf}}$  be the perfection of  $R$ , so that  $R \rightarrow R_{\mathrm{perf}}$  is faithfully flat by our assumptions. Then  $R_{\mathrm{perf}} \otimes_R M = R_{\mathrm{perf}}[F] \otimes_{R[F]} M \in \mathcal{D}(R_{\mathrm{perf}}[F])$ ; since  $M$  was assumed unit, it follows that  $F$  acts invertibly on  $R_{\mathrm{perf}} \otimes_R M$ , i.e., it belongs to the image of the fully faithful restriction functor  $\mathcal{D}(R_{\mathrm{perf}}[F^{\pm 1}]) \rightarrow \mathcal{D}(R_{\mathrm{perf}}[F])$ . The strategy is now to reduce the statement over  $R$  to a statement over  $R_{\mathrm{perf}}$ .

Note that  $\mathrm{Sol}(M)$  belongs to the thick subcategory of  $\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)$  generated by products  $\prod_I \mathbb{G}_a$ ; in fact, this follows because  $M$  has finite projective dimension as an  $R[F]$ -module by assumption. Using the Breen–Deligne resolution, one sees that the map

$$\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)}(\mathrm{Sol}(M), \mathbb{G}_a) \rightarrow \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{R_{\mathrm{perf}}}, \mathbb{F}_p)}(\mathrm{Sol}(R_{\mathrm{perf}} \otimes_R M), \mathbb{G}_a)$$

exhibits the target as the extension of scalars of the source from  $R$  to  $R_{\mathrm{perf}}$  (cf. also Remark 6.15); note here that we regard the source as living in  $\mathcal{D}(R[F])$  and the target in  $\mathcal{D}(R_{\mathrm{perf}}[F])$ , because of the presence of the  $\mathbb{G}_a$ 's. In fact, by a thick subcategory argument, this reduces to the claim that for any set  $I$ , the map

$$(9) \quad \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)}(\prod_I \mathbb{G}_a, \mathbb{G}_a) \rightarrow \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{R_{\mathrm{perf}}}, \mathbb{F}_p)}(\prod_I \mathbb{G}_a, \mathbb{G}_a)$$

exhibits the target as the extension of scalars of the source from  $R$  to  $R_{\mathrm{perf}}$ ; this in turn follows from the Breen–Deligne resolution for  $\prod_I \mathbb{G}_a$ . Alternatively, one observes that the pushforward of  $\mathbb{G}_a$  from  $\mathrm{Sch}_{R_{\mathrm{perf}}}$  to  $\mathrm{Sch}_R$  is  $\mathbb{G}_a \otimes_R R_{\mathrm{perf}}$ ; since  $\prod_I \mathbb{G}_a$  is pseudo-coherent and  $R_{\mathrm{perf}}$  is a filtered colimit of finitely generated free  $R$ -modules by Lazard's theorem, the claim about (9) follows.

Thus, in order to prove that (8) is an equivalence, it suffices to show that the map

$$R_{\mathrm{perf}} \otimes_R M \rightarrow \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{R_{\mathrm{perf}}}, \mathbb{F}_p)}(\mathrm{Sol}(R_{\mathrm{perf}} \otimes_R M), \mathbb{G}_a)$$

is an equivalence. Since  $R_{\mathrm{perf}} \otimes_R M \in \mathcal{D}(R_{\mathrm{perf}}[F^{\pm 1}])$  has bounded projective amplitude (because it has bounded projective amplitude over  $R_{\mathrm{perf}}[F]$  by assumption on  $M$ ), the claim follows from the results of [Bre81] (see Corollary A.16).  $\square$

Before stating and proving Theorem 6.20 below, we need some lemmas about étale sheaves and  $p$ -linear algebra.

**Lemma 6.17.** *Let  $k$  be a separably closed field of characteristic  $p$ . Let  $V$  be a  $k[F]$ -module which is finite-dimensional as a  $k$ -vector space. Then for any extension  $k \subset k'$  of separably closed fields, the map*

$$\mathrm{fib}(F - 1 : V \rightarrow V) \rightarrow \mathrm{fib}(F - 1 : k' \otimes_k V \rightarrow k' \otimes_k V)$$

*is an equivalence.*

*Proof.* By [DK73, Exp. XXII, Prop. 1.2], the operator  $F - 1$  acts surjectively on  $V$  and on  $k' \otimes_k V$ . Thus, it suffices to prove the assertion on  $H^0$ , which follows from [DK73, Exp. XXII, Cor. 1.1.10] (noting also [DK73, Exp. XXII, Eq. (1.0.10)]).  $\square$

**Lemma 6.18.** *Let  $M$  be an  $n$ -by- $n$  matrix with coefficients in the  $\mathbb{F}_p$ -algebra  $R$ . The construction which carries an  $R$ -algebra  $R'$  to the mapping fiber of  $R'^n \xrightarrow{1-M\varphi} R'^n$  carries local homomorphisms of strictly henselian local  $R$ -algebras to equivalences.*

*Proof.* We have already seen that the conclusion of the lemma holds for inclusions of separably closed fields (Lemma 6.17). Thus, it suffices to treat the case of the map from a strictly henselian local  $R$ -algebra  $R'$  to its residue field.

There is a map  $h : \mathbb{A}_R^n \rightarrow \mathbb{A}_R^n$  such that on  $R'$ -points, it is given by the map  $R'^n \xrightarrow{1-M\varphi} R'^n$ ; this map is étale by the Jacobian criterion [Sta22, Tag 02GU]. Now by the lifting property for étaleness, any diagram

$$\begin{array}{ccc} \mathrm{Spec}(R'/\mathfrak{m}) & \longrightarrow & \mathbb{A}_R^n \\ \downarrow & \nearrow & \downarrow h \\ \mathrm{Spec}(R') & \longrightarrow & \mathbb{A}_R^n \end{array}$$

admits a unique dotted arrow that makes the extended diagram commute [Sta22, Tag 08HQ]. Using this, the lemma follows: in fact, for  $\mathfrak{m} \subset R'$  the maximal ideal, we find that the map  $\mathfrak{m}^n \xrightarrow{1-M\varphi} \mathfrak{m}^n$  is an isomorphism.  $\square$

**Lemma 6.19.** *Let  $X$  be any qcqs scheme. Then  $\mathcal{D}_{\mathrm{cons}}^b(X_{\mathrm{et}}, \mathbb{F}_p)$  is generated as a thick subcategory by objects of the form  $f_*(\mathbb{F}_p)$  for  $f : Y \rightarrow X$  a finite, finitely presented morphism.*

*Proof.* Recall that pushforward by finite, finitely presented morphisms preserves constructibility [Sta22, Tag 095R]. Without loss of generality, we may assume (in light of the limit formalism [Sta22, Tag 01ZA]) that  $X$  is of finite type over  $\mathbb{Z}$ , cf. [BM21, Prop. 5.10]. By noetherian induction, we may assume that the lemma is known for any proper closed subscheme of  $X$  and that  $X$  is irreducible (or otherwise we could decompose  $X$  into irreducible components). It thus suffices to show that if  $j : U \subset X$  is the inclusion of an open subset and  $\mathcal{L}$  is a locally constant constructible sheaf on  $U$ , then  $j_!(\mathcal{L})$  belongs to the desired thick subcategory. By the “méthode de la trace” [Sta22, Tag 03SH] and [Sta22, Tag 0A3R], we find that  $j_!(\mathcal{L})$  belongs to the thick subcategory generated by  $h_!(\mathbb{F}_p)$  for  $h : V \rightarrow X$  the composite of a finite étale cover of  $U$  together with the inclusion into  $X$ . By Zariski’s main theorem [Sta22, Tag 02LQ], we can factor  $h$  as the composite of an open immersion  $\tilde{j} : V \rightarrow X'$  together with a finite map  $q : X' \rightarrow X$ . It follows that if  $Z' \subset X'$  is the complementary closed subscheme to  $V \subset X'$  (with the reduced induced structure), then  $h_!(\mathbb{F}_p)$  is the mapping fiber of  $q_*(\mathbb{F}_p) \rightarrow (q|_{Z'})_*(\mathbb{F}_p)$ , whence the result.  $\square$

**Theorem 6.20** ([EK04]). *Suppose  $R$  is a regular noetherian  $\mathbb{F}_p$ -algebra.<sup>7</sup> Then the functor*

$$\mathrm{RH}_{\mathrm{cont}} = \mathrm{RHom}_{\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)}(-, \mathbb{G}_a) : \mathcal{D}_{\mathrm{cons}}^b(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)^{op} \rightarrow \mathcal{D}(R[F])$$

*is fully faithful and has image given precisely by  $\mathcal{D}_{\mathrm{fgu}}^b(R[F])$ .*

*Proof.* Given  $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}^b(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$ , we will show that  $\lambda^*\mathcal{F}$  (with notation as in Definition 6.13) is in the essential image of the fully faithful embedding  $\mathrm{Sol} : \mathcal{D}_{\mathrm{unit}}^{\mathrm{fproj}}(R[F])^{op} \rightarrow \mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)$  of Theorem 6.16; more precisely, there exists  $M \in \mathcal{D}_{\mathrm{fgu}}^b(R[F])$  (of finite projective dimension by Corollary 6.12 and necessarily unique) such that  $\lambda^*\mathcal{F} = \mathrm{Sol}(M)$ . It follows that

$$M = \mathrm{RHom}_{\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)}(\mathcal{F}, \mathbb{G}_a) = \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)}(\lambda^*\mathcal{F}, \mathbb{G}_a),$$

since  $\mathbb{G}_a$  is pushed forward from the big étale site; in particular,  $M = \mathrm{RH}_{\mathrm{cont}}(\mathcal{F})$ , which gives full faithfulness of  $\mathrm{RH}_{\mathrm{cont}}$ , with left inverse given by  $\lambda_*\mathrm{Sol}$ .

<sup>7</sup>In [EK04], the result is stated when  $R$  is smooth over a field; the present extension appears in [BL19], and can also be deduced from the case where  $R$  is smooth over  $\mathbb{F}_p$  by passage to filtered colimits and Popescu’s theorem.

Suppose  $B$  is a finite, finitely presented  $R$ -algebra; let  $g : \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(R)$ . Then  $\lambda^*(g_*(\mathbb{F}_p))$  is given by the functor which sends an  $R$ -algebra  $R'$  to the fiber of  $F - 1$  on  $B \otimes_R^{\mathbb{L}} R'$ , thanks to the Artin–Schreier sequence in the form of Remark 5.1. In light of Proposition 6.5 (and a resolution, using that  $B$  has finite projective dimension [BL19, Lem. 11.3.10]), this is precisely  $\mathrm{Sol}$  of the unitalization of  $\mathrm{RHom}_R(B, R)$  equipped with the linear dual of the Frobenius  $\varphi^* B \rightarrow B$ . Since objects of the form  $g_*(\mathbb{F}_p)$  generate  $\mathcal{D}_{\mathrm{cons}}^b(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$  as a thick subcategory (Lemma 6.19), we conclude that the same holds with  $g_*(\mathbb{F}_p)$  replaced by any object of  $\mathcal{D}_{\mathrm{cons}}^b(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$  as desired.

It remains to identify the essential image of  $\mathrm{RH}_{\mathrm{cont}}$ . Suppose  $M \in \mathcal{D}_{\mathrm{fgu}}^b(R[F])$ . We first show that  $\mathrm{Sol}(M)$  belongs to the image of the pullback  $\lambda^*$  from the small étale site to the big étale site. Without loss of generality, we can suppose that  $M$  is discrete. By assumption and Proposition 6.11,  $M$  arises as the unitalization of some finitely generated  $R$ -module  $M_0$  equipped with a map  $M_0 \rightarrow \varphi^* M_0$ . Since  $M_0$  has finite projective dimension over  $R$  by [BL19, Lem. 11.3.10], we find easily (by taking resolutions of  $M_0, M_0 \rightarrow \varphi^* M_0$ ) that  $M$  belongs to the thick subcategory of  $\mathcal{D}_{\mathrm{fgu}}^b(R[F])$  generated by objects that arise as the unitalization of a finitely generated *free*  $R$ -module  $M_0$  with a map  $M_0 \rightarrow \varphi^* M_0$ . We may therefore assume that  $M$  itself is obtained as a unitalization of  $M_0 \rightarrow \varphi^* M_0$  with  $M_0$  free. Thus,  $M_0^\vee$  is an  $R[F]$ -module which is finitely generated free as an  $R$ -module. Using the universal property of the unitalization again (Proposition 6.5), we find that  $\mathrm{Sol}(M_0^{\mathrm{unit}})$  is the functor carrying an  $R$ -algebra  $R'$  to the fiber of  $F - 1$  on  $R' \otimes_R M_0^\vee$ . This functor evidently commutes with filtered colimits in  $R'$ . The desired rigidity statement (needed to see that  $\mathrm{Sol}(M)$  belongs to the image of  $\lambda^*$  as in Definition 6.13) now follows from Lemma 6.18. Therefore, we conclude  $\mathrm{Sol}(M)$  belongs to the image of the pullback  $\lambda^*$  from the small étale site to the big étale site.

Finally, we need to show that the preimage of  $\mathrm{Sol}(M)$  under  $\lambda^*$  actually belongs to the bounded constructible  $\infty$ -category (and not simply to  $\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$ ). Boundedness is evident, since  $\mathrm{Sol}(M)$  is bounded as  $M$  has finite projective dimension over  $R[F]$  (Corollary 6.12). Moreover, constructibility follows because  $\mathrm{Sol}(M)$  (as a sheaf on  $\mathrm{Sch}_R$ ) belongs to the thick subcategory generated by  $\prod_I \mathbb{G}_a$ , whence is pseudo-coherent in  $\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)$ . This implies that the preimage of  $\mathrm{Sol}(M)$  is compact in any truncation of  $\mathcal{D}(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{F}_p)$ , whence the cohomology sheaves are constructible (cf. Proposition 2.8 and the references in the proof).  $\square$

Note that the proof also establishes that for  $M \in \mathcal{D}_{\mathrm{fgu}}^b(R[F])$ , we have a natural equivalence

$\mathrm{RH}_{\mathrm{cont}}(\lambda_*(\mathrm{Sol}(M))) \simeq \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)}(\lambda^* \lambda_* \mathrm{Sol}(M), \mathbb{G}_a) = \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)}(\mathrm{Sol}(M), \mathbb{G}_a) \simeq M$ , in light of Theorem 6.16 (and as proved in [EK04]).

#### APPENDIX A. A PROOF OF BREEN’S THEOREM

In the appendix, we describe a proof of the theorem of [Bre81] that is independent of explicit computations in algebraic topology (cf. [FLS94, Lem. 0.3] and [Kuh95, Cor. 1.2] for other algebraic proofs). The essential idea of the proof (the use of  $v$ -descent results of [BS17]) was suggested by Scholze. Let us first recall the statement.

**Theorem A.1** (Breen [Bre81]). *Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra. Then  $\mathrm{Ext}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)}^i(\mathbb{G}_a, \mathbb{G}_a) = 0$  for  $i > 0$  and is given by  $R[F^{\pm 1}]$  for  $i = 0$ .*

The strategy is as follows. It suffices to replace the source  $\mathbb{G}_a$  by the big Witt vectors,  $W^{\mathrm{big}}$  and to compute the Ext groups in  $\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}})$ . In fact, this follows from the equivalence  $\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}})}(W, \mathbb{G}_a) = \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)}(\mathbb{G}_a, \mathbb{G}_a)$  and the fact that  $W$  is a retract of  $W^{\mathrm{big}}$ . It will be convenient to use



big Witt vectors for the following reason:  $W^{\text{big}}$  can be approximated by the subsheaf  $W^{\text{rat}}$  of *rational* Witt vectors, for which we can do the calculation explicitly since, in the  $v$ -topology,  $W^{\text{rat}}$  is essentially the free abelian group on  $\mathbb{A}^1$  (with a basepoint at zero). This approach does not recover the more general calculation of extensions of the additive group on all (not necessarily perfect)  $\mathbb{F}_p$ -schemes of [Bre78].

Generalizations of Theorem A.1 have also been obtained in the relatively perfect site by Kato [Kat86] and by Anschütz–Le Bras [ACLB21] in analytic (perfectoid) settings.

**A.1. Comparing rational and big Witt vectors.** Fix a base field  $k$  (which will later be taken to be  $\mathbb{F}_p$ ). Let  $\text{Sch}_k$  be the site of all (qcqs)  $k$ -schemes, equipped with the étale topology. In this subsection, we compare the sheaves of rational Witt vectors and big Witt vectors on  $\text{Sch}_k$ .

**Definition A.2** (Ind-affine schemes). Let  $\text{AffSch}_{k,\text{ft}}^*$  denote the category of pointed affine schemes of finite type over  $k$  (i.e., the opposite to the category of finite type augmented  $k$ -algebras). We consider the ind-category  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$ . Now  $\text{AffSch}_{k,\text{ft}}^*$  admits finite coproducts and finite products, and finite products distribute over finite coproducts. It follows that  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$  admits filtered colimits (and arbitrary coproducts) and finite products, and finite products distribute over filtered colimits (and arbitrary coproducts).

We have a natural functor

$$(10) \quad \text{Ind}(\text{AffSch}_{k,\text{ft}}^*) \rightarrow \text{Shv}(\text{Sch}_k, \text{Set})$$

carrying the class of  $X \in \text{AffSch}_{k,\text{ft}}^*$  to the representable sheaf and then is defined by Kan extension; this construction preserves filtered colimits and finite products. Typically, we will abuse notation and refer to an object of  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$  and the corresponding sheaf on  $\text{Sch}_k$  by the same notation.

**Construction A.3** (Commutative monoids in ind-affine schemes). We consider commutative monoids in  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$ . Note that any such yields a commutative monoid in  $\text{Shv}(\text{Sch}_k, \text{Set})$ , since the functor of (10) preserves finite products.

Given  $X \in \text{AffSch}_{k,\text{ft}}^*$ , we can form the free commutative monoid on  $X$  in the category  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$  (note that this forces the basepoint to be the monoid unit); this is the filtered colimit  $\text{free}_*(X) \stackrel{\text{def}}{=} \varinjlim_i \text{Sym}^i X$  where the inclusion maps are induced by  $*$  (and the colimit is taken in the ind-sense).

Now we specialize to the example of interest.

**Construction A.4** (The free commutative monoid on  $\mathbb{A}_k^1$ ). We let  $\mathbb{A}_k^1$  denote the affine line over  $k$  with basepoint at origin. The  $i$ th symmetric power  $\text{Sym}^i(\mathbb{A}_k^1)$ ,  $i \geq 0$  is given as  $\text{Spec } k[s_1, \dots, s_i]$ , for  $s_1, \dots, s_i$  the elementary symmetric functions. Using the basepoint at the origin, we can form the object of  $\text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$  given as  $W_{\text{rat}}^+ = \text{free}_*(\mathbb{A}_k^1) = \varinjlim_i \text{Sym}^i(\mathbb{A}_k^1) \in \text{Ind}(\text{AffSch}_{k,\text{ft}}^*)$ .

Unwinding the definitions and using the theorem on symmetric functions, we see that for any  $k$ -algebra  $R$ , the commutative monoid  $W_{\text{rat}}^+(R)$  is  $1 + tR[t]$  under multiplication.<sup>8</sup> Given a ring  $R$ , the group completion of  $W_{\text{rat}}^+(R)$  is known as the *rational Witt vectors* of  $R$ , cf. [Alm78, Alm73], and the notation  $W_{\text{rat}}^+$  is chosen for this reason. In the following, we let  $W_{\text{rat}}$  denote the group completion of  $W_{\text{rat}}^+$  as sheaves on  $\text{Sch}_k$  (so  $W_{\text{rat}}$  is the étale sheafification of the rational Witt vectors).

<sup>8</sup>A variant of this fact is used essentially in the Cartier–Dieudonné theory of commutative formal groups, cf. [Cha04, Th. 2.2].

**Construction A.5** (Big Witt vectors). We let  $W^{\text{big}}$  denote the big Witt vector functor (cf. [Haz12, Sec. 17] for an account); it carries a commutative ring  $R$  to the abelian group  $1 + tR[[t]]$  under multiplication, and is representable by a polynomial ring on countably many variables. We have a natural map  $W_{\text{rat}}^+ \rightarrow W^{\text{big}}$  including polynomials inside power series.

The purpose of this subsection is to prove the following result to the effect that maps into  $\mathbb{G}_a$  (or any direct sum of copies of such) does not see the difference between  $W_{\text{rat}}$  and  $W^{\text{big}}$ ; the argument was inspired by the Clausen–Scholze solid formalism [Sch].

**Proposition A.6.** *The map  $W_{\text{rat}} \rightarrow W^{\text{big}}$  in  $\mathcal{D}(\text{Sch}_k)$  induces an equivalence upon applying  $\text{RHom}_{\mathcal{D}(\text{Sch}_k)}(-, \mathbb{G}_a \otimes_k V)$  for any  $k$ -vector space  $V$ .*

**Remark A.7** (Tensors with pointed sets). In the following, we will use that any pointed category with finite coproducts (such as the categories  $\text{Ind}(\text{AffSch}_{k, \text{ft}}^*)$  or commutative monoids in it, or the category of sheaves of abelian monoids on a site) is naturally tensored over the category of finite pointed sets; we denote this tensor by  $\otimes$ . Explicitly, given a pointed set and an object  $X$ , the tensor  $S \otimes X$  is the pushout of the coproduct  $\bigsqcup_S X$  along  $X \rightarrow *$  (where  $X$  includes in  $\bigsqcup_S X$  via the basepoint of  $S$ )<sup>9</sup>

In the next result, we identify an object of  $\text{Ind}(\text{AffSch}_{k, \text{ft}}^*)$  with the corresponding sheaf on  $\text{Sch}_k$ , so that we can talk about the cohomology of a sheaf on an object of  $\text{Ind}(\text{AffSch}_{k, \text{ft}}^*)$ .

**Proposition A.8.** *Let  $A$  be an augmented  $k$ -algebra of finite type, so  $\text{Spec}(A) \in \text{AffSch}_{k, \text{ft}}^*$ . Let  $I_A$  denote the augmentation ideal, and let  $V$  be any  $k$ -vector space.*

*For any pointed finite set  $S$ , there is a natural isomorphism*

$$(11) \quad R\Gamma(S \otimes \text{free}_*(\text{Spec}(A)), \mathbb{G}_a \otimes_k V) \simeq \prod_{i \geq 0} \Gamma^i(\text{Hom}_k(k[\overline{S}], I_A)) \otimes_k V,$$

for  $k[\overline{S}]$  the cokernel of the map  $k \rightarrow k[S]$  arising from the basepoint and  $\Gamma^i$  denoting the  $i$ th divided power functor.

*Proof.* By the universal property, the commutative monoid  $S \otimes \text{free}_*(\text{Spec}(A))$  can equivalently be described as  $\text{free}_*(S \otimes \text{Spec}(A))$ . Now  $S \otimes \text{Spec}(A)$ , the tensor of  $S$  with  $\text{Spec}(A)$  in the category of pointed affine finite type  $k$ -schemes, is exactly the spectrum of the limit of the diagram

$$\begin{array}{ccc} & & k \\ & & \downarrow \\ & A^S & \longrightarrow A \\ & \downarrow & \\ k & \longrightarrow & k^S \end{array}$$

whose augmentation ideal is exactly  $\text{Hom}_k(k[\overline{S}], I_A)$ . It follows (cf. Construction [A.3]) that  $S \otimes \text{free}_*(\text{Spec}(A))$  is computed in  $\text{Ind}(\text{AffSch}_{k, \text{ft}}^*)$  as the filtered colimit of  $\text{Sym}^i(\text{Spec}(k \oplus \text{Hom}_k(k[\overline{S}], I_A)))$ , whence the result.  $\square$

<sup>9</sup>Note that if  $S = S_0 \sqcup *$ , then  $S \otimes X = \bigsqcup_{S_0} X$ ; thus the above construction only requires finite coproducts and not pushouts.

In the next results, we let  $x$  denote the coordinate on  $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$ , and let  $k[x]^+$  denote the augmentation ideal  $(x) \subset k[x]$ . The following result is a consequence of Proposition A.8 for the pointed affine line.

**Proposition A.9.** *For any pointed finite set  $S$  and  $k$ -vector space  $V$ , there is a natural isomorphism (with notation as in Proposition A.8)*

$$(12) \quad R\Gamma(S \otimes W_{\operatorname{rat}}^+, \mathbb{G}_a \otimes_k V) \simeq \prod_{i \geq 0} \Gamma^i(\operatorname{Hom}_k(k[\overline{S}], k[x]^+)) \otimes_k V.$$

**Proposition A.10.** *For any pointed finite set  $S$ , and  $k$ -vector space  $V$ , there is a natural isomorphism (with notation as in Proposition A.8)*

$$(13) \quad R\Gamma(S \otimes W^{\operatorname{big}}, \mathbb{G}_a \otimes_k V) = \bigoplus_{i \geq 0} \Gamma^i(\operatorname{Hom}_k(k[\overline{S}], k[x]^+)) \otimes_k V.$$

With this identification, the map  $S \otimes W^{\operatorname{rat}} \rightarrow S \otimes W^{\operatorname{big}}$  induces the obvious sum-to-product map on  $R\Gamma(-, \mathbb{G}_a \otimes_k V)$ .

*Proof.* Since  $S \otimes W^{\operatorname{big}}$  is affine, we may assume  $V = k$  throughout. The map in question is the map

$$\varinjlim_n \operatorname{Sym}^n(S \otimes \mathbb{A}_k^1) \rightarrow S \otimes W^{\operatorname{big}}.$$

Note that both sides carry  $\mathbb{G}_m$ -actions. We claim that the map  $\operatorname{Sym}^n(S \otimes \mathbb{A}_k^1) \rightarrow S \otimes W^{\operatorname{big}}$  (of affine schemes) induces an isomorphism in weights  $\leq n$  on functions, i.e.,  $R\Gamma(-, \mathbb{G}_a)$ . It follows that functions on  $S \otimes W^{\operatorname{big}}$  have a natural grading, and are the inverse limit (over  $n$ ) in graded vector spaces of functions on  $\operatorname{Sym}^n(S \otimes \mathbb{A}_k^1)$ . From this, the result easily follows. To see the claim, we may reduce to the case where  $S = \{*, 1\}$  (via taking tensor products), for which we need to see that  $\operatorname{Sym}^n(\mathbb{A}_k^1) \rightarrow W^{\operatorname{big}}$  induces an isomorphism in weights  $\leq n$  on functions. This is evident given the explicit description of functions on  $W^{\operatorname{big}}$  as the polynomial ring in all symmetric functions; alternatively,  $\operatorname{Sym}^n(\mathbb{A}_k^1) \rightarrow W^{\operatorname{big}}$  is the natural transformation that sends  $R$  to the map  $(1 + tR[t])_{\leq n} \rightarrow 1 + tR[[t]]$ .  $\square$

Given a pointed simplicial set  $S_\bullet$ , we can (by taking geometric realizations) consider  $S_\bullet \otimes W_{\operatorname{rat}}^+, S_\bullet \otimes W^{\operatorname{big}}$  as sheaves of animated commutative monoids.

**Proposition A.11.** *For any simplicial pointed finite set  $S_\bullet$  which is connected and any  $k$ -vector space  $V$ , the map  $S_\bullet \otimes W_{\operatorname{rat}}^+ \rightarrow S_\bullet \otimes W^{\operatorname{big}}$  induces an equivalence on  $R\Gamma(-, \mathbb{G}_a \otimes_k V)$ .*

*Proof.* Recall that both infinite direct sums and direct products commute with totalizations in the coconnective derived  $\infty$ -category. Therefore, we find that the equations (12) and (13) hold for  $R\Gamma(S_\bullet \otimes W_{\operatorname{rat}}^+, \mathbb{G}_a \otimes_k V)$ ,  $R\Gamma(S_\bullet \otimes W^{\operatorname{big}}, \mathbb{G}_a \otimes_k V)$  with  $k[\overline{S}]$  replaced by reduced singular chains on  $S_\bullet$ , i.e.,  $\overline{C}_*(S_\bullet; k)$  (and with the derived functors of the divided powers). Now the result follows because if  $U \in \mathcal{D}(k)^{\geq 1}$ , then  $\Gamma^i U \in \mathcal{D}(k)^{\geq i}$ ; for example, this follows by duality (and passage to filtered colimits) for the corresponding connectivity assertion for nonabelian derived symmetric powers in the connective case, cf. [Lur18, Prop. 25.2.4.1]. Therefore, because of the connectivity bounds, the map from the direct sum to the direct product (i.e., from the analogs of (13) and (12)) is an equivalence.  $\square$

*Proof of Proposition A.6.* We use throughout that  $BW_{\operatorname{rat}}^+ = BW_{\operatorname{rat}}$  since  $W_{\operatorname{rat}}$  is the group completion of the commutative monoid  $W_{\operatorname{rat}}^+$ ; this is a special case of the group-completion theorem [MS76]. From Proposition A.11 (applied to a simplicial model of a wedge of circles), it follows that for any

finite pointed set  $T$ , the map of classifying stacks  $B(T \otimes W_{\text{rat}}^+) \rightarrow B(T \otimes W^{\text{big}})$  induces an equivalence on  $R\Gamma(-, \mathbb{G}_a \otimes_k V)$ . Note that both  $B(T \otimes W_{\text{rat}}^+) \rightarrow B(T \otimes W^{\text{big}})$  are abelian group objects. By the functorial Breen–Deligne resolution [Sch1, Lec. 4], there is a resolution of  $W_{\text{rat}}[1]$  in  $\mathcal{D}(\text{Sch}_k)$  all of whose terms are finite direct sums of the form  $\mathbb{Z}[B(T \otimes W_{\text{rat}}^+)]$  for various finite pointed sets  $T$ , and an analogous resolution of  $W^{\text{big}}[1]$  all of whose terms are of the form  $\mathbb{Z}[B(T \otimes W^{\text{big}})]$  (with the same finite sets and finite direct sums appearing). Since  $\text{RHom}_{\mathcal{D}(\text{Sch}_k)}(-, \mathbb{G}_a \otimes_k V)$  carries the map  $\mathbb{Z}[B(T \otimes W_{\text{rat}}^+)] \rightarrow \mathbb{Z}[B(T \otimes W^{\text{big}})]$  to an equivalence by Proposition A.11, the result follows.  $\square$

## A.2. Proof of Breen’s theorem.

**Construction A.12** (The  $v$ -site of perfect schemes). Let  $\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}$  denote the site of qcqs perfect  $\mathbb{F}_p$ -schemes, equipped with the  $v$ -topology (or universally subtrusive topology) of [Ryd10, BS17]. Note that this is a subcanonical topology since we are working with perfect schemes, cf. [BM21, Th. 5.17]. In particular, the additive  $\mathbb{G}_a$  is a sheaf on  $\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}$ , with trivial higher cohomology on affines, cf. [BS17, Th. 11.2(2)].

We have a morphism of sites  $\eta : \text{Sch}_{\mathbb{F}_p}^{\text{perf}, v} \rightarrow \text{Sch}_{\mathbb{F}_p}$  arising from the inverse limit perfection functor  $(-)^{\text{perf}} : \text{Sch}_{\mathbb{F}_p} \rightarrow \text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}$ , inducing a geometric morphism of topoi.

**Remark A.13** (Examples of  $\eta$ -pullbacks). First,  $\eta^* W^{\text{big}}$  is simply (by pulling back the representing object) the big Witt vector functor on  $\text{Sch}_{\mathbb{F}_p}^{\text{perf}}$ , which is already a  $v$ -sheaf. Second, note that  $\eta^* W_{\text{rat}}^+$  is the functor carrying a perfect  $\mathbb{F}_p$ -algebra  $R$  to  $1 + tR[t]$ , considered as a monoid under multiplication; this is already a  $v$ -sheaf (since as a set it is a filtered colimit of finite products of copies of  $R$ ).

The next result, which is closely related to [SV96, Lem. 5.16], states that quotients of affine schemes by finite group actions are actually quotients in the  $v$ -topology.

**Proposition A.14.** *Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra with an action of a finite group  $G$ ; let  $Y = \text{Spec}(R)$  and  $X = \text{Spec}(R^G)$ . Then the map  $Y \rightarrow X$  exhibits, on representables  $h_Y \rightarrow h_X$ , the target as the quotient by  $G$  in the category of sheaves of sets on  $\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}$ .*

*Proof.* Since  $R^G \subset R$  is an integral extension, the map of schemes  $Y \rightarrow X$  is integral and surjective, hence a  $v$ -cover [Ryd10, Rem. 2.5]. It follows that the map  $h_Y/G \rightarrow h_X$  (of sheaves of sets) is surjective. To see that  $h_Y/G \rightarrow h_X$  is an isomorphism, it suffices to show that the map  $G \times Y \rightarrow Y \times_X Y$  is a  $v$ -cover, but this follows because it is also a surjective integral map by [Bon98, Th. 2, Sec. V.2.2].  $\square$

**Proposition A.15.** *The object  $\eta^* W_{\text{rat}}^+ \in \text{Shv}(\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}, \text{CMon})$  is the free commutative monoid on the pointed object  $h_{\mathbb{A}^1, \text{perf}}$  (with basepoint at the origin). That is, for any sheaf of commutative monoids  $M$  on  $\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}$ , we have*

$$\text{Hom}_{\text{Shv}(\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}, \text{CMon})}(\eta^* W_{\text{rat}}^+, M) \simeq \text{Hom}_{\text{Shv}(\text{Sch}_{\mathbb{F}_p}^{\text{perf}, v}, \text{Set}_*)}(h_{\mathbb{A}^1, \text{perf}}, M).$$

*Proof.* The  $v$ -sheaf  $\eta^* W_{\text{rat}}^+$  is the filtered colimit  $\varinjlim h_{\text{Sym}^i(\mathbb{A}_{\mathbb{F}_p}^1, \text{perf})}$ . However, these symmetric powers can be either taken at the level of schemes or at the level of  $v$ -sheaves of sets thanks to Proposition A.14, whence the result since the latter construction gives the free commutative monoid sheaf on  $h_{\mathbb{A}_{\mathbb{F}_p}^1, \text{perf}}$ .  $\square$

*Proof of Theorem A.1.* The claim for  $i = 0$  is straightforward (cf. [Bre81, §1.4]) to deduce from the claim about endomorphisms of  $\mathbb{G}_a$  on all  $\mathbb{F}_p$ -algebras, so we treat the vanishing for  $i > 0$ . The derived pushforward of  $\mathbb{G}_a$  from the  $v$ -site to the étale site on perfect schemes is simply  $\mathbb{G}_a$  thanks to [BS17, Th. 11.2(2)]. Thus, by adjunction, it suffices to show that  $\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{perf},v})}(\mathbb{G}_a, \mathbb{G}_a)$  is concentrated in cohomological degree zero. We will show that  $\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{perf},v})}(W^{\mathrm{big}}, \mathbb{G}_a)$  is concentrated in degree zero. This will suffice to prove the result, because  $W$  is a retract of  $W^{\mathrm{big}}$  and  $\mathbb{G}_a = W/p$  on  $\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{perf},v}$ .

By Proposition A.15 and group completion, we find that  $\eta^* W_{\mathrm{rat}}(-)$  is the free abelian group sheaf on  $\mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{perf}}$  with the origin as zero. Thus, also using [BS17, Th. 11.2(2)] again, we have that

$$\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{perf},v})}(\eta^* W_{\mathrm{rat}}, \mathbb{G}_a) = \mathrm{fib} \left( R\Gamma_v(\mathbb{A}_{\mathbb{F}_p}^{1,\mathrm{perf}}, \mathbb{G}_a) \rightarrow \mathbb{F}_p \right) = \left\{ f \in \mathbb{F}_p[x^{1/p^\infty}], f(0) = 0 \right\}$$

is discrete. Thus, it suffices to show that the map  $\eta^* W_{\mathrm{rat}} \rightarrow \eta^* W^{\mathrm{big}} = W^{\mathrm{big}}$  induces an equivalence on  $\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{perf},v})}(-, \mathbb{G}_a)$ . But this follows because the map  $W_{\mathrm{rat}} \rightarrow W^{\mathrm{big}}$  induces an equivalence on  $\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p})}(-, \eta_* \mathbb{G}_a)$ , noting that  $\eta_* \mathbb{G}_a = \varinjlim_{\varphi} \mathbb{G}_a$  (again by [BS17, Th. 11.2(2)]) and using Proposition A.6 (applied with  $V$  a countably-dimensional  $\mathbb{F}_p$ -vector space, since  $\varinjlim_{\varphi} \mathbb{G}_a$  is a mapping cone of an endomorphism of  $\bigoplus_{i=0}^{\infty} \mathbb{G}_a$ ).  $\square$

We also record an equivalent form of Theorem A.1, where the extension groups are taken in the big étale site (rather than the perfect site). For this, we consider the inverse limit perfection  $\varprojlim_{\varphi} \mathbb{G}_a$  in  $\mathcal{D}(\mathrm{Sch}_{\mathbb{F}_p})$  and calculate maps into  $\mathbb{G}_a$ .

**Corollary A.16.** *Let  $R$  be a perfect  $\mathbb{F}_p$ -algebra and let  $I$  be any set. We have*

$$\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)} \left( \prod_I \varprojlim_{\varphi} \mathbb{G}_a, \mathbb{G}_a \right) = \bigoplus_I R[F^{\pm 1}],$$

where the products are calculated in  $\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)$ .

*Proof.* We may replace the étale topology on  $\mathrm{Sch}_R$  with the flat topology since  $\mathbb{G}_a$  satisfies flat descent; then  $\prod_I \varprojlim_{\varphi} \mathbb{G}_a$  becomes a discrete object (representable by a product of copies of the perfection of  $\mathbb{G}_a$ ). Using the Breen–Deligne resolution [Sch, Lec. 4] again in the flat site, one sees that

$$\mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R, \mathbb{F}_p)} \left( \prod_I \varprojlim_{\varphi} \mathbb{G}_a, \mathbb{G}_a \right) = \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)} \left( \prod_I \mathbb{G}_a, \mathbb{G}_a \right) = \bigoplus_I \mathrm{RHom}_{\mathcal{D}(\mathrm{Sch}_R^{\mathrm{perf}}, \mathbb{F}_p)}(\mathbb{G}_a, \mathbb{G}_a),$$

since (for the second claim)  $\mathbb{G}_a$ -cohomology carries cofiltered limits of qcqs schemes along affine transition maps to filtered colimits.  $\square$

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